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On the Gap Conjecture concerning group growth

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Abstract We discuss some new results concerning Gap Conjecture on group growth and present a reduction of it (and its ∗-version) to several special classes of groups. Namely we show that its validity for the classes of simple groups and residually finite groups will imply the Gap Conjecture in full generality. A similar type reduction holds if the Conjecture is valid for residually polycyclic groups and just-infinite groups. The cases of residually solvable groups and right orderable groups are considered as well.

1 Introduction

Growth functions of finitely generated groups were introduced by Schvarz [37] and independently by Milnor [29], and remain popular subject of geometric group theory. Growth of a finitely generated group can be polynomial, exponential or intermediate between polynomial and exponential. The class of groups of polynomial growth coincides with the class of virtually nilpotent groups as was conjectured by Milnor and confirmed by Gromov [24]. Milnor's problem on the existence of groups of intermediate growth was solved by the author in [12,13], where for any prime p an uncountable family of 2-generated torsion p-groups $\mathcal{G}_\omega^{(p)}$ with different types of intermediate growth was constructed. Here ω is a parameter of construction taking values in the space of infinite sequences over the alphabet on $p+1$ letters. All groups $\mathcal{G}_\omega^{(p)}$ satisfy the following lower bound on growth function

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$$\gamma_{\mathcal{G}_\omega}(n) \succeq e^{\sqrt{n}}, \tag{1.1}$$

where $\gamma_G(n)$ denotes the growth function of a group G and $\succeq$ is a natural comparison of growth functions (see the next section for definition). The inequality (1.1) just indicates that growth of a group is not less than the growth of the function $e^{\sqrt{n}}$.

All groups from families $\mathcal{G}_\omega^{(p)}$ are residually finite-p groups (i.e. are approximated by finite p-groups). In [15] the author proved that the lower bound (1.1) is universal for all residually finite-p groups and this fact has a straightforward generalization to residually nilpotent groups, as it is indicated in [28].

The paper [13] also contains an example of a torsion free group of intermediate growth, which happened to be right orderable group, as was shown in [19]. For this group the lower bound (1.1) also holds.

In the ICM Kyoto paper [23] the author raised a question if the function $e^{\sqrt{n}}$ gives a universal lower bound for all groups of intermediate growth. Moreover, later he conjectured that indeed this is the case. The corresponding conjecture is now called the *Gap Conjecture* on group growth. In this note we collect known facts related to the Conjecture and present some new results. A recent paper [22] gives further information about the history and developments around the notion of growth in group theory.

The first part of the note is introductory. The second part begins with the case of residually solvable groups where basically we present some of results of Wilson from [40,42] and a consequence from them. Then we consider the case of right orderable groups, and the final part contains two reductions of the Conjecture (and its $*$-version) to the classes of residually finite groups and simple groups (Theorem 7.4), and to the class of just-infinite groups, modulo its correctness for residually polycyclic groups (Theorem 7.3).

2 Preliminary facts

Let G be a finitely generated group with a system of generators $A = \{a_1, a_2, \dots, a_m\}$ (throughout the paper we consider only infinite finitely generated groups and only finite systems of generators). The *length* $|g| = |g|_A$ of an element $g \in G$ with respect to A is the length n of the shortest presentation of g in the form

$$g = a_{i_1}^{\pm 1} a_{i_2}^{\pm 1} \cdots a_{i_n}^{\pm 1},$$

where a_{i_j} are elements in A. It depends on the set of generators, but for any two systems of generators A and B there is a constant $C \in \mathbb{N}$ such that the inequalities

$$|g|_A \le C|g|_B, \qquad |g|_B \le C|g|_A. \tag{2.1}$$

hold.

The *growth function* of a group G with respect to the generating set A is the function

$$\gamma_G^A(n) = \big|\{g \in G : |g|_A \le n\}\big|,$$

where $|E|$ denotes the cardinality of a set E, and n is a natural number.

If $\Gamma = \Gamma(G, A)$ is the Cayley graph of a group G with respect to the generating set A, then $|g|$ is the combinatorial distance between vertices g and e (the identity element in G), and $\gamma_G^A(n)$ counts the number of vertices at combinatorial distance $\le n$ from e (i.e., it counts the number of elements in the ball of radius n with center at the identity element).

It follows from (2.1) that growth functions $\gamma_G^A(n)$, $\gamma_G^B(n)$ satisfy the inequalities

$$\gamma_G^A(n) \le \gamma_G^B(Cn), \qquad \gamma_G^B(n) \le \gamma_G^A(Cn). \tag{2.2}$$

The dependence of the growth function on generating set is inconvenience and it is customary to avoid it by using the following trick. Two functions on the naturals γ_1 and γ_2 are called *equivalent* (written $\gamma_1 \sim \gamma_2$) if there is a constant $C \in \mathbb{N}$ such that $\gamma_1(n) \le C\gamma_2(Cn)$, $\gamma_2(n) \le C\gamma_1(Cn)$ for all $n \ge 1$. Then according to (2.2), the growth functions constructed with respect to two different systems of generators are equivalent. The class of equivalence $[\gamma_G^A]$ of growth function is called *degree of growth*, or *rate of growth* of G. It is an invariant not only up to isomorphism but also up to weaker equivalence relation called *quasi-isometry* [8].

We will also consider a preoder $\preceq$ on the set of growth functions:

$$\gamma_1(n) \preceq \gamma_2(n) \tag{2.3}$$

if there is an integer $C > 1$ such that $\gamma_1(n) \le \gamma_2(Cn)$ for all $n \ge 1$. This converts the set $\mathcal{W}$ of growth degrees of finitely generated groups into a partially ordered set. The notation $\prec$ will be used in this article to indicate a strict inequality.

Let us remind some basic facts about growth rates that will be used in the paper.

- The power functions n^α belong to different equivalence classes for different $\alpha \ge 0$.
- The polynomial function $P_d(n) = c_d n^d + \cdots + c_1 n + c_0$, where $c_d \ne 0$ is equivalent to the power function n^d.
- All exponential functions λ^n, $\lambda > 1$ are equivalent and belong to the class $[e^n]$.
- All functions of *intermediate type* e^{n^α}, $0 < \alpha < 1$ belong to different equivalence classes.

This is not a complete list of rates of growth that a group may have. Much more is provided in [12] and [3].

It is easy to see that growth of a group coincides with the growth of a subgroup of finite index, and that growth of a group is not smaller than the growth of a finitely generated subgroup or of a factor group. Since a group with m generators can be presented as a quotient group of a free group of rank m, the growth of a finitely generated group cannot be faster than exponential (i.e., it can not be superexponential). Therefore we can split the growth types into three classes:

- *Polynomial* growth. A group G has *polynomial* growth if there are constants $C > 0$ and $d > 0$ such that $\gamma(n) < Cn^d$ for all $n \geq 1$. Minimal d with this property is called the degree of polynomial growth.
- *Intermediate* growth. A group G has *intermediate* growth if $\gamma(n)$ grows faster than any polynomial but slower than any exponent function λ^n, $\lambda > 1$ (i.e. $\gamma(n) \prec e^n$).
- *Exponential growth*. A group G has *exponential* growth if $\gamma(n)$ is equivalent to e^n.

The question on the existence of groups of intermediate growth was raised in 1968 by Milnor [30]. For many classes of groups (for instance for linear groups by Tits alternative [38], or for solvable groups by the results of Milnor [31] and Wolf [43]) intermediate growth is impossible. Milnor's question was answered by author in 1983 [10,12,20], where it was shown that there are uncountably many 2-generated torsion groups of intermediate growth. Moreover, it was shown in [12,13,20] that for any prime p a partially ordered set $\mathcal{W}_p$ of growth degrees of finitely generated torsion p-groups contains uncountable chain and contains uncountable anti-chain. The immediate consequence of this result is the existence of uncountably many quasi-isometry equivalence classes of finitely generated groups (in fact 2-generated groups) [12].

Below we will use several times the following lemma [24, page 59].

Lemma 2.1 (Splitting lemma) *Let G be a finitely generated group of polynomial growth of degree d and $H \lhd G$ be a normal subgroup with quotient G/H being an infinite cyclic group. Then H has polynomial growth of degree $\leq d - 1$.*

3 Gap Conjecture and its modifications

We will say that a group is *virtually nilpotent* (virtually solvable) if it contains nilpotent (solvable) subgroup of finite index. It was observed around 1968 by Milnor, Wolf, Hartly and Guivarc'h that a nilpotent group has polynomial growth and hence a virtually nilpotent group also has polynomial growth. In his remarkable paper [24], Gromov established the converse.

Theorem 3.1 (Gromov 1981) *If a finitely generated group G has polynomial growth, then G contains a nilpotent subgroup of finite index.*

In fact Gromov obtained stronger result about polynomial growth.

Theorem 3.2 *For any positive integers d and k, there exist positive integers R, N and q with the following property. If a group G with a fixed system of generators satisfies the inequality $\gamma(n) \leq kn^d$ for $n = 1, 2, \dots, R$ then G contains a nilpotent subgroup H of index at most q and whose degree of nilpotence is at most N.*

The above theorem implies existence of a function υ growing faster than any polynomial and such that if $\gamma_G \prec \upsilon$, then growth of G is polynomial.

Indeed, taking a sequence $\{k_i, d_i\}_{i=1}^{\infty}$ with $k_i \to \infty$ and $d_i \to \infty$ when $i \to \infty$ and the corresponding sequence $\{R_i\}_{i=1}^{\infty}$, whose existence follows from Theorem 3.2, one can build a function $\upsilon(n)$ which coincides with the polynomial $k_i n^{d_i}$ on the interval $[R_{i-1} + 1, R_i]$ and separates polynomial growth from intermediate. Therefore there is a *Gap* in the scale of rates of growth of finitely generated groups and a big problem is to find the optimal function (or at least to provide good lower and upper bounds for it) which separates polynomial growth from intermediate. The best known result in this direction is the function $n^{(\log\log n)^c}$ (c some positive constant) which appeared recently in the paper of Shalom and Tao [36, Corollary 8.6].

The lower bound of the type $e^{\sqrt{n}}$ for all groups $\mathcal{G}_\omega^{(p)}$ of intermediate growth established in [10,12,13,20] allowed the author to guess that equivalence class of function $e^{\sqrt{n}}$ could be a good candidate for a "border" between polynomial and exponential growth. This guess was further strengthened in 1988 when the author obtained the result published in [15] (see Theorem 5.1). For the first time the Gap Conjecture was formulated in the form of a question in 1991 (see [23]).

Conjecture 1 (Gap Conjecture) *If the growth function $\gamma_G(n)$ of a finitely generated group G is strictly bounded from above by $e^{\sqrt{n}}$ (i.e. if $\gamma_G(n) \prec e^{\sqrt{n}}$), then growth of G is polynomial.*

The question of independent interest is whether there is a group, or more generally a cancellative semigroup, with growth equivalent to $e^{\sqrt{n}}$ (for the role of cancellative semigroups in growth business see [14]).

In [22] the author formulated a number of conjectures relevant to the main Conjecture discussed there and in this note. Let us recall some of them as they will play some role in what follow.

Conjecture 2 (Gap Conjecture with parameter β, $0 < \beta < 1$). *If the growth function $\gamma_G(n)$ of a finitely generated group G is strictly bounded from above by e^{n^β} (i.e. if $\gamma(n) \prec e^{n^\beta}$) then the growth of G is polynomial.*

Thus the Gap Conjecture with parameter $1/2$ is just the Gap Conjecture 1. If $\beta < 1/2$ then the Gap Conjecture with parameter β is weaker than the Gap Conjecture, and if $\beta > 1/2$ then it is stronger than the Gap Conjecture.

Conjecture 3 (Weak Gap Conjecture). *There is a β, $0 < \beta < 1$ such that if $\gamma_G(n) \prec e^{n^\beta}$ then the Gap Conjecture with parameter β holds.*

The gap type conjectures can be formulated for other asymptotic characteristics of groups like return probabilities $P_{e,e}^{(n)}$ (e denotes the identity element) for a non degenerate random walk on a group, Fölner function $\mathcal{F}(n)$, or spectral density $\mathcal{N}(\lambda)$. There is a close relation between them and the Gap Conjecture on growth, which was mentioned in [22]. When writing this note the author realized that to understand better the relation between different forms of the gap type conjectures it is useful to consider in parallel to the Conjecture 2 [which we will denote $C(\beta)$] a stronger version of it, which we will denote $C^*(\beta)$:

Conjecture 4 (Conjecture $C^*(\beta)$) *If a group C is not virtually nilpotent then $\gamma_C(n) \succeq e^{n^\beta}$.*

It is obvious that $C^*(\beta)$ implies $C(\beta)$ but the opposite is not clear. This is related to the fact that there are groups with incomparable growths [12] as the set $\mathcal{W}$ of rates of growth of finitely generated groups is not linear ordered. The motivation for introducing a $*$-version of the Gap Conjecture will be more clear when a second note [21] of the author is submitted to the arXiv.

4 Growth and elementary amenable groups

Amenable groups were introduced by von Neumann in 1929 [39]. Now they play extremely important role in many branches of mathematics. Let AG denote the class of amenable groups. By a theorem of Adelson-Velskii [1], each finitely generated group of subexponential growth belongs to the class AG. This class contains finite groups and commutative groups and is closed under the following operations:

(1) taking a *subgroup*,
(2) taking a *quotient group*,
(3) *extensions*,
(4) *unions* (i.e. if for some net $\{\alpha\}, G_\alpha \in AG$ and $G_\alpha \subset G_\beta$ if $\alpha < \beta$ then $\cup_\alpha G_\alpha \in AG$).

Let EG be the class of *elementary* amenable groups i.e., the smallest class of groups containing finite groups, commutative groups which is closed with respect to the operations (1)–(4). For instance, virtually nilpotent and, more generally, virtually solvable groups belong to the class EG. This concept defined by Day in [6] got further development in the article [5] of Chou who suggested the following approach to study of elementary amenable groups.

For each ordinal α define a subclass EG_α of EG in the following way. EG_0 consists of finite groups and commutative groups. If α is a limit ordinal then

$$EG_\alpha = \bigcup_{\beta \preceq \alpha} EG_\beta.$$

Further, $EG_{\alpha+1}$ is defined as as the class of groups which are extensions of groups from set EG_α by groups from the same set or are direct limits of a family of groups from set EG_α. It is known (and easy to check) that each of the classes EG_α is closed with respect to the operations (1) and (2) [5]. By the *elementary complexity* of a group $G \in EG$ we call the smallest α such that $G \in EG_\alpha$.

It was shown in [5] that class EG does not contain groups of intermediate growth, groups of Burnside type (i.e. finitely generated infinite torsion groups), and finitely generated infinite simple groups. A further study of elementary groups and its generalizations was done by Osin [33].

A larger class SG of subexponentially amenable groups was (implicitly) introduced in [9], and explicitly in [16], and studied in [7] and other papers.

A useful fact about groups of intermediate growth which we will use is due to Rosset [35].

Theorem 4.1 *If G is a finitely generated group which does not grow exponentially and H is a normal subgroup such that G/H is solvable, then H is finitely generated.*

We propose the following generalization of this result.

Theorem 4.2 *Let G be a finitely generated group with no free subsemigroup on two generators and let the quotient G/N be an elementary amenable group. Then the kernel N is a finitely generated group.*

The latter two statements and the chain of further statements of the same spirit that appeared in the literature were initiated by the following lemma of Milnor [31]: if G

is a finitely generated group with subexponential growth, and if $x, y \in G$, then the group generated by the set of conjugates $y, xyx^{-1}, x^2yx^{-2}, \ldots$ is finitely generated.

Proof For the proof of the Theorem 4.2 we will apply induction on elementary complexity α of the quotient group $H = G/N$. If complexity is 0 then the group is either finite or abelian. In the first case N is finitely generated for obvious reason. In the second case we apply the following statements from the paper of Longobardi and Rhemtulla [27, Lemmas 1,2]. □

Lemma 4.3 *If G has no free subsemigroups, then for all $a, b \in G$ the subgroup $\langle a^{b^n}, n \in \mathbb{Z}\rangle$ is finitely generated.*

Lemma 4.4 *Let G be a finitely generated group. If $N \trianglelefteq G, G/N$ is cyclic, and $\langle a^{b^n}, n \in \mathbb{Z}\rangle$ is finitely generated for all $a, b \in G$, then N is finitely generated.*

Assume that the statement of the theorem is correct for quotients $H = G/H$ with complexity $\alpha \leq \beta - 1$ for some ordinal $\beta, \beta \geq 1$. The group H, being finitely generated, allows a short exact sequence

$$\{1\} \to A \to H \to B \to \{1\},$$

where $A, B \in EG_{\beta-1}$. Let $\varphi : G \to G/N$ be the canonical homomorphism and $M = \varphi^{-1}(A)$. Then M is a normal subgroup in G and $G/M \simeq G/N/M/N \simeq H/A \simeq B$. By the inductive assumption M is finitely generated and has no free subsemigroup on two generators. As $M/N \simeq A$, again by induction, N is finitely generated and we are done.

We will discuss just-infinite groups in detail in the last section. But let us prove now a preliminary result which will be used later. Recall that a group is called just-infinite if it is infinite, but every proper quotient is finite (i.e. every nontrivial normal subgroup is of finite index). A group G is called hereditary just-infinite if it is residually finite and every subgroup $H < G$ of finite index (including G itself) is just infinite. Observe that a subgroup of finite index of a hereditary just-infinite group is hereditary just-infinite.

We learned the following result from de Cournulier. A proof is provided here as there is no one in the literature.

Theorem 4.5 *Let G be a finitely generated hereditary just-infinite group, and suppose that G belongs to the class EG of elementary amenable groups. Then G is isomorphic either to the infinite cyclic group $\mathbb{Z}$ or to the infinite dihedral group D_∞.*

Proof If $G \in EG_0$ then G is abelian and hence $G \simeq \mathbb{Z}$. Assume that the statement is correct for all groups from classes $EG_\alpha, \alpha < \beta$ for some ordinal β. Let us prove it for β. Assume $G \in EG_\beta$ and β is smallest with this property. β can not be a limit ordinal because G is finitely generated. Therefore G is the extension of a group A by a group $B = G/A$, where $A, B \in EG_{\beta-1}$. In fact B is a finite group (as G is just-infinite). As a subgroup of finite index in a hereditary just-infinite group, A is hereditary just-infinite and moreover finitely generated (as a subgroup of finite index in a finitely generated group). By inductive assumption A is isomorphic either to the infinite cyclic group $\mathbb{Z}$ or to the infinite dihedral group D_∞. In particular G has a normal subgroup H of finite index isomorphic to $\mathbb{Z}$.

Let G act on H by conjugation. Then we get a homomorphism $\psi : G \to Aut(H) \simeq \mathbb{Z}_2$. If $\psi(G) = \{1\}$, then H is a central subgroup. It is a standard fact in group theory (see for instance [25, Proposition 2.4.4]) that if there is a central subgroup of finite index in G then the commutator subgroup G' is finite. But as G is just-infinite, $G' = \{1\}$ and so G is abelian, hence $G \simeq \mathbb{Z}$ in this case.

If $\psi(G) = Aut(H)$ then $N = ker\psi$ is a centralizer $C_G(H)$ of H in G. Subgroup N has index 2 in G, is just-infinite and hence by the same reason as above $N' = \{1\}$, so N is abelian. Being finitely generated and just infinite implies $N \simeq \mathbb{Z}$.

Let $x \in G, x \notin N$. The element x acts on N by conjugation mapping each element to its inverse. In particular, $x^{-1}(x^2)x = x^{-2}$, so $(x^2)^2 = 1$. But $x^2 \in N$. Since N is torsion free $x^2 = 1$. Therefore

$$G = \langle x, N \rangle = \langle x, y : x^2 = 1, x^{-1}yx = y^{-1} \rangle \simeq D_\infty,$$

where y is a generator of N. □

5 Gap Conjecture for residually solvable groups

Recall that a group G is said to be a residually finite-p group (sometimes also called residually finite p-group) if it is approximated by finite p-groups, i.e., for any $g \in G$ there is a finite p-group H and a homomorphism $\phi : G \to H$ with $\phi(g) \neq 1$. This class is, of course, smaller than the class of residually finite groups, but it is pretty large. For instance, Golod-Shafarevich groups, p-groups $\mathcal{G}_\omega$ from [12,13], and many other groups belong to this class.

Theorem 5.1 [15] *Let G be a finitely generated residually finite-p group. If $\gamma_G(n) \prec e^{\sqrt{n}}$ then G has polynomial growth.*

As was established by the author in a discussion with Lubotzky and Mann during the conference on profinite groups in Oberwolfach in 1990, the same arguments as given in [12] combined with the following lemma from [28].

Lemma 5.2 (Lemma 1.7, [28]) *Let G be a finitely generated residually nilpotent group. Assume that for every prime p the pro-p-closure $G_{\hat{p}}$ of G is p-adic analytic. Then G is linear.*

allows one to prove a stronger version of the above theorem (see the Remark after Theorem 1.8 in [28]):

Theorem 5.3 *Let G be a residually nilpotent finitely generated group. If $\gamma_G(n) \prec e^{\sqrt{n}}$ then G has polynomial growth.*

To be linear means to be isomorphic to a subgroup of the linear group $GL_n(\mathbb{K})$ for some field $\mathbb{K}$. By Tits alternative [38] every finitely generated linear group either contains a free subgroup on two generators or is virtually solvable. Hence the above lemma immediately reduces Theorem 5.3 to Theorem 5.1.

The latter two theorems (where the first one is the corresponding statement from [15] while the second one is a corrected form of what is stated in Remark on page

527 in [28]) show that Gap Conjecture $C(1/2)$ holds for the class of residually finite-p groups and more generally for the class of residually nilpotent groups. In fact, arguments provided in [15,28] allow to prove stronger conjecture $C^*(1/2)$ for these classes of groups.

Let p be a prime and $a_n^{(p)}$ be the n-th coefficient of the power series given by

$$\sum_{n=0}^{\infty} a_n^{(p)} z^n = \prod_{n=1}^{\infty} \frac{1-z^{pn}}{1-z^n}.$$

Then the lower bound $a_n^{(p)} \succeq e^{\sqrt{n}}$ holds. Moreover if a group G is a residually finite-p group and is not virtually nilpotent then for any system of generators A

$$\gamma_G^A(n) \geq a_n^{(p)}, n = 1, 2, \ldots$$

(see the relation (23) and Lemma 8 in [15]). Observe that the latter statement is valid not only in the case when A is a system of elements that generate G as a group but even in a more general case when A is a generating set for the group G considered as a semigroup. In fact, growth function of any group is bounded from below by a sequence of coefficients of Hilbert-Poincaré series of the universal p-enveloping algebra of the restricted Lie p-algebra associated with the group using the factors of the lower p-central series [15].

Theorem 1.8 from [28] contains an interesting approach to polynomial growth type theorems in the case of residually nilpotent groups. Moreover, as is mentioned in [28] in the remark after the theorem, the proof provided there yields the same conclusion under a weaker assumption: $\gamma_G(n) \prec 2^{2^{\sqrt{\log_2 n}}}$.

Surprisingly, in his first paper on the gap type problem [42] Wilson used a similar upper bound $\gamma_G(n) \prec e^{e^{(1/2)\sqrt{\ln n}}}$ to measure size of a gap for residually solvable groups. Wilson's approach is quite different from those that were used before and is based on exploring self-centralizing chief factors in finite solvable groups.

Recall that a chief factor of a group G is a (nontrivial) minimal normal subgroup of some quotient G/N, and that L/M is a self-centralizing chief factor of a group G if M is normal in G, L/M is a minimal normal subgroup of G/M, and $L/M = C_{G/M}(L/M)$. One of the results in [42] is

Theorem 5.4 (Wilson) *Let G be a residually solvable group of subexponential growth whose finite self-centralizing chief factors all have rank at most k. Then G has a residually nilpotent normal subgroup whose index is finite and bounded in terms of k and $\gamma_G(n)$.*

If, in addition $\gamma_G(n) \prec e^{\sqrt{n}}$, then G has a nilpotent normal subgroup whose index is finite and bounded in terms of k and $\gamma_G(n)$.

The proof of this result is based on the following lemma the proof of which uses ultraproducts.

Lemma 5.5 (Lemma 2.1, [42]) *Let k be a positive integer and $\alpha : \mathbb{N} \to \mathbb{R}_+$ a function such that $\alpha(n)/n \to 0$ as $n \to \infty$. Suppose that G is a finite solvable group having* (i) *a self-centralizing minimal normal subgroup V of rank at most k and* (ii) *a generating set A such that* $\gamma_G^A(n) \le e^{\alpha(n)}$ *for all n. Then* $|G/V|$ *is bounded in terms of k and* α *alone.*

One of the almost immediate corollaries of the technique developed in [42] are the facts stated below in Theorems 5.6 and 5.7.

Recall that a group is called supersolvable if it has a finite normal descending chain of subgroups with cyclic quotients. Every finitely generated nilpotent group is supersolvable [34], and the symmetric group $Sym(4)$ is the simplest example of a solvable but not supersolvable group.

Theorem 5.6 *The Gap Conjecture holds for residually supersolvable groups. Moreover, the conjecture* $C^*(1/2)$ *holds for residually supersolvable groups.*

Developing his technique and using the known facts about maximal primitive solvable subgroups of $GL_n(p)$ (p prime) Wilson in [40] proved that the Gap Conjecture with parameter $1/6$ holds for residually solvable groups. In fact what follows from arguments in [42], combined with arguments from [15,28] and with what was written above, can be formulated as

Theorem 5.7 *The conjecture* $C^*(1/6)$ *holds for residually solvable groups.*

There is a hope that eventually the Gap Conjecture and its $*$-version will be proved for residually solvable groups, or at least for residually polycyclic groups (which is the same as to prove it for groups approximated by finite solvable groups, because polycyclic groups are residually finite [34]). If the latter is done, then we will have complete reduction of the Gap Conjecture to just-infinite groups (more on this in the last section).

6 Gap Conjecture for right orderable groups

Recall that a group is called right orderable if there is a linear order on the set of its elements invariant with respect to multiplication on the right. In a similar way are defined left orderable groups. A group is bi-orderable (or totally orderable) if there is a linear order invariant with respect to multiplication on the left and on the right. Every right orderable group is left orderable and vise versa but there are right orderable groups which are not totally orderable (see [26] for examples). As was shown by Machi and the author the class of finitely generated right orderable groups of intermediate growth is nonempty [19]. The corresponding group $\hat{\mathcal{G}}$ was earlier constructed in [16] as an example of a torsion free group of intermediate growth. It was implicitly observed in [19] that the class of countable right orderable groups coincides with the class of groups acting faithfully by homeomorphisms on the line $\mathbb{R}$ (or, what is the same, on the interval $[0, 1]$). Recently Erschler and Bartholdi managed to compute the growth of $\hat{\mathcal{G}}$ which happens to be $e^{\log(n)n^{\alpha_0}}$ where $\alpha_0 = \log 2/\log(2/\rho) \approx 0.7674$, and ρ is the real root of the polynomial $x^3 + x^2 + x - 2$. The question if there exists a finitely generated, totally orderable group of intermediate growth is still open.

The Gap Conjecture and it modifications stated in Sect. 3 are interesting problems even for the class of right orderable groups. Our next result makes some contribution to this topic. The result of Wilson combined with theorems of Morris [32] and Rosset [35] can be used to prove the following statement.

Theorem 6.1 (i) *The Gap Conjecture with parameter* $1/6$, *and, moreover, the conjecture* $C^*(1/6)$ *hold for right orderable groups.*

(ii) *The Gap Conjecture* $C(1/2)$ [*or its* $*$*-version* $C^*(1/2)$] *holds for right orderable groups if it* [*or its* $*$*-version* $C^*(1/2)$] *holds for residually polycyclic groups.*

Proof (i) Let G be a finitely generated right orderable group with growth $\prec e^{n^{1/6}}$. In [32] Morris proved that every finitely generated right orderable amenable group is indicable (i.e. can be mapped onto $\mathbb{Z}$). As by Adelson-Velskii theorem [1] a group of intermediate growth is amenable, we conclude that the abelianization $G_{ab} = G/[G, G]$ is infinite and hence has a decomposition $G_{ab} = G^-_{ab} \oplus G^+_{ab}$ where $G^-_{ab} \simeq \mathbb{Z}^d, d \geq 1$ is a torsion free part of an abelian group and G^+_{ab} is a torsion part. Let $N \lhd G$ be a normal subgroup such that $G/N = G^-_{ab}$. Since the commutator subgroup of a group is a characteristic group and the torsion free part of abelian group also is a characteristic subgroup we conclude that N is a characteristic subgroup of G. By Theorem 4.1 N is a finitely generated group. Therefore we can proceed with N as we did with G. This allows us to get a descending chain

$$G > G_1 > G_2 > \cdots \tag{6.1}$$

(where $G_1 = N$ etc) of characteristic subgroups with the property that $G_i/G_{i+1} \simeq \mathbb{Z}^{d_i}$ if $G_{i+1} \neq \{1\}$, for some sequence $d_i \in \mathbb{N}, i = 1, 2, \ldots$.

If the chain (6.1) terminates after finitely many steps then G is solvable and by the results of Milnor and Wolf [31,43] G is virtually nilpotent in this case.

Suppose that chain (6.1) is infinite and consider the intersection $G_\omega = \bigcap_{i=1}^{\infty} G_i$. If $G_\omega = \{1\}$, then the group G is residually solvable (in fact residually polycyclic), and, because of restriction on growth, by Theorem 5.7, G is virtually nilpotent and hence has polynomial growth of some degree d. But this contradicts Splitting Lemma 2.1. Therefore $G_\omega \neq \{1\}$. G/G_ω is residually polycyclic, has growth not greater than the growth of G and by previous argument is virtually nilpotent. If the degree of polynomial growth of G/G_ω is l then again by Splitting Lemma the length of the chain (6.1) can not be larger than l, and we get a contradiction. The part (i) of the theorem is proven.

Now the proof of part (ii) follows immediately. If we assume that G has growth $\prec e^{\sqrt{n}}$ and that the Gap Conjecture holds for the class of residually polycyclic groups then the arguments from previous part (i) are applicable in the same manner. The only difference is that instead of Theorem 5.7 one should use the assumption that the Gap Conjecture holds for residually polyciclic groups. The same argument works in the case of conjecture $C^*(1/2)$. □

7 Gap Conjecture and just-infinite groups

There is a strong evidence based on considerations presented below that the Gap Conjecture can be reduced to three classes of groups: *simple* groups, *branch* groups and *hereditary just-infinite* groups. These three types of groups appear in a natural partition of the class of just-infinite groups into three subclasses described in Theorem 7.3. The following statement is an easy application of Zorn's lemma.

Proposition 7.1 *Let G be a finitely generated infinite group. Then G has a just-infinite quotient.*

Corollary 7.2 *Let $\mathcal{P}$ be a group theoretical property preserved under taking quotients. If there is a finitely generated group satisfying the property $\mathcal{P}$ then there is a just-infinite group satisfying this property.*

Although the property of a group to have intermediate growth is not preserved when passing to a quotient group (the image may have polynomial growth), by theorems of Gromov [24] and Rosset [35], if the quotient G/H of a group G of intermediate growth has polynomial growth then H is a finitely generated group (of intermediate growth, as the extension of a virtually nilpotent group by a virtually nilpotent group is an elementary amenable group and therefore can not have intermediate growth), and one may look for a just-infinite quotient of H and iterate this process in order to represent G as a consecutive extension of a chain of groups that are virtually nilpotent or just-infinite groups. This observation was used in the previous section for the proof of Theorem 6.1 and is the base of the arguments for Theorems 7.4 and 7.5.

Recall that hereditary just-infinite groups were already defined in Sect. 4. We call a just infinite group near simple if it contains a subgroup of finite index which is a direct product of finitely many copies of a simple group.

Branch groups are groups that have a faithful level transitive action on an infinite spherically homogeneous rooted tree $T_{\bar{m}}$ defined by a sequence $\{m_n\}_{n=1}^{\infty}$ of natural numbers $m_n \geq 2$ (determining the branching number for vertices of level n) with the property that the rigid stabilizer $rist_G(n)$ has finite index in G for each $n \geq 1$. Here by $rist_G(n)$ we mean a subgroup $\prod_{v \in V_n} rist_G(v_n)$ which is a product of rigid stabilizers $rist_G(v_n)$ of vertices v_n taken over the set V_n of all vertices of level n, and $rist_G(v)$ is a subgroup of G consisting of elements fixing the vertex v and acting trivially outside the full subtree with the root at v. For a more detailed discussion of this notion see [4,18]. This is a geometric definition. It follows immediately from the definition that branch groups are infinite. The definition of an algebraically branch group can be found in [4,17]. Every geometrically branch group is algebraically branch but not vice versa. If G is algebraically branch then it has a quotient G/N which is geometrically branch. The difference between two versions of the definitions is not large but still there is no complete understanding how much the two classes differ (it is not clear what can be said about the kernel N, it is believed that it should be central in G). For just-infinite branch groups the algebraic and geometric definitions are equivalent. Not every branch group is just-infinite, but every proper quotient of a branch group is virtually abelian [18]. Therefore branch groups are "almost just-infinite" and most of known finitely generated branch groups are just-infinite. Observe that a finitely generated virtually nilpotent group is not branch. This follows for instance from the fact that a finitely generated nilpotent group satisfies a minimal condition for normal subgroups while a branch group not.

The next theorem was derived by the author from a result of Wilson [41].

Theorem 7.3 [18] *The class of just-infinite groups naturally splits into three subclasses:* (*B*) *branch just-infinite groups,* (*H*) *hereditary just-infinite groups, and* (*S*) *near-simple just-infinite groups.*

It is already known that there are finitely generated branch groups of intermediate growth. For instance, groups $\mathcal{G}_\omega$ of intermediate growth from the articles [11,13] are of this type. In fact, all known examples of groups of intermediate growth are of branch type or are reconstructions on the base of groups of branch type. The question about existence of amenable but non-elementary amenable hereditary just-infinite group is still open (remind that by Theorem 4.5 the only elementary amenable hereditary just-infinite groups are $\mathbb{Z}$ and D_∞).

Problem 1 Are there finitely generated hereditary just-infinite groups of intermediate growth?

Problem 2 Are there finitely generated simple groups of intermediate growth?

The next theorem is a straightforward corollary of the main result of Bajorska and Makedonska from [2] (observe that it was not stated in [2]). Here we suggest a different proof which is adapted to the needs of the proof of the main Theorem 7.5.

Theorem 7.4 *If the Gap Conjecture or conjecture* $C^*(1/2)$ *holds for the classes of residually finite groups and simple groups, then the corresponding conjecture holds for the class of all groups.*

Proof Assume that the Gap Conjecture is correct for residually finite groups and for simple groups. Let G be a finitely generated group with growth $\prec e^{\sqrt{n}}$. By Proposition 7.1 it has just-infinite quotient $\bar{G} = G/N$, which belongs to one of the three types of groups listed in the statement of the Theorem 7.3. The rate of growth of $\bar{G}$ is not greater than the rate of growth of $e^{\sqrt{n}}$. The group $\bar{G}$ can not be near simple because in this case it will have a subgroup H of finite index with infinite finitely generated simple quotient whose rate of growth is $\prec e^{\sqrt{n}}$. This is impossible as a virtually nilpotent group can not be infinite simple.

The group $\bar{G}$ also can not be branch as branch groups are residually finite and finitely generated virtually nilpotent groups are not branch. So we can assume that $\bar{G}$ is hereditary just infinite and hence residually finite. Using the assumption of the theorem we conclude that $\bar{G}$ is virtually nilpotent, and therefore elementary amenable. By Theorem 4.5 $\bar{G}$ is isomorphic either to the infinite cyclic group or to the infinite dihedral group D_∞. By Theorem 4.1 kernel N is finitely generated. As the rate of growth of N is less than $e^{\sqrt{n}}$ we can apply to N the same arguments as for G in order to get a surjective homomorphism either onto $\mathbb{Z}$ or onto D_∞.

If $G/N \simeq \mathbb{Z}$, then we repeat the first step of the proof of Theorem 6.1 replacing N by a finitely generated characteristic subgroup $N_1 \lhd G$ with quotient $G/N_1 \simeq \mathbb{Z}^{d_1}$ for some $d_1 \geq 1$. If $G/N_1 \simeq D_\infty$ then we slightly modify the first step. Namely, in this case G has indicable subgroup H of index 2. Let H_1 be the intersection of groups $H^\phi, \phi \in Aut(G)$. As there are only finitely many subgroups of index 2 in G this intersection involves only finitely many groups and H_1 is a characteristic subgroup in

G of finite index of type 2^t for some $t \in \mathbb{N}$. Moreover, $G/H_1 \simeq \mathbb{Z}_2^t$ as the quotient G/H_1 is isomorphic to a subgroup of a direct product of finitely many copies of group $\mathbb{Z}_2$ of order 2. The subgroup H_1, being a subgroup of index 2^{t-1} in H, is indicable and we can apply the argument of the first step of the proof of Theorem 6.1 getting a finitely generated subgroup $H_2 \trianglelefteq H_1$ characteristic in G with quotient $H_1/H_2 \simeq \mathbb{Z}^{d_1}$ for some $d_1 \in \mathbb{N}$.

Let $G_1 \triangleleft G$ be a subgroup N, H_1 or H_2 depending on the case. Proceed with G_1 in a similar fashion as we did with G, etc. We get a descending chain $\{G_i\}_{i\geq 1}$ of finitely generated subgroups characteristic in G. There are two possibilities.

(1) After finitely many steps we get a group G_i which is hereditary just-infinite and elementary amenable, and hence infinite cyclic or D_∞ (Theorem 4.5). In this case G is polycyclic and we are done in view of the result of Milnor and Wolf on growth of solvable groups.
(2) The process of construction of the chain of subgroups will continue forever. In this case we get a chain with the property that G_i/G_{i+1} is isomorphic either to
  (i) $\mathbb{Z}^{d_i}$, $d_i \in \mathbb{N}$ or to
  (ii) $\mathbb{Z}_2^{t_i}$, $t_i \in \mathbb{N}$. Moreover, each step of type (ii) is immediately followed by a step of type (i).

Let us show that this is impossible. Let G_ω be the intersection $\bigcap_{i\geq 1} G_i$. Then G/G_ω is residually polycyclic and hence residually finite as every polycyclic group is residually finite [34]. Growth of G/G_ω is less than $e^{\sqrt{n}}$. Hence by the assumption of the theorem the group G/G_ω is virtually nilpotent with the rate of polynomial growth of degree d for some $d \in \mathbb{N}$. But this contradicts the splitting lemma as for infinitely many i the quotients G_i/G_{i+1} are isomorphic to $\mathbb{Z}^{d_i}$. This proves the conjecture $C(1/2)$.

In the case of the conjecture $C^*(1/2)$ we proceed in a similar fashion. Only at the beginning we assume that the conjecture $C^*(1/2)$ holds for residually finite groups and for simple groups and that G is a finitely generated group of intermediate growth whose growth does not satisfy inequality $\gamma(n) \succeq e^{n^{1/2}}$. □

Now we state and prove our main result.

Theorem 7.5 (i) *If the Gap Conjecture with parameter* $1/6$ *or its* $*$*-version* $C^*(1/6)$ *holds for just-infinite groups then the corresponding conjecture holds for all groups.*

(ii) *If the Gap Conjecture or its* $*$*-version* $C^*(1/2)$ *holds for residually polycyclic groups and for just-infinite groups then the corresponding conjecture holds for all groups.*

Proof (i) The proof follows the same strategy as the proof of Theorem 7.4. Let G be a finitely generated group with growth $\prec e^{n^{1/6}}$. There can be two possibilities.

(1) G has a finite descending chain $\{G_i\}_{i=1}^k$ of finitely generated characteristic in G groups with consecutive quotients $G_i/G_{i+1} \simeq \mathbb{Z}^{d_i}$ or $G_i/G_{i+1} \simeq \mathbb{Z}_2^{t_i}$, for $i < k$ and $G_k = \{1\}$. In this case G is polycyclic and hence virtually nilpotent

(2) G has an infinite descending chain $\{G_i\}_{i=1}^{\infty}$, with the property that $G_i/G_{i+1} \simeq \mathbb{Z}^{d_i}$ or $G_i/G_{i+1} \simeq \mathbb{Z}_2^{t_i}$, and if $G_i/G_{i+1} \simeq \mathbb{Z}_2^{t_i}$ then $G_{i+1}/G_{i+2} \simeq \mathbb{Z}^{d_{i+1}}$. The group G/G_ω, where $G_\omega = \bigcap_{i\geq 1} G_i$, is residually polycyclic with growth $\prec e^{n^{1/6}}$. Apply in this case the result of Wilson stated in Theorem 5.4 concluding that G/G_ω is virtually nilpotent which is impossible by the splitting lemma.

(ii) Proceed as in (i) with the only difference that in the subcase (2) we apply the assumption that the Gap Conjecture holds for residually polycyclic groups to conclude that this subcase is impossible.

These are arguments for $C(1/2)$ version. The arguments for $*$-version $C^*(1/2)$ are similar. □

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