Crossings and Nestings of Two Edges in Set Partitions

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Key Words: set partitions, crossings, nestings
AMS subject classifications. 05A18, 05A15

Abstract

Let π and λ be two set partitions with the same number of blocks. Assume π is a partition of $[n]$. For any integer $l, m \geq 0$, let $\mathcal{T}(\pi, l)$ be the set of partitions of $[n+l]$ whose restrictions to the last n elements are isomorphic to π, and $\mathcal{T}(\pi, l, m)$ the subset of $\mathcal{T}(\pi, l)$ consisting of those partitions with exactly m blocks. Similarly define $\mathcal{T}(\lambda, l)$ and $\mathcal{T}(\lambda, l, m)$. We prove that if the statistic cr (ne), the number of crossings (nestings) of two edges, coincides on the sets $\mathcal{T}(\pi, l)$ and $\mathcal{T}(\lambda, l)$ for $l = 0, 1$, then it coincides on $\mathcal{T}(\pi, l, m)$ and $\mathcal{T}(\lambda, l, m)$ for all $l, m \geq 0$. These results extend the ones obtained by Klazar on the distribution of crossings and nestings for matchings.

1 Introduction and Statement of Main Result

In a recent paper [5], Klazar studied distributions of the numbers of crossings and nestings of two edges in (perfect) matchings. All matchings form an infinite tree $\mathcal{T}$ rooted at the empty matching $\emptyset$, in which the children of a matching M are the matchings obtained from M by adding to M in all possible ways a new first edge. Given two matchings M and N on $[2n]$, Klazar decided when the number of crossings (nestings) have identical distribution on the levels of the two subtrees of $\mathcal{T}$ rooted at M and N. In the last section of [5] Klazar raised the question as to apply the method to other structures besides matchings. In the present paper we consider set partitions, which have a natural graphic representation by a set of arcs. We establish the Klazar-type results to the distribution of crossings and nestings of two edges in set partitions.

Our approach follows that of Klazar for matchings [5], but is not a straightforward generalization. The structure of set partitions is more complicated than that of matchings. For example,

[3]The second author was supported in part by NSF grant DMS-0245526 and DMS-0653846.

partitions of $[n]$ may have different number of blocks, while every matching of $[2n]$ has exactly n blocks. To get the results we first defined an infinite tree $\mathcal{T}(\Pi)$ on the set of all set partitions, which, when restricted to matchings, is different than the one introduced by Klazar. We state our main result in Theorem 1.1, whose proof and applications are given in Sections 2 and 3. Section 4 is devoted to the enumeration of the crossing/nesting similarity classes. Though the ideas of the proofs are similar to those in [5], in many places we have to supply our own argument to fit in the different structure, and use a variety of combinatorial structures, in particular, Motzkin paths, Charlier diagrams, and binary sequences. We also analyze the joint generating function of the statistics cr and ne over partitions rooted at π, and derive a continued fraction expansion for general π.

We begin by introducing necessary notations. A (set) partition of $[n] = \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets of $[n]$, called blocks, whose union is $[n]$. A matching of $[2n]$ is a partition of $[2n]$ in n two-element blocks, which we also call *edges*. If a partition π has k blocks, we write $|\pi| = k$. A partition π is often represented as a graph on the vertex set $[n]$, drawn on a horizontal line in the increasing order from left to right, whose edge set consists of arcs connecting the elements of each block in numerical order. We write an arc e as a pair (i, j) with $i < j$.

For a partition π of $[n]$, we say that the arcs (i_1, j_1) and (i_2, j_2) form a crossing if $i_1 < i_2 < j_1 < j_2$, and they form a nesting if $i_1 < i_2 < j_2 < j_1$. By $cr(\pi)$ (resp. $ne(\pi)$), we denote the number of crossings (resp. nestings) of π. The distribution of the statistics cr and ne on matchings has been studied in a number of articles, including [2, 5, 6, 7, 8], to list a few. The symmetry of cr and ne for set partitions was established in [4]. In this paper we investigate the distribution of the statistics $cr(\pi)$ and $ne(\pi)$ over the partitions of $[n]$ with a prefixed restriction to the last k elements.

Denote by Π_n the set of all partitions of $[n]$, and by $\Pi_{n,k}$ the set of partitions of $[n]$ with k blocks. For $n = 0$, Π_0 contains the empty partition. Let $\Pi = \cup_{n=0}^{\infty} \Pi_n = \cup_{n=0}^{\infty} \cup_{k \leq n} \Pi_{n,k}$. We define *the tree* $\mathcal{T}(\Pi)$ *of partitions* as a rooted tree whose nodes are partitions such that:

1. The root is the empty partition;
2. The partition π of $[n+1]$ is a child of λ, a partition of [n], if and only if the restriction of π on $\{2, \ldots, n+1\}$ is order-isomorphic to λ.

See Figure 1 for an illustration of $\mathcal{T}(\Pi)$.

Observe that if λ is a partition of $[n]$ with $|\lambda| = k$, then λ has $k+1$ children in $\mathcal{T}(\Pi)$. Let $B_1, \ldots, B_k$ be the blocks of λ ordered in increasing order with respect to their minimal elements. For a set S, let $S + 1 = \{a + 1 : a \in S\}$. We denote the children of λ by $\lambda^0, \lambda^1, \ldots, \lambda^k$ as follows: λ^0 is a partition of $[n+1]$ with $k+1$ blocks,

$$\lambda^{(0)} = \{\{1\}, B_1 + 1, \ldots, B_k + 1\};$$

for $1 \leq i \leq k$, λ^i is a partition of $[n+1]$ with k blocks,

$$\lambda^{(i)} = \{\{1\} \cup (B_i + 1), B_1 + 1, \ldots, B_{i-1} + 1, B_{i+1} + 1, \ldots, B_k + 1\}.$$

For a partition λ, let $\mathcal{T}(\lambda)$ denote the subtree of $\mathcal{T}(\Pi)$ rooted at λ, and let $\mathcal{T}(\lambda, l)$ be the set of all partitions at the l-th level of $\mathcal{T}(\lambda)$. $\mathcal{T}(\lambda, l, m)$ is the set of all partitions on the l-th level of $\mathcal{T}(\lambda)$ with m blocks. Note that $\mathcal{T}(\lambda, l, m) \neq \emptyset$ if and only if $k \leq m \leq k + l$.

Let G be an abelian group and α, β two elements in G. Consider the statistics $s_{\alpha,\beta} : \Pi \to G$ given by $s_{\alpha,\beta}(\lambda) = cr(\lambda)\alpha + ne(\lambda)\beta$. In [5], Klazar defines a tree of matchings and shows that for

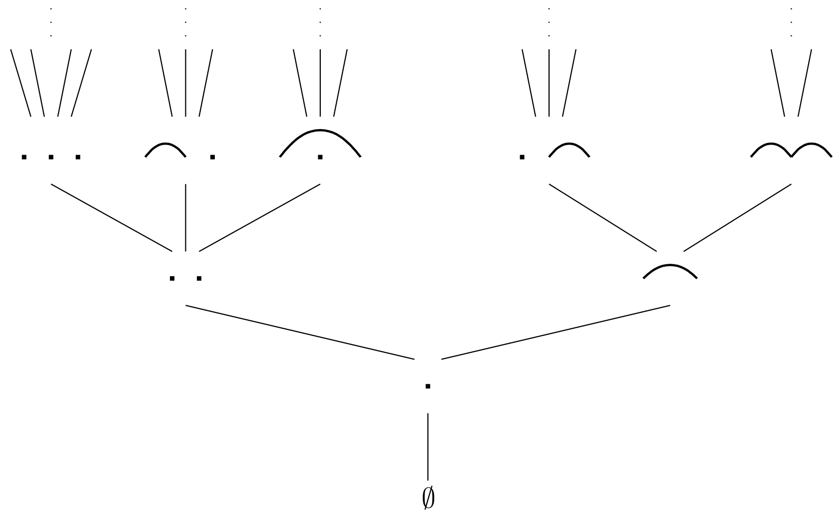


Figure 1: The tree of partitions $\mathcal{T}(\Pi)$

two matchings M and N, if the statistic $s_{\alpha,\beta}$ coincides at the first two levels of $\mathcal{T}(M)$ and $\mathcal{T}(N)$ then it coincides at all levels, and similarly for the pair of statistics $s_{\alpha,\beta}, s_{\beta,\alpha}$. In this article we prove that in the tree of partitions defined above, the same results hold. Precisely,

Theorem 1.1. *Let $\lambda, \pi \in \mathcal{T}(\Pi)$ be two non-empty partitions, and $s_{\alpha,\beta}(T)$ be the multiset containing $\{s_{\alpha,\beta}(t) : t \in T\}$. We have*

(a) If $s_{\alpha,\beta}(\mathcal{T}(\lambda, l)) = s_{\alpha,\beta}(\mathcal{T}(\pi, l))$ for $l = 0, 1$ then $s_{\alpha,\beta}(\mathcal{T}(\lambda, l, m)) = s_{\alpha,\beta}(\mathcal{T}(\pi, l, m))$ for all $l, m \geq 0$.

(b) If $s_{\alpha,\beta}(\mathcal{T}(\lambda, l)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l))$ for $l = 0, 1$ then $s_{\alpha,\beta}(\mathcal{T}(\lambda, l, m)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l, m))$ for all $l, m \geq 0$.

In other words, if the statistic $s_{\alpha,\beta}$ coincides on the first two levels of the trees $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$ then it coincides on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ on all levels, and similarly for the pair of statistics $s_{\alpha,\beta}, s_{\beta,\alpha}$.

Note that the conditions of Theorem 1.1 imply that λ and π have the same number of blocks. But they are not necessarily partitions of the same $[n]$.

At the end of the introduction we would like to point out the major differences between the structure of crossing and nesting of set partitions and that of matchings.

1. The tree of partitions $\mathcal{T}(\Pi)$ and the tree of matchings are different. In $\mathcal{T}(\Pi)$, children of a partition π is obtained by adding a new vertex, instead of adding a first edge. Hence Klazar's tree of matchings is not a sub-poset of $\mathcal{T}(\Pi)$. The definition of $\mathcal{T}(\Pi)$ allows us to define the analogous operators $R_{\alpha,\beta,i}$, as in [5, §2]. Since some descendants of π are obtained by adding isolated points, we need to introduce an extra operator M, (see Definition 2.2), and supply some new arguments to work with our structure and M.

2. The type of a matching is encoded by a Dyck path, while for set partitions, the corresponding structure is restricted bicolored Motzkin paths(RBM), (c.f. Section 3).

3. The enumeration of crossing/nesting similarity classes is different. A crossing-similarity class is determined by a value $cr(M)$ ($cr(\pi)$) and a composition $(a_1, a_2, \ldots, a_m)$ of n. For matchings $cr(M)$ can be any integer between 0 and $1 + a_2 + 2a_3 + \cdots + (m-1)a_m$. But for partitions the possible value of $cr(\pi)$ depends only on m, but not a_i's.

   In matchings there is a bijection between the set of nesting sequences of matchings of $[2n]$ and the set of Dyck paths $\mathcal{D}(n)$. There is no analogous result between set partitions and restricted bicolored Motzkin paths.

4. For matchings every nesting-similarity class is a subset of a crossing-similarity class. This is not true for set partitions.

2 Proof of Theorem 1.1

Throughout this article we will generally adapt Klazar's notation on multisets. Formally a multiset is a pair (A, m), where A is a set, called the underlying set, and $m : A \to \mathbb{N}$ is a mapping that determines the multiplicities of the elements of A. We often write multisets by repeating the elements according to their multiplicities.

For a map $f : X \to Y$ and $Z \subset X$, let $f(Z)$ denote the multiset whose underlying set is $\{f(z) : z \in Z\}$ and in which each element y appears with multiplicity equal to the cardinality of the set $\{z : z \in Z \text{ and } f(z) = y\}$. $\mathcal{S}(X)$ denotes the set of all finite multisets with elements in the set X. Any function $f : X \to S(Y)$ naturally extends to $f : \mathcal{S}(X) \to \mathcal{S}(Y)$ by $f(Z) = \bigcup_{z \in Z}\{f(z)\}$, where $\bigcup$ is union of multisets (the multiplicities of elements are added).

For each $b_i = \min B_i$ of λ define $u_i(\lambda)$ to be the number of edges (p, q) such that $p < b_i < q$ and $v_i(\lambda)$ to be the number of edges (p, q) such that $p < q < b_i$. They satisfy the obvious recursive relations

$$u_i(\lambda^0) = \begin{cases} 0 & \text{if } i = 1 \\ u_{i-1}(\lambda) & \text{if } 2 \le i \le k+1 \end{cases} \tag{2.1}$$

$$v_i(\lambda^0) = \begin{cases} 0 & \text{if } i = 1 \\ v_{i-1}(\lambda) & \text{if } 2 \le i \le k+1 \end{cases} \tag{2.2}$$

$$u_i(\lambda^j) = \begin{cases} 0 & \text{if } i = 1 \\ u_{i-1}(\lambda) + 1 & \text{if } 2 \le i \le j \\ u_i(\lambda) & \text{if } j+1 \le i \le k \end{cases} \tag{2.3}$$

$$v_i(\lambda^j) = \begin{cases} 0 & \text{if } i = 1 \\ v_{i-1}(\lambda) & \text{if } 2 \le i \le j \\ v_i(\lambda) + 1 & \text{if } j+1 \le i \le k \end{cases} \tag{2.4}$$

for $j = 1, \ldots, k$, where $k = |\lambda| \ge 1$. For the statistics $s_{\alpha,\beta} : \Pi \to G$ defined by $s_{\alpha,\beta}(\lambda) =$

$cr(\lambda)\alpha + ne(\lambda)\beta$, we have that

$$s_{\alpha,\beta}(\lambda^0) = s_{\alpha,\beta}(\lambda^1) = s_{\alpha,\beta}(\lambda), \tag{2.5}$$

$$s_{\alpha,\beta}(\lambda^j) = s_{\alpha,\beta}(\lambda) + u_j(\lambda)\alpha + v_j(\lambda)\beta, \qquad j \geq 1. \tag{2.6}$$

For simplicity, we will write λ^{ij} for $(\lambda^i)^j$.

Lemma 2.1. *For* $|\lambda| \geq 1$,

$$s_{\alpha,\beta}(\lambda^{0j})) = \begin{cases} s_{\alpha,\beta}(\lambda) & \text{if } j = 0, 1 \\ s_{\alpha,\beta}(\lambda^{j-1}) & \text{if } j \geq 2, \end{cases} \tag{2.7}$$

and for $i \geq 1$,

$$s_{\alpha,\beta}(\lambda^{ij}) = \begin{cases} s_{\alpha,\beta}(\lambda^i) & \text{if } j = 0, 1 \\ s_{\alpha,\beta}(\lambda^i) + s_{\alpha,\beta}(\lambda^{j-1}) - s_{\alpha,\beta}(\lambda^1) + \alpha & \text{if } 2 \leq j \leq i \\ s_{\alpha,\beta}(\lambda^i) + s_{\alpha,\beta}(\lambda^j) - s_{\alpha,\beta}(\lambda^1) + \beta & \text{if } j \geq i+1. \end{cases} \tag{2.8}$$

Proof. We first show (2.8). The first line in (2.8) follows directly from (2.5). For the other two,

$$\begin{aligned} s_{\alpha,\beta}(\lambda^{ij}) &= s_{\alpha,\beta}(\lambda^i) + u_j(\lambda^i)\alpha + v_j(\lambda^i)\beta \\ &= \begin{cases} s_{\alpha,\beta}(\lambda^i) + u_{j-1}(\lambda)\alpha + \alpha + v_{j-1}(\lambda)\beta & \text{if } 2 \leq j \leq i \\ s_{\alpha,\beta}(\lambda^i) + u_j(\lambda)\alpha + v_j(\lambda)\beta + \beta & \text{if } j \geq i+1 \end{cases} \\ &= \begin{cases} s_{\alpha,\beta}(\lambda^i) + s_{\alpha,\beta}(\lambda^{j-1}) - s_{\alpha,\beta}(\lambda^1) + \alpha & \text{if } 2 \leq j \leq i \\ s_{\alpha,\beta}(\lambda^i) + s_{\alpha,\beta}(\lambda^j) - s_{\alpha,\beta}(\lambda^1) + \beta & \text{if } j \geq i+1. \end{cases} \end{aligned}$$

The first and third equality follow from (2.6) and the second one follows from (2.3) and (2.4). Similarly, (2.7) follows from (2.1), (2.2), (2.5), and (2.6). □

To each partition λ with k blocks, $(k \geq 1)$, we associate a sequence

$$seq_{\alpha,\beta}(\lambda) := s_{\alpha,\beta}(\lambda^1)s_{\alpha,\beta}(\lambda^2)\dots s_{\alpha,\beta}(\lambda^k)$$

The sequence $seq_{\alpha,\beta}(\lambda)$ encodes the information about the distribution of $s_{\alpha,\beta}$ on the children of λ in $\mathcal{T}(\Pi)$, in which $s_{\alpha,\beta}(\lambda^1)$ plays a special role when we analyze the change of $seq_{\alpha,\beta}(\lambda)$ below . This is due to the fact that $s_{\alpha,\beta}(\lambda^1)$ carries information about λ and two children of λ, namely, λ^0 and λ^1.

For an abelian group G, let G_l^* denote the set of finite sequences of length l over G, and $G^* = \cup_{l\geq 1} G_l^*$. If $u = x_1x_2\dots x_k \in G^*$ and $y \in G$, then we use the convention that the sequence $(x_1+y)(x_2+y)\dots(x_k+y)$ is denoted by $x_1x_2\dots x_k + y$.

Definition 2.2. For $\alpha, \beta \in G$ and $i \geq 1$, define $R_{\alpha,\beta,i} : G_l^* \to G_l^*$, $(i \leq l)$ by setting

$$R_{\alpha,\beta,i}(x_1x_2\dots x_l) = x_i(x_1\dots x_{i-1} + (x_i - x_1 + \alpha))(x_{i+1}\dots x_l + (x_i - x_1 + \beta))$$

and $R_{\alpha,\beta} : G^* \to S(G^*)$ by setting

$$R_{\alpha,\beta}(x_1x_2\dots x_l) = \{R_{\alpha,\beta,i}(x_1x_2\dots x_l) : 1 \leq i \leq l\}.$$

In addition, define $M : G^* \to G^*$ by setting

$$M(x_1x_2\dots x_l) = x_1x_1x_2\dots x_l.$$

Lemma 2.1 immediately implies that

$$\begin{aligned} seq_{\alpha,\beta}(\lambda^0) &= M(seq_{\alpha,\beta}(\lambda)), \\ seq_{\alpha,\beta}(\lambda^i) &= R_{\alpha,\beta,i}(seq_{\alpha,\beta}(\lambda)), \qquad \text{for } 1 \le i \le |\lambda|. \end{aligned}$$

For $l \geq 0$, let $E_{\alpha,\beta}(\lambda, l, m) = \{seq_{\alpha,\beta}(\mu) : \mu \in \mathcal{T}(\lambda, l, m)\}$, the multiset of sequences $seq_{\alpha,\beta}(\mu)$ associated to partitions $\mu \in \mathcal{T}(\lambda, l, m)$. Then for $l \geq 1$,

$$E_{\alpha,\beta}(\lambda, l, m) = R_{\alpha,\beta}(E_{\alpha,\beta}(\lambda, l-1, m)) \cup M(E_{\alpha,\beta}(\lambda, l-1, m-1)). \tag{2.9}$$

Next, we define an auxiliary function f which reflects the change of the statistic $s_{\alpha,\beta}$ along $\mathcal{T}(\Pi)$. Then we prove two general properties of f and use these properties to prove Theorem 1.1. For an integer $r \geq 0$ and $\gamma \in G$, the function $f : G^* \to S(G)$ is defined by

$$f^r_\gamma(x_1 x_2 \dots x_l) := \{x_{a_1} + x_{a_2} + \cdots + x_{a_r} - (r-1)x_1 + \gamma : 1 < a_1 < a_2 < \cdots < a_r \leq l\}$$

In particular,

$$\begin{aligned} f^0_0(x_1 x_2 \dots x_l) &= \{x_1\}, \\ f^1_0(x_1 x_2 \dots x_l) &= \{x_2, \dots, x_l\}. \end{aligned}$$

Lemma 2.3. *Let $X, Y \in S(G^*)$ be two multisets such that $f^r_\gamma(X) = f^r_\gamma(Y)$ for every $r \geq 0$ and $\gamma \in G$. Then*

(a) $f^r_\gamma(M(X)) = f^r_\gamma(M(Y))$,

(b) $f^r_\gamma(R_{\alpha,\beta}(X)) = f^r_\gamma(R_{\alpha,\beta}(Y))$,

(c) $f^r_\gamma(R_{\alpha,\beta}(X)) = f^r_\gamma(R_{\beta,\alpha}(Y))$,

for every $r \geq 0$ and $\gamma \in G$.

Proof. (a) The elements in $f^r_\gamma(M(X))$ have the form $y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1 + \gamma$ for some $y_1 y_2 \dots y_{l+1} \in M(X)$, where $y_1 y_2 \dots y_{l+1} = x_1 x_1 x_2 \dots x_l$ for some $x_1 x_2 \dots x_l \in X$. For $r = 0$,

$$f^0_\gamma(M(X)) = \{y_1 + \gamma : y_1 y_2 \dots y_{l+1} \in M(X)\} = \{x_1 + \gamma : x_1 x_2 \dots x_l \in X\} = f^0_\gamma(X).$$

Hence $f^0_\gamma(X) = f^0_\gamma(Y)$ implies $f^0_\gamma(M(X)) = f^0_\gamma(M(Y))$.

For $r \geq 1$, divide the multiset $f^r_\gamma(M(X))$ into two disjoint multisets,

$$A = \{y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1 + \gamma : y_1 y_2 \dots y_{l+1} \in M(X),\ a_1 = 2\}$$

and

$$B = \{y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1 + \gamma : y_1 y_2 \dots y_{l+1} \in M(X),\ a_1 > 2\}.$$

The elements of A can be written as

$$\begin{aligned} y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1 + \gamma &= x_1 + y_{a_2} + \cdots + y_{a_r} - (r-1)x_1 + \gamma \\ &= y_{a_2} + \cdots + y_{a_r} - (r-2)x_1 + \gamma \\ &= x_{a_2-1} + \cdots + x_{a_r-1} - (r-2)x_1 + \gamma. \end{aligned}$$

Since $a_2-1>a_1-1=1$, the multiset A is equal to $f_\gamma^{r-1}(X)$. The elements in B can be written as

$$y_{a_1}+y_{a_2}+\cdots+y_{a_r}-(r-1)y_1+\gamma = x_{a_1-1}+x_{a_2-1}+\cdots+x_{a_r-1}-(r-1)x_1+\gamma.$$

Since $a_1\geq 3$, the indices on the right-hand side run through all the increasing r-tuples $1<a_1-1<a_2-1<\cdots<a_r-1\leq l$. Therefore, B is equal to $f_\gamma^r(X)$. So,

$$f_\gamma^r(M(X)) = f_\gamma^{r-1}(X)\cup f_\gamma^r(X). \tag{2.10}$$

By assumption we have

$$f_\gamma^r(M(X)) = f_\gamma^{r-1}(X)\cup f_\gamma^r(X) = f_\gamma^{r-1}(Y)\cup f_\gamma^r(Y) = f_\gamma^r(M(Y)).$$

(c) We will prove only (c) because the proof of (b) is similar and easier. Since $f_\gamma^r(X)$ is a translation of $f_0^r(X)$ by γ, it is enough to prove the result for $\gamma=0$ only. The elements of $f_0^r(R_{\alpha,\beta}(X))$ have the form $y_{a_1}+y_{a_2}+\cdots+y_{a_r}-(r-1)y_1$, where $y_1y_2\ldots y_l\in R_{\alpha,\beta}(X)$ is equal to $x_i(x_1\ldots x_{i-1}+x_i-x_1+\alpha)(x_{i+1}\ldots x_l+x_i-x_1+\beta)$ for some $x_1x_2\ldots x_l\in X$ and $i\in[l]$.

For $0\leq t\leq r$, let

$$\begin{aligned}C_{t,\alpha,\beta}(X)=\{&y_{a_1}+y_{a_2}+\cdots+y_{a_r}-(r-1)y_1 :\\ &y_1y_2\ldots y_l\in R_{\alpha,\beta,i}(X) \text{ and } a_t\leq i<a_{t+1}, \text{ for some } i\in[l]\}.\end{aligned}$$

An element $y_{a_1}+y_{a_2}+\cdots+y_{a_r}-(r-1)y_1\in C_{t,\alpha,\beta}(X)$ is equal to

$$\begin{aligned}&x_{a_1-1}+\cdots x_{a_t-1}+t(x_i-x_1+\alpha)+x_{a_{t+1}}+\cdots+x_{a_r}+(r-t)(x_i-x_1+\beta)-(r-1)x_i\\ =\;&x_{a_1-1}+\cdots x_{a_t-1}+x_i+x_{a_{t+1}}+\cdots+x_{a_r}-rx_1+t\alpha+(r-t)\beta.\end{aligned} \tag{2.11}$$

Again, we consider two cases, according to the value of a_1. By (2.11), the submultiset of $C_{t,\alpha,\beta}(X)$ for $a_1>2$ is equal to $f^{r+1}_{t\alpha+(r-t)\beta}(X)$, and for $a_1=2$ the corresponding submultiset is equal to $f^{r}_{t\alpha+(r-t)\beta}(X)$. Therefore,

$$C_{t,\alpha,\beta}(X) = f^{r+1}_{t\alpha+(r-t)\beta}(X)\cup f^{r}_{t\alpha+(r-t)\beta}(X). \tag{2.12}$$

Similarly,

$$C_{t,\beta,\alpha}(Y) = f^{r+1}_{t\beta+(r-t)\alpha}(Y)\cup f^{r}_{t\beta+(r-t)\alpha}(Y). \tag{2.13}$$

So,

$$\begin{aligned}f_0^r(R_{\alpha,\beta}(X)) &= \bigcup_{t=0}^{r} C_{t,\alpha,\beta}(X)\\ &= \bigcup_{t=0}^{r} f^{r+1}_{t\alpha+(r-t)\beta}(X)\cup\bigcup_{t=0}^{r} f^{r}_{t\alpha+(r-t)\beta}(X)\\ &= \bigcup_{t=0}^{r} f^{r+1}_{t\alpha+(r-t)\beta}(Y)\cup\bigcup_{t=0}^{r} f^{r}_{t\alpha+(r-t)\beta}(Y)\\ &= \bigcup_{t=0}^{r} f^{r+1}_{(r-t)\alpha+t\beta}(Y)\cup\bigcup_{t=0}^{r} f^{r}_{(r-t)\alpha+t\beta}(Y)\\ &= \bigcup_{t=0}^{r} C_{t,\beta,\alpha}(Y)\\ &= f_0^r(R_{\beta,\alpha}(Y)).\end{aligned}$$

The second and fifth equality follow from (2.12) and (2.13) respectively. The third equality follows from the assumption of the lemma, while the fourth equality is just a reordering of the unions. □

Lemma 2.4. *If $X, Y \in S(G^*)$ are one-element sets such that $f_0^0(X) = f_0^0(Y)$ and $f_0^1(X) = f_0^1(Y)$, then $f_\gamma^r(X) = f_\gamma^r(Y)$ for every $r \geq 0$ and $\gamma \in G$.*

Proof. We need to prove that if $u, v \in G^*$ are two sequences beginning with the same term and having equal numbers of occurrences of each $g \in G$, then $f_\gamma^r(u) = f_\gamma^r(v)$ for every $r \geq 0$ and $\gamma \in G$. It suffices to prove the statement for $\gamma = 0$, because $f_\gamma^r(u)$ is a translation of $f_0^r(u)$ by γ. Let $u = u_1 \dots u_l$ and $v = v_1 \dots v_l$. Since $u_1 = v_1$, it suffices to prove that the multisets $\{u_{a_1} + u_{a_2} + \cdots + u_{a_r} : 1 < a_1 < a_2 < \cdots < a_r \leq l\}$ and $\{v_{a_1} + v_{a_2} + \cdots + v_{a_r} : 1 < a_1 < a_2 < \cdots < a_r \leq l\}$ are equal. That is clear because $\{u_2, \dots, u_l\}$ and $\{v_2, \dots, v_l\}$ are equal as multisets. □

Proof of Theorem 1.1 . We prove (b), the proof of (a) is similar. First, we prove by induction on l that

$$f_\gamma^r(E_{\alpha,\beta}(\lambda, l, m)) = f_\gamma^r(E_{\beta,\alpha}(\pi, l, m)) \text{ for every } r \geq 0 \text{ and } \gamma \in G. \tag{2.14}$$

Before we proceed with the induction, it is useful to observe that the assumption

$$s_{\alpha,\beta}(\mathcal{T}(\lambda, l)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l)) \text{ for } l = 0, 1$$

of Theorem (1.1) (b) is equivalent to

$$s_{\alpha,\beta}(\mathcal{T}(\lambda, l, m)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l, m)) \text{ for } l = 0, 1 \text{ and } k \leq m \leq k + l, \tag{2.15}$$

where $k = |\lambda|$. One direction is clear, the other one follows from the following equations.

$$s_{\alpha,\beta}(\mathcal{T}(\lambda, 1, k+1)) = s_{\alpha,\beta}(\mathcal{T}(\lambda, 0, k)) = s_{\alpha,\beta}(\mathcal{T}(\lambda, 0)),$$
$$s_{\alpha,\beta}(\mathcal{T}(\lambda, 1, k)) = s_{\alpha,\beta}(\mathcal{T}(\lambda, 1)) \backslash s_{\alpha,\beta}(\mathcal{T}(\lambda, 0)),$$

where $\backslash$ is the difference of multisets. For the same reason the assumption of part (a) is equivalent to

$$s_{\alpha,\beta}(\mathcal{T}(\lambda, l, m)) = s_{\alpha,\beta}(\mathcal{T}(\pi, l, m)) \text{ for } l = 0, 1 \text{ and } k \leq m \leq k + l.$$

Now we show (2.14). For $l = 0$ we need to show $f_\gamma^r(E_{\alpha,\beta}(\lambda, 0, k)) = f_\gamma^r(E_{\beta,\alpha}(\pi, 0, k))$. By Lemma 2.4 we only need to check that $f_0^0(X) = f_0^0(Y)$ and $f_0^1(X) = f_0^1(Y)$ for $X = \{seq_{\alpha,\beta}(\lambda)\}$ and $Y = \{seq_{\beta,\alpha}(\pi)\}$. This follows from (2.15), because $f_0^0(X) = s_{\alpha,\beta}(\mathcal{T}(\lambda, 0, k))$ and $f_0^1(X) = s_{\alpha,\beta}(\mathcal{T}(\lambda, 1, k))$.

Suppose $f_\gamma^r(E_{\alpha,\beta}(\lambda, s, m)) = f_\gamma^r(E_{\beta,\alpha}(\pi, s, m))$ for all $0 \leq s < l$ and all m. Then using (2.9), the induction hypothesis, and Lemma 2.3 we have

$$\begin{aligned} f_\gamma^r(E_{\alpha,\beta}(\lambda, l, m)) &= f_\gamma^r(R_{\alpha,\beta}(E_{\alpha,\beta}(\lambda, l-1, m))) \cup f_\gamma^r(M(E_{\alpha,\beta}(\lambda, l-1, m-1))) \\ &= f_\gamma^r(R_{\beta,\alpha}(E_{\beta,\alpha}(\pi, l-1, m))) \cup f_\gamma^r(M(E_{\beta,\alpha}(\pi, l-1, m-1))) \\ &= f_\gamma^r(E_{\beta,\alpha}(\pi, l, m)), \end{aligned}$$

and the induction is completed. This proves (2.14). Now

$$s_{\alpha,\beta}(\mathcal{T}(\lambda, l, m)) = f_0^0(E_{\alpha,\beta}(\lambda, l, m)) = f_0^0(E_{\beta,\alpha}(\pi, l, m)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l, m)).$$

□

3 Applications and Examples

As a direct corollary, we obtain a result of Kasraoui and Zeng [4, Eq.(1.6)].

Corollary 3.1. *The joint distribution of crossings and nestings of partitions is symmetric i.e.*

$$\sum_{\pi\in\Pi_n} p^{cr(\pi)}q^{ne(\pi)} = \sum_{\pi\in\Pi_n} p^{ne(\pi)}q^{cr(\pi)}$$

Proof. Let $G = (\mathbb{Z}\oplus\mathbb{Z}, +), \alpha = (1, 0)$ and $\beta = (0, 1)$. The result follows from the second part of Theorem 1.1 for $\lambda = \pi = \{\{1\}\}$. □

For a partition λ we say that two edges form an alignment if they neither form a crossing nor a nesting. The total number of alignments in λ is denoted by $al(\lambda)$. A stronger result of Kasraoui and Zeng [4, Eq. (1.4)] can also be derived from Theorem 1.1.

Corollary 3.2.

$$\sum_{\pi\in\Pi_n} p^{cr(\pi)}q^{ne(\pi)}t^{al(\pi)} = \sum_{\pi\in\Pi_n} p^{ne(\pi)}q^{cr(\pi)}t^{al(\pi)}$$

Proof. Again we use $G = (\mathbb{Z}\oplus\mathbb{Z}, +)$, $\alpha = (1, 0)$, $\beta = (0, 1)$, and $\lambda = \pi = \{\{1\}\}$. Any partition $\mu \in \Pi_n$ with k blocks has $n-k$ edges. Hence $cr(\mu) + ne(\mu) + al(\mu) = \binom{n-k}{2}$. The result follows from the second part of Theorem 1.1. □

Corollary 3.3. *Let λ and π be two partitions of $[n]$ with same number of blocks k. If the statistic al is equidistributed on the first two levels of $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$, it is equidistributed on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ for all $l, m \geq 0$.*

Proof. Again we use the identity $cr(\mu) + ne(\mu) + al(\mu) = \binom{n-k}{2}$, which holds for any partition $\mu \in \Pi_n$ with k blocks. Moreover, $al(\lambda) = al(\lambda^0)$. Therefore the condition that the statistic al is equidistributed on the first two levels of $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$ implies that the statistic $cr + ne$ is equidistributed on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ for all $l = 0, 1$ and all m. In other words, if we set $G = \mathbb{Z}$ and $\alpha = \beta = 1$ then the the assumption of Theorem 1.1 is satisfied, and hence $cr + ne$ is equidistributed on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ for all $l, m \geq 0$. This, in return, implies that al is equidistributed on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ for all $l, m \geq 0$. □

Example 3.4. *Let $\lambda = \{\{1, 2, 5\}, \{3, 4\}\}$ and $\pi = \{\{1, 2, 4\}, \{3, 5\}\}$. There are as many partitions on $[n]$ with m crossings and l nestings which restricted to the last five points form a partition isomorphic to λ as there are partitions of $[n]$ with l crossings and m nestings which restricted to the last five points form a partition isomorphic to π.*

Proof. Set $G = (\mathbb{Z}\oplus\mathbb{Z}, +), \alpha = (1, 0)$ and $\beta = (0, 1)$, $s_{\alpha,\beta} = (cr, ne)$. The claim follows from part (b) of Theorem 1.1 since $s_{\alpha,\beta}(\lambda) = (0, 1) = s_{\beta,\alpha}(\pi)$ and $s_{\alpha,\beta}(\mathcal{T}(\lambda, 1)) = \{(0, 1), (0, 1), (1, 2)\} = s_{\beta,\alpha}(\mathcal{T}(\pi, 1))$ □

Example 3.5. *Let $\lambda = \{\{1, 7\}, \{2, 6\}, \{3, 4\}, \{5, 8\}\}$ and $\pi = \{\{1, 8\}, \{2, 4\}, \{3, 6\}, \{5, 7\}\}$. There are as many partitions on $[n]$ with m crossings and l nestings which restricted to the last eight points form a partition isomorphic to λ as there are ones which restricted to the last eight points form a partition isomorphic to π.*

Proof. Again set $G = (\mathbb{Z} \oplus \mathbb{Z}, +), \alpha = (1, 0)$ and $\beta = (0, 1)$. Then $s_{\alpha,\beta} = (cr, ne)$. The claim follows from part (a) of Theorem 1.1 since

$$s_{\alpha,\beta}(\lambda) = (2, 3) = s_{\alpha,\beta}(\pi)$$

and

$$s_{\alpha,\beta}(\mathcal{T}(\lambda, 1)) = \{(2, 3), (2, 3), (3, 3), (4, 3), (4, 4)\} = s_{\alpha,\beta}(\mathcal{T}(\pi, 1)).$$

□

4 Number of Crossing and Nesting-similarity Classes

In this section we consider equivalence relations $\sim_{cr}$ and $\sim_{ne}$ on set partitions in the same way Klazar defines them on matchings [5] . We determine the number of crossing-similarity classes in $\Pi_{n,k}$. For $\sim_{ne}$, we find a recurrence relation for the number of nesting-similarity classes in $\Pi_{n,k}$, and compute the total number of such classes in Π_n.

Define an equivalence relation $\sim_{cr}$ on Π_n: $\lambda \sim_{cr} \pi$ if and only if $cr(\mathcal{T}(\lambda, l, m)) = cr(\mathcal{T}(\pi, l, m))$ for all $l, m \geq 0$. The relation $\sim_{cr}$ partitions $\Pi_{n,k}$ into equivalence classes. Theorem 1.1 implies that $\lambda \sim_{cr} \pi$ if and only if $cr(\lambda) = cr(\pi)$ and $f_0^1(seq_{1,0}(\lambda)) = f_0^1(seq_{1,0}(\pi))$. Define $crseq(\lambda) = seq_{1,0}(\lambda) - cr(\lambda)$. For the upcoming computations it is useful to observe that $\lambda \sim_{cr} \pi$ if and only if $cr(\lambda) = cr(\pi)$ and $f_0^1(crseq(\lambda)) = f_0^1(crseq(\pi))$, i.e., λ and π are equivalent if and only if they have the same number of crossings and their sequences $crseq(\lambda)$ and $crseq(\pi)$ are equal as multisets. Denote the multiset consisting of the elements of $crseq(\lambda)$ by $crset(\lambda)$.

Similarly, define $\lambda \sim_{ne} \pi$ if and only if $ne(\mathcal{T}(\lambda, l, m)) = ne(\mathcal{T}(\pi, l, m))$ for all $l, m \geq 0$. Again, from Theorem 1.1 we have that $\lambda \sim_{ne} \pi$ if and only if $ne(\lambda) = ne(\pi)$ and $f_0^1(seq_{0,1}(\lambda)) = f_0^1(seq_{0,1}(\pi))$. Since the sequence $seq_{0,1}(\lambda)$ is nondecreasing, $\lambda \sim_{ne} \pi$ if and only if $ne(\lambda) = ne(\pi)$ and $seq_{0,1}(\lambda) - ne(\lambda) = seq_{0,1}(\pi) - ne(\pi)$. With the notation at the beginning of Section 2, $seq_{0,1}(\lambda) - ne(\lambda) = v_1 \dots v_k$. Denote this sequence by $neseq(\lambda)$.

A Motzkin path $M = (s_1, \dots, s_n)$ is a path from $(0, 0)$ to $(n, 0)$ consisting of steps $s_i \in \{(1, 1), (1, 0), (1, -1)\}$ which does not go below the x-axis. We say that the step s_i is of height l if its left endpoint is at the line $y = l$. A restricted bicolored Motzkin path is a Motzkin path with each horizontal step colored red or blue which does not have a blue horizontal step of height 0. We will denote the steps $(1, 1)$, $(1, -1)$, red $(1, 0)$, and blue $(1, 0)$ by NE (northeast), SE (southeast), RE (red east), and BE (blue east) respectively. The set of all restricted bicolored Motzkin paths of length n is denoted by RBM_n. A Charlier diagram of length n is a pair $h = (M, \xi)$ where $M = (s_1, \dots, s_n) \in RBM_n$ and $\xi = (\xi_1, \dots, \xi_n)$ is a sequence of integers such that $\xi_i = 1$ if s_i is a NE or RE step, and $1 \leq \xi_i \leq l$ if s_i is a SE or BE step of height l. Γ_n will denote the set of Charlier diagrams of length n.

It is well known that partitions are in one-to-one correspondence with Charlier diagrams. Here we use two maps described in [4], which are based on similar constructions in [3, 10]. For our purpose, we reformulate the maps $\Phi_r, \Phi_l : \Gamma_n \to \Pi_n$ as follows. Given $(M, \xi) \in \Gamma_n$, construct $\lambda \in \Pi_n$ step by step. The path $M = (s_1, \dots, s_n)$ determines the type of λ: $i \in [n]$ is

- a minimal but not a maximal element of a block of λ (opener) if and only if s_i is a NE step;
- a maximal but not a minimal element of a block of λ (closer) if and only if s_i is a SE step;

- both a minimal and a maximal element of a bock of λ (singleton) if and only if s_i is a RE step;

- neither a minimal nor a maximal element of a block of λ (transient) if and only if s_i is a BE step.

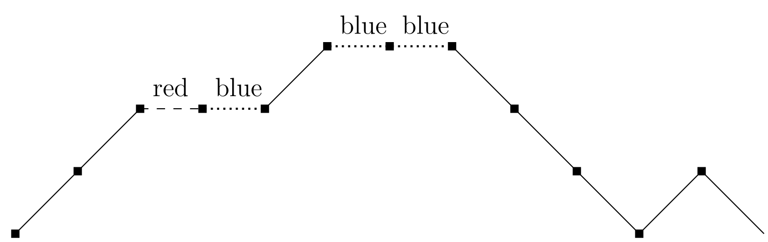
To draw the edges in $\Phi_r((M,\xi))$, we process the closers and transients one by one from left to right. Each time we connect the vertex i that we are processing to the ξ_i-th available opener or transient to the left of i, where the openers and transients are ranked from right to left. If we rank the openers and transients from left to right, we get $\Phi_l((M,\xi))$. It can be readily checked that Φ_r and Φ_l are well defined. Moreover:

Proposition 4.1. *The maps $\Phi_r, \Phi_l : \Gamma_n \to \Pi_n$ are bijections.*

The proof can be found in [3, 4] and their references.

Example 4.2. *If (M,ξ) is the Charlier diagram in Figure 2, then*

$$\Phi_r((M,\xi)) = \{\{1,7,10\},\{2,4,6,8\},\{3\},\{5,9\},\{11,12\}\}$$
$$\Phi_l((M,\xi)) = \{\{1,4,6,7,9\},\{2,10\},\{3\},\{5,8\},\{11,12\}\}$$



i	1	2	3	4	5	6	7	8	9	10	11	12
ξ_i	1	1	1	1	1	2	3	2	2	1	1	1

Figure 2: A Charlier diagram

For $M \in RBM_n$ let d_i be the number of NE and RE steps that start at height i, $(i \geq 0)$. The *profile* of M is the sequence $pr(M) = (d_0, \dots, d_l)$, where $l = \max\{i : d_i \neq 0\}$. Note that this implies that $d_i \geq 1$ for each $i = 0, \dots, l$, and that the path M is of height l or $l+1$. The *semi-type* of $M = (s_1, \dots, s_i)$ is the sequence $st(M) = (\epsilon_1, \dots, \epsilon_n)$ where $\epsilon_i = 0$ if s_i is a NE or RE step, and $\epsilon_i = 1$ if s_i is a SE or BE step. For example, if M is the path in Figure 2, then $pr(M) = (2,1,2)$, and $st(M) = (0,0,0,1,0,1,1,1,1,1,0,1)$.

Let $\lambda \in \Pi_n$ and $\Phi_r^{-1}(\lambda) = (M, \xi^r)$, $\Phi_l^{-1}(\lambda) = (M, \xi^l)$. Define $\varphi(\lambda) = M \in RBM_n$. Note that for a given λ, $\varphi(\lambda)$ can be easily constructed using the four steps above. The next lemma gives the relation between a partition and its corresponding restricted bicolored Motzkin path and Charlier diagram.

Lemma 4.3. *Let Φ_r, Φ_l and φ be the maps defined above and $\Phi_r^{-1}(\lambda) = (M, \xi^r)$, $\Phi_l^{-1}(\lambda) = (M, \xi^l)$.*

(a) The number of blocks of λ is equal to the total number of NE and RE steps of M.

(b) $cr(\lambda) = \sum_{i=1}^{n} (\xi_i^r - 1)$, $ne(\lambda) = \sum_{i=1}^{n} (\xi_i^l - 1)$.

(c) $pr(M) = (d_0, \ldots, d_l)$ if and only if $crset(\lambda) = \{0^{d_0}, \ldots, l^{d_l}\}$.

(d) $neseq(\lambda) = v_1 \ldots v_k$ if and only if the zeros in $st(M) = (\epsilon_1, \ldots, \epsilon_n)$ are in the positions $v_1 + 1, v_2 + 2, \ldots, v_k + k$.

Proof. (a) The result follows from the fact that the number of blocks of λ is equal to the total number of openers and singletons.

(b) Denote by E be the set of arcs of λ. For $e = (i, j) \in E$ let $c_e = |\{(p, q) \in E : i < p < j < q\}|$. Then $cr(\lambda) = \sum_{e \in E} c_e$. Similarly, if $n_e = |\{(p, q) \in E : p < i < j < q\}|$, then $ne(\lambda) = \sum_{e \in E} n_e$. From the definitions of Φ_r and Φ_l it follows that $c_{(i,j)} = \xi_j^r - 1$ and $n_{(i,j)} = \xi_j^l - 1$. Hence the claim.

(c) Using the notation at the beginning of Section 2, we have $crseq(\lambda) = (u_1, \ldots, u_k)$, where u_i is the number of edges (p, q) such that $p < b_i < q$. Here $b_i = \min B_i$, that is, b_i is the i-th opener or singleton from left to right. But the step in M which corresponds to b_i is of height h if and only if $u_i = h$.

(d) It follows directly from the definitions of $neseq(\lambda)$ and $st(M)$. $\square$

A composition of k is an ordered tuple $(d_0, \ldots, d_l)$ of positive integers whose sum is k.

Lemma 4.4. *Let $l \geq 0, k \geq 1$, and $n \geq k$.*

(a) If $\lambda \in \Pi_{n,k}$ and $crset(\lambda) = \{0^{d_0}, \ldots, l^{d_l}\}$, then $(d_0, \ldots, d_l)$ is a composition of k into $l + 1$ parts, where $l \leq n - k$, and $0 \leq cr(\lambda) \leq (n - k - 1)l - \frac{l(l-1)}{2}$.

(b) Given a composition $(d_0, \ldots, d_l)$ of k into $l + 1 \leq n - k + 1$ parts and an integer c such that $0 \leq c \leq (n-k-1)l - \frac{l(l-1)}{2}$, there exists $\lambda \in \Pi_{n,k}$ with $crset(\lambda) = \{0^{d_0}, \ldots, l^{d_l}\}$ and $cr(\lambda) = c$.

Proof. (a) It is clear that $d_0 + \cdots + d_l = k$. It follows that all the d_i's are positive from part (c) of Lemma 4.3. Moreover, λ has at least l openers and, therefore, at least l closers. So, $k + l \leq n$, i.e., $l + 1 \leq n - k + 1$. Let c_i (respectively t_i) be the number of SE (respectively BE) steps at level i, $1 \leq i \leq l + 1$. Then $\sum_{i=1}^{l+1} (c_i + t_i) = n - k$ and $c_i \geq 1$, $1 \leq i \leq l$. Using part (b) of Lemma 4.3, we have $0 \leq cr(\lambda) \leq \sum_{i=1}^{l} (1 + 0)(i - 1) + (n - k - l)l = (n - k - 1)l - \frac{l(l-1)}{2}$.

(b) Suppose first that $l + 1 \leq n - k$. Let $M \in RBM_n$ consist of $d_0 - 1$ RE steps followed by a NE step, then $d_1 - 1$ RE steps followed by one NE step, etc., $d_l - 1$ RE steps followed by a NE step, then $n - k - l - 1$ BE steps, and $l + 1$ SE steps. It is not hard to see that indeed $M \in RBM_n$. The path never crosses the x-axis and all the BE steps, if any, are at hight $l + 1 \geq 1$. Also $pr(M) = (d_0, \ldots, d_l)$. Consider all the sequences $\xi = (\xi_1, \ldots, \xi_n)$ such that (M, ξ) is a Charlier diagram. Then

$$\begin{aligned} \xi_i &= 1, & 1 &\leq i \leq k \\ 1 \leq \xi_i &\leq l + 1, & k + 1 &\leq i \leq n - l - 1 \\ 1 \leq \xi_{n-i+1} &\leq i, & 1 &\leq i \leq l + 1. \end{aligned} \tag{4.1}$$

Hence

$$0 \le \sum_{i=1}^{n} (\xi_i - 1) \le (n-k-l-1)l + l + (l-1) + \cdots + 1 = (n-k-1)l - \frac{l(l-1)}{2}. \tag{4.2}$$

In the case $l = n-k$, construct $M \in RBM_n$ similarly: $d_0 - 1$ RE steps followed by a NE step, then $d_1 - 1$ RE steps followed by one NE step, etc., d_l RE steps, followed by l SE steps. (Note that, unlike in the case $l < n-k$, the path M is of height l) All the sequences $\xi = (\xi_1, \ldots, \xi_n)$ such that (M, ξ) is a Charlier diagram satisfy the following properties:

$$\begin{aligned} \xi_i = 1, &\qquad 1 \le i \le k \\ 1 \le \xi_{n-i+1} \le i, &\qquad 1 \le i \le l. \end{aligned} \tag{4.3}$$

Hence

$$0 \le \sum_{i=1}^{n} (\xi_i - 1) \le (l-1) + \cdots + 1 = (n-k-1)l - \frac{l(l-1)}{2}. \tag{4.4}$$

Because of (4.2) (respectively (4.4)), for any integer c between 0 and $(n-k-1)l - \frac{l(l-1)}{2}$, ξ can be chosen to satisfy the conditions (4.1) (respectively (4.3)) and such that $\sum_{i=1}^{n}(\xi_i - 1) = c$. Since Φ is a bijection, there is $\lambda \in \Pi_{n,k}$ such that $\Phi(\lambda) = (M, \xi)$ and, by part (b) and (c) of Lemma 4.3, $cr(\lambda) = c$ and $crset(\lambda) = \{0^{d_0}, \ldots, l^{d_l}\}$. □

Theorem 4.5. *Let $n \ge k \ge 1$ and $m = \min\{n-k, k-1\}$. Then*

$$|\Pi_{n,k}/\sim_{cr}| = \sum_{l=0}^{m} \binom{k-1}{l} [(n-k-1)l - \frac{l(l-1)}{2} + 1]. \tag{4.5}$$

In particular, if $n \ge 2k-1$,

$$|\Pi_{n,k}/\sim_{cr}| = (n-k-1)(k-1)2^{k-2} + 2^{k-1} - (k-1)(k-2)2^{k-4}. \tag{4.6}$$

Proof. Recall that $\lambda \sim_{cr} \pi$ if and only if $cr(\lambda) = cr(\pi)$ and $crset(\lambda) = crset(\pi)$. Therefore, $|\Pi_{n,k}/\sim_{cr}| = |\{(crset(\lambda), cr(\lambda)) : \lambda \in \Pi_{n,k}\}|$. Using Lemma 4.4 and the fact that the number of compositions of k into $l+1$ parts, $0 \le l \le k-1$, is $\binom{k-1}{l}$, we derive (4.5). In particular, when $n \ge 2k-1$,

$$|\Pi_{n,k}/\sim_{cr}| = \sum_{l=0}^{k-1} \binom{k-1}{l} [(n-k-1)l - \frac{l(l-1)}{2} + 1].$$

But

$$\sum_{l=0}^{k-1}\binom{k-1}{l} = (1+x)^{k-1}|_{x=1} = 2^{k-1},$$
$$\sum_{l=0}^{k-1} l\binom{k-1}{l} = \left(\frac{d}{dx}(1+x)^{k-1}\right)|_{x=1} = (k-1)(1+x)^{k-2}|_{x=1}$$
$$= (k-1)2^{k-2},$$
$$\sum_{l=0}^{k-1} l(l-1)\binom{k-1}{l} = \left(\frac{d^2}{dx^2}(1+x)^{k-1}\right)|_{x=1}$$
$$= (k-1)(k-2)(1+x)^{k-3}|_{x=1} = (k-1)(k-2)2^{k-3},$$

and (4.6) follows. □

Theorem 4.5 implies that there are many more examples of different partitions λ and π for which the statistic cr has same distribution on the the levels of $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$. For example, $|\Pi_{2k,k}| > (2k-1)!! \approx \sqrt{2}\left(\frac{2k}{e}\right)^k$ while $|\Pi_{2k,k}/\sim_{cr}| \approx 3k^2 2^{k-4}$.

Next we analyze the number of nesting-similarity classes. First we derive a recurrence for the numbers $f_{n,k} = |\Pi_{n,k}/\sim_{ne}|$.

Theorem 4.6. *Let $n \geq k \geq 1$. Then*

$$f_{n,1} = 1, \tag{4.7}$$
$$f_{n,k} = \sum_{r=k-1}^{n-1} f_{r,k-1} + (k-1)\binom{n-2}{k}, \qquad k \geq 2. \tag{4.8}$$

Proof. Equation (4.7) is clear since $|\Pi_{n,1}| = 1$.

Recall that $\lambda \sim_{ne} \pi$ if and only if $ne(\lambda) = ne(\pi)$ and $neseq(\lambda) = neseq(\pi)$. By Lemma 4.3, $f_{n,k}$ is equal to the number of pairs (ϵ, c) such that there exists $\lambda \in \Pi_{n,k}$ with $ne(\lambda) = c$ and $st(\varphi(\lambda)) = \epsilon$. It is not hard to see that for a given a sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, there exists $\lambda \in \Pi_{n,k}$ such that $st(\varphi(\lambda)) = \epsilon$ if and only if ϵ has k zeros and $\epsilon_1 = 0$. Denote the set of all such sequences by $S^0_{n,k}$ and denote the set of all $\epsilon \in \{0,1\}^n$ with k zeros by $S_{n,k}$.

For a sequence $\epsilon \in S_{n,k}$ define a bicolored Motzkin path $M = M(\epsilon) = (s_1, \ldots, s_n)$ as follows. For i from n to 1 do:

- If $\epsilon_i = 0$ and s_i is not defined yet, then set s_i to be a RE step;
- If $\epsilon_i = 1$ and there is $j < i$ such that $\epsilon_j = 0$ and s_j is not defined yet, then set s_i to be a SE step and s_{j_0} to be a NE step, where $j_0 = \min\{j : \epsilon_j = 0 \text{ and } s_j \text{ is not defined yet}\}$;
- If $\epsilon_i = 1$ and there is no $j < i$ such that $\epsilon_j = 0$ and s_j has not been defined yet, set s_i to be a BE step.

Note that we build M backwards, from $(n,0)$ to $(0,0)$. Let h_i be the height of s_i and $ne(\epsilon) = \sum(h_i - 1)$, where the sum is over all the indices i such that $\epsilon_i = 1$. For example, if $\epsilon = (0,0,0,1,0,1,1,1,1,1,1,0,1)$, then

$$M(\epsilon) = (NE, NE, NE, BE, NE, BE, BE, SE, SE, SE, RE, SE).$$

The sequence of the heights of all the steps of M is $(0, 1, 2, 3, 3, 4, 4, 4, 3, 2, 1, 1)$ and $ne(\epsilon) = (3 - 1) + (4 - 1) + (4 - 1) + (4 - 1) + (3 - 1) + (2 - 1) + (1 - 1) = 14$.

Although clearly $M(\epsilon)$ stays above the x-axis, it is not necessarily a restricted bicolored Motzkin path. The reason is that any 1 in ϵ before the first zero would produce a BE step on the x-axis. Hence, $M(\epsilon) \in RBM_n$ if and only if $\epsilon_1 = 0$, or equivalently, $\epsilon \in S^0_{n,k}$.

We claim that for a fixed $\epsilon \in S^0_{n,k}$, there is $\lambda \in \Pi_{n,k}$ such that $st(\varphi(\lambda)) = \epsilon$ and $ne(\lambda) = c$ if and only if $0 \le c \le ne(\epsilon)$. To show the if part, one can choose a sequence $\xi = (\xi_1, \dots, \xi_n)$ with $1 \le \xi_i \le h_i$ if $\epsilon_i = 1$, $\xi_i = 1$ if $\epsilon_i = 0$, and $\sum_{i=1}^{n} (\xi_i - 1) = c$. Then Lemma 4.3 implies that $\Phi_l((M, \xi))$ satisfies the requirements. Conversely, suppose $\lambda \in \Pi_{n,k}$ is such that $st(\varphi(\lambda)) = \epsilon$. Let $\Phi_l^{-1}(\lambda) = (M', \xi')$. Then the height $h_i^{'}$ of each BE and SE step of M' satisfies

$$h_i^{'} \le \min\{\#\text{ zeros in } (\epsilon_1, \dots, \epsilon_{i-1}), (\#\text{ ones in } (\epsilon_{i+1}, \dots, \epsilon_n)) + 1\} = h_i.$$

Now, by Lemma 4.3,

$$ne(\lambda) = \sum (\xi_i^{\prime} - 1) \le \sum (h_i^{\prime} - 1) \le \sum (h_i - 1) = ne(\epsilon).$$

The claim is proved. Back to the proof of Theorem 4.6, we have

$$f_{n,k} = \sum_{\epsilon \in S^0_{n,k}} (ne(\epsilon) + 1) = \sum_{\epsilon \in S^0_{n,k}} (\sum (h_i - 1) + 1) = \sum_{\epsilon \in S^0_{n,k}} \sum h_i - (n - k - 1)\binom{n-1}{k-1}.$$

Set

$$g_{n,k} = \sum_{\epsilon \in S^0_{n,k}} \sum h_i \qquad \text{and} \qquad g^*_{n,k} = \sum_{\epsilon \in S_{n,k}} \sum h_i,$$

where the inner sums are taken over all the indices i such that $\epsilon_i = 1$. With this notation,

$$f_{n,k} = g_{n,k} - (n - k - 1)\binom{n-1}{k-1}. \tag{4.9}$$

The sequences $g_{n,k}$ and $g^*_{n,k}$ satisfy the following recurrence relations:

$$g_{n,k} = g_{n-1,k-1} + g^*_{n-2,k-1} + (n - k)\binom{n-2}{k-1}, \tag{4.10}$$

$$g^*_{n,k} = \sum_{r=k}^{n} g_{r,k}. \tag{4.11}$$

To see (4.10), note that if $\epsilon_n = 0$ then $(\epsilon_1, ..., \epsilon_{n-1}) \in S^0_{n-1,k-1}$ and $M(\epsilon)$ is $M(\epsilon_1, ..., \epsilon_{n-1})$ with one RE step appended, and if $\epsilon_n = 1$ then $(\epsilon_2, ..., \epsilon_{n-1}) \in S_{n-2,k-1}$ and $M(\epsilon_2, ..., \epsilon_{n-1})$ is obtained from $M(\epsilon)$ by deleting the first NE and the last SE step. For (4.11), if $\epsilon_1 = \cdots = \epsilon_{r-1} = 1$ and $\epsilon_r = 0$, then $M(\epsilon_r, \dots, \epsilon_n)$ is obtained from $M(\epsilon)$ by deleting the first $r - 1$ BE steps at level 0. Substituting (4.11) into (4.10) gives

$$g_{n,k} = \sum_{r=k-1}^{n-1} g_{r,k-1} + (n - k)\binom{n-2}{k-1}. \tag{4.12}$$

Finally, by substituting $g_{n,k}$ from (4.9) into (4.12) and simplifying, we obtain (4.8). $\square$

Corollary 4.7.

$$|\Pi_1/\sim_{ne}| = 1, \qquad |\Pi_2/\sim_{ne}| = 2$$
$$|\Pi_n/\sim_{ne}| = 2^{n-5}(n^2 - 5n + 22), \qquad n \geq 3$$

Proof. Denote $|\Pi_n/\sim_{ne}|$ by F_n. Using $F_n = \sum_{k=1}^{n} f_{n,k}$, (4.7), and (4.8), we get

$$\begin{aligned} F_n &= 1 + F_1 + \cdots + F_{n-1} + \sum_{k=2}^{n}(k-1)\binom{n-2}{k} \\ &= F_1 + \cdots + F_{n-1} + (n-4)2^{n-3} + 2, \qquad n \geq 2. \end{aligned}$$

This yields the recurrence relation

$$F_n = 2F_{n-1} + (n-3)2^{n-4}, \qquad n \geq 3$$

with initial values $F_1 = 1$ and $F_2 = 2$, which has the solution

$$F_n = 2^{n-5}(n^2 - 5n + 22), \qquad n \geq 3.$$

□

The following tables give the number of crossing/nesting-similarity classes on $\Pi_{n,k}$ for small n and k.

crossing-similarity classes

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	7	4	1	
6	1	5	11	4	5	1

nesting-similarity classes

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	2	1			
4	1	4	3	1		
5	1	7	9	4	1	
6	1	11	22	16	5	1

The two equivalence relations $\sim_{cr}$ and $\sim_{ne}$ on set partitions are not compatible. From the tables it is clear that $\sim_{cr}$ is not a refinement of $\sim_{ne}$. On the other hand, let $\pi = \{\{1,3\},\{2,4\},\{5,6\}\}$ and $\lambda = \{\{1,3,6\},\{2,4\},\{5\}\}$. It is easy to check that $\pi \sim_{ne} \lambda$, but $\pi \not\sim_{cr} \lambda$, as $cr(\pi) = 1$ and $cr(\lambda) = 2$.

5 Generating function for crossings and nestings

In this section we analyze the generating function

$$S_\pi(q,p,z) = \sum_{l \geq 0} \sum_{\lambda \in \mathcal{T}(\pi,l)} q^{cr(\lambda)} p^{ne(\lambda)} z^l$$

for a given partition π, and derive a continued fraction expansion for $S_\pi(q,p,z)$. For this we work with the group $G = \mathbb{Z} \oplus \mathbb{Z}$ and $\alpha = (1,0), \beta = (0,1)$. Fix a partition π with k blocks. Define

$E_{\alpha,\beta}(\pi,l)=\cup_{m=k}^{k+l}E_{\alpha,\beta}(\pi,l,m)$, i.e., $E_{\alpha,\beta}(\pi,l)$ is the multiset of sequences $seq_{\alpha,\beta}(\mu)$ associated to the partitions $\mu\in\mathcal{T}(\pi,l)$. A recurrence analogous to (2.9) holds. Namely, for $l\geq 1$

$$E_{\alpha,\beta}(\lambda,l)=R_{\alpha,\beta}(E_{\alpha,\beta}(\lambda,l-1))\cup M(E_{\alpha,\beta}(\lambda,l-1)). \tag{5.1}$$

For simplicity we write E_l instead of $E_{\alpha,\beta}(\pi,l)$ when there is no confusion. Define $b_{l,r}$ to be the generating function of the multiset $f_0^r(E_l)$, i.e.,

$$b_{l,r}(q,p)=\sum_{(x,y)\in f_0^r(E_l)}q^xp^y,$$

where $(x,y)\in f_0^r(E_l)$ contributes to the sum above according to its multiplicity in $f_0^r(E_l)$. By convention, let $b_{l,r}(q,p)=0$ if $f_0^r(E_l)=\emptyset$, or, one of l,r is negative. For simplicity we write $b_{l,r}$ for $b_{l,r}(q,p)$. Note that $b_{l,0}=\sum_{\lambda\in\mathcal{T}(\pi,l)}q^{cr(\lambda)}p^{ne(\lambda)}$ and hence

$$S_\pi(q,p,z)=\sum_{l\geq 0}b_{l,0}z^l. \tag{5.2}$$

By the formulas (5.1), (2.10), and the proof of part (c) of Lemma 2.3, we get

$$\begin{aligned} f_0^r(E_l) &= f_0^r(M(E_{l-1}))\cup f_0^r(R_{\alpha,\beta}(E_{l-1}))\\ &= f_0^{r-1}(E_{l-1})\cup f_0^r(E_{l-1})\cup\bigcup_{t=0}^{r}f_{t\alpha+(r-t)\beta}^{r+1}(E_{l-1})\cup\bigcup_{t=0}^{r}f_{t\alpha+(r-t)\beta}^{r}(E_{l-1}),\end{aligned}$$

which leads to a recurrent relation for $b_{l,r}$:

$$b_{l,r}=b_{l-1,r-1}+b_{l-1,r}+(\sum_{t=0}^{r}q^tp^{r-t})b_{l-1,r+1}+(\sum_{t=0}^{r}q^tp^{r-t})b_{l-1,r}.$$

Using the standard notation $[r]_{q,p}:=\frac{q^r-p^r}{q-p}$, we can write this as

Proposition 5.1.

$$b_{l,r}=b_{l-1,r-1}+(1+[r+1]_{q,p})b_{l-1,r}+[r+1]_{q,p}b_{l-1,r+1}.$$

If the sequence associated to the partition π is $x_1x_2\dots x_k$, with $x_i=u_i\alpha+v_i\beta,\ 1\leq i\leq k$, then

$$\begin{aligned} b_{0,0} &= q^{u_1}p^{v_1}\\ b_{0,r} &= \sum_{1<i_1<\cdots<i_r\leq k}q^{u_{i_1}+\cdots+u_{i_r}-(r-1)u_1}p^{v_{i_1}+\cdots+v_{i_r}-(r-1)v_1}\quad \text{for } r\geq 1.\end{aligned} \tag{5.3}$$

In particular, $b_{0,r}=0$ if $r\geq k$.

Given l and s, nonnegative integers, consider the paths from $(l,0)$ to $(0,s)$ using steps $(-1,0)$, $(-1,1)$, and $(-1,-1)$ which do not go below the x-axis. Each step $(-1,0)$ $((-1,1)$, $(-1,-1)$ respectively) starting at the line $y=r$ has weight $[r+1]_{q,p}$ $(1+[r+1]_{q,p}$, 1, respectively). The weight $w(M)$ of such a path M is defined to be the product of the weights of its steps. Let

$c_{l,s} = \sum w(M)$, where the sum is over all the paths M described above. Then from Proposition 5.1 one has

$$b_{l,0} = \sum_{0 \le s \le k-1} c_{l,s} b_{0,s}.$$

Set $a_r = [r+1]_{q,p}$ and $c_r = [r+1]_{q,p} + 1$. By the well-known theory of continued fractions (see [3]), $c_{l,s}$ is equal to the coefficient in front of z^l in

$$K_s(z) := J^{/0/}(z) a_0 z J^{/1/}(z) a_1 z \cdots J^{/s/}(z) = \frac{1}{z^s}(Q_{s-1}(z)J(z) - P_{s-1}(z)) \tag{5.4}$$

where

$$J^{/h/}(z) = \cfrac{1}{1 - c_h z - \cfrac{a_h z^2}{1 - c_{h+1} z - \cfrac{a_{h+1} z^2}{\ddots}}}$$

and $\frac{P_k(z)}{Q_k(z)}$ is the k-th convergent of $J(z) := J^{/0/}(z)$. Hence

Theorem 5.2. *Let π be a partition with k blocks whose associated sequence is $x_1 x_2 \dots x_k$, where $x_i = u_i\alpha + v_i\beta$ for $1 \le i \le k$. Then*

$$S_\pi(q, p, z) = \sum_{0 \le s \le k-1} b_{0,s} K_s(z),$$

where $b_{0,s}$ is given by the formula (5.3)*, and $K_s(z)$ is given by* (5.4)*.*

In particular, when $k = 1$, i.e., π is a partition with only one block, then $b_{0,0} = 1$ and $b_{l,0} = c_{l,0} b_{0,0} = c_{l,0}$. Therefore

Corollary 5.3. *If $|\pi| = 1$, then*

$$S_\pi(q, p, z) = \cfrac{1}{1 - ([1]_{q,p} + 1)z - \cfrac{[1]_{q,p} z^2}{1 - ([2]_{q,p} + 1)z - \cfrac{[2]_{q,p} z^2}{\ddots}}}.$$

Remark. Corollary 5.3 leads to a continued fraction expansion for the generating function of crossings and nestings over Π: Just taking π to be the partition of $\{1\}$, and bearing in mind that we are counting the empty partition as well, we get

$$\begin{aligned} \sum_{n \ge 0} \sum_{\lambda \in \Pi_n} q^{cr(\lambda)} p^{ne(\lambda)} z^n &= 1 + z S_{\{1\}}(q, p, z) \\ &= 1 + \cfrac{z}{1 - ([1]_{q,p} + 1)z - \cfrac{[1]_{q,p} z^2}{1 - ([2]_{q,p} + 1)z - \cfrac{[2]_{q,p} z^2}{\ddots}}}. \end{aligned} \tag{5.5}$$

A different expansion was given in [4], as

$$\sum_{n\geq 0}\sum_{\lambda\in\Pi_n} q^{cr(\lambda)}p^{ne(\lambda)}z^n = \cfrac{1}{1-z-\cfrac{z^2}{1-([1]_{q,p}+1)z-\cfrac{[2]_{q,p}z^2}{1-([2]_{q,p}+1)z-\cfrac{[3]_{q,p}z^2}{\ddots}}}}. \tag{5.6}$$

The fractions (5.6) and (5.5) can be transformed into each another by applying twice the following contraction formula for continued fraction, (for example, see [1]):

$$\cfrac{c_0}{1-\cfrac{c_1 z}{1-\cfrac{c_2 z}{\ddots}}} = c_0 + \cfrac{c_0c_1z}{1-(c_1+c_2)z-\cfrac{c_2c_3z^2}{1-(c_3+c_4)z-\cfrac{c_4c_5z^2}{\ddots}}}.$$

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