

**Pseudo-supersymmetry, Consistent Sphere Reduction and
Killing Spinors for the Bosonic String**

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ABSTRACT

Certain supergravity theories admit a remarkable consistent dimensional reduction in which the internal space is a sphere. Examples include type IIB supergravity reduced on S^5 , and eleven-dimensional supergravity reduced on S^4 or S^7 . Consistency means that any solution of the dimensionally-reduced theory lifts to give a solution in the higher dimension. Although supersymmetry seems to play a role in the consistency of these reductions, it cannot be the whole story since consistent sphere reductions of non-supersymmetric theories are also known, such as the reduction of the effective action of the bosonic string in any dimension D on either a 3-sphere or a $(D - 3)$ -sphere, retaining the gauge bosons of $SO(4)$ or $SO(D - 2)$ respectively. We show that although there is no supersymmetry, there is nevertheless a natural Killing spinor equation for the D -dimensional bosonic string. A projection of the full integrability condition for these Killing spinors gives rise to the bosonic equations of motion (just as happens in the supergravity examples). Thus it appears that by extending the notion of supersymmetry to “pseudo-supersymmetry” in this way, one may be able to obtain a broader understanding of a relation between Killing spinors and consistent sphere reductions.

Kaluza-Klein dimensional reduction was introduced in the 1920's in an attempt to unify four-dimensional gravity and electromagnetism into the theory of pure gravity in five dimensions. Its most important applications in physics came after the discovery of string theories, whose natural space-time dimensions are higher than four. As in the original motivation of Kaluza and Klein, dimensional reduction can provide a natural interpretation for lower-dimensional gauge symmetries as general coordinate transformations in the higher dimension. In particular, the gauge group of the lower-dimensional theory is associated with the isometry group of the internal space.

An important question that arises in a Kaluza-Klein reduction is whether the procedure is consistent or not. By consistency, we mean that all solutions of the lower-dimensional theory are also solutions of the higher-dimensional theory. (Consistency is always guaranteed if one retains the full Kaluza-Klein towers for all modes; the issue here, though, is whether there exists a useful consistent truncation to a finite set of modes.) In fact the original proposal of Kaluza and Klein to unify gravity and electromagnetism in five-dimensional pure gravity could be said to be only partially successful, because the consistency of the reduction requires that an additional massless scalar field (the dilaton) must be retained in the reduced four-dimensional theory. The consistency in this, and many cases, can be understood straightforwardly by a group-theoretic argument. If the internal space is symmetric under some group action, then it is consistent to perform a dimensional reduction that retains all the singlets, and only the singlets, under this action [1]. Such a reduction was called a DeWitt reduction in [2]. A simple example is an n -torus reduction; it is consistent to keep all the massless modes, since they are singlets under the action of the $U(1)^n$ isometry group. More complicated examples, introduced by DeWitt [3], involve dimensional reduction on a group manifold G , in which only those modes that are invariant under the left action of the $G \times G$ isometry group are retained.

A much more subtle reduction is exemplified by Pauli's (albeit unsuccessful) attempt in the early 1950's to obtain $SO(3)$ non-abelian gauge fields by reducing six-dimensional gravity on the 2-sphere (see [4, 2]). The inconsistency in this case can be understood by considering the untruncated theory in four dimensions, prior to setting any of the fields in the Kaluza-Klein towers to zero. In this reduction, the $SO(3)$ gauge fields act as sources not only for gravity, but also for certain *massive* spin-2 fields in four dimensions. Thus the massive spin-2 Kaluza-Klein tower cannot be consistently truncated in this reduction. In fact this same problem, of the retained gauge fields acting as sources for massive spin 2 fields that one wants to discard, means that Pauli reductions will, generically, be inconsistent.

Remarkably, however, there do exist certain theories for which a Pauli sphere reduction *is* consistent.

Specifically, it has been demonstrated that in certain supergravities where the theory admits an AdS×Sphere vacuum, it *is* consistent to perform a Pauli reduction on the n -sphere in which all the gauge fields associated with the $SO(n+1)$ isometry group are retained. A notable example is the S^7 reduction of $D=11$ supergravity [5]. Various other examples of consistent sphere reductions of M-theory and type IIB supergravity were obtained in [6]-[14]. The fact that there is a supergravity underlying the higher-dimensional theory in all these examples might suggest that the consistency of the reduction could be intimately related to the supersymmetry of the higher-dimensional theory. Indeed, the demonstration of the consistency of the S^7 reduction of eleven-dimensional supergravity in [5] made extensive use of the Killing spinors that exist in the pure $AdS_4 \times S^7$ background.

As we shall discuss below, there also exist purely bosonic theories that are not contained within any supergravities and that also admit non-trivial consistent Pauli sphere reductions. It is of considerable interest to see if there exists any universal way of characterising bosonic theories that admit Pauli reductions, to encompass both the supersymmetric and the non-supersymmetric examples.

One feature common to all the supergravity examples is that if one looks at the equations for Killing spinors in purely bosonic backgrounds, then by taking certain canonical projections of the integrability conditions for the Killing spinor equations, one can essentially derive the bosonic equations of motion for the theory. For example, the gravitino transformation rule in bosonic backgrounds in eleven-dimensional supergravity is $\delta\psi_M = \hat{D}_M\epsilon$, where

$$\hat{D}_M = D_M - \frac{1}{288}(\Gamma_M^{N_1\dots N_4} F_{N_1\dots N_4} - 8F_{MN_1\dots N_4} \Gamma^{N_1\dots N_4}), \quad (1)$$

Projecting the integrability condition $[\hat{D}_M, \hat{D}_N]\epsilon = 0$ with Γ^M gives an equation of the form $(R_{MN} + \dots)\Gamma^N\epsilon = 0$, where the factor in brackets vanishes by virtue of the bosonic equations of motion. (This is related to the fact that in the supersymmetry variation of the action, the terms coming from varying ψ_M in the gravitino terms must cancel against those coming from varying the bosonic terms in the action.) Thus, one may say that a characterisation of the bosonic equations of motion in the supergravity theories that admit consistent Pauli reductions is that these equations can be derived from an appropriate projection of the integrability condition for Killing spinors.

Recently, it was shown that for a large class of theories admitting AdS×Sphere vacua, encompassing the supergravities mentioned above but including also non-supersymmetric the-

ories, a broader notion of Killing spinors can be introduced [15]. In such non-supersymmetric theories, bosonic backgrounds for which the Killing spinor equations admit solutions were referred to as “pseudo-supersymmetric.”¹ The simplest class of such theories, which admit AdS×Sphere vacua, is provided by Einstein gravity coupled to an n -form field strength, with the Lagrangian

$$\mathcal{L} = \sqrt{-g}(R - \frac{1}{2n!}F_n^2), \quad (2)$$

where $F_n = dA_{(n-1)}$. A Killing spinor equation has been introduced for this system [15], given by

$$D_M \hat{\epsilon} + \frac{\tilde{\alpha}}{(n-1)!} \Gamma^{M_1 \dots M_{n-1}} F_{M M_1 \dots M_{n-1}} \hat{\epsilon} + \frac{\tilde{\beta}}{n!} \Gamma_M{}^{M_1 \dots M_n} F_{M_1 \dots M_n} \hat{\epsilon} = 0, \quad (3)$$

where D_M is the covariant derivative, defined by

$$D_M \hat{\epsilon} \equiv \partial_M \hat{\epsilon} + \frac{1}{4}(\omega_M)^A{}_B \Gamma_A{}^B \hat{\epsilon}. \quad (4)$$

The constants $(\tilde{\alpha}, \tilde{\beta})$ are given by

$$\tilde{\alpha} = i^{[(n+1)/2]} \frac{\sqrt{\Delta}}{4d}, \quad d\tilde{\alpha} + \tilde{d}\tilde{\beta} = 0, \quad (5)$$

where $d = n - 1$, $\tilde{d} = D - n - 1$, and $\Delta = 2d\tilde{d}/(D - 2)$.

Although it was shown that the AdS×Sphere vacuum, and a class of p -brane solutions, are “pseudo-supersymmetric” with respect to this definition of a Killing spinor, the integrability conditions in (3) are not in general consistent, in the sense that the equations of motion following from (2) are necessary but not sufficient to ensure the vanishing of the projected integrability condition. Rather, additional constraints must still be imposed [15]. These additional constraints are absent in certain special cases, such as $n = 4$, $D = 11$, if a suitable $F \wedge F \wedge A$ term is added to the Lagrangian; or in the case $n = 5$, $D = 10$, if the 5-form is restricted to be self-dual. Interestingly enough, these additional terms or restrictions are also precisely what is needed in order to permit a consistent Pauli sphere reduction.

Let us consider the case of ten-dimensional gravity coupled to a 5-form field strength in more detail. It was shown in [15] that if one does not require the 5-form $H_{(5)}$ in $D = 10$ to be self-dual, then the projected integrability condition for the Killing spinor will only vanish upon use of the equations of motion if, in addition, the extra constraints

$$H_{M_1[M_2 M_3 M_4 M_5]} H^{M_1}{}_{N_2 N_3 N_4 N_5} = 0, \quad H_{M_1[M_2 M_3 M_4 M_5]} H_{N_1 N_2 N_3 N_4 N_5} = 0 \quad (6)$$

¹A related notion, referred to as “fake supersymmetry,” was introduced for scalar-gravity theories in [16].

are imposed. In [15], a new class of pseudo-supersymmetric “bubbling AdS geometries” was constructed, that satisfy the constraints (6). In particular, as in the case of the LLM solution [17], the new solution corresponding to the elliptic disc boundary condition is expected to admit a reduction to $D = 5$, with an S^5 internal space. It is therefore of interest to examine whether the extra conditions (6) are related to the consistency of the 5-sphere reduction.

The consistent S^5 reduction for $D = 10$ with a self-dual 5-form was obtained in [12]. The reduction ansatz is given by

$$d\hat{s}_{10}^2 = \Delta^{1/2} ds_5^2 + g^{-2} \Delta^{-1/2} T_{ij}^{-1} D\mu^i D\mu^j, \quad (7)$$

$$\hat{H}_{(5)} = \hat{G}_{(5)} + *\hat{G}_{(5)}, \quad (8)$$

where

$$\begin{aligned} \hat{G}_{(5)} &= -gU \epsilon_{(5)} + g^{-1} (T_{ij}^{-1} *D T_{jk}) \wedge (\mu^k D\mu^i) \\ &\quad - \frac{1}{2} g^{-2} T_{ik}^{-1} T_{j\ell}^{-1} *F_{(2)}^{ij} \wedge D\mu^k \wedge D\mu^\ell, \end{aligned} \quad (9)$$

$$\begin{aligned} *\hat{G}_{(5)} &= \frac{1}{5!} \epsilon_{i_1 \dots i_6} \left[g^{-4} U \Delta^{-2} D\mu^{i_1} \wedge \dots \wedge D\mu^{i_5} \mu^{i_6} \right. \\ &\quad - 5g^{-4} \Delta^{-2} D\mu^{i_1} \wedge \dots \wedge D\mu^{i_4} \wedge DT_{i_5 j} T_{i_6 k} \mu^j \mu^k \\ &\quad \left. - 10g^{-3} \Delta^{-1} F_{(2)}^{i_1 i_2} \wedge D\mu^{i_3} \wedge D\mu^{i_4} \wedge D\mu^{i_5} T_{i_6 j} \mu^j \right], \end{aligned} \quad (10)$$

and

$$\begin{aligned} U &\equiv 2T_{ij} T_{jk} \mu^i \mu^k - \Delta T_{ii}, & \Delta &\equiv T_{ij} \mu^i \mu^j, \\ F_{(2)}^{ij} &= dA_{(1)}^{ij} + g A_{(1)}^{ik} \wedge A_{(1)}^{kj}, & DT_{ij} &\equiv dT_{ij} + g A_{(1)}^{ik} T_{kj} + g A_{(1)}^{jk} T_{ik}, \\ \mu^i \mu^i &= 1, & D\mu^i &\equiv d\mu^i + g A_{(1)}^{ij} \mu^j, \end{aligned} \quad (11)$$

with $\epsilon_{(5)}$ being the volume form on the five-dimensional spacetime. Note that $*\hat{G}_{(5)}$ is derivable from the given expressions (7) and (9). The coordinates μ^i , subject to the constraint $\mu^i \mu^i = 1$, parameterise points in the internal 5-sphere. It was shown in [12] that the reduction is consistent, giving rise to lower-dimensional equations of motion that can be derived from the five-dimensional Lagrangian

$$\begin{aligned} \mathcal{L}_5 &= R * \mathbf{1} - \frac{1}{4} T_{ij}^{-1} *D T_{jk} \wedge T_{k\ell}^{-1} DT_{\ell i} - \frac{1}{4} T_{ik}^{-1} T_{j\ell}^{-1} *F_{(2)}^{ij} \wedge F_{(2)}^{k\ell} - V * \mathbf{1} \\ &\quad - \frac{1}{48} \epsilon_{i_1 \dots i_6} \left(F_{(2)}^{i_1 i_2} F_{(2)}^{i_3 i_4} A_{(1)}^{i_5 i_6} - g F_{(2)}^{i_1 i_2} A_{(1)}^{i_3 i_4} A_{(1)}^{i_5 j} A_{(1)}^{j i_6} + \frac{2}{5} g^2 A_{(1)}^{i_1 i_2} A_{(1)}^{i_3 j} A_{(1)}^{j i_4} A_{(1)}^{i_5 k} A_{(1)}^{k i_6} \right), \end{aligned} \quad (12)$$

where the potential V is given by

$$V = \frac{1}{2} g^2 \left(2T_{ij} T_{ij} - (T_{ii})^2 \right). \quad (13)$$

(In (12), the wedge symbols in the final topological term are omitted to economise on space.)

If instead we do not impose the self-duality condition for the 5-form, so that its reduction ansatz is now given simply by

$$H_{(5)} = \hat{G}_{(5)}, \quad (14)$$

then the reduction will not in general be consistent, since, as was observed in [12], the field equation $d\hat{*}G_{(5)} = 0$ gives rise to the constraint

$$\epsilon_{ijk_1 \dots k_4} F_{(2)}^{k_1 k_2} \wedge F_{(2)}^{k_3 k_4} = 0. \quad (15)$$

The intriguing point is that if we substitute the reduction ansatz (14) into (6), we arrive at exactly the same constraint (15) that arose in [15] from imposing the projected integrability condition for the Killing spinors.²

Thus we find that the extra constraint needed for the projected integrability of the Killing spinor for the $D = 10$, $n = 5$ system is exactly the same as the extra constraint (15) that is required for the consistency of the S^5 reduction. This observation leads us to speculate that the ability of a theory to be consistently reduced, à la Pauli, on a sphere may go hand in hand with its admitting some suitably-defined Killing spinor equation. In some cases, namely certain supergravity theories, the Killing spinor equation is simply the standard one associated with supersymmetry of the bosonic background. In more general situations, however, the Killing spinor equation may be associated with a “pseudo-supersymmetry” that has not hitherto been considered.

There are some further examples that lend support to this idea. Consider pure gravity in $(D + 1)$ dimensions, for which the Lagrangian is

$$\mathcal{L}_{D+1} = \sqrt{-\hat{g}} \hat{R}. \quad (16)$$

The associated Killing spinor equation is simply

$$0 = D_M \hat{\epsilon} \equiv \partial_M \hat{\epsilon} + \frac{1}{4} (\omega_M)^A{}_B \Gamma_A{}^B \hat{\epsilon}, \quad (17)$$

The projected integrability condition is

$$0 = \Gamma^M [D_M, D_N] \hat{\epsilon} = \frac{1}{2} R_{MN} \Gamma^M \hat{\epsilon}, \quad (18)$$

²It is worth pointing out that there are large classes of solutions in five dimensional gauged supergravity that satisfy the condition (15). These solutions can now also be lifted to the non-supersymmetric ten-dimensional theory where the 5-form is not self-dual. A summary of such liftings, together with an explicit example, is presented in appendix A.

which is satisfied by virtue of the Einstein equations of motion. We now perform a Kaluza-Klein reduction on S^1 , with the metric ansatz given by

$$\begin{aligned} ds_{D+1}^2 &= e^{2\alpha\phi} ds_D^2 + e^{2\beta\phi} (dz + A_{(1)})^2, \\ \beta &= -(D-2)\alpha, \quad \alpha^2 = \frac{1}{2(D-1)(D-2)}. \end{aligned} \quad (19)$$

The reduced D -dimensional Lagrangian in D is

$$\mathcal{L}_D = \sqrt{g} (R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{a\phi} F_{(2)}^2), \quad (20)$$

where $F_{(2)} = dA_{(1)}$ and $a = -2(D-1)\alpha$. We can also perform the Kaluza-Klein reduction of (17), to obtain the equations for the D -dimensional Killing spinors:

$$\begin{aligned} D_M \eta + \frac{i}{8(D-2)} e^{\frac{1}{2}a\phi} \left(\Gamma_M^{M_1 M_2} - 2(D-2) \delta_M^{M_1} \Gamma^{M_2} \right) F_{M_1 M_2} \eta &= 0, \\ \Gamma^M \partial_M \phi \eta + \frac{i}{4} a e^{\frac{1}{2}a\phi} \Gamma^{M_1 M_2} F_{M_1 M_2} &= 0. \end{aligned} \quad (21)$$

One can obviously expect that the projected integrability conditions for these equations should be satisfied by virtue of the D -dimensional equations of motion. Indeed the projected integrability conditions following from (21) are given by

$$\begin{aligned} &\left[R_{MN} - \frac{1}{2} \partial_M \phi \partial_N \phi - \frac{1}{2} e^{a\phi} (F_{MN}^2 - \frac{1}{2(D-2)} F^2 g_{MN}) \right] \Gamma^N \eta \\ &- \frac{i}{4(D-2)} e^{\frac{1}{2}a\phi} \nabla_N F_{M_1 M_2} \left(\Gamma_M \Gamma^{N M_1 M_2} - 3(D-2) \delta_M^{[N} \Gamma^{M_1 M_2]} \right) \eta \\ &- \frac{i}{2(D-2)} e^{-\frac{1}{2}a\phi} \nabla_N \left(e^{a\phi} F^N{}_{M_2} \right) \left(\Gamma_M \Gamma^{M_2} - (D-2) \delta_M^{M_2} \right) \eta = 0, \end{aligned} \quad (22)$$

and

$$\begin{aligned} &\left(\nabla^2 \phi - \frac{1}{4} a e^{a\phi} F^2 \right) \eta + \frac{i}{4} a e^{\frac{1}{2}a\phi} \Gamma^{N M_1 M_2} \nabla_N F_{M_1 M_2} \eta \\ &+ \frac{i}{2} a e^{-\frac{1}{2}a\phi} \Gamma^{M_2} \nabla_N \left(e^{a\phi} F^N{}_{M_2} \right) \eta = 0. \end{aligned} \quad (23)$$

Thus the equations of motion imply that the projected integrability conditions are satisfied for any dimension D .

The interesting point is that it is also consistent to perform a Pauli S^2 reduction of the system (20) in any dimension D , yielding a theory in $(D-2)$ dimensions that includes the full set of $SO(3)$ gauge bosons [18]. This can be seen from the fact that it is consistent to perform a (necessarily consistent) DeWitt reduction of the pure gravity theory (16) on $S^3 \sim SU(2)$, viewing it as the $SU(2)$ group manifold and keeping all the singlets of the left-invariant action. Since S^3 can be viewed as a $U(1)$ bundle over S^2 , the reduction can be split into two stages; an S^1 reduction followed by an S^2 Pauli reduction. Thus the

consistency of the DeWitt reduction guarantees the consistency of the Pauli S^2 reduction in this case [18].

Of course, this is a rather simple example. There are in fact further examples of consistent Pauli sphere reductions of non-supersymmetric theories. It was shown in [18] that it is consistent to perform an S^3 or an S^{D-3} Pauli reduction of the effective action of the bosonic string in any dimension D . This leads us to consider the possibility of a defining a Killing spinor equation for the bosonic string.

The Lagrangian for the effective theory of the bosonic string in D -dimensions is given by

$$\mathcal{L}_D = \sqrt{-g} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{a\phi}F_{(3)}^2 \right), \quad (24)$$

where $F_{(3)} = dA_{(2)}$ and $a^2 = 8/(D-2)$. The equations of motion are given by

$$\begin{aligned} \square\phi &= \frac{1}{12}a e^{a\phi}F_{(3)}^2, & dF_{(3)} &= 0 = d(e^{a\phi}*F_{(3)}), \\ R_{MN} &= \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{4}e^{a\phi}\left(F_{MN}^2 - \frac{2}{3(D-2)}F^2g_{MN}\right). \end{aligned} \quad (25)$$

We find that the appropriate equations for defining a Killing spinor in this case are

$$D_M\eta + \frac{1}{96}e^{\frac{1}{2}a\phi}\left(a^2\Gamma_M\Gamma^{NPQ} - 12\delta_M^N\Gamma^{PQ}\right)F_{NPQ}\eta = 0, \quad (26)$$

$$\Gamma^M\partial_M\phi\eta + \frac{1}{12}ae^{\frac{1}{2}a\phi}\Gamma^{MNP}F_{MNP}\eta = 0. \quad (27)$$

The forms of these Killing spinor equations are motivated by generalising the supersymmetry transformation rules for the gravitino and dilatino in $D = 10$, $\mathcal{N} = 1$ supergravity [19]. The coefficients of each term are determined by investigating the projected integrability conditions, whose derivation is presented in appendix B. They are given by

$$\begin{aligned} &\left[R_{MN} - \frac{1}{2}\partial_M\phi\partial_N\phi - \frac{1}{4}e^{a\phi}\left(F_{MN}^2 - \frac{2}{3(D-2)}F^2g_{MN}\right)\right]\Gamma^N\eta \\ &- \frac{1}{6(D-2)}e^{\frac{1}{2}a\phi}\nabla_N F_{M_1M_2M_3}\left(\Gamma_M\Gamma^{NM_1M_2M_3} - 2(D-2)\delta_M^{[N}\Gamma^{M_1M_2M_3]}\right)\eta \\ &- \frac{1}{2(D-2)}e^{-\frac{1}{2}a\phi}\nabla_N\left(e^{a\phi}F^N{}_{M_2M_3}\right)\left(\Gamma_M\Gamma^{M_2M_3} - (D-2)\delta_M^{M_2}\Gamma^{M_3}\right)\eta = 0, \end{aligned} \quad (28)$$

and

$$\begin{aligned} &\left(\nabla^2\phi - \frac{1}{12}ae^{a\phi}F^2\right)\eta + \frac{1}{12}ae^{\frac{1}{2}a\phi}\Gamma^{NM_1M_2M_3}\nabla_N F_{M_1M_2M_3}\eta \\ &+ \frac{1}{4}ae^{-\frac{1}{2}a\phi}\Gamma^{M_2M_3}\nabla_N\left(e^{a\phi}F^N{}_{M_2M_3}\right)\eta = 0. \end{aligned} \quad (29)$$

Thus we see that the projected integrability conditions are satisfied by virtue of the full set of equations of motion. In the special case when $D = 10$, the theory can be supersymmetrised, to give $\mathcal{N} = 1$, $D = 10$ supergravity, and the Killing spinor defined above just reduces to

the usual Killing spinor of the supergravity theory. But the construction we have discussed here works equally well in any spacetime dimension.

Sometimes it is advantageous to work with the theory in the string frame, rather than the Einstein frame we have been using until now. It is defined by rescaling the metric so that $ds_{\text{string}}^2 = e^{-\frac{1}{2}a\phi} ds_{\text{Einstein}}^2$. If we now define $\Phi = -\phi/a$, the Lagrangian becomes

$$\mathcal{L} = e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{1}{12} F_{(3)}^2 \right). \quad (30)$$

The defining equations for the Killing spinors, which are now scaled by the factor $e^{-\frac{1}{8}a\phi}$, are given by

$$D_M(\omega_-)\eta = 0, \quad \Gamma^M \partial_M \Phi \eta - \frac{1}{12} \Gamma^{MNP} F_{MNP} \eta = 0, \quad (31)$$

where ω_- is the torsionful spin connection, given by

$$\omega_{M\pm}{}^{AB} = \omega_M^{AB} \pm \frac{1}{2} F_M{}^{AB}. \quad (32)$$

To conclude, we have observed an intriguing feature common to all the known examples of consistent Pauli sphere reductions. Namely, in all such cases, the higher-dimensional theory admits a natural definition of a Killing spinor. A certain canonical projection of the integrability conditions for the Killing spinor is satisfied by virtue of the equations of motion of the theory. In certain cases, the projected integrability conditions may also impose quadratic algebraic constraints on field strengths in the theory. In such cases, these turn out to be precisely the same as constraints that must be imposed in order to achieve a consistent Pauli reduction.

We discussed various classes of examples that provide support for this relation between consistent Pauli reductions and the existence of a Killing spinor equation. First of all, there are cases such as eleven-dimensional supergravity and type IIB supergravity, where the Killing spinor equation simply reduces to the standard Killing spinor equations associated with supersymmetry. We then considered the example of ten-dimensional gravity coupled to a 5-form field strength with no self-duality constraint. In this case, we saw that both the consistency of the Pauli S^5 reduction and the consistency of the projected integrability conditions for the Killing spinor equations required exactly the same quadratic constraint (15) on the 5-form field. Further examples that we considered included dilatonic gravity coupled to a 2-form field strength in any dimension, and dilatonic gravity coupled to a 3-form field strength in any dimension. The latter example arises as the effective action for the bosonic string. The fact that there exists a natural notion of a Killing spinor for the bosonic string in an arbitrary spacetime dimension suggests that there may some generalised geometric structure still to be uncovered.

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A Lifting of the solutions

As discussed in the paper, the S^5 Pauli reduction of the theory described by the Lagrangian

$$\mathcal{L}_{10} = \sqrt{-g}(R - \frac{1}{240}F_{(5)}^2), \quad (33)$$

where $F_{(5)}$ is not self-dual, is not in general consistent. There is an extra condition (15) that has to be satisfied. However, this also implies that all the solutions of five-dimensional theory (12) that satisfy (15) are also solutions of (33), with the lifting ansatz given in this paper. Thus all the domain wall solutions supported by the scalar fields, which are dual to the Coulomb branch of the boundary conformal theory [20, 21], are solutions of (33). The $U(1)^3$ charged black holes in $D = 5$ supergravity [22] can be embedded not only in type IIB supergravity [23], but also in the theory described by (33). Furthermore, the smooth $U(1)^3$ charged bubbling soliton solutions obtained in [24] can also be lifted into solutions of (33). In particular, the single $U(1)$ charged solution can be lifted to give a pseudo-supersymmetric AdS bubble geometry with an elliptic disc boundary condition, as was constructed in [15]. Five-dimensional rotating black holes do not in general satisfy the supplementary constraint (15), and so they will not lift to give solutions of (33). However, the singly-charged rotating black hole constructed in [25] does satisfy the condition (15), and so in this case a lifting to give a solution of (33) is possible. All such liftings use the reduction ansatz we have given in this paper, and we shall not present them in detail.

Here, we present one simple example in detail, namely the embedding of the five-dimensional Reissner-Nordström black hole in (33). Expressed in the notation we are using in this paper, the five-dimensional Reissner-Nordström solution is given by

$$\begin{aligned} ds_5^2 &= -H^{-2} f dt^2 + H (f^{-1} dr^2 + r^2 d\Omega_3^2), \\ T_{ij} &= \delta_{ij}, \\ A^{12} &= A^{34} = A^{56} = \frac{1}{\sqrt{3}} A, \end{aligned} \quad (34)$$

where

$$A = \frac{\sqrt{q(q+2m)}}{(r^2+q)} dt, \quad H = 1 + \frac{q}{r^2}, \quad f = 1 - \frac{2m}{r^2} + g^2 r^2 H^3. \quad (35)$$

Substituting into (7) and (9), we find that the Reissner-Nordström solution lifts to give the ten-dimensional solution

$$\begin{aligned} d\hat{s}_{10}^2 &= -H^{-2} f dt^2 + H (f^{-1} dr^2 + r^2 d\Omega_3^2) + g^{-2} (d\psi + B - \frac{g}{\sqrt{3}} A)^2 + d\Sigma_2^2, \\ \hat{F}_5 &= 4g \epsilon_5 - \frac{1}{\sqrt{3} g^2} *F \wedge J, \end{aligned} \quad (36)$$

and hence

$$\hat{*}\hat{F}_5 = 2g^{-4} (d\psi + B - \frac{g}{\sqrt{3}} A) \wedge J \wedge J - \frac{1}{\sqrt{3} g^3} (d\psi + B - \frac{g}{\sqrt{3}} A) \wedge F \wedge J, \quad (37)$$

where $d\Sigma_2^2$ is the standard Fubini-Study metric on CP^2 , J is its Kähler form, $dB = 2J$, and ψ is the coordinate on the Hopf fibre over CP^2 , with period 2π . (The proof of these results follows using analogous manipulations to those in appendix B of [26].)

B Projected Integrability Conditions for the Bosonic String

Here, we derive the projected integrability conditions for the Killing spinor equations for the D -dimensional bosonic string. We begin by supposing that the Killing spinor equations take the form

$$D_M \eta + a_2 e^{\frac{1}{2} a \phi} \left(\Gamma_M \Gamma^{M_1 M_2 M_3} - a_1 \delta_M^{M_1} \Gamma^{M_2 M_3} \right) F_{M_1 M_2 M_3} \eta = 0, \quad (38)$$

$$\Gamma^M \partial_M \phi \eta - a_3 e^{\frac{1}{2} a \phi} \Gamma^{M_1 M_2 M_3} F_{M_1 M_2 M_3} \eta = 0. \quad (39)$$

The motivation for these equations is provided by the supersymmetry transformation rules for the gravitino and dilatino in ten-dimensional $\mathcal{N} = 1$ supergravity [19]. The constants a_1 , a_2 and a_3 will be determined below. We also leave the dilaton coupling constant a unspecified for now.

The next step is to compute the projected commutator $\Gamma^M [D_N, D_M]$ acting on η , and then to choose the undetermined coefficients by requiring that it should vanish upon use of the equations of motion. After lengthy calculations, we find

$$\begin{aligned} 0 &= R_{MN} \Gamma^N \eta - 2a_2 e^{\frac{1}{2} a \phi} \nabla_N F_{M_1 M_2 M_3} \left(\Gamma_M^{N M_1 M_2 M_3} - \frac{4}{3} (a_1 - 3) \delta_M^{[N} \Gamma^{M_1 M_2 M_3]} \right) \eta \\ &\quad - 2a_2 e^{-\frac{1}{2} a \phi} \nabla_N \left(e^{a \phi} F^N{}_{M_2 M_3} \right) \left(3\Gamma_M^{M_2 M_3} - 2(a_1 - 3) \delta_M^{M_2} \Gamma^{M_3} \right) \eta \\ &\quad + \frac{aa_1 a_2}{3a_3} \nabla_M \phi \nabla_N \phi \Gamma^N \eta + 2a_2 \left(\frac{2}{3} a_1 - D + 2 \right) \nabla_M \left(e^{\frac{1}{2} a \phi} F_{M_1 M_2 M_3} \right) \Gamma^{M_1 M_2 M_3} \eta \\ &\quad + \left[12(2a_1 - 3(D - 4))a_2^2 + 9aa_2 a_3 \right] e^{a \phi} \Gamma_M^{M_2 M_3 N_2 N_3} F_{M_1 M_2 M_3} F^{M_1 N_2 N_3} \eta \\ &\quad + \left[24(D - 4)a_2^2 - 6aa_2 a_3 \right] e^{a \phi} \Gamma_M F^2 \eta \\ &\quad - \left[4(4a_1 - 3(D - 2))a_2^2 + aa_1 a_2 a_3 \right] e^{a \phi} \Gamma^{M_2 M_3 N_1 N_2 N_3} F_{M M_2 M_3} F_{N_1 N_2 N_3} \eta \end{aligned}$$

$$\begin{aligned}
& - \left[8 (2a_1^2 - 3(D-2)a_1 + 9(D-6)) a_2^2 - 6(6-a_1)aa_2a_3 \right] \\
& \quad \times e^{a\phi} \Gamma^{M_3 N_2 N_3} F_{M_1 M M_3} F^{M_1 N_2 N_3} \eta \\
& - \left[8 (2a_1^2 - 12a_1 + 9(D-2)) a_2^2 - 6aa_1a_2a_3 \right] e^{a\phi} \Gamma^N F_{MN}^2 \eta.
\end{aligned} \tag{40}$$

The vanishing of the $\nabla_M (e^{\frac{1}{2}a\phi} F_{M_1 M_2 M_3})$ term implies

$$a_1 = \frac{3}{2}(D-2), \tag{41}$$

which then leaves

$$\begin{aligned}
0 = & \left[R_{MN} + \frac{aa_2(D-2)}{2a_3} \nabla_M \phi \nabla_N \phi + (24(D-4)a_2^2 - 6aa_2a_3) e^{a\phi} g_{MN} F^2 \right. \\
& - 9(D-2) (4(D-4)a_2^2 - aa_2a_3) e^{a\phi} F_{MN}^2 \left. \right] \Gamma^N \eta \\
& - 2a_2 e^{\frac{1}{2}a\phi} \nabla_N F_{M_1 M_2 M_3} \left(\Gamma_M \Gamma^{N M_1 M_2 M_3} - 2(D-2) \delta_M^{[N} \Gamma^{M_1 M_2 M_3]} \right) \eta \\
& - 6a_2 e^{-\frac{1}{2}a\phi} \nabla_N \left(e^{a\phi} F^N{}_{M_2 M_3} \right) \left(\Gamma_M \Gamma^{M_2 M_3} - (D-2) \delta_M^{M_2} \Gamma^{M_3} \right) \eta \\
& + 9(8a_2^2 + aa_2a_3) e^{a\phi} \Gamma_M{}^{M_2 M_3 N_2 N_3} F_{M_1 M_2 M_3} F^{M_1 N_2 N_3} \eta \\
& - \frac{3}{2}(D-2) (8a_2^2 + aa_2a_3) e^{a\phi} \Gamma^{M_2 M_3 N_1 N_2 N_3} F_{M M_2 M_3} F_{N_1 N_2 N_3} \eta \\
& - 9(D-6) (8a_2^2 + aa_2a_3) e^{a\phi} \Gamma^{M_3 N_2 N_3} F_{M_1 M M_3} F^{M_1 N_2 N_3} \eta
\end{aligned} \tag{42}$$

The terms involving $\nabla_M \phi \nabla_N \phi$, F_{MN}^2 and $\Gamma_M{}^{M_2 M_3 N_2 N_3} F_{M_1 M_2 M_3} F^{M_1 N_2 N_3}$ will then vanish upon use of the equations of motion, provided that we choose

$$\begin{aligned}
a_3 + a a_2 (D-2) &= 0, & 8a_2^2 + a a_2 a_3 &= 0, \\
9(D-2) (4(D-4)a_2^2 - a a_2 a_3) &= \frac{1}{4},
\end{aligned} \tag{43}$$

for which the solution is

$$a^2 = \frac{8}{D-2}, \quad a_2 = \frac{1}{12(D-2)}, \quad a_3 = -\frac{1}{12}a. \tag{44}$$

Acting on (39) with $\Gamma^N \nabla_N$, we have

$$\begin{aligned}
& \nabla^2 \phi \eta = \Gamma^N \Gamma^M \nabla_N \nabla_M \phi \eta = \Gamma^N D_N (\Gamma^M \partial_M \phi \eta) - \Gamma^N \Gamma^M \partial_M \phi D_N \eta \\
= & a_3 e^{\frac{1}{2}a\phi} \Gamma^{N M_1 M_2 M_3} \nabla_N F_{M_1 M_2 M_3} \eta + 3a_3 e^{-\frac{1}{2}a\phi} \Gamma^{M_2 M_3} \nabla_N \left(e^{a\phi} F^N{}_{M_2 M_3} \right) \eta \\
& - 6(D-2 - \frac{2}{3}a_1) a_2 e^{\frac{1}{2}a\phi} \Gamma^{M_2 M_3} F^N{}_{M_2 M_3} \partial_N \phi \eta \\
& + \left[6(3D-2a_1-12)a_2a_3 - \frac{9}{2}aa_3^2 \right] e^{a\phi} \Gamma^{M_2 M_3 N_2 N_3} F_{M_1 M_2 M_3} F^{M_1 N_2 N_3} \eta \\
& - 3 \left[4(D-4)a_2a_3 - aa_3^2 \right] e^{a\phi} F^2 \eta.
\end{aligned} \tag{45}$$

This is also satisfied by the equations of motion, provided that the coefficients a , a_1 , a_2 and a_3 are chosen as in (41) and (44). Thus the Killing spinor equations are given by (26) and (27). The projected integrability conditions are given by (28) and (29).

Finally we would like to remark that we have investigated the Killing spinors for a more general system with the 3-form field strength replaced by an arbitrary n -form. It turns out that that projected integrability condition works only for two cases, the bosonic string (24) and the Kaluza-Klein theory (20).

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