

Supersymmetric Yang-Mills Equations in 10+2 Dimensions

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ABSTRACT

We present a model for supersymmetric Yang-Mills theory in 10+2 dimensions. Our construction uses a constant null vector, and leads to a consistent set of field equations and constraints. The model is invariant under generalized translations and an extra gauge transformation. Ordinary dimensional reduction to ten dimensions yields the usual supersymmetric Yang-Mills equations, while dimensional reduction to 2+2 yields supersymmetric Yang-Mills equations in which the Poincaré supersymmetry is reduced by a null vector. We also give the corresponding formulation in superspace.

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1. Introduction

It has been proposed by Vafa [1] that the Type IIB and the Type I or SO(32) heterotic strings which do not admit a direct M-theory unification in 10+1 dimensions, may in fact arise from a unifying theory in 10 + 2 dimensions, called F-theory. This picture emerged in the analysis of certain Type IIB string vacua. In a somewhat related fashion, Hull [2] has also proposed a twelve dimensional (12D) picture, but in 11 + 1 dimensions. In another development, Tseytlin [3] has suggested that the worldvolume vector fields of the Type IIB Dirichlet three-brane in 10D may provide two extra dimensions to imply a three-brane in 12D with (11, 1) signature. This idea has been realized [4], at least in the bosonic sector, and without the full 12D Poincaré invariance.⁴ However, the relation between the conjectured 12D theories with (11, 1) versus (10, 2) signatures is not clear at present.

As far as a three-brane in 10 + 2 dimensions is concerned, the possibility of its existence was conjectured some time ago by Blencowe and Duff [5], on the basis of matching bose and fermi degrees of freedom. There is a subtlety, however, in 10 + 2 dimensions, namely the minimal chiral superalgebra with 32 component Majorana-Weyl spinors takes the form

$$\{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu})_{\alpha\beta} P_{\mu\nu} + (\gamma^{\mu_1 \dots \mu_6})_{\alpha\beta} Z_{\mu_1 \dots \mu_6} , \quad (1)$$

where the 66 generators $P_{\mu\nu}$ and the 462 generators $Z_{\mu_1 \dots \mu_6}$ have only nonvanishing commutators with the Lorentz generators $M_{\mu\nu}$, and the Dirac γ -matrices are chirally projected. (For a discussion of how this algebra might play a role in the classification of perturbative and nonperturbative multiplets of string theory, see [6].) Now this algebra does not contain the usual translation generators. Therefore, one does not expect a Poincaré invariant field theory of the usual kind, and strictly speaking the usual argument about $d - p - 1$ translational zero modes does not really hold. Nevertheless, there may indeed exist evidence for the conjecture of [5], in view of some recent developments to which we now turn.

In an attempt to give a concrete description of M-theory based on [7], Kutasov and Martinec [8] have proposed an $N = (2, 1)$ superstring theory, where the left movers with local $N = 2$ worldsheet supersymmetry live in 2 + 2 dimensional target space, and right movers with local $N = 1$ worldsheet supersymmetry surprisingly live in 10 + 2 dimensional

⁴Long ago Nahm [9] showed that supergravity theories are impossible in more than 10 + 1 dimensions, and supersymmetric Yang-Mills theories in more than 10 + 1 dimensions, because higher spin fields set in. In [10], it was shown that although the bose and fermi degrees of freedom could be matched in 11 + 1 dimensions, no corresponding supergravity model existed. The lack of full 12D Poincaré invariance can avoid these problems.

target space [7]. Thus the geometrical spacetime can at most be $2 + 2$ dimensional. There is however a null vector in the construction, which can be chosen in two different ways, corresponding to $1 + 1$ or $2 + 1$ dimensional target space. The resulting $1 + 1$ and $2 + 1$ dimensional target space theories are then proposed to be the worldvolume theories for a 10D superstring or 11D supermembrane. Just as there is an underlying $2 + 2$ worldvolume with two kinds of null reductions, the target space of the resulting extended object is also conjectured to be $10 + 2$ dimensional with different kinds of null reductions. In [8], it is furthermore speculated that the relevant worldvolume theory may involve a $10 + 2$ dimensional Yang-Mills theory, presumably reduced to $2 + 2$ dimensions. Motivated by these developments, and as a first step towards the investigation of supersymmetric field theories in $10 + 2$ dimensions, in this paper we present a model of supersymmetric Yang-Mills theory by utilizing a constant null vector. The usage of a null vector n_μ in our construction results in a modified form of the algebra (1), namely

$$\{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu})_{\alpha\beta} n_\nu P_\mu . \quad (2)$$

Due to this constant null vector throughout the formalism, the model will lack the full Poincaré invariance in $10 + 2$ dimensions. The model has a generalized translational and extra gauge invariances, and it reduces to the 10D supersymmetric Yang-Mills system upon ordinary dimensional reduction. If we relax the assumption of ordinary dimensional reduction, the model curiously reduces, modulo global matters that may involve Wilson lines, to the 10D supersymmetric Yang-Mills equations depending only on one combination of the two extra coordinates, in addition to the 10D coordinates.

In what follows, we will first describe the model. We will then examine its dimensional reduction to $9 + 1$ and $2 + 2$ dimensions, followed by the superspace formulation of the model, and our conclusions.

2. Supersymmetric Yang-Mills Model in $10 + 2$ Dimensions

The field content we consider is the same as 10D supersymmetric Yang-Mills multiplet, namely a real vector A_μ^I in the adjoint representation, and a Majorana-Weyl fermion λ^I with the positive chirality, $\gamma_{13}\lambda^I = +\lambda^I$, also in the adjoint representation labelled by the indices I, J, \dots . We propose the following supersymmetry transformation rules:

$$\delta_Q A_\mu^I = \bar{\epsilon} \gamma_\mu \lambda^I , \quad (3)$$

$$\delta_Q \lambda^I = \frac{1}{4} \gamma^{\mu\nu\rho} \epsilon F_{\mu\nu}^I n_\rho , \quad (4)$$

where n^μ is a constant vector. The Dirac γ -matrices obey the $SO(10,2)$ Clifford algebra, and the signature of spacetime is taken to be $(- + + \cdots + -)$. Note that γ^μ flips the chirality, and that the supersymmetry parameter and the gauge fermions have opposite chiralities. The above ansätze for the transformation rules are motivated by the chirality properties of the fermions and the requirement of translation emerging in the commutation of two supersymmetry transformations. These requirements also allow the possibility of taking the supersymmetry parameter and the gauge fermion to have the same chirality, and the introduction of the constant null vector in $\delta_Q A_\mu$. However, we have realized that in this case the closure of the supersymmetry algebra imposes too strong conditions on the fields to allow acceptable set of equations. In particular, we obtain the constraint $n_{[\rho} F_{\mu\nu]} = 0$, which is clearly too strong.

Our guiding principle now is to establish the closure of the supersymmetry transformation rules (3) and (4). In what follows, we shall first describe the full system of equations of motion, constraints and symmetries, which together with (3) and (4) characterize fully our model. Next, we shall explain the derivation of these equations.

To begin with, the constant vector n^μ must satisfy the condition

$$n_\mu n^\mu = 0 \quad . \quad (5)$$

The Yang-Mills field obeys the field equation

$$D_\mu F^\mu{}_{[\rho}{}^I n_{\sigma]} + \frac{1}{4} f^{IJK} \left(\bar{\lambda}^J \gamma_{\rho\sigma} \lambda^K \right) = 0 \quad , \quad (6)$$

and the constraint

$$n^\mu F_{\mu\nu}{}^I = 0 \quad . \quad (7)$$

The gauge fermion obeys the equation

$$\gamma^\mu D_\mu \lambda^I \equiv \gamma^\mu \left(\partial_\mu \lambda^I + f^{IJK} A_\mu{}^J \lambda^K \right) = 0 \quad , \quad (8)$$

and the constraints

$$n^\mu D_\mu \lambda^I = 0 \quad , \quad (9)$$

$$n^\mu \gamma_\mu \lambda^I = 0 \quad . \quad (10)$$

The whole system is invariant under the supersymmetry transformations (3) and (4), the usual Yang-Mills gauge transformations and the following extra local gauge transformation

$$\delta_\Omega A_\mu{}^I = \Omega^I n_\mu \quad , \quad (11)$$

subject to the condition

$$n^\mu D_\mu \Omega^I(x) = 0 \quad . \quad (12)$$

The commutator of two supersymmetry transformations yields a generalized translation, the usual Yang-Mills gauge transformation and an extra gauge transformation with parameters ξ^μ , Λ , Ω , respectively, as follows:

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_\xi + \delta_\Lambda + \delta_\Omega \quad , \quad (13)$$

where the composite parameters are given by

$$\xi^\mu = \bar{\epsilon}_2 \gamma^{\mu\nu} \epsilon_1 n_\nu \quad , \quad (14)$$

$$\Lambda^I = -\xi^\mu A_\mu^I \quad , \quad (15)$$

$$\Omega^I = \frac{1}{2} \bar{\epsilon}_2 \gamma^{\mu\nu} \epsilon_1 F_{\mu\nu}^I \quad . \quad (16)$$

Note that the global part of the algebra (13) is given by (2).

We now explain the derivation of the above equations. First, we note that the supersymmetry transformations close on the gauge field as in (13). Next, we find that the commutator of two supersymmetry transformations on the gauge fermion yields the result

$$\begin{aligned} [\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] \lambda^I &= \xi^\mu D_\mu \lambda^I - \frac{1}{32} (3\gamma_{\rho\sigma}{}^\tau - 8\gamma_\rho \delta_\sigma^\tau) n_\tau \zeta^{\rho\sigma} \not{D} \lambda^I \\ &\quad - \frac{7}{16} \gamma_\rho \not{n} D_\sigma \lambda^I \zeta^{\rho\sigma} + \frac{3}{32} \gamma_{\rho\sigma} n^\mu D_\mu \lambda^I \zeta^{\rho\sigma} \\ &\quad - \frac{1}{32(6!)} \gamma^\nu \gamma^{\rho_1 \dots \rho_6} \left(\not{n} D_\nu \lambda^I - n_\nu \not{D} \lambda^I \right) \xi_{\rho_1 \dots \rho_6} \quad , \end{aligned} \quad (17)$$

where $\xi^{\mu\nu} \equiv (\bar{\epsilon}_2 \gamma^{\mu\nu} \epsilon_1)$, $\xi^{\mu_1 \dots \mu_6} \equiv (\bar{\epsilon}_2 \gamma^{\mu_1 \dots \mu_6} \epsilon_1)$, $\not{n} \equiv \gamma^\mu n_\mu$ and $\not{D} \equiv \gamma^\mu D_\mu$. In obtaining the above result, we have used the Fierz identity for four Majorana-Weyl spinors of the same chirality in 12D:

$$\begin{aligned} (\bar{\psi}_1 \psi_2) (\bar{\psi}_3 \psi_4) &= -\frac{1}{32} \sum_i (\bar{\psi}_1 \mathcal{O}_i \psi_4) (\bar{\psi}_3 \mathcal{O}_i \psi_2) \quad , \\ \{\mathcal{O}_i\} &\equiv \left\{ I, \frac{i}{\sqrt{2!}} \gamma^{\mu\nu}, \frac{1}{\sqrt{4!}} \gamma^{\mu_1 \dots \mu_4}, \frac{i}{\sqrt{6!}} \gamma^{\mu_1 \dots \mu_6} \right\} \quad , \end{aligned} \quad (18)$$

and the following γ -matrix identities

$$\gamma^{\mu\nu\rho} \gamma_{\sigma\tau} \gamma_\mu = 6\gamma_{\sigma\tau}{}^{\nu\rho} + 32\delta_{[\sigma}^{[\nu} \gamma_{\tau]}^{\rho]} - 20\delta_{[\sigma}{}^\nu \delta_{\tau]}{}^\rho \quad , \quad (19)$$

$$\gamma^\mu \gamma_{\nu_1 \dots \nu_6} \gamma_\mu \equiv 0 \quad . \quad (20)$$

From (17) we see that the first term is the translation δ_ξ and gauge transformation δ_Λ consistent with (13), while the second, the third and the last term vanish on-shell, when the

λ -field satisfies the field equation (8). The two terms proportional to $\not{\eta}$ vanish by imposing the constraint (10), and the remaining term proportional to $n^\mu D_\mu \lambda^I$ vanishes by imposing the constraint (9). It follows from (10) that the constant vector n_μ must be null, *i.e.*, it must satisfy (5).

We next check the invariance of the field equation (8) and the constraints (9) and (10) under supersymmetry and extra gauge transformations. By varying (9) we obtain

$$\delta_Q(n^\mu D_\mu \lambda^I) = \frac{1}{4} \gamma^{\rho\sigma\tau} \epsilon \left(-2D_\rho F_{\sigma\mu}{}^I \right) n^\mu n_\tau + f^{IJK} \lambda^K \left(\bar{\epsilon} \not{\eta} \lambda^K \right) , \quad (21)$$

where we have also used the usual Bianchi identity $D_{[\mu} F_{\nu\rho]}{}^I \equiv 0$. Now the second term vanishes under (10), while for the first term to vanish, we need the new constraint (7). Similarly the variation of the constraint (10) yields

$$\delta_Q(n^\mu \gamma_\mu \lambda^I) = \frac{1}{2} \gamma^{\nu\rho} \epsilon F_{\sigma\nu}{}^I n_\rho n^\sigma + \frac{1}{4} \gamma^{\mu\nu} \epsilon F_{\mu\nu}{}^I n_\rho n^\rho , \quad (22)$$

which vanishes due to (7) and (5).

The δ_Q -transformation of the constraint (7) is easily confirmed:

$$\delta_Q(n^\mu F_{\mu\nu}{}^I) = \left(\bar{\epsilon} \gamma_\nu n^\mu D_\mu \lambda^I \right) - \left(\bar{\epsilon} D_\nu \not{\eta} \lambda^I \right) = 0 , \quad (23)$$

thanks to (9) and (10). As for the Yang-Mills field equation (6), it follows from the δ_Q -variation of the λ^I -field equation (8):

$$0 = \delta_Q \left(\gamma^\mu D_\mu \lambda^I \right) = \frac{1}{2} \gamma^{\rho\sigma} \epsilon \left[D^\nu F_{\nu[\rho}{}^I n_{\sigma]} + \frac{1}{4} f^{IJK} \left(\bar{\lambda}^J \gamma_{\rho\sigma} \lambda^K \right) \right] , \quad (24)$$

where we have used the constraint (7), together with the Bianchi identity for $F_{\mu\nu}$. We have checked that a further supersymmetric variation of the Yang-Mills equation (6) vanishes modulo the gauge fermion field equation (8), and the constraints (7), (9) and (10).

At this point we have checked fully the supersymmetry of all the equations of motion and the constraints. However, we still need to check the invariance of these equations under the extra gauge symmetry (11). Obviously, the constraints (9) and (10) are invariant under these transformations, while the variation of the constraint (7) yields

$$\delta_\Omega \left(n^\nu F_{\nu\mu}{}^I \right) = \left(n^\nu D_\nu \Omega^I \right) n_\mu - \left(D_\mu \Omega^I \right) n^\nu n_\nu . \quad (25)$$

The last term vanishes due to (5), while the vanishing of the first term requires the condition (12). The invariance of the gaugino equation (8) is obvious due to the condition (10).

Finally, we have also checked that the Yang-Mills equation is invariant under the extra gauge transformation (11), thanks to the constraints (7) and (12).

To summarize, we have the supersymmetric Yang-Mills multiplet with the supertransformation rule (3) and (4), and their field equations (8) and (6). We need the null-vector n^μ and other constraints on the fields given in (7), (9), (10) and (5). The on-shell closure of the gauge algebra is guaranteed to produce the translation as well as the extra gauge transformation (11) with the condition (12) for its parameter. The extra gauge transformation commutes with supersymmetry.

3. Dimensional Reduction to Ten and Lower Dimensions

In this section we shall use hats for the fields and indices in 12D, to be distinguished from the unhatted ones in 10D. We choose our coordinates to be $(\hat{x}^0, \hat{x}^1, \dots, \hat{x}^9, \hat{x}^{11}, \hat{x}^{12})$ with the metric $(\hat{\eta}_{\hat{\mu}\hat{\nu}}) = \text{diag.} (- + \dots + -)$. The 12D γ -matrices satisfy $\{\hat{\gamma}_{\hat{\mu}}, \hat{\gamma}_{\hat{\nu}}\} = 2\hat{\eta}_{\hat{\mu}\hat{\nu}}$. We can choose the γ -matrix to be

$$\hat{\gamma}_{\hat{\mu}} = \begin{cases} \hat{\gamma}_\mu = \gamma_\mu \otimes \sigma_3 & , \\ \hat{\gamma}_{11} = I \otimes \sigma_1 & , \\ \hat{\gamma}_{12} = -I \otimes i\sigma_2 & . \end{cases} \quad (26)$$

Here I is the 32×32 unit matrix and the σ 's are the Pauli matrices and γ_μ are the 32×32 γ -matrices in $9 + 1$ dimensions. The charge conjugation matrix can be chosen as

$$\hat{C} = -\hat{\gamma}_0 \hat{\gamma}_{12} = \gamma_0 \otimes \sigma_1 \equiv C \otimes \sigma_1 , \quad (27)$$

where we have identified γ_0 with the charge conjugation matrix C in 10D. Both \hat{C} and C are antisymmetric. Next, we define the matrix

$$\hat{\gamma}_{13} \equiv \hat{\gamma}_0 \hat{\gamma}_1 \cdots \hat{\gamma}_9 \hat{\gamma}_{11} \hat{\gamma}_{12} = \gamma_{11} \otimes \sigma_3 , \quad (28)$$

where $\gamma_{11} \equiv \gamma_0 \gamma_1 \cdots \gamma_9$. Finally, we choose the null-vector as

$$\left(\hat{n}^{\hat{\mu}}\right) = (0, 0, \dots, 0, +1, -1) , \quad \left(\hat{n}_{\hat{\mu}}\right) = (0, 0, \dots, 0, +1, +1) . \quad (29)$$

Even though this form looks special, we can show that starting with an arbitrary null-vector $\hat{n}'_{\hat{\mu}}$, we can *always* utilize orthogonal transformations within the two compact sub-manifolds to bring its components to the standard form (29).

We are now ready to analyse the null reduction of our field equations and constraints. To begin with, the constraint (10) implies that $\sigma_+ \hat{\lambda} = 0$, where $\sigma_\pm \equiv (\sigma_1 \pm i\sigma_2)/\sqrt{2}$. Together

with the 12D chirality condition, this implies that we can write $\widehat{\lambda}$ as

$$\widehat{\lambda} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \quad \gamma_{11}\lambda = \lambda. \quad (30)$$

Hence, the gauge fermion λ is a Majorana-Weyl spinor of $SO(9,1)$ as it should be in 10D supersymmetric Yang-Mills theory. For simplicity in notation, we shall use the matrix notation for the gauge field and gauge fermions, in the rest of the paper.

Using (29) and (30), we see that the remaining constraints given in (7) and (9) can be expressed as follows:

$$F_{-+} = 0, \quad F_{-\mu} = 0, \quad D_-\lambda = 0. \quad (31)$$

where we have used the coordinates $x^\pm \equiv (x^{11} \pm x^{12})/\sqrt{2}$. Using (30) and (31), we easily find that the field equations (6) and (8) take the form

$$D^\mu F_{\mu\nu} - \bar{\lambda}\gamma_\nu\lambda = 0, \quad \gamma^\mu D_\mu\lambda = 0. \quad (32)$$

Furthermore, the supersymmetry transformation rules (3) and (4) take the form

$$\delta_Q A_\pm = 0, \quad \delta_Q A_\mu = \bar{\epsilon}\gamma_\mu\lambda, \quad \delta_Q\lambda = \frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu}. \quad (33)$$

Our model is also invariant under the following local Yang-Mills and extra gauge transformations:

$$\delta A_+ = D_+\Lambda + \Omega, \quad D_-\Omega = 0, \quad (34)$$

$$\delta A_- = D_-\Lambda, \quad (35)$$

$$\delta A_\mu = D_\mu\Lambda, \quad \delta\lambda = -[\Lambda, \lambda]. \quad (36)$$

It is interesting to note that we have obtained the 10D supersymmetric Yang-Mills equations (32) and supersymmetry transformation rules (33), without any assumption on the dependence of the fields on the extra two coordinates x^\pm .

Considering the so called ordinary dimensional reduction, we set $\partial_+ = \partial_- = 0$. In that case, using the local Ω transformations we can gauge away A_+ . Then, the first equation in (31) is satisfied, while the second one reduces to $D_\mu A_- = 0$. From this equation we deduce that $A_- = 0$. Then the last equation in (31) is also satisfied, and the full system of equations of motions and symmetry transformations reduces precisely to the 10D supersymmetric Yang-Mills system.

If we do not assume independence of fields of the coordinates x^\pm , we find that we can gauge away the fields A_\pm , *modulo global issues*, by using the first constraint in (31),

together with the Ω gauge transformation (34) and the x^- dependent part of the Λ gauge transformation. Furthermore, using the second and third constraints in (31), we observe that the fields A_μ and λ depend on the 10D spacetime coordinates and x^+ alone.

In summary, we obtain the 10D supersymmetric Yang-Mills system with an arbitrary dependence on the extra coordinate x^+ . From the 10D point of view, we can interpret the coordinate x^+ as the affinization of the Yang-Mills gauge group. We shall comment further on this point at the conclusions.

Let us now consider the ordinary dimensional reduction of the 12D model down to 2+2 dimensions, where we will see that the coordinates x^+ will have a different significance, since they will be treated as part for the 2+2 dimensional spacetime. We label the coordinates as $\hat{x}^{\hat{\mu}} = (x^\mu, x^i)$ with $\mu = 0, 1, 11, 12$ and $i = 2, \dots, 9$, and set $\partial_i = 0$. We work with the Dirac γ -matrices: $\hat{\gamma}_\mu = \gamma_\mu \otimes I$ and $\hat{\gamma}_i = \Gamma_5 \otimes \gamma_i$, where γ_μ and γ_i are the $SO(2,2)$ and $SO(8)$ Dirac matrices, respectively: $\Gamma_5 = \gamma_0 \gamma_1 \gamma_{11} \gamma_{12}$, and I is the $16 \otimes 16$ unit matrix. It follows that $\hat{\gamma}_{13} = \Gamma_5 \otimes \Gamma_9$ with $\Gamma_9 \equiv \gamma_2 \cdots \gamma_9$ and $\hat{C} = C \otimes I$, where C is the antisymmetric charge conjugation matrix in 2+2 dimensions.

The 12D chirality condition on the gauge fermion becomes $\Gamma_5 \Gamma_9 \lambda = \lambda$. This means that in 2+2 dimensions we have a left-handed spinor in the $\mathbf{8}_S$ representation of $SO(8)$, and a right-handed spinor in the $\mathbf{8}_C$ representation. These make up 32 real components. The null-ness condition (10), and the Dirac equation further reduce the number of degrees of freedom to 8, which is to be expected in an $N = 4$ supersymmetric Yang-Mills multiplet in 4D. Note that the chirality condition $\Gamma_5 \Gamma_9 \epsilon = -\epsilon$ implies that the left-handed parameters are in the $\mathbf{8}_C$ representation of $SO(8)$, while the right-handed parameters are in the $\mathbf{8}_S$ representation.

Defining $A_i \equiv \phi^i$ and choosing the null-vector as

$$(n_\mu) = (0, 0, 1, 1) , \quad (n_i) = (0, \dots, 0) , \quad (37)$$

we find that the field equations of 10+2 dimensions reduces to 2+2 dimensions as follows:

$$\begin{aligned} D_\mu F^\mu{}_{[\nu} n_{\rho]} - [\phi_i, D_{[\nu} \phi^i] n_{\rho]} + \frac{1}{2} \bar{\lambda} \gamma_{\nu\rho} \lambda &= 0 , & n^\mu F_{\mu\nu} &= 0 , \\ D_\mu D^\mu \phi^i n_\nu + [\phi_k, [\phi^k, \phi_i]] n_\nu - \bar{\lambda} \gamma_\nu \gamma_i \Gamma_9 \lambda &= 0 , & n^\mu D_\mu \phi_i &= 0 , \\ \gamma^\mu D_\mu \lambda + \gamma^i [\phi_i, \lambda] &= 0 , & n^\mu D_\mu \lambda &= 0 , & n^\mu \gamma_\mu \lambda &= 0 . \end{aligned} \quad (38)$$

These equations are invariant under the following supersymmetry transformations

$$\delta_Q A_\mu = \bar{\epsilon} \gamma_\mu \lambda , \quad \delta_Q \phi_i = \bar{\epsilon} \gamma_i \Gamma_9 \lambda ,$$

$$\delta_Q \lambda = \left(\frac{1}{4} \gamma^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \gamma^\mu \gamma^i \Gamma_9 D_\mu \phi_i + \frac{1}{2} \gamma^{ij} \phi_i \phi_j \right) \not{\epsilon} . \quad (39)$$

In these equations we have suppressed the $SO(8)$ indices. A simple way to make them explicit is to split the fermions into their left and right-handed $SO(2,2)$ chiralities, and then to label the left (right)-handed gauge fermions with A (\dot{A}), and the left (right)-handed supersymmetry parameter with \dot{A} (A), where $A, \dot{A} = 1, \dots, 8$. For example, $\delta\phi^i = \bar{\epsilon}^A(\gamma^i)_{AA} \lambda^{\dot{A}} - \bar{\epsilon}^{\dot{A}}(\gamma^i)_{AA} \lambda^A$, with the $SO(2,2)$ chiralities suppressed, but clearly correlated with the $SO(8)$ chiralities.

The dimensionally reduced model also has an extra gauge symmetry under which only the A_μ transforms: $\delta A_\mu = \Omega n_\mu$. The model is a version of $N = 4$ super Yang-Mills that has an $SO(1,1) \otimes SO(8)$ bosonic symmetry, living in $2 + 2$ dimensional spacetime.

We can perform a different kind of dimensional reduction to $2 + 2$ dimensions, in which again $\partial_i = 0$, but with the null vector chosen as

$$(n_\mu) = (0, 0, 0, 1) , \quad (n_i) = (1, 0, \dots, 0) . \quad (40)$$

This will lead to a version of $N = 4$ super Yang-Mills which has $SO(2,1) \otimes SO(7)$ bosonic symmetry, but lives in $2 + 2$ dimensions. It is straightforward to obtain the analogs of (38) and (39) for this case, but we shall skip the details here.

4. Superspace Formulation

We have thus far established the component formulation for 12D supersymmetric Yang-Mills theory. Our next natural task is to re-formulate the same system in superspace. We have a chiral superspace with coordinates Z^M , and supervielbein E_M^A , where the tangent space indices are $A = (a, \alpha)$ with $a = 0, 1, 2, \dots, 9, 11, 12$ labelling the bosonic directions, and $\alpha = 1, \dots, 32$ labelling the fermionic directions. We may of course have spinor superfields that carry spinor indices of opposite chirality labelled by $\dot{\alpha} = 1, \dots, 32$. We use the (anti) symmetrization symbols such as $[AB]$ with unit strength normalization in superspace. Our superspace conventions are those of [11].

We proceed by defining the torsion super two-form $T^A = dE^A$, as can be read from the superalgebra (2):

$$T^c = e^\alpha \wedge e^\beta (\sigma^{cd})_{\alpha\beta} n_d , \quad T^\alpha = 0 , \quad (41)$$

where the basis one-forms are defined as $e^A = dZ^M E_M^A$. Hence, they satisfy

$$de^c = e^\alpha \wedge e^\beta (\sigma^{cd})_{\alpha\beta} n_d , \quad de^\alpha = 0 . \quad (42)$$

We next define the Yang-Mills curvature super two-form $F = dA + A^2$ as follows

$$F = e^\alpha \wedge e^b [n_b \chi_\alpha - 2(\sigma_b \lambda)_\alpha] + \frac{1}{2} e^a \wedge e^b F_{ba} , \quad (43)$$

where we have introduced the chiral spinor superfield χ_α and the anti-chiral spinor superfield $\lambda_{\dot{\alpha}}$. The presence of the two-form superfield F_{ab} is as expected, and its zero-th order in θ component is the usual Yang-Mills field strength. The presence of the spinor superfield $\lambda_{\dot{\alpha}}$ is also as expected, and its zero-th order in θ component is the gauge fermion. However, the occurrence of the spinor superfield χ is special to our model. It is needed for the closure of the super two-form F , but it drops out of the physical field equations, as we will explain further below.

Our task is to show that $DF = 0$, modulo the field equations and constraints of the previous section. To this end, we also impose the following constraints on various quantities occurring in (43):

$$n^a n_a = 0 , \quad n^a F_{ab} = 0 , \quad (44)$$

$$n^a (\sigma_a)_{\alpha\dot{\beta}} \lambda_{\dot{\beta}} = 0 , \quad n^a \nabla_a \lambda_{\dot{\beta}} = 0 , \quad (45)$$

$$\nabla_\alpha \lambda_{\dot{\beta}} = -\frac{1}{4} (\sigma^{cde})_{\alpha\dot{\beta}} F_{cd} n_e , \quad \nabla_{(\alpha} \chi_{\beta)} = \frac{1}{2} (\sigma^{ab})_{\alpha\beta} F_{ab} , \quad (46)$$

$$\nabla_\gamma F_{ab} = 4 \left(\sigma_{[a} \nabla_{b]} \lambda \right)_\gamma - 2 n_{[a} \nabla_{b]} \chi_\gamma . \quad (47)$$

We now show that $DF = 0$. First, we observe that the terms in DF proportional to $e^\alpha \wedge e^\beta \wedge e^\gamma$ vanish due to the identity

$$(\sigma^a{}^b)_{(\alpha\beta} (\sigma^a)_{\gamma)}{}^{\dot{\delta}} = \frac{1}{12} (\sigma_{cd})_{(\alpha\beta} (\sigma^{cd} \sigma^b)_{\gamma)}{}^{\dot{\delta}} , \quad (48)$$

together with the first of the constraints in (45). The superfield χ_α drops out automatically in this sector. We next examine the terms in DF proportional to $e^\alpha \wedge e^\beta \wedge e^c$. We find that these terms vanish due to the constraints (46). It is in this sector that we find the need to introduce the superfield χ_α . Next, we find that the terms in DF proportional to $e^a \wedge e^b \wedge e^\gamma$ vanish, due to the constraint (47). Finally, the terms proportional to $e^a \wedge e^b \wedge e^c$ vanish trivially, due to the usual Bianchi identity $D_{[a} F_{bc]} = 0$.

The constraints (44)-(47) encode the information about the supersymmetric Yang-Mills field equations. We can use (47) to get the λ -field equation by evaluating the anti-commutator $\{\nabla_\alpha, \nabla_\beta\} \lambda_{\dot{\gamma}}$ in two different ways:

$$T_{\alpha\beta}{}^c \nabla_c \lambda_{\dot{\gamma}} = \{\nabla_\alpha, \nabla_\beta\} \lambda_{\dot{\gamma}} = 2 \nabla_{(\alpha} \left(\nabla_{\beta)} \lambda_{\dot{\gamma}} \right) , \quad (49)$$

from which we find the gauge fermion field equation

$$(\sigma^a)_\beta \dot{\gamma} \nabla_a \lambda_{\dot{\gamma}} = 0 \quad , \quad (50)$$

in agreement with (8). In this process we also see that the superfield χ disappears completely, and it plays only a role of auxiliary superfield.

By taking a spinorial derivative of (50), we can get the Yang-Mills field equation

$$\nabla_a F^a_{[b} n_{c]} + \frac{1}{2} (\sigma_{bc})_{\dot{\alpha}}^{\dot{\beta}} \lambda^{\dot{\alpha}} \lambda_{\dot{\beta}} = 0 \quad . \quad (51)$$

Interestingly, the superfield χ disappeared completely also from this equation. We also note that in deriving (51), we have used the following identity:

$$(\sigma^a)_{(\alpha} \dot{\gamma} (\sigma^a)_{\beta)} \dot{\delta} = \frac{1}{8} (\sigma^{bc})_{\alpha\beta} (\sigma_{bc})^{\dot{\gamma}\dot{\delta}} \quad . \quad (52)$$

Finally, we mention the consistency related to χ -superfield. We can confirm the consistency of $\{\nabla_{(\alpha}, \nabla_{\beta)}\} \chi_{\gamma)} = 2\nabla_{(\alpha} (\nabla_{\beta} \chi_{\gamma)})$ by comparing the two sides, and see they actually coincide after the use of the n -dependent constraints, as well as the identity (48).

5. Conclusions

In this paper we have given an explicit field theoretic realization of the 12D superalgebra (2), in terms of supersymmetric Yang-Mills multiplet. The key to our construction is the use of a constant null vector, as motivated by (2, 1) strings [7, 8]. The resulting model, of course, lacks the full Poincaré invariance in $10 + 2$ dimensions, as expected in F-theory [1], and as is evident from the underlying algebra (2).

We have shown that the ordinary dimensional reduction to 10D yields the usual supersymmetric Yang-Mills equations, while two kinds of ordinary dimensional reductions to $2 + 2$ dimensions yield new kinds of $N = 4$ super Yang-Mills systems where the $SO(2, 2)$ group is broken down to $SO(1, 1)$ or $SO(2, 1)$, depending on the choice of the null vector, in accordance with [7, 8].

We have also found the curious result that if we relax the restriction of ordinary dimensional reduction, we obtain, modulo global matters which may involve Wilson lines, the 10D supersymmetric Yang-Mills system depending on the extra coordinate x^+ . This dependence can be interpreted as a straightforward affinization of the Yang-Mills group, which, however, appears to be redundant. The resolution of this issue may lie in finding a

suitable action principle for our model, which is lacking at present. An appropriate superstring or super three-brane action may also shed light on this issue, by providing the extra information about the dependence of the target space fields on the x_{\pm} coordinates.

We have also established the corresponding superspace formulation, in which we see the necessity of the fermionic superfield χ that eventually disappears in the superfield equations for physical fields.

A future direction to extend the present work is the construction of supergravity theories in $10+2$ dimensions. A hint to the structure of these theories may come from the requirement of κ -symmetry of a possible Green-Schwarz type action in curved superspace. One approach would be to lift the 10D heterotic string action to $10+2$ dimension by utilizing a null vector. However, a more satisfactory approach might be the construction of a three-brane action in which a $2+2$ dimensional worldvolume is embedded in $10+2$ dimensional spacetime. In either case, a curved superspace version of the flat superspace with a null vector described here may be relevant. We are currently investigating these issues.

In conclusion, we stress that the 12D model presented here provides a unified description for a variety of interesting supersymmetric field theories that can be obtained by dimensional reduction. As far as F-theory is concerned, although its low energy limit is not known with certainty at present, we hope that our results will be of relevance to this intriguing problem.

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