Superradiant control of γ-ray propagation by vibrating nuclear arrays

Xiwen Zhang* and Anatoly A. Svidzinsky

Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA
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The collective nature of light interactions with atomic and nuclear ensembles yields the fascinating phenomena of superradiance and radiation trapping. We study the interaction of γ rays with a coherently vibrating periodic array of two-level nuclei. Such nuclear motion can be generated, e.g., in ionic crystals illuminated by a strong driving optical laser field. We find that deflection of the incident γ beam into the Bragg angle can be switched on and off by nuclear vibrations on a superradiant time scale determined by the collective nuclear frequency Ω_a, which is of the order of terahertz. Namely, if the incident γ wave is detuned from the nuclear transition by frequency $\Delta \gg \Omega_a$ it passes through the static nuclear array. However, if the nuclei vibrate with frequency Δ then parametric resonance can yield energy transfer into the Bragg deflected beam on the superradiant time scale, which can be used for fast control of γ rays.

I. INTRODUCTION

Gamma rays are widely used in contemporary technologies for material modification, food sterilization and testing for weak points in welded structures. Medical applications of γ rays include the imaging technique of positron emission tomography and radiation therapies to treat cancerous tumors as well as detecting brain and cardiovascular abnormalities.

Since the discovery of recoilless nuclear resonance by Mössbauer [1, 2], studies of the interaction between γ rays and Mössbauer nuclear ensembles have undergone rapid development and have yielded many real and potential applications in, e.g., Mössbauer spectroscopy [3] and quantum information [4, 5]. Due to their small wavelength, γ rays are naturally suitable for achieving high spatial resolution and for making small quantum photonic circuits [6].

However, control of γ rays still remains a challenging problem. Coherent effects, such as level mixing induced transparency [7], electromagnetically induced transparency [8], γ echo [9, 10], phase modulation [11, 12], and the nuclear lighthouse effect [13], can be adopted to manipulate γ radiation. Modulation of Mössbauer radiation by pulsed laser excitation was demonstrated in [14]. The total reflection of the grazing incidence was used to reflect γ rays, but application of this technique is limited due to small grazing angle. Development of the γ-ray optics led to the design of the Laue lens [15] via nuclear Bragg diffraction [16]. It has also been suggested that γ rays can be manipulated using Delbrück scattering [17].

Effective control of γ rays requires further advancements and innovations. Development of a fast switch of γ rays is important for extending the time resolution of γ-ray sources and for increasing the operating speed of γ-ray quantum information processing. Nanosecond γ-ray switching has been realized by magnetically manipulating

nuclear excitation based on the quantum beat in nuclear Bragg scattering [18]. Picosecond x-ray Bragg switch utilizing laser-generated phonons was proposed [19] and later demonstrated experimentally [20, 21].

In this paper we investigate a way to control propagation of a γ-ray beam through a crystal by controlling its collective absorption and reemission by many nuclei. Collective spontaneous emission from atomic ensembles has been a subject of long-standing interest since the pioneering work of Dicke [22]. The collective nature of light interaction yields fascinating effects such as superradiance and radiation trapping even at the single-photon level. Recent studies focus on collective, virtual and nonlocal effects in such systems [23–40]. The Josephson effect for photons in two weakly linked microcavities is an example of the collective physics in coupled atom-cavity systems [41].

The interaction of light with ordered arrays of nuclei in crystals offers new perspectives. For example, a photon collectively absorbed by a random medium (e.g., gas) will be reemitted in the same direction as the incident photon [23]. However, in the case of a crystal lattice, collective reemission can occur in several directions (Bragg angles). The interaction strength between the γ-ray beam and the crystal depends on the detuning Δ of the photon frequency from the nuclear transition. Here we show that one can redirect a γ-ray beam into a desirable Bragg angle by making the crystal lattice coherently vibrate with frequency Δ which lies, e.g., in the infrared region. Such lattice vibrations are in the combination parametric resonance with the frequency difference between two eigenmodes of the coupled light-nuclear system which results in resonant energy transfer from the incident γ-ray beam to the wave propagating at the Bragg angle. This process is analogous to the parametric frequency mixing in propagating circuits [42].

Nuclear vibrations can be generated by a driving laser pulse and can be turned on and off on a short time scale. γ-ray redirection, produced by parametric resonance, occurs on a time scale determined by the collective nuclear frequency Ω_a which typically lies in the terahertz region. This mechanism allows us to control propagation of high

frequency γ photons by driving the system, e.g., with an infrared laser.

II. THE MODEL AND DERIVATION OF BASIC EQUATIONS

We consider a perfect crystal composed of two-level (a and b) nuclei with transition frequency ω_{ab} as shown in Fig. 1(a). The nuclear transition frequency ω_{ab} typically lies in the hard x-ray or γ-ray region. Nuclei are located at positions $\mathbf{r}_j$ and form a periodic lattice, where the index j labels different nuclei. Typically, the inter nuclei spacing is much larger than the nuclear radiation wave length $\lambda_{ab} = 2\pi c/\omega_{ab}$, where c is the speed of light.
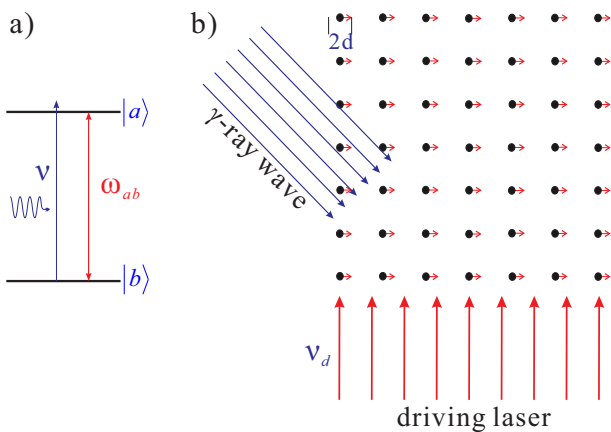

FIG. 1: (Color online) Illustration of the model. (a) Energy diagram of the two-level nuclear system. (b) Present model: an incident γ-ray plane wave interacts collectively with a recoilless nuclear array, while the strong optical laser field produces coherent oscillations of the nuclei with amplitude d and frequency ν_d.

We assume that the lattice is coherently excited so that nuclei oscillate along the direction given by a unit vector $\hat{n}$ around their equilibrium positions $\mathbf{r}_j^0$. The oscillation frequency ν_d lies in the infrared or visible region. In ionic crystals such oscillations can be produced, e.g., by a strong linearly polarized driving laser pulse with frequency ν_d. A typical example is potassium iodide crystal, which has a face-centered-cubic unit cell of iodide ions with potassium ions in octahedral holes. By applying an external driving field one can make ions K^+ and I^- move in opposite directions such that nuclei of the same species will oscillate in unison. Both K and I have Mössbauer isotopes. Namely, ^{40}K has a Mössbauer transition with energy 29.8 keV and spontaneous decay rate $\Gamma = 2.4 \times 10^8$ s^{-1}, while ^{127}I has a transition with energy 58.6 keV and $\Gamma = 5.1 \times 10^8$ s^{-1}.

We consider an interaction of high-frequency (x- or γ-ray) photons with a coherently vibrating nuclear lattice of a particular Mössbauer isotope. The presence of nuclei of another species in the crystal is irrelevant since

they have a very different transition frequency. We assume that the motion of each nuclei j involved in the interaction is given by

$$\mathbf{r}_j(t) = \mathbf{r}_j^0 + \hat{n} f(t), \tag{1}$$

where

$$f(t) = d\sin(\nu_d t). \tag{2}$$

Here $\nu_d \ll \omega_{ab}$ and $d \lesssim \lambda_{ab}$ is the amplitude of the laser induced nuclei oscillations.

In our model a weak, plane, linearly polarized γ-ray wave with the wave vector $\mathbf{k}_1$ and frequency $\nu_1 = ck_1$ detuned from the nuclear transition frequency ω_{ab} by an amount $\Delta_1 \ll \omega_{ab}$ enters the crystal and collectively interacts with the oscillating recoilless nuclei [see Fig. 1(b)]. For the sake of simplicity, we consider only the interaction of the wave with the nuclei and disregard interaction with electrons. Processes such as internal conversion, the photoelectric effect [43], and electron Rayleigh scattering [44, 45] are neglected.

We treat the problem in a semiclassical formalism. Namely, the electromagnetic field $E(t, \mathbf{r})$ of the γ ray is described by the classical Maxwell equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E = \mu_0 \frac{\partial^2 P}{\partial t^2} \tag{3}$$

in which μ_0 is the permeability of free space and the polarization of the medium

$$P = \sum_j \left(d_{ba}\rho_{ab}^j + \text{c.c.}\right) \delta\left(\mathbf{r} - \mathbf{r}_j(t)\right) \tag{4}$$

is determined by the off-diagonal elements of the nuclear density matrix ρ_{ab}^j. In Eq. (4) the summation is taken over nuclei that are treated as point particles located at positions $\mathbf{r}_j(t)$. Assuming that the nuclear transition matrix element d_{ab} is real and introducing the Rabi frequency of the γ-ray field $\Omega_\gamma(t, \mathbf{r}) = d_{ab}E(t, \mathbf{r})/\hbar$, we obtain

$$\left(c^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\Omega_\gamma(t, \mathbf{r})$$
$$= \frac{c^2\mu_0|d_{ab}|^2}{\hbar}\frac{\partial^2}{\partial t^2}\sum_j\left(\rho_{ab}^j + \text{c.c.}\right)\delta(\mathbf{r} - \mathbf{r}_j(t)). \tag{5}$$

Equation (5) must be supplemented by the evolution equation for the nuclear density matrix

$$\frac{\partial \rho_{ab}^j(t)}{\partial t} = -i\omega_{ab}\rho_{ab}^j(t) + i\Omega_\gamma(t, \mathbf{r}_j(t))(1 - 2\rho_{aa}^j). \tag{6}$$

We assume that nuclear excitation remains weak, so the population of the excited state ρ_{aa}^j can be disregarded. We look for a solution in the form

$$\Omega_\gamma(t, \mathbf{r}) = \Omega(t, \mathbf{r})e^{-i\omega_{ab}t} + \text{c.c.}, \tag{7}$$

$$\rho_{ab}^j(t) = \rho^j(t)e^{-i\omega_{ab}t}, \qquad (8)$$

where $\Omega(t, \mathbf{r})$ and $\rho^j(t)$ are slowly varying functions of t as compared to the fast oscillating exponentials. In the slowly varying amplitude approximation, Eqs. (5) and (6) reduce to

$$\left\{ \frac{\partial}{\partial t} + \frac{c^2}{2i\omega_{ab}} \left[\left(\frac{\omega_{ab}}{c}\right)^2 + \nabla^2 \right] \right\} \Omega(t, \mathbf{r})$$
$$= i\frac{\Omega_a^2}{N} \sum_j \rho^j(t)\delta(\mathbf{r} - \mathbf{r}_j(t)), \qquad (9)$$

$$\frac{\partial \rho^j(t)}{\partial t} = i\Omega(t, \mathbf{r}_j(t)), \qquad (10)$$

where

$$\Omega_a = \sqrt{\frac{c^2\mu_0|d_{ab}|^2\omega_{ab}N}{2\hbar}} = \sqrt{\frac{3cN\lambda_{ab}^2\Gamma}{8\pi}} \qquad (11)$$

is the collective nuclei frequency proportional to the square root of the average nuclei density N and Γ is the spontaneous decay rate of the nuclear transition. Physically, Ω_a determines the time scale of the collective resonant absorption of the incident photon by the medium [27, 38, 46] and typically is of the order of terahertz. For example, for a 29.8-keV transition of the ^{40}K Mössbauer isotope that spontaneously decays at the rate $\Gamma = 2.4 \times 10^8$ s^{-1} if we take the nuclei density to be $N = 8 \times 10^{21}$ cm^{-3} we obtain $\Omega_a \sim 3 \times 10^{11}$ s^{-1}.

A crystal is made up of a periodic arrangement of atoms (Bravais lattice) that form an infinite array of discrete points given by $\mathbf{r}_j^0 = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3$, where m_i $(i = 1, 2, 3)$ are any integers and $\mathbf{a}_i$ are the primitive lattice vectors. As a consequence, $\sum_j \delta(\mathbf{r} - \mathbf{r}_j(t))$ is a periodic function of $\mathbf{r}$ with periods $\mathbf{a}_i$ and, thus, it can be expanded in the Fourier series as

$$\sum_j \delta(\mathbf{r} - \mathbf{r}_j(t)) = N \sum_m e^{i\mathbf{K}_m \cdot [\mathbf{r} - \mathbf{r}_j(t)]}$$
$$= N \sum_m e^{i\mathbf{K}_m \cdot [\mathbf{r} - \hat{n}f(t)]}, \qquad (12)$$

where $\mathbf{K}_m = m_1\mathbf{b}_1 + m_2\mathbf{b}_2 + m_3\mathbf{b}_3$, $\mathbf{b}_{1,2,3}$ are the primitive vectors of the reciprocal lattice and N is the average nuclear density.

We look for $\rho^j(t)$ in the form

$$\rho^j(t) = \rho(t)e^{i\mathbf{k}_1 \cdot \mathbf{r}_j^0}. \qquad (13)$$

Multiplying both sides of Eq. (12) by $e^{i\mathbf{k}_1 \cdot \mathbf{r}}$ we obtain

$$\sum_j e^{i\mathbf{k}_1 \cdot \mathbf{r}_j^0}\delta(\mathbf{r} - \mathbf{r}_j(t)) = N \sum_m e^{i(\mathbf{k}_1 + \mathbf{K}_m)\cdot[\mathbf{r} - \hat{n}f(t)]}. \qquad (14)$$

This sum enters the right hand side of Eq. (9). In the Fourier series (14) we are interested in terms that are in resonance with the left hand side of Eq. (9). For simplicity we assume that only two vectors, namely, $\mathbf{k}_1$ and $\mathbf{k}_2 = \mathbf{k}_1 + \mathbf{K}_b$ have absolute values close to ω_{ab}/c, where $\mathbf{K}_b$ is a reciprocal lattice vector, see Fig. 2. The other terms in (14) are off resonance and thus can be disregarded. Therefore, one can write approximately

$$\sum_j e^{i\mathbf{k}_1 \cdot \mathbf{r}_j^0}\delta(\mathbf{r} - \mathbf{r}_j(t))$$
$$\approx Ne^{-i\mathbf{k}_1 \cdot \hat{n}f(t)}e^{i\mathbf{k}_1 \cdot \mathbf{r}} + Ne^{-i\mathbf{k}_2 \cdot \hat{n}f(t)}e^{i\mathbf{k}_2 \cdot \mathbf{r}}. \qquad (15)$$

This approximation implies that the incident wave $\mathbf{k}_1$ is coupled only with one Bragg wave $\mathbf{k}_2$.
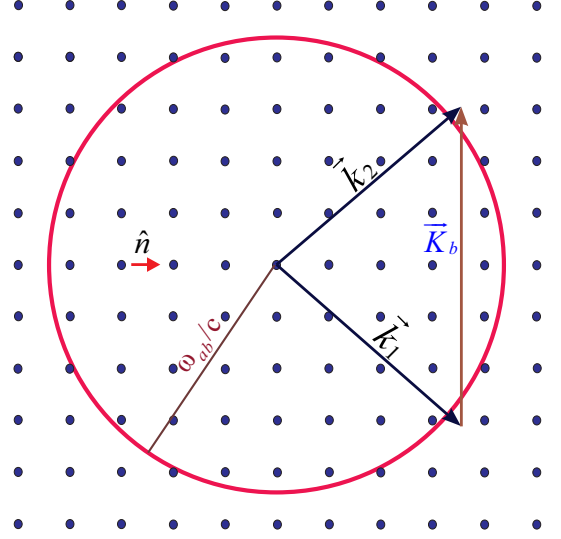

FIG. 2: (Color online) Two dimensional reciprocal lattice of the crystal is shown by dots. The incident γ-ray beam with the wave vector $\mathbf{k}_1$ is detuned from the nuclear transition frequency ω_{ab}. The incident wave is coupled with the Bragg wave that has wave vector $\mathbf{k}_2 = \mathbf{k}_1 + \mathbf{K}_b$, where $\mathbf{K}_b$ is a reciprocal lattice vector.

Equation (15) suggests that one can look for a solution for $\Omega(t, \mathbf{r})$ in the form of a superposition of these coupled waves

$$\Omega(t, \mathbf{r}) = \Omega_1(t)e^{i\mathbf{k}_1 \cdot \mathbf{r}} + \Omega_2(t)e^{i\mathbf{k}_2 \cdot \mathbf{r}}. \qquad (16)$$

Then Eqs. (9) and (10) yield the following equations for $\Omega_1(t)$, $\Omega_2(t)$ and $\rho(t)$ (we take into account that $e^{i\mathbf{k}_1 \cdot \mathbf{r}_j^0} = e^{i\mathbf{k}_2 \cdot \mathbf{r}_j^0}$)

$$\left(\frac{\partial}{\partial t} + i\Delta_1\right)\Omega_1(t) = i\Omega_a^2 e^{-i\mathbf{k}_1 \cdot \hat{n}f(t)}\rho(t), \qquad (17)$$

$$\left(\frac{\partial}{\partial t} + i\Delta_2\right)\Omega_2(t) = i\Omega_a^2 e^{-i\mathbf{k}_2 \cdot \hat{n}f(t)}\rho(t), \qquad (18)$$

$$\frac{\partial \rho(t)}{\partial t} = i\Omega_1(t)e^{i\mathbf{k}_1 \cdot \hat{n}f(t)} + i\Omega_2(t)e^{i\mathbf{k}_2 \cdot \hat{n}f(t)}, \qquad (19)$$

where

$$\Delta_{1,2} = \frac{c^2 k_{1,2}^2 - \omega_{ab}^2}{2\omega_{ab}} \approx \nu_{1,2} - \omega_{ab} \qquad (20)$$

are detunings of the two coupled waves from the nuclear transition frequency ω_{ab}. Taking the time derivative of both sides of Eqs. (17) and (18) and using Eq. (19), we obtain evolution equations for two γ-ray waves $\Omega_1(t)$ and $\Omega_2(t)$:

$$\left(\frac{\partial}{\partial t} + i\mathbf{k}_1 \cdot \hat{n}\dot{f}\right)\left(\frac{\partial}{\partial t} + i\Delta_1\right)\Omega_1$$
$$+ \Omega_a^2\left[\Omega_1 + \Omega_2 e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\hat{n}f(t)}\right] = 0\,, \quad (21)$$

$$\left(\frac{\partial}{\partial t} + i\mathbf{k}_2 \cdot \hat{n}\dot{f}\right)\left(\frac{\partial}{\partial t} + i\Delta_2\right)\Omega_2$$
$$+ \Omega_a^2\left[\Omega_2 + \Omega_1 e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\hat{n}f(t)}\right] = 0\,. \quad (22)$$

Equations (21) and (22) constitute one of our main findings. These equations describe two coupled harmonic oscillators whose parameters periodically change in time. The varying of the parameters drives the system. Namely, nuclei vibrations modulate coupling between two oscillators as indicated by the $\Omega_a^2 e^{\pm i(\mathbf{k}_1-\mathbf{k}_2)\cdot\hat{n}f(t)}$ terms and, in addition, they periodically modulate the oscillator's frequency by means of the Doppler shift $i\mathbf{k}_{1,2} \cdot \hat{n}\dot{f}$.

It is known that parametric oscillators can have parametric resonances when system's parameters are periodically modulated which can lead to exponentially growing oscillations. An interesting question appears in this context: Can Eqs. (21) and (22) yield exponentially growing solutions which would imply that the high-frequency γ-ray field is being generated at the expense of the energy stored in the low-frequency nuclear vibrations? In the Appendix we show that the answer to this question is that nuclear vibrations can not excite nuclear transitions in the present model. Specifically, we show that the sum of the energy of the high frequency field $\Omega(t, \mathbf{r})$ and that stored in the nuclear excitation is conserved no matter how nuclei move.

Nevertheless, parametric resonance can be useful in the present problem. Namely, it can substantially speed up energy transfer from one coupled oscillator to another, that is, from the incident γ-ray beam to the deflected one. This mechanism can be used to control propagation of γ rays on a short time scale, which we discuss next.

III. BEAM DEFLECTION BY COHERENT LATTICE VIBRATION

A. Deflection by static lattice

First we consider the interaction between the γ-ray field and a static nuclear array. In this case there is no nuclear motion, so $f = 0$ and Eqs. (17)−(19) can be solved analytically. In particular, if $\Delta_1 = \Delta_2 = \Delta$ the solution satisfying the initial condition $\Omega_1(0) = A$,

$\Omega_2(0) = 0$ and $\rho(0) = 0$ reads

$$\Omega_1(t) = \frac{Ae^{-i\Delta t}}{2}\left(\frac{\omega_+ e^{i\omega_- t} - \omega_- e^{i\omega_+ t}}{\sqrt{\Delta^2 + 8\Omega_a^2}} + 1\right)\,, \quad (23)$$

$$\Omega_2(t) = \frac{Ae^{-i\Delta t}}{2}\left(\frac{\omega_+ e^{i\omega_- t} - \omega_- e^{i\omega_+ t}}{\sqrt{\Delta^2 + 8\Omega_a^2}} - 1\right)\,, \quad (24)$$

$$\rho(t) = -\frac{Ae^{-i\Delta t}}{\sqrt{\Delta^2 + 8\Omega_a^2}}(e^{i\omega_- t} - e^{i\omega_+ t})\,, \quad (25)$$

where

$$\omega_\pm = \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + 8\Omega_a^2}\right)\,. \quad (26)$$

Equations (23)−(25) yield that on resonance ($\Delta = 0$)

$$\Omega_1 = A\cos^2(\Omega_a t/\sqrt{2})\,, \quad (27)$$

$$\Omega_2 = -A\sin^2(\Omega_a t/\sqrt{2})\,, \quad (28)$$

$$\rho = \frac{iA}{\sqrt{2}\Omega_a}\sin(\sqrt{2}\Omega_a t)\,. \quad (29)$$

That is, energy is periodically transferred back and forth between two coupled waves on a time scale given by the collective nuclear frequency Ω_a which is proportional to the square root of the nuclear density. Typically $1/\Omega_a$ is of the order of picoseconds. Amplitudes of the γ-ray beams undergo collective oscillations [27, 38] with frequency $\sqrt{2}\Omega_a$, as shown in Fig. 3a.

According to Eq. (29), nuclei become excited during the energy transfer between two γ waves. Namely, the incoming γ wave is partially absorbed by the nuclear array. Absorption is followed by the superradiant spontaneous emission of photons into the coupled wave.

In contrast, if the wave frequency is off resonance, i.e. $\Delta \gg \Omega_a$, the energy transfer between beams Ω_1 and Ω_2 occurs over a much longer time

$$t_{\text{tr}}^0 = \frac{\pi}{|\omega_-|} \approx \frac{\pi|\Delta|}{2\Omega_a^2}, \quad (30)$$

as shown in Fig. 3(b).

One should mention that energy oscillations between two γ-ray modes, referred to as the temporal Pendellösung effect due to different hyperfine transition frequencies at different nuclear sites, have been discussed in [47]. In Ref. [18] the Bragg switching of the γ-ray beam was realized using manipulation of nuclear spin states. In our mechanism, oscillations appear due to the collective nature of the interaction between light and the nuclear ensemble. Next we investigate how nuclear motion affects energy transfer between the two coupled γ-ray waves.

B. Beam deflection by oscillating lattice

Here we assume that the nuclear array coherently vibrates with frequency ν_d and the consider transformation
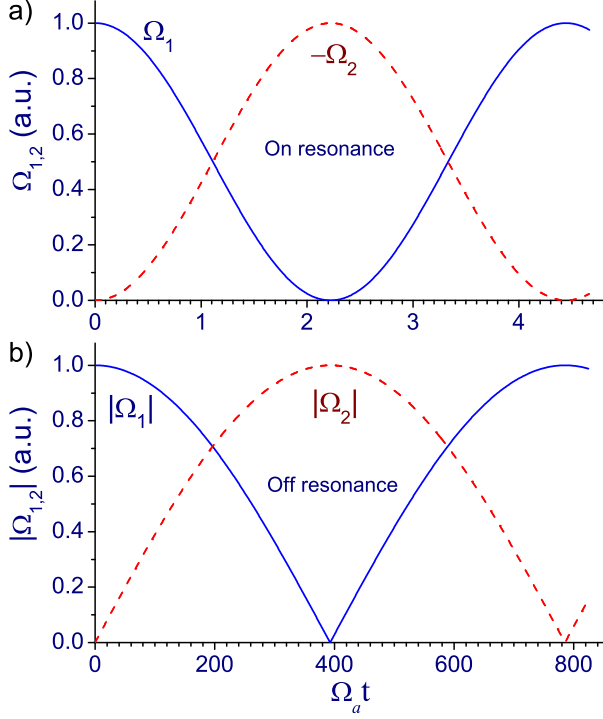
FIG. 3: (Color online) Time evolution of coupled γ-ray waves $\Omega_1(t)$ and $\Omega_2(t)$ produced by interaction with static nuclear array. Initially $\Omega_2(0) = 0$ and nuclei are in the ground state. The solid line represents $|\Omega_1(t)|$ and the dashed line indicates $|\Omega_2(t)|$. (a) The wave frequency is in resonance with the nuclear transition. Energy is transferred back and forth between $\Omega_1(t)$ and $\Omega_2(t)$ with collective frequency $\sqrt{2}\Omega_a$ which typically lies in the terahertz range. (b) Off-resonance interaction with frequency detuning $\Delta = 250\Omega_a$. The energy transfer occurs on a much longer time scale $\pi\Delta/2\Omega_a^2$.

of the incoming wave $\mathbf{k}_1$ into the deflected wave $\mathbf{k}_2$. We assume that both waves are equally detuned from the nuclear transition, that is $\Delta_1 = \Delta_2 = \Delta$. We also assume that nuclei vibrate with amplitude d along the direction $\hat{n}$ perpendicular to $\mathbf{k}_1 - \mathbf{k}_2$, as indicated in Fig. 2. Then $\mathbf{k}_1 \cdot \hat{n} = \mathbf{k}_2 \cdot \hat{n}$. Introducing the dimensionless modulation amplitude

$$\kappa = d\mathbf{k}_1 \cdot \hat{n} \tag{31}$$

Eqs. (21) and (22) reduce to

$$\ddot{\Omega}_1 + i(\Delta + F)\dot{\Omega}_1 + (\Omega_a^2 - \Delta F)\Omega_1 + \Omega_a^2\Omega_2 = 0, \tag{32}$$
$$\ddot{\Omega}_2 + i(\Delta + F)\dot{\Omega}_2 + (\Omega_a^2 - \Delta F)\Omega_2 + \Omega_a^2\Omega_1 = 0, \tag{33}$$

where

$$F(t) = \kappa\nu_d\cos(\nu_d t) \tag{34}$$

is a function that describes the modulation produced by nuclear motion. The amplitude of the nuclei vibrations

d is much smaller than the spacing a between nuclei. However, since the wave length of the nuclear transition is also small compared to a the modulation amplitude κ could be of the order of 1.

The initial conditions for Eqs. (32) and (33) are $\Omega_1(0) = A$ and $\Omega_2(0) = 0$. We assume that initially there is no nuclear excitation [$\rho(0) = 0$], which, according to Eqs. (17) and (18), yield $\dot{\Omega}_1(0) = -i\Delta A$ and $\dot{\Omega}_2(0) = 0$.

Equations (32) and (33) have the integral of motion

$$\Omega_1 = \Omega_2 + Ae^{-i\Delta t}. \tag{35}$$

Plugging this into Eq. (33) and introducing $\tilde{\Omega}_2$ according to

$$\Omega_2 = e^{-i\Delta t}\left(\tilde{\Omega}_2 - \frac{A}{2}\right) \tag{36}$$

we obtain the following equation for $\tilde{\Omega}_2$

$$\frac{d^2\tilde{\Omega}_2}{dt^2} + i(F - \Delta)\frac{d\tilde{\Omega}_2}{dt} + 2\Omega_a^2\tilde{\Omega}_2 = 0 \tag{37}$$

which is an equation of the parametric oscillator. Equation (37) has a solution in terms of special functions, however, such a solution is not very insightful. Instead, we derive an approximate solution that clearly shows the physics behind the parametric speed up of the energy transfer. Introducing the function $u(t)$

$$\frac{d\tilde{\Omega}_2}{dt} = \exp\left(-i\int_0^t (F(t') - \Delta)dt'\right)u(t), \tag{38}$$

one can rewrite Eq. (37) as

$$\frac{du}{dt} = -2\Omega_a^2\exp\left(i\int_0^t (F(t') - \Delta)dt'\right)\tilde{\Omega}_2. \tag{39}$$

Next we expand the exponential factor into the Fourier series

$$\exp\left(i\int_0^t (F(t') - \Delta)dt'\right) = e^{-i\Delta t}e^{i\kappa\sin(\nu_d t)}$$
$$= e^{-i\Delta t}[J_0(\kappa) + 2iJ_1(\kappa)\sin(\nu_d t)$$
$$+ 2J_2(\kappa)\cos(2\nu_d t) + \dots], \tag{40}$$

where $J_n(\kappa)$ are the Bessel functions. We assume that ν_d is close to Δ while $\tilde{\Omega}_2$ and u are slowly varying functions of time on the scale $1/\nu_d$. Then in the Fourier expansion (40) one can keep only the slowly varying term and approximately write

$$\exp\left(i\int_0^t (F(t') - \Delta)dt'\right) \approx J_1(\kappa)e^{i(\nu_d - \Delta)t}. \tag{41}$$

As a result, Eqs. (38) and (39) reduce to

$$\frac{d\tilde{\Omega}_2}{dt} = J_1(\kappa)e^{-i(\nu_d - \Delta)t}u, \tag{42}$$
$$\frac{du}{dt} = -2J_1(\kappa)\Omega_a^2 e^{i(\nu_d - \Delta)t}\tilde{\Omega}_2, \tag{43}$$

which can be solved analytically. Plugging this solution into Eqs. (35) and (36) we finally obtain

$$\Omega_1 = \frac{A e^{-i\Delta t}}{2}\left(\frac{\omega_+ e^{-i\omega_- t} - \omega_- e^{-i\omega_+ t}}{\sqrt{(\nu_d - \Delta)^2 + 8J_1^2(\kappa)\Omega_a^2}} + 1\right) , \quad (44)$$

$$\Omega_2 = \frac{A e^{-i\Delta t}}{2}\left(\frac{\omega_+ e^{-i\omega_- t} - \omega_- e^{-i\omega_+ t}}{\sqrt{(\nu_d - \Delta)^2 + 8J_1^2(\kappa)\Omega_a^2}} - 1\right) , \quad (45)$$

$$(46)$$

where

$$\omega_\pm = \frac{1}{2}\left(\nu_d - \Delta \pm \sqrt{(\nu_d - \Delta)^2 + 8J_1^2(\kappa)\Omega_a^2}\right). \quad (47)$$

When $\nu_d = \Delta$ we find

$$\Omega_1 = A e^{-i\Delta t}\cos^2\left(\frac{J_1(\kappa)}{\sqrt{2}}\Omega_a t\right) , \quad (48)$$

$$\Omega_2 = -A e^{-i\Delta t}\sin^2\left(\frac{J_1(\kappa)}{\sqrt{2}}\Omega_a t\right) . \quad (49)$$

Eqs. (48) and (49) show that the rate of energy transfer between two coupled waves depends on the amplitude of the nuclear vibrations. The optimum value of the modulation amplitude κ corresponds to maximum of $J_1(\kappa)$, that is $\kappa = 1.841$ which gives $J_1(\kappa)/\sqrt{2} = 0.411$. For larger κ the transfer rate oscillates following $J_1(\kappa)$.

For $\kappa \ll 1$ one can use the expansion $J_1(\kappa) \approx \kappa/2$. Then Eq. (49) yields that energy transfer time between two waves is

$$t_{\text{tr}} = \frac{\sqrt{2}\pi}{\kappa\Omega_a}. \quad (50)$$

In Fig. 4 we plot $\Omega_2(t)$ for different values of the modulation amplitude κ obtained by numerical solution of Eqs. (32) and (33). Our analytical result (49) gives essentially the same curves.

When the incident γ wave Ω_1 is off resonance with the nuclear transition, the time it takes for the energy to transfer from Ω_1 into the deflected wave Ω_2 can substantially vary with or without nuclear vibrations. This can be used for fast switching of the wave propagation that can be achieved in the regime $\kappa\Delta \gg \Omega_a$. If the γ-wave detuning Δ is large enough then the wave will pass through the static crystal without deflection. However, if the nuclear vibrations are suddenly turned on with $\nu_d = \Delta$ the incident γ wave will be deflected on a time scale given by Eq. (50), which could be a few picoseconds.

Figure 5 demonstrates the effect for a medium with $\Omega_a = 0.8$ THz assuming that the incident wave is detuned from the nuclear transition by $\Delta = 250\Omega_a$. For a static crystal the fields Ω_1 and Ω_2 are shown by dashed lines. Without nuclear vibrations it takes $t_{\text{tr}}^0 = 491$ ps for the wave Ω_1 to convert into Ω_2. If the crystal size
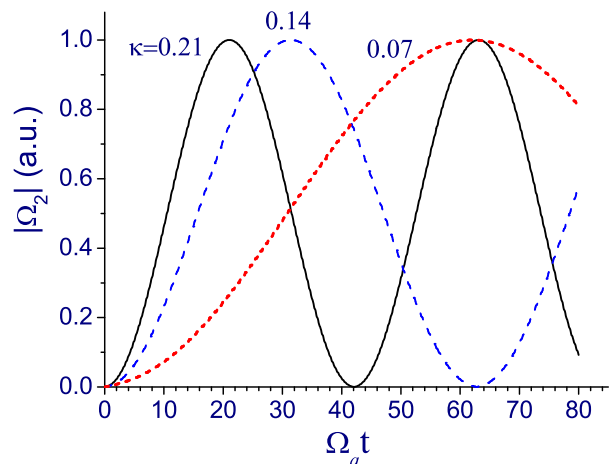

FIG. 4: (Color online) Time evolution of the deflected γ-wave $|\Omega_2(t)|$ obtained by numerical solution of Eqs. (32) and (33) for different values of the modulation amplitude $\kappa = 0.21$, 0.14 and 0.07. In simulations we set $\nu_d = \Delta = 250\Omega_a$. The transformation time between two γ-waves Ω_1 and Ω_2 is $t_{\text{tr}} \approx \sqrt{2}\pi/\kappa\Omega_a$.

is smaller than $ct_{\text{tr}}^0 = 15$ cm the incident wave passes through. However, if the nuclear array vibrates with modulation amplitude $\kappa = 0.21$ the conversion time becomes $t_{\text{tr}} = 26$ ps and, thus, the wave will be deflected at a length of 0.8 cm (solid lines in Fig. 5).
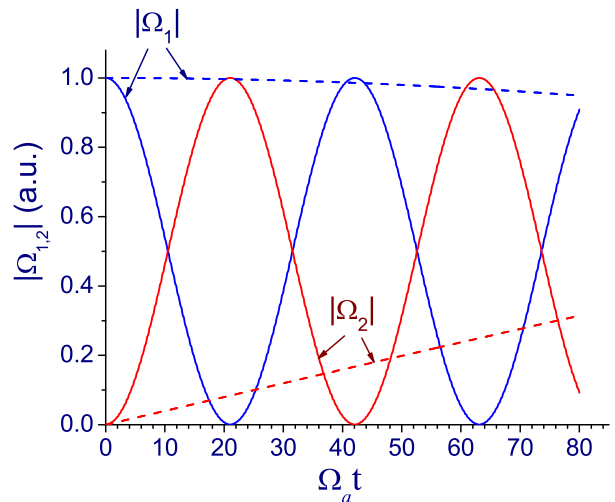

FIG. 5: (Color online) Illustration of the γ-ray switch operation. The incident γ wave Ω_1 is detuned from the nuclear transition by $\Delta = 250\Omega_a$. The dashed lines show the transformation of Ω_1 into Ω_2 for a static crystal and the solid lines are for nuclear array vibrating with frequency $\nu_d = \Delta$ and modulation amplitude $\kappa = 0.21$.

One should note that the γ-ray switch can also operate in the on resonance regime. Namely, when the incident

wave is on resonance with the nuclear transition it converts fast into the deflected wave on a time scale $\pi/\sqrt{2}\Omega_a$. Turning on nuclear vibrations would destroy the resonance interaction and make the wave pass through the crystal.

IV. DISCUSSION

The physics behind the speed up of the energy transfer between two waves can be understood as a parametric resonance in a system of coupled oscillators. A single oscillator whose frequency is periodically modulated provides a simple example of parametric resonance. The motion of such an oscillator is described by the Mathieu's equation

$$\ddot{x} + \omega_0^2 \left[1 + \delta \cos(\nu_d t)\right] x = 0, \qquad (51)$$

where δ is the modulation amplitude. If $\delta = 0$ then the system has two natural frequencies $\pm\omega_0$. If the system's parameters vary with frequency ν_d equal to the difference between natural frequencies, that is $\nu_d = 2\omega_0$, the oscillator phase locks to the parametric variation and undergoes a parametric resonance absorbing energy at a rate proportional to the energy it already has.

A similar situation takes place if the system has several natural frequencies (normal modes). To achieve parametric resonance the modulation frequency ν_d must match the difference between two normal mode frequencies [48]. This is known as the difference combination resonance [49]. In the present problem the natural frequencies of the coupled light-nuclear system are determined from the solution (23) and (24) obtained for the static lattice. Thus, if the frequency of the nuclear vibrations matches the frequency difference, namely, $\nu_d = \omega_+ - \omega_- = \sqrt{\Delta^2 + 8\Omega_a^2}$ the system undergoes parametric resonance, which speeds up the energy transfer between two γ waves. This phenomenon is analogous to parametric frequency mixing in propagating circuits [42], in which power can flow back and forth between the two coupled circuits if the coupling reactance variation frequency matches their frequency difference.

Combination parametric resonance at the frequency difference between two normal modes of the coupled light-atom system is the essence of the QASER [50], a device that can generate high-frequency (e.g., XUV) coherent light by driving an atomic medium with a low frequency (e.g., infrared) field [51]. Contrary to the laser, the QASER does not require any atomic population in the excited state and yields high-frequency light amplification. In the case of the QASER the external field drives the atomic transition which produces modulation of the atom-field coupling strength and yields gain at high frequency. In the present model, unlike the QASER, modulation is produced by the nuclear motion, which does not yield amplification of the high-frequency (γ) field. However, parametric resonance and collective effects of the light interaction with a nuclear array enhance the rates of the radiation absorption and reemission. As a result, energy transfer between two γ waves occurs on a much shorter superradiant time scale determined by the collective nuclear frequency Ω_a which is of the order of terahertz. The combination of the Dicke superradiance, Bragg diffraction, and combination parametric resonance introduces interesting features to our problem and allows us to achieve fast manipulation of the γ-ray propagation.

If the incident γ wave is far detuned from the nuclear transition by the amount $\Delta \gg \Omega_a$ then the light-nuclear interaction is weak and the γ wave passes through the static nuclear array. We found that if we make the nuclei vibrate coherently with frequency Δ then the combination parametric resonance effectively enhances the light-nuclear interaction strength. As a result, the incident γ wave undergoes deflection into a wave propagating at the Bragg angle on a short superradiant time scale. The maximum energy transfer rate is achieved for the amplitude of nuclear oscillations $d \sim \lambda_{ab}/2\pi$, where λ_{ab} is the wavelength of the nuclear transition. Since λ_{ab} is typically much smaller than the spacing between nuclei in crystals the required nuclear vibrations are also small. Such nuclear motion can be realized, e.g., in ionic crystals illuminated by a strong driving optical laser field.

Our findings could be used for manipulation of the propagation direction of γ rays on a picosecond time scale.

Appendix A: Conservation of high frequency energy component during light interaction with vibrating nuclei

We start from Eqs. (9) and (10) describing light propagation through a moving crystal lattice

$$\left(\frac{\partial}{\partial t} + \frac{c^2}{2i\omega_{ab}}\left[\left(\frac{\omega_{ab}}{c}\right)^2 + \nabla^2\right]\right)\Omega(t,\mathbf{r})$$
$$= i\frac{\Omega_a^2}{N}\sum_j \rho^j(t)\delta(\mathbf{r} - \mathbf{r}_j(t)), \qquad (A1)$$

$$\frac{\partial \rho^j(t)}{\partial t} = i\Delta\omega_{ab}(t)\rho^j(t) + i\Omega(t, \mathbf{r}_j(t)). \qquad (A2)$$

In Eq. (A2) we introduced an additional time-dependent frequency shift $\Delta\omega_{ab}(t)$ to include possible external modulation of the nuclear transition frequency. Multiplying

both sides of Eq. (A1) by $\Omega^*(t,\mathbf{r})$ we obtain

$$\Omega^*(t,\mathbf{r})\left(\frac{\partial}{\partial t}+\frac{c^2}{2i\omega_{ab}}\left[\left(\frac{\omega_{ab}}{c}\right)^2+\nabla^2\right]\right)\Omega(t,\mathbf{r})$$
$$=i\frac{\Omega_a^2}{N}\sum_j\rho^j(t)\Omega^*(t,\mathbf{r}_j(t))\delta\left(\mathbf{r}-\mathbf{r}_j(t)\right). \quad (A3)$$

While Eq. (A2) yields

$$i\rho^j(t)\Omega^*(t,\mathbf{r}_j(t))=-\rho^j(t)\dot{\rho}^{j*}(t)-i\Delta\omega_{ab}(t)|\rho^j(t)|^2. \quad (A4)$$

Plugging Eq. (A4) into Eq. (A3) gives

$$\Omega^*(t,\mathbf{r})\left(\frac{\partial}{\partial t}+\frac{c^2}{2i\omega_{ab}}\left[\left(\frac{\omega_{ab}}{c}\right)^2+\nabla^2\right]\right)\Omega(t,\mathbf{r})$$
$$+\frac{\Omega_a^2}{N}\sum_j\rho^j(t)\dot{\rho}^{j*}(t)\delta(\mathbf{r}-\mathbf{r}_j(t))$$
$$+i\Delta\omega_{ab}(t)\frac{\Omega_a^2}{N}\sum_j|\rho^j(t)|^2\delta(\mathbf{r}-\mathbf{r}_j(t))=0. \quad (A5)$$

Adding to Eq. (A5) its complex conjugate we obtain

$$\frac{\partial}{\partial t}|\Omega(t,\mathbf{r})|^2+\frac{c^2}{2i\omega_{ab}}\nabla\left[\Omega^*(t,\mathbf{r})\nabla\Omega(t,\mathbf{r})-c.c\right]$$
$$+\frac{\Omega_a^2}{N}\sum_j\delta\left(\mathbf{r}-\mathbf{r}_j(t)\right)\frac{\partial}{\partial t}|\rho^j(t)|^2=0. \quad (A6)$$

Integrating Eq. (A6) over space and taking into account that for weakly excited nuclei $|\rho^j|^2=\rho_{aa}^j$, where ρ_{aa}^j is the excited state population of the nucleus j, we find

$$\int d\mathbf{r}|\Omega(t,\mathbf{r})|^2+\frac{\Omega_a^2}{N}\sum_j\rho_{aa}^j(t)=\text{const}. \quad (A7)$$

Equation (A7) shows that the sum of the energy of the high frequency field $\Omega(t,\mathbf{r})$ and that stored in nuclear excitation is conserved no matter how nuclei move. This implies that nuclear motion can not amplify high-frequency field.

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