

# Anomaly Cancellation and Smooth Non-Kähler Solutions in Heterotic String Theory

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## Abstract

We show that six-dimensional backgrounds that are  $T^2$  bundle over a Calabi–Yau two-fold base are consistent smooth solutions of heterotic flux compactifications. We emphasize the importance of the anomaly cancellation condition which can only be satisfied if the base is  $K3$  while a  $T^4$  base is excluded. The conditions imposed by anomaly cancellation for the  $T^2$  bundle structure, the dilaton field, and the holomorphic stable bundles are analyzed and the solutions determined. Applying duality, we check the consistency of the anomaly cancellation constraints with those for flux backgrounds of M-theory on eight-manifolds.

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## 1. Introduction

Since their discovery, almost ten years ago, tractable flux compactifications in string theory have become a very active area of research. The reasons for this are numerous but they share the common feature of putting the connection between string theory and realistic models of particle phenomenology into a new focus. Some of the most vexing problems in high energy physics, like the cosmological constant problem, moduli stabilization or the hierarchy problem, have found a natural description within string theory once fluxes are taken into account.

However, besides intense work on flux backgrounds in string theory, the properties of the spacetime geometry, is largely an uncharted territory. The conditions imposed by supersymmetry have been understood in detail, however, less is known about the background geometries, especially for the interesting case of the heterotic string with fluxes. Generically, the presence of  $H$ -flux in compactifications of the heterotic string is required due to the presence of  $\alpha'$  corrections in the Bianchi identity which are needed for anomaly cancellation.<sup>1</sup>

From the supersymmetry constraints [2] (see also [3] and [4]) it becomes evident that the  $H$ -field has the geometrical meaning of torsion of the  $SU(3)$  holonomy connection. Moreover, the  $H$ -field is the obstruction for the background metric to be Kähler and in particular the metric turns out to be conformally balanced<sup>2</sup> instead of Calabi–Yau [6,7,8,9,10]. Not being Calabi–Yau, many theorems of Kähler geometry do not apply which makes their analysis more arduous. The existence of smooth solutions has not been proven so far. It is the purpose of this paper to fill in this gap.

In this paper, smooth flux backgrounds for the heterotic string are constructed. The orbifold limit of these manifolds has been described previously in the literature

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<sup>1</sup> In Calabi–Yau compactifications of the heterotic string the spin connection is embedded into the gauge connection. This has received the name ‘standard embedding’ in the traditional string theory literature, which is more a misnomer as there is nothing standard about this embedding. In general the spin connection is not embedded into the gauge connection, so that  $H$ -flux is required to satisfy the Bianchi identity. In the context of heterotic M-theory, solutions with non-standard embedding have been discussed in [1] and references therein.

<sup>2</sup> See for example [5] for a mathematical discussion of the balanced condition.

[11,12,13,14]. The manifolds described herein are  $T^2$  bundles over a smooth four-dimensional Calabi–Yau base. We explicitly solve the conditions imposed by supersymmetry. Moreover, consistency requires the background to be a solution of the anomaly cancellation condition. Of all the constraints on the background fields, the anomaly cancellation is the most difficult one to satisfy. It constrains the geometry as well as the gauge bundle leading to topological restrictions. We will see that this condition requires the base to be  $K3$  and that a  $T^4$  base is prohibited for a flux compactification.

The anomaly cancellation of the heterotic theory is a highly non-linear differential equation for the dilaton. The existence of a smooth solution of this equation has recently been proven in [15]. We will briefly describe the method used for the proof and describe the limits placed on the dilaton field. Although our results are derived completely within the context of the heterotic theory, they exhibit features that should also be applicable to flux compactifications of type II theories as string dualities map our solution to flux backgrounds of M-theory on  $K3 \times K3$  as well as the F-theory duals discussed in [16,11,17].

The outline of the paper is as follows. In section 2, we set up our notation by reviewing the supersymmetry constraints imposed on the background in flux compactifications of heterotic strings. It is particularly important to pay proper attention to the sign conventions, as this is a delicate point that will have a strong implication on the existence proof (see [18] for a careful discussion on sign conventions.). It has been pointed out by Gauntlett et al. [8] that the “Iwasawa solution” presented in [7] is not a valid solution due to a sign error in the torsional equation presented in the next section. This can easily be seen from our derivation, as the Iwasawa solution is a  $T^2$  bundle over a  $T^4$  base, that will be excluded once the Bianchi identity is taken into account. In section 3, we introduce the conformally balanced metric ansatz and we motivate this background using string duality which relates it to flux compactifications of M-theory on  $K3 \times K3$ . We describe the solutions for the heterotic gauge field and show that it solves the Hermitian-Yang-Mills equation. Section 4 is devoted to showing that the solution presented in section 3 solves the anomaly cancellation condition. We write down the necessary topological constraints and explain the method used in [15] to establish existence of smooth solutions. In section 5, the properties of our solutions as well as some concrete examples are presented. Open

problems and future directions are presented in the conclusion.

This paper is a companion paper to [15] where some of the mathematical results described here, in particular the existence of a smooth dilaton solution, are proven rigorously.

## 2. Torsional constraints

In order to set up our conventions we begin by summarizing the supersymmetry constraints for an  $\mathcal{N} = 1$  compactification of the heterotic string to four dimensions. The bosonic part of the ten-dimensional supergravity action in the string frame is

$$\mathcal{S} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \left[ R + 4|\partial\phi|^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{4}\text{tr}(|F|^2) \right], \quad (2.1)$$

where  $\phi$  is the dilaton,  $R$  is the curvature scalar, and  $F_{MN}$  is the gauge field strength which we take to be hermitian, *i.e.*

$$F = dA - iA \wedge A. \quad (2.2)$$

The three-form  $H$  is defined in terms of a two-form potential  $B$  and the Chern–Simons three-form  $\Omega(A)$  according to

$$H = dB + \frac{\alpha'}{4}\Omega(A) \quad \text{where} \quad \Omega(A) = \text{tr} \left( A \wedge dA - i\frac{2}{3}A \wedge A \wedge A \right). \quad (2.3)$$

This leads to the tree-level Bianchi identity

$$dH = \frac{\alpha'}{4}\text{tr}(F \wedge F). \quad (2.4)$$

Note that beyond tree level there is an additional contribution to  $H$ , namely the Lorentz Chern-Simons term, which depends on the spin connection. This higher-derivative term is important for anomaly cancellation [19] and will play a crucial role in our analysis. The choice of connection to be used in the Lorentz Chern–Simons form is a subtle issue and will be discussed in more detail in section 4.

After including the contributions from the fermionic fields the supergravity action is invariant under the  $\mathcal{N} = 1$  supersymmetry transformations

$$\begin{aligned} \delta\psi_M &= \nabla_M \epsilon + \frac{1}{4}\mathbf{H}_M \epsilon, \\ \delta\lambda &= \not{\partial}\phi \epsilon + \frac{1}{2}\mathbf{H} \epsilon, \\ \delta\chi &= 2\mathbf{F} \epsilon, \end{aligned} \quad (2.5)$$

where  $\psi_M$  is the gravitino,  $\lambda$  is the dilatino and  $\chi$  is the gaugino. A background is supersymmetric if a non-vanishing spinor  $\epsilon$  satisfying  $\delta_\epsilon(\text{fermi}) = 0$  can be found. These constraints were worked out in [2]. We are interested in six-dimensional Poincaré invariant compactifications preserving an  $\mathcal{N} = 1$  supersymmetry in four dimensions. Since the supersymmetry transformations (2.5) are written in the string frame the background is a direct product of a four-dimensional space-time and an internal six-dimensional manifold.

Unbroken supersymmetry implies the vanishing of the four-dimensional cosmological constant and as a result the external space is Minkowski. Moreover, the constraints imposed by (2.5) imply the following conditions on the internal manifold  $X$

- (1) It is complex and the metric is hermitian. As a result we can choose the standard local coordinates where the complex structure  $J_m{}^n$  takes the form

$$J_a{}^b = i \delta_a{}^b \quad \text{and} \quad J_{\bar{a}}{}^{\bar{b}} = -i \delta_{\bar{a}}{}^{\bar{b}} . \quad (2.6)$$

The hermitian (1,1)-form is then related to the hermitian metric by  $J_{a\bar{b}} = i g_{a\bar{b}}$ .

- (2) It is non-Kähler in the presence of a non-vanishing  $H$ -field, which is related to the derivative of  $J$  by the torsional constraint<sup>3</sup>

$$H = i(\bar{\partial} - \partial) J . \quad (2.7)$$

This condition can be conveniently written in the form

$$H = d^c J , \quad (2.8)$$

where we have used the operator  $d^c = i(\bar{\partial} - \partial)$  which is standard in the mathematics literature (see *e.g.* [20]). The first equation in (2.5) implies that in these backgrounds spinors can be found which are covariantly constant, not with respect to the usual Christoffel connection, but with respect to the ‘Strominger connection’ which includes the  $H$ -flux.

- (3) Moreover, there exists a holomorphic (3,0)-form which we shall denote by  $\Omega$  and which is the three-form fermion bilinear scaled by a factor  $e^{2\phi}$ . Its norm is proportional to the exponential of the dilaton field,

$$\|\Omega\|^2 = e^{-4(\phi+\phi_0)} , \quad (2.9)$$

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<sup>3</sup> This corrects a sign error in [2].

for some constant  $\phi_0$ . The dilaton is, in turn, related to the metric by the condition that  $X$  is conformally balanced, *i.e.*

$$d(\|\Omega\| J \wedge J) = d(e^{-2\phi} J \wedge J) = 0 . \quad (2.10)$$

(4) The gauge field satisfies the Hermitian-Yang-Mills conditions

$$F^{(2,0)} = F^{(0,2)} = F_{mn} J^{mn} = 0 . \quad (2.11)$$

The torsional constraint on the  $H$ -field can be derived using the equations of motion. The action (2.1) leads to the equations of motion

$$\begin{aligned} \delta g^{MN} : \quad & R_{MN} + 2 \nabla_M \nabla_N \phi - \frac{1}{4} H_{MPQ} H_N{}^{PQ} - \frac{\alpha'}{4} \text{tr}(F_{MQ} F_N{}^Q) = 0 , \\ \delta \phi : \quad & R - 4|\partial\phi|^2 + 4 \nabla_P \partial^P \phi - \frac{1}{2} |H|^2 - \frac{\alpha'}{4} \text{tr}(|F|^2) = 0 , \\ \delta B^{MN} : \quad & \nabla_P (e^{-2\phi} H^P{}_{MN}) = 0 , \\ \delta A^M : \quad & D_N (e^{-2\phi} F^N{}_M) - \frac{1}{2} e^{-2\phi} H_{MNP} F^{NP} = 0 , \end{aligned} \quad (2.12)$$

where we have used the  $\delta B^{MN}$  and  $\delta\phi$  equations to simplify the  $\delta A^M$  equation of motion and Einstein equations respectively. The trace of the Einstein equation is then

$$\nabla_M \nabla^M e^{-2\phi} - e^{-2\phi} |H|^2 - \frac{\alpha'}{4} e^{-2\phi} \text{tr}(|F|^2) = 0 . \quad (2.13)$$

This can be integrated over  $X$  and implies

$$\int_X e^{-2\phi} H \wedge \star H + \frac{\alpha'}{4} \int_X e^{-2\phi} \text{tr}(F \wedge \star F) = 0 . \quad (2.14)$$

Note that if there are no additional contributions, each term in (2.14) being positive semi-definite must vanish identically. But since there are  $\alpha' R^2$  corrections to the action that will shortly be taken into account and that give a negative contribution to this equation, we shall formally proceed assuming  $H$  and  $F$  are non-zero. Using the fact that a Hermitian-Yang-Mills field strength satisfies  $\star F = -J \wedge F$  and applying (2.3), the previous equation can be rewritten as

$$\int_X e^{-2\phi} H \wedge \star H - \int_X e^{-2\phi} dH \wedge J = 0 . \quad (2.15)$$

Integrating by parts we find

$$\star H = e^{2\phi} d(e^{-2\phi} J) . \quad (2.16)$$

This is another way of expressing the result for  $H$  which using (2.10) can be shown to be equivalent to (2.7).

Together, (2.3) and (2.7) imply

$$i \partial \bar{\partial} J = \frac{\alpha'}{8} \text{tr}(F \wedge F) . \quad (2.17)$$

Moreover, as mentioned above, beyond tree level, the anomaly cancellation requires an additional contribution of order  $\alpha' R^2$  on the right hand side of (2.17). After taking this contribution into account (2.17) takes the form

$$i \partial \bar{\partial} J = \frac{\alpha'}{8} \left[ \text{tr}(\hat{R} \wedge \hat{R}) - \text{tr}(\mathcal{F} \wedge \mathcal{F}) \right] . \quad (2.18)$$

Here and in the following we will be using conventions which are standard in the mathematics literature. Namely the curvature two-form is given by

$$\hat{R} = d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} , \quad (2.19)$$

where  $\hat{\omega}$  is the spin connection which will be described in more detail in section 4 and the gauge field

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} , \quad (2.20)$$

are both anti-hermitian. We introduce here the calligraphic symbol  $\mathcal{F}$  to distinguish it from the hermitian gauge field  $F$  which is more commonly used in the physics literature. After integrating over a four-cycle, (2.18) requires that the first Pontryagin numbers of the gauge and tangent bundles agree, *i.e.*

$$\frac{p_1(E)}{2} = \frac{p_1(M)}{2} . \quad (2.21)$$

### 3. Solution ansatz and its M-theory dual origin

In the following we present the ansatz for the metric and gauge bundle and describe the M-theory dual of this solution.

### 3.1. The metric

We study the class of supersymmetric solutions that are topologically  $T^2$  bundles over a four-dimensional base manifold  $S$ . The metric on this space can be written in the form

$$ds^2 = e^{2\phi} ds_S^2 + (dx + \alpha_1)^2 + (dy + \alpha_2)^2 . \quad (3.1)$$

Here  $\phi$  depends on the coordinates of the base manifold  $S$  only,  $(x, y)$  are the fiber coordinates and  $\alpha = \alpha_1 + i\alpha_2$  is a one-form which will be further constrained below. Introducing complex coordinates  $z = x + iy$  and defining  $\theta = dz + \alpha$ , which is required to be a  $(1, 0)$  form, we can write

$$ds^2 = e^{2\phi} ds_S^2 + |dz + \alpha|^2 . \quad (3.2)$$

To preserve supersymmetry, we require the base manifold  $S$  to be a Calabi–Yau manifold and we denote its Kähler form with  $J_S$ . The hermitian  $(1, 1)$ -form on  $X$  can then be expressed through  $J_S$  according to

$$\begin{aligned} J &= e^{2\phi} J_S + (dx + \alpha_1) \wedge (dy + \alpha_2) \\ &= e^{2\phi} J_S + \frac{i}{2} \theta \wedge \bar{\theta} . \end{aligned} \quad (3.3)$$

Moreover, as we will see below, the condition of having a conformally balanced metric requires the two-form

$$\omega = \omega_1 + i\omega_2 = d\alpha = (\partial + \bar{\partial}) \alpha = \omega_S + \omega_A , \quad (3.4)$$

to be primitive on the base, *i.e.*

$$\omega \wedge J_S = 0 . \quad (3.5)$$

In the above expression,  $\omega_S$  is the self-dual  $(2, 0)$  part of  $\omega$  and  $\omega_A$  is its anti-self-dual  $(1, 1)$  part. The holomorphic  $(3, 0)$ -form on  $X$  is then determined to be

$$\Omega = \Omega_S \wedge \theta, \quad (3.6)$$

where  $\Omega_S$  is the holomorphic  $(2, 0)$ -form on the base.



Using the previous equations, the metric can be readily checked to satisfy the conformally balanced condition (2.10)

$$\begin{aligned}
d(e^{-2\phi} J \wedge J) &= d(e^{-2\phi} [e^{4\phi} (J_S \wedge J_S) + i e^{2\phi} \theta \wedge \bar{\theta} \wedge J_S]) \\
&= i d(\theta \wedge \bar{\theta}) \wedge J_S \\
&= i \bar{\theta} \wedge \omega \wedge J_S - i \theta \wedge \bar{\omega} \wedge J_S \\
&= 0 ,
\end{aligned} \tag{3.7}$$

where in the second line we have used that  $\phi$  depends on the base coordinates only and that  $\omega$  is primitive on the base. Note that the  $e^{2\phi}$  factor in the metric precisely cancels the  $e^{-2\phi}$  factor in the conformally balanced condition.

For an  $\mathcal{N} = 1$  compactification with non-zero  $H$ -flux, the  $T^2$  bundle has to be non-trivial. A non-twisted  $T^2$  fiber would result in  $\mathcal{N} = 2$  supersymmetry in four dimensions. Moreover, to ensure that the metric in (3.2) is globally-defined, we impose

$$\tilde{\omega}_i = \frac{\omega_i}{2\pi\sqrt{\alpha'}} \in H^2(S, \mathbb{Z}) , \tag{3.8}$$

that is  $\tilde{\omega}_1$  and  $\tilde{\omega}_2$  represent a non-trivial integral cohomology class on  $S$ . The normalization is due to taking the periodicity of the torus coordinates to be

$$x \sim x + 2\pi\sqrt{\alpha'} \quad \text{and} \quad y \sim y + 2\pi\sqrt{\alpha'} . \tag{3.9}$$

Note that  $\tilde{\omega} = \tilde{\omega}_1 + i\tilde{\omega}_2$  is the curvature two-form of the  $T^2$  fiber. The quantization condition is equivalent to the requirement of the first Chern class for each  $S^1$  bundle to be integral.

The non-trivial twisting has an effect on the de Rham cohomology of the compactification manifold  $X$  [21,13]. Assuming that  $\omega_1$  is not proportional to  $\omega_2$ , the second Betti number satisfies  $b_2(X) = b_2(S) - 2$ . The harmonic two-forms on  $X$  are those on  $S$  modded out by  $\omega_1$  and  $\omega_2$ , since these two elements of  $H^2(S)$  are exact in  $X$ . The reason is that  $\omega_1 = d(dx + \alpha_1)$  and similarly for  $\omega_2$ . Importantly, the area element of the  $T^2$  fiber  $\theta \wedge \bar{\theta}$  also does not constitute a harmonic two-form in  $X$ . By Poincaré duality, this implies that the volume form of  $K3$  is also not an element in  $H^4(X)$ . Or equivalently, the  $K3$  base is not a four-cycle of  $X$ . The twisting has effectively made the volume element of  $K3$  trivial in the de Rham cohomology of  $X$ .

### 3.2. Gauge Bundle

With the manifold being a  $T^2$  bundle, we can easily construct Hermitian-Yang-Mills bundles on the total space  $X$  by taking those on the base  $S$  and pulling them back to  $X$ . Indeed, suppose we have a Hermitian-Yang-Mills gauge bundle  $\mathcal{F}^S$  on the base  $S$ . Then it satisfies  $\mathcal{F}_{mn}^S J_S^{mn} = 0$ , which is equivalent to

$$\mathcal{F}^S \wedge J_S = 0 . \quad (3.10)$$

This in fact implies that  $\mathcal{F}_{mn}^S$  is also Hermitian-Yang-Mills on  $X$  since

$$\mathcal{F}_{mn}^S J^{mn} = \star (\mathcal{F}^S \wedge J \wedge J) = \star (\mathcal{F}^S \wedge [e^{4\phi} (J_S \wedge J_S) + i e^{2\phi} J_S \wedge \theta \wedge \bar{\theta}]) = 0 , \quad (3.11)$$

where (3.10) has been used.

The obvious question is therefore whether all Hermitian-Yang-Mills connections on  $X$  are those that are lifted from the base. To answer this, we first point out the relation between Hermitian-Yang-Mills connections and gauge bundles which are stable. In general, for a compact hermitian manifold  $X$ , a holomorphic gauge bundle  $E$  with field strength  $\mathcal{F}$  is called stable if and only if all coherent subsheaves<sup>4</sup>  $E'$  of  $E$  satisfy the condition

$$\text{slope}(E') < \text{slope}(E) , \quad (3.12)$$

where the slope of  $E$  is defined using the degree of  $E$

$$\text{slope}(E) = \frac{\text{deg } E}{\text{rank } E} = \frac{1}{\text{rank } E} \left( \frac{1}{2\pi} \int_X \text{tr}(\mathcal{F}) \wedge \tilde{J}^2 \right) , \quad (3.13)$$

with the rank being the dimension of the fiber. Here,  $\tilde{J}$  is the Gauduchon hermitian form which in six dimensions satisfies [23]

$$\partial \bar{\partial} \tilde{J}^2 = 0 . \quad (3.14)$$

The balanced condition (2.10) implies that the Gauduchon two-form is given by  $\tilde{J} = e^{-\phi} J$ .

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<sup>4</sup> Sheaves generalize the notion of vector bundles and allow the type of the fiber to change (or even degenerate) over the base. For an accessible account, see [22].

Now, due to a theorem of Li and Yau [24], it turns out that a vector bundle admits a Hermitian-Yang-Mills connection if and only if it is stable (see also [25]). Thus, finding all Hermitian-Yang-Mills connections on  $X$  is equivalent to categorizing the stable gauge bundles on  $X$ . Moreover, since we have  $\star(\mathcal{F}_{mn}J^{mn}) = \mathcal{F} \wedge J^2 = 0$ , we are specifically interested in stable bundles of degree zero.<sup>5</sup> As shown in section 5.3, using the anomaly cancellation condition and also allowing for possible holonomy along the fibers [26], the relevant stable bundles for the  $T^2$  bundle over the Calabi–Yau base consist only of the stable bundles on  $S$  tensored with a holomorphic line bundle on  $X$ , i.e.  $\mathcal{F} = \mathcal{F}^S \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{F}^L$ .

### 3.3. M-theory dual

The construction of a conformally balanced metric for a heterotic flux background was first obtained via duality from M-theory compactifications on  $K3 \times K3$ . The metric was first written down in the orbifold limit in [11] and such backgrounds have since been studied extensively in [12,13,14,10]. The metric and the  $H$ -flux are derived by applying a chain of supergravity dualities valid only at the orbifold limit of  $K3 \times K3$ . The resulting geometry in the heterotic theory is a  $T^2$  bundle over the  $K3$  orbifold  $T^4/\mathbb{Z}_2$ . The orbifold limit has the advantage that the form of the metric can be written down explicitly, but has the drawback that the geometry and the  $H$ -field are singular at the 16 orbifold fixed points. Analyses are then typically separated into consideration far from the singularities and that at the singularities.

The class of heterotic metrics (3.2) can be motivated via duality from M-theory. For  $S = K3$ , the heterotic solution is dual to M-theory on  $K3 \times K3$ , with the second  $K3$  taken as a  $T^4/\mathbb{Z}_2$  orbifold. To be precise, the metric is conformal to  $K3 \times K3$ . Starting from M-theory on  $Y = K3 \times K3$  with non-zero flux, the series of dualities leading to the heterotic solution are roughly as follows. Treat the second  $K3 = T^4/\mathbb{Z}_2$  as an elliptic fibration over  $CP_1$ . Reducing the  $T^2$  fiber to zero size, we obtain the type IIB theory on  $K3 \times T^2/\mathbb{Z}_2$ , where  $\mathbb{Z}_2 = \Omega(-1)^{FL} I_{89}$  with  $I_{89} : (x, y) \rightarrow (-x, -y)$  and  $\Omega$  being the world sheet parity operator. Applying further two T-dualities, one in each direction of  $T^2/\mathbb{Z}_2$ ,

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<sup>5</sup> To be precise, for zero degree stable bundle, the stability requirement (3.12) should be modified to  $\text{slope}(E') \leq \text{slope}(E)$ . This is known as the semistable condition.

results in the dual type I theory on  $K3$  with a  $T^2$  bundle. Finally, an S-duality takes the type I background to the above heterotic solution.

The knowledge of the dual backgrounds in type IIB and M-theory is very useful in providing insights into the heterotic flux background. From the the dual type IIB theory on  $K3 \times T^2/\mathbb{Z}_2$ , we see the origin of the twisting of the  $T^2$  bundle. Here, the  $T^2$  metric is not twisted. However, there are non-zero  $B$ -fields present that under the two T-dualities are absorbed into the metric and thus twist the  $T^2$  fiber. In the type II theory, the three-form  $H = dB$  satisfies the Dirac quantization condition. This condition leads to the requirement that  $\tilde{\omega}_i$  are in integral cohomology classes. Specifically, the  $B$ -field has the form

$$B = \frac{1}{2}(\alpha \wedge d\bar{z} + \bar{\alpha} \wedge dz) \quad \text{where} \quad d\alpha = \omega_1 + i\omega_2 . \quad (3.15)$$

As in the notation of the heterotic solution,  $z$  is the complex coordinate on the  $T^2$ . Dirac quantization requires that the corresponding three-form satisfies

$$\frac{1}{(2\pi)^2\alpha'} \int_{\Gamma} H \in \mathbb{Z} \quad \text{where} \quad \Gamma \in H_3(K3 \times T^2/\mathbb{Z}_2, \mathbb{Z}) . \quad (3.16)$$

Notice that (3.16) contains  $\alpha'$  and thus the quantization of  $d\alpha$  is relative to the length-scale set by  $\alpha'$ .

From the dual M-theory  $G$ -flux background, we can obtain insights on the heterotic anomaly cancellation equation. Indeed, on the M-theory side the four-form  $G$  is constrained by supersymmetry and the Bianchi identity [27]. Under duality it maps to the heterotic three-form and Yang-Mills gauge fields. The equation of motion associated with  $G$  takes the form,

$$d \star G = -\frac{1}{2}G \wedge G - \beta X_8 \quad \text{with} \quad X_8 = \frac{1}{(2\pi)^4} \frac{1}{4!} \left[ \frac{1}{8} \text{tr} R^4 - \frac{1}{32} (\text{tr} R^2)^2 \right] , \quad (3.17)$$

where  $\beta = 2\kappa_{11}^2 T_{M_2}$  is expressed in terms of the membrane tension  $T_{M_2}$  and the eleven-dimensional gravitational constant  $\kappa_{11}$ . Under duality this gives rise to the anomaly cancellation equation on the heterotic side [28].

Equation (3.17) can be integrated over the compact Calabi–Yau four-fold  $Y$  to give the condition

$$\frac{1}{2} \int_Y G \wedge G = \frac{\chi}{24} , \quad (3.18)$$

where  $\chi$  is the Euler character and we have  $\beta = 1$ . Additional M2-brane sources lead to a contributions  $+N$  on the left hand side of (3.18), where  $N$  is the number of M2-branes. Since supersymmetry requires  $G$  to be a primitive (2,2)-form, which implies self-duality, we have

$$\int_Y G \wedge G \geq 0, \quad (3.19)$$

and it vanishes only if  $G = 0$ . As a result a non-zero  $G$ -flux is consistent with a  $K3 \times K3$  compactification geometry in M-theory. However, notice that the duality mapping at each step described above does not affect the base manifold. Hence, if the base manifold on the heterotic side is taken to be  $T^4$ , then the corresponding dual M-theory background geometry would be  $T^4 \times K3$ . Since  $\chi(T^4 \times K3) = 0$  it cannot support non-zero  $G$ -flux. Thus, from the duality perspective, there is no consistent heterotic flux background solution with base  $S = T^4$ .

Finally, we describe how the heterotic gauge fields arise from duality mapping. As discussed in [12], the gauge fields can be traced back to the  $G$ -flux in M-theory. Their appearance can be seen most transparently in the dual type IIB theory on  $K3 \times T^2/\mathbb{Z}_2$ . Present at each of the four fixed points of  $T^2/\mathbb{Z}_2$  are four  $D7$ -branes and one  $O7$ -plane. Each set of four  $D7$ 's supports at most a  $U(4)$  gauge bundle which is broken down to  $SO(8)$  by the projection of the  $O7$ -plane. These bundles are localized on  $T^2/\mathbb{Z}_2$  and hence only have dependence on the  $K3$  coordinates. Applying the duality mapping, the gauge bundles in the heterotic theory from duality at the orbifold limit have the maximal gauge group  $SO(8)^4$  and dependence only on the base coordinates.

#### 4. Solving the anomaly cancellation

In this section we demonstrate that the heterotic metric ansatz (3.2) satisfies the anomaly cancellation condition

$$dH = 2i \partial \bar{\partial} J = \frac{\alpha'}{4} \left[ \text{tr} \left( \hat{R} \wedge \hat{R} \right) - \text{tr} \left( \mathcal{F} \wedge \mathcal{F} \right) \right]. \quad (4.1)$$

However, in writing this condition there is a subtlety related to the choice of connection  $\hat{\omega}$  since anomalies can be cancelled independently of the choice [29]. Different connections

correspond to different regularization schemes in the two-dimensional worldsheet non-linear sigma model. Hence the background fields given for a particular choice of connection must be related to those for a different choice by a field redefinition [30]. In the following we will be using the hermitian connection [2]. The advantage of this choice is that it implies that  $\text{tr}(\hat{R} \wedge \hat{R})$  is a  $(2, 2)$ -form while the  $(3, 1)$  and  $(1, 3)$  contributions vanish. This is consistent with the other two terms in (4.1) which are both  $(2, 2)$ -forms. We will use the hermitian connection below and denote the hermitian curvature two-form simply as  $R$ .

To evaluate the constraints imposed by the anomaly cancellation condition (4.1), it is convenient to rewrite the flux and curvature dependent terms. First we notice that the flux dependent term can be rewritten as

$$\begin{aligned} dH &= 2i \partial \bar{\partial} e^{2\phi} \wedge J_S + \omega_S \wedge \bar{\omega}_S - \omega_A \wedge \bar{\omega}_A \\ &= 2i \partial \bar{\partial} e^{2\phi} \wedge J_S + \omega_S \wedge \star \bar{\omega}_S + \omega_A \wedge \star \bar{\omega}_A \\ &= 2i \partial \bar{\partial} e^{2\phi} \wedge J_S + (\|\omega_S\|^2 + \|\omega_A\|^2) \frac{J_S^2}{2!} , \end{aligned} \tag{4.2}$$

where we have used the definition of  $\|\omega\|$  given in the appendix. For  $\text{tr} R \wedge R$  we refer to the calculation presented in [15] which gives <sup>6</sup>

$$\text{tr} R \wedge R = \text{tr} R_S \wedge R_S + 2 \partial \bar{\partial} [e^{-2\phi} \text{tr} (\bar{\partial} \mathcal{B} \wedge \partial \mathcal{B}^\dagger g_S^{-1})] + 16 \partial \bar{\partial} \phi \wedge \partial \bar{\partial} \phi , \tag{4.3}$$

where  $R_S$  and  $g_S$  are respectively the hermitian curvature tensor and the metric on  $S$ , and we have defined a column vector  $\mathcal{B}$  locally given by

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix} \quad \text{with} \quad \bar{\partial}(\mathcal{B}_1 dz^1 + \mathcal{B}_2 dz^2) = \omega_A . \tag{4.4}$$

Here  $(dz^1, dz^2)$  is the basis of  $(1, 0)$ -forms on  $S$ .

#### 4.1. Topological conditions

Using the previous results we can now derive constraints on the allowed flux background solutions. These constraints can be obtained by integrating the anomaly cancellation equation over  $X$  or the base  $S$ . Indeed, we can apply to (4.1) an exterior product

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<sup>6</sup> Note that our metric convention differs slightly from that of [15], i.e.  $g_{a\bar{b}} = (1/2)(g_{a\bar{b}})_{FY}$ .

with the hermitian form  $J$  and integrate over the six-manifold  $X$ . The three terms that contribute can be written as follows. First, the contribution coming from the flux takes the form

$$\int_X 2i \partial \bar{\partial} J \wedge J = \frac{1}{2} \int_X e^{-4\phi} (\|\omega_S\|^2 + \|\omega_A\|^2) J^3 . \quad (4.5)$$

The term involving the curvature takes the form

$$\int_X \text{tr} R \wedge R \wedge J = \int_X \text{tr} R_S \wedge R_S \wedge J , \quad (4.6)$$

since the  $\partial \bar{\partial}$ -exact terms in (4.2) and (4.3) when wedged with  $J$  integrate to zero over  $X$ .

For the gauge field term we use the six-dimensional identity

$$\star \mathcal{F} = \frac{1}{4} (J \wedge J) J_{mn} \mathcal{F}^{mn} - \frac{1}{2} J \wedge \tilde{\mathcal{F}}, \quad \text{where} \quad \tilde{\mathcal{F}}_{mn} = 2J_{mr} J_{ns} \mathcal{F}^{rs} . \quad (4.7)$$

If we now impose the supersymmetry requirement that  $\mathcal{F}$  is a  $(1, 1)$ -form, we can rewrite  $\tilde{\mathcal{F}}_{a\bar{b}} = 2\mathcal{F}_{a\bar{b}}$  and  $\star \mathcal{F} = -J \wedge \mathcal{F}$  to obtain

$$\int_X \text{tr} \mathcal{F} \wedge \mathcal{F} \wedge J = - \int_X \text{tr} \mathcal{F} \wedge \star \mathcal{F} > 0 . \quad (4.8)$$

This expression is positive because  $\text{tr} \mathcal{F} \wedge \star \mathcal{F}$  is negative semi-definite since  $\mathcal{F}$  is anti-hermitian. Altogether, we obtain the inequality

$$\int_X \text{tr} R_S \wedge R_S \wedge J = \frac{2}{\alpha'} \int_X e^{-4\phi} (\|\omega_S\|^2 + \|\omega_A\|^2) J^3 - \int_X \text{tr} \mathcal{F} \wedge \star \mathcal{F} > 0 , \quad (4.9)$$

which gives a constraint for  $\text{tr} R_S \wedge R_S$ , a four-form defined on the base manifold. Both terms on the right hand of this equation are bigger than zero for a non-trivial solution. As a result, backgrounds with a  $T^4$  base only lead to trivial solutions for which the fluxes, gauge fields and the twist vanish, because for  $T^4$  the curvature vanishes  $R_S = 0$ . In particular this implies that the Iwasawa manifold is not a consistent heterotic flux background. Moreover, since the base is required to be a Calabi–Yau manifold, it can only be  $K3$ . This result is dual to the M-theory tadpole constraint where non-vanishing fluxes are only allowed on manifolds with non-zero Euler characteristic.

By integrating (4.1) over the base manifold  $K3$  we obtain the topological constraint on which the existence proof of [15] is based. Using (4.2) and (4.3) the integrated equation takes the form

$$\frac{1}{\alpha'} \int_S (\|\omega_S\|^2 + \|\omega_A\|^2) J_S \wedge J_S = \frac{1}{2} \int_S \text{tr} R_S \wedge R_S - \text{tr} \mathcal{F} \wedge \mathcal{F} . \quad (4.10)$$

Multiplying both sides by a factor of  $(1/4\pi^2)$  we obtain

$$\int_S (\|\tilde{\omega}_S\|^2 + \|\tilde{\omega}_A\|^2) J_S \wedge J_S = -p_1(S) + p_1(E) > 0 , \quad (4.11)$$

since the integral on the left hand side is positive definite. For a base manifold  $S = K3$  the characteristic classes satisfy

$$2 c_2(K3) = -p_1(K3) = 48 . \quad (4.12)$$

Therefore from (4.11) an important equation that is at the heart of the existence theorem derived in [15] can be obtained

$$-\frac{p_1(E)}{2} + \int_S (\|\tilde{\omega}_S\|^2 + \|\tilde{\omega}_A\|^2) \frac{J_S \wedge J_S}{2!} = 24 . \quad (4.13)$$

As we will discuss below, as long as this equation is satisfied the existence of a smooth solution for the dilaton can be established. Note that

$$p_1(E) = 2 \text{ch}_2(E) = c_1^2(E) - 2c_2(E) , \quad (4.14)$$

and for a gauge bundle admitting spinors,  $c_1(E)$  is divisible by two [31]. The norm of  $\tilde{\omega}_i$  appearing in (4.13) can be found from the intersection numbers of  $K3$ . Since  $p_1(E) < 0$ , the number of different allowed gauge bundles is finite and we can write the possible solutions in terms of the data  $(\tilde{\omega}_1, \tilde{\omega}_2, E)$ . Below we will explicitly construct examples of backgrounds satisfying (4.13).

#### 4.2. Differential equation and the elliptic condition

The anomaly condition leads to the differential equation

$$\frac{2i}{\alpha'} \partial \bar{\partial} e^{2\phi} \wedge J_S - \frac{1}{2} \partial \bar{\partial} [e^{-2\phi} \text{tr} (\bar{\partial} \mathcal{B} \wedge \partial \mathcal{B}^\dagger g_S^{-1})] - 4 \partial \bar{\partial} \phi \wedge \partial \bar{\partial} \phi + \psi J_S^2 / 2 = 0 , \quad (4.15)$$

where we have defined  $\psi$  according to

$$\psi J_S^2 = \frac{1}{\alpha'} (\|\omega_S\|^2 + \|\omega_A\|^2) J_S^2 - \frac{1}{2} (\text{tr} R_S \wedge R_S - \text{tr} \mathcal{F} \wedge \mathcal{F}) . \quad (4.16)$$

From the topological constraint (4.10), we see that  $\psi$  integrates to zero on  $K3$ , *i.e.*  $\int_{K3} \psi = 0$ . The  $\psi$  term can be heuristically treated as a source term contribution to



the differential equation. Equation (4.15) is then the differential equation that determines the functional form for the background dilaton field  $\phi$ . We will now describe the existence proof showing that the dilaton differential equation does indeed have a solution.

To prove that a solution to (4.13) indeed exists, an elliptic condition given below is imposed. From the mathematical point of view, this allows the application of powerful techniques for solving elliptic non-linear partial differential equations. However, such a condition can also be motivated from the physics point of view. Indeed, consider the deformation of the dilaton field  $\phi \rightarrow \phi + \delta\phi$  with all other background fields fixed. For an infinitesimal variation, the deformation is studied by linearizing (4.15) with respect to  $\delta\phi$ . We expect the number of independent deformations to be finite and thus it is natural to require that the resulting second-order linear partial differential equation for  $\delta\phi$  to be elliptic.<sup>7</sup> Here ellipticity means that the coefficient matrix of the second order derivative of  $\delta\phi$  is positive. From the variation of  $\delta\phi$ , we obtain the condition

$$\frac{4}{\alpha'} e^{2\phi} J_S - i e^{-2\phi} \text{tr} (\bar{\partial}\mathcal{B} \wedge \partial\mathcal{B}^\dagger g_S^{-1}) + 8i \partial\bar{\partial}\phi > 0 . \quad (4.17)$$

In addition, as a convention, we will choose to normalize the volume of  $K3$  to be one, *i.e.*  $\int_{K3} J_S^2/2 = 1$  and define the constant  $A$  according to

$$A = \left( \int_S e^{-8\phi} \frac{J_S \wedge J_S}{2!} \right)^{1/4} . \quad (4.18)$$

Below we will see that the solutions are labelled by different values of  $A$ .

### 4.3. Existence and a priori bounds

The existence of a smooth solution for  $\phi$  in the differential equation (4.15) is proven in [15] using the standard continuity method.<sup>8</sup> The idea is to connect via a parameter  $t \in [0, 1]$ , a difficult non-linear differential equation at  $t = 1$  to a simpler one at  $t = 0$  with

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<sup>7</sup> A simple example of an elliptic equation is the Laplace equation on a torus, whose solution is a constant. As opposed to this, the wave equation is hyperbolic and the solutions are given by an infinite number of propagating modes.

<sup>8</sup> This is the same method that established the existence of the Calabi–Yau metric in [32].

known solution. Specifically for (4.15), consider the one parameter family of differential equations

$$L_t(\phi_t) = \frac{2i}{\alpha'} \partial \bar{\partial} e^{2\phi_t} \wedge J_S - \frac{t}{2} \partial \bar{\partial} [e^{-2\phi_t} \text{tr} (\bar{\partial} \mathcal{B} \wedge \partial \mathcal{B}^\dagger g_S^{-1})] - 4 \partial \bar{\partial} \phi_t \wedge \partial \bar{\partial} \phi_t + t \psi J_S^2 / 2 = 0 . \quad (4.19)$$

At  $t = 0$ , the solution is given by the constant  $\phi_0 = -\frac{1}{2} \ln A$  which satisfies the normalization (4.18). The goal is to show that there also exists a solution  $\phi_t$  at  $t = 1$  which is the differential equation (4.15).

To do so define the set

$$T = \{t \in [0, 1] \mid L_t(\phi_t) = 0 \text{ has a solution}\}, \quad (4.20)$$

consisting of values of the parameter  $t$  for which a solution exists. Having already a solution for  $t = 0$ , the existence of a solution at  $t = 1$  (that is  $t = 1 \in T$ ) is guaranteed if we can show that the set  $T$  is both open and closed. This is because the only non-empty subset with  $t \in [0, 1]$  that is both open and closed is the whole set  $t = [0, 1]$  which contains  $t = 1$ . Below we briefly describe the standard method to show that  $T$  is open and closed.

Demonstrating openness is usually not difficult. We need to show that for any point  $t_0 \in T$  its neighboring points  $t + \delta t$  is also in  $T$ . Here, we can re-express (4.19) as a function of both  $t$  and  $\phi_t$ ,

$$\tilde{L}(t, \phi_t) = \star_S \left\{ \frac{2i}{\alpha'} \partial \bar{\partial} e^{2\phi_t} \wedge J_S - \frac{t}{2} \partial \bar{\partial} [e^{-2\phi_t} \text{tr} (\bar{\partial} \mathcal{B} \wedge \partial \mathcal{B}^\dagger g_S^{-1})] - 4 \partial \bar{\partial} \phi_t \wedge \partial \bar{\partial} \phi_t + t \psi J_S^2 / 2 \right\}, \quad (4.21)$$

where the Hodge  $\star_S$  is with respect to the base  $S = K3$ . Assuming now that  $\tilde{L}(t_0, \phi_{t_0}) = 0$  is a solution, we need to show that the first order partial derivative  $\partial \tilde{L} / \partial \phi|_{(t_0, \phi_{t_0})}$  is invertible (*i.e.* isomorphic between function spaces). If so, then the implicit function theorem (see for example [33]) implies the existence of a connected open neighborhood around  $t_0$  that also satisfy  $\tilde{L}(t, \phi_t) = 0$  and hence openness. Note that since  $\partial \tilde{L} / \partial \phi|_{(t_0, \phi_{t_0})}$  is a linearized differential the elliptic condition is important for demonstrating invertibility.

The major task of the existence proof in [15] is to demonstrate closedness by deriving the delicate estimates for  $\phi_t$ . Recall that the set  $T$  is closed if for any convergent sequence  $\{t_i\}$  in  $T$ , the limit point  $t'$  is also contained in  $T$ . Since the sequence  $\{t_i\}$  is in  $T$ ,

there is a corresponding sequence of functions  $\{\phi_{t_i}\}$  that are solutions, *i.e.*  $L(t_i, \phi_{t_i}) = 0$ . Proving  $T$  is closed therefore requires that the sequence of functions  $\{\phi_{t_i}\}$  converges in some Banach space to some function  $\phi'$  and that  $L(t', \phi') = 0$ , *i.e.*  $\phi' = \phi_{t'}$ . The sequence  $\{\phi_{t_i}\}$  will converge if we can show that any solution  $\phi_t$  must satisfy certain bounds that are  $t$  independent. More explicitly, the norm (in some suitable Banach space) of  $\phi_t$  and derivatives of  $\phi_t$  should have finite upper bounds. These bounds on the solutions are called a priori estimates since they are obtained prior to and without any explicit solution. The bounds are characteristics of the differential equation and do depend on  $\psi$ , and  $A$ . To show that the solution is smooth requires only the existence of bounds up to the third derivatives of  $\phi_t$ . Higher derivatives bounds can then be obtained by applying Schauder's interior estimates (see for example Chapter 6 in [34]). With the required boundedness, the Arzela–Ascoli theorem (see for example [35]) then implies that the sequence  $\{\phi_{t_i}\}$  must contain a uniformly convergent subsequence. The corresponding convergent subsequence in  $\{t_i\}$  necessarily converges to  $t'$  and the limit of the subsequence  $\phi'$  becomes just  $\phi_{t'}$ . Thus, closedness is established once the difficult estimate bounds are obtained. We refer the reader to [15] for details of these important estimate calculations.

## 5. Analysis of the Solutions

The existence proof demonstrates that the  $T^2$  bundle over a  $K3$  base leads to a flux background for the heterotic string as long as the topological condition (4.13) is satisfied. Below, we describe how the anomaly cancellation constraint restricts the dilaton field, the twists in the  $T^2$  bundle, and the stable gauge bundles of the solution.

### 5.1. Dilaton

As worked out in [15] (see Proposition 21), a sufficient condition for ensuring the validity of the estimates necessary to prove the existence of a solution is

$$A \ll 1. \tag{5.1}$$

As a result this is also the sufficient condition to demonstrate the existence of a smooth

dilaton field.<sup>9</sup> This condition corresponds to a lower bound (see Proposition 20 in [15]) for

$$e^{2\phi} \gg 1 . \tag{5.2}$$

Note that  $e^{2\phi}$  is the conformal factor in the metric (3.2) for the  $K3$  base. A large conformal factor implies that the volume of the base is large. This is consistent with duality since a large conformal factor corresponds to a large warp factor and a large Calabi–Yau volume in the dual M- and type IIB pictures. This warp factor is often not taken into account since in the large volume limit it is constant to leading order. Here, via duality, the results on the heterotic side imply the existence of a warp factor function away from the large volume limit in type II and M-theories.

Moreover the string coupling constant on the heterotic side  $g_s = e^\phi$  is large. A large  $g_s$  background for the  $SO(32)$  heterotic can be equivalently considered as a small  $g_s$  background in the S-dual type I theory. If we consider the  $E_8 \times E_8$  heterotic instead, then the dual is M-theory on  $S^1/\mathbb{Z}_2$  with the radius of the  $S^1$  proportional to the coupling  $g_s$ . We note that the existence proof for  $\phi$  holds for both heterotic theories independently of the gauge group.

It is worthwhile to point out that there is a one parameter family of solutions. Indeed, the supersymmetry constraints (2.7)-(2.11) are invariant under a constant shift of  $\phi \rightarrow \phi + c$ . However, this constant shift is not an invariant of the anomaly equation (4.15). Nevertheless, there is still a one parameter family of solutions for  $\phi$  labelled by  $A$  in (5.1). For each value of  $A \ll 1$ , there exists a smooth solution of the dilaton for the metric ansatz (3.2). A variation of  $A$  will result in a non-constant variation of  $\phi$ . But notice that for our specific metric ansatz (3.2), the supersymmetry constraints (2.7)-(2.11) are in fact invariant for any functional variation of  $\phi$ .

## 5.2. Solutions with trivial gauge fields

The background solutions must satisfy the topological constraint (4.13) which we write

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<sup>9</sup> More specifically,  $A$  is required to be smaller than certain complicated bounds. Eq. (5.1) effectively ensures that the those bounds for  $A$  are satisfied. In other words,  $A \ll 1$  need not be necessary but is sufficient to guarantee a solution.

as

$$-\frac{p_1(E)}{2} + N = 24 \quad \text{where} \quad N = \int_{K3} (\tilde{\omega}_S \wedge \bar{\tilde{\omega}}_S - \tilde{\omega}_A \wedge \bar{\tilde{\omega}}_A) . \quad (5.3)$$

On  $K3$  there is a standard basis of two-forms in the integral cohomology class which we denote with  $\tilde{\omega}_I$  with  $I = 1, \dots, 22$ . The intersection matrix is given by the integral

$$d_{IJ} = \int_{K3} \tilde{\omega}_I \wedge \tilde{\omega}_J . \quad (5.4)$$

The matrix  $d_{IJ}$  is the metric of the even self-dual lattice with Lorentzian signature  $(3, 19)$  given by

$$(-E_8) \oplus (-E_8) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (5.5)$$

where

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad (5.6)$$

is the Cartan matrix of  $E_8$  Lie algebra. With the lattice being even,  $N$  is an even positive integer and allowed to have the maximum value of  $N = 24$  if the gauge bundle is trivial.

To be more explicit, we shall write  $\tilde{\omega}_S$  and  $\tilde{\omega}_A$  in terms of a basis of integral two-forms. First, for  $\tilde{\omega}_S$ , it must be proportional to the unique holomorphic  $(2, 0)$ -form  $\Omega_S$  on  $K3$ . Therefore, we can write

$$\tilde{\omega}_S = m \Omega_S = m (\Omega_{S1} + i \Omega_{S2}) , \quad (5.7)$$

where  $m = m_1 + i m_2$  is a gaussian integer and we have decomposed  $\Omega_S$  into its real and imaginary parts. Since  $\tilde{\omega}_S$  is in the integral class, the holomorphic  $(2, 0)$ -form must be normalized as follows.

$$\int_{K3} \Omega_S \wedge \bar{\Omega}_S = \int_{K3} (\Omega_{S1} \wedge \bar{\Omega}_{S1} + \Omega_{S2} \wedge \bar{\Omega}_{S2}) = 4 . \quad (5.8)$$

We can similarly express  $\tilde{\omega}_A = \tilde{\omega}_{A1} + i \tilde{\omega}_{A2}$  and decompose

$$\tilde{\omega}_{Ai} = \sum_{I=1}^{19} n_i^I K_I , \quad (5.9)$$

where  $i = 1, 2$  and  $K_I$  is a basis generating the integral anti-self-dual  $(1, 1)$ -forms. We note that such a basis is only present for Kummer  $K3$  surfaces [20]. All together, we have for  $N$  the condition

$$N = 4(m_1^2 + m_2^2) - \sum_{IJi} d_{IJ} n_i^I n_i^J, \quad (5.10)$$

where now the intersection matrix  $d_{IJ}$  for the integral anti-self-dual forms is just

$$d_{IJ} = (-E_8) \oplus (-E_8) \oplus -2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.11)$$

Many solutions can be found for the 40 integers combinations  $(m_1, m_2, n_1^I, n_2^I)$  for  $N \leq 24$ . As an example, for the case of trivial gauge bundle

$$(m_1, m_2, n_1^{19}, n_2^{19}) = (\pm 2, \pm 1, \pm 1, \pm 1), \quad (5.12)$$

give  $N = 24$ . We note that having trivial gauge bundle requires at least one  $n_i^I$  is non-zero.

### 5.3. Solutions with non-trivial gauge fields

We now consider solutions with non-trivial gauge fields. The gauge fields are Hermitian-Yang-Mills which as mentioned are in one-to-one correspondence with stable bundles. The anomaly cancellation equation further restricts the type of bundles to those with zero field strength in the directions of the  $T^2$  fiber. This implies that the stable vector bundles on  $T^2$  bundle over  $K3$  are the stable bundles on  $K3$  tensored with a line bundle on  $X$ . The line bundle simply comes from the flat connections (with possible twisting) on the torus fiber. The arguments below are similar to those given in [26].

We first introduce the vielbeins,  $\theta^1, \theta^2$ , and  $\theta^3 = \theta$ , which provide a local basis of orthonormal  $(1, 0)$ -forms. The hermitian form is then written simply as

$$J = \frac{i}{2} \sum_{i=1,2} \theta^i \wedge \bar{\theta}^i + \frac{i}{2} \theta \wedge \bar{\theta}. \quad (5.13)$$

In this basis, the  $(1, 1)$ -form gauge field strength decomposes as follows:

$$\begin{aligned} \mathcal{F} = & \frac{i}{2} \sum_{i=1,2} a_i (\theta^i \wedge \bar{\theta}^i) + \frac{i}{2} b (\theta^1 \wedge \bar{\theta}^2 + \theta^2 \wedge \bar{\theta}^1) \\ & + \frac{i}{2} \sum_{i=1,2} b_i (\theta^i \wedge \bar{\theta} + \theta \wedge \bar{\theta}^i) + \frac{i}{2} a (\theta \wedge \bar{\theta}), \end{aligned} \quad (5.14)$$

where the coefficients  $a, a_i, b$ , and  $b_i$  take values in the Lie algebra of the gauge group. Now consider the four-form  $\text{tr } \mathcal{F} \wedge \mathcal{F}$ . From the anomaly cancellation equation and the explicit calculations of the terms  $dH$  and  $\text{tr } R \wedge R$ ,  $\text{tr } \mathcal{F} \wedge \mathcal{F}$  can not have any dependence on  $\theta$  or  $\bar{\theta}$ . Denoting  $J'_S = \frac{i}{2} \sum_{i=1,2} \theta^i \wedge \bar{\theta}^i$ , we thus have the condition

$$\text{tr } (\mathcal{F} \wedge \mathcal{F}) \wedge J'_S = -\frac{i}{4} \text{tr} [-b_1^2 - b_2^2 + a(a_1 + a_2)] (\theta^1 \wedge \bar{\theta}^1 \wedge \theta^2 \wedge \bar{\theta}^2 \wedge \theta \wedge \bar{\theta}) = 0 . \quad (5.15)$$

Using the Hermitian-Yang-Mills condition,  $\mathcal{F}_{mn} J^{mn} = 0$ , which with (5.13) and (5.14) imply  $a_1 + a_2 + a = 0$ , we have the condition

$$\text{tr} [-b_1^2 - b_2^2 - a^2] = 0 . \quad (5.16)$$

With the gauge generators being anti-hermitian, the trace of each term is non-negative and therefore, we have  $a = b_1 = b_2 = 0$ . Referring back to (5.14), we find that  $\mathcal{F}$  does not have any non-zero components tangential to the  $T^2$  fiber, *i.e.*  $\mathcal{F}_{zm} = \mathcal{F}_{\bar{z}m} = \mathcal{F}_{z\bar{z}} = 0$ .

However, with  $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$  on the fiber, we can have  $U(1)$  line bundles which are non-trivial on the base coordinates. These gauge fields can take the form  $\mathcal{A} = ip(dx + \alpha_1) + iq(dy + \alpha_2)$  implying  $\mathcal{F} = ip\omega_1 + iq\omega_2$  where  $p$  and  $q$  are constants.<sup>10</sup> Note that the field strength does not have any components in the fiber direction. (With the holomorphic condition, we will require that  $\omega_1$  and  $\omega_2$  consist only of the anti-self-dual (1,1)-part.) Tensoring these  $U(1)$  line bundle with the stable bundle from the  $K3$  surface gives the most general stable bundle on  $X$ .

Below, we give some examples of solutions that satisfy the topological constraint (4.13). We will utilize degree zero stable bundles on  $K3$ . A sufficient condition for the existence of a stable bundle  $E$  with  $(r, c_1^2(E), c_2(E))$  on  $K3$  is given by the inequality [36,37]

$$2r c_2(E) - (r - 1) c_1^2(E) - 2r^2 \geq -2 , \quad (5.17)$$

where  $r$  is the rank of the bundle.<sup>11</sup> From this condition, many possible gauge groups

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<sup>10</sup> For arbitrary constants  $p$  and  $q$ , these gauge fields have non-trivial holonomy along the  $T^2$  bundle. However, imposing the topological condition (4.13),  $p$  and  $q$  must then be quantized and the holonomy along  $T^2$  becomes trivial.

<sup>11</sup> Note that the stable bundle with  $c_1 = 0$  has zero degree. If a stable bundle with field strength  $\mathcal{F}$  has non-zero degree, then we can obtain a zero degree semistable bundle by considering  $\mathcal{F} - \frac{1}{r} \text{tr}(\mathcal{F}) \mathbf{1}$ .

are allowed. With non-trivial gauge bundle and twisting, the solutions can be described by the following parameters  $(r, c_1^2, c_2, m_1, m_2, n_1^I, n_2^I)$  satisfying the topological constraint (5.3) (inserting (4.14) and (5.10))

$$\left(c_2 - \frac{c_1^2}{2}\right) + 4(m_1^2 + m_2^2) - \sum_{I,J,i} d_{IJ} n_i^I n_i^J = 24, \quad (5.18)$$

where the intersection matrix  $d_{IJ}$  is that in (5.11) and moreover (5.17) implies

$$c_2 - \frac{c_1^2}{2} \geq r - \frac{2 + c_1^2}{2r}. \quad (5.19)$$

Thus for instance, consider a degree zero  $SU(4)$  stable bundle on  $K3$  with  $(r, c_1^2, c_2) = (4, 0, 4)$ . The constraint (5.18) can be satisfied by the twisting  $(m_1, m_2) = (\pm 2, \pm 1)$ . If we consider instead  $(r, c_1^2, c_2) = (4, 0, 20)$ , then for example we can have  $(m_1, m_2) = (\pm 1, 0)$ .

## 6. Conclusion

In this paper, we have constructed and discussed the properties of a class of smooth compact flux backgrounds for heterotic string theory. This is the first such solution which is tractable and is formulated away from the orbifold limit. The existence of a smooth dilaton solution has been proven if the topological constraint (4.13) is satisfied. We have discussed in detail the properties of the solutions, in particular those of the geometry as well as the gauge fields. It turns out that the gauge fields do not satisfy the ‘standard embedding’ condition and this raises the interesting possibility of enlarging the class of gauge symmetry breaking patterns of heterotic strings that leads to standard-model like models. We have presented concrete examples in which the solutions of the Hermitian-Yang-Mills equation are given by  $SU(4)$  gauge groups but larger groups like  $SU(5)$  certainly also provide solutions. This represents a way of breaking  $E_8$  down to groups like  $SO(10)$  or  $SU(5)$  rather than  $E_6$  and could have very interesting applications to phenomenology. We leave the exploration of these ideas to future work.

In the following, we will discuss additional open questions and future directions. First, it would be interesting to study generalizations of the class of solutions presented in this paper. So for example, we have considered a metric ansatz (3.1) which is a torus bundle



over a  $K3$  base. These solutions are special since the complex structure of the torus has been set to a constant. A natural generalization would be to consider non-constant  $\tau$  given, for example, as the solution of

$$\bar{\partial}\tau(z_i, \bar{z}_i) = 0 , \tag{6.1}$$

where  $\tau$  depends only on two of the coordinates of the base which we denote by  $z$  and  $\bar{z}$ . However, as discussed in [38], the solutions will necessarily be singular resulting in a decompactified solution. Whether solutions with a non-constant  $\tau$  exist remains an open question.

Next, it would be interesting to describe supersymmetric cycles within the torsional background geometry. These can probably be found by representing  $K3$  as an elliptic fibration over a two-sphere. The torus fiber together with one of the circles of the torus representing the fiber of  $X$  over the base  $S = K3$  is a candidate for a supersymmetric three-cycle. Performing three T-dualities fiberwise may give rise to a mirror symmetric background along the lines of [39].

Also, it would be desirable to find a precise description of the coordinates on the moduli space for the torsional backgrounds. However, the most interesting models are perhaps torsional backgrounds with no moduli at all. Indeed, the existence of such backgrounds could be motivated by using the duality map to M-theory compactified to three dimensions. It was observed in [40] that for generic flux compactifications of M-theory on  $K3 \times K3$ , all the moduli can be fixed by a combination of fluxes and instanton effects. Studying the instanton effects on torsional backgrounds and fixing all the moduli should be very interesting for the construction of realistic models of particle phenomenology with predictive power.

To conclude, it is believed that the moduli spaces of Calabi-Yau manifolds form a connected web with the connection points given by conifold singularities. These singularities should correspond to points in which supersymmetric cycles collapse. Are torsional backgrounds a part of this web? Can we describe conifold transitions in Calabi-Yau manifolds which lead to backgrounds with vanishing  $b_2$ ? Can the transitions be described by the torsional backgrounds analyzed herein? At this moment torsional backgrounds are mainly terra incognita in the string theory landscape. The answer to these questions may lead us

to the path which connects string theory to our four-dimensional world.

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### Appendix

In this appendix we summarize our notation and conventions

- For  $p$ -form tensor fields  $F_{N_1 \dots N_p}$  we define

$$|F|^2 = \frac{1}{p!} F_{N_1 \dots N_p} F_{M_1 \dots M_p} g^{N_1 M_1} \dots g^{N_p M_p} ,$$

and

$$\begin{aligned} \mathbf{F} &= \frac{1}{p!} F_{N_1 \dots N_p} \gamma^{N_1 \dots N_p} , \\ \mathbf{F}_N &= \frac{1}{(p-1)!} F_{N N_1 \dots N_{p-1}} \gamma^{N_1 \dots N_{p-1}} , \\ &\vdots \end{aligned}$$

where

$$\gamma^{N_1 \dots N_p} = \frac{1}{p!} (\gamma^{N_1} \dots \gamma^{N_p} \pm \text{permutations}) .$$

- The gauge field  $F_{MN}$  can be written in terms of the hermitian generators  $\lambda^a$  in the vector representation of the  $G = SO(32)$  gauge group

$$F_{MN} = F_{MN}^a \lambda^a \quad \text{with} \quad a = 1, \dots, \dim(G) . \quad (6.2)$$

This gives the generator independent result  $\text{tr}(F_{MN} F^{MN}) = 2F_{MN}^a F^{aMN}$ . Here we have used the normalization  $\text{tr}(\lambda^a \lambda^b) = 2\delta^{ab}$  for generators in the vector representation of  $SO(32)$ . If  $\lambda^a$  are in the adjoint representation,  $\text{tr}$  is replaced by  $\frac{1}{30}\text{Tr}$  since  $\frac{1}{30}\text{Tr}(\lambda^a \lambda^b) = \text{tr}(\lambda^a \lambda^b)$ . For the case that the gauge group is  $E_8 \times E_8$ , the generators are in the adjoint representation.

- $R$  is the Ricci scalar constructed from the metric  $g_{MN}$  using the Christoffel connection. We are using Lorentzian signature  $(-, +, +, \dots, +)$ . We will be denoting the curvature tensors constructed using the Christoffel connection with  $R, R_{MN}$ , etc.
- We have introduced the covariant derivative

$$D_N = \nabla_N - i[A_N, \ ] .$$

- We follow the convention standard in the mathematics literature for the Hodge star operator. In particular,  $(\star H)_{mnp} = \frac{1}{3!} H_{rst} \epsilon^{rst}_{mnp}$  with  $\epsilon_{mnpqrst}$  being the Levi-Civita tensor.
- We use the definition for  $\|\omega\|^2$

$$\omega \wedge (\star_S) \bar{\omega} = \|\omega\|^2 \frac{J_S^2}{2!} .$$

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