Supersymmetric Domain Walls from Metrics of Special Holonomy

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ABSTRACT

Supersymmetric domain-wall spacetimes that lift to Ricci-flat solutions of M-theory admit generalized Heisenberg (2-step nilpotent) isometry groups. These metrics may be obtained from known cohomogeneity one metrics of special holonomy by taking a "Heisenberg limit," based on an Inönü-Wigner contraction of the isometry group. Associated with each such metric is an Einstein metric with negative cosmological constant on a solvable group manifold. We discuss the relevance of our metrics to the resolution of singularities in domain-wall spacetimes and some applications to holography. The extremely simple forms of the explicit metrics suggest that they will be useful for many other applications. We also give new but incomplete inhomogeneous metrics of holonomy $SU(3)$, G_2 and Spin(7), which are T_1, T_2 and T_3 bundles respectively over hyper-Kähler four-manifolds.

Contents

1 Introduction

Spaces with special holonomy are natural candidates for the extra dimensions in string and M-theory, since they provide a simple geometrical mechanism for reducing the number of supersymmetries. Complete non-singular examples on non-compact manifolds have been constructed where the Ricci-flat metrics can be given explicitly. Attention has mostly focused on cases of cohomogeneity one that are asymptotically conical (AC) or asymptotically locally conical (ALC). The AC examples include the Eguchi-Hanson metric in $D = 4$ [1], deformed or resolved conifolds in $D = 6$ [2], and G_2 and Spin(7) holonomy metrics in $D = 7$ and 8 [3, 4]. Four-dimensional ALC solutions have been also known for some time; they are the Taub-NUT [5] and Atiyah-Hitchin metrics [6]. Supersymmetric higher-dimensional ALC solutions have been elusive, until the recent explicit constructions of ALC metrics with Spin(7) holonomy [7] and G_2 holonomy [8]. One characteristic of those manifolds is that they all have non-abelian isometry groups.

Another situation where special holonomies are encountered is in BPS solutions in lower-dimensional supergravities that are supported by fields originating purely from the gravitational sector of a higher-dimensional theory. After oxidising the solutions back to the higher-dimension, they give rise to Ricci-flat metrics. Since the BPS solutions partially break supersymmetry, while retaining a certain number of Killing spinors, it follows that the Ricci-flat metrics will have special holonomy. In [9, 10, 11], a geometrical interpretation of these domain walls as Ricci-flat spaces with toroidal fibre bundle level surfaces was given.

Amongst the BPS solutions are a special class of domain-walls (($D - 2$)-branes in D dimensions) that have the property of *scale invariance*. Technically, this means that they possess *homotheties*, i.e. conformal Killing symmetries where the conformal scaling factor is a constant.

The class of scale-invariant domain walls has appeared in another context, namely the possibility of blowing up the singularities into regular manifolds. An example of this is given by a singular limit of K3 that produces the transverse and internal dimensions of the oxidation of an eight-dimensional 6-brane to $D = 11$ [11]. Since metrics on K3 are not known explicitly, the discussion was necessarily a highly implicit one. For our present purposes, however, the salient properties of the K3 degeneration for this identification with the domain wall were the appearance of a Heisenberg symmetry in the singular limit, as well as a characteristic rate of growth of the volume of the manifold as one recedes from the singularity. Higher-dimensional examples with more internal directions, related to higher-dimensional Calabi-Yau manifolds, were also considered in [11]. The associated domain walls

have generalised Heisenberg symmetries. Since these Heisenberg groups are homothetically invariant, they fall into the class of scale-invariant domain walls that we are concerned with here.

Four-dimensional supergravity domain walls arising from matter superpotentials have been extensively studied in [12, 13, 14, 15, 16]. Domain walls can also exist in maximal supergravities. For example, the D8-brane of massive type IIA supergravity [17] was discussed in [18]. Generalised Scherk-Schwarz reductions give rise to lower-dimensional massive supergravities that admit domain-wall solutions. It was shown that the D7-brane of type IIB and the D8-brane of massive type IIB are T-dual, via a generalised Scherk-Schwarz S^1 reduction [19]. A large class of domain walls arising from Scherk-Schwarz reduction was obtained in [23]. A complete classification of such domain walls in maximal supergravities was given in [24].

In lower-dimensional maximal supergravities, the "cosmological potential" associated with the construction of supersymmetric domain walls can arise either by generalised Kaluza-Klein reduction on spheres, or by generalised toroidal reductions, where in both cases internal fluxes are turned on. The former give cosmological potentials with at least two exponential terms, whilst the latter can give potentials with a single (positive) exponential. Importantly for our purposes, the latter have the feature that the potential has no intrinsic scale, and so the associated domain walls are scale invariant.

One motivation for the present work was to study the possibility of smooth resolutions of Hořava-Witten type geometries. The idea would be to seek everywhere smooth solutions of eleven-dimensional supergravity that resemble two domain walls at the ends of a finite interval. This was discussed in the context of domain walls based on the ur-Heisenberg group in Ref. [11]. In that reference, it was shown that the singularity arising from the vanishing of the harmonic function could be resolved by replacing the four-dimensional hyper-Kähler metric $\mathcal{M}_4$ by a smooth complete everywhere non-singular hyper-Kähler metric $\mathcal{M}^{\text{resolved}}$ on the complement in $\mathbb{CP}^2$ of a smooth cubic. The smooth non-singular metric $\mathcal{M}^{\text{resolved}}$ (called the BKTY metric in [11]) is non-compact and has a single "end" (i.e. a single connected infinite region) which is given by $\mathcal{M}_4$ up to small terms as one goes away from the domain-wall source. This was referred to in [11] as a "single-sided domain wall."

The question naturally arises whether two such single-sided domain walls may be joined together by an extended "neck" to form a complete non-singular compact manifold $\mathcal{M}^{\text{compact}}$ which resembles the Hořava-Witten type geometry. For these purposes, we need not restrict ourselves to four-dimensional manifolds and shall consider any dimension less than eleven.

To answer this question, we need to make some further assumptions about the geometry of the neck region. In the light of the previous example, it seems reasonable to require that the neck region be of cohomogeneity one, perhaps with the group being one of the generalised Heisenberg or Nilpotent groups that arise in the known supersymmetric domain walls of M-theory. We could, of course, assume a more general group or more generally drop the assumption that the neck region is invariant under any group action. However, it does seem reasonable to assume that the neck region is covered by a coordinate patch in which the metric takes the form

$$ds^2 = dt^2 + g_{ij}(x,t)dx^i dx^j + \ldots \tag{1}$$

where t is the proper distance along the neck. In the cohomogeneity one case $g_{ij}(x,t)dx^i dx^j$ is a left-invariant metric on G/H, and the ellipsis denotes extra terms which grow at very large $|t|$ and which may break G-invariance, corresponding to corrections to the metric arising from the smooth resolutions at either end of the interval.

If $\mathcal{M}^{\text{compact}}$ is Ricci-flat, or more generally if R_{tt} is non-negative, a simple consistency check immediately arises. The curves $x^i = \text{const}$, with tangent vectors $T = \frac{\partial}{\partial t}$, constitute a congruence of geodesics of the metric (1). A congruence of curves in a $(d+1)$-dimensional manifold is a d-dimensional family of curves, one passing through every point of the manifold. A congruence is *hypersurface-orthogonal* (or vorticity-free) if the curves are orthogonal to a family of d-dimensional surfaces. The congruence we are considering is clearly hypersurface-orthogonal, since every curve is orthogonal to the surfaces $t =$ constant.

Now let

$$V(x,t) = \sqrt{\det g_{ij}} \tag{2}$$

and let $\Theta = \frac{\dot{V}}{V} = g^{ij}\frac{\partial g_{ij}}{\partial t}$. Then $\Theta(t, x^i)$ is the expansion rate of the geodesic congruence, and is therefore subject to the Raychaudhuri equation,[1] which then reads

$$\frac{d\Theta}{dt} \leq -\frac{1}{d}\Theta^2 - 2\Sigma^2. \tag{3}$$

where $\Sigma^2 = \frac{1}{2}\Sigma_{ij}\Sigma^{ij}$ and $\Sigma_{ij} = \frac{\partial g_{ij}}{\partial t} - \frac{1}{d}g^{rs}\frac{\partial g_{rs}}{\partial t}g_{ij}$ with $d+1 = \dim\mathcal{M}^{\text{compact}}$. The quantity Σ^2 is a measure of the shear of the geodesic congruence given by $x^i = \text{const}$. It is an easy consequence of (3) that if Θ is negative, it remains negative and moreover tends to minus infinity in finite proper time t. This means that if the volume V of the neck is decreasing at one value of t,it is always decreasing. This simple result, which may be verified in our explicit examples, indicates that neck geometries in which V increases as one goes outward

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[1]For a brief review of the Raychaudhuri equation, see Appendix 1.

to the resolved regions in both directions are excluded. They also show that resolving periodic arrays of domain walls in such a way as to make the metric $g_{ij}(x, t)$ periodic in the proper distance variable t are excluded.

In this paper, we shall study the relationship between scale-invariant domain walls with Heisenberg symmetries, and complete non-singular manifolds of special holonomy. The simplest example, which we discuss in section 4, involves the oxidation of the 6-brane in $D = 8$ to $D = 11$. We show that the associated four-dimensional Ricci-flat space (which has $SU(2)$ holonomy and thus is self-dual) can be obtained from the Eguchi-Hanson metric, by taking a rescaling limit in which the $SU(2)$ isometry degenerates to the Heisenberg group. (For this limit, one has to take the version of Eguchi-Hanson where the curvature singularity appears in the manifold. The relation to the non-singular Eguchi-Hanson requires an additional analytic continuation in the scale parameter.) The other examples, corresponding to higher-dimensional Ricci-flat metrics, are similarly obtained as Heisenberg limits of higher-dimensional metrics of special holonomy.

It turns out that each of our scale-invariant Ricci flat metrics with nilpotent isometry group acting on orbits of co-dimension one is closely related to a complete homogeneous Einstein manifold with negative cosmological constant with a solvable isometry group. The simplest case is flat space, $\mathbb{E}^n$ with metric $ds^2 = dt^2 + dx^\mu \, dx_\mu$, which is related to the hyperbolic space H^n with metric $ds^2 = dt^2 + e^{2k\,t} \, dx^\mu \, dx_\mu$. In some cases these metrics have been used as replacements for the hyperbolic space H^n in the AdS/CFT correspondence [25]. A striking feature is the degenerate nature of the conformal boundary. For this reason we shall include a discussion of these metrics and some of their properties.

2 Four-dimensional manifolds with $SU(2)$ holonomy

2.1 The basic domain-wall construction

We consider a domain wall solution in eight-dimensional maximal supergravity, supported by the 0-form field strength $\mathcal{F}^1_{(0)23}$ coming from the dimensional reduction of the Kaluza-Klein vector in $D = 10$.[2] The metric is given by

$$ds_8^2 = H^{1/6} \, dx^\mu \, dx_\mu + H^{7/6} \, dy^2 \,, \tag{4}$$

[2]In this paper, we adopt the notation of [27, 28], where the lower dimensional maximal supergravities were obtained by consecutive S^1 reduction with the indices $i = 1, 2, \ldots$ denoting the i'th coordinate in the reduction.

where $H = 1 + m\,|y|$. Oxidised back to $D = 11$ using the standard KK rules, we find that the eleven-dimensional metric is given by $ds_{11}^2 = dx^\mu\,dx_\mu + ds_4^2$ where

$$ds_4^2 = H\,dy^2 + H^{-1}\,(dz_1 + m\,z_3\,dz_2)^2 + H\,(dz_2^2 + dz_3^2)\,. \tag{5}$$

Since the CJS field $F_{(4)}$ is zero, ds_4^2 must be Ricci flat. The solution preserves $1/2$ of the supersymmetry, implying that (5) has $SU(2)$ holonomy, i.e. it is a Ricci-flat Kähler metric. The eleven-dimensional solution was obtained in [10], where domain wall charge quantisation through topological constraints was discussed. It was used in [29] to argue that M-theory compactified on a T^2 bundle over S^1 is dual to the massive type IIA string theory. The solution has a singularity at $y = 0$. In [11], it was shown that the metric (5) is the asymptotic form of a complete non-singular hyper-Kähler metric on the complement in $\mathbb{CP}^2$ of a smooth cubic curve. The metric (5) was obtained by means of a double T-duality transformation of the D8-brane solution of the massive IIA theory in [19].

In the orthonormal basis

$$
\begin{aligned}
e^0 &= H^{1/2}\,dy\,, &\qquad e^1 &= H^{-1/2}\,(dz_1 + m\,z_3\,dz_2)\,, \\
e^2 &= H^{1/2}\,dz_2\,, &\qquad e^3 &= H^{1/2}\,dz_3\,,
\end{aligned}
\tag{6}
$$

it is easily verified that the 2-form,

$$J \equiv e^0 \wedge e^1 - e^2 \wedge e^3 = dy \wedge (dz_1 + m\,z_3\,dz_2) - H\,dz_2 \wedge dz_3\,, \tag{7}$$

is closed, and in fact covariantly constant. It is a privileged Kähler form amongst the 2-sphere of complex structures.

If we define the holomorphic and antiholomorphic projectors $P_i{}^j = \frac{1}{2}(\delta_i^j - \mathrm{i}\,J_i{}^j)$ and $Q_i{}^j = \frac{1}{2}(\delta_i^j + \mathrm{i}\,J_i{}^j)$, then complex coordinates ζ^μ must satisfy the differential equations

$$Q_i{}^j\,\partial_j\zeta^\mu = 0\,. \tag{8}$$

The integrability of these equations is assured from the fact that our metric is already established to be Kähler.

Let $x = y + \frac{1}{2}m\,y^2$, so that $H\,dy = dx$. Then (8) can be shown to reduce to just the following pair of independent equations:

$$
\begin{aligned}
\frac{\partial\zeta^\mu}{\partial x} + \mathrm{i}\,\frac{\partial\zeta^\mu}{\partial z_1} &= 0\,, \\
\frac{\partial\zeta^\mu}{\partial z_3} + \mathrm{i}\,\frac{\partial\zeta^\mu}{\partial z_2} + \tfrac{1}{2}m\,z_3\left(\frac{\partial\zeta^\mu}{\partial x} - \mathrm{i}\,\frac{\partial\zeta^\mu}{\partial z_1}\right) &= 0\,.
\end{aligned}
\tag{9}
$$

Solutions of these differential equations define the complex coordinates ζ^μ in terms of the real coordinates.

Any pair of independent solutions to the above equations gives a valid choice of complex coordinates. A convenient choice is

$$
\begin{aligned}
\zeta_1 &= z_3 + \mathrm{i}\, z_2 \,, \\
\zeta_2 &= x + \mathrm{i}\, z_1 - \tfrac{1}{4} m (z_2^2 + z_3^2) + \tfrac{\mathrm{i}}{2}\, m\, z_2\, z_3 \,,
\end{aligned}
\tag{10}
$$

implying that the metric becomes

$$
ds_4^2 = H\, |d\zeta_1|^2 + H^{-1}\, |d\zeta_2 + \tfrac{1}{2} m\, \bar{\zeta}_1\, d\zeta_1|^2 \,,
\tag{11}
$$

with

$$
H = \left[1 + m\, (\zeta_2 + \bar{\zeta}_2) + \tfrac{1}{2} m^2\, |\zeta_1|^2 \right]^{1/2} \,.
\tag{12}
$$

This agrees, up to coordinate redefinitions, with results in [11].

The metric in (11) has the characteristic Hermitean form

$$
ds^2 = 2 g_{\mu\bar{\nu}}\, d\zeta^\mu\, d\bar{\zeta}^{\bar{\nu}} \,,
\tag{13}
$$

and in fact

$$
g_{\mu\bar{\nu}} = \frac{\partial^2 K}{\partial \zeta^\mu\, \partial \bar{\zeta}^{\bar{\nu}}} \,,
\tag{14}
$$

where the Kähler function K given by $K = 2H^3/(3m^2)$.

The metric (4) with $H = 1 + m\,|y|$ physically represents a domain wall located at $y = 0$. This is constructed by patching two sides, each of which is part of a smooth but incomplete metric in which H can instead be taken to have the form $H = m\,y$. The metric (5) with $H = m\,y$ has a scaling symmetry generated by the dilatation operator

$$
D = y\, \frac{\partial}{\partial y} + 2 z_1\, \frac{\partial}{\partial z_1} + z_2\, \frac{\partial}{\partial z_2} + z_3\, \frac{\partial}{\partial z_3}
\tag{15}
$$

As we shall discuss in section 2.4, this is a homothetic Killing vector. In addition, (5) is invariant under the linear action of $U(1)$ on (z_2, z_3), preserving the 2-form $dz_3 \wedge dz_2$.

2.2 Domain-wall as Heisenberg contraction of Eguchi-Hanson

As discussed in [11], the isometry group of the metric (5) is the Heisenberg group, and it acts tri-holomorphically. In other words, it leaves invariant all three of the 2-sphere's worth of complex structures. The Heisenberg group may be obtained as the Inönü-Wigner contraction of $SU(2)$. It is not unreasonable, therefore, to expect to obtain (5) as a limit

of the Eguchi-Hanson metric, which is the only complete and non-singular hyper-Kähler 4-metric admitting a tri-holomorphic $SU(2)$ action. One could consider the larger class of triaxial metrics admitting a tri-holomorphic $SU(2)$ action considered in [26], but our metric is symmetric under the interchange of z_2 and z_3, and so it is only necessary to consider the biaxial case.

The Eguchi-Hanson metric is

$$ds_4^2 = \left(1 + \frac{Q}{r^4}\right)^{-1} dr^2 + \tfrac{1}{4} r^2 \left(1 + \frac{Q}{r^4}\right) \sigma_3^2 + \tfrac{1}{4} r^2 \left(\sigma_1^2 + \sigma_2^2\right), \tag{16}$$

where the σ_i are the left-invariant 1-forms of $SU(2)$, satisfying

$$d\sigma_1 = -\sigma_2 \wedge \sigma_3 \,, \quad d\sigma_2 = -\sigma_3 \wedge \sigma_1 \,, \quad d\sigma_3 = -\sigma_1 \wedge \sigma_2 \,. \tag{17}$$

If we define rescaled 1-forms $\tilde{\sigma}_i$, according to

$$\sigma_1 = \lambda \, \tilde{\sigma}_1 \,, \quad \sigma_2 = \lambda \, \tilde{\sigma}_2 \,, \quad \sigma_3 = \lambda^2 \, \tilde{\sigma}_3 \,, \tag{18}$$

then after taking the limit $\lambda \longrightarrow 0$, we find that (17) becomes[3]

$$d\tilde{\sigma}_1 = 0 \,, \quad d\tilde{\sigma}_2 = 0 \,, \quad d\tilde{\sigma}_3 = -\tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \,. \tag{19}$$

This is the same exterior algebra as in the 1-forms appearing in the domain-wall metric (5), as can be seen by making the associations

$$\tilde{\sigma}_1 = m \, dz_2 \,, \quad \tilde{\sigma}_2 = m \, dz_3 \,, \quad \tilde{\sigma}_3 = m \, (dz_1 + m \, z_3 \, dz_2) \,. \tag{20}$$

To see how the Eguchi-Hanson metric (16) limits to the domain-wall solution, we should combine the rescaling (18) with

$$r = \lambda^{-1} \, \tilde{r} \,, \qquad Q = \lambda^{-6} \, \widetilde{Q} \,, \tag{21}$$

under which (16) becomes

$$ds_4^2 = \left(\lambda^4 + \frac{\widetilde{Q}}{\tilde{r}^4}\right)^{-1} d\tilde{r}^2 + \tfrac{1}{4} \tilde{r}^2 \left(\lambda^4 + \frac{\widetilde{Q}}{\tilde{r}^4}\right) \tilde{\sigma}_3^2 + \tfrac{1}{4} \tilde{r}^2 \left(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2\right) \,. \tag{22}$$

If we now define $\widetilde{Q} = 16 m^{-4}$, $H = \tfrac{1}{4} m^2 \tilde{r}^2$ and take the Heisenberg limit $\lambda \longrightarrow 0$, we find, after making the association (20), that (22) becomes precisely (5) after a further coordinate change $y = \tfrac{1}{4} m \tilde{r}^2$.

The metric (16) has a curvature singularity at $r = 0$. If $Q < 0$, this is not part of the Eguchi-Hanson manifold, which includes only $r \geq -Q$. Intuitively, one may consider that

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[3]Heisenberg limits for more general groups are discussed in Appendix B.

the singularity is hidden behind a bolt in this case. If $Q > 0$, the singularity at $r = 0$ is "naked." In order to take the limit $\lambda \longrightarrow 0$ in the rescaled metric (22), we must let $\tilde{Q} > 0$. The near-singularity behaviour of the resulting metric is similar to that of (16), but with the $SU(2)$ orbits flattened to Heisenberg orbits.

2.3 Heisenberg limit of the superpotential

One may take the Heisenberg limit at the level of the equations of motion. Thus the 4-dimensional metric

$$ds_4^2 = (a\,b\,c)^2\,d\eta^2 + a^2\,\sigma_1^2 + b^2\,\sigma_2^2 + c^2\,\sigma_3^2\,, \tag{23}$$

where a, b and c are functions of η, will be Ricci-flat if $\alpha \equiv \log a$, $\beta \equiv \log b$ and $\gamma \equiv \log c$ satisfy the equations of motion coming from the Lagrangian

$$L = \dot\alpha\,\dot\beta + \dot\beta\,\dot\gamma + \dot\gamma\,\dot\alpha - V\,, \tag{24}$$

with

$$V = \tfrac{1}{4}(a^4 + b^4 + c^4 - 2b^2\,c^2 - 2c^2\,a^2 - 2a^2\,b^2)\,. \tag{25}$$

A superpotential is given by

$$W = a^2 + b^2 + c^2 - 2\lambda_1\,b\,c - 2\lambda_2\,c\,a - 2\lambda_3\,a\,b\,, \tag{26}$$

for any choice of the constants λ_i that satisfy the three equations

$$\lambda_1 = \lambda_2\,\lambda_3\,, \qquad \lambda_2 = \lambda_3\,\lambda_1\,, \qquad \lambda_3 = \lambda_1\,\lambda_2\,. \tag{27}$$

If $\lambda_1 = \lambda_2 = \lambda_3 = 0$, we get the equations of motion for hyper-Kähler metrics with tri-holomorphic $SU(2)$ action, solved in [26]. It is possible to rescale the variables a, b and c to obtain a superpotential giving the equations of motion for metrics admitting a tri-holomorphic action of the Heisenberg group. One sets

$$a = \lambda^{-1}\,\tilde{a}\,, \quad b = \lambda^{-1}\,\tilde{b}\,, \quad c = \lambda^{-2}\,\tilde{c}\,, \tag{28}$$

together with $\eta = \lambda^4\,\tilde{\eta}$ and $W = \lambda^{-4}\,\widetilde{W}$, giving

$$\widetilde{W} = \tilde{c}^2\,. \tag{29}$$

This gives rise to the first-order equations

$$\frac{d\tilde\alpha}{d\tilde\eta} = \frac{d\tilde\beta}{d\tilde\eta} = -\frac{d\tilde\gamma}{d\tilde\eta} = e^{2\tilde\gamma}\,, \tag{30}$$

from which one can easily rederive the domain-wall solution (5).

2.4 Hypersurface-orthogonal homotheties

The large-radius behaviour of the Eguchi-Hanson metric is that of a Ricci-flat cone over $\mathbb{RP}^3$ [26]. In this limit, the metric becomes scale-invariant: scaling the metric by a constant factor λ^2 is equivalent to performing the diffeomorphism $r \longrightarrow \lambda\, r$. This transformation is generated by the "Euler vector"

$$E \equiv r\, \frac{\partial}{\partial r}\,, \tag{31}$$

which satisfies

$$\nabla_\mu\, E_\nu = g_{\mu\nu}\,. \tag{32}$$

The vector E^μ is a special kind of conformal Killing vector K^μ, which would in general satisfy

$$\nabla_\mu\, K_\nu + \nabla_\nu\, K_\nu = 2f\, g_{\mu\nu}\,. \tag{33}$$

If f is constant, then K^μ generates a homothety, and if $\nabla_\mu\, K_\nu = \nabla_\nu\, K_\mu$ then K_μ is a gradient, and hence it is hypersurface-orthogonal. The Euler vector E^μ in (31) is an example of such a hypersurface-orthogonal homothetic conformal Killing vector [30].

The Heisenberg limit of the Eguchi-Hanson metric is also scale-invariant. In other words, the metric near the singularity is scale-free. Thus it is unchanged, up to a diffeomorphism, by the transformation

$$z_1 \longrightarrow \lambda^2\, z_1\,, \quad z_2 \longrightarrow \lambda\, z_2\,, \quad z_3 \longrightarrow \lambda\, z_3\,, \quad y \longrightarrow \lambda\, y\,, \tag{34}$$

where in this section we are taking $H = m\, y$. This is generated by the homothetic Killing vector

$$D = y\, \frac{\partial}{\partial y} + 2z_1\, \frac{\partial}{\partial z_1} + z_2\, \frac{\partial}{\partial z_2} + z_3\, \frac{\partial}{\partial z_3}\,. \tag{35}$$

In contrast to the Euler vector (31) for the cone, the homothety (35) is neither a gradient nor is it proportional to a gradient, and so it is not hypersurface-orthogonal.

The existence of the homothety generated by (34) depends crucially on the fact that the Heisenberg algebra, represented in the Cartan-Maurer form in (19), is invariant under the scaling

$$\tilde\sigma_1 \longrightarrow \lambda\, \tilde\sigma_1\,, \quad \tilde\sigma_2 \longrightarrow \lambda\, \tilde\sigma_2\,, \quad \tilde\sigma_3 \longrightarrow \lambda^2\, \tilde\sigma_3\,. \tag{36}$$

By contrast, the original $SU(2)$ algebra, represented in (17), is of course not invariant under (36).

In [30], it was shown that (5) arises as the large-distance limit of a non-compact Calabi-Yau 3-fold. In the large-distance limit, the metric becomes scale-invariant.

2.5 Multi-instanton construction

The four-dimensional metric (5) can be related to the general class of multi-instantons obtained in [31], which are constructed as follows. Let x^i denote Cartesian coordinates on $\mathbb{R}^3$. We the write the metric

$$ds_4^2 = V^{-1} (d\tau + A_i \, dx^i)^2 + V \, dx^i \, dx^i \,, \tag{37}$$

where V and A_i depend only on the x^i. In the orthonormal basis

$$e^0 = V^{-1/2} (d\tau + A_i \, dx^i) \,, \qquad e^i = V^{1/2} \, dx^i \,, \tag{38}$$

the spin connection is then given by

$$\omega_{0i} = \tfrac{1}{2} V^{-3/2} \left[-\partial_i V \, e^0 + F_{ij} \, e^j \right] \,, \qquad \omega_{ij} = \tfrac{1}{2} V^{-3/2} \left[\partial_j V \, e^i - \partial_i V \, e^j - F_{ij} \, e^0 \right] \,, \tag{39}$$

where we have defined $F_{ij} \equiv \partial_i A_j - \partial_j A_i$. It is convenient to introduce a "dual" $\widetilde{\omega}_{0i}$ of the spin-connection components ω_{ij}, as $\widetilde{\omega}_{0i} \equiv \tfrac{1}{2} \epsilon_{ijk} \, \omega_{jk}$. It is easily seen that if the spin connection is self-dual or anti-self-dual, in the sense that $\widetilde{\omega}_{0i} = \pm \omega_{0i}$, then the curvature 2-forms Θ are self-dual or anti-self-dual, not only in the analogous sense $\Theta_{0i} = \pm \tfrac{1}{2} \epsilon_{ijk} \Theta_{jk}$, but also in the normal sense $\Theta_{ab} = \pm * \Theta_{ab}$. In particular, when this condition is satisfied, the metric is Ricci flat.

It is easy to see from (39) that the spin-connection satisfies $\widetilde{\omega}_{0i} = \pm \omega_{0i}$ if and only if the metric functions satisfy

$$\vec{\nabla} V = \pm \vec{\nabla} \times \vec{A}, \tag{40}$$

where the expressions here are the standard ones of three-dimensional Cartesian coordinates. In other words, $\partial_i V = \pm \epsilon_{ijk} \, \partial_j A_k$. Thus (40) is the condition for (37) to be Ricci flat, and furthermore, self-dual or anti-self-dual. In particular, taking the divergence of (40) we get $\nabla^2 V = 0$, so V should be harmonic, and then $\vec{A}$ can be solved for (modulo a gauge transformation) using (40).

The multi-centre instantons [31] are obtained by taking

$$V = c + \sum_\alpha \frac{q_\alpha}{|\vec{x} - \vec{x}_\alpha|} \,, \tag{41}$$

where c and q_α are constants. If $c = 0$ one gets the multi Taub-NUT metrics, while if $c \neq 0$ (conveniently one chooses $c = 1$), the metrics are instead multi Eguchi-Hanson.

If we take a uniform distribution of charges spread over a two-dimensional plane of radius R, then at a perpendicular distance y from the centre of the disc the potential V is

given by

$$
\begin{aligned}
V &= c + q \int_0^R \frac{r\,dr}{(y^2 + r^2)^{1/2}}, \\
&= c + q \left[(R^2 + y^2)^{1/2} - |y| \right], \\
&= c + q\,R - q\,|y| + \mathcal{O}(1/R).
\end{aligned}
\tag{42}
$$

Thus if we send R to infinity, while setting $q = -m$ and adjusting c such that $c + q\,R = 1$, we obtain

$$
V = 1 + m\,|y|.
\tag{43}
$$

It is easy to establish from (40) that a solution for $\vec{A}$ is then

$$
\vec{A} = (0, m\,z_3, 0),
\tag{44}
$$

(where we take the Cartesian coordinates to be $\vec{x} = (y, z_2, z_3)$), and so we have arrived back at our original metric (5). It can therefore be described as a continuum of Taub-NUT instantons distributed uniformly over a two-dimensional plane. (This is essentially the construction of [19].)

A more physical picture of this limit is that there can be multi-instanton generalisations of an AC manifold, and a uniform distribution would turn the non-abelian isometry group into an abelian $U(1)$ group.

3   Orientifold planes and the Atiyah-Hitchin metric

In addition to D-branes, which have positive tension, string theory admits orientifold planes which have negative tension. Since orientifold planes are not dynamical, the negative tension does not lead to instabilities as it would if the tension of an ordinary D-brane were negative. This is because they are pinned in position: the inversion symmetry employed in the orientifold projection excludes translational zero modes.

In M-theory, an orientifold plane corresponds to the product of the Atiyah-Hitchin metric [6, 20] with seven-dimensional Minkowski space-time [21]. The Atiyah-Hitchin metric is a smooth nonsingular hyper-Kähler 4-metric and hence BPS. It is invariant under $SO(3)$ acting on principal orbits of the form $SO(3)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ is realised as the group of diagonal $SO(3)$ matrices. Near infinity, it is given approximately by the Taub-NUT metric divided by CP, taken with a *negative* ADM mass. The CP quotient symmetry here takes (ψ, x^i) to $(-\psi, -x^i)$, where ψ is the Kaluza-Klein coordinate. As in string theories,

the negative mass does not lead to instabilities because this quotient symmetry of the asymptotic metric eliminates translational zero modes.

One might think that being BPS and having negative ADM mass would be inconsistent because of the Positive Mass Theorem. However the Positive Mass Theorem for ALF spaces such as the Atiyah-Hitchin metric is rather subtle [22]. Suffice it to say that one needs to solve the Dirac equation subject to boundary conditions at infinity as an essential ingredient in the proof. If the manifold is simply connected there is a unique spin structure but because a neighbourhood of infinity where one imposes the boundary conditions is not simply-connected, it is not obvious that a suitable global solution exists in the unique spin structure . In the Atiyah-Hitchin case it seems clear that it does not.

This example shows that in principle, gravity can resolve singularities associated with branes of negative tension. However, to make this more precise we would need to consider the unresolved spacetime and its relationship to the Atiyah-Hitchin metric. To lowest order, one might consider this to be the flat metric on $S^1 \times \mathbb{R}^3$ with coordinates $(\psi, \mathbf{x})$. This clearly has a singularity at $(0, 0, 0, 0)$. However this approximation ignores the Kaluza-Klein magnetic field generated by the orientifold plane. If we maintain spherical symmetry we would be led at the next level of approximation to the Taub-NUT metric with negative mass:

$$ds^2 = 4(1 + \frac{2M}{r})^{-1}(d\psi + \cos\theta d\phi)) + (1 + \frac{2M}{r})(dr^2 + r^2(\theta^2 + \sin\theta d\phi^2). \qquad (45)$$

CP acts as $(\psi, \theta, \phi) \to (-\pi, \pi - \theta, \phi + \pi)$ and the ADM mass M is negative. Clearly the Taub-NUT approximation breaks down at small positive r because if $r < -2M$, the metric signature is $----$ rather than $++++$.

The full non-singular Atiyah-Hitchin metric can be written as

$$ds^2 = dt^2 + a^2(t)\sigma_1^2 + b^2(t)\sigma_2^2 + c^2(t)\sigma_3^2, \qquad (46)$$

where $\sigma_1, \sigma_2, \sigma_3$ are Cartan-Maurer forms for $SU(2)$ and the allowed range of angular coordinates is restricted by the fact that CP should act as the identity. For large t we have $a \to b$ and the metric tends to the Taub-NUT metric.

The Atiyah-Hitchin metric and Taub-NUT metric are members of a general family of locally $SU(2)$-invariant hyper-Kähler metrics in which the three Kähler forms transform as a triplet. They satisfy a set of first order differential equations coming from a superpotential as described in section 2.3. The Eguchi-Hanson metric discussed above is a member of another family of locally $SU(2)$-invariant hyper-Kähler metrics in which the three Kähler forms

transform as singlets [4]. They satisfy a different set of first order differential equations also given in section 2.3. One may check that the Heisenberg limits of these two sets of equations are identical and the solutions are precisely the metrics (5) in which the Heisenberg group acts triholomorphically. Thus in the Heisenberg limit the triplet becomes three singlets.

One may also take the Heisenberg limit directly in the asymptotic Taub-NUT metric. This also leads to the metric (5). However in order to build in the projection under CP one may identify (z_1, z_2, z_3) with $(-z_1, -z_2, z_3)$. Thus by analogy with the construction of section 2.5 we propose that the singular unresolved metric for a stack of orientifold planes analogous to the metric of a stack of D6-branes is (5) with this additional identification.

4 Domain-walls from pure gravity

The eight-dimensional domain-wall example of the previous section lifted to a solution of eleven-dimensional supergravity with vanishing CJS field $F_{(4)}$. Its transverse coordinate y, together with the three toroidal coordinates (z_1, z_2, z_3) of the reduction from $D = 11$, gave the four coordinates of the cohomogeneity one Ricci-flat Heisenberg metric, which could be viewed as a limit of the Eguchi-Hanson metric. The principal orbits in the Heisenberg limit were T^1 bundles over T^2.

In this section we shall generalise this construction by considering domain-wall solutions in maximal supergravities that lift to give purely geometrical solutions in eleven dimensions. Thus if we begin with such a domain-wall in D-dimensional supergravity, the metric after lifting to eleven dimensions will be $ds_{11}^2 = dx^\mu dx_\mu + ds_{12-D}^2$, where ds_{12-D}^2 is a Ricci-flat metric of cohomogeneity one, with principal orbits that are again of the form of torus bundles.

The cases that we shall consider here arise from domain-wall solutions in $D = 7$, 6 and 5. Correspondingly, these give rise to Ricci-flat Heisenberg metrics of dimensions 6, 7 and 8. Since each domain wall preserves a fraction of the supersymmetry, it follows that the associated Ricci-flat metrics admit certain numbers of covariantly-constant spinors. In other words, they are metrics of special holonomy. This property generalises the special holonomy of the 4-dimensional Ricci-flat Kähler metric in the previous example in section 2.1. Specifically, for the Ricci-flat metrics in dimension 6, 7 and 8 we shall see that the special holonomies $SU(3)$, G_2 and $\mathrm{Spin}(7)$ arise.

---

[4]One says that $SU(2)$ acts triholomorphically in this case

4.1 Six-dimensional manifolds with $SU(3)$ holonomy

4.1.1 T^1 bundle over T^4

In this first six-dimensional example, the principal orbits form an T^1 bundle over T^4. It arises if we take a domain wall solution in six spacetime dimensions, supported by the two 0-form field strengths $\mathcal{F}^1_{(0)23}$ and $\mathcal{F}^1_{(0)45}$, carrying equal charges m. In what follows, we shall adhere to the terminology "charges," although in some circumstances it may be more appropriate to think of "fluxes." The domain-wall metric is

$$ds^2 = H^{1/2}\, dx^\mu\, dx_\mu + H^{5/2}\, dy^2\,, \tag{47}$$

where $H = 1 + m\,|y|$. Oxidising back to $D = 11$, we get the eleven-dimensional metric

$$d\hat{s}^2 = dx^\mu\, dx_\mu + H^{-2}\,[dz_1 + m\,(z_3\, dz_2 + z_5\, dz_4)]^2 + H^2\, dy^2 + H\,(dz_2^2 + \cdots + dz_5^2)\,. \tag{48}$$

Thus we conclude that the six-dimensional metric

$$ds_6^2 = H^{-2}\,[dz_1 + m\,(z_3\, dz_2 + z_5\, dz_4)]^2 + H^2\, dy^2 + H\,(dz_2^2 + \cdots + dz_5^2) \tag{49}$$

is Ricci flat. Since the solution carries two charges it preserves $\frac{1}{4}$ of the supersymmetry, and so this 6-metric must have $SU(3)$ holonomy. Thus it is a Ricci-flat Kähler 6-metric.

Define an orthonormal basis by

$$e^0 = H\, dy\,, \qquad e^1 = H^{-1}\,[dz_1 + m\,(z_3\, dz_2 + z_5\, dz_4)]\,,$$
$$e^2 = H^{1/2}\, dz_2\,, \qquad e^3 = H^{1/2}\, dz_3\,, \qquad e^4 = H^{1/2}\, dz_4\,, \qquad e^5 = H^{1/2}\, dz_5\,. \tag{50}$$

It can then be seen that the Kähler form is given by

$$J = e^0 \wedge e^1 - e^2 \wedge e^3 - e^4 \wedge e^5\,. \tag{51}$$

Following the same strategy as in the previous section, we can obtain the differential equations whose solutions define complex coordinates ζ^μ in terms of the real coordinates. First, define a new real coordinate x in place of y, such that $H^2\, dy \equiv dx$, and hence $y + m\, y^2 + \frac{1}{3}m^2\, y^3 = x$. This implies that $H = (1 + 3m\, x)^{1/3}$. After straightforward algebra, we find that a suitable choice for the definition of the complex coordinates is

$$\zeta_1 = z_3 + \mathrm{i}\, z_2\,, \qquad \zeta_2 = z_5 + \mathrm{i}\, z_4\,,$$
$$\zeta_3 = x + \mathrm{i}\, z_1 + \tfrac{1}{4}m\,(z_2^2 + z_4^2 - 3z_3^2 - 3z_5^2 - 2\mathrm{i}\, z_2\, z_3 - 2\mathrm{i}\, z_4\, z_5)\,. \tag{52}$$

The metric then takes the form

$$ds_6^2 = H\,(|d\zeta_1|^2 + |d\zeta_2|^2) + H^{-2}\,|d\zeta_3 + \tfrac{1}{2}m\,(\bar{\zeta}_1\, d\zeta_1 + \bar{\zeta}_2\, d\zeta_2)|^2\,, \tag{53}$$

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the harmonic function H is given by

$$H = \left[1 + \tfrac{3}{2}m\,(\zeta_3 + \bar{\zeta}_3) + \tfrac{3}{4}m^2\,(|\zeta_1|^2 + |\zeta_2|^2)\right]^{1/3},\tag{54}$$

and the Kähler function is $K = H^4/m^2$.

In the case that $H = m\,y$, there is a scaling invariance of the metric (49) generated by the homothetic Killing vector

$$D = y\,\frac{\partial}{\partial y} + 3z_1\,\frac{\partial}{\partial z_1} + \tfrac{3}{2}z_2\,\frac{\partial}{\partial z_2} + \tfrac{3}{2}z_3\,\frac{\partial}{\partial z_3} + \tfrac{3}{2}z_4\,\frac{\partial}{\partial z_4} + \tfrac{3}{2}z_5\,\frac{\partial}{\partial z_5}\,.\tag{55}$$

In addition, (49) is invariant under the linear action of $U(2)$ on (z_2, z_3, z_4, z_5) preserving the self-dual 2-form $dz_3 \wedge dz_2 + dz_5 \wedge dz_4$.

4.1.2 T^2 bundle over T^3

There is a second type of 2-charge domain wall in $D = 6$, again supported by two 0-form field strengths coming from the Kaluza-Klein reduction of the eleven-dimensional metric. A representative example is given by using the field strengths $(\mathcal{F}^1_{(0)34}, \mathcal{F}^2_{(0)35})$. The domain-wall metric is again given by (47), but now, upon oxidation back to $D = 11$, we obtain $d\hat{s}^2 = dx^\mu\,dx_\mu + ds_6^2$ with the Ricci-flat 6-metric now given by

$$\begin{aligned}ds_6^2 &= H^2\,dy^2 + H^{-1}\,(dz_1 + m\,z_4\,dz_3)^2 + H^{-1}\,(dz_2 + m\,z_5\,dz_3)^2 \\ &\quad + H^2\,dz_3^2 + H\,(dz_4^2 + dz_5^2)\,.\end{aligned}\tag{56}$$

Defining the orthonormal basis

$$\begin{aligned}&e^0 = H\,dy\,, \qquad e^1 = H^{-1/2}\,(dz_1 + m\,z_4\,dz_3)\,, \qquad e^2 = H^{-1/2}\,(dz_2 + m\,z_5\,dz_3)\,, \\ &e^3 = H\,dz_3\,, \qquad e^4 = H^{1/2}\,dz_4\,, \qquad e^5 = H^{1/2}\,dz_5\,,\end{aligned}\tag{57}$$

we find that the torsion-free spin connection is given by

$$\begin{aligned}\omega_{01} &= \lambda\,e^1\,, & \omega_{02} &= \lambda\,e^2\,, & \omega_{03} &= -2\lambda\,e^3\,, \\ \omega_{04} &= -\lambda\,e^4\,, & \omega_{05} &= -\lambda\,e^5\,, & \omega_{12} &= 0\,, \\ \omega_{13} &= -\lambda\,e^4\,, & \omega_{14} &= \lambda\,e^3\,, & \omega_{15} &= 0\,, \\ \omega_{23} &= -\lambda\,e^5\,, & \omega_{24} &= 0\,, & \omega_{25} &= \lambda\,e^3\,, \\ \omega_{34} &= \lambda\,e^1\,, & \omega_{35} &= \lambda\,e^2\,, & \omega_{45} &= 0\,,\end{aligned}\tag{58}$$

where $\lambda \equiv \tfrac{1}{2}m\,H^{-2}$.

From this, it is easily established that the following 2-form is covariantly constant:

$$J = e^0 \wedge e^3 + e^1 \wedge e^4 + e^2 \wedge e^5 \,. \tag{59}$$

This is the Kähler form. From this, using the same strategy as we used in previous sections, we can deduce that the following are a suitable set of complex coordinates:

$$\zeta_1 = z_1 + i\,H\,z_4\,, \qquad \zeta_2 = z_2 + i\,H\,z_5\,, \qquad \zeta_3 = y + i\,z_3\,. \tag{60}$$

The Hermitean metric tensor $g_{\mu\bar\nu}$ can then be derived from the Kähler function

$$K = -\frac{1}{4H}\left[(\zeta_1 - \bar\zeta_1)^2 + (\zeta_2 - \bar\zeta_2)^2\right] + \tfrac{1}{48}\,|\zeta_3|^2\left[8(H^2 + H + 1) - m^2\,|\zeta_3|^2\right]\,. \tag{61}$$

In the case that $H = m\,y$, there is a scaling invariance of the metric (56) generated by the homothetic Killing vector

$$D = y\,\frac{\partial}{\partial y} + \tfrac{5}{2}z_1\,\frac{\partial}{\partial z_1} + \tfrac{5}{2}z_2\,\frac{\partial}{\partial z_2} + z_3\,\frac{\partial}{\partial z_3} + \tfrac{3}{2}z_4\,\frac{\partial}{\partial z_4} + \tfrac{3}{2}z_5\,\frac{\partial}{\partial z_5}\,. \tag{62}$$

In addition, (56) is invariant under the linear action of $SO(2)$ on (z_1, z_2) and (z_4, z_5).

4.2 Seven-dimensional manifolds with G_2 holonomy

4.2.1 T^2 bundle over T^4

Now consider a 4-charge domain wall in $D = 5$. Take the charges to be carried by the following 0-form field strengths: $(\mathcal{F}^1_{(0)34}, \mathcal{F}^1_{(0)56}, \mathcal{F}^2_{(0)35}, \mathcal{F}^2_{(0)46})$. Note that here, unlike the 2-charge cases in $D = 6$, it matters what the relative signs of the charges are here, in order to get a supersymmetric solution. Specifically, the Bogomol'nyi matrix $\mathcal{M}$ is given here by [27]

$$\mathcal{M} = \mu\,\mathbb{1} + q_1\,\Gamma_{y134} + q_2\,\Gamma_{y156} + q_3\,\Gamma_{y235} + q_4\,\Gamma_{y246}\,, \tag{63}$$

where

$$\mu = \sum_i |q_i|\,. \tag{64}$$

Having fixed a set of conventions, the signs of the first three charges (q_1, q_2, q_3) can be arbitrary for supersymmetry, but only for one sign of the fourth charge q_4 is there supersymmetry. With our conventions, it must be negative. Thus we may consider:

$$q_1 = q_2 = q_3 = -q_4 = m\,. \tag{65}$$

The domain-wall metric in $D = 5$ is

$$ds^2 = H^{4/3}\,dx^\mu\,dx_\mu + H^{16/3}\,dy^2\,. \tag{66}$$

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Oxidising back to $D = 11$ in the standard way, we get $d\hat{s}^2 = dx^\mu \, dx_\mu + ds_7^2$, where the seven-dimensional metric is given by

$$
\begin{aligned}
ds_7^2 &= H^4 \, dy^2 + H^{-2} \left[dz_1 + m \left(z_4 \, dz_3 + z_6 \, dz_5 \right) \right]^2 + H^{-2} \left[dz_2 + m \left(z_5 \, dz_3 - z_6 \, dz_4 \right) \right]^2 \\
&\quad + H^2 \left(dz_3^2 + \cdots + dz_6^2 \right).
\end{aligned} \tag{67}
$$

Note that the minus sign in the term involving $-z_6 \, dz_5$ is a reflection of the fact that the charge associated with $\mathcal{F}_{(0)46}^2$ is negative for supersymmetry. On account of the supersymmetry, we conclude that the Ricci-flat metric ds_7^2 admits one covariantly-constant spinor, and thus it must have G_2 holonomy.

Define the orthonormal basis

$$
e^0 = H^2 \, dy \,, \qquad e^3 = H \, dz_3 \,, \qquad e^4 = H \, dz_4 \,, \qquad e^5 = H \, dz_5 \,, \qquad e^6 = H \, dz_6 \,,
$$
$$
e^1 = H^{-1} \left[dz_1 + m \left(z_4 \, dz_3 + z_6 \, dz_5 \right) \right] \,, \qquad e^2 = H^{-1} \left[dz_2 + m \left(z_5 \, dz_3 - z_6 \, dz_4 \right) \right]. \tag{68}
$$

In this basis, the torsion-free spin connection is given by

$$
\begin{aligned}
\omega_{01} &= 2\lambda \, e^1 \,, & \omega_{02} &= 2\lambda \, e^2 \,, & \omega_{03} &= -2\lambda \, e^3 \,, \\
\omega_{04} &= -2\lambda \, e^4 \,, & \omega_{05} &= -2\lambda \, e^2 \,, & \omega_{06} &= -2\lambda \, e^6 \,, \\
\omega_{12} &= 0 \,, & \omega_{13} &= -\lambda \, e^4 \,, & \omega_{14} &= \lambda \, e^3 \,, \\
\omega_{15} &= -\lambda \, e^6 \,, & \omega_{16} &= \lambda \, e^5 \,, & \omega_{23} &= -\lambda \, e^5 \,, \\
\omega_{24} &= \lambda \, e^6 \,, & \omega_{25} &= \lambda \, e^3 \,, & \omega_{26} &= -\lambda \, e^4 \,, \\
\omega_{34} &= \lambda \, e^1 \,, & \omega_{35} &= \lambda \, e^2 \,, & \omega_{36} &= 0 \,, \\
\omega_{45} &= 0 \,, & \omega_{46} &= -\lambda \, e^2 \,, & \omega_{56} &= \lambda \, e^1 \,,
\end{aligned} \tag{69}
$$

where $\lambda \equiv \frac{1}{2} m \, H^{-3}$.

It can now be verified that the following 3-form is covariantly constant:

$$
\begin{aligned}
\psi_{(3)} &\equiv e^0 \wedge e^1 \wedge e^2 - e^1 \wedge e^4 \wedge e^6 + e^1 \wedge e^3 \wedge e^5 \\
&\quad - e^2 \wedge e^5 \wedge e^6 - e^2 \wedge e^3 \wedge e^4 - e^0 \wedge e^3 \wedge e^6 - e^0 \wedge e^4 \wedge e^5 .
\end{aligned} \tag{70}
$$

The existence of such a 3-form is characteristic of a 7-manifold with G_2 holonomy. In fact the components ψ_{ijk} are the structure constants of the multiplication table of the seven imaginary unit octonions γ_i:

$$
\gamma_i \, \gamma_j = -\delta_{ij} + \psi_{ijk} \, \gamma_k \,. \tag{71}
$$

Note that if the sign of the $z_6 \, dz_4$ term in (67) had been taken to be $+$ instead of $-$ (while keeping all other conventions unchanged), then there would not exist a covariantly-constant

3-form. This is another reflection of the fact that the occurrence of supersymmetry is dependent up on sign of the fourth charge. (At the same time as the 8-dimensional spinor representation of the $SO(7)$ tangent-space group decomposes as $8 \longrightarrow 7+1$ under G_2, the 35-dimensional antisymmetric 3-index representation decomposes as $35 \longrightarrow 27+7+1$. It is the singlet in each case that corresponds to the covariantly-constant spinor (Killing spinor) and 3-form.)

Note that we can write the 3-form $\psi_{(3)}$ as

$$\psi_{(3)} = e^0 \wedge e^1 \wedge e^2 - e^0 \wedge K_0 - e^1 \wedge K_1 - e^2 \wedge K_2 , \tag{72}$$

where

$$K_0 \equiv e^3 \wedge e^6 + e^4 \wedge e^5 , \qquad K_1 \equiv e^4 \wedge e^6 - e^3 \wedge e^5 , \qquad K_2 \equiv e^5 \wedge e^6 + e^3 \wedge e^4 . \tag{73}$$

The three 2-forms K_0, K_1 and K_2 are self-dual with respect to the metric in the $(3, 4, 5, 6)$ directions. Thus in the entire construction, both of the 7-dimensional metric ds_7^2 and the covariantly-constant 3-form $\psi_{(3)}$, the flat 4-torus metric $dz_3^2 + \cdots + dz_6^2$ can be replaced by any hyper-Kähler 4-metric. (The potential terms that "twist" the fibres in the z_1 and z_2 directions are now replaced by potentials for the self-dual 2-forms K_2 and K_1. See section 6.2 below.)

In the case that $H = m\,y$, there is a scaling invariance of the metric (67) generated by the homothetic Killing vector

$$D = y\,\frac{\partial}{\partial y} + 4z_1\,\frac{\partial}{\partial z_1} + 4z_2\,\frac{\partial}{\partial z_2} + 2z_3\,\frac{\partial}{\partial z_3} + 2z_4\,\frac{\partial}{\partial z_4} + 2z_5\,\frac{\partial}{\partial z_5} + 2z_6\,\frac{\partial}{\partial z_6} . \tag{74}$$

In addition, (67) is invariant under the linear action of $SU(2)$ on (z_3, z_4, z_5, z_6) that preserves the two 2-forms $dz_4 \wedge dz_3 + dz_6 \wedge dz_5$ and $dz_5 \wedge dz_3 - dz_6 \wedge dz_4$. The 6-dimensional nilpotent algebra in this case is the complexification of the standard 3-dimensional ur-Heisenberg algebra.

4.2.2 T^3 bundle over T^3

There is an inequivalent class of domain-wall solutions in five-dimensional spacetime, for which a representative example is supported by the three fields $\{\mathcal{F}^1_{(0)56}, \mathcal{F}^2_{(0)46}, \mathcal{F}^3_{(0)45}\}$. This gives the Ricci-flat 7-metric

$$\begin{aligned} ds_7^2 &= H^3\,dy^2 + H^{-1}\,(dz_1 + m\,z_6\,dz_5)^2 + H^{-1}\,(dz_2 - m\,z_6\,dz_4)^2 \\ &\quad + H^{-1}\,(dz_3 + m\,z_5\,dz_4)^2 + H^2(dz_4^2 + dz_5^2 + dz_6^2) . \end{aligned} \tag{75}$$

20

In the obvious orthonormal basis $e^0 = H^{3/2} \, dy$, $e^2 = H^{-1/2} \, (dz_1 + m \, z_6 \, dz_5)$, *etc.*, the covariantly-constant associative 3-form is given by

$$
\begin{aligned}
\psi_{(3)} \;&=\; e^0 \wedge e^1 \wedge e^4 + e^0 \wedge e^2 \wedge e^5 + e^0 \wedge e^3 \wedge e^6 \\
&\quad + e^1 \wedge e^2 \wedge e^6 + e^2 \wedge e^3 \wedge e^4 + e^3 \wedge e^1 \wedge e^5 - e^4 \wedge e^5 \wedge e^6 \,.
\end{aligned}
\tag{76}
$$

In the case that $H = m \, y$, there is a scaling invariance of the metric (75) generated by the homothetic Killing vector

$$
D = y \, \frac{\partial}{\partial y} + 3z_1 \, \frac{\partial}{\partial z_1} + 3z_2 \, \frac{\partial}{\partial z_2} + 3z_3 \, \frac{\partial}{\partial z_3} + \tfrac{3}{2} z_4 \, \frac{\partial}{\partial z_4} + \tfrac{3}{2} z_5 \, \frac{\partial}{\partial z_5} + \tfrac{3}{2} z_6 \, \frac{\partial}{\partial z_6} \,.
\tag{77}
$$

In addition, (75) is invariant under $SO(3)$ acting linearly on (z_1, z_2, z_3) and (z_3, z_4, z_5, z_6).

4.3 Eight-dimensional manifolds with Spin(7) holonomy

4.3.1 T^3 bundle over T^4

Now consider a 6-charge domain wall solution in $D = 4$, supported by the 0-form field strengths $(\mathcal{F}^1_{(0)45}, \mathcal{F}^1_{(0)67}, \mathcal{F}^2_{(0)46}, \mathcal{F}^2_{(0)57}, \mathcal{F}^3_{(0)47}, \mathcal{F}^3_{(0)56})$. As in the previous case, the signs of the charges must be appropriately chosen in order to have a supersymmetric solution. In $D = 4$, the domain wall metric, with all charges chosen equal in magnitude, is

$$
ds^2 = H^3 \, dx^\mu \, dx_\mu + H^9 \, dy^2 \,.
\tag{78}
$$

Oxidising back to $D = 11$ gives the eleven-dimensional metric $d\hat{s}^2 = dx^\mu \, dx_\mu + ds_8^2$, where the Ricci-flat 8-metric is given by

$$
\begin{aligned}
ds_8^2 \;&=\; H^6 \, dy^2 + H^{-2} \, [dz_1 + m \, (z_5 \, dz_4 + z_7 \, dz_6)]^2 + H^{-2} \, [dz_2 + m \, (z_6 \, dz_4 - z_7 \, dz_5)]^2 \\
&\quad + H^{-2} \, [dz_3 + m \, (z_7 \, dz_4 + z_6 \, dz_5)]^2 + H^3 \, (dz_4^2 + \cdots + dz_7^2) \,.
\end{aligned}
\tag{79}
$$

Since the solution preserves $1\tfrac{1}{16}$ of the supersymmetry, it follows that this Ricci-flat 8-metric must have Spin(7) holonomy.

Let us choose the natural orthonormal basis,

$$
\begin{aligned}
&e^0 = H^3 \, dy \,, \qquad e^1 = H^{-1} \, [dz_1 + m \, (z_5 \, dz_4 + z_7 \, dz_6)] \,, \\
&e^2 = H^{-1} \, [dz_2 + m \, (z_6 \, dz_4 - z_7 \, dz_5)] \,, \qquad e^3 = H^{-1} \, [dz_3 + m \, (z_7 \, dz_4 + z_6 \, dz_5)] \,, \\
&e^4 = H^{3/2} \, dz_4 \,, \qquad e^5 = H^{3/2} \, dz_5 \,, \qquad e^6 = H^{3/2} \, dz_6 \,, \qquad e^7 = H^{3/2} \, dz_7 \,.
\end{aligned}
\tag{80}
$$

The spin connection is then given by $\omega_{ij} = \tfrac{1}{2}(c_{ijk} + c_{ikj} - c_{kji}) \, e^k$, where the non-vanishing connection coefficients $c_{ijk} = -c_{jik}$ are specified by

$$
c_{01}{}^1 = c_{45}{}^1 = c_{67}{}^1 = c_{02}{}^2 = c_{46}{}^2 = -c_{57}{}^2 = c_{03}{}^3 = c_{47}{}^3 = c_{56}{}^3 = 2\lambda \,,
\tag{81}
$$

$$
c_{04}{}^4 = c_{05}{}^5 = c_{06}{}^6 = c_{07}{}^7 = -3\lambda \,,
$$

and $\lambda \equiv \frac{1}{2} m \, H^{-4}$ here.

From this, it is straightforward to show that the following 4-form is covariantly constant:

$$
\begin{aligned}
\Psi_{(4)} \;=\; & -(e^0 \wedge e^1 + e^2 \wedge e^3) \wedge (e^4 \wedge e^5 + e^6 \wedge e^7) - (e^0 \wedge e^2 - e^1 \wedge e^3) \wedge (e^4 \wedge e^6 - e^5 \wedge e^7) \\
& -(e^0 \wedge e^3 + e^1 \wedge e^2) \wedge (e^4 \wedge e^7 + e^5 \wedge e^6) + e^0 \wedge e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6 \wedge e^7 \quad (82)
\end{aligned}
$$

The existence of this 4-form, which is self-dual, is characteristic of 8-manifolds with $\mathrm{Spin}(7)$ holonomy.

In the case that $H = m \, y$, there is a scaling invariance of the metric (79) generated by the homothetic Killing vector

$$
D = y \, \frac{\partial}{\partial y} + 5 z_1 \, \frac{\partial}{\partial z_1} + 5 z_2 \, \frac{\partial}{\partial z_2} + 5 z_3 \, \frac{\partial}{\partial z_3} + \tfrac{5}{2} z_4 \, \frac{\partial}{\partial z_4} + \tfrac{5}{2} z_5 \, \frac{\partial}{\partial z_5} + \tfrac{5}{2} z_6 \, \frac{\partial}{\partial z_6} + \tfrac{5}{2} z_7 \, \frac{\partial}{\partial z_7} \, . \quad (83)
$$

In addition, (79) is invariant under the linear action of $SU(2)$ on (z_4, z_5, z_6, z_7) that preserves the three 2-forms $dz_5 \wedge dz_4 + dz_7 \wedge dz_6$, $dz_6 \wedge dz_4 - dz_7 \wedge dz_5$ and $dz_7 \wedge dz_4 + dz_6 \wedge dz_5$.

4.4 Further examples and specialisations

The various domain walls that we have obtained above are the most natural ones to consider, since they possess the maximum number of charges in each case, and after lifting to $D = 11$ they are irreducible. It is, nevertheless, of interest also to study some of the other possible examples.

4.4.1 T^1 bundle over T^3

In seven-dimensional maximal supergravity, the largest number of allowed charges for domain walls is 1 (see, for example, [24]). The metric is given by

$$
ds_7^2 = H^{1/5} \, dx^\mu \, dx_\mu + H^{6/5} \, dy^2 \, , \quad (84)
$$

where $H = 1 + m \, |y|$. After lifting to $D = 11$, this gives $ds_{11}^2 = dx^\mu \, dx_\mu + ds_5^2$, where ds_5^2 is the Ricci-flat metric

$$
ds_5^2 = H \, dy^2 + H^{-1} \, (dz_1 + m \, z_3 \, dz_2)^2 + H \, (dz_2^2 + dz_3^2) + dz_4^2 \, . \quad (85)
$$

This is clearly reducible, being nothing but the direct sum of the four-dimensional Ricci-flat metric (5) and a circle. It is for this reason that we omitted this 5-dimensional example in our enumeration above. It can be viewed as a T^1 bundle over T^3, but since the bundle is trivial over a T^1 factor in the base, it would be more accurate to describe it as T^1 times a T^1 bundle over T^2.

4.4.2 Ricci-flat metrics with fewer charges

There are many possibilities for obtaining other Ricci-flat metrics, by turning on only subsets of the charges in the metrics we have already obtained. We shall illustrate this by considering the example of the 8-dimensional metric (79). If we introduce parameters ϵ_i, where $\epsilon_i = 1$ if the i'th of the six charges listed above (78) is turned on, an $\epsilon_i = 0$ if the i'th charge is turned off. After lifting the resulting domain wall from $D = 4$ to $D = 11$, we get a Ricci-flat 8-metric given by

$$
\begin{aligned}
ds_8^2 &= H^{\sum_i \epsilon_i}\, dy^2 + H^{-\epsilon_1-\epsilon_2}\, h_1^2 + H^{-\epsilon_3-\epsilon_4}\, h_2^2 + H^{-\epsilon_5-\epsilon_6}\, h_3^2 \\
&\quad + H^{\epsilon_1+\epsilon_3+\epsilon_5}\, h_4^2 + H^{\epsilon_1+\epsilon_4+\epsilon_6}\, h_5^2 + H^{\epsilon_2+\epsilon_3+\epsilon_6}\, h_6^2 + H^{\epsilon_2+\epsilon_4+\epsilon_5}\, h_7^2 \,,
\end{aligned} \tag{86}
$$

$$
h_1 = dz_1 + \epsilon_1\, z_5\, dz_4 + \epsilon_2\, z_7\, dz_6\,, \quad h_2 = dz_2 + \epsilon_3\, z_6\, dz_4 - \epsilon_4\, z_7\, dz_5\,, \tag{87}
$$

$$
h_3 = dz_3 + \epsilon_5\, z_7\, dz_4 + \epsilon_6\, z_6\, dz_5\,, \quad h_4 = dz_4\,, \quad h_5 = dz_5\,, \quad h_6 = dz_6\,, \quad h_7 = dz_7\,.
$$

4.4.3 $SU(4)$ holonomy in $D = 8$

There are other possibilities, which involve a lesser number of charges which are not themselves a subset of the maximal set. For example, we can consider the following 8-dimensional Ricci-flat metric that comes from lifting a 3-charge four-dimensional domain wall, supported by the fields $\mathcal{F}^1_{(0)23}$, $\mathcal{F}^1_{(0)45}$, $\mathcal{F}^1_{(0)67}$. This gives

$$
ds_8^2 = H^3\, dy^2 + H^{-3}\, (dz_1 + z_3\, dz_2 + z_5\, dz_4 + z_7\, dz_6)^2 + H\, (dz_2^2 + \cdots + dz_7^2)\,. \tag{88}
$$

This metric has $SU(4)$ holonomy, and it can be viewed as a Heisenberg limit of a complex line bundle over a six-dimensional Einstein-Kähler space such as $S^2 \times S^2 \times S^2$, or $\mathbb{CP}^3$.

5 Heisenberg limits of complete metrics of special holonomy

In this section, we generalise the discussion of the Heisenberg limit of the Eguchi-Hanson metric that we gave in section 2.2, and show how the various Ricci-flat metrics that we obtained from domain-wall solutions in section 4 can be viewed as arising as Heisenberg limits of complete metrics of special holonomy.

5.1 Contractions of Ricci-flat Kähler 6-metrics

5.1.1 Contractions of T^1 bundles over Einstein-Kähler

The contraction to the Heisenberg limit of the Eguchi-Hanson metric was discussed in section 2.2 at the level of the metric itself, and in section 2.3 at the level of the equations of

motion and superpotential. This contraction procedure can be easily generalised to higher dimensions. In particular, we may obtain the six-dimensional Ricci-flat Heisenberg metric (49) as a contraction of a Ricci-flat metric on a line bundle over an Einstein-Kähler 4-metric with positive scalar curvature, such as $\mathbb{CP}^2$. If we consider the more general case of a line bundle over $\mathbb{CP}^n$, the starting point will be the metric

$$ds^2_{2n+2} = dt^2 + a^2 \, \sigma^\alpha \, \bar\sigma_\alpha + c^2 \, \nu^2 \,, \tag{89}$$

where the left-invariant 1-forms of $SU(n+1)$ are defined in appendix B.2.4. The conditions for Ricci-flatness for the line bundle over $\mathbb{CP}^n$ then follow from the Lagrangian $L = T - V$, where

$$T = 2\alpha' \, \gamma' + (2n - 1) \, \alpha'^2 \,, \qquad V = a^{4n-4} \, c^4 + 2(n+1) \, a^{4n-2} \, c^2 \,, \tag{90}$$

$d\eta = dt/(a^{2n} \, c)$, $a = e^\alpha$ and $b = e^\beta$. The superpotential W is given by

$$W = a^{2n-2} \, c^2 + \frac{n+1}{n} \, a^{2n} \,. \tag{91}$$

The scalings (206) induce the following scalings in the metric coefficients:

$$a \longrightarrow \lambda^{-1} \, a \,, \qquad c \longrightarrow \lambda^{-2} \, c \,. \tag{92}$$

After sending λ to zero, the rescaled superpotential becomes

$$W = a^{2n-2} \, c^2 \,. \tag{93}$$

Solving the resulting first-order equations gives

$$a \propto t^{1/(n+1)} \,, \qquad c \propto t^{-n/(n+1)} \,. \tag{94}$$

In particular, for $n = 1$ we recover the 4-dimensional Heisenberg metric (5), and for $n = 2$ we recover the 6-dimensional metric (49).

It should be remarked that we could in fact obtain the same Heisenberg contractions if the $\mathbb{CP}^n$ metrics are replaced by any other $(2n)$-dimensional homogeneous Einstein-Kähler metrics of positive scalar curvature. In section 6.1, we shall give a version of this construction for inhomogeneous Einstein-Kähler manifolds.

5.1.2   Contraction of $T^* S^{n+1}$

Starting from the left-invariant $SO(n+2)$ 1-forms in the notation of (182), the ansatz that gives rise to the Stenzel [32] metrics on $T^* S^{n+1}$ is [33]

$$ds^2 = dt^2 + a^2 \, \sigma_i^2 + b^2 \, \tilde\sigma_i^2 + c^2 \, \nu^2 \,. \tag{95}$$

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The Ricci-flat equations can be derived from the Lagrangian $L = T - V$ with

$$
\begin{aligned}
T &= \alpha'\,\gamma' + \beta'\,\gamma' + n\,\alpha'\,\beta' + \tfrac{1}{2}(n-1)\left(\alpha'^2 + \beta'^2\right), \\
V &= \tfrac{1}{4}(a\,b)^{2n-2}\left(a^4 + b^4 + c^4 - 2a^2\,b^2 - 2n\left(a^2 + b^2\right)c^2\right),
\end{aligned}
\tag{96}
$$

and V can be obtained from the superpotential [33]

$$
W = \tfrac{1}{2}(a\,b)^{n-1}\left(a^2 + b^2 + c^2\right).
\tag{97}
$$

Solutions of the associated first-order equations give the Ricci-flat Kähler Stenzel metrics on T^*S^{n+1} [33].

After applying the scalings (187), which imply $(a, b, c) \longrightarrow (\lambda^{-2}\,a, \lambda^{-1}\,b, \lambda^{-1}\,c)$, and then sending λ to zero, we obtain the superpotential

$$
W = \tfrac{1}{2}a^{n+1}\,b^{n-1}\,.
\tag{98}
$$

This leads to the first-order equations [33]

$$
\dot{a} = -\frac{a^2}{2b\,c}\,, \qquad \dot{b} = \frac{a}{2c}\,, \qquad \dot{c} = \frac{n\,a}{2b}\,,
\tag{99}
$$

and after defining a new radial variable by $dt = 2b\,c\,d\rho$, we obtain the Ricci-flat Heisenberg metric

$$
ds^2 = \rho^{2n+2}\,d\rho^2 + \frac{1}{\rho}\,\sigma_i^2 + \rho\,\tilde{\sigma}_i^2 + \rho^n\,\nu^2\,,
\tag{100}
$$

where the left-invariant 1-forms satisfy the exterior algebra (188). Setting $n = 2$, it is easily seen after a coordinate transformation that we reproduce the Ricci-flat metric (56).

5.2 Contractions of 7-metrics of G_2 holonomy

In the present section, we shall show how the two seven-dimensional Heisenberg metrics (67) and (75) can be obtained as contraction limits of complete G_2 metrics of cohomogeneity one. Later, in section 6.2, we shall give a version of this construction using inhomogeneous hyper-Kähler 4-metrics.

5.2.1 Contraction of $\mathbb{R}^3$ bundle over S^4

The complete metric of G_2 holonomy is [3, 4]

$$
ds_7^2 = \left(1 + \frac{Q}{r^4}\right)^{-1} dr^2 + r^2\left(1 + \frac{Q}{r^4}\right)\left(R_1^2 + R_2^2\right) + \tfrac{1}{2}\,r^2\,P_\alpha^2\,,
\tag{101}
$$

where R_1, R_2 and σ_α are given in terms of the left-invariant 1-forms of $SO(5)$ in appendix B.2.3. After implementing the rescalings given in (202), together with

$$r \longrightarrow \lambda^{-2}\, r\,, \qquad Q \longrightarrow \lambda^{-12}\, Q\,, \tag{102}$$

then after sending λ to zero we get the Heisenberg metric

$$ds_7^2 = \frac{r^4}{Q}\, dr^2 + \frac{Q}{r^2}\, (\nu_1^2 + \nu_2^2) + \tfrac{1}{2}\, r^2\, \sigma_\alpha^2\,, \tag{103}$$

where ν_1, ν_2 and σ_α satisfy the contracted algebra given in (203). After a coordinate transformation, this can be seen to be equivalent to the Heisenberg metric (67).

5.2.2 Contraction of $\mathbb{R}^4$ bundle over S^3

The complete metric of G_2 holonomy is [3, 4]

$$ds_7^2 = \left(1 + \frac{Q}{r^3}\right)^{-1} dr^2 + \tfrac{1}{9}r^2 \left(1 + \frac{Q}{r^3}\right) \nu_i^2 + \tfrac{1}{12}\, r^2\, \sigma_i^2\,, \tag{104}$$

in the notation of (A.2.2). After taking the scaling limit (192), together with

$$r \longrightarrow \lambda^{-1}\, r\,, \qquad Q \longrightarrow \lambda^{-5}\, Q\,, \tag{105}$$

and then sending λ to zero, we obtain the following Heisenberg limit of the metric (104):

$$ds_7^2 = \frac{r^3}{Q}\, dr^2 + \frac{Q}{9r}\, \nu_i^2 + \tfrac{1}{12}\, r^2\, \sigma_i^2\,, \tag{106}$$

where ν_i and σ_i now satisfy the contracted exterior algebra given in (194). After a coordinate transformation, this can be seen to be equivalent to the Heisenberg metric (75).

5.3 Contraction of 8-metric of Spin(7) holonomy

Here, we shall show how the eight-dimensional Heisenberg metric (79) can be obtained as a contraction limits of a complete Spin(7) metric of cohomogeneity one. Later, in section 6.3, we shall give a version of this construction using inhomogeneous hyper-Kähler 4-metrics.

The complete metric of Spin(7) holonomy is [3, 4]

$$ds_8^2 = \left(1 + \frac{Q}{r^{10/3}}\right)^{-1} dr^2 + \frac{9r^2}{100} \left(1 + \frac{Q}{r^{10/3}}\right) R_i^2 + \frac{9r^2}{20}\, P_\alpha^2\,, \tag{107}$$

where L_i and P_α are given in terms of the left-invariant 1-forms of $SO(5)$ in appendix (A.2.3). Implementing the scalings in (200), together with

$$r \longrightarrow \lambda^{-2}\, r\,, \qquad Q \longrightarrow \lambda^{-32/3}\, Q\,, \tag{108}$$

then after sending λ to zero the metric (107) becomes

$$ds_8^2 = \frac{r^{10/3}}{Q}\, dr^2 + \frac{9Q}{100 r^{4/3}}\, \nu_i^2 + \frac{9r^2}{20}\, \sigma_\alpha^2\,, \tag{109}$$

where ν_i and σ_α now satisfy the contracted algebra (201). After a coordinate transformation, this can be seen to be equivalent to the $D = 8$ Heisenberg metric (79).

6   More general constructions of special-holonomy manifolds in 6, 7 and 8 dimensions

It is clear from the structure of the Ricci-flat Heisenberg metrics in dimensions 6, 7 and 8 in section 4 that in each case where the principal orbits are torus bundles over T^4, this 4-torus can itself be replaced by an arbitrary Ricci-flat Kähler 4-metric. In other words, we can allow the 4-manifold to be any hyper-Kähler space. Such a space admits a triplet of covariantly-constant 2-forms J^a, which satisfy the multiplication rules of the imaginary unit quaternions:

$$J_{ij}^a\, J_{jk}^b = -\delta_{ab}\, \delta_{ij} + \epsilon_{abc}\, J_{ik}^c\,. \tag{110}$$

In this section, we shall consider this more general construction in each of the dimensions 6, 7 and 8.

6.1   6-metric of $SU(3)$ holonomy from T^1 bundle over hyper-Kähler

6.1.1   Description in real coordinates

Let ds_4^2 be a hyper-Kähler 4-metric, and then consider the following 6-metric:

$$d\hat{s}_6^2 = H^2\, dy^2 + H^{-2}\, (dz_1 + A_{(1)})^2 + H\, ds_4^2\,, \tag{111}$$

where $H = y$, and $dA_{(1)} = J$, a Kähler form on ds_4^2 (we take $m = 1$ here). In the orthonormal frame

$$\hat{e}^0 = H\, dy\,, \qquad \hat{e}^1 = H^{-1}\, (dz_1 + A_{(1)})\,, \qquad \hat{e}^i = H^{1/2}\, e^i\,, \tag{112}$$

where e^i is an orthonormal frame for ds_4^2, we find that the spin connection is given by

$$\hat{\omega}_{01} = H^{-2}\, \hat{e}^1\,, \qquad \hat{\omega}_{0i} = -\tfrac{1}{2} H^{-2}\, \hat{e}^i\,, \qquad \hat{\omega}_{1i} = \tfrac{1}{2} H^{-2}\, J_{ij}\, \hat{e}^j\,,$$

$$\hat{\omega}_{ij} = \omega_{ij} - \tfrac{1}{2} H^{-2}\, J_{ij}\, \hat{e}^1\,, \tag{113}$$

where ω_{ij} is the spin connection for ds_4^2.

From this, it follows that the curvature 2-forms are given by

$$
\begin{aligned}
\Theta_{01} &= -3H^{-4}\,\hat{e}^0 \wedge \hat{e}^1 + \tfrac{3}{2}H^{-4}\,J_{ij}\,\hat{e}^i \wedge \hat{e}^j \,, \\
\Theta_{0i} &= \tfrac{3}{4}H^{-4}\,\hat{e}^0 \wedge \hat{e}^i + \tfrac{3}{4}H^{-4}\,J_{ij}\,\hat{e}^1 \wedge \hat{e}^j \,, \\
\hat{\Theta}_{1i} &= \tfrac{3}{4}H^{-4}\,\hat{e}^1 \wedge \hat{e}^i - \tfrac{3}{4}H^{-4}\,\hat{e}^0 \wedge \hat{e}^j \,, \\
\hat{\Theta}_{ij} &= \Theta_{ij} - \tfrac{1}{4}H^{-4}\left(\delta_{ik}\,\delta_{j\ell} + J_{ik}\,J_{j\ell} + J_{ij}\,J_{k\ell}\right)\hat{e}^k \wedge \hat{e}^\ell \,,
\end{aligned}
\tag{114}
$$

where Θ_{ij} is the curvature 2-form for ds_4^2. From these, we can read off that the Ricci tensor vanishes.

The Kähler form for the 6-dimensional metric is given by

$$
\hat{J} = \hat{e}^0 \wedge \hat{e}^1 + H\,J \,.
\tag{115}
$$

The Lorentz-covariant exterior derivative $\hat{D}$ acting on a spinor ψ is given by

$$
\begin{aligned}
\hat{D}\,\psi &\equiv d\psi + \tfrac{1}{4}\hat{\omega}^{AB}\,\hat{\Gamma}_{AB}\,\psi \,, \\
&= D\,\psi + \tfrac{1}{2}H^{-2}\,(\hat{\Gamma}_{01} - \tfrac{1}{4}J_{ij}\,\hat{\Gamma}_{ij})\,\psi\,\hat{e}^1 - \tfrac{1}{4}H^{-2}\,(\hat{\Gamma}_{0i} - J_{1j}\,\hat{\Gamma}_{1j})\,\psi\,\hat{e}^j \,,
\end{aligned}
\tag{116}
$$

where $D \equiv d + \tfrac{1}{4}\omega^{ij}\,\hat{\Gamma}_{ij}$ is the Lorentz-covariant exterior derivative on ds_4^2 (except that the gamma matrices are the six-dimensional ones).

It follows from this expression for $\hat{D}$ that a Killing spinor $\hat{\eta}$ must satisfy the conditions

$$
D\,\hat{\eta} = 0 \,, \qquad \hat{\Gamma}_{0i}\,\eta = J_{ij}\,\hat{\Gamma}_{1j}\,\hat{\eta} \,.
\tag{117}
$$

6.1.2 Description in complex coordinates

The above discussion made use of real coordinates on the six-dimensional Ricci-flat Kähler manifold. The structure of the metric in the complex notation (53), and of the Kähler potential in the form (54), suggest the natural generalisation for the construction in a complex notation. Thus we are led to the following:

Let ds^2 be a Ricci-flat Kähler metric of complex dimension n, with Kähler function K, and Kähler form $J = i\,\partial\bar{\partial}\,K$. Then

$$
d\tilde{s}^2 = H\,ds^2 + H^{-n}\,|d\zeta_{n+1} + A|^2
\tag{118}
$$

is a Ricci-flat Kähler metric of complex dimension $(n+1)$, where

$$
A = \frac{1}{n+1}\,\partial K \,, \qquad H = \phi^{1/(n+1)} \,,
\tag{119}
$$

and we have defined

$$\phi \equiv 1 + \zeta_{n+1} + \bar{\zeta}_{n+1} + \frac{1}{n+1}\, K\,. \tag{120}$$

(The "1" is inessential here, of course.) The Kähler function for $d\tilde{s}^2$ is given by

$$\widetilde{K} = \frac{(n+1)^2}{n+2}\,\phi^{(n+2)/(n+1)} = \frac{(n+1)^2}{n+2}\,H^{n+2}\,, \tag{121}$$

and its Kähler form is given by

$$\widetilde{J} = H\,J + \mathrm{i}\,H^{-n}\,(d\zeta_{n+1} + A) \wedge (d\bar{\zeta}_{n+1} + \bar{A})\,. \tag{122}$$

The proof is as follows. First, note that calculating $\widetilde{J}$ from the Kähler function $\widetilde{K}$ given above, we get

$$
\begin{aligned}
\widetilde{J} &= \mathrm{i}\,\partial\bar{\partial}\,\widetilde{K} = \mathrm{i}\,\partial\bar{\partial}\left(\frac{(n+1)^2}{n+2}\,\phi^{(n+2)/(n+1)}\right), \\
&= \mathrm{i}\,(n+1)\,\partial\,(\phi^{1/(n+1)}\,\bar{\partial}\,\phi) = \mathrm{i}\,(n+1)\,\phi^{1/(n+1)}\,\partial\bar{\partial}\,\phi + \mathrm{i}\,\phi^{-n/(n+1)}\,\partial\phi \wedge \bar{\partial}\phi\,, \\
&= \mathrm{i}\,\phi^{1/(n+1)}\,\partial\bar{\partial}\,K + \mathrm{i}\,\phi^{-n/(n+1)}\,(d\zeta_{n+1} + A) \wedge (d\bar{\zeta}_{n+1} + \bar{A})\,, \\
&= H\,J + \mathrm{i}\,H^{-n}\,(d\zeta_{n+1} + A) \wedge (d\bar{\zeta}_{n+1} + \bar{A})\,. \tag{123}
\end{aligned}
$$

Bearing in mind that the Kähler form is related to the metric by $\widetilde{J} = \mathrm{i}\,\widetilde{g}_{\mu\bar{\nu}}\,d\zeta^\mu \wedge \bar{d}\zeta^{\bar{\nu}}$, we see that this does indeed agree with the metric given in (118).

This shows that the metric (118) is indeed Kähler. Finally, to show that it is Ricci-flat, we calculate the determinant:

$$\det(\tilde{g}) = H^{2n}\,H^{-2n}\,\det(g) = \det(g)\,. \tag{124}$$

Since the Ricci form is given by $\widetilde{\mathcal{R}} = \mathrm{i}\,\partial\bar{\partial}\,\log\det(\tilde{g})$, it follows that if the Ricci form $\mathcal{R}$ for the metric ds^2 is zero (which was the initial assumption), then the Ricci form $\widetilde{\mathcal{R}}$ for $d\tilde{s}^2$ is zero also.

It is easily seen that the 6-metric metric we obtained in section 4.1.1 is an example of this type, since the Kähler function for the flat 4-torus can be taken to be $K = |\zeta_1|^2 + |\zeta_2|^2$.

6.2   7-metric of G_2 holonomy from T^2 bundle over hyper-Kähler

Let ds_4^2 be a hyper-Kähler metric, with a triplet of Kähler forms J^a, with associated 1-form potentials $A_{(1)}^a$:

$$J^a = dA_{(1)}^a\,, \qquad \nabla\,J^a = 0\,. \tag{125}$$

It turns out to be convenient to let a range over the values 0,1,2.

Consider the metric

$$d\hat{s}_7^2 = H^4 \, dy^2 + H^{-2} \sum_{\alpha=1}^{2} (dz^\alpha + A_{(1)}^\alpha)^2 + H^2 \, ds_4^2, \tag{126}$$

where $H = y$. (We have set $m = 1$.) Define vielbeins by

$$\hat{e}^0 = H^2 \, dy, \qquad \hat{e}^\alpha = H^{-1} \, (dz^\alpha + A_{(1)}^\alpha), \qquad \hat{e}^i = H \, e^i, \tag{127}$$

where $a = (0, \alpha)$, $i = (3, 4, 5, 6)$, and e^i is a vielbein for the hyper-Kähler 4-metric ds_4^2. Then it can be verified that the following 3-form is closed:

$$\psi_{(3)} \equiv \hat{e}^0 \wedge \hat{e}^1 \wedge \hat{e}^2 + H^2 \, \hat{e}^0 \wedge J^0 - \epsilon_{\alpha\beta} \hat{e}^\alpha \wedge J^\beta. \tag{128}$$

In fact this 3-form is covariantly constant, as can be verified using the expressions for the spin connection:

$$\hat{\omega}_{0\alpha} = H^{-3} \, \hat{e}^\alpha, \qquad \hat{\omega}_{0i} = -H^{-3} \, \hat{e}^i, \qquad \hat{\omega}_{\alpha\beta} = 0,$$

$$\hat{\omega}_{\alpha i} = \tfrac{1}{2} H^{-3} \, J_{ij}^\alpha \, \hat{e}^i, \qquad \hat{\omega}_{ij} = \omega_{ij} - \tfrac{1}{2} H^{-3} \, J_{ij}^\alpha \, \hat{e}^\alpha, \tag{129}$$

where J_{ij}^a denotes the components of J^a with respect to the vielbein e^a for ds_4^2, and ω_{ij} is the spin-connection for the vielbein e^i. The covariant-constancy of $\psi_{(3)}$ proves that the metric $d\hat{s}_7^2$ has G_2 holonomy, and it is also therefore Ricci flat.

From (129) is is also straightforward to calculate the vielbein components of the Riemann tensor for the metric $d\hat{s}_7^2$. We find

$$\hat{R}_{0\alpha0\beta} = -4H^{-6} \, \delta_{\alpha\beta}, \qquad \hat{R}_{0\alpha ij} = 2H^{-6} \, J_{ij}^\alpha,$$

$$\hat{R}_{0i0i} = 2H^{-6} \, \delta_{ij}, \qquad \hat{R}_{0i\alpha j} = H^{-6} \, J_{ij}^\alpha,$$

$$\hat{R}_{\alpha\beta\gamma\delta} = -H^{-6} \, (\delta_{\alpha\gamma} \, \delta_{\beta\delta} - \delta_{\alpha\delta} \, \delta_{\beta\gamma}), \qquad \hat{R}_{\alpha\beta ij} = \tfrac{1}{2} H^{-6} \, \epsilon_{a\beta} \, J_{ij}^0,$$

$$\hat{R}_{\alpha i\beta j} = \tfrac{5}{4} H^{-6} \, \delta_{\alpha\beta} \, \delta_{ij} + \tfrac{1}{4} H^{-6} \, \epsilon_{\alpha\beta} \, J_{ij}^0, \tag{130}$$

$$\hat{R}_{ijk\ell} = H^{-2} \, R_{ijk\ell} - \tfrac{1}{4} H^{-6} \, (J_{ik}^\alpha \, J_{j\ell}^\alpha - J_{i\ell}^\alpha \, J_{jk}^\alpha + 2J_{ij}^\alpha \, J_{k\ell}^\alpha + 4\delta_{ik} \, \delta_{j\ell} - 4\delta_{i\ell} \, \delta_{jk}),$$

where $R_{ijk\ell}$ is the Riemann tensor of the hyper-Kähler metric ds_4^2. It is easily verified from these expressions that the Ricci tensor $\hat{R}_{AB}$ of the metric $d\hat{s}_7^2$ is zero.

6.3   8-metric of Spin(7) holonomy from T^3 bundle over hyper-Kähler

In a similar fashion, we can give the general construction for 8-metrics, in terms of a hyper-Kähler base metric ds_4^2. This time we shall have

$$d\hat{s}_8^2 = H^6 \, dy^2 + H^{-2} \sum_{a=1}^{3} (dz^a + A_{(1)}^a)^2 + H^3 \, ds_4^2. \tag{131}$$

Note that here, it is convenient to label the three Kähler forms of ds_4^2 by $J^a = dA_{(1)}^a$ with $a = 1, 2, 3$. We then define the vielbeins

$$\hat{e}^0 = H^3 \, dy \,, \qquad \hat{e}^a = H^{-1} \left(dz^a + A_{(1)}^a \right) \,, \qquad \hat{e}^i = H^{3/2} \, e^i \,, \qquad (132)$$

where here $i = 4, 5, 6, 7$, and e^i is a vielbein for the hyper-Kähler metric ds_4^2. It can then be verified that the self-dual 4-form $\Psi_{(4)}$ given by

$$\Psi_{(4)} = \hat{e}^0 \wedge \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3 + \tfrac{1}{6} H^6 \, J^a \wedge J^a + H^3 (\hat{e}^0 \wedge \hat{e}^a + \tfrac{1}{2} \epsilon_{abc} \, \hat{e}^b \wedge \hat{e}^c) \wedge J^a \qquad (133)$$

is closed. (Note that $\tfrac{1}{6} J^a \wedge J^a$ is just another way of writing the volume form of ds_4^2.)

In fact it can also be verified that $\Psi_{(4)}$ is covariantly constant, by making use of the following results for the spin connection of the 8-metric:

$$\hat{\omega}_{0a} = H^{-4} \, \hat{e}^a \,, \qquad \hat{\omega}_{0i} = -\tfrac{3}{2} H^{-4} \, \hat{e}^i \,, \qquad \hat{\omega}_{ab} = 0 \,,$$
$$\hat{\omega}_{ai} = \tfrac{1}{2} H^{-4} \, J_{ij}^a \, \hat{e}^j \,, \qquad \hat{\omega}_{ij} = \omega_{ij} - \tfrac{1}{2} H^{-4} \, J_{ij}^a \, \hat{e}^a \,. \qquad (134)$$

Calculating the curvature from this, we find

$$\hat{R}_{0a0b} = -5 H^{-8} \, \delta_{ab} \,, \qquad \hat{R}_{0aij} = \tfrac{5}{2} H^{-8} \, J_{ij}^a \,,$$
$$\hat{R}_{0i0j} = \tfrac{15}{4} \, H^{-8} \, \delta_{ij} \,, \qquad \hat{R}_{0iaj} = \tfrac{5}{4} H^{-8} \, J_{ij}^a \,,$$
$$\hat{R}_{abcd} = -H^{-8} \left(\delta_{ac} \, \delta_{bd} - \delta_{ad} \, \delta_{bc} \right) \,, \qquad \hat{R}_{abij} = \tfrac{1}{2} H^{-8} \, \epsilon_{abc} \, J_{ij}^c \,, \qquad (135)$$
$$\hat{R}_{aibj} = \tfrac{7}{4} H^{-8} \, \delta_{ab} \, \delta_{ij} + \tfrac{1}{4} H^{-8} \, \epsilon_{abc} \, J_{ij}^c \,,$$
$$\hat{R}_{ijk\ell} = H^{-2} \, R_{ijk\ell} - \tfrac{1}{4} H^{-8} \left(J_{ik}^a \, J_{j\ell}^a - J_{i\ell}^a \, J_{jk}^a + 2 J_{ij}^a \, J_{k\ell}^a + 9 \delta_{ik} \, \delta_{j\ell} - 9 \delta_{i\ell} \, \delta_{jk} \right) \,.$$

It is easily verified that the Ricci tensor $\hat{R}_{AB}$ for the 8-dimensional metric $d\hat{s}_8^2$ vanishes.

7 Cosmological resolutions

There is an alternative approach to resolving the various Heisenberg metrics that we have been discussing in this paper. This involves modifying the requirement of Ricci-flatness, so that instead the metrics are now required to satisfy the Einstein condition with a negative Ricci tensor. It turns out in all the previous examples, we can now obtain complete and non-singular non-compact metrics. In each case, this is achieved by replacing the various powers of H appearing as prefactors of the terms $(dz_i + \cdots)^2$ by arbitrary functions of the radial variable, and then solving the Einstein equations.

In all the cases we consider, the homothetic conformal Killing vector D of the original Ricci-flat metric is replaced by a true Killing vector of the associated Einstein metric.

This, together with the generators of the nilpotent Heisenberg group generate a solvable group, which acts simply-transitively on the Einstein manifold, which may thus be taken to be a solvable group manifold Solv. In addition, all our metrics admit some manifest compact symmetries, which act linearly on the Heisenberg manifold. We have also identified some non-linearly acting symmetries. In all cases, we can express the manifold as $G/H =$ Solv, where the non-compact group G has maximal compact subgroup H. The group H contains the linearly-realised compact symmetries. This is quite striking because a theorem of Alekseevskii and Kimel'fel'd [34] states that any homogeneous non-compact Ricci flat Riemannian metric must be flat [35]. In fact a theorem of Dotti states that a left-invariant Einstein metric on a unimodular solvable group must be flat, so our solvable groups cannot be unimodular, that is the trace of the structure constants of the Lie algebra cannot vanish [36].[5]

The simplest example is when the Ricci-flat manifold is flat space, and the associated solvable group manifold is hyperbolic space. This has been encountered in studies of the AdS/CFT correspondence and is related to the ideas of Ref. [37], in which the fifth dimension corresponds to the Liouville mode of a non-critical string theory which thus becomes dynamical. The idea is that the string coordinates appear in the effective action multiplied by a function of the Liouville field. This function should vanish at large negative values of the Liouville field, in order to enforce a "zig-zag symmetry." To achieve this and to fix the functional form, the effective Lagrangian for the string is taken to include a piece invariant under both Poincaré transformations and dilatations. As a result of imposing the dilatation symmetry, an exponential function of the Liouville mode multiplies the coordinates of the string. The vanishing of this function corresponds to the horizon in AdS spacetime. From the ten-dimensional point of view, one must take the product metric AdS$_5 \times S^5$, where the $SO(6)$ R-symmetry group arises from the isometry group of the S^5 factor.

Our metrics arise by replacing the usual commuting translations of the string by non-commuting translations satisfying a Heisenberg algebra. This may be relevant when considering strings in constant background fields. It is a striking fact that the obvious nilpotent symmetry is, as in the standard AdS case, enhanced to a much larger group G.

A common feature of all of our Ricci-flat metrics is that the size of the toric fibres goes to zero as a negative power of distance as we go to infinity, while the size of the base expands as a positive power. By contrast, for our Einstein metrics both directions expand exponentially as one goes to infinity, but the toric fibre directions expand more rapidly

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[5]We shall demonstrate this explicitly below for all our examples.

than the base. In some cases the exponential expansion of some of the directions in the base is different from that of other directions. This has the consequence that the conformal geometry on the boundary is singular. If one were to use as conformal factor the scale-size of the fibres, then the metric on the base would tend to zero. If one used the metric on the smallest-growing base direction, then the metric on the fibres would diverge. In the case of a single scale-factor, the resulting metric is referred to by mathematicians as a Carnot-Carathéodory metric [38]. This behaviour has been commented on in Ref. [39] and [40] in the case of the four-dimensional Bergman metric on $SU(2,1)/(SU(2) \times U(1))$. In these references the metric was written in coordinates adapted to the maximal compact subgroup. At constant radius the metric is a squashed 3-sphere, where the ratio of lengths on the $U(1)$ fibres compared with the S^2 base diverges as one approaches infinity. In fact, as we shall illustrate below, one may also write the metric in Heisenberg-horospherical coordinates and obtain the same behaviour. In other words, the Bergman metric is of cohomogeneity one with respect to *both* $SU(2)$ and its contraction to the ur-Heisenberg group.

In order to get a solution of Type IIB theory in ten dimensions, one needs a five-dimensional rather than a four-dimensional metric. In what follows we shall present a new complete five-dimensional Einstein metric on a solvable group manifold, which may be used to obtain a Euclidean-signatured solution of Type IIB supergravity in ten dimensions, with a complex self-dual 5-form. (This theory is obtained by Wick rotation from the Lorentzian IIB theory, and the components of the 5-form with a time index are purely imaginary.) In general, these solutions need have no real Lorentzian sections. Although the five-dimensional metric may well have appeared before in general mathematical classification schemes, it and all of our metrics that are not symmetric spaces have not, as far as we are aware, been previously written down explicitly, nor have they been used in the construction of supergravity solutions.

Physically, the unusual behaviour of the boundary appears to be related to a mismatch in dimension between the boundary theory and the dimension of the bulk theory minus one. This behaviour presumably arises because, from the Kaluza-Klein point of view, a Heisenberg isometry gives rise to a background magnetic field. Systems in strong magnetic fields are well known to exhibit a reduction in dimensionality. The size of the toric fibre here is inversely proportional to the electric charge, and so as we go to infinity in our metrics the charge goes to zero, at constant magnetic field. Equivalently, the magnetic field goes to infinity, at fixed electric charge.

In some instances, we can give a more complete interpolation between a Ricci-flat Heisen-

berg metric and the associated "cosmological resolution;" these are constructed in section 7.3. Specifically, for the four-dimensional metric (5), and for the two six-dimensional metrics (49) and (56), we can obtain more general solutions with both a charge parameter and a cosmological constant. Indeed, in the case of (5) and (49) these "Heisenberg-de Sitter" metrics are themselves specialisations of already known metrics with cosmological constants. Thus the four-dimensional Heisenberg-de Sitter metric is a contraction limit of the Eguchi-Hanson-de Sitter metric [41, 42], and the six-dimensional Heisenberg-de Sitter generalisation of (49) is a contraction of a complete metric with cosmological constant on the complex line bundle over $\mathbb{CP}^2$. General results for such cosmological metrics on line bundles over Einstein-Kähler spaces were obtained in [43]. We expect that the Heisenberg-de Sitter generalisation of the six-dimensional metric (56) that we obtain in section 7.3 may similarly be a contraction limit of a "Stenzel-de Sitter" metric.

7.1 The Einstein metrics

We shall begin by listing the results Einstein metrics for all the cases. As remarked above, in the original Ricci-flat metrics the lengths of the Kaluza-Klein fibre directions go to zero at large y while the base space expands. By contrast, in the related Einstein metrics both the fibre and base-space directions expand exponentially as one goes to infinity. In fact the fibre directions now expand faster than the base. After each metric, we give its Ricci tensor, and also the algebra of exterior derivatives of the vielbein 1-forms. We choose the obvious basis, with $ds^2 = e^a \otimes e^a$, and $e^0 = dt$, etc.

$D = 4$; T^1 bundle over T^2:

$$
\begin{aligned}
ds_4^2 &= dt^2 + 4k^2\, e^{4k\,t}\,(dz_1 + z_3\, dz_2)^2 + e^{2k\,t}\,(dz_2^2 + dz_3^2)\,, &(136)\\
R_{ab} &= -6k^2\, g_{ab}\,,\\
de^0 &= 0\,, \quad de^1 = 2k\,(e^0 \wedge e^1 - e^2 \wedge e^3)\,, \quad de^2 = k\, e^0 \wedge e^2\,, \quad de^3 = k\, e^0 \wedge e^3\,.
\end{aligned}
$$

$D = 5$; T^1 bundle over T^3:

$$
\begin{aligned}
ds_4^2 &= dt^2 + 22k^2\, e^{8k\,t}\,(dz_1 + z_3\, dz_2)^2 + e^{4k\,t}\,(dz_2^2 + dz_3^2) + e^{6k\,t}\, dz_4^2\,, &(137)\\
R_{ab} &= -22k^2\, g_{ab}\,,\\
de^0 &= 0\,, \quad de^1 = 4k\,(e^0 \wedge e^1 - e^2 \wedge e^3)\,, \quad de^2 = 2k\, e^0 \wedge e^2\,, \quad de^3 = 2k\, e^0 \wedge e^2\,,\\
de^4 &= 3k\, e^0 \wedge e^4\,.
\end{aligned}
$$

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$D = 6$; T^1 bundle over T^4:

$$ds_6^2 = dt^2 + 4k^2 e^{4kt} (dz_1 + z_3\,dz_2 + z_5\,dz_4)^2 + e^{2kt} (dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2), \quad (138)$$

$$R_{ab} = -8k^2\,g_{ab},$$

$$de^0 = 0, \quad de^1 = 2k\,(e^0 \wedge e^1 - e^2 \wedge e^3 - e^4 \wedge e^5),$$

$$de^2 = k\,e^0 \wedge e^2, \quad de^3 = k\,e^0 \wedge e^3, \quad de^4 = k\,e^0 \wedge e^4, \quad de^5 = k\,e^0 \wedge e^5.$$

$D = 6$; T^2 bundle over T^3:

$$ds_6^2 = dt^2 + 36k^2 e^{10kt} [(dz_1 + z_4\,dz_3)^2 + (dz_2 + z_5\,dz_3)^2]$$
$$+ e^{4kt}\,dz_3^2 + e^{6kt} (dz_4^2 + dz_5^2), \quad (139)$$

$$R_{ab} = -18k^2\,g_{ab},$$

$$de^0 = 0, \quad de^1 = 5k\,(e^0 \wedge e^1 - e^3 \wedge e^4), \quad de^2 = 5k\,(e^0 \wedge e^2 - e^3 \wedge e^5),$$

$$de^3 = k\,e^0 \wedge e^3, \quad de^4 = k\,e^0 \wedge e^4, \quad de^5 = k\,e^0 \wedge e^5.$$

$D = 7$; T^2 bundle over T^4:

$$ds_7^2 = dt^2 + 4k^2 e^{4kt} [(dz_1 + z_4\,dz_3 + z_6\,dz_5)^2 + (dz_2 + z_5\,dz_3 - z_6\,dz_4)^2]$$
$$+ e^{2kt} (dz_3^2 + dz_4^2 + dz_5^2 + dz_6^2), \quad (140)$$

$$R_{ab} = -12k^2\,g_{ab},$$

$$de^0 = 0, \quad de^1 = 2k\,(e^0 \wedge e^1 - e^3 \wedge e^4 - e^5 \wedge e^6),$$

$$de^2 = 2k\,(e^0 \wedge e^2 - e^3 \wedge e^5 - e^6 \wedge e^4), \quad de^3 = k\,e^0 \wedge e^3, \quad de^4 = k\,e^0 \wedge e^4,$$

$$de^5 = k\,e^0 \wedge e^5, \quad de^6 = k\,e^0 \wedge e^6.$$

$D = 7$; T^3 bundle over T^3:

$$ds_7^2 = dt^2 + 6k^2 e^{4kt} [(dz_1 + z_6\,dz_5)^2 + (dz_2 + z_6\,dz_4)^2 + (dz_3 + z_5\,dz_4)^2]$$
$$+ e^{2kt} (dz_4^2 + dz_5^2 + dz_6^2), \quad (141)$$

$$R_{ab} = -15k^2\,g_{ab},$$

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$$de^0 = 0, \quad de^1 = 2k\,(e^0 \wedge e^1 - e^5 \wedge e^6), \quad de^2 = 2k\,(e^0 \wedge e^2 - e^4 \wedge e^6),$$

$$de^3 = 2k\,(e^0 \wedge e^3 - e^4 \wedge e^5), \quad de^4 = k\,e^0 \wedge e^4,$$

$$de^5 = k\,e^0 \wedge e^5, \quad de^6 = k\,e^0 \wedge e^6.$$

$D = 8$; T^3 bundle over T^4:

$$
\begin{aligned}
ds_8^2 &= dt^2 + 4k^2\,e^{4kt}\,[(dz_1 + z_5\,dz_4 + z_7\,dz_6)^2 + (dz_2 + z_6\,dz_4 - z_7\,dz_5)^2 \\
&\qquad + (dz_3 + z_7\,dz_4 + z_6\,dz_5)^2] \\
&\quad + e^{2kt}\,(dz_4^2 + dz_5^2 + dz_6^2 + dz_7^2)\,, \\
R_{ab} &= -16k^2\,g_{ab}\,, \\
de^0 &= 0, \quad de^1 = 2k\,(e^0 \wedge e^1 - e^4 \wedge e^5 - e^6 \wedge e^7), \\
de^2 &= 2k\,(e^0 \wedge e^2 - e^4 \wedge e^6 + e^5 \wedge e^7), \quad de^3 = 2k\,(e^0 \wedge e^3 - e^4 \wedge e^7 - e^5 \wedge e^6), \\
de^4 &= k\,e^0 \wedge e^4, \quad de^5 = k\,e^0 \wedge e^5, \quad de^6 = k\,e^0 \wedge e^6, \quad de^7 = k\,e^0 \wedge e^7\,.
\end{aligned}
\tag{142}
$$

Note that the algebras of exterior derivatives are all of the form $de^a = -\frac{1}{2}c^a{}_{bc}\,e^b \wedge e^c$, where the $c^a{}_{bc}$ are constants. These are in fact the structure constants of the corresponding solvable groups. Observe that these are indeed not traceless, $c^a{}_{ba} \neq 0$, as is required by Dotti's theorem.

In the next subsection, we shall discuss these solvable groups as coset spaces, exploiting the Iwasawa decomposition.

7.2 Coset constructions

7.2.1 $SU(n,1)/U(n) = \widetilde{\mathbb{CP}^n} = \mathrm{H}^n_{\mathbb{C}}$

Those examples above whose principal orbits are of the form of T^1 bundles over T^p are in fact Bergman metrics on the non-compact forms of $\widetilde{\mathbb{CP}^n}$, with $p = 2n - 2$. These are nothing but the Fubini-Study metrics with the opposite sign for the cosmological constant. They are obtained by starting from coordinates Z^A on $\mathbb{C}^{n+1}$, with the constraint

$$\eta_{AB}\,Z^A\,\bar{Z}^B = -1\,, \tag{143}$$

where η_{AB} is diagonal with $\eta_{00} = -1$, $\eta_{ab} = 1$, where $1 \leq a \leq n$. The Hopf fibration of this AdS_{2n+1} by $U(1)$ (taken to be timelike) then gives the Bergman metric.

We can express the Bergman metric in a "horospherical" form [44], by introducing real coordinates $(\tau, \phi, \chi, x_i, y_i)$, in terms of which we parametrize the Z^A that satisfy (143) as

$$
\begin{aligned}
Z^0 &= e^{\frac{i}{2}\tau}\left(\cosh\tfrac{1}{2}\phi + \tfrac{1}{8}e^{\frac{1}{2}\phi}(4\mathrm{i}\,\chi + x_i^2 + y_i^2)\right), \\
Z^n &= e^{\frac{i}{2}\tau}\left(\sinh\tfrac{1}{2}\phi - \tfrac{1}{8}e^{\frac{1}{2}\phi}(4\mathrm{i}\,\chi + x_i^2 + y_i^2)\right), \\
Z^i &= \tfrac{1}{2}e^{\frac{i}{2}\tau + \frac{1}{2}\phi}(x_i + \mathrm{i}\,y_i),
\end{aligned}
\tag{144}
$$

where $1 \le i \le n-1$. Substituting into the metric $d\hat{s}^2 = \eta_{AB}\,dZ^A\,d\bar{Z}^B$ on AdS_{2n+1}, we find

$$
d\hat{s}^2 = -\tfrac{1}{4}\left(d\tau + e^{\phi}\,[d\chi + \tfrac{1}{2}(y_i\,dx_i - x_i\,dy_i)]\right)^2 + d\Sigma_{2n}^2,
\tag{145}
$$

where

$$
d\Sigma_{2n}^2 = \tfrac{1}{4}d\phi^2 + \tfrac{1}{4}e^{\phi}(dx_i^2 + dy_i^2) + \tfrac{1}{4}e^{2\phi}[d\chi + \tfrac{1}{2}(y_i\,dx_i - x_i\,dy_i)]^2.
\tag{146}
$$

Thus if we fibre AdS_{2n+1} by the $U(1)$ whose coordinate τ is the time parameter, we obtain the Bergman metric (146) on $\widetilde{\mathbb{CP}^n}$. Comparing with (136) and (138), we see that these correspond to $\widetilde{\mathbb{CP}^2}$ and $\widetilde{\mathbb{CP}^3}$ respectively.

If $n = 3$, the denominator group $U(3)$ contains the linearly-realised $U(2)$ noted in section 4.1.1, and similarly if $n = 2$ the denominator group $U(2)$ contains the linearly-realised $U(1)$ noted in section 2.1.

In the case $n = 2$ one can regard this solution as a special case of the Taub-NUT-de Sitter metrics, which have been applied to the AdS/CFT correspondence in [45]. (For more general higher-dimensional metrics of this type, see [46].) The case $n = 2$ is of further interest because, while the Bergman metric is not conformal to the associated Ricci-flat metric[6], there is a metric which is conformal to our Ricci-flat metric that is distinguished by the property that it is essentially the only non-trivial complete homogeneous hyper-Hermitean metric [47]. The conformally related metric is

$$
ds^2 + z^{-4}(d\tau + xdy)^2 + z^{-2}(dx^2 + dy^2 + dz^2)
\tag{147}
$$

It would be interesting to know whether our other Ricci-flat metrics are conformal to similarly-distinguished metrics.

7.2.2 $Sp(n,1)/(Sp(n)\cdot Sp(1)) = \widetilde{\mathbb{HP}^n} = \mathrm{H}_{\mathbb{H}}^n$

A similar construction can be given for the non-compact versions of the quaternionic projective spaces, $\widetilde{\mathbb{HP}^n}$. Now, we start from $n+1$ quaternionic coordinates Q^A, subject to the

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[6]It is impossible for two Riemannian Einstein metrics to be conformal with non-constant conformal factor.

constraint

$$\eta_{AB} \, Q^A \, \bar{Q}^B = -1 \,, \tag{148}$$

where again η_{AB} is diagonal with $\eta_{00} = -1$, $\eta_{ab} = 1$, where $1 \le a \le n$. This restricts us to a spacetime of anti-de Sitter type, except that now we have three timelike coordinates.

We can again introduce real horospherical coordinates. The three times appear as the Euler angles of $SU(2)$, and in fact we can just introduce them implicitly via the $Sp(1) = SU(2)$ quaternionic U. In addition, we introduce real coordinates $(\phi, \chi_\alpha, x_i, y_i^\alpha)$, where $1 \le \alpha \le 3$ and $1 \le i \le n-1$. We shall denote the imaginary unit quaternions by $\iota_\alpha = (\mathrm{i}, \mathrm{j}, \mathrm{k})$ (in terms of which U can be written as $U = e^{\frac{\mathrm{k}}{2} t_1} e^{\frac{\mathrm{i}}{2} t_2} e^{\frac{\mathrm{k}}{2} t_3}$). We then parametrize the quaternions Q^A that satisfy (148) as

$$\begin{aligned}
Q^0 &= U \left(\cosh \tfrac{1}{2}\phi + \tfrac{1}{8} e^{\frac{1}{2}\phi} (4\iota_\alpha \chi_\alpha + x_i^2 + (y_i^\alpha)^2) \right), \\
Q^n &= U \left(\sinh \tfrac{1}{2}\phi - \tfrac{1}{8} e^{\frac{1}{2}\phi} (4\iota_\alpha \chi_\alpha + x_i^2 + (y_i^\alpha)^2) \right), \\
Q^i &= \tfrac{1}{2} U \, e^{\frac{1}{2}\phi} (x_i + \iota_\alpha y_i^\alpha).
\end{aligned} \tag{149}$$

These are closely analogous to (144) for the complex case. Substituting into the metric $d\hat{s}^2 = \eta_{AB} \, dQ^A \, d\bar{Q}^B$ on the three-timing AdS$_{4n+3}$, we find

$$d\hat{s}^2 = - \left| U^{-1} \, dU + e^\phi \, \iota_\alpha \, [d\chi_\alpha + \tfrac{1}{2}(x_i \, dy_i^\alpha - y_i^\alpha \, dx_i)] \right|^2 + d\Xi_{4n}^2 \,, \tag{150}$$

where

$$d\Xi_{4n}^2 = \tfrac{1}{4} d\phi^2 + \tfrac{1}{4} e^\phi \, (dx_i^2 + (dy_i^\alpha)^2) + \tfrac{1}{4} \, e^{2\phi} \, [d\chi_\alpha + \tfrac{1}{2}(x_i \, dy_i^\alpha - y_i^\alpha \, dx_i)]^2 \,. \tag{151}$$

Thus if we project orthogonally to the $SU(2)$ timelike fibres, we obtain (151) as the metric on $\widetilde{\mathbb{HP}^n}$. It is the coset $Sp(n,1)/(Sp(n) \cdot Sp(1))$.

Comparing with (142), we see that our 8-dimensional Einstein metric is precisely the non-compact "quaternionic Bergman metric" on $\widetilde{\mathbb{HP}^2}$, which is the coset $Sp(2,1)/(Sp(2) \cdot Sp(1))$. In this case the denominator group contains the linearly-realised $Sp(1) \equiv SU(2)$ noted in section 4.4.1.

In all cases, one may check that the exponential expansion of the $SU(2)$ fibres is more rapid than that of the $\widetilde{\mathbb{HP}^n}$ base, as one goes to infinity.

7.3   Heisenberg-de Sitter metrics

So far, we have considered Ricci-flat Heisenberg metrics, and also "cosmological resolutions" that do not have an immediate mathematical relation to the Heisenberg metrics. In certain

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cases, at least, we can find a more general solution that encompasses both the Heisenberg metric and the cosmological metric, as certain limits. In fact, at least in some of these examples, we know that there exist "de Sitterised" versions of the complete non-singular Ricci-flat metrics, even before the Heisenberg limit is taken.

A case in point is the Eguchi-Hanson-de Sitter metric, given by

$$ds_4^2 = F^{-1} dt^2 + \tfrac{1}{4} r^2 F \sigma_3^2 + \tfrac{1}{4} r^2 (\sigma_1^2 + \sigma_2^2) \,, \tag{152}$$

where

$$F = 1 + \frac{Q}{r^4} - \tfrac{1}{6} \Lambda r^2 \,. \tag{153}$$

It is an Einstein metric, with $R_{ab} = \Lambda g_{ab}$, and so Λ is the cosmological constant.

If we now take the Heisenberg limit, as in section 2.2, we obtain the Heisenberg-de Sitter metric

$$ds_4^2 = F^{-1} dt^2 + \tfrac{1}{4} r^2 F (dz_1 + z_3 dz_2)^2 + \tfrac{1}{4} r^2 (dz_2^2 + dz_3^2) \,, \tag{154}$$

where we are dropping the tildes used to denote the rescaled quantities in section 2.2, and now we have

$$F = \frac{Q}{r^4} - \tfrac{1}{6} \Lambda r^2 \,. \tag{155}$$

Note that Λ does not suffer any rescaling in the taking of this limit, so the cosmological constant is still Λ.

Having obtained the Heisenberg de-Sitter 4-metric, we can note that if we set Λ to zero, we recover (after an obvious coordinate transformation) the Heisenberg metric (5). On the other hand, if we set Q to zero, then after an obvious coordinate transformation, (154) gives the cosmological resolution (136). If we keep both Q and Λ non-vanishing, we have a more general Einstein metric that encompasses both the Heisenberg and cosmological metrics discussed previously.

We can now attempt a generalisation of the above to other cases. Analogues of the Eguchi-Hanson-de Sitter metric are known for all Ricci-flat metrics on complex line bundles over Einstein-Kähler bases [43]. Thus we can expect to be able to get generalised Heisenberg-de Sitter metrics for all the cases where the principal orbits are T^1 bundles over T^{2n}. For example, in $D = 6$ we can get a Heisenberg-de Sitter metric for the case of T^1 bundle over T^4, namely

$$ds_6^2 = F^{-1} dr^2 + r^2 F [dz_1 + m (z_3 dz_2 + z_5 dz_4)]^2 + \tfrac{1}{2} r^2 (dz_2^2 + \cdots + dz_5^2) \,, \tag{156}$$

where

$$F = \frac{Q}{r^6} - \tfrac{1}{8} \Lambda r^2 \,. \tag{157}$$

This is equivalent to (49) if Λ is set to zero. On the other hand, if Q is instead set to zero, it is equivalent to the cosmological resolution metric (138).

A slightly more complicated example is the second of the two $D = 6$ Heisenberg metrics, given in (56). Here, we find that the following is an Einstein metric, with cosmological constant Λ:

$$
\begin{aligned}
ds_6^2 &= F^{-1} dr^2 + r^2 F^{2/3} [(dz_1 + m\, z_4\, dz_3)^2 + (dz_2 + m\, z_5\, dz_3)^2] \\
&\quad + r^2 F^{-1/3} dz_3^2 + \tfrac{1}{2} r^2 (dz_4^2 + dz_5^2) ,
\end{aligned}
\tag{158}
$$

where

$$
F = \frac{Q}{r^6} - \tfrac{1}{8} \Lambda\, r^2 .
\tag{159}
$$

This again has the appropriate limits, yielding (56) if Λ is set to zero, and yielding (139) if instead Q is set to zero.

Having obtained these Heisenberg-de Sitter metrics, we can observe that they provide a way to "cap off" the large-radius portions of the Ricci-flat Heisenberg metrics that are transverse to the domain-wall spacetimes. In this respect, they appear to be conjugate to the constructions of [11], which by contrast resolve the curvature singularities in the small-radius portions of the domain-wall spacetimes.

Let us illustrate this by considering the example of the 4-metric (5), transverse to the domain wall in $D = 8$ supergravity. If we take the Heisenberg-de Sitter metric (154), with both Q and the cosmological constant Λ in (155) positive, we see that the metric running from the singularity at $r = 0$ reaches a natural endpoint at $r = r_0$, where the function F vanishes, i.e. at

$$
r_0^6 = \frac{6Q}{\Lambda} .
\tag{160}
$$

To study the behaviour of the metric near $r = r_0$, we introduce a new coordinate ρ defined by $r = r_0 - \rho^2$. In terms of this, the metric near $r = r_0$ takes the form

$$
ds_4^2 \sim \frac{2r_0^5}{3Q} \left(d\rho^2 + \frac{r_0^2}{16} \left(\frac{6Q}{r_0^5} \right) \rho^2 (dz_1 + z_3\, dz_2)^2 \right) + \tfrac{1}{4} r_0^2 (dz_2^2 + dz_3^2) .
\tag{161}
$$

This will be regular at $r = r_0$ provided that z_1 has a period given by

$$
\Delta z_1 = \frac{4\pi\, r_0^4}{3Q} .
\tag{162}
$$

8 Conclusions

As mentioned in the introduction, one of the motivations for the present study was the possibility of resolving some of the BPS domain walls along the lines of the "single-sided"

domain-wall construction described in [11]. The case studied in detail in that reference was four-dimensional, and the resolution was a certain non-compact degeneration of a K3 surface, but generalisations to higher dimensions were also indicated there which would correspond, for example, to non-compact degenerations of Calabi-Yau complex 3-folds and 4-folds. For example, the metric based on a circle bundle over a K3 surface seems to be related to the metric one would obtain by solving the Monge-Ampére equation on the complement of a quartic surface in $\mathbb{CP}^3$. In the case of Calabi-Yau metrics, one has various proofs showing the existence of smooth resolved metrics, but these give very little detailed information about the explicit forms of the metrics. The information one gets is mainly about the asymptotic form of the metric near infinity. This is where our work may help identify the resolutions. In the case of K3 surfaces, as well as asymptotically "nil-manifolds" based on generalised Heisenberg groups, one also encounters asymptotically "solv-manifolds," based on solvable groups. It may be that our work in this paper and generalisations of it may be relevant to higher-dimensional Calabi-Yau spaces. A more challenging problem would be to relate our metrics to compact Calabi-Yau manifolds. For reasons explained in the introduction, it is not easy to see how to do this using cohomogeneity one Ricci-flat metrics. One may, of course, give qualitative discussions [48] but it is extremely hard to make quantitative progress with present-day techniques.

In the case of metrics with exceptional holonomy the situation is much less clear than in the case of Calabi-Yau metrics, because existence theorems have been studied to a much lesser extent. However, the important work of Joyce described in his recent book [49] encourages us to believe that our results will prove applicable in that case as well.

Acknowledgements

A   The Raychaudhuri equation

Let T^α be a unit vector tangent to a congruence of curves,

$$T^\alpha T_\alpha = 1 \Rightarrow T_{\alpha;\beta} \, T^\alpha = 0 \ . \tag{163}$$

Let us decompose the covariant derivative of T_α perpendicular and parallel to T_α using the projection operator $h_{\alpha\beta} = g_{\alpha\beta} - T_\alpha T_\beta$, such that $h_{\alpha\beta} T^\beta = 0$. One has the decomposition

$$T_{\alpha;\beta} = \Theta_{\alpha\beta} + \omega_{\alpha\beta} + T_\beta \, a_\alpha \ , \tag{164}$$

where $\Theta_{\alpha\beta} = \Theta_{\beta\alpha}$ is a symmetric expansion tensor and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ is an antisymmetric vorticity tensor, both of which are orthogonal to T_α,

$$\Theta_{\alpha\beta} T^\beta = \omega_{\alpha\beta} T^\beta = 0. \tag{165}$$

The vector $a_\alpha = T_{\alpha;\beta} \, T^\beta$ is the *acceleration* vector, and would vanish if the congruence is geodesic.

Since $T_\alpha \, a^\alpha = 0$, we have

$$T^\alpha_{;\alpha} = \Theta_{\alpha\beta} \, h^{\alpha\beta} = \Theta_{\alpha\beta} \, g^{\alpha\beta}. \tag{166}$$

This is the *expansion.* We may then set

$$\Theta_{\alpha\beta} = \frac{\Theta h_{\alpha\beta}}{d} + \Sigma_{\alpha\beta} \ , \tag{167}$$

where $\Sigma_{\alpha\beta}$ is the *shear*, satisfying $\Sigma_{\alpha\beta} h^{\alpha\beta} = \Sigma_{\alpha\beta} g^{\alpha\beta} = 0$. Thus we have

$$T_{\alpha;\beta} T^{\beta;\alpha} = \Theta_{\alpha\beta} \Theta^{\alpha\beta} - \omega_{\alpha\beta} \omega^{\alpha\beta} \ . \tag{168}$$

Note that only the vorticity term is negative.

From the Ricci identity, one now has

$$T^\alpha_{;\mu;\nu} - T^\alpha_{;\nu;\mu} = -R^\alpha{}_{\beta\mu\nu} \, T^\beta \ , \tag{169}$$

so

$$\frac{d\Theta}{dt} = (T^\alpha_{;\beta} T^\beta)_{;\alpha} - R_{\alpha\beta} \, T^\alpha T^\beta - T_{\alpha;\beta} \, T^{\beta;\alpha} \tag{170}$$

and hence

$$\frac{d\Theta}{dt} = a^\alpha_{;\alpha} - R_{\alpha\beta} \, T^\alpha \, T^\beta - \Theta_{\alpha\beta} \, \Theta^{\alpha\beta} + \omega_{\alpha\beta} \, \omega^{\alpha\beta} \ . \tag{171}$$

The assumption that the congruence is geodesic implies $a_\alpha = 0$. The assumption that it is hypersurface orthogonal implies $T_{[\alpha;\beta}T_{\nu]} = 0 \Rightarrow \omega_{[\alpha\beta}T_{\nu]} = 0 \Rightarrow \omega_{\alpha\beta} = 0$. Thus,

$$\frac{d\Theta}{dt} = -\Theta_{\alpha\beta}\,\Theta^{\alpha\beta} - R_{\alpha\beta}\,T^\alpha T^\beta \ , \tag{172}$$

and so

$$\frac{d\Theta}{dt} = -\frac{1}{d}\,\Theta^2 - 2\Sigma_{\alpha\beta}\,\Sigma^{\alpha\beta} - R_{\alpha\beta}\,T^\alpha\,T^\beta \ . \tag{173}$$

Finally, we set $R_{\alpha\beta} = 0$, and find $\frac{d\Theta}{dt} < 0$.

B   Generalised Heisenberg Groups

As we discussed in the introduction, we are interested in the relation between the holonomy spaces with non-abelian isometry groups and those with nilpotent Heisenberg groups. In this section, we describe Heisenberg groups as contractions of semi-simple groups.

B.1   Definition

We may define a generalised Heisenberg group as a (nilpotent) central extension of an abelian group, with Lie algebra generated by $\mathbf{e}_\alpha$ and $\mathbf{q}_m$, with the commutation relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = F^m_{\alpha\beta}\,\mathbf{q}_m\,, \qquad [\mathbf{e}_a, \mathbf{q}_m] = 0\,, \qquad [\mathbf{q}_m, \mathbf{q}_n] = 0\,. \tag{174}$$

We shall suppose the original abelian algebra to be q-dimensional, and the centre to be p-dimensional. Thus $1 \le \alpha \le q$ and $1 \le m \le p$. An appropriate left-invariant basis of 1-forms is (dx^α, ν^m), where

$$\nu^m = dy^m - \tfrac{1}{2}F^m_{\alpha\beta}\,x^\alpha\,dx^\beta\,, \tag{175}$$

and we have

$$d\nu^m = -\tfrac{1}{2}F^m_{\alpha\beta}\,dx^\alpha \wedge dx^\beta\,. \tag{176}$$

The right-invariant Killing vectors R_α and R_m, which generate left translations, are given by

$$R_\alpha = \partial_\alpha + \tfrac{1}{2}F_{\alpha\beta}\,x^\beta\,\partial_m\,, \qquad R_m = \partial_m\,. \tag{177}$$

These satisfy

$$[R_\alpha, R_\beta] = -F^m_{\alpha\beta}\,R_m\,, \quad [R_\alpha, R_m] = 0\,, \quad]R_m, R_n] = 0\,. \tag{178}$$

There is an obvious Kaluza-Klein interpretation for the central coordinates y^m. The quantities $F^m_{\alpha\beta}$ are just q constant $U(1)$ field strengths defined over $\mathbb{E}^p$, and (175) can be

viewed as the quantity $\nu^m = dy^m - A^m$, where $A_\alpha^m = \frac{1}{2}F_{\alpha\beta}\,x^\beta$ is the Kaluza-Klein vector potential for $F_{\alpha\beta}^m$.

The Baker-Campbell-Hausdorff formula gives

$$e^{a\cdot\mathbf{e}}\,e^{b\cdot\mathbf{e}}\,e^{-a\cdot\mathbf{e}}\,e^{-b\cdot\mathbf{e}} = e^{\mathbf{q}_m\,F^m(a,b)}\,, \tag{179}$$

where we have defined $a\cdot\mathbf{e} \equiv a^\alpha\,\mathbf{e}_\alpha$ and $F^m(a,b) \equiv F_{\alpha\beta}^m\,a^\alpha\,b^\beta$. In practice, we want to consider the case where the original abelian algebra is a torus T^q, whose coordinates x^α therefore live on a lattice. Because of (179), the coordinates y^m associated to the centre must also be identified consistently. Suppose that x^α and $x^\alpha + a^\alpha$ are to be identified, and that x^α and $x^\alpha + b^\alpha$ are to be identified. The associated group elements $e^{a\cdot\mathbf{e}}$ and $e^{b\cdot\mathbf{e}}$ can taken to be the identity only if the group element on the right-hand side of (179) is also the identity. This means that $F^m(a,b)$ must be an integer multiple of one of the periods of the coordinates y^m, for all lattice vectors a^α and b^α. This places conditions on the $F_{\alpha\beta}^m$. The case we are mainly interested in is when the resulting group may be thought of as a T^p bundle over T^q. The simplest example is when $p = 1$ and the consistency conditions reduce to the Dirac quantisation conditions for a $U(1)$ bundle over T^q. Thus in this case $F(a,b)$ is the magnetic flux through the cycle spanned by a^α and b^α.

B.2   Heisenberg Groups as Contractions

Heisenberg algebras may arise as contractions of semi-simple algebras. The simplest example is the Inönü-Wigner contraction of $SO(3)$ to the *ur* Heisenberg algebra. The former is

$$[\mathbf{L}_3, \mathbf{L}_\pm] = \pm\mathbf{L}_\pm\,, \qquad [\mathbf{L}_+, \mathbf{L}_-] = 2\mathbf{L}_3\,. \tag{180}$$

Writing $\mathbf{e}_1 = \lambda\,\mathbf{L}_1$, $\mathbf{e}_2 = \lambda\,\mathbf{L}_2$ and $\mathbf{q} = \lambda^2\,\mathbf{L}_3$, and taking the limit where the constant λ goes to zero, we obtain the latter:

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{q}\,, \qquad [\mathbf{e}_1, \mathbf{q}] = [\mathbf{e}_2, \mathbf{q}] = [\mathbf{q}, \mathbf{q}] = 0\,. \tag{181}$$

From the Kaluza-Klein point of view, $SO(3)$ is the Dirac T^1 bundle over S^2. Contraction gives a magnetic field over $\mathbb{E}^2$, which if we identify to make a torus, gives an T^1 bundle over T^2.

Since all the AC manifolds we are considering here have isometry groups of the type $SO(n)$ or $SU(n)$, we shall consider the contractions of these groups.

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B.2.1 Contractions of $SO(n+2)$

This example extends straightforwardly to the case of $SO(n + 2)$. It is convenient to work with the left-invariant basis of 1-forms, which we shall denote by L_{AB}. These satisfy $dL_{AB} = L_{AC} \wedge L_{CB}$. Splitting the index as $A = (1, 2, i)$, and defining

$$L_{ij} = M_{ij}, \qquad L_{1i} = \sigma_i, \qquad L_{2i} = \tilde{\sigma}_i, \qquad L_{12} = \nu, \tag{182}$$

then after the scalings

$$M_{ij} \longrightarrow \lambda\, M_{ij}, \qquad \sigma_i \longrightarrow \lambda\, \sigma_i, \qquad \tilde{\sigma}_i \longrightarrow \lambda\, \tilde{\sigma}_i, \qquad \nu \longrightarrow \lambda^2\, \nu \tag{183}$$

we have

$$d\sigma_i = \lambda^2\, \nu \wedge \tilde{\sigma}_i + \lambda\, M_{ij} \wedge \sigma_j, \quad d\tilde{\sigma}_i = -\lambda^2\, \nu \wedge \sigma_i + \lambda\, M_{ij} \wedge \tilde{\sigma}_j, \quad d\nu = -\sigma_i \wedge \tilde{\sigma}_i,$$

$$dM_{ij} = \lambda\, M_{ik} \wedge M_{kj} - \lambda\, \sigma_i \wedge \sigma_j - \lambda\, \tilde{\sigma}_i \wedge \tilde{\sigma}_j. \tag{184}$$

Taking the limit where λ goes to zero, we obtain the generalised Heisenberg algebra with

$$d\sigma_i = d\tilde{\sigma}_i = 0, \qquad d\nu = -\sigma_i \wedge \tilde{\sigma}_i, \qquad dM_{ij} = 0. \tag{185}$$

The $\frac{1}{2}(n+1)(n+2)$-dimensional $so(n+2)$ simple algebra has decomposed in the $\lambda \longrightarrow 0$ limit as the direct sum of a $(2n + 1)$-dimensional generalised Heisenberg algebra, spanned by $(\sigma_i, \tilde{\sigma}_i, \nu)$, and a $\frac{1}{2}n(n-1)$-dimensional completely abelian piece, spanned by the M_{ij}. It is consistent to identify the Heisenberg group in such as way as to obtain a T^1 bundle over T^{2n}. One has $2n$ coordinates x^α on the base, whose differentials dx^α give σ_i and $\tilde{\sigma}_i$. The real coordinates can be grouped into n complex coordinates z^i, with

$$z^i = x^i + i\, x^{i+n}, \qquad dz^i = \sigma_i + i\, \tilde{\sigma}_i. \tag{186}$$

The field strength $F_{\alpha\beta}$ is proportional to the standard Kähler form on $\mathbb{C}^n$.

A different contraction of $so(n + 2)$ may be obtained by instead applying the scalings

$$M_{ij} \longrightarrow \lambda\, M_{ij}, \qquad \sigma_i \longrightarrow \lambda^2\, \sigma_i, \qquad \tilde{\sigma}_i \longrightarrow \lambda\, \tilde{\sigma}_i, \qquad \nu \longrightarrow \lambda\, \nu \tag{187}$$

After sending λ to zero, we obtain

$$d\sigma_i = \nu \wedge \tilde{\sigma}_i, \qquad d\tilde{\sigma}_i = 0, \qquad d\nu = 0, \qquad dM_{ij} = 0. \tag{188}$$

The $\frac{1}{2}(n + 1)(n + 2)$-dimensional $so(n + 2)$ simple algebra has again decomposed in the $\lambda \longrightarrow 0$ limit as the direct sum of a $(2n + 1)$-dimensional generalised Heisenberg algebra, spanned by $(\sigma_i, \tilde{\sigma}_i, \nu)$, and a $\frac{1}{2}n(n-1)$-dimensional completely abelian piece, spanned by the M_{ij}. However now, it is consistent to identify the Heisenberg group in such as way as to obtain a T^n bundle over T^{n+1}. One has $(n + 1)$ coordinates x^α on the base manifold T^{n+1} whose differentials give $\tilde{\sigma}_i$ and ν. The field strengths $F^i_{\alpha\beta}$ are now all simple; $F^i = \nu \wedge \tilde{\sigma}_i$.

B.2.2 Contraction of $SO(4)$

The $so(4)$ algebra is the direct sum of two $so(3)$ algebras:

$$d\Sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\,\Sigma_j \wedge \Sigma_k\,, \qquad d\widetilde{\Sigma}_i = -\tfrac{1}{2}\epsilon_{ijk}\,\widetilde{\Sigma}_j \wedge \widetilde{\Sigma}_k\,. \tag{189}$$

Let

$$\nu_1 = \widetilde{\Sigma}_1 - \tfrac{1}{2}\Sigma_1\,, \qquad \nu_2 = \widetilde{\Sigma}_2 - \tfrac{1}{2}\Sigma_2\,, \qquad \nu_3 = \widetilde{\Sigma}_3 - \tfrac{1}{2}\Sigma_3\,,$$

$$\sigma_1 = \Sigma_1\,, \qquad \sigma_2 = \Sigma_2\,, \qquad \sigma_3 = \Sigma_3\,. \tag{190}$$

These give

$$d\nu_1 = -\nu_2 \wedge \nu_3 - \tfrac{1}{2}\nu_2 \wedge \Sigma_3 + \tfrac{1}{2}\nu_3 \wedge \Sigma_2 + \tfrac{1}{4}\Sigma_2 \wedge \Sigma_3\,, \quad \text{and cyclic on } \{123\}\,,$$

$$d\sigma_1 = -\sigma_2 \wedge \sigma_3\,, \qquad \text{and cyclic}\,. \tag{191}$$

We now implement the constant rescalings

$$(\nu_1, \nu_2, \nu_3) \longrightarrow \lambda^2\,(\nu_1, \nu_2, \nu_3)\,, \qquad (\sigma_1, \sigma_2, \sigma_3) \longrightarrow \lambda\,(\sigma_1, \sigma_2, \sigma_3)\,. \tag{192}$$

Now, we find that (191) becomes

$$d\nu_1 = -\lambda^2\,\nu_2 \wedge \nu_3 - \tfrac{1}{2}\lambda\,\nu_2 \wedge \sigma_3 + \tfrac{1}{2}\lambda\,\nu_3 \wedge \sigma_2 + \tfrac{1}{4}\sigma_2 \wedge \sigma_3\,, \qquad \text{and cyclic}\,,$$

$$d\sigma_1 = -\lambda\,\sigma_2 \wedge \sigma_3\,, \qquad \text{and cyclic}\,. \tag{193}$$

The Heisenberg limit now corresponds to sending λ to zero, implying

$$d\nu_1 = \tfrac{1}{4}\sigma_2 \wedge \sigma_3\,, \qquad d\nu_2 = \tfrac{1}{4}\sigma_3 \wedge \sigma_1\,, \qquad d\nu_3 = \tfrac{1}{4}\sigma_1 \wedge \sigma_2\,,$$

$$d\sigma_1 = 0\,, \qquad d\sigma_2 = 0\,, \qquad d\sigma_3 = 0\,. \tag{194}$$

In this limit, we may introduce coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) such that

$$\nu_1 = dy_1 - \tfrac{1}{4}x_3\,dx_2\,, \qquad \nu_2 = dy_2 - \tfrac{1}{4}x_1\,dx_3\,, \qquad \nu_3 = dy_3 - \tfrac{1}{4}x_2\,dx_1\,,$$

$$\sigma_1 = dx_1\,, \qquad \sigma_2 = dx_2\,, \qquad \sigma_3 = dx_3\,, \tag{195}$$

It is consistent to make identifications to give a T^3 bundle over T^3. The three field strengths F^i are given by

$$F^i_{jk} = -\tfrac{1}{2}\epsilon_{ijk}\,. \tag{196}$$

B.2.3 Contraction of $SO(5)$

Our next example is for $SO(5)$. Splitting the $SO(5)$ index as $A = (\alpha, 4)$, and then splitting $\alpha = (0, i)$ with $i = 1, 2, 3$, we define

$$L_{\alpha 4} \equiv P_\alpha = \lambda^2 \, \sigma_\alpha \,, \quad L_{0i} + \tfrac{1}{2}\epsilon_{ijk} L_{jk} \equiv R_i = \lambda^4 \, \nu^i \,, \quad L_{0i} - \tfrac{1}{2}\epsilon_{ijk} L_{jk} \equiv L_i = \lambda^3 \, J^i \,, \quad (197)$$

we obtain, after sending λ to zero,

$$d\nu^i = -\sigma_0 \wedge \sigma_i - \tfrac{1}{2}\epsilon_{ijk} \, \sigma_j \wedge \sigma_k \,, \qquad d\sigma_\alpha = 0 \,, \qquad dJ_i = 0 \,. \qquad (198)$$

The ten-dimensional $so(5)$ algebra has thus been decomposed as a direct sum of a seven-dimensional generalised Heisenberg algebra spanned by σ_α and ν^i, and a three-dimensional completely abelian piece spanned by the J^i. It is now consistent to identify the Heisenberg group in such as way as to obtain a T^3 bundle over T^4. There are four coordinates x^α on the base whose differentials give the σ_α. The three field strengths $F^i_{\alpha\beta}$ give three self-dual 2-forms on T^4, which endow it with a hyper-Kähler structure. The left-invariant 1-forms can be written as

$$\sigma_0 = dx_0 \,, \quad \sigma_1 = dx_1 \,, \quad \sigma_2 = dx_2 \,, \quad \sigma_3 = dx_3 \,, \qquad (199)$$

$$\nu_1 = dy_1 - x_0 \, dx_1 - x_2 \, dx_3 \,, \quad \nu_2 = dy_2 - x_0 \, dx_2 - x_3 \, dx_1 \,, \quad \nu_3 = dy_3 - x_0 \, dx_3 - x_1 \, dx_2 \,.$$

There is in fact a different contraction of $so(5)$, in which the J^i act non-trivially. This is achieved by using the scalings

$$L_{\alpha 5} = \lambda^2 \, \sigma_\alpha \,, \qquad L_{0i} + \tfrac{1}{2}\epsilon_{ijk} L_{jk} = \lambda^4 \, \nu^i \,, \qquad L_{0i} - \tfrac{1}{2}\epsilon_{ijk} L_{jk} = J^i \,, \qquad (200)$$

After sending λ to zero, we now find

$$d\nu^i = -\sigma_0 \wedge \sigma_i - \tfrac{1}{2}\epsilon_{ijk} \, \sigma_j \wedge \sigma_k \,,$$
$$dJ^i = \tfrac{1}{2}\epsilon_{ijk} \, J^j \wedge J^k \,,$$
$$d\sigma_0 = \tfrac{1}{2} J^i \wedge \sigma_i \,, \qquad d\sigma_i = -\tfrac{1}{2} J^i \wedge \sigma_0 + \tfrac{1}{2}\epsilon_{ijk} \, J^j \wedge \sigma_k \,. \qquad (201)$$

This algebra is the semi-direct sum of the previously-obtained generalised Heisenberg algebra with $so(3)$, spanned by the J^i.

A further contraction of $SO(5)$ is possible, leading to a seven-dimensional Heisenberg algebra in which is the direct sum of a six-dimensional Heisenberg algebra and a one-dimensional summand. We obtain this by implementing a further singular scaling of ν_3.

Equivalently, we can obtain the algebra directly as a limit of the $so(5)$ algebra, by making the rescalings

$$P_\alpha = \lambda^2 \, \sigma_\alpha \,, \quad R_1 = \lambda^4 \, \nu_1 \,, \quad R_2 = \lambda^4 \, \nu_2 \,, \quad R_3 = \lambda^3 \, \nu_3 \,. \tag{202}$$

After sending λ to zero, we get the contracted algebra

$$d\nu_1 = -\sigma_0 \wedge \sigma_1 - \sigma_2 \wedge \sigma_3 \,, \qquad d\nu_2 = -\sigma_0 \wedge \sigma_2 - \sigma_3 \wedge \sigma_1 \,, \qquad d\nu_3 = 0 \,, \qquad d\sigma_\alpha = 0 \,. \tag{203}$$

The left-invariant 1-forms can be written as

$$\sigma_0 = dx_0 \,, \quad \sigma_1 = dx_1 \,, \quad \sigma_2 = dx_2 \,, \quad \sigma_3 = dx_3 \,, \tag{204}$$

$$\nu_1 = dy_1 - x_0 \, dx_1 - x_2 \, dx_3 \,, \quad \nu_2 = dy_2 - x_0 \, dx_2 - x_3 \, dx_1 \,, \quad \nu_3 = dy_3 \,.$$

B.2.4 Contraction of $SU(n+1)$

The left-invariant 1-forms $L_A{}^B$ of $SU(n+1)$ satisfy the Maurer-Cartan relations $dL_A{}^B = L_A{}^C \wedge L_C{}^B$. Splitting the index as $A = (0, \alpha)$, we make the following definitions:

$$L_0{}^0 = \nu \,, \qquad L_0{}^\alpha = \sigma^\alpha \,, \qquad L_\alpha{}^\beta = M_\alpha{}^\beta \,. \tag{205}$$

After the scalings

$$\nu \longrightarrow \lambda^2 \, \nu \,, \qquad \sigma^\alpha \longrightarrow \lambda \, \sigma^\alpha \,, \qquad M_\alpha{}^\beta \longrightarrow \lambda \, M_\alpha{}^\beta \,, \tag{206}$$

and sending λ to zero, we obtain

$$d\nu = \sigma^\alpha \wedge \bar\sigma_\alpha \,, \qquad d\sigma^\alpha = 0 \,, \qquad dM_\alpha{}^\beta = 0 \,. \tag{207}$$

The generalised Heisenberg algebra spanned by σ^α, $\bar\sigma_\alpha$ and ν corresponds to an T^1 bundle over T^{2n}. It is in fact identical to the algebra (185) that we obtained earlier as a contraction of $so(n+2)$.

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