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Classification of p-branes, NUTs, Waves and Intersections

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ABSTRACT

We give a full classification of the multi-charge supersymmetric p-brane solutions in the massless and massive maximal supergravities in dimensions $D \geq 2$ obtained from the toroidal reduction of eleven-dimensional supergravity. We derive simple universal rules for determining the fractions of supersymmetry that they preserve. By reversing the steps of dimensional reduction, the p-brane solutions become intersections of p-branes, NUTs and waves in $D = 10$ or $D = 11$. Having classified the lower-dimensional p-branes, this provides a classification of all the intersections in $D = 10$ and $D = 11$ where the harmonic functions depend on the space transverse to all the individual objects. We also discuss the structure of U-duality multiplets of p-brane solutions, and show how these translate into multiplets of harmonic and non-harmonic intersections.

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1 Introduction

The BPS-saturated soutions in supergravity theories play an important rôle in M-theory or string theory, since it is believed that they will survive at the full quantum level, and they can therefore provide information about the non-perturbative structure. The easiest way to find such BPS solutions is by looking for extremal p-brane solitons, either in the $D = 11$ or $D = 10$ supergravities themselves, or in their Kaluza-Klein reductions to lower dimensions. Since the Kaluza-Klein reduction procedure itself preserves all of the original supersymmetry, in the case of toroidal compactifications, it follows that the lower-dimensional p-brane solitons can be re-interpreted back in the higher dimension as supergravity solutions that preserve the same fractions of supersymmetry as they do in the lower dimension. The simplest p-brane solitons preserve one half of the original supersymmetry. These may be characterised as solutions that can be supported by a single charge, carried by a single field strength in the supergravity theory. The metric functions, and all the other non-vanishing fields in the solution, are expressed in terms of a single function that is harmonic in the space transverse to the p-brane's world-volume. There also exist families of related solutions, obtained by acting with U-duality [1] transformations, which are the discretised Cremmer-Julia (CJ) [2, 3] global symmetry groups of the lower-dimensional supergravities. These families are in general more complicated, with more than one field strength becoming active. However, since U-duality commutes with supersymmetry, they will still preserve the same one half fraction of the supersymmetry. At the classical level, the U-duality transformed solutions are effectively just more complicated presentations of the same single-field-strength solutions, and so it is useful to introduce the notion of a "simple" single-charge p-brane as one where a single field strength carries the charge. In general, the higher-dimensional solution that one arrives at by taking such a lower-dimensional p-brane, and retracing the steps of Kaluza-Klein reduction, or "oxidising" the solution, will not necessarily have the form of a p-brane soliton. It may instead be like a continuum of p-branes distributed over a hypersurface in the compactifying space, or else like a gravitational wave or a Taub-NUT-like monopole solution. Nonetheless, the oxidised solution will still preserve one half of the supersymmetry, and so it is still a configuration that should enjoy a preferred status in the full quantum theory.

There also exist more complicated p-brane solitons in the lower-dimensional theories, which carry more than one kind of charge. In its simplest form, such an N-charge solution is characterised by N independent harmonic functions on the transverse space, one for each of the charges. The solution is "diagonalised," in the sense that each charge is associated with a particular field strength, and each of these is expressed purely in terms of the harmonic function for its charge. Again, more complicated solutions can be found, by acting with U-duality transformations. It is therefore useful again to introduce the notion of "simple" N-charge p-branes, meaning the ones that are in the diagonal form described above. The fractions of supersymmetry that are preserved by the simple N-charge p-branes are smaller than for the single-charge cases. Those with $N=2$ preserve $\frac{1}{4}$, those with $N=3$ preserve $\frac{1}{8}$, and the story becomes more involved for those with $N \geq 4$. If multi-charge p-branes are oxidised back to $D=10$ or $D=11$, they describe more complicated kinds of configurations than do the oxidations of the single-charge examples discussed above. They can, however, in general be interpreted as intersections [4, 5] of the basic p-branes, waves and NUTs mentioned previously. Again, of course, the intersecting solutions preserve the same fractions of supersymmetry as do their lower-dimensional p-brane reductions.

Without the guidance of the lower-dimensional p-branes, it would not be easy to obtain an orderly understanding and characterisation of supersymmetric solutions in $D=10$ or $D=11$. In particular, in the lower dimensions the global E_{11-D} symmetries of the maximal massless supergravities can be used to generate families of supersymmetric solutions from the simple multi-charge p-branes. Such symmetries are non-manifest in $D=10$ or $D=11$, and it would, for example, be no easy task to recognise families of eleven-dimensional solutions that are related, from the four-dimensional viewpoint, by E_7 U-duality transformations of four-dimensional multi-charge p-brane solutions. The best way to benefit from the organising power of the U-duality symmetries is to study the higher-dimensional supersymmetric solutions from the lower-dimensional viewpoint. Large classes of these lower-dimensional solutions are provided by the p-branes that we discussed above, and it is these that will provide the focus of our attention in this paper.

We shall present a full classification of all the p-brane solutions in all the maximal (massless and massive) supergravities in dimensions $2 \leq D \leq 11$ that are obtained by dimensional reduction of eleven-dimensional supergravity. We discuss their oxidations to $D=10$ and $D=11$, and present simple rules for obtaining the higher-dimensional solutions. We also give a detailed discussion of the supersymmetry of all the solutions, and derive simple rules for determining the fractions of preserved supersymmetry.

We begin in section 2 by giving the bosonic Lagrangians for the dimensionally-reduced massless maximal supergravities in $D \geq 3$, and summarising the form of the simple multi-charge p-brane solutions. In section 3, we show how these solutions can be oxidised back to $D=10$ or $D=11$. In fact the end-product of the oxidation of a multi-charge solution is easily obtained by the mechanical application of elementary rules. In section 4, we give a complete classification of all the simple multi-charge p-branes in all the massless and massive supergravities in $D \geq 3$. This is extended in section 5 with a derivation of the bosonic sectors of the maximal two-dimensional massless and massive supergravities, and a classification of their multi-charge p-brane solutions. Simple N-charge solutions with different values of N belong to different U-duality multiplets. The classification of simple multi-charge solutions hence subsumes the classification of different U-duality p-brane multiplets. In section 6 we give an analysis of the supersymmetry of all the solutions in $2 \leq D \leq 11$. Included in this discussion is a detailed study of how the successive introduction of additional charges affects the supersymmetry. In general, when a new charge is added in an existing N-charge solution to give a simple $(N+1)$-charge solution, it can have the following effects on the preserved supersymmetry. One possibility is that the new charge does not further break the supersymmetry. In this case, the same charge but with the opposite sign will break all the supersymmetry. The one remaining possible effect is that the introduction of the charge leads to a halving of the supersymmetry of the N-charge solution. In this case, the sign of the new charge is immaterial. Note that in all cases the new solution is still extremal, and there is no force between the charges. In section 7, we summarise the results of the classification of harmonic intersections in M-theory and string theory, corresponding to the oxidations of the simple multi-charge p-branes obtained earlier in the paper. In section 8, we extend the discussion of p-brane solutions to include the multiplets that are filled out by acting with U-duality on the simple solutions. In particular, we consider examples in $D = 9$ and $D = 8$ that are related to simple solutions by means of $SL(2, \mathbb{R})$ transformations. We obtain their oxidations in $D = 10$ and $D = 11$, and show that they can be viewed as non-harmonic intersections of the basic p-branes, waves and NUTs. We also discuss solutions for all possible pairs of field strengths, showing that there are three categories. We end our paper with a conclusion in section 9. In an appendix, we list all the field configurations for all the simple ($N \geq 3$) charge solutions using 1-form field strengths.

2 Review of maximal supergravities and p-branes

2.1 Maximal supergravities in $D \leq 11$

In this section, we present the toroidal dimensional reductions of the bosonic sector of $D = 11$ supergravity, whose Lagrangian takes the form [6]

$$\mathcal{L} = \hat{e}\hat{R} - \tfrac{1}{48}\hat{e}\,\hat{F}_4^2 + \tfrac{1}{6} * (\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3) \ . \tag{2.1}$$

The subscripts on the potential A_3 and its field strength $F_4 = dA_3$ indicate the degrees of the differential forms. We may reduce the theory to D dimensions in a succession of 1-step compactifications on circles. At each stage in the reduction, say from $(D+1)$ to D dimensions, the metric is reduced according to the standard Kaluza-Klein prescription

$$ds_{D+1}^2 = e^{2\alpha\varphi}\, ds_D^2 + e^{-2(D-2)\alpha\varphi}\,(dz + \mathcal{A}_1)^2 \ , \tag{2.2}$$

where the D dimensional metric, the Kaluza-Klein vector potential $\mathcal{A}_1 = \mathcal{A}_M\, dx^M$ and the dilatonic scalar φ are taken to be independent of the additional coordinate z on the compactifying circle. The constant α is given by $\alpha^{-2} = 2(D-1)(D-2)$, and the parameterisation of the metric is such that a pure Einstein action is reduced again to a pure Einstein action. (This choice is possible for the descent down as far as $D = 3$, but when $D = 2$ it is no longer possible to choose an Einstein-frame parametrisation of the metric. We shall discuss this special case in section 5.) Gauge potentials are reduced according to the prescription $A_n(x,z) = A_n(x) + A_{n-1}(x) \wedge dz$, implying that

$$F_n \longrightarrow dA_{n-1} + dA_{n-2} \wedge dz = dA_{n-1} - dA_{n-2} \wedge \mathcal{A}_1 + dA_{n-2} \wedge (dz + \mathcal{A}_1) \ . \tag{2.3}$$

Thus while the dimensionally-reduced field strength F_{n-1} is defined by $F_{n-1} = dA_{n-2}$, the reduction of F_n is defined by $F_n = dA_{n-1} - dA_{n-2} \wedge \mathcal{A}_1$, and it is this gauge-invariant field strength that appears in the lower-dimensional gauge-field kinetic term. These Kaluza-Klein modifications to the lower-dimensional field strengths become progressively more complicated as the descent through the dimensions continues. Their presence significantly restricts the possible solutions for p-branes and intersecting p-branes, as we shall see later.

It is easy to see that the original eleven-dimensional fields g_{MN} and A_{MNP} in (2.1) will give rise to the following fields in D dimensions:

$$\begin{array}{rclllll} g_{MN} & \longrightarrow & g_{MN}\ , & \vec{\phi}\ , & \mathcal{A}_1^{(i)}\ , & \mathcal{A}_0^{(ij)}\ , & \\ A_3 & \longrightarrow & A_3\ , & A_2^{(i)}\ , & A_1^{(ij)}\ , & A_0^{(ijk)}\ , & \end{array} \tag{2.4}$$

where the indices $i, j, k = 1, \ldots, 11-D$ run over the $11-D$ internal toroidally-compactified dimensions, starting from $i = 1$ for the step from $D = 11$ to $D = 10$. The potentials $A_1^{(ij)}$ and $A_0^{(ijk)}$ are automatically antisymmetric in their internal indices, whereas the 0-form potentials $\mathcal{A}_0^{(ij)}$ that come from the subsequent dimensional reductions of the Kaluza-Klein vector potentials $\mathcal{A}_1^{(i)}$ are defined only for $j > i$. The quantity $\vec{\phi}$ denotes the $(11-D)$-vector of dilatonic scalar fields coming from the diagonal components of the internal metric. The Lagrangian for the bosonic D-dimensional toroidal compactification of eleven-dimensional supergravity then takes the form [7]

$$\begin{aligned} \mathcal{L} &= eR - \tfrac{1}{2}e\,(\partial\vec{\phi})^2 - \tfrac{1}{48}e\,e^{\vec{a}\cdot\vec{\phi}}\,F_4^2 - \tfrac{1}{12}e\sum_i e^{\vec{a}_i\cdot\vec{\phi}}\,(F_3^{(i)})^2 - \tfrac{1}{4}e\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}\,(F_2^{(ij)})^2 \\ &\quad - \tfrac{1}{4}e\sum_i e^{\vec{b}_i\cdot\vec{\phi}}\,(\mathcal{F}_2^{(i)})^2 - \tfrac{1}{2}e\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}\,(F_1^{(ijk)})^2 - \tfrac{1}{2}e\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,(\mathcal{F}_1^{(ij)})^2 + \mathcal{L}_{FFA}\ , \end{aligned} \qquad (2.5)$$

where the "dilaton vectors" $\vec{a}$, $\vec{a}_i$, $\vec{a}_{ij}$, $\vec{a}_{ijk}$, $\vec{b}_i$, $\vec{b}_{ij}$ are constants that characterise the couplings of the dilatonic scalars $\vec{\phi}$ to the various gauge fields. They are given by [7]

$$\begin{array}{lll} & F_{MNPQ} & \text{vielbein} \\ 4-\text{form}: & \vec{a} = -\vec{g}\ , & \\ 3-\text{forms}: & \vec{a}_i = \vec{f}_i - \vec{g}\ , & \\ 2-\text{forms}: & \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g}\ , & \vec{b}_i = -\vec{f}_i\ , \\ 1-\text{forms}: & \vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g}\ , & \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j\ , \\ 0-\text{forms}: & \vec{a}_{ijk\ell} = \vec{f}_i + \vec{f}_j + \vec{f}_k + \vec{f}_\ell - \vec{g}\ , & \vec{b}_{ijk} = -\vec{f}_i + \vec{f}_j + \vec{f}_k\ . \end{array} \qquad (2.6)$$

The explicit expressions for the vectors $\vec{g}$ and $\vec{f}_i$, which have $(11-D)$ components in D dimensions, are given in [7]. For our purposes, it is sufficient to note that they satisfy the relations

$$\vec{g}\cdot\vec{g} = \tfrac{2(11-D)}{D-2}, \qquad \vec{g}\cdot\vec{f}_i = \tfrac{6}{D-2}\ , \qquad \vec{f}_i\cdot\vec{f}_j = 2\delta_{ij} + \tfrac{2}{D-2}\ . \qquad (2.7)$$

We have also included the dilaton vectors $\vec{a}_{ijk\ell}$ and $\vec{b}_{ijk}$ for "0-form field strengths" in (2.6), although they do not appear in (2.5), because they fit into the same general pattern and they will arise later when we consider $(D-2)$-brane solutions (*i.e.* domain walls) [8] in section 4.2. The field strengths are associated with the gauge potentials in the obvious way; for example F_4 is the field strength for A_3, $F_3^{(i)}$ is the field strength for $A_2^{(i)}$, *etc.* The complete expressions for the Kaluza-Klein modifications to the various field strengths are given in [7], as are the cubic Wess-Zumino terms $\mathcal{L}_{FFA}$ coming from the $F_4\wedge F_4\wedge A_3$ term in the eleven-dimensional Lagrangian (2.1).

In the subsequent sections, we shall be making extensive use of the results presented here, in order to discuss various aspects of p-brane solitons in toroidally-compactified M-theory and type II strings.

2.2 Review of supersymmetric p-branes

In this subsection, we review the form of p-brane soliton solutions in maximal supergravities in all the dimensions $3 \le D \le 11$; we shall derive the analogous results in $D=2$ later in section 5. These various p-brane solutions preserve certain fractions of the supersymmetry, owing to which it is believed that they also will also be contained in the spectrum of type II strings or M-theory. As is well known, massless maximal supergravities in D dimensions have E_{11-D} global symmetries [2, 3], which provide a powerful organising principle for the solitonic solutions. However, not all the solutions of a given dimension form one single multiplet under E_{11-D}. For example, solutions that preserve different fractions of the supersymmetry must clearly belong to different multiplets, since the global symmetry commutes with supersymmetry.

It is non-trivial to solve the general equations of motion following from the D-dimensional supergravity Lagrangians presented in the previous subsection. Moreover, one wishes to avoid laboriously solving for solutions that are nothing but U-duality transformations of already obtained solutions. Thus we shall simplify the problem by starting with truncated Lagrangians that contain just the fields that will play a rôle in the construction of the particular solitonic p-branes under consideration. The p-brane solutions of these truncated Lagrangians that we shall construct will also be solutions of the original theory. (Of course not all solutions of the truncated Lagrangian will be solutions of the original one, since the truncation of the theory itself is not in general a consistent one, and so there is a non-trivial check to verify that a p-brane solution of the truncated system is indeed also a solution of the original one; we shall discuss this later.) The truncated Lagrangians are of the following form, comprising dilatonic scalar fields and n-index antisymmetric tensor field strengths [9]:

$$\mathcal{L} = eR - \tfrac{1}{2}e(\partial\vec{\phi})^2 - \frac{1}{2n!}e\sum_{\alpha=1}^{N} e^{\vec{c}_\alpha\cdot\vec{\phi}}(F_n^\alpha)^2 \ , \tag{2.8}$$

where $F_n^\alpha = dA_{n-1}^\alpha$. In writing the truncated Lagrangian in this form, it is understood that the field strengths F_n^α are taken from some subset of the field strengths appearing in (2.5), possibly with dualisations. When a particular field strength F_n in (2.8) is exactly the same as a field strength appearing in (2.5), the corresponding dilaton vector is given by (2.6). However, a field strength might be related by dualisation to one of the original fields in (2.5). In such a case, an original field F_{D-n} with kinetic term $e^{\vec{c}\cdot\vec{\phi}}F_{D-n}^2$ in (2.5) would be represented by a new field $F_n = e^{\vec{c}\cdot\vec{\phi}} * F_{D-n}$, where $*$ is the Hodge dual, with kinetic term $e^{-\vec{c}\cdot\vec{\phi}}F_n^2$ in (2.8). Thus the dilaton vector for this field F_n in (2.8) will be of the opposite sign to the dilaton vector of the original field F_{D-n}, in (2.5). Later, in sections 4 and 5, we shall still use the original fields to label the sets of field configurations for multi-charge p-brane solutions. We shall use a "$*$" to indicate that dualisation of that particular field is to be performed in obtaining the associated truncated Lagrangian. This implies that the dilaton vector associated with the starred field strength has the opposite sign to the one given in (2.6) for the original field.

We shall show later that the p-brane solutions for this truncated system, where in particular there are no Wess-Zumino terms or Kaluza-Klein modifications to the field strengths, are also solutions of the original system described by (2.5) provided that the dilaton vectors $\vec{c}_\alpha$ satisfy the dot product relations [9]

$$M_{\alpha\beta} = \vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta} - \frac{2(n-1)(D-n-1)}{D-2} \ . \tag{2.9}$$

We can then obtain electric $(n-2)$-branes or magnetic $(D-n-2)$-branes, with metrics

$$ds^2 = \Big(\prod_{\alpha=1}^N H_\alpha\Big)^{-\frac{\tilde{d}}{D-2}} dx^\mu dx^\nu \eta_{\mu\nu} + \Big(\prod_{\alpha=1}^N H_\alpha\Big)^{\frac{d}{D-2}} dy^m dy^m \ , \tag{2.10}$$

where the H_α are harmonic functions depending on the coordinates y^m of the $(D-d)$-dimensional space transverse to the d-dimensional world-volume, and $\tilde{d} = D - d - 2$. Note that one can only find solutions for the truncated Lagrangian (2.8) involving N independent harmonic functions if the dilaton vectors satisfy the conditions (2.9) [9]. The p-branes are supported by field strengths F_n^α that all carry either electric or magnetic charges:

$$\begin{aligned} \text{electric}: &\qquad F_n^\alpha = d^d x \wedge dH_\alpha^{-1} \ , \\ \text{magnetic}: &\qquad F_n^\alpha = e^{-\vec{c}_\alpha\cdot\vec{\phi}} * (d^d x \wedge dH_\alpha^{-1}) \ . \end{aligned} \tag{2.11}$$

The dilatonic scalars $\vec{\phi}$ are given by

$$\vec{\phi} = \tfrac{1}{2}\epsilon \sum_\alpha \vec{c}_\alpha \log H_\alpha \ , \tag{2.12}$$

where $\epsilon = 1$ for electric solutions and $\epsilon = -1$ for magnetic solutions. Although in terms of the field strengths F_n^α appearing in the truncated Lagrangian (2.8) the solutions are either purely electric or purely magnetic, in terms of the original fields in (2.5) the solutions will carry both electric and magnetic charges if dualisations of the kind we discussed below (2.8) have been performed. (This can be seen from the fact that an electric p-brane solution supported by F_n is identical to the magnetic p-brane solution supported by its original field F_{D-n}, and *vice versa* [10].) These may be called dyonic solutions of the second kind [7], describing the situation where each individual field strength carries purely an electric or purely a magnetic charge. There are also dyonic solutions of the first kind, where a given field strength carries both electric and magnetic charges. In fact, the solutions discussed above cover all possible simple multi-charge p-brane solutions that can arise from supergravities, with the exception of dyonic solutions of the first kind. However, only one such solution arises that is supersymmetric, namely the dyonic string in $D=6$ [11], whose structure is well understood.

The solutions given above range from (-1)-branes (instantons) to $(D-2)$-branes (domain walls). For an isotropic p-brane, the harmonic functions are given by $H_\alpha = 1+|Q_\alpha|\, r^{-\tilde d}$ where $r=\sqrt{y^m y^m}$, and the ADM mass per unit p-volume is $M=\sum_\alpha |Q_a|$, where Q_α are the charges carried by the field strengths F_n^α. These formulae assume that the dilatonic scalars vanish asymptotically at infinity. If instead they approach the constant values $\vec\phi_0$ asymptotically, then we will have

$$H_\alpha = 1 + \frac{|Q_\alpha|\, e^{-\frac12 \epsilon\, \vec c_\alpha\cdot\vec\phi_0}}{r^{\tilde d}}\ , \qquad M=\sum_\alpha^N |Q_\alpha|\, e^{-\frac12 \epsilon \vec c_\alpha\cdot\vec\phi_0}\ . \tag{2.13}$$

Owing to the quadratic nature of field strength kinetic terms in the Lagrangian, for each charge there is a sign choice to be made, which determines whether it contributes positively or negatively to the mass. We always make the choice where it contributes positively, so that the solution is free from naked singularities. Note that a p-brane with positive mass but the opposite charge can be viewed as an anti-p-brane.

When all the N charges Q_α are equal (we consider the case where $\vec\phi_0=0$ for simplicity), the harmonic functions H_α in (2.10) become equal. Under these circumstances, it is easy to see from (2.12) that all except one combination of the dilatonic scalars, namely $\vec\phi = N^{-1}\sum_\alpha \vec c_\alpha\, \phi$, will become zero, and at the same time all the participating field strengths will become equal, $F^\alpha = F/\sqrt N$. The resulting single-scalar configuration is a solution of the truncated Lagrangian

$$\mathcal{L} = eR - \tfrac12 e\,(\partial\phi)^2 - \frac{1}{2n!} e\, e^{a\phi} F^2\ , \tag{2.14}$$

and is given by [10]

$$\begin{aligned} ds^2 &= H^{-\frac{4\tilde d}{\Delta(D-2)}}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}}\,(dr^2 + r^2\, d\Omega^2)\ ,\\ e^{\frac{\epsilon\Delta}{2a}\phi} &= H\ , \end{aligned} \tag{2.15}$$

where $\Delta = 4/N$, and

$$a^2 = (\vec c_\alpha)^2 = \Delta - \frac{2d\tilde d}{D-2}\ . \tag{2.16}$$

In maximal supergravities (massless or massive), single-charge solutions all have $\Delta=4$. In fact the $\Delta=4/N$ solutions (2.15) can be viewed as bound states of $\Delta=4$ solutions, with zero-binding energy [12,13,14]. To construct N-charge solutions using $\Delta=4$ building blocks, each associated with a harmonic function, the dilaton vector dot products (2.9) must be satisfied. Extremal p-brane solutions (2.15) in various supergravities in different dimensions were constructed in the past [15, 16, 17, 18, 19, 20, 21, 22, 23, 10, 7].

The various p-brane solutions obtained above have distinct behaviours with respect to supersymmetry, which are best characterised by looking at the eigenvalues of the Bogomol'nyi matrix, *i.e.* the anticommutator of the $D = 11$ supercharges, which we shall discuss in detail in section 6. In particular, if there are k zero eigenvalues, then the solution preserves a fraction $k/32$ of the original $D = 11$ supersymmetry. The non-zero eigenvalues provide additional information that characterises the solutions. This is because the eigenvalues are invariant under U-duality, and so this provides a way of recognising families of p-branes that lie in different U-duality multiplets. In particular, this leads to the conclusion that solutions for different values of N belong to different U-duality multiplets. Acting with U-duality on these simple solutions, we can fill out complete U-duality multiplets. The U-duality transformations of the simple solutions will always give solutions involving N' field strengths with $N' \geq N$. If $N' = N$, then the new solution also satisfies the equations of motion following from a truncated Lagrangian of the form (2.8) (but with a different set of field strengths retained in the truncation). In fact these sets of solutions with $N' = N$ form multiplets under the Weyl subgroup of the U-duality group [24]. On the other hand if $N' > N$, then the solutions will be of a more complicated form, where the contributions from the Kaluza-Klein modifications and Wess-Zumino terms cannot be ignored. The cases where $N' = N$ are characterised by the fact that the number of non-zero charges is equal to the number of independent harmonic functions in the solution. We refer to these as the "simple" multi-charge solutions. Thus the classification of simple multi-charge solutions subsumes the classification of the different U-duality p-brane multiplets. Some related discussions of solution multiplets and their supersymmetry have been given in [25]. We shall for now concentrate on the Weyl-group multiplets of simple multi-charge solutions; these are the ones that are directly associated with the harmonic intersections of p-branes. We return to the discussion of general U-duality multiplets in section 8.

3 Oxidation rules for p-branes

The classification of supersymmetric p-branes in all dimensions $D \leq 11$ is an important problem in its own right, since these BPS-saturated solutions are expected to describe the perturbative and non-perturbative states of the fully quantised string theories. In addition, by reversing the Kaluza-Klein reduction procedure and "oxidising" them back to $D = 10$ or $D = 11$, they provide a convenient classification of BPS-saturated solutions in the original higher-dimensional theories. In special cases, the oxidation of an isotropic single-charge p-brane in D dimensions will again give rise to an isotropic single-charge solution in $D = 10$ or $D = 11$. In other cases, the end product of the oxidation can be a line, or more generally a hyperplane, of p-branes. More complicated possibilities also arise when the charge of the single-charge p-brane in D dimensions is carried by a field strength derived from the higher-dimensional metric in the Kaluza-Klein reduction process. In such cases, the end product of the oxidation will be a wave-like solution or a Taub-NUT like solution, rather than what one would normally regard as a p-brane. Nevertheless, in all these cases the higher-dimensional end product preserves the same fraction $\frac{1}{2}$ of the supersymmetry as does the D-dimensional p-brane from which it is derived. This is because the Kaluza-Klein reduction procedure itself breaks none of the supersymmetry. Thus all these higher-dimensional end products deserve to be considered in their own right, since they will describe quantum-protected states in the $D = 10$ string or $D = 11$ M-theory.

More complicated situations arise when we begin with multi-charge solutions in D dimensions. These will give rise to oxidation end products that can be described as intersections [4, 5] of the various p-branes, waves and NUTs mentioned above. The particular combinations of these basic ingredients that arise in any given situation are governed by the details of the set of charges in the lower-dimensional solution. To be more precise, one can read off the combination in the final end product by looking first at the individual sets of end products associated with each individual charge in the D-dimensional p-brane solution. The oxidation end product of the entire multi-charge solution will then be described in terms of intersections of these sets of ingredients. Again, the fraction of supersymmetry that is preserved by the intersecting solution in $D = 10$ or $D = 11$ will be the same as the fraction that is preserved by the multi-charge solution in D dimensions.

3.1 Intersections in M-theory

Let us first consider oxidations to $D = 11$. All the lower-dimensional single-charge solutions give rise to one of the following four kinds of solution:

$$
\begin{aligned}
\text{Membrane}: \quad & ds_{11}^2 = H^{-\frac{2}{3}}\,(-dt^2 + d\vec{x}^2 + d\vec{z}_W^2) + H^{\frac{1}{3}}\,(d\vec{y}^2 + d\vec{z}_T^2)\ ,\\
\text{5-brane}: \quad & ds_{11}^2 = H^{-\frac{1}{3}}\,(-dt^2 + d\vec{x}^2 + d\vec{z}_W^2) + H^{\frac{2}{3}}\,(d\vec{y}^2 + d\vec{z}_T^2)\ , \qquad (3.1)\\
\text{Wave}: \quad & ds_{11}^2 = -H^{-1}\,dt^2 + H\,(dz_W + (H^{-1} - 1)\,dt)^2 + (d\vec{y}^2 + d\vec{z}_T^2)\ ,
\end{aligned}
$$

$$\text{NUT}: \quad ds_{11}^2 = -dt^2 + d\vec{x}^2 + d\vec{z}_W^2 + H^{-1}\,\omega^2 + H\, d\vec{y}^2\ .$$

The notation here is as follows. The $(11-D)$ internal coordinates z^i are divided into two categories, $z^i = (\vec{z}_W, \vec{z}_T)$, namely those that acquire the interpretation of world-volume coordinates in $D=11$ and those that become transverse space coordinates. The harmonic functions depend on the $\vec{y}$ transverse coordinates only. (These were the transverse coordinates in the original D-dimensional p-brane solution.) Note that here we have generalised the concept of world-volume and transverse space to include waves and NUTs.

In the first two cases in (3.1), the world-volume dimensions are 3 and 6 respectively, with p of the spatial coordinates being the original ones $\vec{x}$ of the D-dimensional solution. The membrane and 5-brane solutions arise when the D-dimensional p-brane is supported by an electric or magnetic charge respectively for a field strength originating from F_4 in $D=11$. The wave solution arises when the D-dimensional p-brane is a black hole or instanton carrying an electric charge for a Kaluza-Klein vector. The NUT solution arises when a field strength coming from the metric carries a magnetic charge. There are in fact three distinct subclasses to consider, depending on whether the p-brane in D dimensions is a $(D-4)$-brane, a $(D-3)$-brane or a $(D-2)$-brane. These will be supported by a field strength of degree 2, 1 or 0 coming from the $D=11$ metric. (The last case is associated with a more general kind of Kaluza-Klein reduction which we shall discuss in section 4.2.) The 1-form ω in (3.1) is given by

$$(D-4)\text{-brane}: \qquad \omega = dz_T + Q\,\cos\theta\, d\varphi\ , \quad d\vec{y}^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2\ , \tag{3.2}$$

$$(D-3)\text{-brane}: \qquad \omega = dz_T^1 + Q\, y^1\, dz_T^2\ , \qquad d\vec{y}^2 = (dy^1)^2 + (dy^2)^2\ , \tag{3.3}$$

$$(D-2)\text{-brane}: \qquad \omega = dz_T^1 + Q z_T^3\, dz_T^2\ , \qquad d\vec{y}^2 = (dy^1)^2\ . \tag{3.4}$$

Thus one, two or three internal coordinates respectively acquire the interpretation of lying in the transverse space, with the remainder lying in the world-volume.

Having given the possible forms of end products of the oxidation of single-charge p-branes to $D=11$, it remains to present the rules that determine the precise end products for each lower-dimensional single-charge p-brane. We derive these by noting that in a single step of double dimensional reduction, the degree of the field strength reduces by 1 if it carries an electric charge, while remaining unchanged if it instead carries a magnetic charge. When a step of double dimensional reduction is reversed, the compactification coordinate joins the higher-dimensional world-volume. Conversely, in a step of vertical reduction, the degree of the field strength is unchanged if it carries an electric charge, but is reduced by 1 if it carries a magnetic charge. Upon reversing the vertical reduction, the compactification coordinate joins the transverse space. The complete results can now be presented in the form of two tables, Table (1a) for the p-branes supported by field strengths derived from F_4 in $D = 11$, and Table (1b) for p-branes supported by field strengths coming from the metric in $D = 11$.[1]

	F_4	F_3^i	F_2^{ij}	F_1^{ijk}	$F_0^{ijk\ell}$	Endpoint
Electric $\vec{z}_W =$	–	z^i	z^i, z^j	z^i, z^j, z^k	N/A	Membrane
Magnetic $\vec{z}_T =$	–	z^i	z^i, z^j	z^i, z^j, z^k	z^i, z^j, z^k, z^ℓ	5-brane

Table (1a): Oxidations to M-branes

The tables indicates how the internal compactification coordinates divide between the world-volume and the transverse space after the D-dimensional p-brane is oxidised to $D = 11$. Where world-volume coordinates $\vec{z}_W$ are listed, the remaining unlisted coordinates $\vec{z}_T$ are associated with the transverse space, and *vice versa*. The indices $i, j, \ldots$ on the internal coordinates run from 1 to $(11 - D)$, starting with $i = 1$ for the reduction from $D = 11$ to $D = 10$. The "0-form field strengths" $F_0^{(ijk\ell)}$ in Table (1a) and $\mathcal{F}_0^{(ijk)}$ in Table (1b) are like cosmological terms in the D-dimensional Lagrangian, and arise from generalised Kaluza-Klein reductions, as we shall discuss in section 4.2.

	$\mathcal{F}_2^i$	$\mathcal{F}_1^{(ij)}$	$\mathcal{F}_0^{(ijk)}$	Endpoint
Electric $\vec{z}_W =$	z^i	z^i, z^j	N/A	Wave
Magnetic $\vec{z}_T =$	z^i	z^i, z^j	z^i, z^j, z^k	NUT

Table (1b): Oxidations to M-waves and M-NUTs

[1]The electric solutions supported by 1-form field strengths are instantons, or "(-1)-branes." In these cases, there is no world-volume in the D-dimensional configuration, and all D dimensions are spatial transverse coordinates. In the present discussion, this situation arises when one of the compactification coordinates z^i is actually the time coordinate of the eleven-dimensional theory, implying that the lower-dimensional theory is then formulated in a space of Euclidean signature. The reduction on the time coordinate automatically results in the 1-form field strengths appearing with the opposite sign to the usual one in the D-dimensional Lagrangian, a feature that is in fact necessary in order to describe the instanton solutions. Note that in this discussion, there is never any need to perform any Euclideanisation by hand; the positive-definite metric signature arises from compactification of the time coordinate. It is interesting that the only theory where an explicit Euclideanisation would be needed in order to be able to construct single-charge instanton solutions is the ten-dimensional type IIB supergravity. However, the status of such a Euclideanisation is rendered problematical by the fact that the self-duality condition on the 5-form field strength requires a Lorentzian signature for the spacetime.

The asymmetry between the membranes and 5-branes in Table (1a), and between the waves and NUTs in Table (1b), arises because the electric solutions supported by $F_0^{(ijk\ell)}$ or $\mathcal{F}_0^{(ijk)}$ would be (-2)-branes, which do not seem to have any meaning. The metric and the field strength F_4 in $D = 11$ can be easily determined by retracing the steps of the dimensional reduction given in (2.2) and (2.3).

Having discussed the oxidation of single-charge solutions to $D = 11$, we are now in a position to discuss the multi-charge solutions. It is manifest from the form (2.10) and (2.12) of the D-dimensional N-charge solution, and the structure (2.2) of the Kaluza-Klein reduction of the metric, that the oxidation to $D = 11$ will give a metric where each non-vanishing metric component will be a product $\prod_\alpha H_\alpha^{m_\alpha}$ of certain specific powers m_α of the N independent harmonic functions H_α. An easy way to calculate these powers is by first considering the oxidations of the N individual single-charge components, corresponding to all harmonic functions being 1 except for the one associated with the single charge under consideration. These single-charge oxidations immediately give the exponents m_α in the products in the metric components.

We shall illustrate the above procedure with a few examples. First, consider a dyonic string solution in $D = 6$ [11]. It is easy to verify that such 2-charge solutions can be constructed provided that the electric charge Q_e and the magnetic charge Q_m are carried by the same 3-form field strength in $D = 6$. For definiteness, we shall consider the case where it is $F_3^{(1)}$ that carries these charges, *i.e.*

$$Q_e = \int e^{\vec{a}_1\cdot\vec{\phi}} * F_3^{(1)}\ , \qquad Q_m = \int F_3^{(1)}\ , \tag{3.5}$$

where the dilaton vector $\vec{a}_1$ is given by (2.6). From (2.10) and (2.12), we see that the dyonic solution in $D = 6$ is given by

$$\begin{aligned}
ds_6^2 &= (H_e\, H_m)^{-\frac{1}{2}}\,(-dt^2 + dx^2) + (H_e\, H_m)^{\frac{1}{2}}\, d\vec{y}^2\ ,\\
\vec{\phi} &= \tfrac{1}{2}\vec{a}_1\, \log(H_e/H_m)\ ,\\
F_3^{(1)} &= d^2x \wedge dH_e^{-1} + e^{-\vec{a}_1\cdot\vec{\phi}} * (d^2x \wedge dH_m^{-1})\ ,
\end{aligned} \tag{3.6}$$

where the harmonic functions are given by $H_e = 1 + Q_e/y^2$ and $H_m = 1 + Q_m/y^2$, and $y^2 = \vec{y}\cdot\vec{y}$.

If $H_m = 1$ or $H_e = 1$, the solution is purely electric or purely magnetic, and from (3.1) and the oxidation rules in Table (1a) we see that the corresponding end products in $D = 11$ have metrics $ds_{11}^2(e)$ and $ds_{11}^2(m)$ given by

$$ds_{11}^2(e) = H_e^{-\frac{2}{3}}\,(-dt^2 + dx^2 + dz_1^2) + H_e^{\frac{1}{3}}\,(d\vec{y}^2 + dz_2^2 + \cdots + dz_5^2)\ , \tag{3.7}$$

$$ds^2_{11}(m) \;=\; H_m^{-\frac{1}{3}}\,(-dt^2 + dx^2 + dz_2^2 + \cdots + dz_5^2) + H_m^{\frac{2}{3}}\,(d\vec{y}^2 + dz_1^2)\ . \tag{3.8}$$

They respectively describe a membrane distributed uniformly over the 4-plane (z_2, z_3, z_4, z_5) and a 5-brane distributed uniformly over the line z_1. Since there is a unique answer for the products of harmonic functions for each coordinate direction in the oxidation of the dyon, it must be that they are simply the products of the H_e and H_m factors for each coordinate direction in the two limits above. Thus the dyonic string oxidises to give

$$\begin{aligned} ds^2_{11} \;=\;& H_e^{-\frac{2}{3}}\, H_m^{-\frac{1}{3}}\,(-dt^2 + dx^2) + H_e^{\frac{1}{3}}\, H_m^{\frac{2}{3}}\, d\vec{y}^2 \\ &+ H_e^{-\frac{2}{3}}\, H_m^{\frac{2}{3}}\, dz_1^2 + H_e^{\frac{1}{3}}\, H_m^{-\frac{1}{3}}\,(dz_2^2 + \cdots + dz_5^2)\ , \end{aligned} \tag{3.9}$$

which describes a membrane intersecting a 5-brane.

For another example, consider a 3-charge extremal black hole in $D = 5$. As we shall show in section 4.1, there are two different kinds of configuration of charges that can support this solution. One involves electric charges carried by $\{F_2^{(ij)}, F_2^{(k\ell)}, F_2^{(mn)}\}$, where the internal indices are all different (there are actually 15 different sub-cases here, corresponding to all possible choices of index values). The other configuration, which we can denote by $\{F_2^{(ij)}, {*F_3^{(i)}}, \mathcal{F}_2^{(j)}\}$, where i and j are different, has electric charges carried by $F_2^{(ij)}$ and $\mathcal{F}_2^{(j)}$, and a magnetic charge carried by $F_3^{(i)}$ (here, there are 30 sub-cases corresponding to different choices for the internal indices). In both cases, if the three charges are set equal, the solution reduces to the Reissner-Nordstrøm extremal black hole in $D = 5$. The oxidation of the first case to $D = 11$ is straightforward, and gives three intersecting membranes, whose spatial world-volume coordinates are (z^i, z^j), (z^k, z^ℓ) and (z^m, z^n) respectively. The second case oxidises to give an intersection of a membrane, a 5-brane and a wave. Consider the example $\{F_2^{(12)}, {*F_3^{(1)}}, \mathcal{F}_2^{(2)}\}$, with charges denoted by Q_e, Q_m and Q_w respectively; from the oxidations of the individual components given in (3.1), and from Tables (1a) and (1b), we immediately see that this 3-charge solution oxidises to give

$$\begin{aligned} ds^2_{11} \;=\;& -H_e^{-\frac{2}{3}}\, H_m^{-\frac{1}{3}}\, H_w^{-1}\, dt^2 + H_e^{-\frac{2}{3}}\, H_m^{\frac{2}{3}}\, dz_1^2 + H_e^{\frac{1}{3}}\, H_m^{\frac{2}{3}}\, d\vec{y}^2 \\ &+ H_e^{-\frac{2}{3}}\, H_m^{-\frac{1}{3}}\, H_w\,(dz_2 + (H_w^{-1} - 1)dt)^2 + H_e^{\frac{1}{3}}\, H_m^{-\frac{1}{3}}\,(dz_3^2 + \cdots + dz_6^2)\ . \end{aligned} \tag{3.10}$$

As the above examples illustrate, the procedure of oxidising a given multi-charge p-brane in D dimensions back to intersections in $D = 11$ is a completely straightforward and mechanical one. These intersections are of a type where all the harmonic functions depend on the $\vec{y}$ coordinates that are transverse to the world-volumes of all the constituents.[2] Thus

[2]There are also other kinds of intersections, where each harmonic function only depends on the "relative

the task of classifying all such intersections in $D=11$ is subsumed by the task of classifying all D-dimensional multi-charge p-branes. In turn, this latter classification problem reduces simply to the task of finding all possible sets of dilaton vectors, defined in (2.6), that satisfy the dot-product relations (2.9). There is only one remaining subtlety, namely that our discussion so far has been restricted to reductions down to $D=3$. Since some intersections in $D=11$ can only be described in terms of oxidations of multi-charge solutions in $D=2$, we will only have a complete classification scheme after having obtained a construction for p-branes in $D=2$. We shall address this in section 5. Note that although we have focussed attention on the simple multi-charge p-branes where the number of non-zero charges is equal to the number of independent harmonic functions, it is straightforward also to oxidise the more complicated solutions that are related to the simple ones by U-duality rotations that lie outside the Weyl group. We shall discuss this in more detail in section 8.

3.2 Intersections in type IIA string theory

So far, we have considered oxidations of lower-dimensional p-brane solutions to eleven-dimensional M-theory. It is also of interest to view the lower-dimensional solutions instead from the standpoint of ten-dimensional string theory. For example, we can categorise lower-dimensional p-brane solutions according to whether they are supported by NS-NS, or R-R, or mixed sets of ten-dimensional fields. In particular, p-branes carrying R-R charges acquire the interpretation of being Dp-branes [34]. While M-theory is intrinsically non-perturbative, the oxidation of p-branes to ten-dimensional string theory allows us to distinguish between perturbative and non-perturbative string states.

Massless type IIA supergravity can be viewed as the first step in the dimensional reduction of $D=11$ supergravity. Thus it is convenient to describe its fields, and their subsequent dimensional reduction, in the same notation as we used for the reductions of eleven-dimensional supergravity itself. From the viewpoint of the type IIA string, these divide into NS-NS fields g_{MN}, ϕ and $A_2^{(1)}$, and R-R fields A_3 and $\mathcal{A}_1^{(1)}$. This separation into NS-NS and R-R fields is preserved under the subsequent steps of dimensional reduction. It

transverse space" of coordinates transverse only to the constituent associated with the harmonic [26, 28, 29, 31, 32, 33]. Thus in these intersections, each harmonic function depends on totally non-overlapping subsets of transverse-space coordinates, and so these solutions do not dimensionally reduce to p-branes. For related reasons, it is not clear that any notion of mass or tension can be given for such configurations. Thus their relevance as quantum states in string theory is unclear. In this paper, unless indicated otherwise, intersections will be assumed to be of the kind that do dimensionally reduce to p-branes.

follows that in D dimensions, the breakdown of fields into NS-NS and R-R is as follows [24]:

$$\mathrm{NS-NS}: \quad A_2^{(1)} \quad A_1^{(1\alpha)} \quad A_0^{(1\alpha\beta)} \quad \mathcal{A}_1^{(\alpha)} \quad \mathcal{A}_0^{(\alpha\beta)} \quad \vec{\phi} \quad g_{\mu\nu}\ , \tag{3.11}$$

$$\mathrm{R-R}: \quad A_3 \quad A_2^{(\alpha)} \quad A_1^{(\alpha\beta)} \quad A_0^{(\alpha\beta\gamma)} \quad \mathcal{A}_1^{(1)} \quad \mathcal{A}_0^{(1\alpha)}\ , \tag{3.12}$$

where we have decomposed the internal index i as $i = (1, \alpha)$. All the lower-dimensional single-charge p-brane solutions, upon oxidation back to $D = 10$, will therefore give rise to one of the following eight kinds of solution, which we subdivide into four NS-NS and 4 R-R:

NS-NS single-charge endpoints:

$$\begin{aligned}
\text{String}: \quad ds_{10}^2 &= H^{-\frac{3}{4}}\,(-dt^2 + d\vec{x}^2 + d\vec{z}_W^2) + H^{\frac{1}{4}}\,(d\vec{y}^2 + d\vec{z}_T^2)\ ,\\
\text{5-brane}: \quad ds_{10}^2 &= H^{-\frac{1}{4}}\,(-dt^2 + d\vec{x}^2 + d\vec{z}_W^2) + H^{\frac{3}{4}}\,(d\vec{y}^2 + d\vec{z}_T^2)\ ,\\
\text{Wave}: \quad ds_{10}^2 &= -H^{-1}\,dt^2 + H\,(dz_W + (H^{-1} - 1)\,dt)^2 + (d\vec{y}^2 + d\vec{z}_T^2)\ ,\\
\text{NUT}: \quad ds_{10}^2 &= -dt^2 + d\vec{x}^2 + d\vec{z}_W^2 + H^{-1}\,\omega^2 + H\,d\vec{y}^2\ .
\end{aligned} \tag{3.13}$$

R-R single-charge endpoints:

$$\begin{aligned}
\text{D0-brane}: \quad ds_{10}^2 &= H^{-\frac{7}{8}}\,(-dt^2 + d\vec{x}^2 + d\vec{z}_W^2) + H^{\frac{1}{8}}\,(d\vec{y}^2 + d\vec{z}_T^2)\ ,\\
\text{D6-brane}: \quad ds_{10}^2 &= H^{-\frac{1}{8}}\,(-dt^2 + d\vec{x}^2 + d\vec{z}_W^2) + H^{\frac{7}{8}}\,(d\vec{y}^2 + d\vec{z}_T^2)\ ,\\
\text{D2-brane}: \quad ds_{10}^2 &= H^{-\frac{5}{8}}\,(-dt^2 + d\vec{x}^2 + d\vec{z}_W^2) + H^{\frac{3}{8}}\,(d\vec{y}^2 + d\vec{z}_T^2)\ ,\\
\text{D4-brane}: \quad ds_{10}^2 &= H^{-\frac{3}{8}}\,(-dt^2 + d\vec{x}^2 + d\vec{z}_W^2) + H^{\frac{5}{8}}\,(d\vec{y}^2 + d\vec{z}_T^2)\ ,
\end{aligned} \tag{3.14}$$

The $(10 - D)$ internal coordinates z^α are divided into world-volume coordinates $\vec{z}_W$ and transverse coordinates $\vec{z}_T$. The rules for how each lower-dimensional single-charge solution oxidises to $D = 10$ can be summarised in the following tables, in which the division of the internal coordinates between world-volume and transverse is given. Table (2a) gives the rules for NS-NS oxidations, and Table (2b) gives the rules for R-R oxidations.

	$F_3^{(1)}$	$F_2^{(1\alpha)}$	$F_1^{(1\alpha\beta)}$	$F_0^{(1\alpha\beta\gamma)}$	Endpoint
Electric $\vec{z}_W =$	–	z^α	z^α, z^β	N/A	String
Magnetic $\vec{z}_T =$	–	z^α	z^α, z^β	$z^\alpha, z^\beta, z^\gamma$	5-brane
	$\mathcal{F}_2^{(\alpha)}$	$\mathcal{F}_1^{(\alpha\beta)}$	$\mathcal{F}_0^{(\alpha\beta\gamma)}$		
Electric $\vec{z}_W =$	z^α	z^α, z^β	N/A		Wave
Magnetic $\vec{z}_T =$	z^α	z^α, z^β	z^a, z^β, z^γ		NUT

Table (2a): NS-NS oxidations to $D = 10$

	F_4	$F_3^{(\alpha)}$	$F_2^{(\alpha\beta)}$	$F_1^{(\alpha\beta\gamma)}$	$F_0^{(\alpha\beta\gamma\delta)}$	Endpoint
Electric $\vec{z}_W =$	–	z^α	z^α, z^β	$z^\alpha, z^\beta, z^\gamma$	N/A	D2-brane
Magnetic $\vec{z}_T =$	–	z^α	z^α, z^β	$z^\alpha, z^\beta, z^\gamma$	$z^\alpha, z^\beta, z^\gamma, z^\delta$	D4-brane
	$\mathcal{F}_2^{(1)}$	$\mathcal{F}_1^{(1\alpha)}$	$\mathcal{F}_0^{(1\alpha\beta)}$			
Electric $\vec{z}_W =$	–	z^α	N/A			D0-brane
Magnetic $\vec{z}_T =$	–	z^α	z^α, z^β			D6-brane

Table (2b): R-R oxidations to $D = 10$

The notation in the tables carries over, *mutatis mutandis*, from the notation that we described previously for the oxidations to $D = 11$. In exactly the same way as in for $D = 11$, having given the oxidation rules for single-charge p-branes in D dimensions, it is a straightforward and purely mechanical process to deduce the oxidation endpoints in $D = 10$ when starting from simple multi-charge p-branes in D dimensions. We shall not present any further examples here, since no new issues of principle arise. We may note, however, that the two examples given in our discussion of oxidations to $D = 11$, namely the dyonic string in $D = 6$, and the 3-charge black-hole in $D = 5$, become intersections of a string with a 5-brane, and intersections of a string, 5-brane and a wave, respectively.

Again, the classification of all the resulting intersections in $D = 10$ is subsumed by a classification of all possible lower-dimensional multi-charge p-branes.

4 Classification of p-branes in maximal supergravities

In this section, we shall address the problem of classifying all multi-charge p-brane solutions in maximal supergravities in $D \geq 3$. We shall discuss the $D = 2$ case in section 5. This is an important problem in its own right, since these extremal BPS saturated solutions are expected to survive as quantum states in compactified string theory or M-theory. In addition, as we have seen in the previous section, their classification also provides a classification of multiple intersections in M-theory or string theory, where the harmonic functions all depend on the coordinates transverse to the individual world-volumes.

Our discussion here will divide into two parts. The first applies to the p-branes in D dimensions that are supported by 4-form, 3-form, 2-form or 1-form field strengths. Such solutions can all be viewed as solutions of the standard massless maximal supergravities that are derived from $D = 11$ supergravity by ordinary Kaluza-Klein dimensional reduction. The second part of our discussion will be concerned with p-branes in D dimensions that are supported by "0-form field strengths." These solutions are not seen in the ordinary massless supergravities, and in fact the 0-form field strengths are really like cosmological terms in massive supergravities. In fact such massive theories, still maximally supersymmetric, do arise as consistent Kaluza-Klein reductions of $D = 11$ supergravity. However, they are obtained by making a generalised reduction of the Scherk-Schwarz type [35, 36, 8, 37, 38]. The p-brane solutions supported by the cosmological terms in these massive theories are all $(D-2)$-branes, and are commonly known as domain walls. Since some new features arise in these cases, we shall discuss them separately.

4.1 p-branes in massless supergravities

In section 2, we gave a review of the p-brane solitons, supported by certain n-rank field strengths, that arise in D-dimensional supergravities. In the massless maximal supergravities, we encounter field strengths of degrees $n = 4, 3, 2, 1$. In D dimensions, an n-form field strength is dual to a $(D-n)$-form, and in this paper, such a dualisation will always be performed if $D - n < n$. The resulting versions of the supergravities may be called "fully dualised." Thus in these versions the 4-form exists for $D \geq 8$; the 3-forms exist for $D \geq 6$; 2-forms exist for $D \geq 4$ and 1-forms exist for $D \geq 2$.

4.1.1 p-branes from 4-form and 3-form field strengths

There is only one 4-form field strength, and it gives rise to electric membrane or magnetic $(D-6)$-brane solutions. In $D = 8$, there exists a dyonic solution where the 4-form field strength carries both electric and magnetic charges [39]. However, in this solution the contribution from the $\mathcal{L}_{FFA}$ Wess-Zumino term to the equations of motion does not vanish, and the solution is nothing but a perturbative $SL(2,\mathbb{R})$ transformation of a singly-charged purely electric or magnetic membrane. The solution preserves half the supersymmetry, as in the case of the purely electric or purely magnetic solutions. As we discussed in section 2, such a solution can really be regarded, from the classical point of view, as a single-charge solution. We shall return to this example later, in section 8.

In $D \geq 6$ dimensions there are $(11 - D)$ 3-forms in the fully-dualised supergravities. However, it is not possible to construct simple solutions (in the sense defined in section 2) using more than one 3-form field strength. This can be seen from the fact that the dilaton vectors $\vec{a}_i$ for the 3-form field strengths $F_3^{(i)}$ satisfy the dot-product relation [7]

$$\vec{a}_i \cdot \vec{a}_j = 2\delta_{ij} - \frac{2(D-6)}{D-2} \ , \tag{4.1}$$

which is not of the form given by (2.9). Of course, there will exist solutions which are merely U-duality transformations of singly-charged solutions, and these can involve more than one 3-form field strength, but we may again view these as being singly-charged, for the same reasons as we discussed before. In particular this implies that extremal solutions supported only by 3-forms will always preserve half of the supersymmetry, when $D \ge 7$. In $D = 6$, however, there exist dyonic string solutions, where a single 3-form field strength carries both electric and magnetic charges. (In fact we discussed this solution in section 3.) The dyonic string preserves $\frac{1}{4}$ of the supersymmetry, a characteristic of all simple 2-charge solutions. There are in all five 3-form field strengths in $D = 6$, giving five possible dyonic string solutions, denoted by the participating field strengths [7]:

$$\{*F_3^{(i)}, F_3^{(i)}\}_5 \ , \tag{4.2}$$

where $i = 1, \ldots, 5$. We have introduced here a notation that we shall use throughout the paper, in which a simple p-brane solution is characterised by the list of non-vanishing field strengths that support it. In general a field strength labelled with a $*$ signifies that it carries a magnetic charge if the unstarred field strengths carry electric charges, and *vice versa*. The subscript attached to the list indicates the multiplicity of such solutions, corresponding to the different solutions that can be obtained by making different choices for the internal index values on the participating field strengths. Thus in the case of the dyonic string, the multiplicity of 5 arises because the index i can take five possible values.

4.1.2 Multi-charge p-branes from 2-form field strengths

Simple multi-charge p-brane solutions involving multiple field strengths of degrees 2 or 1 do exist, and their Weyl multiplet structures were discussed in [24]. First let us discuss the case of 2-form solutions, in $D \ge 4$. They can be either black holes or $(D-4)$-branes, and can involve up to four participating field strengths, in sufficiently low dimensions. Not surprisingly, the multiplicities for these p-brane solutions grow with decreasing dimension D (since the range of the internal indices grows). These multiplicities, obtained in [24], are given in Table 3 below. Note that the subscripts indicate the fractions of preserved supersymmetry. We shall derive these fractions in section 6.

Dim.	$N=1$	$N=2$	$N=3$	$N=4$
$D=10$	$1_{\frac{1}{2}}$			
$D=9$	$1_{\frac{1}{2}}+2_{\frac{1}{2}}$	$2_{\frac{1}{4}}$		
$D=8$	$6_{\frac{1}{2}}$	$6_{\frac{1}{4}}$		
$D=7$	$10_{\frac{1}{2}}$	$15_{\frac{1}{4}}$		
$D=6$	$16_{\frac{1}{2}}$	$40_{\frac{1}{4}}$		
$D=5$	$27_{\frac{1}{2}}$	$135_{\frac{1}{4}}$	$45_{\frac{1}{8}}$	
$D=4$	$56_{\frac{1}{2}}$	$756_{\frac{1}{4}}$	$2520_{\frac{1}{8}}$	$630_{\frac{1}{8}}$

Table 3: Multiplicities for N-charge supersymmetric 2-form solutions

The $N=1$ solutions can be easily classified, since each field strength can give rise to one p-brane that is either electrically-charged or magnetically-charged. All the singly-charged solutions in a given dimension preserve half the supersymmetry, and they form an irreducible multiplet under the Weyl subgroup of the U-duality group [24]. Acting on these solutions with the full U-duality transformations, we obtain a full multiplet of solutions that preserve half the supersymmetry. When $N \geq 2$ the classification becomes more complicated, since one cannot obtain simple N-charge solutions using an arbitrary set of 2-form field strengths; only sets whose dilaton vectors satisfy the relations (2.9) will admit such solutions. It is a straightfoward matter, given the expressions (2.6) and (2.7), to enumerate the sets of fields which can lead to multi-charge solutions. We shall now discuss these for each dimension $3 \leq D \leq 9$.

$D=9$

In $D=9$, although there are three 2-form field strengths, their associated three dilaton vectors $\vec{a}_{12}$, $\vec{b}_1$ and $\vec{b}_2$ do not all satisfy (2.9). However, two of the three possible pairs of dilaton vectors, namely $\{\vec{a}_{12}, \vec{b}_1\}$ and $\{\vec{a}_{12}, \vec{b}_2\}$, do satisfy (2.9). Thus the maximum number of 2-form field strengths for simple solutions in $D=9$ is $N_{\max}=2$, given by

$$\{F_2^{(12)}, \mathcal{F}_2^{(1)}\}\ , \qquad \{F_2^{(12)}, \mathcal{F}_2^{(2)}\}\ . \tag{4.3}$$

They form a doublet under S_2, the Weyl group of the CJ group $SL(2,\mathbb{R})$. They give rise to 2-charge electric black holes, or 2-charge magnetic 5-branes. These, and indeed all simple 2-charge p-brane solutions, preserve $\frac{1}{4}$ of the supersymmetry. Acting with $SL(2,\mathbb{R})$, one obtains the full $SL(2,\mathbb{R})$ multiplet of solutions that preserve $\frac{1}{4}$ of the supersymmetry. This fraction of preserved supersymmetry distinguishes the multiplet from the one that would be obtained by acting with $SL(2,\mathbb{R})$ on a single-charge p-brane solution, which would instead preserve half the supersymmetry (see section 8).

$D = 8, 7, 6$

From Table 3, in all these dimensions we have $N_{\rm max} = 2$, and the associated solutions have multiplicities $M = 6$, 15 and 40 in $D = 8$, 7 and 6. These are also the dimensions of the irreducible representations of the associated U Weyl groups [24]. The pairs of field strengths whose dilaton vectors satisfy (2.9) are easily identified, and are given by

$$
\begin{aligned}
D=8: &\quad \{F_2^{(ij)}, \mathcal{F}_2^{(i)}\}_6 \ , && (4.4)\\
D=7: &\quad \{F_2^{(ij)}, \mathcal{F}_2^{(i)}\}_{12} \ , \quad \{F_2^{(ij)}, F_2^{(k\ell)}\}_3 \ , && (4.5)\\
D=6: &\quad \{F_2^{(ij)}, \mathcal{F}_2^{(i)}\}_{20} \ , \quad \{F_2^{(ij)}, F_2^{(k\ell)}\}_{15} \ , \quad \{\mathcal{F}_2^{(i)}, {*F_4}\}_5 \ , && (4.6)
\end{aligned}
$$

where the indices $(i, j, k, \ldots)$ are all different, and run from 1 to $11 - D$. The subscripts, as usual, denote the multiplicities. Note that in $D = 7$ and $D = 6$ the multiplicities $M = 15$ and $M = 40$ arise from more than one kind of structure for the possible pairs of field strengths. This phenomenon, which occurs in general in lower dimensions, is a reflection of the fact that the various field strengths here are characterised by $SL(11-D,\mathbb{R})$ indices $i, j \ldots$, but in the fully-dualised supergravities they assemble into E_{11-D} multiplets [40]. Thus the multiplets of multi-charge solutions, although irreducible under the E_{11-D} CJ groups, decompose into reducible representations under $SL(11-D,\mathbb{R})$.

Recall that when a field strength carries a $*$, this indicates that it is related by dualisation to the corresponding field in the truncated Lagrangian (2.8). Thus for example the last combination in (4.6) corresponds to a truncated Lagrangian with two 2-form field strengths F_2^α, one chosen from the five $\mathcal{F}_2^{(i)}$, and the other being the dualised field $e^{\vec a\cdot\vec\phi} * F_4$. In this example, the two field strengths F_2^α either both carry electric charges or both carry magnetic charges, corresponding to a 2-charge black-hole or 2-charge membrane respectively. In terms of the original fields, the black hole carries an electric charge for one of the $\mathcal{F}_2^{(i)}$ fields, and a magnetic charge for the F_4 field. The charge complexions are reversed in the case of the membrane. Thus in terms of the original variables, the solutions could be viewed as being dyonic (of the second kind).

$D = 5$

In five dimensions, we have $N_{\rm max} = 3$. The $N = 2$ solutions have multiplicity $M = 135$, and form a 135-dimensional irreducible representation of the E_6 Weyl group [24]. The

$N = 3$ solutions have multiplicity $M = 45$, and likewise form a 45-dimensional irreducible representation under the E_6 Weyl group. The sets of allowed participating field strengths for these solutions are given by

$$
\begin{aligned}
N=2: &\quad \{F_2^{(ij)}, \mathcal{F}_2^{(i)}\}_{30}\ , \quad \{F_2^{(ij)}, F_2^{(k\ell)}\}_{45}\ , \quad \{*F_3^{(i)}, \mathcal{F}_2^{(j)}\}_{30}\ , \quad \{*F_3^{(i)}, F_2^{(ij)}\}_{30}\ , \\
N=3: &\quad \{F_2^{(ij)}, F_2^{(k\ell)}, F_2^{(mn)}\}_{15}\ , \quad \{*F_3^{(i)}, \mathcal{F}_2^{(j)}, F_2^{(ij)}\}_{30}\ ,
\end{aligned}
\tag{4.7}
$$

where the indices $(i, j, \ldots)$ are all different, and run from 1 to $11 - D = 6$.

$D = 4$

In $D = 4$, the maximal number of participating 2-form field strengths is $N_{\max} = 4$. The $N = 2, 3, 4$ solutions form irreducible representations of the E_7 Weyl group with dimensions 756, 2520 and 630 [24]. The participating field strengths are given by

$$
\begin{aligned}
N=2: &\quad \{F_2^{(ij)}, \mathcal{F}_2^{(i)}\}_{42+42}\ , \qquad \{F_2^{(ij)}, F_2^{(k\ell)}\}_{105+105}\ , \qquad \{*F_2^{ij}, F_2^{(ik)}\}_{210}\ , \\
&\quad \{*F_2^{(ij)}, \mathcal{F}_2^{(k)}\}_{105+105}\ , \qquad \{*\mathcal{F}_2^{(i)}, \mathcal{F}_2^{(j)}\}_{42}\ ,
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
N=3: &\quad \{F_2^{(ij)}, F_2^{(k\ell)}, F_2^{(mn)}\}_{105+105}\ , \quad \{F_2^{(ij)}, F_2^{(k\ell)}, *F_2^{(ik)}\}_{420+420}\ , \\
&\quad \{F_2^{(ij)}, F_2^{(k\ell)}, *\mathcal{F}_2^{(m)}\}_{315+315}\ , \qquad \{F_2^{(ij)}, *F_2^{(ik)}, \mathcal{F}_2^{(j)}\}_{210+210}\ , \\
&\quad \{F_2^{(ij)}, \mathcal{F}_2^{(i)}, *\mathcal{F}_2^{(k)}\}_{210+210}\ ,
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
N=4: &\quad \{F_2^{(ij)}, F_2^{(k\ell)}, F_2^{(mn)}, *\mathcal{F}_2^{p}\}_{105+105}\ , \quad \{F_2^{(ij)}, *F_2^{(ik)}, \mathcal{F}_2^{(j)}, *\mathcal{F}_2^{(k)}\}_{210}\ , \\
&\quad \{F_2^{(ij)}, F_2^{(k\ell)}, *F_2^{(ik)}, *F_2^{(j\ell)}\}_{210}\ ,
\end{aligned}
\tag{4.10}
$$

where the indices $(i, j, \ldots)$ are all different, and run from 1 to $11 - D = 7$. The pairs of numbers in the multiplicity subscripts indicate that the 0-brane solutions can be dualised to give an equal number of new solutions that are again 0-branes. Although the 2-form solutions in higher dimensions could also be dualised, in those cases the solutions dual to 0-branes would be $(D-4)$-branes, and thus there is no doubling of the multiplicities of p-branes with a given p.

4.1.3 Comments on 2-form solutions

1) In the previous subsection, we presented the complete results for simple multi-charge p-brane solutions supported by 2-form field strengths in $D \ge 4$, by listing all their multiple field strength configurations. The exact solutions are given by (2.10) and (2.12), where the dilaton vectors $\vec{c}_\alpha$ for each multiple field configuration can be read off from (2.6). *A priori*, we know only that these solutions satisfy the equations of motion coming from a truncated

Lagrangian of the form (2.8). In fact, they also satisfy the full equations of motion coming from the complete D-dimensional supergravity Lagrangian (2.5). In other words, in these simple solutions, owing to the specific combinations of multiple field strengths that are involved, the Kaluza-Klein modifications to field strengths, and the Wess-Zumino terms $\mathcal{L}_{FFA}$, make no contribution in the equations of motion. It is necessary, and non-trivial, to verify this point, since our criterion for recognising valid sets of field strengths for multi-charge solutions was based only on the criterion that their dilaton vectors should satisfy (2.9).

In order to show that this criterion is in fact necessary and sufficient, we first note that all our solutions form irreducible multiplets of the Weyl subgroup of the E_{11-D} CJ group [24]. In particular, the single-charge solutions form highest-weight representations of the Weyl group [40]. In any simple single-charge solution, all the axions vanish, as do all field strengths other than the specific 2-form that carries the charge. This will continue to be true after acting with any element of the Weyl group, since it simply permutes field strengths, together with their dilaton vectors [24]. The complete multiplets of multi-charge solutions that we listed above can be generated from any given member of the multiplet by acting with Weyl group. Since the action of the Weyl group is the same regardless of the number of charges N in a particular solution, it follows that we only need verify that one member of the Weyl group multiplet has the simple form of solution where all non-charge-carrying field strengths vanish, in order to establish that all members of the multiplet have this property. This is a much simpler task than verifying the point for each member of the multiplet, and indeed one can easily check that it is true. It should be emphasised that since the Weyl group preserves the dot products between dilaton vectors [24], it follows that the criterion that a set of N field strengths have dilaton vectors satisfying (2.9) is not only necessary, but also sufficient, as a procedure for generating all simple N-charge solutions.

2) The following provides another argument which establishes that simple multi-charge 2-form solutions exist if and only if the dilaton vectors of the participating field strengths satisfy (2.9). The potentially dangerous terms in the Lagrangian that could spoil the existence of the simple solution are either interactions of the form $\chi\, F_2^\alpha \cdot F_2^\beta$, which we may call Kaluza-Klein type, or interactions of the form $A \wedge F_2^\alpha \wedge F_2^\beta$, which we may call Wess-Zumino type. Here χ is an axion, A is a $(D-4)$-form potential, and F_2^α and F_2^β are two of the field strengths that participate in the solution. In fact the Wess-Zumino type interactions will always give contributions to the equations of motion that vanish in the background of the putative simple multi-charge solution, because the field strengths $F_2^\alpha, F_2^\beta, \ldots$ involved in the solution either all carry electric charges or else all carry magnetic charges, given by (2.11).[3] Thus only interactions of the Kaluza-Klein type would cause trouble, since the equation of motion for the field χ in such a term would forbid us from setting it to zero, spoiling the existence of the simple solution. One can show that there is a one-to-one relation between such cubic terms and the summation rules for the dilaton vectors of the three fields [40]. Specifically, it is an easily verified rule that every term in the D-dimensional Lagrangian of the Kaluza-Klein type cubic form above has the property that the associated dilaton vectors satisfy $\vec{c}_\alpha - \vec{c}_\beta = \pm\vec{c}_\chi$, where $\vec{c}_\chi$ is the dilaton vector for the axion χ, and in fact this sum rule gives the necessary and sufficient condition for the occurrence of this cubic interaction [40]. Since dilaton vectors for axions always satisfy $\vec{c}_\chi.\vec{c}_\chi = 4$ (see (2.6) and (2.7)), it follows that if $\vec{c}_\alpha$ and $\vec{c}_\beta$ satisfy (2.9), then for these particular field strengths the worrisome cubic terms cannot be present in the Lagrangian, and so the existence of the simple multi-charge solution cannot be spoiled by the Kaluza-Klein type of interaction terms. Conversely, if a pair of dilaton vectors $\vec{c}_\alpha$ and $\vec{c}_\beta$ do not satisfy (2.9), then the sum rule *is* satisfied (as can easily be verified from (2.6) and (2.7)), implying that the cubic term will occur in the Lagrangian, and so the simple multi-charge solution involving these field strengths will not exist.

We have shown that in terms of the field strengths F^α appearing in the truncated Lagrangian (2.8), the requirement that their dilaton vectors satisfy (2.9) implies that the only cubic interactions that can arise are of the Wess-Zumino type, and furthermore the contributions that these make in the equations of motion vanish owing to the purely electric or purely magnetic nature of the charges carried by the field strengths F^α, as given by (2.11). Of course, we should further verify that all the interactions of higher than cubic order also give no contributions in the equations of motion, for the solutions we discussed above. In fact, the higher-order interaction terms are also governed by dilaton vector sum-rules, which satisfy a chain-rule relation [40]. This implies that if the solution is immune from all the cubic interactions, it is immune from all higher-order interactions as well.

It is worth remarking that applying the same argument to the case of two 3-form field strengths, whose dilaton vectors always satisfy (4.1), we find that a cubic interaction $\chi\, F_3^\alpha \cdot F_3^\beta$ *does* exist, which explains why simple multi-charge solutions involving two or more

[3] Recall that, as we discussed in section 2.2, the field strengths F_n^α in the truncated Lagrangian (2.8) are not necessarily the same as the ones coming by direct reduction from $D = 11$. In cases where some of the field strengths in a list for a multi-charge solutions are denoted with a $*$, the corresponding F_n^α's are the duals of the directly-reduced fields. Thus although the F_n^α fields themselves all carry electric charges, or all carry magnetic charges, in terms of the original supergravity fields the solutions can be dyonic.

3-form field strengths cannot occur.

3) It is also worth remarking that although we enumerated all of the $N \geq 2$ combinations of field strength configurations allowed by (2.9), in fact the combination rules are already completely encoded by the allowed 2-charge configurations. This is because any $N \geq 3$ charge configuration will be allowed by (2.9) if and only if all pairwise sub-combinations of two field strengths are allowed. The utility of nevertheless listing the $N \geq 3$ combinations explicitly is that the above rule, although easily stated, is not necessarily easy to implement by hand in practice. This point becomes more acutely apparent in the case of solutions for 1-form or 0-form field strengths, as we shall see presently.

4) Note that all the $N = 2$ solutions preserve $\frac{1}{4}$ of the supersymmetry and all the $N = 3, 4$ solutions preserve $\frac{1}{8}$ of the supersymmetry. Thus we see that for the cases $N = 1, 2, 3$, each additional charge breaks one half of the remaining supersymmetry, but the introduction of the fourth charge does not further break the supersymmetry [7] (although it does, however, modify the structure of non-vanishing eigenvalues in the Bogomol'nyi matrix [9]), if the sign of the new charge is appropriately chosen. For the other choice of the sign, it will break all the supersymmetry [9, 42]. We shall explain this in section 6. Purely electric or purely magnetic 2-form solutions can have a maximum of only $N = 3$ charges; all the $N = 4$ solutions are dyonic in the sense that some of the four participating (original) field strengths carry electric charges and the others carry magnetic charges. The Reissner-Nordstrøm black hole solutions arise when all four of these charges are set equal. They occur with multiplicity $M = 756$. Note that in the second and third field configurations given by (4.10), the solution contains two electric and two magnetic charges. Reissner-Nordstrøm black holes of a similar kind arise also in the compactified heterotic string. However, the first of the field configurations in (4.10) is of a different kind, in that three of the charges are electric and one is magnetic (or *vice versa*); solutions of this type do not occur in the heterotic string. This can be understood from the fact that in this case at least two R-R fields are needed, whereas there are none in the heterotic string. In $D = 5$, there are in total 135 3-charge black holes, which all give rise to Reissner-Nordstrøm black holes when the charges are set equal.

5) All the supersymmetric dyonic solutions in $D = 4$ are of the second kind [7], in that it is *different* field strengths that carry the electric and the magnetic charges, rather than having one field strength carrying electric and magnetic charges simultaneously. In fact, there does exist a 2-charge solution in $D = 4$ in which a single field strength carries both electric and magnetic charges [41]. This is a dyonic black hole of the first kind. However, it breaks all the supersymmetry. In fact it can be viewed as a bound state with positive binding energy [41]. When its electric and magnetic charges are equal, the solution reduces to a $D = 4$ Reissner-Nordstrøm black hole. Thus although the dyonic solution breaks all the supersymmetry in general, the supersymmetry is fully restored at the horizon, which is $\mathrm{AdS}_2 \times S^2$, as well as in the asymptotically Minkowskian region near infinity.

4.1.4 Multi-charge p-branes from 1-form field strengths

We now turn to the case of p-brane solutions using 1-form field strengths, which exist in the fully-dualised supergravities for all $2 \le D \le 9$, although in this section we shall consider only $D \ge 3$. The discussion for $D = 2$ will be given in section 5. 1-form field strengths can support either electric instantons in a Euclidean-signature space or magnetic $(D-3)$-branes in the usual Lorentzian-signature spacetime. In the latter case, the transverse space is two-dimensional. It turns out that multi-charge solutions can involve up to $N_{\max} = 8$ participating field strengths, although as usual, the maximum number depends upon the dimension D. We find that the multiplicities M of the 1-form solutions are given by

Dim.	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
$D=9$	$1_{\frac{1}{2}}$							
$D=8$	$4_{\frac{1}{2}}$	$3_{\frac{1}{4}}$						
$D=7$	$10_{\frac{1}{2}}$	$15_{\frac{1}{4}}$						
$D=6$	$20_{\frac{1}{2}}$	$70_{\frac{1}{4}}$	$60_{\frac{1}{8}}$	$15_{\frac{1}{8}}$				
$D=5$	$36_{\frac{1}{2}}$	$270_{\frac{1}{4}}$	$540_{\frac{1}{8}}$	$135_{\frac{1}{8}}$				
$D=4$	$63_{\frac{1}{2}}$	$945_{\frac{1}{4}}$	$3780_{\frac{1}{8}}$ $315_{\frac{1}{8}}$	$945_{\frac{1}{8}}$ $3780_{\frac{1}{16}}$	$2835_{\frac{1}{16}}$	$945_{\frac{1}{16}}$	$135_{\frac{1}{16}}$	
$D=3$	$120_{\frac{1}{2}}$	$3780_{\frac{1}{4}}$	$37800_{\frac{1}{8}}$	$9450_{\frac{1}{8}}$ $113400_{\frac{1}{16}}$	$113400_{\frac{1}{16}}$	$56700_{\frac{1}{16}}$	$16200_{\frac{1}{16}}$	$2025_{\frac{1}{16}}$

Table 4: Multiplicities for supersymmetric 1-form solutions

In the above table, we list the multiplicities M of the possible field strength configurations. The dimension of the Weyl group representation is given by $2^N M$. (The reason for the extra 2^N factor, which did not arise in the case of 2-form solutions, is because of a special feature of 1-form field strengths, related to the fact that their dilaton vectors, together with the negatives of the dilaton vectors, form the roots of the E_{11-D} algebra [24].) Again, the subscripts on the multiplicities indicate the fractions of preserved supersymmetry.

The classification of single-charge p-branes for 1-form field strengths is completely straightforward since any one of them can give rise to such a solution. As can be seen from the multiplicities listed in Table 4, the classification of multi-charge solutions rapidly becomes rather complicated. This is merely because of the profusion of combinatoric possibilities, and the underlying structure is still very simple: any set of N 1-form field strengths whose dilaton vectors satisfy (2.9) will give rise to a simple N-charge solution. As we discussed in section 3.1.2, the essential combination rules are in fact already encoded in the results for 2-charge solutions, since the dilaton vectors for a set of N field strengths will satisfy (2.9) if and only if all pairwise combinations of dilaton vectors satisfy (2.9). Accordingly, we shall only present the explicit listings of 2-charge combinations in this section. The full listings, together with their individual multiplicities, are relegated to the appendix. The sums of these individual multiplicities make up the total multiplicities M given in Table 4.

In all the listings, the indices $(i, j, k, \ldots)$ are understood to be all different, and to run from 1 to $(11 - D)$.

$D = 8, 7$

In both $D = 8$ and 7 dimensions, the maximum number of 1-form field strengths that can satisfy (2.9) is $N_{\rm max} = 2$, with total multiplicities $M = 3$ and 15 respectively:

$$D = 8: \qquad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}\}_3 \ , \tag{4.11}$$

$$D = 7: \qquad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}\}_{12} \ , \qquad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}\}_3 \ . \tag{4.12}$$

$D = 6$

In this dimension, we have $N_{\rm max} = 4$, The $N = 2$ solutions, numbering 70 in total, are given by

$$\{F_1^{(ijk)}, F_1^{i\ell m}\}_{15} \ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}\}_{30} \ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}\}_{10} \ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}\}_{15} \ . \tag{4.13}$$

$D = 5$

As in $D = 6$, we have $N_{\rm max} = 4$ in $D = 5$. The $N = 2$ solutions, of which there are 270 in total, are given by

$$\begin{gathered} \{F_1^{(ijk)}, F_1^{i\ell m}\}_{90} \ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}\}_{60} \ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}\}_{60} \ , \\ \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}\}_{45} \ , \quad \{*F_4, \mathcal{F}_1^{(ij)}\}_{15} \ . \end{gathered} \tag{4.14}$$

Note that in this case there is an additional 1-form field strength $*F_4$, coming from the dualisation of the 4-form F_4.

$D = 4$

In $D = 4$, there are a total of 63 1-form field strengths: 35 $F_1^{(ijk)}$, 21 $\mathcal{F}_1^{(ij)}$ and 7 $*F_3^{(i)}$ which come from the dualisation of the 3-forms. There can be a up to $N_{\rm max} = 7$ 1-form field strengths that satisfy (2.9). For $N = 2$, there are a total of $M = 945$ possibilities:

$$\begin{aligned} &\{F_1^{(ijk)}, *F_3^{(i)}\}_{105}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}\}_{315}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}\}_{105}\ , \\ &\{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}\}_{210}\ , \quad \{\mathcal{F}_1^{(ij)}, *F_3^{(k)}\}_{105}\ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}\}_{105}\ . \end{aligned} \tag{4.15}$$

$D = 3$

There are a total of 120 1-form field strengths in $D = 3$: 56 $F_1^{(ijk)}$ coming from dimensional reduction of the F_4 in $D = 11$, 28 $\mathcal{F}_1^{(ij)}$ coming from the metric, and in addition, 28 $*F_2^{(ij)}$ and 8 $*\mathcal{F}_2^{(i)}$ coming from dualisation. The maximal number of 1-forms that can satisfy (2.9) in $D = 3$ is $N_{\rm max} = 8$. The $N = 2$ solutions, with total multiplicity $M = 3780$, are given by

$$\begin{aligned} &\{*F_2^{(ij)}, *F_2^{(k\ell)}\}_{210}\ , \quad \{*F_2^{(ij)}, *\mathcal{F}_2^{(i)}\}_{56}\ , \quad \{F_1^{(ijk)}, *F_2^{(i\ell)}\}_{840}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}\}_{840} \\ &\{F_1^{(ijk)}, *\mathcal{F}_2^{(\ell)}\}_{280}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}\}_{168}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}\}_{560}\ , \quad \{\mathcal{F}_1^{(ij)}, *F_2^{(ij)}\}_{48}\ , \\ &\{\mathcal{F}_1^{(ij)}, *F_2^{(k\ell)}\}_{420}\ , \quad \{\mathcal{F}_1^{(ij)} * \mathcal{F}_2^{(k)}\}_{168}\ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}\}_{210}\ , \end{aligned} \tag{4.16}$$

All the $N \geq 3$ solutions are presented in the appendix.

4.1.5 Comments on 1-form solutions

1) As in the case of 2-form solutions, so far we have enumerated the lists of 1-form field strengths whose dilaton vectors satisfy the condition (2.9). Again, it is necessary now to verify that all these combinations of field strengths do indeed admit the construction of simple multi-charge solutions, and furthermore, that these combinations represent all of the possible simple multi-charge solutions. Although the multiplicities of the multi-charge solutions can be very large (see Table 4), the solutions form irreducible multiplets under the Weyl groups of the E_{11-D} CJ groups. The axions (which are the potentials for the 1-form field strengths) and the dilatons parametrise homogeneous coset spaces E_{11-D}/H_{11-D}, where H_{11-D} is the maximal compact subgroup of E_{11-D}. In particular, the dilaton vectors associated with the axions, together with their negatives, are precisely the root vectors of E_{11-D} [24]. This implies that the axions are equivalent under the Weyl group, which permutes the root vectors. Thus verifying that any one member of a Weyl multiplet is a genuine solution of the full D-dimensional supergravity equations of motion implies that the entire Weyl multiplet are also genuine solutions. The task is thus reduced to a simple one, and we have checked by this means that all the combinations of field strengths that we list do indeed give rise to multi-charge solutions. Since the dot-products between dilaton vectors are preserved under the Weyl group, it also follows that the listed combinations represent all the possible simple multi-charge solutions. In fact the same argument that we gave in section 4.1.3 can be applied here, to show that the potentially dangerous interaction terms in the Lagrangian that might spoil the simple multi-charge 1-form solutions are absent if and only if the participating field strengths have dilaton vectors that satisfy (2.9). The argument again involves showing that the dilaton sum rules governing cubic interactions forbid the occurrence of these dangerous terms for the sets of field strengths that we are using. (In the case of 1-form field strengths there is actually another way to choose a set of field strengths, whose dilaton vectors do not satisfy (2.9), for which there are again no interaction terms that contribute in the equations of motion in the p-brane solution backgrounds. This is done by choosing a set of 1-forms whose dilaton vectors form the simple roots of the E_{11-D} algebra [43, 44]. However, these N-charge solutions are not expressible in terms of N independent harmonic functions, and although they can be extremal, the solutions are not supersymmetric. In fact they can be viewed as bound states with negative binding energy [43].)

2) The discussion of the fractions of supersymmetry that are preserved by the multi-charge 1-form solutions is straightforward for $N \le 3$. In these cases, just as for 2-form solutions, the fraction of preserved supersymmetry is 2^{-N}, so that the addition of each extra charge halves the remaining supersymmetry. For 4-charge 1-form solutions, it turns out that there are now two possibilities. In some cases, the introduction of the fourth charge does not break the supersymmetry any further, and $\frac{1}{8}$ of the original supersymmetry is preserved. This is the same as the situation for 4-charge 2-form solutions that we discussed in section 3.1.2. In other cases, when other kinds of combinations of four field strengths are involved, the 4-charge 1-form solutions instead preserve $\frac{1}{16}$ of the original supersymmetry. For $5 \le N \le 8$ charges, all the solutions preserve $\frac{1}{16}$ of the original supersymmetry. We shall return to the discussion of supersymmetry in section 6.

So far, we have completed the classification of simple multi-charge p-branes for all the massless supergravities in $D \ge 3$. We shall discuss the details of dimensional reduction to $D = 2$ in section 5. It is interesting to note that in massless supergravities, the minimum non-vanishing fraction of preserved supersymmetry for any p-brane solution is $\frac{1}{16}$.

4.2 Domain-walls in massive supergravities

In the previous subsection 3.1, we obtained the simple multi-charge p-brane solutions for $n = 4, 3, 2$ and 1-form field strengths. For the solutions of massless maximal supergravities, these results are complete. However, we are interested in obtaining all the BPS solutions in $D = 11$, and these do not only come from the oxidations of p-brane solutions in lower-dimensional *massless* supergravities. Some BPS solutions in $D = 11$ come instead from the oxidation of p-brane solutions of the *massive* maximal supergravities that can also be obtained as consistent dimensional reductions from $D = 11$. The standard toroidal compactifications of eleven-dimensional supergravity can be generalised, by allowing one or more axions in $(D + 1)$ dimensions to be linearly dependent on the the compactifying coordinates [36, 8, 38]. The constants of proportionality become cosmological terms in D dimensions. The consistency of the reduction is not spoiled, since the axions that are involved in the generalised reduction enter the $(D+1)$-dimensional equations of motion only through their derivatives. The cosmological terms can be viewed as 0-form field strengths, labelled using the same scheme as we have adopted for the higher-degree field strengths. In D dimensions, there can be a total of $(11 - D)!/(4!(7 - D)!)$ of the form $F_0^{(ijk\ell)}$ and $(11 - D)!/(3!(8 - D)!)$ of the form $\mathcal{F}_0^{(ijk)}$, with associated dilaton vectors $\vec{a}_{ijk\ell}$ and $\vec{b}_{ijk}$ respectively, defined in (2.6). In addition, there can be D-forms in D dimensions, which can be dualised to give further cosmological terms. Note that unlike the field strengths in massless supergravities, these 0-form field strengths cannot all coexist simultaneously in one single Lagrangian; there are many different massive supergravities, each of which contains a subset of the above list of possible cosmological terms [8, 38]. This is because a 0-form field strength is really an integration constant in the Lagrangian, and it either vanishes or it doesn't. It is not like the situation with a higher-degree field strength, for which the choice as to whether or not it will carry a charge remains as yet unsettled in the Lagrangian. Thus whilst in the usual massless cases the question of what possible combinations of field strengths may carry charges need be decided only at the stage of considering solutions, in the massive theories the "charges" are already present in the Lagrangian, and the restrictions on possible non-vanishing combinations are already operative in the construction of the Lagrangian itself. From the eleven-dimensional point of view, however, all the solutions of these different massive supergravities are equally important in that they are solutions of the $D = 11$ theory.

The p-brane solutions supported by 0-form field strengths can only be magnetic $(D-2)$-branes, since an electric solution would have to be a (-2)-brane, which does not exist. Thus we need consider only magnetic $(D-2)$-branes, which are also known as domain walls. These are more difficult to study than the p-branes discussed in the previous subsection, in that from the lower-dimensional point of view, the domain-wall solutions can belong to large numbers of different massive theories. Furthermore, the CJ groups of the massless supergravities are broken in the massive theories [8]. However, there is a simple criterion to decide whether a domain wall solution is possible or not. First of all, each of the 0-form field strengths can give a single-"charged" domain-wall solution. For solutions with $N \geq 2$ charges, the selection rule is in fact the same as in the previous section, namely that the dilaton vectors of the N 0-form field strengths must satisfy the dot product relation (2.9), with $n = 0$. Thus it simply reduces to the usual mechanical process of enumerating all the possible combinations of 0-form field configurations that satisfy (2.9), using their associated dilaton vectors as given in (2.6). For domain-wall solutions, it turns out that we have $N_{\max} = 8$ when descend down to $D = 3$. The multiplicities M for each number of charges N in $3 \leq D \leq 8$ are presented in Table 5.

Dim.	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
$D=8$	$1_{\frac{1}{2}}$							
$D=7$	$5_{\frac{1}{2}}$	$4_{\frac{1}{4}}$						
$D=6$	$15_{\frac{1}{2}}$	$31_{\frac{1}{4}}$						
$D=5$	$35_{\frac{1}{2}}$	$211_{\frac{1}{4}}$	$271_{\frac{1}{8}}$	$54_{\frac{1}{8}}$				
$D=4$	$71_{\frac{1}{2}}$	$1001_{\frac{1}{4}}$	$3871_{\frac{1}{8}}$	$777_{\frac{1}{8}}$ $3136_{\frac{1}{16}}$	$1332_{\frac{1}{16}}$	$316_{\frac{1}{16}}$	$36_{\frac{1}{16}}$	
$D=3$	$134_{\frac{1}{2}}$	$3836_{\frac{1}{4}}$	$32088_{\frac{1}{8}}$	$6384_{\frac{1}{8}}$ $82632_{\frac{1}{16}}$	$49232_{\frac{1}{16}}$ $56928_{\frac{1}{32}}$	$16376_{\frac{1}{16}}$ $48800_{\frac{1}{32}}$	$3120_{\frac{1}{16}}$ $14768_{\frac{1}{32}}$	$240_{\frac{1}{16}}$ $624_{\frac{1}{32}}$

Table 5: Multiplicities for domain-wall solutions

Note that the occurrence of large prime factors in some of the multiplicities in the list is consistent with the fact that these solutions do not in general form multiplets under any group. As usual, the subscripts on the multiplicities indicate the fractions of preserved supersymmetry.

As we observed previously in the case of 2-form and 1-form solutions, the combination rules for sets of N field strengths whose dilaton vectors satisfy the condition (2.9) are already encoded in the $N = 2$ combination rules. Thus we shall present here in this section the lists of 2-charge solutions for 0-form field strengths, for $3 \le D \le 7$. We shall also give the lists for the maximal numbers of field strengths in each dimension, since they lead to the maximal numbers of intersections that can be achieved in $D = 10$ or $D = 11$. The multiplicities for intermediate numbers of charges are given in Table 5.

$D = 7$

$N = 2$ is the maximum number of charges allowed in $D = 7$, and there is a multiplet of four 2-charge combinations, given by

$$\{F_0^{(ijk\ell)}, \mathcal{F}_0^{(ijk)}\}_4 \ , \tag{4.17}$$

$D = 6$

In this case, we also have $N_{\rm max} = 2$, but with 31 solutions:

$$\{F_0^{(ijk\ell)}, \mathcal{F}_0^{(ijk)}\}_{20} \ , \quad \{\mathcal{F}_0^{(ijk)}, \mathcal{F}_0^{(i\ell m)}\}_3 \ , \quad \{\mathcal{F}_0^{(jik)}, \mathcal{F}_0^{(\ell im)}\}_8 \ . \tag{4.18}$$

Note that one must be careful, in the case of the fields $\mathcal{F}_0^{(ijk)}$, to take account of the fact that although they can be taken to be antisymmetric in jk, the index i has a distinguished rôle, and furthermore they are defined only for $i < j$ and $i < k$.

$D = 5$

In five dimensions, $N_{\rm max}$ is equal to 4. The $N = 2$ combinations are given by

$$\begin{aligned} &\{F_0^{(ijk\ell)}, F_0^{(ijmn)}\}_{45} \ , \quad \{F_0^{(ijk\ell)}, \mathcal{F}_0^{(ijk)}\}_{60} \ , \quad \{F_0^{(ijk\ell)}, \mathcal{F}_0^{(mjn)}\}_{40} \ , \\ &\{\mathcal{F}_0^{(ijk)}, \mathcal{F}_0^{(i\ell m)}\}_{18} \ , \quad \{\mathcal{F}_0^{(jik)}, \mathcal{F}_0^{(\ell im)}\}_{48} \ . \end{aligned} \tag{4.19}$$

The combinations for $N = N_{\rm max} = 4$ are

$$\{F_0^{(ijkm)}, F_0^{(i\ell mn)}, \mathcal{F}_0^{(jkm)}, \mathcal{F}_0^{(\ell mn)}\}_{48} \ , \quad \{\mathcal{F}_0^{(ik\ell)}, \mathcal{F}_0^{(imn)}, \mathcal{F}_0^{(jkm)}, \mathcal{F}_0^{(j\ell n)}\}_6 \ . \tag{4.20}$$

$D = 4$

In four dimensions, the maximum number of charges is $N_{\rm max}=7$. The allowed combinations for $N=2$ are given by

$$\begin{aligned}
&\{F_0^{(ijk\ell)}, F_0^{(ijmn)}\}_{315}\ , \quad \{F_0^{(ijk\ell)}, \mathcal{F}_0^{(ijk)}\}_{140}\ , \quad \{F_0^{(ijk\ell)}, \mathcal{F}_0^{(mjn)}\}_{280}\ ,\\
&\{\mathcal{F}_0^{(ijk)}, {*F_4}\}_{35}\ , \quad \{\mathcal{F}_0^{(ijk)}, \mathcal{F}_0^{(i\ell m)}\}_{63}\ , \quad \{\mathcal{F}_0^{(jik)}, \mathcal{F}_0^{(\ell im)}\}_{168}\ .
\end{aligned} \tag{4.21}$$

The allowed combinations for $N=N_{\rm max}=7$ are

$$\begin{aligned}
&\{F_0^{(ijk\ell)}, F_0^{(ijmn)}, F_0^{(ikmp)}, F_0^{(i\ell np)}, F_0^{(jknp)}, F_0^{(j\ell mp)}, F_0^{(k\ell mn)}\}_{30}\ ,\\
&\{\mathcal{F}_0^{(i\ell m)}, \mathcal{F}_0^{(inp)}, \mathcal{F}_0^{(j\ell n)}, \mathcal{F}_0^{(jmp)}, \mathcal{F}_0^{(k\ell p)}, \mathcal{F}_0^{(kmn)}, {*F_4}\}_6\ .
\end{aligned} \tag{4.22}$$

$D=3$

In three dimensions, the maximal allowed number of charges is $N_{\rm max}=8$. The possible combinations for $N=2$ are

$$\begin{aligned}
&\{F_0^{(ijk\ell)}, F_0^{(ijmn)}\}_{1260}\ , \quad \{F_0^{(ijk\ell)}, \mathcal{F}_0^{(ijk)}\}_{280}\ , \quad \{F_0^{(ijk\ell)}, \mathcal{F}_0^{(mjn)}\}_{1120}\ ,\\
&\{\mathcal{F}_0^{(ijk)}, {*F_3^{(\ell)}}\}_{280}\ , \quad \{F_0^{(ijk\ell)}, {*F_3^{(i)}}\}_{280}\ , \quad \{\mathcal{F}_0^{(ijk)}, \mathcal{F}_0^{(i\ell m)}\}_{168}\ ,\\
&\{\mathcal{F}_0^{(ijk)}, \mathcal{F}_0^{(i\ell m)}\}_{448}\ .
\end{aligned} \tag{4.23}$$

The combinations for $N=N_{\rm max}=8$ are

$$\begin{aligned}
&\{F_0^{(ijk\ell)}, F_0^{(ijmn)}, F_0^{(ijpq)}, F_0^{(ikmp)}, F_0^{(iknq)}, F_0^{(i\ell mq)}, F_0^{(i\ell np)}, {*F_3^{(i)}}\}_{240}\ ,\\
&\{F_0^{(ijkn)}, F_0^{(ikpq)}, F_0^{(i\ell np)}, F_0^{(imnq)}, \mathcal{F}_0^{(jkn)}, \mathcal{F}_0^{(\ell np)}, \mathcal{F}_0^{(mnq)}, {*F_3^{(i)}}\}_{384}\ .\\
&\{F_0^{(j\ell nq)}, F_0^{(jmpq)}, F_0^{(kmnq)}, F_0^{(k\ell pq)}, \mathcal{F}_0^{(ijk)}, \mathcal{F}_0^{(i\ell m)}, \mathcal{F}_0^{(inp)}, {*F_3^{(q)}}\}_{240}\ .
\end{aligned} \tag{4.24}$$

4.2.1 Comments on 0-form solutions

1) As in the cases of solutions for higher-degree forms that we discussed previously, again here for 0-forms it is still necessary to show that the configurations that we have listed do indeed give rise to simple multi-charge solutions of the full equations of motion of the D-dimensional supergravities. In other words, again we have to make sure that interaction terms in the Lagrangian do not spoil the solutions, by preventing us from setting to zero all the other fields in the theory. The Weyl group arguments that we used previously do not help us here, since the standard CJ supergravity symmetries of the massless theories are broken by the generalised Scherk-Schwarz reductions. We can, however, still use the other argument that we presented previously, based on the fact that interaction terms in the Lagrangian occur if and only if the dilaton vectors of the interacting fields satisfy appropriate sum rules. It is straighforward to verify that for sets of 0-form field strengths whose dilaton vectors satisfy (2.9), cubic interactions of the form $\chi F_0^\alpha F_0^\beta$ are forbidden, and hence the simple N-charge 0-form solutions of (2.8) are indeed solutions of the full dimensionally-reduced massive supergravity theories.

2) We presented the listings of allowed field strength combinations for 2-charge 0-form solutions, the $N > 2$ charge solutions can be deduced from these by selecting sets of N fields for which all pairs satisfy the $N = 2$ conditions. We also presented the field combinations for $N = N_{\rm max}$ in each dimension. The intermediate-N cases, although easily generated in principle by a mechanical process, become complicated when the multiplicities are large. We have enumerated all these by computer, and the multiplicities are presented in Table 5.

3) As in the case of solutions involving higher-degree field strengths, the 0-form solutions with $N \le 3$ charges all preserve a fraction 2^{-N} of the original supersymmetry. For $N = 4$, some preserve $\frac{1}{8}$ whilst others preserve $\frac{1}{16}$ of the supersymmetry. For $5 \le N \le 8$ charges, some solutions preserve $\frac{1}{16}$ whilst others preserve $\frac{1}{32}$. In section 6, we shall study the supersymmetry of all the p-branes, and give precise rules that determine the fraction of preserved supersymmetry for all multi-charge solutions. With these rules, all the multi-charge p-branes, and their supersymmetry, will be derivable purely from the knowledge of the 2-charge solutions.

5 $D = 2$ supergravities and their p-brane solutions

So far in the paper, our discussions have been restricted to supergravities in dimensions $D \ge 3$. As can be seen from (2.7), the Kaluza-Klein reduction scheme that we have been using degenerates when $D = 2$. This is because we cannot any longer choose to work with a metric that is in the Einstein frame once we descend to $D = 2$.

5.1 Kaluza-Klein reduction from $D = 3$ to $D = 2$

We shall make the following choice for the Kaluza-Klein reduction of the three-dimensional metric:

$$ds_3^2 = e^{\varphi}\, ds_2^2 + e^{2\varphi}\, (dz_9 + \mathcal{A}_1^{(9)})^2 \ , \tag{5.1}$$

where φ is the new dilatonic scalar, and $\mathcal{A}_1^{(9)}$ is the new Kaluza-Klein vector potential. All other fields in the three-dimensional theory will still be reduced according to $A_n(x, z_9) \to$

$A_n(x) + A_{n-1}(x) \wedge dz_9$. Thus kinetic terms in $D = 3$ will reduce to $D = 2$ according to the following rules:

$$\begin{aligned}
-\tfrac{1}{12} e\, F_3^2 &\longrightarrow -\tfrac{1}{4} e\, e^{-2\varphi}\, F_2^2 \ , \\
-\tfrac{1}{4} e\, F_2^2 &\longrightarrow -\tfrac{1}{4} e\, F_2^2 - \tfrac{1}{2} e\, e^{-\varphi}\, F_1^2 \ , \\
-\tfrac{1}{2} e\, F_1^2 &\longrightarrow -\tfrac{1}{2} e\, e^{\varphi}\, F_1^2 - \tfrac{1}{2} e\, F_0^2 \ , \\
-\tfrac{1}{2} e\, F_0^2 &\longrightarrow -\tfrac{1}{2} e\, e^{2\varphi}\, F_0^2 \ .
\end{aligned} \tag{5.2}$$

The Einstein-Hilbert and dilaton kinetic terms of $D = 3$ reduce according to

$$eR - \tfrac{1}{2} e\, (\partial\vec{\phi})^2 \longrightarrow e\, e^{\varphi}\, R + e\, e^{\varphi}\, (\partial\varphi)^2 - \tfrac{1}{4} e\, e^{2\varphi}\, \mathcal{F}^2 - \tfrac{1}{2} e\, e^{\varphi}\, (\partial\vec{\phi})^2 \ . \tag{5.3}$$

Having established the dimensional reduction rules for all the fields, we can in principle write down all the $D = 2$ supergravity Lagrangians from the ones in $D = 3$. The 2-form field strength in $D = 2$ is not dynamical and can be dualised to a cosmological term. There can also exist a massless supergravity in $D = 2$, which has E_9 global symmetry, whose Lagrangian is given by

$$\begin{aligned}
e^{-1}\mathcal{L} \;=\;& e^{\varphi}\, R + e^{\varphi}\, (\partial\varphi)^2 - \tfrac{1}{2} e^{\varphi}\, (\partial\vec{\phi})^2 - \tfrac{1}{2} \sum_{i<j<k\le 8} e^{\vec{a}_{ijk}\cdot\vec{\phi}+\varphi}\, (F_1^{(ijk)})^2 \\
& -\tfrac{1}{2} \sum_{i<j\le 8} e^{\vec{a}_{ij}\cdot\vec{\phi}-\varphi}\, (F_1^{(ij9)})^2 - \tfrac{1}{2} \sum_{i<j\le 8} e^{\vec{b}_{ij}\cdot\vec{\phi}+\varphi}\, (\mathcal{F}_1^{(ij)})^2 \\
& -\tfrac{1}{2} \sum_{i\le 8} e^{\vec{b}_i\cdot\vec{\phi}-\varphi}\, (\mathcal{F}_1^{(i9)})^2 - \tfrac{1}{144} \epsilon^{\mu\nu}\, \partial_\mu A_0^{(ijk)}\, \partial_\nu d A_0^{(\ell mn)}\, A_0^{(pq9)}\, \epsilon_{ijk\ell mnpq} \ ,
\end{aligned} \tag{5.4}$$

where $\vec{a}_{ijk}$, $\vec{b}_{ij}$, $\vec{a}_{ij}$ and $\vec{b}_i$ are the dilaton vectors in three dimensions, given by (2.6) and (2.7) with $D = 3$. The field strengths $F_1^{(ij9)}$ and $\mathcal{F}_1^{(i9)}$ are the dimensional reductions of the three-dimensional 2-forms $F_2^{(ij)}$ and $\mathcal{F}_2^{(i)}$ respectively. All the field strengths are reduced according to the scheme given in (5.2), and their Kaluza-Klein modifications are given by the standard formulae obtained in [7].

Of course, there are numerous massive supergravities in $D = 2$, where the theories contain cosmological terms.

5.2 Instantons in $D = 2$

There are two types of p-branes in $D = 2$, namely instanton solutions using 1-form field strengths and black hole (domain wall) solutions using cosmological terms. The instanton solutions can arise in massless supergravity in $D = 2$, whose bosonic Lagrangian is given by (5.4). As in the higher-dimensional cases that we discussed earlier, we may consider a truncated Lagrangian of the form

$$\begin{aligned} \mathcal{L} &= e\, e^{\varphi}\, R + e\, e^{\varphi}\, (\partial\varphi)^2 - \tfrac12 e\, e^{\varphi}\, (\partial\vec\phi)^2 \\ &\quad + \tfrac12 e \sum_\alpha e^{\vec c_\alpha\cdot\vec\phi+\varphi}\, (F^\alpha)^2 + \tfrac12 e \sum_a e^{\vec d_a\cdot\vec\phi-\varphi}\, (F^a)^2 \ , \end{aligned} \tag{5.5}$$

where $F^\alpha = d\chi^\alpha$ and $F^a = d\chi^a$ are 1-form field strengths. The kinetic terms for the axions χ^α and χ^a have the opposite sign to the normal ones in a Lorentzian-signature spacetime. This is because, in order to obtain instanton solutions, we need to work with a space of Euclidean signature. This unusual sign for the kinetic terms can arise naturally if one obtains the Euclidean-signature theory in $D=2$ by a dimensional reduction from $D=11$ in which the original time coordinate becomes one of the compactified directions.

The equations of motion following from the truncated Lagrangian (5.5) are

$$\begin{aligned} & R_{\mu\nu} = \nabla_\mu\partial_\nu\varphi + \tfrac12 \partial_\mu\vec\phi\cdot\partial_\nu\vec\phi - \tfrac12 \sum_\alpha e^{\vec c_\alpha\cdot\vec\phi}\, \partial_\mu\chi^\alpha\, \partial_\nu\chi^\alpha \\ &\qquad - \tfrac12 \sum_a e^{\vec d_a\cdot\vec\phi-2\varphi} \Big(\partial_\mu\chi^a\,\partial_\nu\chi^a - g_{\mu\nu}\,(\partial\chi^a)^2\Big) \ , \\ & \Box\varphi + (\partial\varphi)^2 = 0 \ , \\ & \Box\vec\phi + \partial^\mu\varphi\,\partial_\mu\vec\phi = -\tfrac12 \sum_\alpha \vec c_\alpha\, e^{\vec c_\alpha\cdot\vec\phi}\,(\partial\chi^\alpha)^2 - \tfrac12 \sum_a \vec d_a\, e^{\vec d_a\cdot\vec\phi-2\varphi}\,(\partial\chi^a)^2 \ , \\ & \nabla^\mu(e^{\vec c_\alpha\cdot\vec\phi+\varphi}\,\partial_\mu\chi^\alpha) = 0 \ , \qquad \nabla^\mu(e^{\vec d_a\cdot\vec\phi-\varphi}\,\partial_\mu\chi^a) = 0 \ . \end{aligned} \tag{5.6}$$

Note that the equation of motion for the field φ has no sources involving the axionic fields, and so for extremal instanton solutions we may just set $\varphi=0$. (This can only be done after varying the action, however.) The field strengths F^α and F^a are on an equal footing after setting $\varphi=0$, and hence the supersymmetric solutions from the two types of fields will have the same structure, and their dilaton vectors must satisfy the same conditions. Without loss of generality, we can therefore study the conditions for the existence of multi-charge solutions using the fields F^α, and then trivially extend the discussion to include the F^a fields afterwards. We find that multi-charge instanton solutions exist if the dilaton vectors $\vec c_\alpha$ satisfy $\vec c_\alpha\cdot\vec c_\beta = 4\delta_{\alpha\beta}$, which is precisely the usual requirement (2.9) for multi-charge 1-form solutions. The $D=2$ multi-instantons are given by

$$\begin{aligned} ds_2^2 &= dr^2 + r^2\, d\theta^2 \ , \qquad \varphi = 0 \ , \\ \vec\phi &= \tfrac12 \epsilon \sum_\alpha \vec c_\alpha\, \log H_\alpha \ , \end{aligned} \tag{5.7}$$

where $H_\alpha = 1 + |Q_\alpha|\,\log r$, and $\epsilon=+1$ for electric instantons, and $\epsilon=-1$ for magnetic instantons. The axions are given by $\chi^\alpha = H_\alpha^{-1}$ in the electric case, and $\chi^\alpha = Q_\alpha\,\theta$ in the magnetic case. (As usual, when we present a truncated Lagrangian, we choose to make the necessary dualisations so that all the field strengths F^α carry electric charges, or all of them carry magnetic charges. In terms of the original fields in (5.4), some charges may be electric and others magnetic.) Following similar arguments to those given earlier in $D \ge 3$, we may verify that the multi-instanton solutions of the truncated Lagrangian (5.5) are also solutions for the full two-dimensional massless Lagrangian (5.4).

We have now established the rules that determine the field strength configurations for multi-charge instanton solutions in $D = 2$, namely that the associated three-dimensional dilaton vectors $\vec{c}_\alpha$ have to satisfy (2.9), $\vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta}$. Pairwise configurations of field strengths in $D = 3$ that satisfy this dilaton vector dot-product condition are listed in (4.16). In $D = 2$, there are additional dyonic solutions of the second kind, involving dualisations of the field strengths, since the signs of the dilaton vectors do not affect the conditions $\vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta}$. This gives all the pairs of fields in $D = 2$, from which all the higher N-charge solutions can then be obtained. Since the $\pm$ choices of the signs of the dilaton vectors $\vec{c}_\alpha$ do not affect the dot product conditions $\vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta}$, it follows that if a given axion carries electric charge in one multi-charge solution, there is another multi-charge solution where instead the given axion carries a magnetic charge, while all the other axions remain unchanged. Thus the multiplicities for an N-charge instanton solution in $D = 2$ are 2^N times those listed in Table 4 for $D = 3$ solutions. The maximum number of charges in a given solution is $N_{\rm max} = 8$.

The $D = 2$ multi-charge instanton solutions can oxidise back to give either instanton or black hole solutions in $D = 3$, which were classified in section 4.1, or to give intersections of instantons and black holes in $D = 3$. Note that since $c_\alpha \cdot (-\vec{c}_\alpha) = -4$, it follows that there can be no dyonic solutions of the first kind in $D = 2$, where a single field strength would carry both electric and magnetic charge.

The supersymmetry of the multi-charge instantons in $D = 2$ can be established using the procedures that we shall discuss in section 6.

5.3 Black holes in $D = 2$

Black hole solutions arise in 2-dimensional massive supergravities. There are a total of three categories of multi-charge black hole solutions in $D = 2$. The first comprises those which are the vertical dimensional reduction of black holes in $D = 3$ or the double dimensional reduction of strings in $D = 3$. All these solutions have already been completely classified in section 4 for $D = 3$. The second category comprises multi-charge solutions where some charges are carried by field strengths that were already 0-forms in $D = 3$, while the rest are carried by 0-forms coming from the dimensional reduction of 1-forms in $D = 3$. These solutions will oxidise back to intersections of black holes and strings in $D = 3$. The third category comprises multi-charge solutions where one charge is carried by the Kalazu-Klein vector coming from the $D = 3$ to $D = 2$ reduction (which is dualised to a cosmological term), and the rest are carried by the 0-forms that were already 0-forms in $D = 3$. These solutions will oxidise back to the intersections of strings with a wave in $D = 3$. To see explicitly how these three categories of solutions arise in $D = 2$, we need to consider the relevant dimensionally-reduced $D = 2$ massive Lagrangians.

The general class of two-dimensional Lagrangians that we shall be concerned with take the form

$$\begin{aligned}\mathcal{L} &= e\, e^{\varphi}\, R + e\, e^{\varphi}\, (\partial\varphi)^2 - \tfrac12 e\, e^{\varphi}\, (\partial\vec\phi)^2 \\ &\quad -\tfrac12 e\, e^{2\varphi} \sum_\alpha m_\alpha^2\, e^{\vec c_\alpha\cdot\vec\phi} - \tfrac12 e \sum_a \tilde m_a^2\, e^{\vec d_a\cdot\vec\phi} - \tfrac12 e\, m_0^2\, e^{-2\varphi}\ ,\end{aligned}\tag{5.8}$$

where the three kinds of cosmological term arise as follows. Those with dilaton vectors $\vec c_\alpha$ correspond to the reductions of existing 0-forms in $D = 3$. Those with dilaton vectors $\vec d_a$ correspond to the reductions of 1-forms in $D = 3$. Finally, the last cosmological term in (5.8) comes from the dualisation of the Kaluza-Klein vector $\mathcal{A}_1^{(9)}$ in (5.1). It should be understood here that it is not necessarily the case that all the cosmological terms displayed in (5.8) can coexist simultaneously, for the reasons that we have already discussed in section 4. However, any set of cosmological terms which can be used to construct multi-charge solutions can be present simultaneously in the Lagrangian. Thus we will suppose that the various mass parameters (*i.e.* charges) m_α, $\tilde m_a$ and m_0 can be turned on or off at will, to give whichever permitted non-vanishing set we wish to consider at any time.

The equations of motion following from (5.8) are

$$\begin{aligned}&R_{\mu\nu} = \nabla_\mu\partial_\nu\varphi + \tfrac12\partial_\mu\vec\phi\cdot\partial_\nu\vec\phi + \tfrac14\Big(\sum_\alpha m_\alpha^2\, e^{\vec c_\alpha\cdot\vec\phi+\varphi} + \sum_a \tilde m_a^2\, e^{\vec d_a\cdot\vec\phi-\varphi} - 3m_0^2\, e^{-3\varphi}\Big)\, g_{\mu\nu}\ ,\\ &\Box\varphi + (\partial\varphi)^2 = -\tfrac12\sum_\alpha m_\alpha^2\, e^{\vec c_\alpha\cdot\vec\phi+\varphi} - \tfrac12\sum_a \tilde m_a^2\, e^{\vec d_a\cdot\vec\phi-\varphi} - \tfrac12 m_0^2\, e^{-3\varphi}\ ,\\ &\Box\vec\phi + \partial^\mu\varphi\,\partial_\mu\vec\phi = \tfrac12\sum_\alpha m_\alpha^2\, \vec c_\alpha\, e^{\vec c_\alpha\cdot\vec\phi+\varphi} + \tfrac12\sum_a \tilde m_a^2\, \vec d_a\, e^{\vec d_a\cdot\vec\phi-\varphi}\ .\end{aligned}\tag{5.9}$$

Making the metric ansatz $ds_2^2 = -e^{2A}\, dt^2 + e^{2B}\, dy^2$, the equations of motion following from (5.9) are

$$A'' + A'^2 - A'\, B' + A'\, \varphi' = \tfrac34 S - \tfrac14\sum_\alpha S_\alpha + \tfrac14\sum_a \widetilde S_a\ ,$$

$$
\begin{aligned}
&A'' + A'^2 - A'\,B' + \varphi'' - B'\,\varphi' + \tfrac{1}{2}\vec\phi' \cdot \vec\phi' = \tfrac{3}{4}S - \tfrac{1}{4}\sum_\alpha S_\alpha + \tfrac{1}{4}\sum_a \widetilde S_a \ ,\\
&\varphi'' + \varphi'^2 + A'\,\varphi' - B'\,\varphi' = -\tfrac{1}{2}S - \tfrac{1}{2}\sum_\alpha S_\alpha - \tfrac{1}{2}\sum_a \widetilde S_a \ ,\\
&\vec\phi'' + \vec\phi'\,(\varphi' + A' - B') = \tfrac{1}{2}\sum_\alpha \vec c_\alpha\, S_\alpha + \tfrac{1}{2}\sum_a \vec d_a\, \widetilde S_a \ ,
\end{aligned} \tag{5.10}
$$

where $S_\alpha = m_\alpha^2\, e^{\vec c_\alpha\cdot\vec\phi + 2B + \varphi}$, $\widetilde S_a = \tilde m_a^2\, e^{\vec d_a\cdot\vec\phi + 2B - \varphi}$ and $S = m_0^2\, e^{2B - 3\varphi}$. It is straightforward to show that these equations admit two different classes of black-hole solutions. Firstly, we can find solutions with $m_0 = 0$, of the form

$$
\begin{aligned}
ds_2^2 &= -(\prod_a \widetilde H_a)^{-1/2}\,(\prod_\alpha H_\alpha)^{1/2}\,dt^2 + (\prod_a \widetilde H_a)^{1/2}\,(\prod_\alpha H_\alpha)^{3/2}\,dy^2 \ ,\\
\vec\phi &= -\tfrac{1}{2}\sum_\alpha \vec c_\alpha\,\log H_\alpha - \tfrac{1}{2}\sum_a \vec d_a\,\log \widetilde H_a \ ,\\
\varphi &= = \tfrac{1}{2}\sum_a \log H_\alpha + \tfrac{1}{2}\sum_a \log \widetilde H_a \ ,
\end{aligned} \tag{5.11}
$$

where $H_\alpha = 1 + m_\alpha\,|y|$ and $\widetilde H_a = 1 + \tilde m_a\,|y|$ are the independent harmonic functions for the charges m_α and $\tilde m_a$, and the dilaton vectors $\vec c_\alpha$ and $\vec d_a$ satisfy the relations

$$
\vec c_\alpha\cdot\vec c_\beta = 4\delta_{\alpha\beta} + 4\ , \qquad \vec d_a\cdot\vec d_b = 4\delta_{ab}\ , \qquad \vec c_\alpha\cdot\vec d_a = 2\ . \tag{5.12}
$$

These solutions encompass the first category mentioned above (when all charges $m_\alpha = 0$ or else when all charges $\tilde m_a = 0$), and the second category (when charges of both the m_α type and the $\tilde m_a$ type are non-vanishing). Note that the conditions on $\vec c_\alpha\cdot\vec c_\beta$ and $\vec d_a\cdot\vec d_b$ in (5.12) are precisely the usual conditions (2.9) in $D=3$, for dilaton vectors for 0-form fields and 1-form fields respectively.

A second class of black-hole solutions to the equations (5.10) can be obtained by setting all the $\tilde m_a$ charges to zero, while having non-vanishing m_0 and non-vanishing m_α charges, with these latter being associated with dilaton vectors $\vec c_\alpha$ that satisfy

$$
\vec c_\alpha\cdot\vec c_\beta = 4\delta_{\alpha\beta} + 4\ . \tag{5.13}
$$

A simple calculation shows that in this case the solutions take the form

$$
\begin{aligned}
ds_2^2 &= -(\prod_\alpha H_\alpha)^{1/2}\,H^{-3/2}\,dt^2 + (\prod_\alpha H_\alpha)^{3/2}\,H^{-1/2}\,dy^2 \ ,\\
\vec\phi &= -\tfrac{1}{2}\sum_\alpha \vec c_\alpha\,\log H_\alpha\ , \qquad \varphi = \tfrac{1}{2}\sum_\alpha \log H_\alpha + \tfrac{1}{2}\log H\ ,
\end{aligned} \tag{5.14}
$$

where the harmonic functions $H_\alpha = 1 + m_\alpha\,|y|$ are the same as in the previous solutions, and $H = 1 + m_0\,|y|$ is the harmonic function for the Kaluza-Klein charge m_0. Solutions of this kind constitute the third category that we mentioned at the beginning of this subsection.

5.3.1 Comments on black holes in $D = 2$

1) There are no simple solutions that involve both m_0 and $\tilde{m}_a$ charges. This implies that there are no intersections between black holes and waves in $D = 3$. This can be understood from the fact that vertical dimensional reductions of black holes in $D = 3$ are necessarily of the Scherk-Schwarz type, where the axions that support the black hole solutions are linearly proportional to the compactifying coordinate. In such Scherk-Schwarz reductions, the Kaluza-Klein vector becomes massive [8], and hence cannot participate in supporting simple multi-charge p-brane solutions.

2) Having established the necessary requirements for multi-charge black hole solutions, we may now enumerate all the possible solutions in $D = 2$. As we mentioned, all $D = 2$ black hole solutions can be oxidised back to $D = 3$, to become intersecting strings and black holes, together with a wave when $m_0 \neq 0$. Since the criterion for the $D = 2$ solution is expressed in (5.12) and (5.13), which are dot product rules for $D = 3$ dilaton vectors, it is more convenient to characterise the $D = 2$ solutions in terms of their $D = 3$ fields. For the first category, the solutions are fully classified in $D = 3$, since these are just solutions of strings and black holes in $D = 3$. Thus the 2-charge pairs for these solutions are listed in (4.16) and (4.23). For the third category, the solutions are also fully classified, in that all the string solutions in $D = 3$ can intersect with a three-dimensional wave. In other words, we can take the dimensional reduction to $D = 2$ of any of the multi-charge string solutions in $D = 3$, and add an extra charge, namely that of the new Kaluza-Klein vector, together with its associated harmonic function.

It remains for us to classify the second category of solutions. In terms of the $D = 3$ solutions, these are the intersections of strings and black holes. Thus in terms of three-dimensional fields, the combinations of allowed field configurations involve both 1-form and 0-form field strengths, with dilaton vectors $\vec{d}_a$ and $\vec{c}_\alpha$ that satisfy (5.12). As in the previous case, the $N = 2$ charge solutions encode all the combination rules for $N \geq 3$ solutions. We find that they are given by

$$\begin{aligned}
&\{*F_2^{(ij)}, *F_3^{(k)}\}_{168}\ , \quad \{F_1^{(ijk)}, *F_3^{(\ell)}\}_{280}\ , \quad \{F_1^{(ijk)}, F_0^{(ij\ell m)}\}_{1680}\ , \quad \{*\mathcal{F}_2^{(i)}, F_0^{(jk\ell m)}\}_{280}\ ,\\
&\{\mathcal{F}_1^{(ij)}, *F_3^{(i)}\}_{28}\ , \quad \{\mathcal{F}_1^{(ij)}, F_0^{(jk\ell m)}\}_{560}\ , \quad \{*\mathcal{F}_2^{(i)}, \mathcal{F}_0^{(jk\ell)}\}_{280}\ , \quad \{*\mathcal{F}_2^{(i)}, *F_3^{(i)}\}_{8}\ ,\\
&\{*F_2^{(ij)}, F_0^{(ik\ell m)}\}_{1120}\ , \quad \{*F_2^{(ij)}, \mathcal{F}_0^{(ijk)}\}_{112}\ , \quad \{*F_2^{(ij)}, \mathcal{F}_0^{(k\ell m)}\}_{560}\ ,\\
&\{F_1^{(ijk)}, \mathcal{F}_0^{(ijk)}\}_{56}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_0^{(\ell jm)}\}_{1120}\ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_0^{(ik\ell)}\}_{210}\ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_0^{(kj\ell)}\}_{350}\ .
\end{aligned} \tag{5.15}$$

This gives a total of 6812 possible ways for a black hole to intersect a string in $D = 3$. In computing the multiplicities, recall that the i index on $\mathcal{F}_0^{(ijk)}$ must be such that $i<j$ and $i<k$.

Thus taken together with the 2-charge combination rules (4.16), (4.23) and (5.15), we now have an enumeration of all the multi-charge black hole solutions in all three categories in $D=2$.

3) The maximal number of field strengths that can participate in simple multi-charge solutions is $N_{\rm max}=9$. This can be achieved by solutions in the third category, taking the Kaluza-Klein field strength $\mathcal{F}_2^{(9)}$, which is dualised to a cosmological term m_0, together with the any of the 8-field-strength combinations listed in (4.24).

6 Supersymmetry of multi-charge p-branes

We have classified all the simple multi-charge p-brane solutions in all dimensions $D\ge 2$. We have also seen that this implies a classification of the associated intersections of p-branes, waves and NUTs in any higher dimension, since the set of intersections in a given dimension are nothing but the oxidations of all the lower-dimensional p-branes. It is important to establish what fractions of supersymmetry are preserved by the various solutions. Supersymmetry is fully preserved by the Kaluza-Klein reduction procedure itself. This means that the fraction of the original supersymmetry that is preserved by a particular lower-dimensional p-brane is the same as the fraction that is preserved by its oxidation to any higher dimension.

One way, albeit very clumsy, to determine the fraction of supersymmetry that is preserved by a D-dimensional p-brane is to examine the supersymmetry transformation rules of the D-dimensional maximal supergravity, and look for Killing spinors in the background of the p-brane, since these correspond to components of unbroken supersymmetry. This method is especially unattractive in low dimensions, where the multiplicities of the possible non-vanishing field strengths becomes very large. Furthermore, it requires that one know the explicit transformation rules for the maximal supergravity in question, and these have not in general been obtained for the many massive supergravities. An easier method is to oxidise the lower-dimensional p-brane to $D=10$ or $D=11$. At least in the case of solutions supported only by the antisymmetric tensors of $D=10$ or $D=11$, this gives a simpler system of equations for the Killing spinors.

Fortunately, there is a much easier procedure for determining the fraction of supersymmetry that is preserved by any p-brane solution. All that is necessary is to construct the

Nester form N^{AB} in $D=11$, which arises as the anti-commutator of $D=11$ supercharges, $\{Q_{\epsilon_1}, Q_{\epsilon_2}\} = \int_{\partial\Sigma} N^{AB}\, d\Sigma_{AB}$, and dimensionally reduce it to D dimensions. Since it is a purely bosonic object, this is a very simple procedure. The Bogomol'nyi matrix $\mathcal{M}$, defined by $\epsilon_1^\dagger \mathcal{M} \epsilon_2 = \int_{\partial\Sigma} N^{0r}\, r^{\tilde{d}+1}\, d\Omega_{\tilde{d}+1}$ in the asymptotic $r \to \infty$ limit, is then a 32×32 Hermitean matrix each of whose zero eigenvalues corresponds to a component of unbroken supersymmetry. It is given in terms of the mass per unit p-volume, and the charges, by [7]

$$\begin{aligned} \mathcal{M} \;=\;& m\mathbb{1} + u\,\Gamma_{012} + u_i\,\Gamma_{01i} + \tfrac{1}{2} u_{ij}\,\Gamma_{0ij} + \tfrac{1}{6} u_{ijk}\,\Gamma_{ijk} + \tfrac{1}{24} u_{ijk\ell}\,\Gamma_{ijk\ell} \\ & + v\,\Gamma_{\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}} + v_i\,\Gamma_{\hat{1}\hat{2}\hat{3}\hat{4}i} + \tfrac{1}{2} v_{ij}\,\Gamma_{\hat{1}\hat{2}\hat{3}ij} + \tfrac{1}{6} v_{ijk}\,\Gamma_{\hat{1}\hat{2}ijk} + \tfrac{1}{24} v_{ijk\ell}\,\Gamma_{\hat{1}ijk\ell} \\ & + p_i\Gamma_{0i} + \tfrac{1}{2} p_{ij}\,\Gamma_{ij} + \tfrac{1}{6} p_{ijk}\,\Gamma_{ijk} + q_i\,\Gamma_{\hat{1}\hat{2}\hat{3}i} + \tfrac{1}{2} q_{ij}\,\Gamma_{\hat{1}\hat{2}ij} + \tfrac{1}{6} q_{ijk}\,\Gamma_{\hat{1}ijk}\ . \end{aligned} \tag{6.1}$$

The prefactors of the Γ matrices are the various electric and magnetic charges associated with the various field strengths in D dimensions, according to the following scheme:

	F_4	$F_3^{(i)}$	$F_2^{(ij)}$	$F_1^{(ijk)}$	$F_0^{(ijk\ell)}$	$\mathcal{F}_2^{(i)}$	$\mathcal{F}_1^{(ij)}$	$\mathcal{F}_0^{(ijk)}$
Electric	u	u_i	u_{ij}	u_{ijk}	$u_{ijk\ell}$	p_i	p_{ij}	p_{ijk}
Magnetic	v	v_i	v_{ij}	v_{ijk}	$v_{ijk\ell}$	q_i	q_{ij}	q_{ijk}

(6.2)

where u's and p's are electric charges, and v's and q's are magnetic charges. For a given degree n of antisymmetric tensor field strength, only the terms with the corresponding charges, as indicated in (6.2), will occur in (6.1). The indices $0, 1, \ldots$ run over the dimension of the p-brane world-volume, $\hat{1}, \hat{2}, \ldots$ run over the transverse space of the y^m coordinates, and $i, j, \ldots$ run over the dimensions that were compactified in the Kaluza-Klein reduction from 11 to D dimensions. Note that the electric charges $u_{ijk\ell}$ and p_{ijk} would be associated with (-2)-branes, which presumably have no meaning. All the other Γ-matrix combinations appearing in (6.1) are Hermitean, with the exception of the Γ_{ijk} and Γ_{ij} combinations, which are anti-Hermitean. However, these are associated with instantons, whose existence requires that the "spacetime" have Euclidean signature. There will then be an extra i factor coming from the electric charges in such cases, which restores the hermiticity of the Bogomol'nyi matrix.

Determining the supersymmetry of any p-brane solution in any dimension D is now reduced to a matter of elementary algebra. All that is needed is to substitute the relevant N charges of the solution, and its mass m, into the Bogomol'nyi matrix (6.1), and then to evaluate its 32 eigenvalues. The number k of zero eigenvalues implies that a corresponding fraction $k/32$ of the original supersymmetry is preserved by the solution. It is very easy to see that any single-charge solution will give 16 zero eigenvalues, and hence will preserve $\frac{1}{2}$ the supersymmetry. Similarly, any 2-charge solution will preserve $\frac{1}{4}$, and any 3-charge solution will preserve $\frac{1}{8}$. (Of course only sets of charges that correspond to combinations of field strengths allowed by the dilaton-vector conditions (2.9) are to be considered.) For $N \geq 4$ charges, as we have indicated in earlier discussion, the fraction of preserved supersymmetry in general depends on the particular combinations of field strengths involved. For example, although all 4-charge 2-form solutions preserve $\frac{1}{8}$ of the supersymmetry, in the case of 1-form or 0-form 4-charge solutions, some preserve $\frac{1}{8}$ whilst others preserve $\frac{1}{16}$.

It should be noted that since the kinetic terms for field strengths are quadratic, there are actually 2^N different possibilities for the signs of the charges Q_α in a simple N-charge p-brane solution, where the mass is still given as the sum of the N positive quantities $|Q_\alpha|$. It turns out that when $N \geq 4$ these 2^N solutions, although equivalent from a purely bosonic point of view, can have different properties as far as supersymmetry is concerned. (This is because the field strengths enter linearly in the supersymmetry transformation rules.) To be precise, for an N-charge p-brane solution that can preserve a fraction $2^{-\tilde{N}}$ of the supersymmetry, then of the 2^N possible sign choices for the charges, $2^{\tilde{N}}$ will give solutions that do in fact preserve the fraction $2^{-\tilde{N}}$ of supersymmetry, and the remaining $2^N - 2^{\tilde{N}}$ sign choices will give solutions that preserve no supersymmetry. In other words, if an N-charge solution preserves a fraction 2^{-N} of the supersymmetry (*i.e.* the successive introduction of each of the N charges breaks a half of the remaining supersymmetry), then the sign of each of the N charges is immaterial. If now a new charge can be introduced to give an $(N+1)$-charge solution that does not further break the supersymmetry, then the same charge introduced with the opposite sign will cause *all* the supersymmetry to be broken. In other words, if a p-brane breaks $\frac{1}{2}$ of the remaining supersymmetry, so will the anti-p-brane. On the other hand, if a p-brane does not break any further supersymmetry, then the anti-p-brane will break it all, and *vice versa*. We shall present a proof of these statements below. Since, as we have noted previously, the smallest non-vanishing fraction of preserved supersymmetry in any p-brane solution is $\frac{1}{32}$, it follows that we always have $\tilde{N} \leq 5$. Consequently, for simple N-charge p-branes where N is large, the overwhelming majority break *all* the supersymmetry, even though they are extremal and related merely by sign changes of their charges to solutions that are supersymmetric. For example, if we consider 8-charge solutions that preserve $\frac{1}{32}$ of the supersymmetry, then of the 256 possible choices for the signs of the eight charges, 32 will give supersymmetric solutions, while 224 will give solutions that break all the supersymmetry. It is worth remarking that although these solutions are non-supersymmetric, there is still a no-force condition between the individual charges.

To see in detail how the supersymmetry depends on the choice of charges, we now give a complete analysis based on the Bogomol'nyi matrix.

6.1 The Bogomol'nyi matrix and supersymmetry

To begin, we note that since the mass m per unit p-volume for a simple p-brane with N charges Q_α is given by $m = \sum_\alpha |Q_\alpha|$, it follows from (6.1) that its Bogomol'nyi matrix is just the sum of the individual Bogomol'nyi matrices for each of its associated single-charge components: $\mathcal{M} = \sum_\alpha \mathcal{M}_\alpha$, where

$$\mathcal{M}_\alpha = |Q_\alpha| + Q_\alpha\, \Gamma_{(\alpha)}\ , \tag{6.3}$$

and we denote by $\Gamma_{(\alpha)}$ the particular unit-strength Γ-matrix product associated with the charge Q_α, as given by (6.1). One can easily show that the individual Bogomol'nyi matrices commute, $[\mathcal{M}_\alpha, \mathcal{M}_\beta] = 0$.

We may now give an elementary proof of the previous statements about the fractions of preserved supersymmetry. Since the individual $\mathcal{M}_\alpha$ matrices commute, it follows that they may be simultaneously diagonalised. Thus the set of k Killing spinors ϵ^a ($a = 1, \ldots, k$) for an N-charge p-brane solution, defined by $\mathcal{M}\epsilon^a = 0$, can also be chosen to be eigenstates of the individual $\mathcal{M}_\alpha$ matrices,

$$\mathcal{M}_\alpha\, \epsilon^a = \lambda^a_\alpha\, \epsilon^a\ . \tag{6.4}$$

Now the eigenvalues of each $\mathcal{M}_\alpha$ must all be non-negative [45], since otherwise there would be naked singularities in the solution. Thus in particular we must have $\lambda^a_\alpha \ge 0$, and so we have $0 = \mathcal{M}\epsilon^a = \sum_\alpha \mathcal{M}_\alpha\, \epsilon^a = \sum_\alpha \lambda^a_\alpha\, \epsilon^a$, implying that $\lambda^a_\alpha = 0$, and hence the Killing spinors ϵ^a all satisfy $\mathcal{M}_\alpha\, \epsilon^a = 0$. Each $\mathcal{M}_\alpha$ has the form given by (6.3). Since $\mathrm{tr}\, \Gamma_{(\alpha)} = 0$ and $(\Gamma_{(\alpha)})^2 = 1$, it follows that each $\mathcal{M}_\alpha$ has sixteen zero eigenvalues and sixteen non-zero eigenvalues $2|Q_\alpha|$. In an N-charge solution, there will be N individual Bogomol'nyi matrices $\mathcal{M}_\alpha$, with their associated $\Gamma_{(\alpha)}$ matrices. We must now distinguish between two cases. If the $\Gamma_{(\alpha)}$ matrices are all independent, in the sense that none of them can be written in terms of products of any of the rest, then it follows that the number of zero eigenvalues in $\mathcal{M}$ is 2^{5-N}. This can be seen from the fact that in the diagonalised basis, any such set of N independent $\Gamma_{(\alpha)}$ matrices can be chosen from the set

$$\begin{aligned}
&s_1 = \sigma \times 1\!\!1 \times 1\!\!1 \times 1\!\!1 \times 1\!\!1\ , \qquad s_2 = 1\!\!1 \times \sigma \times 1\!\!1 \times 1\!\!1 \times 1\!\!1\ , \qquad s_3 = 1\!\!1 \times 1\!\!1 \times \sigma \times 1\!\!1 \times 1\!\!1\ ,\\
&s_4 = 1\!\!1 \times 1\!\!1 \times 1\!\!1 \times \sigma \times 1\!\!1\ , \qquad s_5 = 1\!\!1 \times 1\!\!1 \times 1\!\!1 \times 1\!\!1 \times \sigma\ ,
\end{aligned} \tag{6.5}$$

where σ is the Pauli matrix σ_3, and $\mathbb{1}$ denotes the 2×2 unit matrix. The eigenvalues μ of $\mathcal{M}$ are therefore given by

$$\mu = \sum_\alpha |Q_\alpha| \pm Q_1 \pm Q_2 \cdots \pm Q_N \ , \tag{6.6}$$

where the sign choices are all independent. In particular, a fraction 2^{-N} of the 32 eigenvalues are zero. Note that it is manifest, for example from (6.5), that the maximum possible number of independent $\Gamma_{(\alpha)}$ matrices is 5, leading to a fraction $\frac{1}{32}$ of preserved supersymmetry.

If not all the $\Gamma_{(\alpha)}$ matrices are independent, in the sense that some can be expressed as products of others, then let us assume that $\tilde{N}$ of them *are* independent. It then follows that the solution either preserves a fraction $2^{-\tilde{N}}$ of the supersymmetry, or it preserves none at all. Which of these occurs depends upon the signs of the charges. To see this, consider an N-charge solution that preserves a fraction $k/32$ of the supersymmetry, with Killing spinors ϵ^a. If we now introduce an $(N+1)$'th charge Q_{N+1}, with its individual Bogomol'nyi matrix $\mathcal{M}_{N+1} = |Q_{N+1}| + Q_{N+1}\, \Gamma_{(N+1)}$, where $\Gamma_{(N+1)}$ is expressible as a product of some of the previous $\Gamma_{(\alpha)}$ matrices, $\Gamma_{(N+1)} = \prod_{\beta\in\{\alpha\}} \Gamma_{(\beta)}$, then $\mathcal{M}_{N+1}$ can be expressed as

$$\mathcal{M}_{N+1} = |Q_{N+1}| + Q_{N+1} \prod_{\beta\in\{\alpha\}} \frac{1}{Q_\beta} \left(\mathcal{M}_\beta - |Q_\beta|\right) . \tag{6.7}$$

Thus for one sign choice for Q_{N+1}, the matrix $\mathcal{M}_{N+1}$ is expressed as polynomials in the $\mathcal{M}_\beta$ with no term of zero'th order in the $\mathcal{M}_\beta$. For this sign choice, the original Killing spinors ϵ^a of the N-charge solution will also satisfy $\mathcal{M}_{N+1}\epsilon^a = 0$, and hence they will all continue to be Killing spinors in the $(N+1)$-charge solution. In this case, there is no further breaking of supersymmetry when the $(N+1)$'th charge is introduced. On the other hand, if the Q_{N+1} charge is chosen with the opposite sign, the previous Killing spinors ϵ^a will satisfy $\mathcal{M}_{N+1}\epsilon^a = 2|Q_{N+1}|\,\epsilon^a$, and thus all the supersymmetry will be broken when the $(N+1)$'th charge is introduced. (It is worth remarking that in simple mult-charge solutions, if a gamma matrix $\Gamma_{(\alpha)}$ is not indepedent, it is always a product of three other gamma matrices associated with the charges in this solution. This explains why N-charge solutions with $N \le 3$ always preserve 2^{-N} of the supersymmetry, and it is only when $N \ge 4$ that the complications set in.)

Iterating the above argument, we see that an N-charge solution for which $\tilde{N}$ of the $\Gamma_{(\alpha)}$ matrices are independent will preserve a fraction $2^{-\tilde{N}}$ of the supersymmetry for $2^{\tilde{N}}$ out of the total of 2^N sign choices for the charges, and it will preserve no supersymmetry for the remaining sign choices.

Having understood that a given N-charge extremal p-brane may have versions that break all the supersymmetry, as well as versions that preserve a fraction $2^{-\tilde{N}}$ of the supersymmetry, we shall in general assume that the sign choices for the charges are made so that the supersymmetric versions are obtained, unless we have specific reasons for wanting to discuss the non-supersymmetric versions.

6.2 Comments on supersymmetry

1) In the previous section, we showed that the fraction of preserved supersymmetry is given by $2^{-\tilde{N}}$, where $\tilde{N}$ is the number of independent $\Gamma_{(\alpha)}$ matrices associated with the N charges. This provides us with a simple way to determine the fraction of preserved supersymmetry without needing to perform any explicit computation of the eigenvalues of the Bogomol'nyi matrix. The Γ-matrix products in the Bogomol'nyi matrix are governed by the internal coordinate indices of the charges of the p-brane solution. All one needs to do is to identify the maximal subset of $\tilde{N}$ independent $\Gamma_{(\alpha)}$ matrices, implying that a fraction $2^{-\tilde{N}}$ of the supersymmetry is preserved.

To illustrate this, let us consider 4-charge 1-form solutions in $D = 3$, for which the allowed field strength combinations are listed in (A.4). We shall consider two specific examples, using the fields $\{F_1^{(123)}, *F_2^{(34)}, *\mathcal{F}_2^{(4)}, F_1^{(12)}\}$, or instead $\{F_1^{(123)}, *F_2^{(34)}, *\mathcal{F}_2^{(4)}, F_1^{(56)}\}$. (Note that the star indicate that the 2-forms are dualised to 1-forms in $D = 3$.) We can use these to give 4-charge black hole solutions, where the unstarred field strengths carry magnetic charges, and the starred ones carry electric charges. From (6.1), the Bogomol'nyi matrix for the first example is

$$\mathcal{M} = \sum_\alpha |Q_\alpha| + Q_1\,\Gamma_{\hat{1}\hat{2}\tilde{1}\tilde{2}\tilde{3}} + Q_2\,\Gamma_{0\tilde{3}\tilde{4}} + Q_3\,\Gamma_{0\tilde{4}} + Q_4\,\Gamma_{\hat{1}\hat{2}\tilde{1}\tilde{2}}\ , \tag{6.8}$$

where we denote explicit internal index values by $\tilde{1}, \tilde{2}, \ldots$. It is easy to verify that any of the $\Gamma_{(\alpha)}$ matrices can be expressed as a product of the other three, implying that this 4-charge black hole preserves $2^{-3} = \frac{1}{8}$ of the original supersymmetry. (Of course, in line with our earlier discussion, eight of the possible sign choices for the four charges will give solutions with this $\frac{1}{8}$ supersymmetry, while the other eight choices will give solutions with no supersymmetry.)

The Bogomol'nyi matrix for the second example is given by

$$\mathcal{M} = \sum_\alpha |Q_\alpha| + Q_1\,\Gamma_{\hat{1}\hat{2}\tilde{1}\tilde{2}\tilde{3}} + Q_2\,\Gamma_{0\tilde{3}\tilde{4}} + Q_3\,\Gamma_{0\tilde{4}} + Q_4\,\Gamma_{\hat{1}\hat{2}\tilde{5}\tilde{6}}\ , \tag{6.9}$$

and in this case we see that no product of two or three of the Γ matrices can give the fourth. In this case, therefore, the solution preserves $2^{-4} = \frac{1}{16}$ of the supersymmetry, and the sign choices for the charges have no effect on the supersymmetry. Note that the only difference between these two examples is that the fourth field strength has different internal indices. This illustrates the fact that one can read off the supersymmetry of a p-brane just by inspecting the internal indices on the field strengths that support the solution.

2) Some observations related to the above were presented in [31,32] for the case of intersections purely involving M-branes or D-branes, but no waves or NUTs. These involved looking at the contributions of the antisymmetric tensor terms in the Killing spinor equations in $D = 11$ or $D = 10$, and recognising certain projection-operator combinations involving the associated Γ matrices that lead to halvings of the numbers of Killing spinors. Our proofs in this section provide a complete analysis of the supersymmetry for all simple multi-charge p-branes, including the cases where there are waves and NUTs in the higher-dimensional intersections, and for all choices of the signs of the charges.

3) The supersymmetry of any simple multi-charge p-brane can also be found by means of the following rule, which we shall refer to as "casting out charges," and is formulated in the following five steps:

1) *Using the rules given in Tables (1a) and (1b), list the world-volume and the transverse space directions z^i of the compactification coordinates for each of the N single-charge solutions obtained by setting in turn all but one of the N charges to zero.*
2) *If any charge is such that its removal contracts the total list of world-volume z^i directions or transverse-space z^i directions, then delete this charge, and accumulate a factor of $\frac{1}{2}$.*
3) *If the removal of no single charge can achieve the above contraction, then delete an arbitrarily chosen charge that is associated with a z coordinate that appears only twice,*[4] *and accumulate a factor of 1.*
4) *Repeat the above steps on the remaining $N-1$ charges, until eventually all have been removed.*
5) *The product of the accumulated $\frac{1}{2}$ and 1 factors gives the fraction of preserved supersymmetry for the original N-charge solution.*

We obtained the fractions of preserved supersymmetry using the above rules for all the p-branes, and we verified that they are consistent with the explicit computations of the

[4]Note that this can always be done.

eigenvalues of the Bogomol'nyi matrix. The results for the preserved supersymmetry are summarised in Tables 3, 4, and 5. The advantage of the casting out charges rule is that the determination of the supersymmetry of a simple multi-charge solution can be done by inspection of the configuration of participating field strengths, rather than by computing the eigenvalues of a 32×32 matrix.

We may illustrate this "casting out charges" rule with some examples. First, consider the example of the dyonic string given by (3.6). From Table (1a), we see that the compactification coordinates z^i divide between the world-volume and the transverse space as follows:

$$
\begin{array}{ccc}
 & \text{World-volume} & \text{Transverse Space} \\
F_3^{(1)} & 1 & 2,\ 3,\ 4,\ 5 \\
{*F}_3^{(1)} & 2,3,4,5 & 1
\end{array}
\tag{6.10}
$$

We see that either of the charges satisfies rule 2, and so we cast out one of them, accumulating a factor of $\frac{1}{2}$. Casting out the remaining one gives another $\frac{1}{2}$, implying that the fraction of supersymmetry preserved by the dyonic string is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

For a more complicated example, consider the two 4-charge black holes solutions in $D = 3$ discussed in comment (1) above. For the first case, we see from Tables (1a) and (1b) that we have

$$
\begin{array}{ccc}
 & \text{World-volume} & \text{Transverse Space} \\
F_1^{(123)} & 4,5,6,7,8 & 1,2,3 \\
{*F}_2^{(34)} & 3,4 & 1,2,5,6,7,8 \\
{*\mathcal{F}}_2^{(4)} & 4 & 1,2,3,5,6,7,8 \\
\mathcal{F}_1^{(12)} & 3,4,5,6,7,8 & 1,2
\end{array}
\tag{6.11}
$$

There is no single charge that can be deleted so as to contract the total list of world-volume or transverse-space directions. Thus we apply rule 3, and arbitrarily delete a charge, say number four, accumulating a factor of 1. Deleting the first charge the removes 5 (and 6) from the list of world-volume directions, and so we accumulate a factor of $\frac{1}{2}$ by rule 2. Deleting the second charge now removes 3 from the list of world-volume directions, and so we accumulate another $\frac{1}{2}$ factor. Deletion of the last two charges accumulates two more factors of $\frac{1}{2}$, leading to the conclusion that this 4-charge black hole preserves $1 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ of the supersymmetry, in agreement with our previous derivation using the Bogomol'nyi matrix.

Taking the other example for a 4-charge black hole, we have

$$
\begin{array}{ccc}
 & \text{World-volume} & \text{Transverse Space} \\
F_1^{(123)} & 4,5,6,7,8 & 1,2,3 \\
*F_2^{(34)} & 3,4 & 1,2,5,6,7,8 \\
*\mathcal{F}_2^{(4)} & 4 & 1,2,3,5,6,7,8 \\
\mathcal{F}_1^{(56)} & 1,2,3,4,7,8 & 5,6
\end{array}
\tag{6.12}
$$

Here, deleting the first charge removes 6 from the total list of world-volume directions, giving a $\frac{1}{2}$ factor. Deleting the third charge then removes 3 from the transverse space, giving another factor of $\frac{1}{2}$. Deleting the remaining two charges gives two more factors of $\frac{1}{2}$. Therefore, this 4-charge black hole solution preserves a fraction $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$ of the supersymmetry, again agreeing with the result we previously derived using the Bogomol'nyi matrix.

Both the 4-charge $D = 3$ black hole solutions (6.11) and (6.12) can be interpreted as intersections of a membrane, a 5-brane, a wave and a NUT in $D = 11$. The different fractions of supersymmetry that the two solutions preserve is related to the different ways in which these eleven-dimensional objects intersect each other. Although in both examples, the full list of transverse-space directions is the same, the list of world-volume directions in (6.12) contains z^5 and z^6, which are not contained in the world-volume list for (6.11). This shows that more supersymmetry is broken by intersecting M-branes, waves or NUTs when they occupy more directions in either the world-volume or the transverse space. In other words, the intersecting objects tend to preserve less supersymmetry if they are spread over more directions in either the world-volume or the transverse space, and conversely they preserve more supersymmetry if they are confined to fewer directions.

Let us consider another pair of examples, namely two configurations of quadruple intersections of 5-branes in $D = 11$. They can both be reduced to $D = 4$, where they become 4-charge magnetic strings, with field strengths given by $\{F_1^{(123)}, F_1^{(145)}, F_1^{(246)}, F_1^{(167)}\}$, and $\{F_1^{(123)}, F_1^{(145)}, F_1^{(246)}, F_1^{(356)}\}$ respectively. Clearly the second solution describes quadruple intersections of 5-branes that fit into fewer transverse directions, in that the total list of transverse-space coordinates is smaller (z^7 is not in the list of transverse-space directions of any of the four 5-branes). In the first solution, the total list of transverse-space coordinates is larger, since it includes z^7 as well. And indeed, the first solution preserves only $\frac{1}{16}$ of the supersymmetry, whilst the second preserves $\frac{1}{8}$.

Thus the "casting out charges" rule exploits the relation between the spacetime geometrical structures of intersecting objects and the supersymmetries they preserve.

7 Harmonic intersections in M-theory or type II strings

In the previous sections, we have obtained a complete classification of all the simple N-charge extremal p-branes in all dimensions $2 \le D \le 11$, and given procedures for determining the fractions of the original $D = 11$ supersymmetry that each of them preserves. We have also shown how they can be oxidised back to $D = 11$ (or $D = 10$), by retracing the steps of the Kaluza-Klein dimensional reductions to the D-dimensional maximal supergravities. The oxidation rules are very simple, and are summarised in (3.1) and Tables (1a) and (1b) (for $D = 11$), and in (3.13), (3.14) and Tables (2a) and (2b) (for $D = 10$). In $D = 11$ or $D = 10$, the lower-dimensional N-charge p-branes become intersections of p-branes, waves and NUTs (*i.e.* monopoles). Thus the classification of the lower-dimensional multi-charge p-branes also provides a classification of the intersections in M-theory or string theory where the harmonic functions depend only on the coordinates transverse to all the individual world-volumes. The classification of more general intersections (*i.e.* including those that do not dimensionally reduce to p-branes) has been studied in [31, 32, 33].

Just as the rules for N-charge p-branes in lower dimensions are encoded in the rules for 2-charge p-branes, so the rules for the intersections of N objects are encoded in the basic rules for the intersections of all possible pairs of objects. In $D = 11$, there are four basic objects, namely the membrane, 5-brane, wave and NUT. (The NUTs subdivide into three sub-categories, NUT$_i$ given by equations (3.2–3.4).) The solutions for all allowed pairwise intersections are characterised by the overlap of the spatial world-volume directions of the two basic objects. For example, the intersection of a membrane and a 5-brane is given in (3.9), and we see that they have the one common spatial world-volume coordinate x. This particular example came from the oxidation of a dyonic string in $D = 6$; one can easily verify that all other 2-charge p-branes that oxidise to an intersection of a membrane with a 5-brane exhibit the same feature, of an overlap of one spatial world-volume coordinate. Thus, knowing that the intersection of a membrane and a 5-brane always shares one common spatial world-volume coordinate, the solution is uniquely determined, up to the relabelling of coordinates. We may summarise all the required information for constructing arbitrary multiple intersections of all the basic objects in a table listing the world-volume overlaps of all allowed pairs. This is given in Table 6.

	Membrane	5-brane	Wave	NUT_1	NUT_2	NUT_3
Membrane	0	1	1	2	2	2
5-brane		3	1	5	5	5
Wave			–	1	1	1
NUT_1				–	–	–
NUT_2					4	4
NUT_3						4

Table 6. Spatial world-volume overlaps of harmonic intersections in $D = 11$

It should be emphasised that the rules in Table 6 are for pairwise intersections that can dimensionally reduce to p-branes. Such pairwise intersection rules in the type IIA theory can be obtained by dimensional reduction of the ones for $D = 11$. This was done in [31, 32, 33]. From these intersection rules, one can derive all the possible intersections of N basic objects. Their supersymmetry can be determined by the "casting out charges" rule described in section 6.2.

7.1 Comments on intersections

1) The maximum possible number of intersecting M-branes is $N = 8$, achieved as the intersection of seven 5-branes and one membrane. The solutions can be dimensionally reduced to $D = 3$, where they become 8-charge string solutions in maximal massive supergravities, as discussed in section 4.2. For intersections of M-branes to be reduced instead to solutions of *massless* supergravities in lower dimensions, the maximum number of intersections is $N = 7$, namely four 5-branes and three membranes, or seven 5-branes. The solutions can be reduced to 7-charge black holes in $D = 3$.

The maximum possible number of intersecting objects is $N = 9$ [33]. For example, the above eight intersecting M-branes can further intersect with a wave. These solutions reduce to two-dimensional 9-charge black holes, as we discussed in section 5. There can also be intersections of one membrane, four 5-branes, one wave and three NUTs.

2) In order to oxidise the solutions we obtained in the previous sections to the type IIA theory, we need to split the internal index $i = (1, \alpha)$ and distinguish between the NS-NS and R-R fields, as explained in section 3.2. The perturbative solutions in lower dimensions are those which are supported only by NS-NS field strengths carrying electric charges. In Minkowskian-signature spacetime, we note that all the $N \geq 3$ charge solutions are non-perturbative. The only perturbative $N = 2$ charge solutions are those described by intersections between the NS-NS string and a wave.

In particular, this implies that two NS-NS strings cannot intersect in Minkowskian-signature spacetime. This can be seen from the fact that the spatial world-volume overlap of two intersecting membranes, as listed in Table 6, is zero, and hence the only way to obtain intersections of two NS-NS strings is by compactifying the intersecting membranes on the time coordinate, giving a $D = 10$ Euclidean-signature space. In fact intersections in string theory are typically a non-perturbative phenomenon.

3) The type IIA and type IIB theories are related by T-duality, and so all the intersections in the type IIA theory can be mapped into intersections in the type IIB theory by this duality transformation. If we wish to oxidise the lower-dimensional solutions to the $D = 10$ type IIB theory directly, we need to split the internal index as $i = (1, 2, \alpha)$, since M-theory compactified on a two-torus is T-dual to the type IIB theory compactified on a circle, and the fields of the two theories can then be identified by T-duality transformations.

For example, there are five dyonic string solutions in $D = 6$, using the field strengths $\{F_3^{(i)}, *F_3^{(i)}\}$. All five of them become intersections of a membrane and a 5-brane in $D = 11$. Oxidising instead to the $D = 10$ type IIA theory, the $i = 1$ solution gives an intersection of a string and a 5-brane; while for $i = 2, 3, 4, 5$, they are intersections of a membrane and a 4-brane. Oxidising back to $D = 10$ type IIB instead, the $i = 1$ solution gives an intersection of an NS-NS string and an NS-NS 5-brane; for $i = 2$, it is the intersection of a R-R string and a R-R 5-brane; and for $i = 3, 4, 5$, they are nothing but intersections of two self-dual 3-branes.

This illustrates that once all the lower-dimensional simple multi-charge p-brane solutions are classified, all the associated intersections in higher dimensions are also completely understood. The oxidations of the lower-dimensional solutions follow a few simple rules, given in section 3.

8 U multiplets and non-harmonic intersections

In the previous sections, we obtained all the simple multi-charge solutions in maximal supergravity theories. These solutions can be viewed as harmonic intersections of p-branes, waves and NUTs, in M-theory or type II strings. In lower dimensional supergravities, there are in general global symmetries that can be used to generate multiplets from any given solution. In particular, in massless maximal supergravities, there are E_{11-D} CJ global symmetry groups. Acting on the simple solutions we obtained in the previous sections, we obtain full multiplets of solutions (see [46] for a discussion of the analogous spectrum-generating symmetries at the quantum level). These can also be oxidised back to $D = 10$ or $D = 11$, where they acquire interpretations as intersections of the basic objects. However, in this case, the number of harmonic functions involved in the intersections is less than the number of basic objects that are involved. We shall refer to these as "non-harmonic intersections."

To find simple examples, we shall study p-brane multiplets of an $SL(2,\mathbb{R})$ global symmetry. In $D = 9$, the two Kaluza-Klein vectors $\mathcal{A}_1^{(1)}$ and $\mathcal{A}_1^{(2)}$ form a doublet under the $SL(2,\mathbb{R})$ factor of the $GL(2,\mathbb{R})$ global symmetry group. The relevant bosonic Lagrangian is given by

$$\begin{aligned} e^{-1}\mathcal{L} &= R - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}(\partial\chi)^2 e^{-2\phi} \\ &\quad -\tfrac{1}{4}e^{-\phi+3\varphi/\sqrt{7}}(\mathcal{F}_2^{(1)})^2 - \tfrac{1}{4}e^{\phi+3\varphi/\sqrt{7}}(\mathcal{F}_2^{(2)})^2 \ , \end{aligned} \tag{8.1}$$

where $\chi = -\mathcal{A}_0^{(12)}$, $\mathcal{F}_2^{(1)} = d\tilde{\mathcal{A}}_1^{(1)} - \chi d\tilde{\mathcal{A}}_1^{(2)} \equiv d\mathcal{A}_1^{(1)} - d\chi \wedge \mathcal{A}_1^{(2)}$ and $\mathcal{F}_2^{(2)} = d\tilde{\mathcal{A}}_1^{(2)} \equiv d\mathcal{A}_1^{(2)}$. The dilatonic scalar fields ϕ and φ are related to the usual ϕ_1 and ϕ_2 fields appearing in (2.5) as follows [24]:

$$\begin{pmatrix} \phi \\ \varphi \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{7}}{4} \\ -\frac{\sqrt{7}}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \ . \tag{8.2}$$

The Lagrangian (8.1) is invariant under the $SL(2,\mathbb{R})$ transformations

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d} \ , \quad \begin{pmatrix} \tilde{\mathcal{A}}_1^{(1)} \\ \tilde{\mathcal{A}}_1^{(2)} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{A}_1^{(1)\prime} \\ \mathcal{A}_1^{(2)\prime} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{A}}_1^{(1)} \\ \tilde{\mathcal{A}}_1^{(2)} \end{pmatrix} \ , \tag{8.3}$$

where $\tau = \chi + ie^{\phi}$ and $ad - bc = 1$. Starting from the simple single-charge black hole solution supported by the field strength $\mathcal{F}_2^{(1)}$,

$$\begin{aligned} &ds_9^2 = -H^{-\frac{6}{7}}\, dt^2 + H^{\frac{1}{7}}\, d\vec{y}^2 \ , \\ &e^{\phi} = H^{-\frac{1}{2}} \ , \quad e^{\varphi} = H^{\frac{3}{2\sqrt{7}}} \ , \quad \chi = 0 \ , \quad \mathcal{A}_1^{(1)} = H^{-1}\, dt \ , \end{aligned} \tag{8.4}$$

we make an $SL(2,\mathbb{R})$ transformation using (8.3) to obtain a new solution, where the metric and the scalar φ are unchanged, but the other fields become

$$\begin{aligned} e^{\phi} &= \frac{H^{\frac{1}{2}}}{c^2 + d^2 H} \ , \qquad \chi = -\frac{ac + bdH}{c^2 + d^2 H} \ , \\ \mathcal{A}_1^{(1)} &= \frac{d}{c^2 + d^2 H}\, dt \ , \qquad \mathcal{A}_1^{(2)} = cH^{-1}\, dt \ . \end{aligned} \tag{8.5}$$

It is straightforward to oxidise this solution back to $D=10$ and $D=11$, leading to the metrics

$$
\begin{aligned}
ds_{10}^2 &= -(c^2+d^2H)^{\frac{1}{8}}H^{-1}\,dt^2 + (c^2+d^2H)^{-\frac{7}{8}}H(dz_2+\mathcal{A}_1^{(2)})^2 + (c^2+d^2H)^{\frac{1}{8}}\,d\vec{y}^2\ ,\\
ds_{11}^2 &= -H^{-1}dt^2 + (c^2+d^2H)^{-1}H\,(dz_2+\mathcal{A}_1^{(2)})^2\\
&\quad +(c^2+d^2H)(dz_1+\mathcal{A}_1^{(1)}-\chi dz_2)^2 + d\vec{y}^2\ ,
\end{aligned}
\tag{8.6}
$$

The $SL(2,\mathbb{R})$ transformation interpolates between two basic objects, namely a wave and a D0-brane in $D=10$, or between two waves in $D=11$. Thus the above two metrics describe non-harmonic intersections of a wave and a D0-brane in $D=10$, and two waves in $D=11$. Of course since they are simply related to the oxidations of a simple single-charge black-hole by an $SL(2,\mathbb{R})$ transformation, the transformed $D=9$ solution and its oxidations all preserve the same fraction $\frac{1}{2}$ of the supersymmetry as does the simple $D=9$ solution itself. In the above example, we have considered the electric black hole solutions. If instead, we consider magnetic 5-brane solutions in $D=9$, then they will oxidise to become non-harmonic intersections of two NUTs in $D=11$.

For another example, consider the string solutions in $D=9$, which again form a doublet under $SL(2,\mathbb{R})$. The relevant part of the $D=9$ Lagrangian is

$$
\begin{aligned}
e^{-1}\mathcal{L} &= R - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}(\partial\chi)^2 e^{-2\phi}\\
&\quad -\tfrac{1}{12}e^{\phi-\varphi/\sqrt{7}}(F_3^{(1)})^2 - \tfrac{1}{12}e^{-\phi+\varphi/\sqrt{7}}(F_3^{(2)})^2\ ,
\end{aligned}
\tag{8.7}
$$

where $F_3^{(1)} = dA_2^{(1)}$ and $F_3^{(2)} = dA_2^{(2)} + \chi\, dA_2^{(1)}$. This is invariant under $SL(2,\mathbb{R})$ transformations where ϕ and χ transform as in (8.3), and the 2-form potentials transform in the contragedient fashion

$$
\begin{pmatrix} A_2^{(1)} \\ A_2^{(2)} \end{pmatrix} \longrightarrow \begin{pmatrix} A_2^{(1)\prime} \\ A_2^{(2)\prime} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} A_2^{(1)} \\ A_2^{(2)} \end{pmatrix}\ ,
\tag{8.8}
$$

Starting from a simple single-charge string solution using the field strength $F_3^{(1)}$, and applying the $SL(2,\mathbb{R})$ transformation, we obtain the more general string solution

$$
\begin{aligned}
&ds_9^2 = H^{-\frac{5}{7}}\,dx^\mu\,dx_\mu + H^{\frac{2}{7}}\,d\vec{y}^2\ , \qquad e^{\varphi} = H^{-\frac{1}{2\sqrt{7}}}\ ,\\
&e^{\phi} = \frac{H^{\frac{1}{2}}}{d^2+c^2H}\ , \qquad \chi = -\frac{bd+acH}{d^2+c^2H}\ ,\\
&A_2^{(1)} = (d^2x\wedge dH^{-1})\,d\ , \qquad A_2^{(2)} = -(d^2x\wedge dH^{-1})\,b\ .
\end{aligned}
\tag{8.9}
$$

It is straightforward to oxidise this to $D=10$ and $D=11$, where we obtain the metrics

$$
ds_{10}^2 = H^{-\frac{3}{4}}\,(d^2+c^2H)^{\frac{1}{8}}\,dx^\mu\,dx_\mu + H^{\frac{1}{4}}\,(d^2+c^2H)^{\frac{1}{8}}\,d\vec{y}^2
$$

$$
\begin{aligned}
&\quad +H^{\frac{1}{4}}\,(d^2+c^2H)^{-\frac{7}{8}}\,dz_2^2\ ,\\
ds_{11}^2 &= H^{-\frac{2}{3}}\,dx^\mu\,dx_\mu + H^{\frac{1}{3}}\,d\vec{y}^2 + H^{\frac{1}{3}}\,(d^2+c^2H)^{-1}\,dz_2^2\\
&\quad +H^{-\frac{2}{3}}\,(d^2+c^2H)\,(dz_1-\chi\,dz_2)^2\ . \qquad (8.10)
\end{aligned}
$$

In $D=10$, this solution interpolates between a string and a membrane. In $D=11$, it interpolates between two membranes. In all cases, the solution preserves $\frac{1}{2}$ of the supersymmetry. Again, if we instead oxidise the magnetic 4-brane solutions in $D=9$, they will become non-harmonic intersections of 5-branes in $D=11$.

Finally, let us consider a dyonic membrane in $D=8$, where the relevant part of the Lagrangian is given by

$$
e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}(\partial\chi)^2 e^{-2\phi} - \tfrac{1}{48}e^{\phi}F_4^2 + \tfrac{1}{48}\chi F_4\cdot *F_4\ , \qquad (8.11)
$$

with $\chi = A_0^{(123)}$ and $\phi = \vec{a}\cdot\vec{\phi}$. The dyonic solution was obtained in [39], and is given by

$$
\begin{aligned}
&ds_8^2 = H^{-\frac{1}{2}}\,dx^\mu\,dx_\mu + H^{\frac{1}{2}}\,d\vec{y}^2\ ,\\
&e^{\phi} = \frac{H^{\frac{1}{2}}}{d^2+c^2H}\ , \qquad \chi = \frac{bd+acH}{d^2+c^2H}\ , \qquad (8.12)\\
&F_4 = (d^3x\wedge dH^{-1})\,d - H^{\frac{1}{2}}\ *(d^3x\wedge dH^{-1})\,c\ ,
\end{aligned}
$$

where a, b, c and d are the parameters of the $SL(2,\mathbb{R})$ factor in the $SL(3,\mathbb{R})\times SL(2,\mathbb{R})$ global symmetry group, satisfying $ad-bc=1$. Performing the oxdations to $D=10$ and $D=11$, we obtain the metrics

$$
\begin{aligned}
ds_{10}^2 &= H^{-\frac{5}{8}}(d^2+c^2H)^{\frac{1}{4}}\,dx^\mu\,dx_\mu + H^{\frac{3}{8}}(d^2+c^2H)^{\frac{1}{4}}d\vec{y}^2\\
&\quad +H^{\frac{3}{8}}(d^2+c^2H)^{-\frac{3}{4}}(dz_2^2+dz_3^2)\ , \qquad (8.13)\\
ds_{11}^2 &= H^{-\frac{2}{3}}(d^2+c^2H)^{\frac{1}{3}}\,dx^\mu\,dx_\mu + H^{\frac{1}{3}}(d^2+c^2H)^{\frac{1}{3}}d\vec{y}^2\\
&\quad +H^{\frac{1}{3}}(d^2+c^2H)^{-\frac{2}{3}}(dz_1^2+dz_2^2+dz_3^2)\ . \qquad (8.14)
\end{aligned}
$$

In $D=10$, this solution interpolates between a membrane and a 4-brane. In $D=11$, it interpolates between a membrane and 5-brane. In all cases, the solution preserves $\frac{1}{2}$ of the supersymmetry.

8.1 Further comments

1) All the 2-charge solutions (simple or non-simple) in lower dimensions can be characterised by the dilaton vectors of the two field strengths that are involved. Defining

$$
\Delta = \tfrac{1}{4}(\vec{c}_a+\vec{c}_\beta)^2 + \frac{2(n-1)(D-n-1)}{D-2}\ , \qquad (8.15)
$$

where $\vec{c}_\alpha$ and $\vec{c}_\beta$ are the dilaton vectors and n is the degree of the field strengths, *all* the 2-charge solutions can be categorised into three types, namely $\Delta = 1, 2$ and 3. The $\Delta = 2$ configurations give rise to simple 2-charge solutions, as discussed earlier, where the solutions involve two independent harmonic functions. All the $\Delta = 3$ configurations give rise to solutions that are $SL(2, \mathbb{R})$ rotations of simple single-charge solutions, and hence the solutions involve only one harmonic function. (Note that $\Delta = 3$ solutions arise only in supergravities with global symmetries that contain $SL(2, \mathbb{R})$ subgroups.) The $\Delta = 1$ type solutions were discussed in [43], where the equations of motion were cast into the form of 1-dimensional $SL(3, \mathbb{R})$ Toda equations; these solutions involve no harmonic functions. The masses (at the self-dual point), the charges, and the eigenvalues of Bogomol'nyi matrix for the above three types of 2-charge solutions are given by

$$\begin{aligned}
\Delta = 1: &\quad m = (Q_1^{2/3} + Q_2^{2/3})^{3/2} , &\quad \mu = m \pm \sqrt{Q_1^2 + Q_2^2} , \\
\Delta = 2: &\quad m = |Q_1| + |Q_2| , &\quad \mu = m \pm Q_1 \pm Q_2 , \\
\Delta = 3: &\quad m = \sqrt{Q_1^2 + Q_2^2} , &\quad \mu = m \pm \sqrt{Q_1^2 + Q_2^2} .
\end{aligned} \tag{8.16}$$

Thus the $\Delta = 3$ solutions preserve $\frac{1}{2}$ of the supersymmetry, and can be viewed as bound states with positive binding energy. The $\Delta = 2$ solutions preserve $\frac{1}{4}$ of the supersymmetry, and can be viewed as bound states of zero binding energy. The eigenvalues of Bogomol'nyi matrix for $\Delta = 1$ solutions are all positive definite, and hence all the supersymmetry is broken. They can be viewed as bound states with negative binding energy [41, 43].

Having obtained all structures for p-brane solutions for all possible pairs of field strengths, it is straightforward to generalise to multi-field strength solutions for all possible sets of field-strength configurations. For example, in an N-field-strength solution, if there are $\tilde{N}$ field strengths that are pairwise of the $\Delta = 2$ type, and the rest are of the $\Delta = 3$ type, then this N-charge solution is a U-duality transformation of a simple $\tilde{N}$-charge solution.

2) Note that as we listed in Table 6, there are no harmonic intersections of two waves. The example of the $SL(2, \mathbb{R})$ multiplets of black hole solutions in $D = 9$ suggests that there can, however, be non-harmonic intersections of waves. Similarly, there can be no world-volume spatial overlap in harmonic intersecting membranes, however, as we saw earlier, for non-harmonic intersections the world-volume spatial overlap can be 1. These suggest that the intersections that are harmonically impossible can be supplemented by non-harmonic intersections. The classification of all possible pair-wise intersections in $D = 11$ is of course subsumed by the classification of all possible pairwise 2-charge solutions described in comment 2. This leads to two more types of possible intersections in $D = 11$, namely non-harmonic but supersymmetric intersections ($\Delta = 3$), presented in table 7, and non-supersymmetric intersections ($\Delta = 1$) presented in Table 8.

	Membrane	5-brane	Wave	NUT_1	NUT_2	NUT_3
Membrane	1	2	0	1	1	1
5-brane		4	0	4	4	4
Wave			0	–	0	0
NUT_1				5	–	–
NUT_2					5	5
NUT_3						5

Table 7. Spatial world-volume overlaps of non-harmonic intersections in $D = 11$

	Membrane	5-brane	Wave	NUT_1	NUT_2	NUT_3
Membrane	–	0	–	–	1	1
5-brane		–	–	–	4	4
Wave			–	0	0	0
NUT_1				–	–	–
NUT_2					5	5
NUT_3						5

Table 8. Spatial world-volume overlaps of non-supersymmetric intersections in $D = 11$

Tables 6, 7 and 8 give to all possible pairwise intersections in $D = 11$ that can dimensionally reduce to p-branes.

9 Conclusion

In this paper, we have performed a classification of all simple N-charge p-brane solutions in the massless and massive supergravities that arise from the toroidal dimensional reduction of $D = 11$ supergravity. These solutions are extremal and expressed in terms of N independent harmonic functions. (Note that they all admit non-extremal generalisations where the mass becomes an additional parameter, independent of the charges [47].) We have also discussed in detail the procedures for determining the fractions of supersymmetry that are preserved by each such extremal solution. Included in this discussion was the question of how the supersymmetry fractions are affected by the possible sign choices for the charges.

A by-product of the classification of lower-dimensional p-branes is a classification of certain kinds of intersections in $D = 10$ or $D = 11$, namely those where the harmonic functions all depend on a common set of transverse coordinates. In fact there are distinct advantages in classifying them in terms of the lower-dimensional p-branes, since it then becomes much easier to study the multiplets of solutions that are related by U-duality transformations. There are also other kinds of intersections [26, 28, 29] that do not dimensionally reduce to p-branes, although they can still give rise to some lower-dimensional solutions. Their significance in string theory is less clear. However the lower-dimensional solutions are supersymmetric, and it would be interesting to give a classification of them, if they do indeed play a rôle in the spectrum of the string.

We also discussed the structure of the multiplets that are generated by acting with U-duality transformations on the simple multi-charge solutions. In particular, any supersymmetric solution involving two charges is either a simple 2-charge solution, which preserves $\frac{1}{4}$ of the supersymmetry, or it is an $SL(2,\mathbb{R})$ rotation of a simple single-charge solution, which preserves $\frac{1}{2}$ of the supersymmetry. (The $SL(2,\mathbb{R})$ is in general a subgroup of the global symmetry group of the supergravity theory.) A third kind of 2-charge solution, which preserves no supersymmetry, can also arise in certain special cases [41, 43, 44]. Any pair of field strengths of equal degree will give rise to solutions of one of these three types.

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A $(N \ge 3)$-charge 1-form solutions

In this appendix, we give all the 1-form field strength configurations for $(N \ge 3)$ charge solutions for $D \ge 2$. Together with the single-charge and 2-charge solutions listed in section 4.1, we have all the simple multi-charge solutions using 1-form field strengths in $D \ge 2$. The $(N \ge 3)$-charge solutions arise in $D \le 6$. Note that all the $N = 3, 4'$ solutions preserve $\frac{1}{8}$ of the supersymmetry and all the $N = 4, 5, 6, 7, 8$ solutions preserve $\frac{1}{16}$.

$D = 6$

In this dimension, we have $N_{\max} = 4'$, The field strength configurations are given by

$$
\begin{aligned}
N = 3: &\quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}\}_{30}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}\}_{30}\ , \qquad (A.1)\\
N = 4': &\quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{jk}, \mathcal{F}_1^{(\ell m)}\}_{15}\ . \qquad (A.2)
\end{aligned}
$$

$D = 5$

As in the case of $D = 6$, we have $N_{\max} = 4'$ in $D = 5$:

$$
\begin{aligned}
N = 3: &\ \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}\}_{120}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}\}_{180}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}\}_{180}\ ,\\
&\ \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}\}_{15}\ , \quad \{{*F_4}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}\}_{45}\ , \qquad (A.3)\\
N = 4': &\ \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, F_1^{(kmn)}\}_{30}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{jk}, \mathcal{F}_1^{(\ell m)}\}_{90}\ .\\
&\ \{{*F_4}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}\}_{30} \qquad (A.4)
\end{aligned}
$$

$D = 4$

For $N = 3$, we have $M = 4095$, given by

$$
\begin{aligned}
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, {*F_3^{(i)}}\}_{315}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}\}_{105}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}\}_{840}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}\}_{630}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(np)}\}_{315}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, {*F_3^{(k)}}\}_{105}\ ,\\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}, {*F_3^{(k)}}\}_{630}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}\}_{630}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(pq)}\}_{105}\ ,\\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, {*F_3^{(m)}}\}_{315}\ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}\}_{105}\ . \qquad (A.5)
\end{aligned}
$$

For $N = 4'$, we have $M = 945$, given by

$$
\begin{aligned}
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, F_1^{(kmn)}\}_{210}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(np)}, {*F_3^{(i)}}\}_{315}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}\}_{315}\ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, {*F_3^{(p)}}\}_{105}\ , \qquad (A.6)
\end{aligned}
$$

For $N = 4$ solutions,we have $M = 3780$, given by

$$
\begin{aligned}
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, {*F_3^{(i)}}\}_{105}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}{}_{840}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}\}_{315}\ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, {*F_3^{(i)}}\}_{630}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(np)}\}_{630}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}, {*F_3^{(k)}}\}_{630}\ ,\\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, {*F_3^{(i)}}\}_{315}\ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}\}_{315}\ . \qquad (A.7)
\end{aligned}
$$

The detail for the $N = 5$ solutions is given by

$$
\begin{aligned}
&N = 5 \; , \qquad M = 2835 \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}, F_1^{(jmp)}\}_{630} \; , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, *F_1^{(i)}\}_{315} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}\}_{315} \; , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, *F_1^{(i)}\}_{315} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(np)}, *F_1^{(i)}\}_{630} \; , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}\}_{315} \; , \\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_3^{(k)}\}_{315} \; ,
\end{aligned} \tag{A.8}
$$

There are 945 cases of $N = 6$ solutions, given by

$$
\begin{aligned}
&N = 6 \; , \qquad M = 945 \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}, F_1^{(jmp)}, F_1^{(k\ell p)}\}_{210} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, *F_3^{(i)}\}_{315} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}\}_{105} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_1^{(i)}\}_{315} \; ,
\end{aligned} \tag{A.9}
$$

Finally in $D = 4$, there are a total of 135 of $N = 7$ solutions, given by

$$
\begin{aligned}
&N = 7 \; , \qquad M = 135 \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}, F_1^{(jmp)}, F_1^{(k\ell p)}, F_1^{(kmn)}\}_{30} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_3^{(i)}\}_{105} \; ,
\end{aligned} \tag{A.10}
$$

$D = 3$

The $N = 3$ solutions have a multiplicity 37800, with 23 different field strength configurations, given by

$$
\begin{aligned}
&N = 3 \; , \qquad M = 37800 \; , \\
&\{*F_2^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}\}_{420} \; , \quad \{F_1^{(ijk)}, *F_2^{(i\ell)}, *F_2^{(jm)}\}_{3360} \; , \quad \{F_1^{(ijk)}, *F_2^{(i\ell)}, *\mathcal{F}_1^{(\ell)}\}_{840} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, *F_2^{(in)}\}_{2520} \; , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, *F_2^{(i\ell)}\}_{3360} \; , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}\}_{840} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}\}_{3360} \; , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, *\mathcal{F}_2^{(n)}\}_{2520} \; , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}\}_{1680} \; , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(np)}\}_{2520} \; , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, *F_2^{(k\ell)}\}_{840} \; , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(mn)}, *F_2^{(k\ell)}\}_{5040} \; , \\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, *\mathcal{F}_2^{(\ell)}\}_{840} \; , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(mn)}, *\mathcal{F}_2^{(\ell)}\}_{1680} \; , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}\}_{1680} \; , \\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}\}_{840} \; , \quad \{\mathcal{F}_1^{(ij)}, *F_2^{(ij)}, *F_2^{(\ell m)}\}_{420} \; , \quad \{\mathcal{F}_1^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}\}_{1260} \; , \\
&\{\mathcal{F}_1^{(ij)}, *F_2^{(k\ell)}, *\mathcal{F}_2^{(k)}\}_{840} \; , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, *F_2^{(ij)}\}_{420} \; , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, *F_2^{(mn)}\}_{1260} \; , \\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, *\mathcal{F}_2^{(m)}\}_{420} \; , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}\}_{420} \; ,
\end{aligned} \tag{A.11}
$$

The $N = 4'$ results are the following:

$$
\begin{aligned}
&N = 4' \ , \qquad M = 9450 \ : \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, *F_2^{(j\ell)}, *F_2^{(km)}\}_{1680} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell 6)}, F_1^{(kmn)}\}_{840} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, *\mathcal{F}_2^{(q)}\}_{840} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(np)}, *F_2^{(iq)}\}_{2520} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}\}_{840} \ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, *F_2^{(k\ell)}, *\mathcal{F}_2^{(\ell)}\}_{840} \ , \\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *\mathcal{F}_2^{(q)}\}_{840} \ , \quad \{\mathcal{F}_1^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}, *F_2^{(pq)}\}_{420} \ , \\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, *F_2^{(ij)}, *F_2^{(k\ell)}\}_{210} \ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, *F_2^{(pq)}\}_{420} \ .
\end{aligned}
\tag{A.12}
$$

The $N = 4$ results are the following:

$$
\begin{aligned}
&N = 4 \ , \qquad M = 113400 \ : \\
&\{*F_2^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}, *F_2^{(pq)}\}_{105} \ , \quad \{F_1^{(ijk)}, *F_2^{(i\ell)}, *F_2^{(jm)}, *F_2^{(kn)}\}_{3360} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, *F_2^{(in)}, *F_2^{(j\ell)}\}_{10080} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, *F_2^{(in)}, *\mathcal{F}_2^{(n)}\}_{2520} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, *F_2^{(iq)}\}_{840} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, *F_2^{(in)}\}_{10080} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}\}_{6720} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, *\mathcal{F}_2^{(p)}\}_{6720} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}\}_{2520} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, \mathcal{F}_1^{(pq)}\}_{3360} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, *F_2^{(in)}\}_{5040} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(np)}, *F_2^{(j\ell)}\}_{10080} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, *\mathcal{F}_2^{(n)}\}_{5040} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(np)}, *\mathcal{F}_2^{(q)}\}_{2520} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(np)}\}_{5040} \ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}, *F_2^{(in)}, *F_2^{(jp)}\}_{10080} \ , \\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}, *F_2^{(in)}, *\mathcal{F}_2^{(n)}\}_{5040} \ , \quad \{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}, *F_2^{(kn)}\}_{5040} \ , \\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_2^{(iq)}\}_{2520} \ , \quad \{F_1^{(ijk)} . \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}, *\mathcal{F}_2^{(n)}\}_{5040} \ , \\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}\}_{2520} \ , \quad \{\mathcal{F}_1^{(ij)}, *F_2^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}\}_{1260} \ , \\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, *F_2^{(ij)}, *F_2^{(mn)}\}_{2520} \ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, *F_2^{(mn)}, *F_2^{(pq)}\}_{630} \ , \\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, *F_2^{(mn)}, *\mathcal{F}_2^{(m)}\}_{2520} \ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, *F_2^{(ij)}\}_{1260} \ , \\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, *\mathcal{F}_2^{(p)}\}_{840} \ , \quad \{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, \mathcal{F}_1^{(pq)}\}_{105} \ ,
\end{aligned}
\tag{A.13}
$$

The $N = 5$ results are the following:

$$
\begin{aligned}
&N = 5 \ , \qquad M = 113400 \ : \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, *F_2^{(in)}, *F_2^{(j\ell)}, *F_2^{(km)}\}_{5040} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, *F_2^{(in)}, *F_2^{(jm)}\}_{10080} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, *F_2^{(iq)}, *\mathcal{F}_2^{(q)}\}_{840} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, F_1^{(kmn)}, *F_2^{(in)}\}_{2520} \ , \\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}, F_1^{(jmp)}\}_{5040} \ , \quad \{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}, *\mathcal{F}_2^{(q)}\}_{6720} \ ,
\end{aligned}
$$

$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(j\ell n)},F_1^{(kmn)},\mathcal{F}_2^{(p)}\}_{1680}\ ,\quad \{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(j\ell n)},F_1^{(kmn)},\mathcal{F}_1^{(pq)}\}_{840}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(inp)},\mathcal{F}_1^{(jk)},*F_2^{(iq)}\}_{2520}\ ,\quad \{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(j\ell n)},\mathcal{F}_1^{(pq)},*F_2^{(in)}\}_{10080}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(inp)},\mathcal{F}_1^{(jk)},*\mathcal{F}_2^{(q)}\}_{2520}\ ,\quad \{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(inp)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(\ell m)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(np)},*F_2^{(iq)},*F_2^{(j\ell)}\}_{10080}\ ,\quad \{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(np)},*F_2^{(j\ell)},*F_2^{(km)}\}_{5040}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(jk)},*F_2^{(ip)},*\mathcal{F}_2^{(p)}\}_{5040}\ ,\quad \{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(np)},*F_2^{(iq)},*\mathcal{F}_2^{(q)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(\ell m)},*F_2^{(in)}\}_{2520}\ ,\quad \{F_1^{(ijk)}.F_1^{(i\ell m)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(np)},*F_2^{(iq)}\}_{5040}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(\ell m)},*\mathcal{F}_2^{(n)}\}_{2520}\ ,\quad \{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(np)},*\mathcal{F}_2^{(q)}\}_{5040}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(\ell m)},\mathcal{F}_1^{(np)}\}_{2520}\ ,\quad \{F_1^{(ijk)},\mathcal{F}_1^{(\ell m)},*F_2^{(in)},*F_2^{(jp)},*F_2^{(kq)}\}_{3360}\ ,$$
$$\{F_1^{(ijk)},\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(\ell m)},*F_2^{(mn)},*\mathcal{F}_2^{(n)}\}_{5040}\ ,\quad \{F_1^{(ijk)},\mathcal{F}_1^{(\ell m)},\mathcal{F}_1^{(np)},*F_2^{(iq)},*\mathcal{F}_2^{(q)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(\ell m)},\mathcal{F}_1^{(np)},*F_2^{(kq)}\}_{2520}\ ,\quad \{\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(ij)},*F_2^{(k\ell)},*F_2^{(mn)},*F_2^{(pq)}\}_{420}\ ,$$
$$\{\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(k\ell)},*F_2^{(ij)},*F_2^{(k\ell)},*F_2^{(mn)}\}_{1260}\ ,\quad \{\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(k\ell)},*F_2^{(ij)},*F_2^{(mn)},*F_2^{(pq)}\}_{1260}\ ,$$
$$\{\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(k\ell)},\mathcal{F}_1^{(mn)},*F_2^{(ij)},*F_2^{(k\ell)}\}_{1260}\ ,\quad \{\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(k\ell)},\mathcal{F}_1^{(mn)},*F_2^{(ij)},*F_2^{(pq)}\}_{1260}\ ,$$
$$\{\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(k\ell)},\mathcal{F}_1^{(mn)},*F_2^{(pq)},*\mathcal{F}_2^{(p)}\}_{840}\ ,\quad \{\mathcal{F}_1^{(ij)},\mathcal{F}_1^{(k\ell)},\mathcal{F}_1^{(mn)},\mathcal{F}_1^{(pq)},*F_2^{(ij)}\}_{420}\ , \qquad \text{(A.14)}$$

The $N=6$ results are the following:

$$N=6\ ,\qquad M=56700\ :$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(j\ell n)},*F_2^{(in)},*F_2^{(jm)},*F_2^{(k\ell)}\}_{3360}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(j\ell n)},F_1^{(kmn)},*F_2^{(in)},*F_2^{(jm)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(inp)},F_1^{(j\ell n)},F_1^{(jmp)},F_1^{(k\ell p)}\}_{1680}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(inp)},F_1^{(j\ell n)},F_1^{(jmp)},*\mathcal{F}_2^{(q)}\}_{5040}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(j\ell n)},F_1^{(kmn)},\mathcal{F}_1^{(pq)},*F_2^{(in)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(j\ell n)},\mathcal{F}_1^{(pq)},*F_2^{(in)},*F_2^{(jm)}\}_{10080}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(inp)},\mathcal{F}_1^{(jk)},*F_2^{(iq)},\mathcal{F}_2^{(q)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(j\ell n)},F_1^{(kmn)},\mathcal{F}_1^{(pq)}\}_{840}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(inp)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(\ell m)},*\mathcal{F}_2^{(q)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(inp)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(\ell m)},\mathcal{F}_1^{(np)}\}_{840}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},F_1^{(1np)},\mathcal{F}_1^{(jk)},*\mathcal{F}_2^{(q)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(np)},*F_2^{(iq)},*F_2^{(j\ell)},*F_2^{(km)}\}_{5040}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(\ell m)},*F_2^{(in)},*\mathcal{F}_2^{(n)}\}_{2520}\ ,$$
$$\{F_1^{(ijk)},F_1^{(i\ell m)},\mathcal{F}_1^{(jk)},\mathcal{F}_1^{(np)},*F_2^{(iq)},*\mathcal{F}_2^{(q)}\}_{5040}\ ,$$

$$
\begin{aligned}
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_2^{(iq)}\}_{2520}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *\mathcal{F}_2^{(q)}\}_{2520}\ ,\\
&\{F_1^{(ijk)}, \mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_2^{(kq)}, *\mathcal{F}_2^{(q)}\}_{2520}\ ,\\
&\{\mathcal{F}_1^{(ij)} .\mathcal{F}_1^{(k\ell)}, *F_2^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}, *F_2^{(pq)}\}_{630}\ ,\\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, *F_2^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}\}_{1680}\ ,\\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, \mathcal{F}_1^{(pq)}, *F_2^{(ij)}, *F_2^{(k\ell)}\}_{630}\ ,
\end{aligned}
\tag{A.15}
$$

The $N = 7$ results are the following:

$$
\begin{aligned}
&N = 7\ , \qquad M = 16200\ :\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, F_1^{(kmn)}, *F_2^{(in)}, *F_2^{(jm)}, *F_2^{(k\ell)}\}_{840}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}, F_1^{(jmp)}, F_1^{(k\ell p)}, F_1^{(kmn)}\}_{240}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}, F_1^{(jmp)}, F_1^{(k\ell p)}, *\mathcal{F}_2^{(q)}\}_{1680}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, F_1^{(kmn)}, \mathcal{F}_1^{(pq)}, *F_2^{(in)}, *F_2^{(jm)}\}_{2520}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, \mathcal{F}_1^{(pq)}, *F_2^{(in)}, *F_2^{(jm)}, *F_2^{(k\ell)}\}_{3360}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, *F_2^{(iq)}, *\mathcal{F}_2^{(q)}\}_{2520}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_2^{(iq)}\}_{840}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *\mathcal{F}_2^{(q)}\}_{840}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_2^{(iq)}, *\mathcal{F}_2^{(q)}\}_{2520}\ ,\\
&\{\mathcal{F}_1^{(ij)} .\mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, *F_2^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}, *F_2^{(pq)}\}_{420}\ ,\\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, \mathcal{F}_1^{(pq)}, *F_2^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)}\}_{420}\ ,
\end{aligned}
\tag{A.16}
$$

The $N = 8$ results are the following:

$$
\begin{aligned}
&N = 8\ , \qquad M = 2025\ :\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, F_1^{(j\ell n)}, F_1^{(jmp)}, F_1^{(k\ell p)}, F_1^{(kmn)}, *\mathcal{F}_2^{(q)}\}_{240}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(j\ell n)}, F_1^{(kmn)}, \mathcal{F}_1^{(pq)}, *F_2^{(in)}, *F_2^{(jm)}, *F_2^{(k\ell)}\}_{840}\ ,\\
&\{F_1^{(ijk)}, F_1^{(i\ell m)}, F_1^{(inp)}, \mathcal{F}_1^{(jk)}, \mathcal{F}_1^{(\ell m)}, \mathcal{F}_1^{(np)}, *F_2^{(iq)}, *\mathcal{F}_2^{(8)}\}_{840}\ ,\\
&\{\mathcal{F}_1^{(ij)}, \mathcal{F}_1^{(k\ell)}, \mathcal{F}_1^{(mn)}, \mathcal{F}_1^{(pq)}, *F_2^{(ij)}, *F_2^{(k\ell)}, *F_2^{(mn)},\ *F_2^{(pq)}\}_{105}\ ,
\end{aligned}
\tag{A.17}
$$

All of the solutions preserve $\frac{1}{16}$ of the supersymmetry.

$D = 2$

As we discussed in section 2, all the possible field configurations for $D = 2$ multi-charge instanton solutions are given by the 1-form field configurations of $D = 3$, except that in the case of $D = 2$, each field strength in the list can either be dualised or un-dualised.

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