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Research Article

Solution for Nonlinear Three-Dimensional Intercept Problem with Minimum Energy

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Classical orbit intercept applications are commonly formulated and solved as Lambert-type problems, where the time-of-flight (TOF) is prescribed. For general three-dimensional intercept problems, selecting a meaningful TOF is often a difficult and an iterative process. This work overcomes this limitation of classical Lambert's problem by reformulating the intercept problem in terms of a minimum-energy application, which then generates both the desired initial interceptor velocity and the TOF for the minimum-energy transfer. The optimization problem is formulated by using the classical Lagrangian f and g coefficients, which map initial position and velocity vectors to future times, and a universal time variable x. A Newton-Raphson iteration algorithm is introduced for iteratively solving the problem. A generalized problem formulation is introduced for minimizing the TOF as part of the optimization problem. Several examples are presented, and the results are compared with the Hohmann transfer solution approaches. The resulting minimum-energy intercept solution algorithm is expected to be broadly useful as a starting iterative for applications spanning: targeting, rendezvous, interplanetary trajectory design, and so on.

1. Introduction

A fundamental problem of astrodynamics is concerned with computing intercept trajectories or interplanetary mission orbit for objects in space [1, 2]. These calculations are often performed assuming a predetermined time-of-flight (TOF). This is the well-known Lambert's problem [3–6]. Applications of Lambert's problem are common in interplanetary trajectory design, spacecraft intercept, rendezvous, ballistic missile targeting, and so on. These problems are formulated by specifying the initial position vectors of an interceptor and target satellite. When the TOF is specified, the initial velocity vector for the interceptor is an unknown implicit function of the local gravity field and can only be recovered by a successive approximation strategy. Other authors have considered alternative problem formulations for a specified TOF that have included minimum-fuel and multiple-impulse strategies [7, 8]. This work overcomes the limitations of these approaches by introducing a minimum-energy approach that simultaneously generates both the TOF and the initial velocity vector for the interceptor.

For the special case of coplanar orbits, the Hohmann transfer algorithm generates a two-impulse minimum-energy orbit transfer by using tangential burns [3–5]. This technique provides a reference orbit transfer for various space applications. For direct applications of the Hohmann transfer to interplanetary orbit transfer, the position vectors of the target planet and initial departure planet are specified assuming a prescribed TOF. When the spacecraft is far from the initial position, one must be alert to the possibility that a multiorbit maneuver may be required.

Clearly, the TOF is a critical parameter for various applications. Once a TOF is determined, the rest of the procedure is solved readily by the solution of Lambert's problem. This work addresses the problem that there are no adequate methods available for determining a TOF, especially, in general three-dimensional (3D) cases. The problem of finding an optimal TOF only becomes well defined when one specifies a minimization criteria. To this end, a minimum-energy version of classical interceptor problem is formulated for recovering the TOF for a 3D orbit transfer. The results of this calculation are useful as a reference value for interplanetary trajectory design, spacecraft intercept, rendezvous, ballistic missile targeting, and so on. Of course, one can also bound the range of achievable transfer trajectory times by solving for the minimum TOF consistent with the maximum energy that can be generated. Yielding a mission design space that spans the range of TOF consists in the range $[\mathrm{TOF}_{\min}, \mathrm{TOF}_{\max}]$.

The design goal for the optimization problem is to simultaneously recover the required initial interceptor velocity and the TOF for the intercept. The mathematical advantage of this approach is that the problem has a unique optimal solution, rather than the family of solutions that characterize the classical Lambert's problem. Mathematically, the problem is defined by a constrained optimization algorithm. Particular care is exercised in formulating the problem for handling the near-parabolic orbits that arise in intercept applications. Analytically, this is handled in a comprehensive way by introducing a universal variable that permits a single TOF equation to be developed that is valid for all conic orbits.

This work is organized in three sections. First, Kepler's equation is used to define the TOF equation. This is followed by a description of the universal variable used for the problem formulation. For completeness, Lambert's problem is briefly described. Second, the minimum-energy problem for the intercept problem is introduced and solved. Third, simulation results are presented which compare the TOF obtained for an interplanetary trajectory design with a trajectory developed using the Hohmann transfer methodology and interceptor design solution approaches.

2. Mathematical Review

A fundamental approach for determining the TOF for spacecraft starts with Kepler's equation that is given by

$$M = n(t - T) = E - e\sin E, \tag{1}$$

where M is the mean anomaly, E and e denote the eccentric anomaly and the eccentricity, respectively, T is the time of periapsis passage, t is the TOF, n is the mean motion defined as $\sqrt{\mu/a^3}$, μ denotes the gravitational constant, and a is the semimajor axis of orbit.

As $e \sim 1$, the solution for Kepler's equation becomes more difficult to obtain. This problem is overcome by introducing the universal variable given by [3]

$$\dot{x} = \frac{\sqrt{\mu}}{r}, \tag{2}$$

where r is the position of spacecraft. As shown in [3], by introducing the universal variable defined by (2), one can express Kepler's equation and the radial spacecraft coordinate in the following form:

$$\begin{aligned}\sqrt{\mu}t = {} & a\left[x - \sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right)\right] \\ & + a\frac{\mathbf{r}_0\cdot\mathbf{v}_0}{\sqrt{\mu}}\left[1 - \cos\left(\frac{x}{\sqrt{a}}\right)\right] + r_0\sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right),\end{aligned} \tag{3}$$

$$r = a + a\left[\frac{\mathbf{r}_0\cdot\mathbf{v}_0}{\sqrt{\mu a}}\sin\left(\frac{x}{\sqrt{a}}\right) + \left(\frac{r_0}{a} - 1\right)\cos\left(\frac{x}{\sqrt{a}}\right)\right], \tag{4}$$

where T is assumed to be zero without loss of generality and $\mathbf{r}_0$ and $\mathbf{v}_0$ are the initial position and velocity vectors of spacecraft, respectively. These necessary conditions describe the position and velocity of an orbiting object as a function of time. If the value of the universal variable from (3) is known, the position of the spacecraft at that time is evaluated. Even though (3) is transcendental in x, a Newton's iteration technique is used to successfully solve for x when the TOF, t, is given.

Assuming that there are no external forces, then the four vectors $\mathbf{r}_0$, $\mathbf{v}_0$, $\mathbf{r}$, and $\mathbf{v}$ are assumed to be governed by Keplerian motion. To compute $\mathbf{v}$ and $\mathbf{r}$ in terms of $\mathbf{v}_0$, $\mathbf{r}_0$, and x, the position and velocity vectors of spacecraft at time t are described as [9]

$$\begin{aligned}\mathbf{r} &= f\mathbf{r}_0 + g\mathbf{v}_0, \\ \mathbf{v} &= \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0,\end{aligned} \tag{5}$$

where f, g, $\dot{f}$, and $\dot{g}$ are scalar time-dependent constants, which are subject to the following constraint:

$$f\dot{g} - \dot{f}g = 1, \tag{6}$$

where

$$\begin{aligned} f &= 1 - \frac{a}{r_0}\left[1 - \cos\left(\frac{x}{\sqrt{a}}\right)\right], \\ g &= t - \frac{a}{\sqrt{\mu}}\left[x - \sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right)\right], \\ \dot{f} &= -\frac{\sqrt{\mu a}}{r r_0}\sin\left(\frac{x}{\sqrt{a}}\right), \\ \dot{g} &= 1 - \frac{a}{r}\left[1 - \cos\left(\frac{x}{\sqrt{a}}\right)\right]. \end{aligned} \tag{7}$$

The energy minimum form of Lambert's problem is solved by introducing the classical Lagrangian coefficients and universal variable in the problem necessary conditions.

3. Time-of-Flight for Minimum-Energy Orbit Transfer

The major objective in this paper is to compute (i) the TOF and (ii) the initial velocity for an interceptor object for two

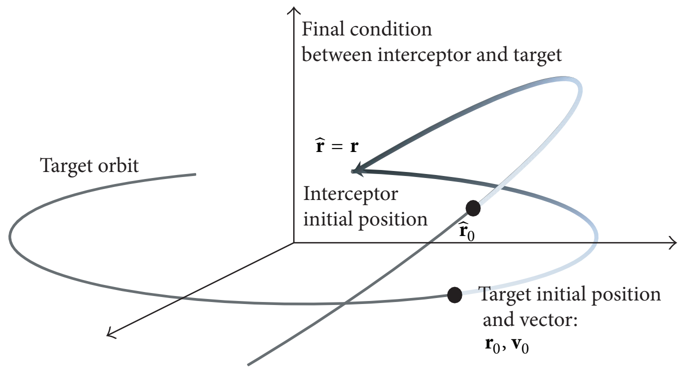

FIGURE 1: Geometry of the minimum-energy problem for a TOF.

arbitrary given position vectors so that the transfer orbit energy is a minimum. This problem differs from the classical Lambert's problem, which fixes a TOF and only recovers a solution for the initial velocity for the interceptor object, given initial and final position vectors of spacecraft.

The optimization problem is formulated by assuming that a target and an interceptor exist in arbitrary orbits, respectively. The problem geometry is illustrated in Figure 1, where $\mathbf{r}_0$ and $\mathbf{v}_0$ denote the initial position and velocity vectors of the target, respectively, and $\hat{\mathbf{r}}_0$ and $\hat{\mathbf{v}}_0$ represent the initial position and velocity vectors of interceptor, respectively.

The unknowns for the problem are the TOF and initial velocity correction for the interceptor. The goal of the trajectory optimization is to reduce the displacement position vector locating the interceptor relative to the target to zero values at the TOF, while minimizing the orbit energy of the interceptor. The problem is formulated as a nonlinear optimization problem.

4. Constrained Optimization Problem

For given $\mathbf{r}_0$, $\mathbf{v}_0$, and $\hat{\mathbf{r}}_0$, find t and $\hat{\mathbf{v}}_0$ by minimizing the performance index defined as the interceptor's orbit energy, $\mathcal{J}$, defined as

$$\mathcal{J} = \frac{\hat{v}_0^2}{2} - \frac{\mu}{\hat{r}_0} \tag{8}$$

subject to

$$\boldsymbol{\eta}(x, \hat{x}, \hat{\mathbf{v}}_0, t) = \begin{bmatrix} \eta(x, t) \\ \hat{\eta}(\hat{x}, \hat{\mathbf{v}}_0, t) \end{bmatrix} = 0, \qquad \hat{\mathbf{r}} - \mathbf{r} = 0, \tag{9}$$

where x and $\hat{x}$ (9) denote the universal variables for the target and the interceptor, respectively, and t is the TOF to be determined.

The displacement vectors for the target and interceptor are expressed using f and g as follows:

$$\mathbf{r} - \hat{\mathbf{r}} = (f\mathbf{r}_0 + g\mathbf{v}_0) - (\hat{f}\hat{\mathbf{r}}_0 + \hat{g}\hat{\mathbf{v}}_0). \tag{10}$$

As a constraint vector, $\boldsymbol{\eta} \in \mathcal{R}^2$, (3) for x and $\hat{x}$ is rewritten as

$$\begin{aligned} \eta(x, t) &= a\left[x - \sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right)\right] + r_0\sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right) \\ &\quad + a\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\left[1 - \cos\left(\frac{x}{\sqrt{a}}\right)\right] - \sqrt{\mu}t, \\ \hat{\eta}(\hat{x}, \hat{\mathbf{v}}_0, t) &= \hat{a}\left[\hat{x} - \sqrt{\hat{a}}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right] + \hat{r}_0\sqrt{\hat{a}}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right) \\ &\quad + \hat{a}\frac{\hat{\mathbf{r}}_0 \cdot \hat{\mathbf{v}}_0}{\sqrt{\mu}}\left[1 - \cos\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right] - \sqrt{\mu}t. \end{aligned} \tag{11}$$

Note that the augmented variables to be obtained are $\hat{\mathbf{v}}_0$, x, $\hat{x}$, and t.

5. Optimal Necessary Conditions

Since the second term of the energy is constant, it does not affect the performance index so that the index is redefined, without loss of generality, as [10]

$$\mathcal{J}(\hat{\mathbf{v}}_0) = \frac{1}{2}\hat{\mathbf{v}}_0^T\hat{\mathbf{v}}_0. \tag{12}$$

The Hamiltonian is formed by appending the constraints of (9) with Lagrange multipliers as follows:

$$H = \mathcal{J}(\hat{\mathbf{v}}_0) + \boldsymbol{\lambda}^T\boldsymbol{\eta}(x, \hat{x}, \hat{\mathbf{v}}_0, t) + \boldsymbol{\phi}^T(\mathbf{r} - \hat{\mathbf{r}}), \tag{13}$$

where $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^T$ and $\boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3]^T$. To minimize the performance index with respect to the augmented variables, the necessary conditions provide the following [11]:

$$\frac{\partial H}{\partial x} = \lambda_1 r + \boldsymbol{\phi}^T\frac{\partial \mathbf{r}}{\partial x} = 0, \tag{14}$$

$$\frac{\partial H}{\partial \hat{x}} = \lambda_2 \hat{r} - \boldsymbol{\phi}^T\frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}} = 0, \tag{15}$$

$$\frac{\partial H}{\partial t} = -\sqrt{\mu}\lambda_1 - \sqrt{\mu}\lambda_2 + \boldsymbol{\phi}^T(\mathbf{v}_0 - \hat{\mathbf{v}}_0) = 0, \tag{16}$$

$$\frac{\partial H}{\partial \hat{\mathbf{v}}_0} = \hat{\mathbf{v}}_0^T + \lambda_2\frac{\partial \hat{\eta}}{\partial \hat{\mathbf{v}}_0} - \boldsymbol{\phi}^T\frac{\partial \hat{\mathbf{r}}}{\partial \hat{\mathbf{v}}_0} = 0, \tag{17}$$

where $\partial\hat{\eta}/\partial\hat{\mathbf{v}}_0 \in \mathscr{R}^{1\times 3}$ represents a row vector, $\partial\hat{\mathbf{r}}/\partial\hat{\mathbf{v}}_0 \in \mathscr{R}^{3\times 3}$ is a matrix (refer to the Appendix for detail derivation), and

$$\frac{\partial \eta}{\partial x} = r, \qquad \frac{\partial \hat{\eta}}{\partial \hat{x}} = \hat{r}. \tag{18}$$

The necessary conditions of (14)–(17) are simplified by the following manipulations. First, from (15), the Lagrange multiplier λ_2 is obtained as

$$\lambda_2 = \frac{1}{\hat{r}}\boldsymbol{\phi}^T \frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}}. \tag{19}$$

Second, substituting (19) into (17) yields

$$\hat{\mathbf{v}}_0^T + \boldsymbol{\phi}^T\left(\frac{1}{\hat{r}}\frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{\mathbf{v}}_0} - \frac{\partial \hat{\mathbf{r}}}{\partial \hat{\mathbf{v}}_0}\right) = 0 \tag{20}$$

which can be solved for $\boldsymbol{\phi}$, leading to

$$\boldsymbol{\phi}^T = \hat{\mathbf{v}}_0^T\left(\frac{\partial \hat{\mathbf{r}}}{\partial \hat{\mathbf{v}}_0} - \frac{1}{\hat{r}}\frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{\mathbf{v}}_0}\right)^{-1}. \tag{21}$$

Third, by using (14), the Lagrange multiplier λ_1 follows as

$$\lambda_1 = -\frac{1}{r}\boldsymbol{\phi}^T \frac{\partial \mathbf{r}}{\partial x}. \tag{22}$$

Collecting the Lagrange multiplier solutions from (19) and (22), introducing the results into (16), one obtains

$$\frac{\sqrt{\mu}}{r}\boldsymbol{\phi}^T\frac{\partial \mathbf{r}}{\partial x} - \frac{\sqrt{\mu}}{\hat{r}}\boldsymbol{\phi}^T\frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}} + \boldsymbol{\phi}^T\left(\mathbf{v}_0 - \hat{\mathbf{v}}_0\right) = 0. \tag{23}$$

This equation is further simplified by recalling the terminal constraint $r = \hat{r}$, leading to

$$\boldsymbol{\phi}^T\left[\left(\frac{\partial \mathbf{r}}{\partial x} - \frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}}\right) + \frac{r}{\sqrt{\mu}}\left(\mathbf{v}_0 - \hat{\mathbf{v}}_0\right)\right] = 0. \tag{24}$$

Substituting (21) into (3) yields the final necessary condition required for finding the TOF for the intercept problem:

$$\hat{\mathbf{v}}_0^T\hat{L}\left[\left(\frac{\partial \mathbf{r}}{\partial x} - \frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}}\right) + \frac{r}{\sqrt{\mu}}\left(\mathbf{v}_0 - \hat{\mathbf{v}}_0\right)\right] = 0, \tag{25}$$

where the new matrix is defined for simplicity as

$$\hat{L} = \left(\frac{\partial \hat{\mathbf{r}}}{\partial \hat{\mathbf{v}}_0} - \frac{1}{\hat{r}}\frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{\mathbf{v}}_0}\right)^{-1}. \tag{26}$$

Satisfaction of the new equation implies that the interceptor can meet the target with minimum energy in a time provided by the computed TOF, not by a predetermined time.

6. Summary

The approach for obtaining the nonlinear 3D intercept problem by using the classical Lagrangian f and g coefficients is summarized. Computing the TOF and the interceptor's initial velocity is the goal to meet the requirement that the final distance between the two spacecrafts becomes zero. Firstly, one can simply obtain a and r_0 using the following:

$$r_0 = \left\|\mathbf{r}_0\right\|, \qquad a = -\frac{\mu}{2\mathscr{E}}, \tag{27}$$

where the orbital energy is given by

$$\mathscr{E} = \frac{v_0^2}{2} - \frac{\mu}{r_0}. \tag{28}$$

Then, one can find the universal variables, the initial velocity for the interceptor, and the TOF using the following equations:

$$\begin{gathered}\boldsymbol{\eta}\left(x, \hat{x}, \hat{\mathbf{v}}_0, t\right) = 0, \\ f\mathbf{r}_0 + g\mathbf{v}_0 - \hat{f}\hat{\mathbf{r}}_0 - \hat{g}\hat{\mathbf{v}}_0 = 0, \\ \hat{\mathbf{v}}_0^T\hat{L}\left[\left(\frac{\partial \mathbf{r}}{\partial x} - \frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}}\right) + \frac{r}{\sqrt{\mu}}\left(\mathbf{v}_0 - \hat{\mathbf{v}}_0\right)\right] = 0,\end{gathered} \tag{29}$$

where the semimajor axis $\hat{a}$ of the interceptor can be iteratively computed with estimated $\hat{\mathbf{v}}_0$. The Newton-Raphson iteration algorithm is applied to solve the previous equations. Next, one can compute all of the f and g expressions using (7). Then, (5) is applied to obtain the final position and velocity vectors.

There are many feasible performance indices to specify a TOF. For example, consider the candidate performance index

$$\mathscr{J}\left(\hat{\mathbf{v}}_0, t\right) = \frac{1}{2}\hat{\mathbf{v}}_0^T\hat{\mathbf{v}}_0 + \alpha t, \tag{30}$$

where α is nonnegative weight. By adding the time as one part of the performance index, the TOF is expected to be shortened with respect to the variation of α. In a similar manner with the minimum-energy procedure in the previous section, the optimization solution to this problem is readily determined. The partial derivative of the Hamiltonian H with respect to t is given by

$$\frac{\partial H}{\partial t} = -\sqrt{\mu}\lambda_1 - \sqrt{\mu}\lambda_2 + \boldsymbol{\phi}^T\left(\mathbf{v}_0 - \hat{\mathbf{v}}_0\right) + \alpha = 0. \tag{31}$$

Finally, a cost-effective equation weighted to the time is obtained as

$$\hat{\mathbf{v}}_0^T\hat{L}\left[\left(\frac{\partial \mathbf{r}}{\partial x} - \frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}}\right) + \frac{r}{\sqrt{\mu}}\left(\mathbf{v}_0 - \hat{\mathbf{v}}_0\right)\right] + \frac{r}{\sqrt{\mu}}\alpha = 0. \tag{32}$$

Numerical convergences based on different methods and their overall computational cost depend on the chosen parameterization, the initial guess, and the numerical technique used for solving the resulting equation. It is known that singularities exist when solving Lambert's problem that prevent some algorithms from converging for particular cases or make convergence extremely slow. For example, Lambert's method fails when the transfer angle is 180 degrees [9]. Therefore, the features of the suggested method must be analyzed. However, this is out of the scope of this paper, which is focused on approaches to determine the TOF and initial velocity of the interceptor.

7. Application Examples

The specification of a TOF for an intercept problem is generally not unique, and a family of solutions are possible when the initial trust level is variable. As a result, the process of determining a useful TOF requires experimentation and iteration. The minimum-energy optimization approach of this work finds a unique value for the TOF. The solution for the intercept problem simultaneously determines the initial interceptor velocity vector and TOF. Numerical examples are presented that compare and contrast the classical Hohmann transfer with the proposed method.

Let us briefly review the Hohmann transfer and compare the minimum-energy problem with it. The geometry of the Hohmann transfer is illustrated in Figure 2. The distances of the departure and arrival orbits are denoted as r_1 and r_2, respectively.

The semimajor axis for the elliptic orbit and the energy are given by [3]

$$2a_h = r_1 + r_2, \tag{33}$$

$$\mathscr{E}_h = -\frac{\mu}{r_1 + r_2}, \tag{34}$$

and, then, the departure velocity of the transfer orbit is readily obtained as

$$v_{h_0} = \sqrt{2\left(\frac{\mu}{r_1} + \mathscr{E}_h\right)}. \tag{35}$$

Since the velocity of the departure orbit is given by

$$v_1 = \sqrt{\frac{\mu}{r_1}}, \tag{36}$$

the velocity change for the Hohmann transfer is calculated as

$$\Delta v = v_{h_0} - v_1, \tag{37}$$

and the TOF of the Hohmann transfer is written as

$$t = \pi\sqrt{\frac{a_h^3}{\mu}}, \tag{38}$$

where a_h is obtained from (33). Two circular orbits are assumed with the radii of $r_1 = 4000$ km and $r_2 = 6000$ km, respectively. Then, the velocity of the departure circular orbit is $v_1 = 9.9825$ km/s, and the remaining parameters for the Hohmann transfer are obtained as $t = 1759.3$ sec, $v_{h0} = 10.9353$ km/s, and $\Delta v = 0.9528$ km/s. To navigate to the final position of the arrival orbit by the Hohmann transfer, the initial position and velocity vectors are assumed to be given by

$$\mathbf{r}_1 = [0, -4000]^T, \qquad \mathbf{r}_2 = [4097.2993, -4383.1653]^T,$$
$$\mathbf{v}_1 = [9.9825, 0]^T, \qquad \mathbf{v}_2 = [5.9543, 5.5660]^T. \tag{39}$$

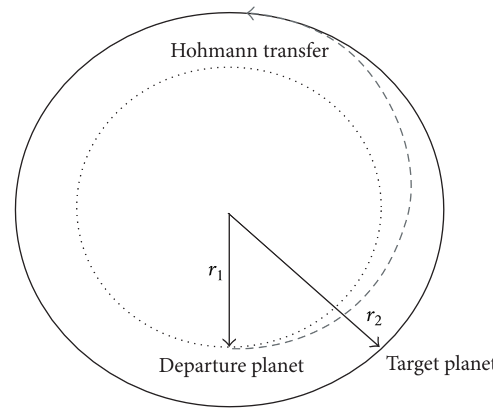

Figure 2: Geometry of the Hohmann transfer.

Table 1: Numerically computed transfer orbits.

Transfer orbit	$[v_x, v_y]^T$ (km/s)	Δv (km/s)	Time (sec)
Hohmann	$[10.9353,\ 0]^T$	0.9528	1759.3
Case 1	$[10.8784, 0.8351]^T$	1.2248	1943.3
Case 2	$[10.8440, -1.0574]^T$	1.3639	1479.0
Case 3	$[10.3372, -2.6878]^T$	2.7111	1063.9

Four cases including the Hohmann case are analyzed. Initial velocities, velocity changes, and TOF obtained by the solution of the proposed minimum-energy problem are arranged in Table 1. Also, the initial positions of target spacecraft and their resultant transfer trajectories are displayed in Figure 3.

As shown in Table 1, the result for the Hohmann case is nearly identical to the output from the classical approach in (35)–(38) with a small numerical error. It proves that the proposed approach provides optimal solutions we are looking for. Moreover, it is obvious that it gives the minimum velocity change, which is tangential with the trajectory, compared with the other cases. If the target is positioned at case 1, 2, or 3, relative to the interceptor's initial position and it is required to start the orbit transfer mission at this time, it would be a great advantage to have a reference minimum-energy trajectory to accomplish the mission. Fortunately, the results in Table 1 can be utilized, since they represent the minimum velocity in each case. This means that there are no more efficient trajectories in these cases than the transfer orbit listed in Table 1. When the target is positioned forward compared to the Hohmann transfer, the phase angle, sometimes called flight-path-angle, at departure should be negative to meet the optimal trajectory requirement. When the target, on the contrary, is positioned backward, the flight-path-angle should be positive.

Even if a circular orbit is selected for the comprehensive analysis by comparing with the Hohmann transfer, the application of the proposed approach is not limited. Therefore, an illustrative example in Figure 4 is conducted to demonstrate the performance of space maneuver of the interceptor. There are two arbitrary elliptic orbits, which are not coplanar. The initial positions of the target and interceptor orbit are depicted in Figure 4. By solving the nonlinear 3D

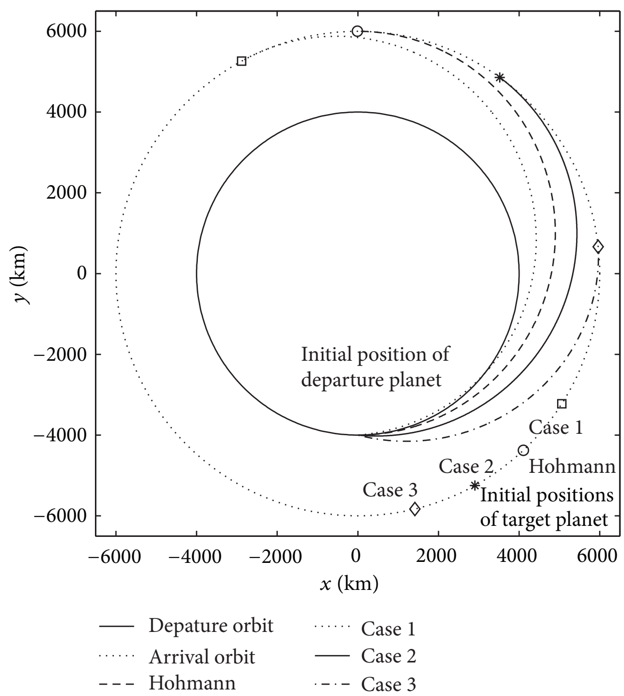

Figure 3: Trajectory generation by initial positions of target.

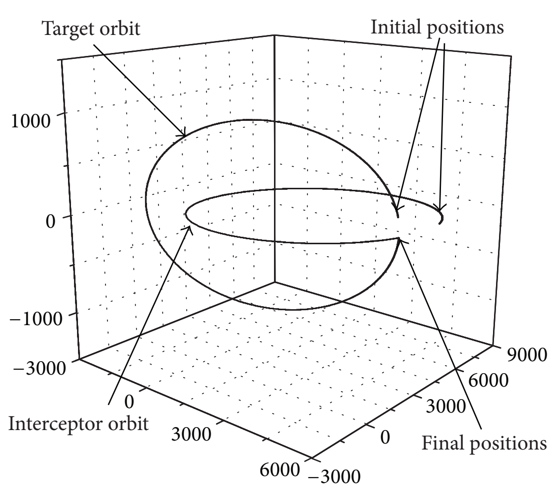

Figure 4: An illustrative example for the intercept problem.

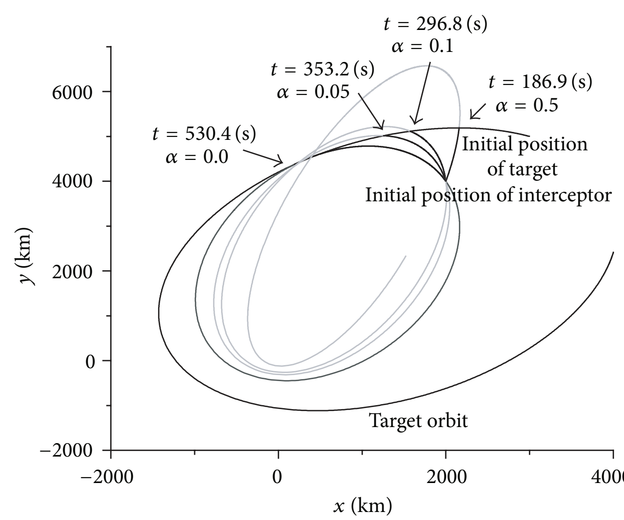

Figure 5: TOF for intercept due to variation of α.

intercept problem through minimizing energy, the TOF and initial velocity is computed. Then, applying minimum-energy velocity obtained for the interceptor confirms that the final distance between the two orbits is zero at the computed TOF. Note that the problems formulated by the universal variable and f and g expressions in this paper are solved in 3D space for supporting the design of arbitrary intercept problems with minimum energy.

The intercept problem is easily generalized by introducing a time weighting factor in the definition of the optimal control problem, which allows a systematic exploration of the intercept design space as a function of the allowed transfer energy. In this example, the outer orbit is selected as a target orbit to be captured as illustrated in Figure 5. The initial position of the interceptor is at the inner orbit. The TOF obtained from the optimization problem is shortened when α increases, and the results are illustrated in Figure 5. As expected, the longest TOF is obtained when α is zero. If α approaches one in this simulation case, the interceptor can hit the target in a very short time. It means that shortening the TOF is the optimal solution to minimize the chosen cost function.

8. Conclusions

A general algorithm is presented for generalizing the classical Lambert's transfer problem, where the determination of time-of-flight (TOF) for a spacecraft intercept, in arbitrary three-dimensional orbit, is addressed. A constrained optimization technique is introduced and iteratively solved to find both the TOF and the initial intercept velocity vector. The proposed algorithm provides a benchmark minimum-energy solution that provides an optimal reference trajectory. A significant advantage of this approach is that the TOF is unique when the energy is minimized. This implies that the interceptor with lower energy than the evaluated minimum energy cannot meet the target. Numerical results are presented, and they compare the intercept solutions with those obtained using the classical Hohmann transfer technique. The proposed algorithm is expected to be broadly useful for all classes of intercept problem that have a Lambert-like character.

Appendix

The Partial Derivatives

The partial derivatives $\mathbf{r}$ and $\hat{\mathbf{r}}$ with respect to x and $\hat{x}$, respectively, are given by

$$\frac{\partial \mathbf{r}}{\partial x} = \frac{\partial f}{\partial x}\mathbf{r}_0 + \frac{\partial g}{\partial x}\mathbf{v}_0,$$
$$\frac{\partial \hat{\mathbf{r}}}{\partial \hat{x}} = \frac{\partial \hat{f}}{\partial \hat{x}}\hat{\mathbf{r}}_0 + \frac{\partial \hat{g}}{\partial \hat{x}}\hat{\mathbf{v}}_0. \quad (A.1)$$

Applying (4) and using (7), the partial derivatives with respect to the universal variable are written as

$$\frac{\partial f}{\partial x} = -\frac{\sqrt{a}}{r_0}\sin\left(\frac{x}{\sqrt{a}}\right),$$
$$\frac{\partial g}{\partial x} = -\frac{a}{\sqrt{\mu}}\left[1-\cos\left(\frac{x}{\sqrt{a}}\right)\right],$$
$$\frac{\partial \hat{f}}{\partial \hat{x}} = -\frac{\sqrt{\hat{a}}}{\hat{r}_0}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right),$$
$$\frac{\partial \hat{g}}{\partial \hat{x}} = -\frac{\hat{a}}{\sqrt{\mu}}\left[1-\cos\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right]. \quad (A.2)$$

The orbit energy has the following relationship:

$$\frac{\hat{\mathbf{v}}_0^T\hat{\mathbf{v}}_0}{2} - \frac{\mu}{\hat{r}_0} = -\frac{\mu}{2\hat{a}}. \quad (A.3)$$

Since $\hat{r}_0$ is a constant in this case, differentiating both sides with respect to $\hat{\mathbf{v}}_0$ yields

$$\hat{\mathbf{v}}_0^T = \frac{\mu}{2\hat{a}^2}\frac{\partial \hat{a}}{\partial \hat{\mathbf{v}}_0}. \quad (A.4)$$

The partial derivative of $\hat{a}$ with respect to $\hat{\mathbf{v}}_0$ is readily written as

$$\frac{\partial \hat{a}}{\partial \hat{\mathbf{v}}_0} = \frac{2\hat{a}^2}{\mu}\hat{\mathbf{v}}_0^T. \quad (A.5)$$

The row vector, partial derivative of $\hat{\eta}$ with respect to $\hat{\mathbf{v}}_0$, is given by

$$\frac{\partial \hat{\eta}}{\partial \hat{\mathbf{v}}_0} = \frac{\partial \hat{\eta}(\hat{a})}{\partial \hat{a}}\frac{\partial \hat{a}}{\partial \hat{\mathbf{v}}_0} + \left.\frac{\partial \hat{\eta}(\hat{\mathbf{v}}_0)}{\partial \hat{\mathbf{v}}_0}\right|_{\hat{a}}, \quad (A.6)$$

where

$$\begin{aligned}\frac{\partial \hat{\eta}(\hat{a})}{\partial \hat{a}} &= \hat{x} - \sqrt{\hat{a}}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right) \\ &+ \frac{\hat{a}}{2}\left[\frac{\hat{x}}{\hat{a}}\cos\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right) - \frac{1}{\sqrt{\hat{a}}}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right] \\ &+ \frac{\hat{\mathbf{r}}_0^T\hat{\mathbf{v}}_0}{\sqrt{\mu}}\left[1-\cos\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right) - \frac{\hat{x}}{2\sqrt{\hat{a}}}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right] \\ &+ \frac{\hat{r}_0}{2}\left[\frac{1}{\sqrt{\hat{a}}}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right) - \frac{\hat{x}}{\hat{a}}\cos\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right],\end{aligned}$$
$$\left.\frac{\partial \hat{\eta}(\hat{\mathbf{v}}_0)}{\partial \hat{\mathbf{v}}_0}\right|_{\hat{a}} = \frac{\hat{a}}{\sqrt{\mu}}\left[1-\cos\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right]\hat{\mathbf{r}}_0^T. \quad (A.7)$$

Next, the partial derivative of $\hat{\mathbf{r}}$ with respect to $\hat{\mathbf{v}}_0$ follows as

$$\frac{\partial \hat{\mathbf{r}}}{\partial \hat{\mathbf{v}}_0} = \left(\frac{\partial \hat{f}}{\partial \hat{a}}\hat{\mathbf{r}}_0 + \frac{\partial \hat{g}}{\partial \hat{a}}\hat{\mathbf{v}}_0\right)\frac{\partial \hat{a}}{\partial \hat{\mathbf{v}}_0} + \left.\frac{\partial \hat{\mathbf{r}}}{\partial \hat{\mathbf{v}}_0}\right|_{\hat{a}}, \quad (A.8)$$

where

$$\frac{\partial \hat{f}}{\partial \hat{a}} = -\frac{1}{\hat{r}_0}\left[1-\cos\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right) - \frac{\hat{x}}{2\sqrt{\hat{a}}}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right],$$
$$\frac{\partial \hat{g}}{\partial \hat{a}} = -\frac{1}{\sqrt{\mu}}\left[\hat{x} - \frac{3\sqrt{\hat{a}}}{2}\sin\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right) + \frac{\hat{x}}{2}\cos\left(\frac{\hat{x}}{\sqrt{\hat{a}}}\right)\right], \quad (A.9)$$
$$\left.\frac{\partial \hat{\mathbf{r}}}{\partial \hat{\mathbf{v}}_0}\right|_{\hat{a}} = \hat{g}I_{3\times 3}.$$

The matrix, $\hat{L}$, in (26), consists of the combination of two vectors and one scaled identity matrix. If $\hat{g}$ is not zero, the matrix would have a full rank of three. The position and velocity vectors in orbit are in general not parallel. Since the condition that $\hat{g}$ is zero means that $\mathbf{r}$ and $\mathbf{v}$ are parallel, this is impossible in orbit. It could guarantee the existence of the inverse of the matrix.

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References

[1] M. Hawkins, Y. Guo, and B. Wie, "Spacecraft guidance algorithms for asteroid intercept and rendezvous missions," *International Journal of Aeronautical and Space Sciences*, vol. 13, no. 2, pp. 154–169, 2012.

[2] T. S. No, J. M. Lee, G. E. Jeon, D. Lee, and G. Kim, "A study on earth-moon transfer orbti design," *International Journal of Aeronautical and Space Sciences*, vol. 13, no. 1, pp. 106–116, 2012.

[3] R. R. Bate, D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics*, Dover Books on Aeronautical Engineering, ch 4–7, Dover, Mineola, NY, USA, 1st edition, 1971.

[4] D. A. Vallado and W. D. McClain, *Fundamentals of Astrodynamics and Applications*, Microcosm Press/Springer, El Segundo, Calif, USA, 2007.

[5] V. A. Chobotov, *Orbital Mechanics*, AIAA Education, ch 4, AIAA, Reston, Va, USA, 3rd edition, 2002.

[6] A. Prado and R. A. Broucke, "The minimum delta-V Lambert's problem," *Sociedade Brasileira de Automática*, vol. 7, no. 2, pp. 84–90, 1996.

[7] J. E. Prussing and J.-H. Chiu, "Optimal multiple-impulse time-fixed rendezvous between circular orbits," *Journal of Guidance, Control, and Dynamics*, vol. 9, no. 1, pp. 17–22, 1986.

[8] H. Shen and P. Tsiotras, "Optimal two-impulse rendezvous using multiple-revolution lambert solutions," *Journal of Guidance, Control, and Dynamics*, vol. 26, no. 1, pp. 50–62, 2003.

[9] R. H. Battin and R. M. Vaughan, "An elegant Lambert algorithm," *Journal of Guidance, Control, and Dynamics*, vol. 7, no. 6, pp. 662–670, 1984.

[10] H. Leeghim and B. A. Jaroux, "Energy-optimal solution to the lambert problem," *Journal of Guidance, Control, and Dynamics*, vol. 33, no. 3, pp. 1008–1009, 2010.

[11] A. E. Bryson, Jr. and Y. C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*, ch 1, Taylor & Francis, 1975.