

On elastic-electromagnetic mathematical equivalences

Luc T. Ikelle

CASP Project, Department of Geology and Geophysics, Texas A&M University, College Station, TX 77843-3115, USA. E-mail: ikelle@geos.tamu.edu

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SUMMARY

Elastic-electromagnetic mathematical equivalences, which here consist of recasting Maxwell's equations and the constitutive relations for an electromagnetic medium into a form that is mathematically identical with that of the basic equations for elastic waves, are presented. These equivalences allow us, for example, to use a finite-difference modelling code developed for elastic wave propagation to model electromagnetic wave propagation. The reverse process, which consists of using a finite-difference modelling code developed for electromagnetic wave propagation to model elastic wave propagation, is valid only for particular forms of elastic wave equations, such as acoustic wave equations in a 2-D medium (i.e. an infinitely long line source in the cross-line direction and acoustic properties invariant in the cross-line direction).

Key words: Electromagnetic theory; Marine electromagnetics; Controlled source seismology; Theoretical seismology; Wave propagation; Acoustic properties.

1 INTRODUCTION

Now that the controlled-source electromagnetic (CSEM) acquisition technique has taken hold as an oil and gas exploration and production tool, there is a need to develop modelling and inversion methods for analysing CSEM data, and even to revamp classical petroleum-seismology classes to include electromagnetic methods. These mathematical developments and numerical coding processes will greatly benefit the progress made in the last four decades in seismic modelling and inversion and in centuries of electromagnetic-wave studies when mathematical equivalences between Maxwell equations and elastic field equations are possible. These equivalences consist of recasting Maxwell's equations and the constitutive relations for an electromagnetic medium into a form of the basic equations for elastic waves, and vice versa. We here present a derivation of these equivalences and a discussion of their limitations. Our derivation here differs from that of Jain & Kanwal (1975) and of Carcione & Cavallini (1995) in that it is not limited to *SH*-wave propagation.

Let us emphasize that the elastic-electromagnetic equivalences described in this paper are only mathematical-based; they are not equivalences from a physics point of view. An electromagnetic phenomenon and a seismic phenomenon can share the same partial differential equation, or any another mathematical structures, while representing two totally different physical phenomena which operate at different frequency ranges. Furthermore, because mathematical structures can be translated into a numerical code, an electromagnetic phenomenon and a seismic phenomenon can share the same numerical code while representing two totally different physical phenomena. The physical differences in numerical code are controlled by the construction of inputs and the requirements associated with the inputs on one side, and on the other side by the selection of outputs and the analysis of these outputs. In other words, the effective use of the elastic-electromagnetic mathematical equivalences and associated numerical codes requires that one be well familiar with the physics of seismic and/or electromagnetic wave propagation, especially their differences. A discussion of the physical differences between seismic wave propagation and electromagnetic wave propagation is not the subject of this tutorial. We refer the reader to Andreis & MacGregor (2008) for the physics of electromagnetic systems and Ikelle & Amundsen (2005) for the physics of elastic systems.

In this paper, the position is specified by the coordinates

$$\mathbf{x} = [x_1, x_2, x_3]^T \quad (1)$$

with respect to a fixed orthonormal Cartesian reference frame with origin O and three mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$, in which each vector has unit length. The symbol T indicates a transpose. In our definitions of elastic and electromagnetic wave equations and of the elastic-electromagnetic equivalences, the subscript notation for vectors and tensors as well as the Einstein summation convention (also known as the summation over repeated indices) will be used. Lowercase Latin subscripts are employed for this purpose (e.g. v_k, τ_{pq}); they are to be assigned the values 1, 2 and 3. Boldface symbols (e.g. $\mathbf{v}, \boldsymbol{\tau}$) will be used to indicate vectors or tensors. Partial differentiation with respect to x_i is denoted by ∂_i ; ∂_t is a reserved symbol for partial differentiation with respect to time t . We will also use the Kronecker delta function δ_{pq} in our definitions of wave equation; the Kronecker δ_{pq} is zero unless $p = q$. Note also that we will use the Levi-Civita symbol in

our definition of cross-products. We denote it by ϵ_{ijk} and define it as follows:

$$\epsilon_{ijk} = \frac{(j-i)(k-i)(k-j)}{2}; \tag{2}$$

$\epsilon_{ijk} = 1$ if ijk is an even permutation, $\epsilon_{ijk} = -1$ if ijk is an odd permutation and $\epsilon_{ijk} = 0$ otherwise. One can verify that the cross product of two vectors \mathbf{a} and \mathbf{b} (i.e. $\mathbf{c} = \mathbf{a} \times \mathbf{b}$) is

$$c_i = \epsilon_{ijk} a_j b_k, \tag{3}$$

and the curl of \mathbf{a} ($\mathbf{d} = \text{curl } \mathbf{a} = \nabla \times \mathbf{a}$) is

$$d_i = \epsilon_{ijk} \partial_j a_k. \tag{4}$$

These notations will be used for the curl of the magnetic and electric fields in the Maxwell's equations.

2 ELASTIC WAVE EQUATIONS

The fields involved in the elastic wave equations are the stress and strain tensors, particle momentum (also known as the mass flow density rate) and the particle velocity vectors. The strain field, \mathbf{e} , and the stress field, $\boldsymbol{\tau}$, can be represented as follows:

$$e_{nm} \Leftrightarrow \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} \quad \text{and} \quad \tau_{kl} \Leftrightarrow \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix}. \tag{5}$$

Note that these tensors are symmetric. The particle momentum, Φ , and the particle velocity, \mathbf{v} , are defined as follows:

$$\Phi_i \Leftrightarrow \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} \quad \text{and} \quad v_q \Leftrightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{6}$$

Using these quantities, the equations of elastic wave propagation can be written as (e.g. Aki & Richards 1980; de Hoop 1995; Gangi 2000; Ikelle & Amundsen 2005)

$$-\Delta_{ijkl}^+ \partial_j \tau_{kl}(\mathbf{x}, t, \mathbf{x}_s) + \partial_t \Phi_i(\mathbf{x}, t, \mathbf{x}_s) = f_i(\mathbf{x}, t, \mathbf{x}_s) \tag{7}$$

$$\Delta_{nmpq}^+ \partial_p v_q(\mathbf{x}, t, \mathbf{x}_s) - \partial_t e_{nm}(\mathbf{x}, t, \mathbf{x}_s) = h_{nm}(\mathbf{x}, t, \mathbf{x}_s), \tag{8}$$

with

$$\Delta_{ijkl}^+ = \frac{1}{2}(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \tag{9}$$

where \mathbf{f} is the volume source density of external forces and \mathbf{h} is the volume source density of the external strain-source rate; 'external' here indicates actions of external sources to the solid under consideration. The position of these sources is specified by \mathbf{x}_s . Δ_{ijkl}^+ is the unit tensor of rank four. It satisfies the following symmetry relations: $\Delta_{ijkl}^+ = \Delta_{ijlk}^+ = \Delta_{jilk}^+ = \Delta_{jikl}^+$ and $\Delta_{ijkl}^+ = \Delta_{klij}^+$. This tensor is used to extract the symmetrical part of a tensor of rank two; that is, for example,

$$\Delta_{ijkl}^+ \tau_{kl} = \frac{1}{2}(\tau_{ij} + \tau_{ji}). \tag{10}$$

The system made of eqs (7) and (8) actually contains nine scalar equations with 18 scalar unknowns, since \mathbf{f} and \mathbf{h} are assumed to be known. The nine additional independent scalar equations are needed to render this system well posed. These additional equations are the constitutive relations and can be written as

$$e_{nm}(\mathbf{x}, t, \mathbf{x}_s) = s_{nmpq}(\mathbf{x}) \tau_{pq}(\mathbf{x}, t, \mathbf{x}_s) \tag{11}$$

and

$$\Phi_i(\mathbf{x}, t, \mathbf{x}_s) = \rho_{ij}(\mathbf{x}) v_j(\mathbf{x}, t, \mathbf{x}_s), \tag{12}$$

where \mathbf{s} is the elastic compliance tensor of the fourth rank, and ρ_{ij} is the mass density of the medium. The compliance tensor \mathbf{s} is symmetric at each point \mathbf{x} ; that is, it satisfies $s_{nmpq} = s_{mnpq} = s_{mnpq} = s_{nmpq}$ in addition to $s_{nmpq} = s_{pqmn}$. We also assume that the tensorial mass density is symmetric; that is, $\rho_{ij} = \rho_{ji}$ at each point \mathbf{x} . Note that we have considered in eq. (12) that the mass density can be anisotropic. Studies of composite materials (e.g. Willis 1985) have confirmed that the mass density can be indeed anisotropic. Substituting the constitutive relations into eqs (7) and (8), we arrive at

$$-\Delta_{ijkl}^+ \partial_j \tau_{kl}(\mathbf{x}, t, \mathbf{x}_s) + \rho_{ir}(\mathbf{x}) \partial_t v_r(\mathbf{x}, t, \mathbf{x}_s) = f_i(\mathbf{x}, t, \mathbf{x}_s) \tag{13}$$

$$\Delta_{nmpq}^+ \partial_p v_q(\mathbf{x}, t, \mathbf{x}_s) - s_{nmkl}(\mathbf{x}) \partial_l \tau_{kl}(\mathbf{x}, t, \mathbf{x}_s) = h_{nm}(\mathbf{x}, t, \mathbf{x}_s). \tag{14}$$

3 MAXWELL'S EQUATIONS

The fields involved in Maxwell's equations are the electric field, electric displacement, magnetic field and magnetic induction. If \mathbf{E} , \mathbf{D} , \mathbf{H} and \mathbf{B} are the electric field, electric displacement, magnetic field and magnetic induction vectors, respectively, we can define them as follows

$$E_n \Leftrightarrow \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad D_i \Leftrightarrow \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}, \quad H_k \Leftrightarrow \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}, \quad B_j \Leftrightarrow \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}. \quad (15)$$

Using these fields, we can write Maxwell's equations as follows (e.g. de Hoop 1995):

$$-\epsilon_{ijk} \partial_j H_k(\mathbf{x}, t, \mathbf{x}_s) + \partial_t D_i(\mathbf{x}, t, \mathbf{x}_s) = -J_i(\mathbf{x}, t, \mathbf{x}_s) \quad (\text{Ampere-Maxwell's law}), \quad (16)$$

and

$$\epsilon_{nmp} \partial_m E_p(\mathbf{x}, t, \mathbf{x}_s) + \partial_t B_n(\mathbf{x}, t, \mathbf{x}_s) = -K_n(\mathbf{x}, t, \mathbf{x}_s) \quad (\text{Faraday's law}). \quad (17)$$

The quantities \mathbf{J} and \mathbf{K} are the volume density of material electric current and the volume density of material magnetic current, respectively. In a vacuum domain, \mathbf{J} and \mathbf{K} are zero. Again, ϵ_{ijk} is the Levi-Civita symbol introduced early. The system made of eqs (16) and (17) has six scalar equations with 12 scalar unknowns, since \mathbf{J} and \mathbf{K} are assumed to be known. The six additional equations are the constitutive relations, which can be written as

$$D_i(\mathbf{x}, t, \mathbf{x}_s) = \epsilon_0^{(ip)}(\mathbf{x}) E_p(\mathbf{x}, t, \mathbf{x}_s) \quad (18)$$

$$B_n(\mathbf{x}, t, \mathbf{x}_s) = \mu_0^{(nq)}(\mathbf{x}) H_q(\mathbf{x}, t, \mathbf{x}_s), \quad (19)$$

where $\epsilon_0^{(ip)}(\mathbf{x})$ and $\mu_0^{(nq)}(\mathbf{x})$ are the permittivity and permeability tensors, respectively. These equations are quite general, as we have included the anisotropy in the electric property of an electromagnetic medium. Substituting the constitutive relations into eqs (16) and (17) gives

$$-\epsilon_{ijk} \partial_j H_k(\mathbf{x}, t, \mathbf{x}_s) + \epsilon_0^{(il)}(\mathbf{x}) \partial_l E_i(\mathbf{x}, t, \mathbf{x}_s) = -J_i(\mathbf{x}, t, \mathbf{x}_s) \quad (20)$$

$$\epsilon_{nmp} \partial_m E_p(\mathbf{x}, t, \mathbf{x}_s) + \mu_0^{(nr)}(\mathbf{x}) \partial_r H_r(\mathbf{x}, t, \mathbf{x}_s) = -K_n(\mathbf{x}, t, \mathbf{x}_s). \quad (21)$$

Note that the electromagnetic system in eqs (20) and (21) has 12 scalar equations only, whereas the equivalent elastic system in eqs (13) and (14) has 18 scalar equations. This difference will play a significant role in the discussion of the elastic-electromagnetic equivalences that we will describe in the next section.

4 ELASTIC-ELECTROMAGNETIC EQUIVALENCES

The purpose of this section is to present an analysis of the equivalence between the basic equations governing elastic wave propagation and electromagnetic wave propagation. We will seek to establish the mathematical similarities of and the differences between the physical quantities in the two cases such that every quantity in one case is shown to have a counterpart in the other case.

Let us start by introducing the following magnetic field, magnetic induction and magnetic current tensors:

$$\tilde{\mathbf{H}} = \begin{bmatrix} 0 & H_3 & -H_2 \\ -H_3 & 0 & H_1 \\ H_2 & -H_1 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \frac{1}{2} \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix},$$

and

$$\tilde{\mathbf{K}} = \begin{bmatrix} 0 & K_3 & -K_2 \\ -K_3 & 0 & K_1 \\ K_2 & -K_1 & 0 \end{bmatrix}. \quad (22)$$

Note that, in contrast to the tensors for elastic stress, strain, and strain-source tensors which are symmetric, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{K}}$ are antisymmetric (i.e. $\tilde{\mathbf{H}}^T = -\tilde{\mathbf{H}}$, $\tilde{\mathbf{B}}^T = -\tilde{\mathbf{B}}$, and $\tilde{\mathbf{K}}^T = -\tilde{\mathbf{K}}$). Using these new tensors, eqs (16) and (17) can be written as

$$-\Delta_{ijkl}^+ \partial_j \tilde{H}_{kl}(\mathbf{x}, t, \mathbf{x}_s) + \partial_t D_i(\mathbf{x}, t, \mathbf{x}_s) = -J_i(\mathbf{x}, t, \mathbf{x}_s) \quad (23)$$

$$\Delta_{nmpq}^- \partial_p E_q(\mathbf{x}, t, \mathbf{x}_s) + \partial_t \tilde{B}_{nm}(\mathbf{x}, t, \mathbf{x}_s) = -\tilde{K}_{nm}(\mathbf{x}, t, \mathbf{x}_s), \quad (24)$$

with

$$\Delta_{nmpq}^- = \frac{1}{2}(\delta_{np} \delta_{mq} - \delta_{nq} \delta_{mp}), \quad (25)$$

where \tilde{H}_{kl} , \tilde{B}_{nm} and \tilde{K}_{nm} are the components of the tensors $\tilde{\mathbf{H}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{K}}$, respectively. Δ_{nmpq}^- is another unit tensor of rank four, just like the one

in eq. (9). But contrary to Δ_{nmpq}^+ , Δ_{nmpq}^- is antisymmetric; that is, it satisfies the following symmetry relations: $\Delta_{nmpq}^- = -\Delta_{nmqp}^- = \Delta_{mnpq}^- = -\Delta_{mnpq}^-$ and $\Delta_{nmpq}^- = \Delta_{pqnm}^-$. This tensor is used to extract the antisymmetrical part of a tensor of rank two; that is, for example,

$$\Delta_{nmpq}^- \tau_{pq} = \frac{1}{2} (\tau_{nm} - \tau_{mn}). \tag{26}$$

We can see that, except for a difference in the unit tensors of eqs (8) and (24), the system made of eqs (7) and (8) is identical in form to one made of eqs (23) and (24). This sign difference is due to the fact that the strain tensor \mathbf{e} is symmetric, whereas its counterpart in the electromagnetic system, $\tilde{\mathbf{B}}$, is antisymmetric. We now need the constitutive relations to completely describe in eqs (23) and (24). After some algebra, we can verify that the relation in eq. (19) can be rewritten with the tensors $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{B}}$, as follows:

$$\tilde{B}_{ij} = \tilde{\mu}_{ijkl} \tilde{H}_{kl}, \tag{27}$$

with

$$\tilde{\mu}_{ijkl} = \epsilon_{ijp} \epsilon_{klq} \mu_0^{(pq)}, \tag{28}$$

where $\tilde{\mu}_{ijkl}$ are the components of the fourth-rank permeability tensor. If $\mu_0^{(pq)}$ is symmetric, this tensor satisfies the following symmetries:

$$\tilde{\mu}_{ijkl} = -\tilde{\mu}_{jikl} = \tilde{\mu}_{klij}. \tag{29}$$

Substituting the constitutive relations in (18) and (27) in eqs (23) and (24), we arrive at

$$-\Delta_{ijkl}^+ \partial_j \tilde{H}_{kl}(\mathbf{x}, t, \mathbf{x}_s) + \epsilon_0^{(ir)}(\mathbf{x}) E_r(\mathbf{x}, t, \mathbf{x}_s) = -J_i(\mathbf{x}, t, \mathbf{x}_s) \tag{30}$$

$$\Delta_{nmpq}^- \partial_p E_q(\mathbf{x}, t, \mathbf{x}_s) + \tilde{\mu}_{nmkl}(\mathbf{x}) \partial_t \tilde{H}_{kl}(\mathbf{x}, t, \mathbf{x}_s) = -\tilde{K}_{nm}(\mathbf{x}, t, \mathbf{x}_s). \tag{31}$$

Again, we note that, except for a difference in the unit tensors in eqs (14) and (31), the systems made of eqs (30) and (31) is identical in form to eqs (13) and (14). Table 1 summarizes this elastic-electromagnetic equivalence.

As we mentioned earlier, the mathematical equivalences between elastic and electromagnetic systems can be used to facilitate the use of concepts and techniques from one area to the other. One such technique is the numerical coding of the finite-difference modelling that we also mentioned earlier. In addition to solving the basic partial-differential equations described in this tutorial, the coding of the finite-difference modelling requires the application of boundary conditions at the free surface. Therefore, it is useful to show that the equivalences that we have just derived hold for boundary conditions; that is our next task.

Let us denote by \mathcal{S} the interface separating two elastic media. We assume that for any point \mathbf{x} on \mathcal{S} , there is a unique tangent plane. We denote by $n_i = n_i(\mathbf{x})$ the components of the unit vector along the normal to the tangent plane at \mathbf{x} . As described in Ikelle & Amundsen (2005), for example, one of the boundary condition is that the traction vector in the direction of the normal vector \mathbf{n} is continuous across the interface \mathcal{S} ; that is,

$$T_i(\mathbf{x}_1, t, \mathbf{x}_s) = T_i(\mathbf{x}_2, t, \mathbf{x}_s) \iff \Delta_{ijkl}^+ n_j(\mathbf{x}_1) \tau_{kl}(\mathbf{x}_1, t, \mathbf{x}_s) = \Delta_{ijkl}^+ n_j(\mathbf{x}_2) \tau_{kl}(\mathbf{x}_2, t, \mathbf{x}_s), \tag{32}$$

where T_i are the components of the traction vector, \mathbf{x}_1 and \mathbf{x}_2 are two neighbouring points which lie on opposite sides of the interface \mathcal{S} . The other boundary condition is that the particle velocity \mathbf{v} is continuous across the interface \mathcal{S} ; that is,

$$v_i(\mathbf{x}_1, t, \mathbf{x}_s) = v_i(\mathbf{x}_2, t, \mathbf{x}_s) \iff \Delta_{ijpq}^+ n_p(\mathbf{x}_1) v_q(\mathbf{x}_1, t, \mathbf{x}_s) = \Delta_{ijpq}^+ n_p(\mathbf{x}_2) v_q(\mathbf{x}_2, t, \mathbf{x}_s). \tag{33}$$

Suppose now that the \mathcal{S} is the interface between two media with different electromagnetic properties. As described in de Hoop (1995), for example, the boundary conditions are

$$\epsilon_{ijk} n_j(\mathbf{x}_1) H_k(\mathbf{x}_1, t, \mathbf{x}_s) = \epsilon_{ijk} n_j(\mathbf{x}_2) H_k(\mathbf{x}_2, t, \mathbf{x}_s) \tag{34}$$

Table 1. Equivalence between the elastic and electromagnetic systems.

	Elastic systems	Electromagnetic systems
Basic equations	$-\Delta_{ijkl}^+ \partial_j \tau_{kl} + \Phi_i = f_i$ $\Delta_{nmpq}^+ \partial_p v_q - \partial_t e_{nm} = h_{nm}$	$-\Delta_{ijkl}^+ \partial_j \tilde{H}_{kl} + \partial_t D_i = -J_i$ $\Delta_{nmpq}^- \partial_p E_q + \partial_t \tilde{B}_{nm} = -\tilde{K}_{nm}$
Constitutive relations	$e_{nm} = s_{nmpq} \tau_{pq}$ $\Phi_i = \rho_{ir} v_r$	$\tilde{B}_{nm} = \tilde{\mu}_{nmpq} \tilde{H}_{pq}$ $D_i = \epsilon_0^{(ir)} E_r$
Free surface	$\Delta_{ijkl}^+ n_j(\mathbf{x}_1) \tau_{kl}(\mathbf{x}_1, t, \mathbf{x}_s) = 0$ No restrictions on \mathbf{v}	$\Delta_{ijkl}^+ n_j(\mathbf{x}_1) \tilde{H}_{kl}(\mathbf{x}_1, t, \mathbf{x}_s) = 0$ No restrictions on the electric field
Fields	Stress: τ_{kl} Strain: e_{nm} Particle momentum: Φ_i Particle velocity: v_r	Magnetic field: \tilde{H}_{ij} Magnetic induction: \tilde{B}_{nm} Electric displacement: D_i Electric field: E_r
Source terms	External forces: f_i External strain-source rate: h_{nm}	Electrical current: $-J_i$ Magnetic current: $-\tilde{K}_{nm}$
Material parameters	Mass density: ρ_{ij} Elastic compliances: s_{nmpq}	Permittivity: $\epsilon_0^{(ij)}$ Permeability: $\tilde{\mu}_{nmpq}$

and

$$\epsilon_{nmp} n_m(\mathbf{x}_1) E_p(\mathbf{x}_1, t, \mathbf{x}_s) = \epsilon_{nmp} n_m(\mathbf{x}_1) E_p(\mathbf{x}_1, t, \mathbf{x}_s). \quad (35)$$

By using the unit tensors in (9) and (25) and the magnetic field tensor in (22), these two boundaries can alternatively be written as

$$\Delta_{ijkl}^+ n_j(\mathbf{x}_1) \tilde{H}_{kl}(\mathbf{x}_1, t, \mathbf{x}_s) = \Delta_{ijkl}^+ n_j(\mathbf{x}_2) \tilde{H}_{kl}(\mathbf{x}_2, t, \mathbf{x}_s) \quad (36)$$

and

$$\Delta_{nmpq}^- n_p(\mathbf{x}_1) E_q(\mathbf{x}_1, t, \mathbf{x}_s) = \Delta_{nmpq}^- n_p(\mathbf{x}_2) E_q(\mathbf{x}_2, t, \mathbf{x}_s). \quad (37)$$

We can see that the boundary conditions of the electromagnetic system can also be recast in the form of the boundary conditions of the elastic system.

For the case in which \mathcal{S} is a free surface (i.e. \mathcal{S} is an interface between a solid and air), the boundary condition (32) becomes

$$T_i(\mathbf{x}_1, t, \mathbf{x}_s) = \Delta_{ijkl}^+ n_j(\mathbf{x}_1) \tau_{kl}(\mathbf{x}_1, t, \mathbf{x}_s) = 0, \quad (38)$$

where \mathbf{x}_1 is on the solid side of the interface \mathcal{S} . No restrictions are imposed on the particle velocity in this case. There are two possible electromagnetic equivalences for this boundary condition. One possibility is \mathcal{S} as the interface between an electrically and magnetically penetrable medium (i.e. its permittivity and permeability are not too large) and a magnetically impenetrable medium (i.e. its permeability goes to infinity). The electromagnetic boundary condition in (36) becomes

$$\Delta_{ijkl}^+ n_j(\mathbf{x}_1) \tilde{H}_{kl}(\mathbf{x}_1, t, \mathbf{x}_s) = 0, \quad (39)$$

with no restrictions on the electric field and where x_1 is located in the penetrable medium. This is the boundary condition included in Table 1.

The other possible electromagnetic equivalences for this boundary condition is to consider \mathcal{S} as the interface between an electrically and magnetically penetrable medium and an electrically impenetrable medium (i.e. its permittivity goes to infinity). The electromagnetic boundary condition in (37) becomes

$$\Delta_{nmpq}^- n_p(\mathbf{x}_1) E_q(\mathbf{x}_1, t, \mathbf{x}_s) = 0 \quad (40)$$

with no restrictions on the magnetic field.

So the elastic-electromagnetic equivalence that we have just described allows us to use any theoretical or numerical solutions developed for the elastic system made of eqs (13) and (14) to solve the electromagnetic systems made of eqs (20) and (21). For example, a finite-difference modelling computer code developed for solving the equations in (13) and (14) can also be used to solve the electromagnetic eqs (20) and (21) by simply adjusting the code for the sign of the unit tensor in eq. (21). However, the reverse is not possible; that is, a finite-difference modelling computer code developed for solving eqs (20) and (21) can be directly used for solving eqs (13) and (14). The problem in the reverse operation comes from the fact that tensors $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{B}}$ contain zero components that are not necessarily zero in $\boldsymbol{\tau}$ or in $\boldsymbol{\epsilon}$. This problem stems from the fact that we have fewer equations in the electromagnetic case (i.e. 12 scalar equations) than the elastic case (i.e. 18 scalar equations). However, there are some particular forms of the elastic wave equation, where we have fewer equations than in the electromagnetic system. The acoustic case is one example.

In the acoustic case, the source-strain rate, the stress tensor and the compliance tensors can be written as $h_{ij} = h_v \delta_{ij}$, $\tau_{ij} = -p \delta_{ij}$ and $s_{ijkl} = \kappa \delta_{ij} \delta_{kl}$, respectively, where h_v is the volume density of the volume injection, p is the pressure, and κ is the compressibility. By using these definitions, eqs (13) and (14) reduce to

$$\rho_{il}(\mathbf{x}) \partial_t v_l(\mathbf{x}, t, \mathbf{x}_s) + \partial_i p(\mathbf{x}, t, \mathbf{x}_s) = f_i(\mathbf{x}, t, \mathbf{x}_s), \quad (41)$$

$$\kappa(\mathbf{x}) \partial_i p(\mathbf{x}, t, \mathbf{x}_s) + \partial_k v_k(\mathbf{x}, t, \mathbf{x}_s) = h_v(\mathbf{x}, t, \mathbf{x}_s). \quad (42)$$

To recast these equations in a form identical to that of the Maxwell's equations in (20) and (21), we assume that the mass density and the compressibility are invariant along the x_2 -axis and that source terms describe an infinitely long line source along the x_2 -axis. This assumption implies that the pressure, p , and the components of the particle velocity, v_1 and v_3 , are invariant along the x_2 -axis and that v_2 is zero. Based on this assumption, we can rewrite eqs (41) and (42) as follows:

$$\epsilon_{ijk} \partial_j \zeta_k(\mathbf{x}, t, \mathbf{x}_s) + \rho_{il}(\mathbf{x}) \partial_t w_l(\mathbf{x}, t, \mathbf{x}_s) = f_i(\mathbf{x}, t, \mathbf{x}_s) \quad (43)$$

$$\epsilon_{nmp} \partial_m w_p(\mathbf{x}, t, \mathbf{x}_s) + \kappa(\mathbf{x}) \partial_t \zeta_n(\mathbf{x}, t, \mathbf{x}_s) = b_n(\mathbf{x}, t, \mathbf{x}_s), \quad (44)$$

with

$$\zeta_k \Leftrightarrow \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix}, \quad w_l \Leftrightarrow \begin{bmatrix} -v_3 \\ 0 \\ v_1 \end{bmatrix}, \quad \text{and} \quad b_n \Leftrightarrow \begin{bmatrix} 0 \\ h_v \\ 0 \end{bmatrix}. \quad (45)$$

Table 2. Equivalence between the acoustic and electromagnetic systems.

	Acoustic systems	Electromagnetic systems
Fields	Pressure: p Particle velocity: v_1 Particle velocity: v_3	Magnetic field: H_2 Electric field: E_3 Electric field: $-E_1$
Source terms	External forces: f_i Volume injection rate: h_v	Electrical current: $-J_i$ Magnetic current: K_2
Materials parameters	Mass density: ρ_{ij} Compressibility: κ	Permittivity: $\epsilon_0^{(ij)}$ Permeability: $\mu_0^{(22)}$

We have an acoustic-electromagnetic equivalence between the system made of eqs (43) and (44) and the system made of eqs (20) and (21) if we assume that

$$E_n \Leftrightarrow \begin{bmatrix} E_1 \\ 0 \\ E_3 \end{bmatrix}, \quad H_k \Leftrightarrow \begin{bmatrix} 0 \\ H_2 \\ 0 \end{bmatrix} \quad \text{and} \quad K_k \Leftrightarrow \begin{bmatrix} 0 \\ K_2 \\ 0 \end{bmatrix}. \tag{46}$$

Note that this equivalence is limited to one polarization of the electromagnetic field and to 2-D acoustic media only. Table 2 summarizes the acoustic-electromagnetic equivalence.

As described in Ikelle & Amundsen (2005), for example, the fluid-air boundary condition is $p(\mathbf{x}_1, t, \mathbf{x}_s) = 0$ (with \mathbf{x}_1 in the fluid), and the corresponding electromagnetic equivalences for the interface between an electrically and magnetically penetrable medium and a magnetically impenetrable medium is $H_2(\mathbf{x}_1, t, \mathbf{x}_s) = 0$ (with \mathbf{x}_1 in the penetrable medium).

Let us also consider the wave propagation of *SH*-waves, which is another particular case of elastic wave propagation encountered in seismology studies, especially in earthquake seismology. If we assume that the material properties are invariant in the x_2 -direction, the wave propagation in the (x_1, x_3) -plane is decoupled from the propagation in the (x_1, x_2) -plane. The *P* waves and *SV* waves propagate in the (x_1, x_3) -plane and the *SH* waves propagate in the (x_1, x_2) -plane. The components of the stress tensor and particle velocity, $\tau_{11}, \tau_{22}, \tau_{33}, \tau_{13}, v_1$, and v_3 , are also decoupled from the components v_2, τ_{12} , and τ_{13} . So if we assume that the wave propagation is taking place in the (x_1, x_2) -plane and that the material properties are invariant in the x_2 -direction, then the equations of motions in (13) and (14) reduce to

$$-\frac{\partial \tau_{21}(\mathbf{x}, t, \mathbf{x}_s)}{\partial x_1} - \frac{\partial \tau_{23}(\mathbf{x}, t, \mathbf{x}_s)}{\partial x_3} + \rho_{22}(\mathbf{x}) \frac{\partial v_2(\mathbf{x}, t, \mathbf{x}_s)}{\partial t} = f_2(\mathbf{x}, t, \mathbf{x}_s) \tag{47}$$

$$\frac{\partial v_2(\mathbf{x}, t, \mathbf{x}_s)}{\partial x_1} - s_{2121}(\mathbf{x}) \frac{\partial \tau_{21}(\mathbf{x}, t, \mathbf{x}_s)}{\partial t} - s_{2123}(\mathbf{x}) \frac{\partial \tau_{23}(\mathbf{x}, t, \mathbf{x}_s)}{\partial t} = h_{21}(\mathbf{x}, t, \mathbf{x}_s) \tag{48}$$

$$\frac{\partial v_2(\mathbf{x}, t, \mathbf{x}_s)}{\partial x_3} - s_{2321}(\mathbf{x}) \frac{\partial \tau_{21}(\mathbf{x}, t, \mathbf{x}_s)}{\partial t} - s_{2323}(\mathbf{x}) \frac{\partial \tau_{23}(\mathbf{x}, t, \mathbf{x}_s)}{\partial t} = h_{23}(\mathbf{x}, t, \mathbf{x}_s). \tag{49}$$

Just like in the acoustic case, we can recast these equations in a form identical to that of the Maxwell's equations in (20) and (21) as follows:

$$\epsilon_{ijk} \partial_j \zeta'_k(\mathbf{x}, t, \mathbf{x}_s) + \rho_{il}(\mathbf{x}) \partial_l w'_i(\mathbf{x}, t, \mathbf{x}_s) = f_i(\mathbf{x}, t, \mathbf{x}_s) \tag{50}$$

$$\epsilon_{nmp} \partial_m w'_p(\mathbf{x}, t, \mathbf{x}_s) + s'_{nq}(\mathbf{x}) \partial_t \zeta'_q(\mathbf{x}, t, \mathbf{x}_s) = b'_n(\mathbf{x}, t, \mathbf{x}_s), \tag{51}$$

with

$$\zeta'_k \Leftrightarrow \begin{bmatrix} -\tau_{23} \\ 0 \\ \tau_{21} \end{bmatrix}, \quad w'_l \Leftrightarrow \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix}, \quad b'_n \Leftrightarrow \begin{bmatrix} -h_{23} \\ 0 \\ h_{21} \end{bmatrix} \tag{52}$$

and

$$s'_{nm} \Leftrightarrow \begin{bmatrix} -s_{2323} & 0 & s_{2321} \\ 0 & 0 & 0 \\ s_{2123} & 0 & -s_{2121} \end{bmatrix}. \tag{53}$$

We have an *SH*-elastic-electromagnetic equivalence between the system made of eqs (43) and (44) and the system made of eqs (20) and (21) if we assume that

$$E_n \Leftrightarrow \begin{bmatrix} 0 \\ E_2 \\ 0 \end{bmatrix}, \quad H_k \Leftrightarrow \begin{bmatrix} H_1 \\ 0 \\ H_3 \end{bmatrix} \quad \text{and} \quad K_k \Leftrightarrow \begin{bmatrix} K_1 \\ 0 \\ K_3 \end{bmatrix}. \tag{54}$$

Table 3. Equivalence between the *SH*-wave elastic and electromagnetic systems.

	Elastic systems	Electromagnetic systems
Fields	Stress: τ_{21}	Magnetic field: H_3
	Stress: τ_{23}	Magnetic field: $-H_1$
	Particle velocity: v_2	Electric field: E_2
Source terms	External forces: f_i	Electrical current: $-J_i$
	External strain-source rate: h_{21}	Magnetic current: K_3
	External strain-source rate: h_{23}	Magnetic current: $-K_1$
Materials parameters	Mass density: ρ_{ij}	Permittivity: $\epsilon_0^{(ij)}$
	Elastic compliances: s'_{nq}	Permeability: $\mu_0^{(nq)}$

Table 3 summarizes the *SH*-elastic-electromagnetic equivalence.

In more general terms, elastic-electromagnetic equivalences can be established as long as there is isomorphy between the elastic energy density,

$$W_{\text{el}} = \frac{1}{2} [s_{ijkl} e_{ij} e_{kl} + \rho_{pq} v_p v_q] \quad (55)$$

and the electromagnetic energy density

$$W_{\text{em}} = \frac{1}{2} [\mu_0^{(ij)} B_i B_j + \epsilon_0^{(pq)} E_p E_q]. \quad (56)$$

We can see that when the compliances can be grouped into a two-rank tensor, as in the case of *SH*-waves in the 2-D medium (i.e. $W_{\text{el}} = \frac{1}{2} [s_{2j2l} e_{2j} e_{2l} + \rho_{pq} v_p v_q]$), we will have isomorphy between the elastic energy density and the electromagnetic energy density.

5 DISCUSSION

5.1 The duality of the two Maxwell's equations

One of the remarkable features of the Maxwell's equations is that they stay the same under the duality transformation $\mathbf{E} \rightarrow \mathbf{H}$, $\mathbf{H} \rightarrow -\mathbf{E}$, $\mathbf{J} \rightarrow \mathbf{K}$ and $\mathbf{K} \rightarrow -\mathbf{J}$. We can verify this feature by replacing \mathbf{E} with \mathbf{H} , \mathbf{H} with $-\mathbf{E}$, \mathbf{J} with \mathbf{K} and \mathbf{K} with $-\mathbf{J}$ in eqs (20) and (21). This observation implies that there exist alternative forms of the elastic-electromagnetic mathematical equivalences described in Tables 1–3. These alternative forms can be obtained by replacing the electric field and electric-displacement tensors with the magnetic ones. Based on the duality of the two Maxwell's equations, we provide in Table 4, as an example, the alternative acoustic-electromagnetic equivalence to the one in Table 2. In Table 5, we provide an alternative *SH*-elastic-electromagnetic equivalence to the one in Table 3.

The existence of alternative equivalences to those in Tables 1–3 reinforces the notion that these equivalences are primarily mathematical, although some physical interpretation can be used to prefer one to another. Such preferences are likely to be driven by the applications under investigation.

As we mentioned earlier in the introduction section, Carcione & Cavallini (1995) have derived an elastic-electromagnetism equivalence for *SH*-wave propagation. However, their equivalences are different from those in Table 3 but identical to those in Table 5 for all the quantities, except for the volume density of the material magnetic current, which is considered zero in their derivations.

5.2 Conductivity in acoustic-electromagnetic equivalence

The source term \mathbf{J} in eq. (20) can be decomposed into parts

$$J_i = \sigma_{iq} E_q + J_i^{(\text{ext})}. \quad (57)$$

Table 4. An alternative equivalence between the acoustic and electromagnetic systems based on the duality of the two Maxwell's equations.

	Acoustic systems	Electromagnetic systems
Fields	Pressure: p	Magnetic field: $-E_2$
	Particle velocity: v_1	Electric field: H_3
	Particle velocity: v_3	Electric field: $-H_1$
Source terms	External forces: f_i	Electrical current: $-K_i$
	Volume injection rate: h_v	Magnetic current: $-J_2$
Materials parameters	Mass density: ρ_{ij}	Permeability: $\mu_0^{(ij)}$
	Compressibility: κ	Permittivity: $\epsilon_0^{(22)}$

Table 5. An alternative equivalence between the *SH*-wave elastic and electromagnetic systems based on the duality of the two Maxwell’s equations.

	<i>SH</i> -wave elastic systems	Electromagnetic systems
Fields	Stress: τ_{21}	Magnetic field: E_3
	Stress: τ_{23}	Magnetic field: $-E_1$
	Particle velocity: v_2	Electric field: H_2
Source terms	External forces: f_i	Electrical current: $-K_i$
	External strain-source rate: h_{21}	Magnetic current: J_3
	External strain-source rate: h_{23}	Magnetic current: $-J_1$
Materials parameters	Mass density: ρ_{ij}	Permittivity: $\mu_0^{(ij)}$
	Elastic compliances: s'_{nq}	Permeability: $\epsilon_0^{(nq)}$

$J_i^{(ext)}$ is the source that generates the field, and the part $\sigma_{iq}E_q$ describes the reaction of matter to the presence of an electromagnetic field, where σ_{iq} is the conductivity tensor. In other words, the Maxwell’s eq. (20) can alternatively be written as

$$-\epsilon_{ijk}\partial_j H_k(\mathbf{x}, t, \mathbf{x}_s) + \epsilon_0^{(il)}(\mathbf{x})\partial_t E_l(\mathbf{x}, t, \mathbf{x}_s) + \sigma_{iq}E_q(\mathbf{x}, t, \mathbf{x}_s) = -J_i^{(ext)}(\mathbf{x}, t, \mathbf{x}_s). \tag{58}$$

By taking the Fourier transform of eq. (58) with respect to time, we can rewrite this equation in the same form as eq. (20) by introducing an effective complex-valued permittivity, as follows:

$$-\epsilon_{ijk}\partial_j H_k(\mathbf{x}, \omega, \mathbf{x}_s) - i\omega\hat{\epsilon}_0^{(il)}(\mathbf{x}, \omega)E_l(\mathbf{x}, \omega, \mathbf{x}_s) = -J_i^{(ext)}(\mathbf{x}, \omega, \mathbf{x}_s), \tag{59}$$

with

$$\hat{\epsilon}_0^{(il)}(\mathbf{x}, \omega) = \epsilon_0^{(il)}(\mathbf{x}) + i\frac{\sigma_{il}(\mathbf{x})}{\omega}, \tag{60}$$

where $\hat{\epsilon}_0^{(il)}(\mathbf{x}, \omega)$ is the effective complex-valued permittivity. We can see that the conductivity can be taken into account in eq. (20) by simply considering a complex-valued permittivity in which the imaginary part characterizes the conductivity. Note that, rather than defining a new symbol to express this physical quantity after it has been Fourier-transformed, we have used the same symbol with different arguments, as the context unambiguously indicates the quantity currently under consideration. For example, $H_k(\mathbf{x}, \omega, \mathbf{x}_s)$ is the Fourier transform of $H_k(\mathbf{x}, t, \mathbf{x}_s)$ with respect to time. We will use this convention for the remaining part of the paper.

The question that we now want to address is where the conductivity tensor fits in the acoustic-electromagnetic equivalence described in Table 2, for example. The answer to this question is that we have modified the acoustic in eq. (41) to include a term that characterizes wave attenuation in addition to the scattering and geometrical spreading effect. We can write the modified acoustic equation as follows:

$$\rho_{il}(\mathbf{x})\partial_l v_l(\mathbf{x}, t, \mathbf{x}_s) + \partial_i p(\mathbf{x}, t, \mathbf{x}_s) + \eta_{iq}v_q(\mathbf{x}, t, \mathbf{x}_s) = f_i(\mathbf{x}, t, \mathbf{x}_s), \tag{61}$$

where η_{iq} is the tensorial coefficient of the attenuation parameter. By taking the Fourier of eq. (61) with respect to time, we can rewrite this equation with an effective density as follows:

$$-i\omega\hat{\rho}_{il}(\mathbf{x}, \omega)v_l(\mathbf{x}, \omega, \mathbf{x}_s) + \partial_i p(\mathbf{x}, \omega, \mathbf{x}_s) = f_i(\mathbf{x}, \omega, \mathbf{x}_s), \tag{62}$$

with

$$\hat{\rho}_{il}(\mathbf{x}, \omega) = \rho_{il}(\mathbf{x}) + i\frac{\eta_{il}(\mathbf{x})}{\omega}. \tag{63}$$

We can see that the acoustic eq. (62) compares to the electromagnetic eq. (59) in the same way that the acoustic eq. (41) compares to the electromagnetic eq. (20). In other words, the equivalences in Table 2, and even Table 1, hold even if the conductivity is not zero, as long as the density is a complex-valued tensor of the form described in eq. (63). These results also show the importance of the source terms in the derivations of the elastic-electromagnetic equivalences.

5.3 The physics of elastic-electromagnetic equivalences

Our focus in this review has been on mathematical equivalences between elastic and electromagnetic systems, and the potential use of these equivalences for translating numerical solutions and techniques designed for electromagnetic systems to elastic systems, or vice versa. Let us draw attention to the fact that these types of equivalences, especially acoustic-electromagnetic equivalences, have been studied in physics and optics for a long time (e.g. MacCullagh 1839; Thomson 1847; Maxwell 1861, 1865) for a different purpose. The primary focus in these studies of elastic-electromagnetic equivalences was to find a solid medium for which elastic equations reproduce precisely the solutions of Maxwell’s equations; this objective is clearly different from that of this paper. The motivation of these studies was that the electromagnetic theory is too abstract, especially the concepts of electric and magnetic fields. By developing analogies between elasticity and electromagnetism, we can gain some understanding of these concepts and of the electromagnetic wave propagation. The famous ether (also known as aether) medium was born of these studies. Ether is a hypothetical linear incompressible elastic medium for transmitting light and heat (radiation), filling all unoccupied space. In 19th-century physics, all waves are propagated through a medium; for example, water waves through water, sound waves through air. When Maxwell developed his electromagnetic theory of light, the 19th century physicists postulated ether as the medium that

transmitted electromagnetic waves. Ether was held to be invisible, without odour, and of such a nature that it did not interfere with the motions of bodies through space. The concept was intended to connect the elastic wave theory with Maxwell's field theory. However, all attempts to demonstrate the existence of ether, most notably the experiment reported by Michelson & Morley (1887), produced negative results and stimulated a vigorous debate that was not ended until the special theory of relativity, proposed by Einstein (1905), became accepted. In other words, the theory of relativity eliminated the need for a light-transmitting medium, so that today the term *ether* is used only in a historical context.

Despite the failure of the ether model, the analogy between the propagation of electromagnetic waves in a vacuum and the propagation of acoustic waves in ether is known to have played a crucial role in the origins of the electromagnetic theory of light (Whittaker 1952). The term *electric displacement* in electromagnetism actually has its origin in this analogy.

Although more efforts to modify the ether models is continuing (e.g. Boulanger & Hayes 2003; Dmitriyev 2004), the focus in modern physics regarding the elastic-electromagnetic analogy has shifted from the view of the electromagnetic energy propagation as an elastic wave phenomenon. Nowadays, most of the focus in the physics of elastic-electromagnetic equivalences has reversed to the application of electromagnetic concepts and techniques to systems involving elastic-wave propagation (Auld 1969; Oliner 1969, 1984; Peng 1973; Li & Chan 2004; Torrent & Sanchez-Dehesa 2008; Lee *et al.* 2009). This modern focus is motivated by the need to take advantage of recent developments of powerful techniques in the electromagnetic field, especially in the areas of the optics of composite materials and in electromagnetic microwave engineering, for the design and development of electro-mechanical transducers, acoustic waveguides and filters, and other applications which are equally useful for systems involving elastic waves. An example of this new focus that a number of seismologists are aware of is the extension of the so-called perfect-match-layer absorbing boundary conditions, which was originally designed for electromagnetic systems, to elastic systems. There are numerical modelling techniques, such as finite-difference modelling, based on eqs (20) and (21), can be used to model electromagnetic synthetic data for a given electromagnetic model. For the specific case of finite-difference modelling, we need to introduce absorbing boundaries to accommodate for the fact that the subsurface is a half-space with infinite lateral boundaries, in addition to the computation of partial differential of the Maxwell's equations in both time and space. The perfectly matched layer (PML) absorbing conditions described in Berenger (1994) is probably the most powerful way of implementing the absorbing boundaries today. Because of the equivalences in Table 2, it is straightforward to use the PML solution of electromagnetic systems for acoustic systems. However, the extension of Berenger's solution to elastic systems has taken years (Chew & Liu 1996) because an electromagnetic finite-difference modelling computer code with PML boundary conditions, as most codes developed for solving eqs (20) and (21), cannot be directly used for solving eqs (13) and (14), as we discussed earlier. However, if the original formulation of PML was designed for elastic systems, the extension to EM systems will have been made straightforward by using Table 1.

6 CONCLUSIONS

We have shown how equivalences between an elastic system and an electromagnetic system can be derived. These mathematical equivalences can be used to take advantage of formulations and techniques from one area to the other one. For example, one can use these equivalences to model electromagnetic wave propagation from the computer code originally designed to model elastic wave propagation. Similarly, a computer code designed for modelling code electromagnetic waves can be used to model acoustic wave propagation when the problem is limited to a 2-D medium.

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