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Research Article

Matrix Theory, AdS/CFT, and Gauge/Gravity Correspondence

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With $N \to \infty$ being fixed, $R \to \infty$, the free energy of the Matrix theory on a supergravity background F is a functional of F, $W = W(R, F)$. We try to relate this functional with $S_{\text{eff}}(R, F)$, the effective action of F, where F is translation invariant along x^-. The vertex function is then associated with the connected correlation function of the current densities. From $W(R, F)$, one can construct an effective action $\Gamma(Y)$ for the arbitrary matrix configuration Y. $\Gamma(Y)$ is R, and thus p^+ independent. If $W(R, F) = S_{\text{eff}}(R, F)$, $\Gamma(Y)$ will give the supergravity interactions among M theory objects with no light-cone momentum exchange. We then discuss the Matrix theory dual of the $11d$ background generated by branes with the definite p^+ as well as the gauge theory dual of the $10d$ background arising from the x^- reduction. Finally, for SYM_4 with the $4d$ background field F_0, we give a possible way to induce the radial dependent $5d$ field $F(\sigma)$.

1. Introduction

Up to present, two kinds of nonperturbative formulation of M/string theory are developed. The one is Matrix theory. The typical examples are BFSS Matrix model [1] and the plane wave Matrix model (PWMM) [2], describing a sector of M theory with the definite light-cone momentum on flat background and pp-wave background, respectively. M theory on a generic weakly curved background is described by BFSS Matrix model with the corresponding vertex operator perturbations added [3, 4]. PWMM could just be derived in this way [5]. For backgrounds that cannot be taken as the perturbations of the flat spacetime, the corresponding Matrix models are also known provided that certain amount of supersymmetries is preserved. The other is the AdS/CFT correspondence [6–9], for which the $\text{AdS}_5/\text{SYM}_4$ correspondence is intensively-studied. SYM_4 gives a nonperturbative description of string theory on $\text{AdS}_5 \times S^5$. It is natural to expect that SYM_4 with the $4d$ vertex operator perturbations added then describes the string theory on $\text{AdS}_5 \times S^5$ with the corresponding $5d$ field perturbations turned on. Although Matrix theory and AdS/CFT are obtained in entirely different ways, both of them use a gauge theory to describe M/string theory on a particular background. If the M theory background is $R_{11-n} \times T^n$, the dual gauge theory will become SYM_{n+1}, which is Matrix theory compactified on T^n [10–12].

As the nonperturbative description of M/string theory on a particular background, the Hilbert space of the gauge theory should be isomorphic to the Hilbert space of the M/string theory on that background. For $\text{AdS}_5/\text{SYM}_4$, the one-to-one correspondence should exist between the state (spectrum) of the second-quantized string theory on $\text{AdS}_5 \times S^5$ and the state (spectrum) of SYM_4. For Matrix theory, it is easier to establish the correspondence between configurations. The transition amplitude between the matrix configurations should be equal to the transition amplitude between their M theory counterparts. When $N \to \infty$, Matrix theory is the discrete regularization of the supermembrane theory in light-cone gauge [13]. The matching is explicit. It is natural that the transition amplitude for membranes is defined in the same way as the transition amplitude for strings, since the former, when wrapping S^1 with the vanishing radius, reduces to the latter. Matrix theory compactified on S^1 gives the Matrix string theory [14, 15]. The off-diagonal degrees of freedom are KK modes of membrane along S^1 [16]. In strong coupling limit (the radius of S^1 approaches 0), matrices commute (KK modes could be dropped), and Matrix string theory reduces to the second quantized type IIA string theory in light-cone gauge. The configurations are multistring configurations with the transition amplitude given by the integration of all intermediate string joining and splitting processes [14, 15].

Since the state, the spectrum, and the transition amplitude are all in one-to-one correspondence, the partition function of the gauge theory equals the partition function of the M/string theory. For SYM_4, we have [8, 9]

$$Z_{\mathrm{SYM}_4}(\beta,\phi_0)=Z_{\mathrm{AdS}_5\times S^5}(\beta,\phi_0), \tag{1}$$

where $\beta = 1/T$ is the radius of the time direction. The zero temperature partition function is only the functional of ϕ_0, for which [8, 9]

$$Z_{\mathrm{SYM}_4}(\phi_0)=e^{W[\phi(\phi_0)]}. \tag{2}$$

$\phi(\phi_0)$ is the on-shell supergravity solution with the boundary value ϕ_0. W is the type IIB supergravity action on $\mathrm{AdS}_5\times S^5$. If the gravity dual of SYM_4 with the source ϕ_0 added is the type IIB string theory on $\mathrm{AdS}_5\times S^5$ with the background field $\phi(\phi_0)$ turned on, in zero temperature limit, the partition function will only contain the contribution of the ground state geometry, so $Z_{\mathrm{SYM}_4}(\phi_0)=e^{W[\phi(\phi_0)]}$.

Except for (2), $Z_{\mathrm{SYM}_4}(\phi_0)$ also has another expression on $\mathrm{AdS}_5\times S^5$. Let W_s be the free energy of the type IIB string theory on $\mathrm{AdS}_5\times S^5$ and Z_{SYM_4} the partition function of SYM_4, it is expected that

$$Z_{\mathrm{SYM}_4}=e^{W_s}, \tag{3}$$

where $W=\sum_{n_h=0}W_{n_h}$. When the background field ϕ_0 varies, SYM_4 undergoes a change of the coupling constants. Correspondingly, there is also a change of the coupling constants for strings living in $\mathrm{AdS}_5\times S^5$, which is in fact a modification of the $5d$ background. For (3) to be valid, a one-to-one correspondence $\phi_0\leftrightarrow\phi(\phi_0)$ should exist:

$$Z_{\mathrm{SYM}_4}(\phi_0)=e^{W_s[\phi(\phi_0)]}. \tag{4}$$

On AdS_5 side, the string free energy $W_s[\phi(\phi_0)]$ cannot be defined without a definite $5d$ background $\phi(\phi_0)$. Also, for the state (spectrum) correspondence to be valid, the definite $5d$ background is necessary; otherwise, it is impossible to determine the string spectrum. If (2) and (4) both hold,

$$W[\phi(\phi_0)]=W_s[\phi(\phi_0)]. \tag{5}$$

The free energy of the string theory on a given background equals the effective action of the background fields.

For Matrix theory, similarly,

$$Z_{\mathrm{Matrix}}(\beta,p^+,F)=Z_M(\beta,p^+,F). \tag{6}$$

F is the $11d$ supergravity background and is the translation invariant along the x^- direction. $Z_M(\beta,p^+,F)$ is the partition function of the M theory sector with the light-cone momentum p^+, which is supposed to be the supermembrane. Equation (6) is trivially satisfied since Matrix theory is just the regularization of the supermembrane with the definite light-cone momentum. When $\beta=\infty$,

$$Z_{\mathrm{Matrix}}(p^+,F)=e^{W(p^+,F)}. \tag{7}$$

$W(p^+,F)$ is the p^+-parameterized functional of F with the $10d$ covariance.

In string theory, one can also calculate the free energy of the strings on a given background F:

$$W_s(F)=\sum_{m=0}^{\infty}e^{\phi(2-2m)}\int_{M_{2-2m}}[dX]\,e^{-S_F}. \tag{8}$$

In [17, 18], it was shown that, for F satisfying the free field equation, $W_s(F)$ could be taken as the effective action of the renormalized background field $\widetilde{F}(F)$; that is, $W_s(F)=S_{\mathrm{eff}}[\widetilde{F}(F)]$.

It is tempting to establish a relation between $W(p^+,F)$ and the effective action of the $11d$ supergravity. However, Matrix theory, no matter if it was taken as the DLCQ formulation of the M theory or as the discrete regularization of the supermembrane theory in light-cone gauge, only describes the sub-Hilbert space of the M theory with the definite light-cone momentum p^+ without capturing all the information of the covariant theory. For different p^+, $W(p^+,F)$ is different. Nevertheless, let $F=\{F_0,F_-,F_{--}\}$, where F_0,F_-,F_{--} represent fields with zero, one, and two x^- indices, respectively; one can find that $W(p^+,F)=W(F_0,F_-/p^+,F_{--}/p^{+2})$. Let $p^+=N/R$ with R being the radius of x^-, $N\to\infty$, $R\to\infty$, and $W(R,F)=W(F_0,RF_-,R^2F_{--})$. On the other hand, for $11d$ supergravity fields F that are translation invariant along x^-, suppose $S_{\mathrm{eff}}(R,F)$ is the effective action of F, and there is also $S_{\mathrm{eff}}(R,F)=S_{\mathrm{eff}}(F_0,RF_-,R^2F_{--})$. The R dependence of $W(R,F)$ is consistent with $S_{\mathrm{eff}}(R,F)$.

One may want to consider the complete partition function with all light-cone momentum taken into account, which is roughly $e^{W(F)}=\int dp^+e^{W(p^+,F)}$. Each $W(p^+,F)$ only differs by a rescaling of (F_-,F_{--}), so the summation does not give more information. It is enough to consider $W(p^+,F)$ with the definite p^+. In fact, F is the translation invariant along x^-; as a result, a sector with the definite p^+ has the enough degrees of freedom to produce $S_{\mathrm{eff}}(R,F)$. The complete M theory degrees of freedom including sectors with all light-cone momentum is necessary only when F is the field with the $11d$ spacetime dependence.

For the arbitrary matrix configuration Y, we may define $\Gamma(Y)$ via

$$\begin{aligned}e^{\Gamma(Y)}&=\int\left[d\widetilde{Y}\right]e^{-S_{R,F}(\widetilde{Y})+S_{R,F}(Y)}\\&=e^{W(R,F)+S_{R,F}(Y)},\end{aligned} \tag{9}$$

with $F(x)$ solved from

$$-\left.\frac{\delta W(R,F)}{\delta F(x)}\right|_F=\left.\frac{\delta S_{R,F}(Y)}{\delta F(x)}\right|_F. \tag{10}$$

It is easy to see that $\Gamma(Y)$ is R independent.

To describe the supergravity interactions among M theory objects with no light-cone momentum exchange, and one may define $\Gamma_g(Y)$ through

$$e^{\Gamma_g(Y)} = \int [dF]\, e^{S_{R,F}(Y)-S_{\rm cla}(R,F)}, \tag{11}$$

with F being the zero mode of the $11d$ supergravity along x^- and $S_{\rm cla}$ the classical action of $11d$ supergravity. The integrating out of F induces the effective action for the M theory object Y with the supergravity interaction (without transferring the light-cone momentum) taken into account.

$$\begin{aligned}\Gamma_g(Y) = \sum_n \frac{1}{n!}\int d^{10}x_1 \cdots \int d^{10}x_n G_{\rm Fc}(x_1,\ldots,x_n)\\ \times V_{F(Y)}(x_1)\cdots V_{F(Y)}(x_n),\end{aligned} \tag{12}$$

where $G_{\rm Fc}(x_1,\ldots,x_n)$ is the connected Green's function of supergravity in light-cone gauge with the zero light-cone momentum and $V_{F(Y)}(x)$ is the current density of the configuration Y coupling with the supergravity field $F(x)$. Under a Legendre transformation, $\Gamma_g(Y)$ could be written as

$$\begin{aligned}\Gamma_g(Y) &= S_{\rm eff}(R,F)\\ &\quad+\int d^{10}x\, F(x)\, V_{R,F(Y)}(x) \sim S_{\rm eff}(R,F) + S_{R,F}(Y),\end{aligned} \tag{13}$$

with $F(x)$ solved from

$$-\left.\frac{\delta S_{\rm eff}(R,F)}{\delta F(x)}\right|_F = V_{R,F(Y)}(x) = \left.\frac{\delta S_{R,F}(Y)}{\delta F(x)}\right|_F. \tag{14}$$

So, if $W(R,F) = S_{\rm eff}(R,F)$, $\Gamma(Y) = \Gamma_g(Y)$.

Although M theory/Matrix theory correspondence and AdS/CFT correspondence are very different, it is possible to construct the connection between the two. In [19–21], it was shown that PWMM expanded around the certain 1/2 BPS states gives $\mathrm{SYM}_{R\times S^2}$, $\mathrm{SYM}_{R\times S^3/Z_k}$, and $\mathrm{SYM}_{R\times S^3}$, while the backreaction of the corresponding 1/2 BPS states on pp-wave produces the gravity dual. We will investigate the correspondence in more detail.

In (7), F is the generic $11d$ supergravity field with the x^- isometry. $W(R,F)$, if is indeed equal to $S_{\rm eff}(R,F)$, gives the effective action of the field F. On the other hand, in (2), only the $4d$ field $\phi_0(x)$ is given, from which the $5d$ field $\phi(x,r)$ is obtained from the equation of motion or from the RG flow. $W[\phi(\phi_0)]$ is the action of $\phi(x,r)$. This is the holography of AdS/CFT. One may want to turn on the arbitrary $\phi(x,r)$ on AdS_5 and try to find the corresponding $4d$ gauge dual. The dual gauge theory may not be SYM_4, since ϕ_0 can only encode a subset of $5d$ fields, which are in one-to-one correspondence with the $4d$ fields. In fact, since the transverse space of $\mathrm{AdS}_5\times S^5$ is S^5 other than R^6, the gauge theory dual may have the scalar fields $\widehat{X}^I$ other than X^I. Suppose the coordinate of $\mathrm{AdS}_5\times S^5$ is (x,r,Ω), for a $10d$ scalar $H_n(x,r,\Omega) = h_n(x,r)Y^n(\Omega)$ with $Y^n(\Omega) = C_{a_1\cdots a_n}(y^{a_1}\cdots y^{a_n})/|y|^n$ the spherical harmonic of S^5; the operator counterpart is

$$H_n\left(x,r,\widehat{X}\right) = h_n(x,r)\, C_{a_1\cdots a_n}\left(\widehat{X}^{a_1}\cdots\widehat{X}^{a_n}\right). \tag{15}$$

$h_n(x,r)$ can be the arbitrary $5d$ function. In SYM_4, we only have $h_n(x)C_{a_1\cdots a_n}(X^{a_1}\cdots X^{a_n})$ to represent such fields. Nevertheless, for SYM_4 with the scalar field X and the $4d$ background $F_0(x)$, a $X \rightarrow \widehat{X}$, $F_0(x) \rightarrow F(x,r)$ transformation can be made, under which the partition function remains invariant. If the SYM_4 with the scalar field $\widehat{X}$ and the $5d$ background $F(x,r)$ is the gauge theory description of the string theory on $\mathrm{AdS}_5\times S^5$ with the background $F(x,r)$, its partition function will then equal $e^{W[F(x,r)]}$ with W being the $10d$ supergravity action. So we arrive at (2). $X \rightarrow \widehat{X}$ is a Weyl transformation, under which $F_0(x)$ must evolve as $F(x,r)$ to preserve the partition function. We will show that, for such $F(x,r)$, $\delta W[F(x,r)]/\delta F(x,r) = 0$, so if $W[F(x,r)]$ is the action of the supergravity, $F(x,r)$ will be the on-shell solution. The discussion can also be extended to SCFT_3 and SCFT_6. With no source term added, under the $X \rightarrow \widehat{X}$ transformation, the induced fields give the near horizon geometry of $M2$ and $M5$, respectively. The holography in AdS/CFT is very similar to the holography in noncritical string coupling with $2d$ being the gravity and with $g \rightarrow \widehat{g}$ replaced by $X \rightarrow \widehat{X}$.

This paper is organized as follows. In Section 2, we consider the free energy of the Matrix theory on supergravity background F that is translation invariant along the x^- direction, and its relation with the $11d$ effective action of F. In Section 3, we consider Matrix theory on the configuration representing branes and its gravity dual. The discussion will then be specified to the PWMM, from which, $\mathrm{SYM}_{R\times S^2}$, $\mathrm{SYM}_{R\times S^3/Z_k}$, and $\mathrm{SYM}_{R\times S^3}$ can be obtained [19–21]. In Section 4, we give a possible way to induce the radial dependent $5d$ fields from the $4d$ background fields in SYM_4.

2. Free Energy and the Effective Action of Supergravity

In this section, we will consider $W(R,F)$, the free energy of the Matrix theory on a generic $11d$ supergravity background F that is translation invariant along x^-. Since the supermembrane action in light-cone gauge only contains one free parameter p^+, as the discrete regularization of the supermembrane action, Matrix theory action $S_{R,F}(Y)$ also has one free parameter which could be taken as R, the radius of x^-. $p^+ = N/R$, and $N \rightarrow \infty$ is fixed. The concrete R dependence of $S_{R,F}(Y)$ is $S_{R,F}(Y) = S_{F_0,RF_-,R^2F_{--}}(Y)$, where F_0, F_-, F_{--} represent fields with zero, one, and two x^- indices. As a result, $W(R,F) = W(F_0, RF_-, R^2F_{--})$. On the other hand, due to the coordinate invariance, the R dependence of the effective action of the supergravity field F is also $S_{\rm eff}(R,F) = S_{\rm eff}(F_0, RF_-, R^2F_{--})$. From $W(R,F)$, we can define $\Gamma(Y) = S_{R,F}(Y) + W(R,F)$, with F solved through $\delta[S_{R,F}(Y) + W(R,F)]/\delta F = 0$. $\Gamma(Y)$ is R, or equivalently, p^+, independent. $\Gamma(Y)$ could be taken as the effective action of the matrix configuration Y. In fact, at the one-loop level, $\Gamma(Y)$ and the standard effective action of the Matrix theory coincide. If $W(R,F) = S_{\rm eff}(R,F)$, we will have $\Gamma(Y) = \Gamma_g(Y)$ with $\Gamma_g(Y)$ being the effective action describing the supergravity interactions among the M theory objects with the zero light-cone momentum exchange.

2.1. The Action of the Matrix Theory on a Generic Background. The Matrix theory action in flat spacetime is

$$S_R = R\int dx^+ \times \operatorname{Tr}\left(\frac{1}{2R^2}D_+X^ID_+X^I + \frac{1}{4l_p^6}\left[X^I,X^J\right]^2 - \frac{i}{R}\theta D_+\theta + \frac{1}{l_p^3}\theta\gamma_I\left[X^I,\theta\right]\right), \tag{16}$$

where X^I, θ, and A_0 are $N\times N$ hermitian matrices with $N \to \infty$, and $I = 1\cdots 9$. $R \to \infty$, $p^+ = N/R$, l_p is the Planck length.

X^I, x^+, and R have the dimension of length, so each commutator is multiplied by a factor $1/l_p^3$ to make the action dimensionless. With the replacement $X^I \to l_pX^I$, $x^+ \to l_px^+$, $R \to l_pR$, we get the action

$$S_R = R\int dx^+ \times \operatorname{Tr}\left(\frac{1}{2R^2}D_+X^ID_+X^I + \frac{1}{4}\left[X^I,X^J\right]^2 - \frac{i}{R}\theta D_+\theta + \theta\gamma_I\left[X^I,\theta\right]\right), \tag{17}$$

in which l_p is cancelled and X^I, θ, x^+, and R are all dimensionless. In the following, we will still adopt this convention, so l_p will not appear explicitly.

The $11d$ supergravity field, after the gauge fixing, has the nonzero components $(g_{+-}, g_{++}, g_{I+}, g_{IJ})$, (C_{IJ+}, C_{IJK}), and (ψ_+, ψ_I) [22]. Based on the Hamiltonian in [22], one can write down the action of the bosonic membrane on such supergravity background:

$$S^b_{F_0} = \int dx^+d^2\sigma\frac{g_{+-}}{P^+} \times\left[\frac{P^{+2}D_+X^ID_+X^J}{2g_{+-}^2}g_{IJ} + \frac{P^+D_+X^I}{g_{+-}}\left(C^J + \frac{P^+g_+^J}{g_{+-}}\right)g_{IJ} + \frac{P^{+2}g_{++}}{2g_{+-}^2} - \left\{X^I,X^J\right\}\left\{X^K,X^L\right\}g_{IK}g_{JL} + \frac{P^+C_+}{g_{+-}}\right], \tag{18}$$

where $C^I = \{X^J, X^K\}C^I_{JK}$ and $C_+ = \{X^J, X^K\}C_{+JK}$. Note that it is g_{+-}/P^+ that appears. The Matrix theory version is

$$S^b_{F_0} = \int dx^+ \times\operatorname{Tr}\left(Rg_{+-}\left\{\frac{D_+X^ID_+X^J}{2(Rg_{+-})^2}g_{IJ} + \frac{D_+X^I}{Rg_{+-}}\left(C^J + \frac{g_+^J}{Rg_{+-}}\right)g_{IJ} + \frac{1}{4}\left[X^I,X^J\right]\left[X^K,X^L\right]g_{IK}g_{JL} + \frac{g_{++}}{2(Rg_{+-})^2} + \frac{C_+}{Rg_{+-}}\right\}\right), \tag{19}$$

where $C^I = -(i/2)[X^J, X^K]C^I_{JK}$ and $C_+ = -(i/2)[X^J, X^K]C_{+JK}$. Without the gauge fixing, g_{I-}, g_{--}, C_{-IJ}, C_{+-I}, $\psi_- \neq 0$, so terms involving them can also be added:

$$S^b_{g_{I-}} = \int dx^+ \times\operatorname{Tr}\left(Rg_{+-}\left\{\frac{D_+X^ID_+X^JD_+X^K}{2(Rg_{+-})^3}g_{JK} - \frac{D_+X^I}{4Rg_{+-}}\left[X^K,X^L\right]\left[X^M,X^N\right] \times g_{KM}g_{LN} + \frac{D_+X^L}{Rg_{+-}}\left[X^J,X^M\right]\left[X^I,X^N\right] \times g_{LJ}g_{MN}\right\} \times Rg_{I-}\right) \tag{20}$$

and similarly for $S^b_{g_{--}}$, which is the sum of the 4th order terms $(D_+X)^4$, $(D_+X)^2[X,X]^2$, and $[X,X]^4$. The current densities for $g_{--}, g_{I-}, C_{-IJ}, C_{+-I}, \psi_-$ were derived in [3, 23–25] through the calculation of the one-loop effective action. Otherwise, in the light-cone quantization of the membrane, one may add $D_+X^- = (1/2)D_+X^ID_+X_I + (1/\nu^2)\{X^I, X^J\}\{X_I, X_J\}$ and $D_aX^- = D_+X^ID_aX_I$ into the action, which could couple with the x^- indexed fields.

In all these terms, it is Rg_{+-}, Rg_{I-}, R^2g_{--}, RC_{-IJ}, RC_{+-I}, $R\psi_-$ that are involved. (Similarly, in supermembrane action, it is g_{+-}/P^+, g_{I-}/P^+, g_{--}/P^{+2}, C_{-IJ}/P^+, C_{+-I}/P^+, ψ_-/P^+ that will appear.) R is the radius of x^-. Under the parameterization transformation $x^+ = f(x'^+)$, $x^- = \alpha x'^- \Leftrightarrow R = \alpha R'$,

$$\begin{gathered} g'_{++} = f'^2\left(x'^+\right) g_{++}, \qquad g'_{+-} = \alpha f'\left(x'^+\right) g_{+-}, \\ g'_{+I} = f'\left(x'^+\right) g_{+I}, \qquad g'_{-I} = \alpha g_{-I}, \\ g'_{--} = \alpha^2 g_{--}, \\ C'_{+-J} = \alpha f'\left(x'^+\right) C_{+-J}, \qquad C'_{+IJ} = f'\left(x'^+\right) C_{+IJ}, \\ C'_{-IJ} = \alpha C_{-IJ}, \end{gathered} \tag{21}$$

the action is invariant. This is consistent with the coordinate transformation

$$\begin{aligned} ds^2 &= g_{++} df\left(x'^+\right) df\left(x'^+\right) + 2 g_{I+} dx^I df\left(x'^+\right) \\ &\quad + g_{IJ} dx^I dx^J + 2\alpha g_{+-} df\left(x'^+\right) dx'^- \\ &\quad + 2\alpha g_{I-} dx^I dx'^- + \alpha^2 g_{--} dx'^- dx'^- \\ &= g_{++} f'^2\left(x'^+\right) dx'^+ dx'^+ + 2 f'\left(x'^+\right) g_{I+} dx^I x'^+ \\ &\quad + g_{IJ} dx^I dx^J + 2\alpha g_{+-} f'\left(x'^+\right) dx'^+ dx'^- \\ &\quad + 2\alpha g_{I-} dx^I dx'^- + \alpha^2 g_{--} dx'^- dx'^-, \\ C &= C_{IJK} dx^I \wedge dx^J \wedge dx^K + C_{IJ+} dx^I \wedge dx^J \wedge df\left(x'^+\right) \\ &\quad + \alpha C_{IJ-} dx^I \wedge dx^J \wedge dx'^- \\ &\quad + \alpha C_{I+-} dx^I \wedge df\left(x'^+\right) \wedge dx'^- \\ &= C_{IJK} dx^I \wedge dx^J \wedge dx^K \\ &\quad + C_{IJ+} f'\left(x'^+\right) dx^I \wedge dx^J \wedge dx'^+ \\ &\quad + \alpha C_{IJ-} dx^I \wedge dx^J \wedge dx^- \\ &\quad + \alpha C_{I+-} f'\left(x'^+\right) dx^I \wedge dx'^+ \wedge dx^-. \end{aligned} \tag{22}$$

$S_{R,F}$, the supersymmetric extension of (19) and (20), is not constructed yet. With $S_{R,F}$ given, the current density for F can be defined as

$$V_{F(Y)}(x) = \left.\frac{\delta S_{R,F}(Y)}{\delta F(x)}\right|_F. \tag{23}$$

In particular, when F is the flat background, for which, the only nonvanishing fields are $\overline{g}_{+-} = -1$ and $\overline{g}_{IJ} = \delta_{IJ}$, $V_{F(Y)}(x)$ reduces to the localized vertex operator of the supergravity field F. In [3, 4, 23–25], the vertex operators for various supergravity fields are constructed. Although the exact $S_{R,F}$ is unknown, with the vertex operators at hand, one can write down the Matrix theory action on weakly curved background in linear gravity approximation [3, 4, 23, 24]:

$$\begin{aligned} S_{R,F} = S_R - R\int dx^+ & \\ \times \operatorname{STr}\Big[& V_{h_{IJ}(X)} h_{IJ}(X) + V_{h_{I+}(X)} h_{I+}(X) \\ & + V_{h_{++}(X)} h_{++}(X) + V_{h_{I-}(X)} h_{I-}(X) \\ & + V_{h_{+-}(X)} h_{+-}(X) + V_{h_{--}(X)} h_{--}(X) \\ & + V_{C_{IJK}(X)} C_{IJK}(X) + V_{C_{+JK}(X)} C_{+JK}(X) \\ & + V_{C_{IJ-}(X)} C_{IJ-}(X) + V_{C_{+-K}(X)} C_{+-K}(X) \\ & + V_{\psi(X)} \psi(X) + V_{\psi_+(X)} \psi_+(X) \\ & + V_{\psi_-(X)} \psi_-(X) \Big], \end{aligned} \tag{24}$$

where, for example,

$$V_{h_{I+}(X)} h_{I+}(X) = \left(\frac{D_+ X^I}{R} - \frac{1}{4}\theta\gamma^{IJ}\theta\frac{\partial}{\partial X^J}\right)\frac{h_{I+}(X)}{R}. \tag{25}$$

$h_{IJ}(X), \ldots, \psi_-(X)$ are $11d$ background fields with the c-number coordinate x replaced by the matrix coordinate X. The background fields are only the functions of $(x^+, x^1, \ldots, x^9)$; they are the zero modes of the $11d$ supergravity along x^-.

The vertex operator can take three different forms. First, it can be the operator defined in a $1d$ SYM theory, just as that in AdS/CFT. Then, the background fields, like $h_{I+}(X)$, should be expanded as the Taylor series:

$$\begin{aligned} & h_{I+}\left(x^+, X^1, \ldots, X^9\right) \\ & \quad = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\partial_{k_1}\cdots\partial_{k_n} h_{I+}\right)\left(x^+, 0, \ldots, 0\right) X^{k_1}\cdots X^{k_n}, \end{aligned} \tag{26}$$

with $(\partial_{k_1}\cdots\partial_{k_n} h_{I+})(x^+, 0, \ldots, 0)$ the derivative of $h_{I+}(x^+, \vec{x})$ at $(x^+, 0)$ [3, 23–25].

$$\begin{aligned} & \operatorname{STr}\left[V_{h_{I+}(X)} h_{I+}(X)\right] \\ & \quad = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\partial_{k_1}\cdots\partial_{k_n} h_{I+}\right)\left(x^+, 0, \ldots, 0\right) \\ & \qquad \times \operatorname{STr}\left[\frac{1}{R}\left(\frac{D_+ X^I}{R} - \frac{1}{4}\theta\gamma^{IJ}\theta\frac{\partial}{\partial X^J}\right) X^{k_1}\cdots X^{k_n}\right] \\ & \quad = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\partial_{k_1}\cdots\partial_{k_n} h_{I+}\right)\left(x^+, 0, \ldots, 0\right) I^{k_1\cdots k_n}_{h_{I+}}\left(x^+\right). \end{aligned} \tag{27}$$

Since all background fields enter into the action in the form of $f(x^+, 0, \ldots, 0)$, the $U(N)$ SYM$_1$ lives at $(x^+, 0, \ldots, 0)$. When

$$\begin{gathered} X^i \longrightarrow X^i - a^i, \\ f\left(x^+, 0, \ldots, 0\right) \longrightarrow f\left(x^+, a^1, \ldots, a^9\right), \end{gathered} \tag{28}$$

SYM_1 undergoes a translation. Nothing specifies where the SYM_1 should be, so one may put it at any point in R^9. Similar to AdS/CFT, there is a one-to-one correspondence between operators and fields. However, no holography is present here. Fields living on SYM_1 are Taylor series coefficients, which uniquely determines the $11d$ background. The background fields are arbitrary and are not necessarily on shell. On the other hand, in AdS/CFT, fields living on CFT are boundary values, from which the full background is solved through the equations of motion or the RG flow. In contrast to the chiral primary operators $\Phi^{k_1\cdots k_n}$ in AdS/CFT, the moment operators $I^{k_1\cdots k_n}$ do not need to be traceless. As a result, $\{1, I^{k_1}, \ldots, I^{k_1\cdots k_n}, \ldots\}$ couples with the $10d$ field, while $\{1, \Phi^{k_1}, \ldots, \Phi^{k_1\cdots k_n}, \ldots\}$ only couples with the $9d$ field.

In the second form, the vertex operator is defined in $10d$ spacetime:

$$\begin{aligned}\frac{\delta}{\delta F(x)} F\left(x^+, X^1, \ldots, X^9\right) &= \delta\left(x^0 - x^+\right)\delta^9\left(x^I - X^I\right) \\ &= \delta^{10}\left(x^\mu - X^\mu\right),\end{aligned} \tag{29}$$

where X^I are $N \times N$ matrices and for uniformity; we have set $X^0 = x^+ \mathbb{1}_{N\times N}$. This is a matrix generalization of the δ-function. In special situations, when all of the X^I are diagonal, that is, $X^I = \operatorname{diag}\{x_1^I, x_2^I, \ldots, x_N^I\}$, (29) becomes

$$\begin{aligned}&\frac{\delta}{\delta F(x)} F\left(x^+, X^1, \ldots, X^9\right) \\ &\quad = \operatorname{diag}\left\{\delta^{10}\left(x^\mu - x_1^\mu\right), \delta^{10}\left(x^\mu - x_2^\mu\right), \ldots, \delta^{10}\left(x^\mu - x_N^\mu\right)\right\}.\end{aligned} \tag{30}$$

With the generalized δ-function, it is straightforward to write down the current densities for various fields. For h_{I+}, we have

$$\begin{aligned}V_{h_{I+}}(x) &= \frac{\delta}{\delta h_{I+}(x)}\left\{-R\int dx^+ \operatorname{STr}\left[V_{h_{I+}(X)} h_{I+}(X)\right]\right\} \\ &= -R\int dx^+ \operatorname{STr}\left[V_{h_{I+}(X)}\delta^{10}(x - X)\right].\end{aligned} \tag{31}$$

It is not convenient to deal with the δ-function. One may want to do a Fourier transformation, which gives the third representation of the vertex operator:

$$\begin{aligned}V_{h_{I+}}(p) &= \int d^{10}x V_{h_{I+}}(x) e^{ipx} \\ &= -R\int dx^+ \operatorname{STr}\left[V_{h_{I+}(X)} e^{ipX}\right] \\ &= -R\int dx^+ \operatorname{STr}\left\{\left[-\frac{D_+ X^I}{R} + \frac{i}{4}\theta\gamma^{IJ}\theta p_J\right] e^{ipX}\right\}.\end{aligned} \tag{32}$$

2.2. Partition Function of Matrix Theory on Curved Background. Suppose the exact form of S_F is given, and the partition function of Matrix theory on supergravity background is

$$\begin{aligned}Z(R,F) &= e^{W(R,F)} \\ &= \int \left[dX\, d\theta\, dA_0\right] e^{-S_{R,F}(X,\theta,A_0)} \\ &= \int [dY]\, e^{-S_{R,F}(Y)},\end{aligned} \tag{33}$$

where F is the $11d$ background field (g_{+-}, g_{++}, g_{I+}, g_{IJ}, g_{--}, g_{I-}), (C_{IJ+}, C_{IJK}, C_{IJ-}, C_{I+-}), and (ψ_+, ψ_I, ψ_-) mentioned before, and Y collectively represents (X^I, θ, A_0).

In gauge/string correspondence, partition function is an important quantity, the value of which should be equal on both sides. For AdS_5/CFT_4 correspondence, it is expected that

$$Z_{SYM_4} = \exp\{-F\} \tag{34}$$

should hold, where $F = \sum_{n_h=0} F_{n_h}$ is the free energy of the strings on $AdS_5 \times S^5$, Z_{SYM_4} and $\exp\{-F\}$ are the partition function of SYM_4 and the partition function of the second quantized type IIB string theory on $AdS_5 \times S^5$, respectively. For the present situation, the comparison is relatively trivial. On one side, we have a gauge theory with the partition function given by (33); on the other side, the M-theory sector with the light-cone momentum p^+ is described by the Matrix model with $p^+ = N/R$, for which the partition function is again (33).

Suppose $S^M_{P^+,F}(Y)$ is the membrane action on background F. $S^M_{P^+,F}(Y)$ should be general covariant, so, for $P^+ = P'^+/\alpha$, $x^+ = f(x'^+)$, and $X = g(X')$, there is $S^M_{P^+,F}(Y) = S^M_{P'^+,F'}(Y')$, where F' is the field coming from the coordinate transformation:

$$x^- = \alpha x'^-, \qquad x^+ = f\left(x'^+\right), \qquad x = g\left(x'\right). \tag{35}$$

For the bosonic action, we can see this is indeed the case. If $[dY] = [dY']$, that is, the path integral measure is coordinate independent, we will have

$$\int [dY]\, e^{-S^M_{P^+,F}(Y)} = \int \left[dY'\right] e^{-S^M_{P'^+,F'}(Y')}. \tag{36}$$

After the matrix regularization, the membrane configurations $Y(x^+, \sigma^1, \sigma^2)$ and $Y'(x^+, \sigma^1, \sigma^2)$ become the matrix configurations $Y(x^+)$ and $Y'(x^+)$, and there is

$$\begin{aligned}Z(R,F) = e^{W(R,F)} &= \int [dY]\, e^{-S_{R,F}(Y)} \\ &= \int \left[dY'\right] e^{-S_{R',F'}(Y')} = e^{W(R',F')} \\ &= Z\left(R', F'\right).\end{aligned} \tag{37}$$

$W(R,F) = W(R',F')$. At least restricted to (35), $W(R,F)$ is the diffeomorphism invariant functional of F.

Let F_0, F_-, F_{--} represent fields with zero, one, and two x^- indices. Since

$$S_{R,F}(Y) = S_{F_0,RF_-,R^2F_{--}}(Y), \quad (38)$$

$$W(R,F) = W\left(F_0, RF_-, R^2F_{--}\right). \quad (39)$$

For the $11d$ supergravity fields (F_0, F_-, F_{--}), which are translation invariant along the x^- direction, $x^- \in [0, 2\pi R)$, the $11d$ supergravity effective action is $S_{\text{eff}}(R; F_0, F_-, F_{--})$. S_{eff} is invariant under the coordinate transformation:

$$x^- \longrightarrow \frac{x^-}{R}, \qquad F_0 \longrightarrow F_0, \qquad F_- \longrightarrow RF_-, \qquad F_{--} \longrightarrow R^2 F_{--}, \quad (40)$$

so

$$S_{\text{eff}}\left(R; F_0, F_-, F_{--}\right) = S_{\text{eff}}\left(1; F_0, RF_-, R^2F_{--}\right). \quad (41)$$

The radius of x^- is absorbed in F_-, F_{--}, as is in (39). In this respect, W is consistent with S_{eff}.

Let $F = \overline{F} + \widetilde{F}$, where $\overline{F}$ is the flat background with $\overline{g}_{+-} = -1$, $\overline{g}_{IJ} = \delta_{IJ}$, and the rest fields being zero. $W(R, \overline{F}) = 0$:

$$\begin{aligned} W(R,F) &= \int d^{10}x \Gamma^c_{R,F}(x) \widetilde{F}(x) + \frac{1}{2!}\int d^{10}x_1 \times \int d^{10}x_2 \Gamma^c_{R,F}(x_1,x_2)\widetilde{F}(x_1)\widetilde{F}(x_2) + \cdots \\ &= \sum_n \frac{1}{n!}\int d^{10}x_1 \cdots \int d^{10}x_n \Gamma^c_{R,F}(x_1,\ldots,x_n) \times \widetilde{F}(x_1)\cdots\widetilde{F}(x_n), \end{aligned} \quad (42)$$

where (F collectively represents supergravity fields. For example, $\Gamma^c_{R,g_{++},g_{--}}(x_1,x_2) = \delta^2 W(R,F)/\delta g_{++}(x_1)\delta g_{--}(x_2)|_{F=\overline{F}}$.)

$$\Gamma^c_{R,F}(x_1,\ldots,x_n) = \left.\frac{\delta^n W(R,F)}{\delta F(x_1)\cdots\delta F(x_n)}\right|_{F=\overline{F}}. \quad (43)$$

For the same F, the $11d$ supergravity effective action $S_{\text{eff}}(R,F)$ is

$$\begin{aligned} S_{\text{eff}}(R,F) = \sum_n \frac{1}{n!}\int_0^{2\pi R} dx_1^- &\times \int dx_1^+ d^9\mathbf{x}_1 \cdots \int_0^{2\pi R} dx_n^- \int dx_n^+ d^9\mathbf{x}_n \\ &\times \Gamma^c_F\left(x_1^-, x_1^+, \mathbf{x}_1; \ldots; x_n^-, x_n^+, \mathbf{x}_n\right) \\ &\times \widetilde{F}\left(x_1^+, \mathbf{x}_1\right)\cdots\widetilde{F}\left(x_n^+, \mathbf{x}_n\right). \end{aligned} \quad (44)$$

$\Gamma^c_F(x_1^-, x_1^+, \mathbf{x}_1; \ldots; x_n^-, x_n^+, \mathbf{x}_n)$ is the vertex function of the $11d$ supergravity. If $S_{\text{eff}}(R,F) = W(R,F)$, there will be

$$\begin{aligned} \Gamma^c_{R,F}&(x_1,\ldots,x_n) \\ &= \Gamma^c_{R,F}\left(x_1^+, \mathbf{x}_1; \ldots; x_n^+, \mathbf{x}_n\right) \\ &= \int_0^{2\pi R} dx_1^- \cdots \int_0^{2\pi R} dx_n^- \Gamma^c_F\left(x_1^-, x_1^+, \mathbf{x}_1; \ldots; x_n^-, x_n^+, \mathbf{x}_n\right). \end{aligned} \quad (45)$$

In linear gravity approximation, $\delta V_F(x)/\delta F(y) = 0$, where V_F is the current density coupling with $F(x)$ as is defined in (23):

$$\Gamma^c_{R,F}(x_1,\ldots,x_n) = (-1)^n \left\langle V_F(x_1)\cdots V_F(x_n)\right\rangle_c. \quad (46)$$

$\langle V_F(x_1)\cdots V_F(x_n)\rangle_c$ is the connected correlation function of the current density, in contrast to

$$\begin{aligned} \left\langle V_F(x_1)\cdots V_F(x_n)\right\rangle &= \int \left[dX\, d\theta\, dA_0\right] e^{-S_0} V_F(x_1)\cdots V_F(x_n) \\ &= \left.\frac{(-1)^n \delta^n Z(R,F)}{\delta F(x_1)\cdots\delta F(x_n)}\right|_{F=\overline{F}}, \end{aligned} \quad (47)$$

which is the correlation function of the current density. In (42), $W(R,F)$ is expanded around the flat background $\overline{F}$. One can of course expand $W(R,F)$ on a different background, giving rise to the different $\Gamma^c_{R,F}(x_1,\ldots,x_n)$.

In terms of $V_F(p)$,

$$\begin{aligned} \left\langle V_F(x_1)\cdots V_F(x_n)\right\rangle_c &= \int d^{10}p_1 \cdots \int d^{10}p_n e^{-i(p_1x_1+\cdots+p_nx_n)} \\ &\quad \times \left\langle V_F(p_1)\cdots V_F(p_n)\right\rangle_c. \end{aligned} \quad (48)$$

Note that $V_{F(X)}(x^+, x^I - a^I) = V_{F(X+a)}(x^+, x^I)$, $S_0(X+a) = S_0(X)$, so

$$\left\langle V_F(x_1)\cdots V_F(x_n)\right\rangle_c = \left\langle V_F(x_1+a_1)\cdots V_F(x_n+a_n)\right\rangle_c. \quad (49)$$

The correlation function is translation invariant. $\langle V_F(p_1)\cdots V_F(p_n)\rangle_c = 0$ if $p_1 + \cdots + p_n \neq 0$.

Let us first consider the one point function

$$\left\langle V_F(x)\right\rangle = -\left.\frac{\delta W(R,F)}{\delta F(x)}\right|_{F=\overline{F}} = \overline{v}_F. \quad (50)$$

$\overline{v}_F$ is the vacuum expectation value of the current density, when the background field is $\overline{F}$, and is a constant in spacetime. S_R is $SO(9)$ invariant, so $\overline{v}_F$ should also be $SO(9)$ invariant. As a result,

$$\begin{aligned} \overline{v}_{g_{I+}} &= \overline{v}_{g_{I-}} = \overline{v}_{C_{IJK}} = \overline{v}_{C_{IJ+}} \\ &= \overline{v}_{C_{IJ-}} = \overline{v}_{C_{I+-}} = \overline{v}_{\psi_I} = 0. \end{aligned} \quad (51)$$

$\overline{\nu}_{g_{IJ}} = 0$ if the traceless condition is imposed. $\overline{\nu}_{\psi_+} = \overline{\nu}_{\psi_-} = 0$ due to the supersymmetry. The nonvanishing current densities are $\overline{\nu}_{h_{++}}$, $\overline{\nu}_{h_{--}}$, and $\overline{\nu}_{h_{+-}}$. In particular, $\overline{\nu}_{h_{++}} = p^+/L^9$ is the vacuum expectation value of the light-cone momentum density. For the generic value of F,

$$\begin{aligned}\langle V_F(x)\rangle_F &= -\left.\frac{\delta W(R,F)}{\delta F(x)}\right|_{F=\overline{F}+\widetilde{F}} = \nu_F(x) \\ &= \sum_n -\frac{1}{(n-1)!} \\ &\quad\times \int d^{10}x_1 \cdots \int d^{10}x_{n-1}\ \Gamma^c_{R,F}(x, x_1, \ldots, x_{n-1}) \\ &\quad\times \widetilde{F}(x_1) \cdots \widetilde{F}(x_{n-1})\end{aligned} \tag{52}$$

is the vacuum expectation value of the current density in presence of the background field. In string theory, the vanishing of the one point function, the tadpole, for vertex operators gives the equations of motion for background fields. Similarly, here, if $W(R,F)$ is the effective action of the supergravity fields, on SUGRA solution background, there will be $\langle V_F(x)\rangle_F = 0$ except for h_{++}, whose vertex operator is the same as the tachyon in bosonic string. We will return to this problem later.

2.3. Another Effective Action of Matrix Theory. For the given $F(x)$, $\nu_F(x)$ is uniquely determined. Conversely, different $F(x)$ may result in the same $\nu_F(x)$. This is quite like the source-gravity coupled system. For the given gravity field, the density of the source can be obtained through $\delta S_g/\delta g^{\mu\nu} = -T_{\mu\nu}$. On the other hand, with the given source, the gravity solution is not unique. Nevertheless, with the proper boundary condition imposed, there is always a privileged solution. We will choose the boundary condition so that, for $\nu_F(x) = 0$, $\widetilde{F}(x) = 0$. $\widetilde{F}(x)$ could be interpreted as the field generated by the current density $\nu_F(x)$. Other boundary conditions correspond to adding the external supergravity background, for example, the plane wave background, in addition to fields generated by source. We will discuss this situation later.

Then, there is a one-to-one correspondence between ν_F and F, and so a Legendre transformation is possible. Before that, we will first define $\Gamma(Y)$:

$$\Gamma(Y) = S_{R,F}(Y) + W(R,F), \tag{53}$$

$$e^{\Gamma(Y)} = \int \left[d\widetilde{Y}\right] e^{-S_{R,F}(\widetilde{Y})+S_{R,F}(Y)}. \tag{54}$$

F is solved from the equation

$$\begin{aligned}&\int \left[d\widetilde{Y}\right] \frac{\delta}{\delta F(x)} e^{-S_{R,F}(\widetilde{Y})+S_{R,F}(Y)} \\ &= \int \left[d\widetilde{Y}\right] e^{-S_{R,F}(\widetilde{Y})+S_{R,F}(Y)} \left[-\frac{\delta S_{R,F}(\widetilde{Y})}{\delta F(x)} + \frac{\delta S_{R,F}(Y)}{\delta F(x)}\right] = 0\end{aligned} \tag{55}$$

or equivalently,

$$\int \left[d\widetilde{Y}\right] e^{-S_{R,F}(\widetilde{Y})} \frac{\delta S_{R,F}(\widetilde{Y})}{\delta F(x)} = \frac{\delta S_{R,F}(Y)}{\delta F(x)} = -\frac{\delta W(R,F)}{\delta F(x)}. \tag{56}$$

In some sense, $F(x)$ is the field generated by Y. Take a derivative of (54) with respect to Y; using (55), we get

$$\frac{\delta\Gamma(Y)}{\delta Y} = \frac{\delta S_{R,F}(Y)}{\delta Y}, \tag{57}$$

where F is solved from (55). The variation on the right-hand side of (57) only acts on Y with F being fixed.

Recall that the R dependence of $S_{R,F}(Y)$ only comes from RF_- and R^2F_{--}, so if the solution of (56) is (F_0, F_-, F_{--}), with R replaced by αR, the solution will become $(F_0, F_-/\alpha, F_{--}/\alpha^2)$. $S_{R,F}(Y)$, $W(R,F)$, and $\Gamma(Y)$ remain invariant. $\Gamma(Y)$ is R-independent. In supermembrane picture, suppose there are two supermembrane theories with the light-cone momentum P^+ and P'^+, and consider the fields generated by the same configuration $Y(x^+, \sigma^1, \sigma^2)$. Let $x'^- \to P'^+x'^-/P^+$, $P'^+ \to P^+$, in this frame, the generated fields are the same, changing back, we get the relation $F_0 = F_0$, $F_-/P^+ = F'_-/P'^+$, $F_{--}/P^{+2} = F'_{--}/P'^{+2}$. When plugged into $S^M_{P^+,F}(Y)$, the R dependence is canceled, so the obtained $\Gamma(Y)$ is the same.

With these properties collected, we may consider the possible interpretation of $\Gamma(Y)$ and $W(R,F)$. If $W(R,F)$ is the effective action of supergravity, since $S_{R,F}(Y)$ is the matrix theory action on background F, $\Gamma(Y)$ will be the action of the source-gravity coupled system and is on shell with respect to supergravity and thus could be taken as the effective action of the configuration Y. $\delta\Gamma(Y)/\delta Y = \delta S_{R,F}(Y)/\delta Y = 0$ is the quantum corrected equation of motion, which differs from the classical equation of motion $\delta S_0(Y)/\delta Y = 0$ in that the background fields in former are generated by the configuration Y itself, while the background fields in latter are given. For time-independent Y, the stationary point of the effective action gives the vacuum configuration. In this case, $\delta\Gamma(Y)/\delta Y = \delta S_{R,F}(Y)/\delta Y = 0$ is equivalent to the requirement that the branes should not exert force to each other, which is the no force condition for BPS configurations [26]. All of the above statements are based on the assumption that $W(R,F)$ is the effective action of supergravity, which, however, is unproved.

Let us continue to explore the properties of $\Gamma(Y)$ and $W(R,F)$. For simplicity, let $R = 1$, or, in other words, let the x^- indexed fields absorb R. In a weakly curved background, $S_F(Y) = S_0(Y) + \int d^{10}x\widetilde{F}(x)V_{F(Y)}(x)$, $\Gamma(Y) = S_0(Y) + \widetilde{\Gamma}(\nu_F)$, where

$$\widetilde{\Gamma}(\nu_F) = \int d^{10}x\widetilde{F}(x)\,\nu_F(x) + W\left(\overline{F} + \widetilde{F}\right) \tag{58}$$

is the Legendre transformation of $W(F)$:

$$\begin{aligned}-\left.\frac{\delta W(F)}{\delta\widetilde{F}(x)}\right|_{\widetilde{F}} &= \nu_F = V_{F(Y)}(x), \\ \left.\frac{\delta\widetilde{\Gamma}(\nu_F)}{\delta\nu_F(x)}\right|_{\nu_F} &= \widetilde{F}(x).\end{aligned} \tag{59}$$

Let

$$\Lambda_F(x_1,\ldots,x_n) = \left.\frac{\delta^n\widetilde{\Gamma}(v_F)}{\delta v_F(x_1)\cdots\delta v_F(x_n)}\right|_{v_F=0}, \quad (60)$$

then

$$\begin{aligned}\Gamma(Y) &= S_0(Y) + \sum_n \frac{1}{n!} \\ &\quad \times \int d^{10}x_1\cdots\int d^{10}x_n\ \Lambda_F(x_1,\ldots,x_n) \\ &\quad \times v_F(x_1)\cdots v_F(x_n) \\ &= S_0(Y) + \sum_n \frac{1}{n!} \\ &\quad \times \int d^{10}x_1\cdots\int d^{10}x_n\ \Lambda_F(x_1,\ldots,x_n) \\ &\quad \times V_{F(Y)}(x_1)\cdots V_{F(Y)}(x_n).\end{aligned} \quad (61)$$

For the given $v_F(x)$,

$$\begin{aligned}F(x) &= \sum_n \frac{1}{(n-1)!} \\ &\quad \times \int d^{10}x_1\cdots\int d^{10}x_{n-1}\Lambda_F(x,x_1,\ldots,x_{n-1}) \\ &\quad \times v_F(x_1)\cdots v_F(x_{n-1}).\end{aligned} \quad (62)$$

Equation (52) gives the current density $v_F(x)$ generated by the field $F(x)$; (62) gives the field $F(x)$ generated by the current density $v_F(x)$. If Λ_F is the connected Green's function of supergravity, $F(x)$ will be the vacuum expectation value of the supergravity field F in presence of the source v_F. In classical level, $F(x)$ could be calculated by $\delta S_g/\delta F(x) = -v_F(x)$ with S_g the classical action of supergravity:

$$\begin{aligned}\Gamma^c_F(x-z) &= -\left.\frac{\delta v_F(x)}{\delta\widetilde{F}(z)}\right|_{\widetilde{F}=0}, \\ \Lambda_F(z-y) &= \left.\frac{\delta^2\widetilde{\Gamma}(v_F)}{\delta v_F(z)\,\delta v_F(y)}\right|_{v_F=0} = \left.\frac{\delta\widetilde{F}(z)}{\delta v_F(y)}\right|_{v_F=0},\end{aligned} \quad (63)$$

so

$$-\int d^{10}z\Gamma^c_F(x-z)\Lambda_F(z-y) = \delta^{10}(x-y). \quad (64)$$

$-\Lambda_F(x_1-x_2)$ is the inverse of $\Gamma^c_F(x_1-x_2)$. Subsequently,

$$\begin{aligned}\Lambda_F(x_1,x_2,x_3) &= \int d^{10}y_1\int d^{10}y_2 \\ &\quad \times \int d^{10}y_3\ \Lambda_F(y_1-x_1)\Lambda_F(y_2-x_2) \\ &\quad \times \Lambda_F(y_3-x_3)\Gamma^c_F(y_1,y_2,y_3), \\ \Gamma^c_F(x_1,x_2,x_3) &= -\int d^{10}y_1\int d^{10}y_2 \\ &\quad \times \int d^{10}y_3\Gamma^c_F(y_1-x_1)\Gamma^c_F(y_2-x_2) \\ &\quad \times \Gamma^c_F(y_3-x_3)\Lambda_F(y_1,y_2,y_3). \\ \Lambda_F(x_1,x_2,x_3,x_4) &= \int d^{10}y_1\int d^{10}y_2\int d^{10}y_3 \\ &\quad \times \int d^{10}y_4\Lambda_F(y_1-x_1)\Lambda_F(y_2-x_2) \\ &\quad \times \Lambda_F(y_3-x_3)\Lambda_F(y_4-x_4) \\ &\quad \times \Big[\Gamma^c_F(y_1,y_2,y_3,y_4) \\ &\quad + \int d^{10}z_1 \\ &\quad \times \int d^{10}z_2\Lambda_F(z_1-z_2) \\ &\quad \times (\Gamma^c_F(z_1,y_1,y_4) \\ &\quad \times \Gamma^c_F(z_2,y_2,y_3) \\ &\quad + \Gamma^c_F(z_1,y_2,y_4) \\ &\quad \times \Gamma^c_F(z_2,y_1,y_3) \\ &\quad + \Gamma^c_F(z_1,y_3,y_4) \\ &\quad \times \Gamma^c_F(z_2,y_1,y_2))\Big], \\ \Gamma^c_F(x_1,x_2,x_3,x_4) &= \int d^{10}y_1\int d^{10}y_2\int d^{10}y_3 \\ &\quad \times \int d^{10}y_4\Gamma^c_F(y_1-x_1)\Gamma^c_F(y_2-x_2) \\ &\quad \times \Gamma^c_F(y_3-x_3)\Gamma^c_F(y_4-x_4) \\ &\quad \times \Big[\Lambda_F(y_1,y_2,y_3,y_4) \\ &\quad + \int d^{10}z_1 \\ &\quad \times \int d^{10}z_2\ \Gamma^c_F(z_1-z_2) \\ &\quad \times (\Lambda_F(z_1,y_1,y_4) \\ &\quad \times \Lambda_F(z_2,y_2,y_3) \\ &\quad + \Lambda_F(z_1,y_2,y_4) \\ &\quad \times \Lambda_F(z_2,y_1,y_3) \\ &\quad + \Lambda_F(z_1,y_3,y_4) \\ &\quad \times \Lambda_F(z_2,y_1,y_2))\Big].\end{aligned} \quad (65)$$

The relation between Λ_F and Γ_F^c shows that Λ_F and Γ_F^c are the connected Green's function and the vertex function of a particular quantum field theory. $\Lambda_F(z-y) = (\delta\widetilde{F}(z)/\delta v_F(y))|_{v_F=0}$ gives the change of the supergravity field with respect to the current density, so it is natural to take it as the propagator of the supergraviton. We will see some evidence for it.

In (54), let $\widetilde{Y} = Y + \eta$,

$$e^{\Gamma(Y)} = \int [d\eta]\, e^{-S_F(Y+\eta)+S_F(Y)}.$$

$$\begin{aligned}
-S_F(Y+\eta) + S_F(Y) &= -\int d\tau \frac{\delta S_F(Y)}{\delta Y(\tau)}\eta(\tau) \\
&\quad - \frac{1}{2}\int d\tau\, d\tau' \frac{\delta^2 S_F(Y)}{\delta Y(\tau)\,\delta Y(\tau')} \\
&\quad \times \eta(\tau)\,\eta(\tau') + \cdots \\
&= -S_F^{(1)}(Y,\eta) - S_F^{(2)}(Y,\eta) + \cdots .
\end{aligned} \tag{66}$$

S_F could be written as (We only write ∂_τ^2, but one should be aware that the fermionic kinetic term ∂_τ also exists.)

$$\begin{aligned}
S_F(Y) &= \int d\tau \left[\frac{1}{2} Y \partial_\tau^2 Y + V_0(Y) + V(F,Y)\right] \\
&= S_0(Y) + \int d\tau V(F,Y)
\end{aligned} \tag{67}$$

with $V(F,Y)$ taking the form as that in (27). For the given Y, solve F_0 from $\delta S_{F_0}(Y)/\delta Y = 0$, and expand $S_F^{(n)}(Y,\eta)$ around F_0. $F = F_0 + f$. Consider

$$\begin{aligned}
S_F^{(1)}(Y,\eta) &= \left.\frac{\delta S_F^{(1)}(Y,\eta)}{\delta F}\right|_{F_0} f + \cdots \\
S_F^{(2)}(Y,\eta) &= S_{F_0}^{(2)}(Y,\eta) + \left.\frac{\delta S_F^{(2)}(Y,\eta)}{\delta F}\right|_{F_0} f + \cdots .
\end{aligned} \tag{68}$$

$f = f(Y)$ is solved as the functional of Y. Let $\Gamma(Y) = \Gamma_1(Y) + \Gamma_2(Y) + \cdots$ since

$$\begin{aligned}
-S_F(Y+\eta) + S_F(Y) &= -S_{F_0}^{(2)}(Y,\eta) + \cdots , \\
\Gamma_1(Y) = \ln\left\{\int [d\eta]\, e^{-S_{F_0}^{(2)}(Y,\eta)}\right\} &= \ln\left(\det A_{F_0}\right)^{1/2}
\end{aligned} \tag{69}$$

with

$$A_{F_0}(\tau,\tau') = \left[\partial_\tau^2 + \frac{\partial^2 V_0(Y)}{\partial Y^2} + \frac{\partial^2 V(F_0,Y)}{\partial Y^2}\right]\delta(\tau-\tau'). \tag{70}$$

In particular, when $\delta S_0(Y)/\delta Y = 0$, $F_0 = 0$, A_{F_0} reduces to

$$A(\tau,\tau') = \left[\partial_\tau^2 + \frac{\partial^2 V_0(Y)}{\partial Y^2}\right]\delta(\tau-\tau'). \tag{71}$$

The corresponding $\Gamma_1(Y)$ is the same as the one-loop contribution $\Gamma_{\text{eff}}^1(Y)$ of the standard effective action $\Gamma_{\text{eff}}(Y)$ in background gauge:

$$e^{\Gamma_{\text{eff}}(Y)} = \int [d\eta]\, e^{-S_0(Y+\eta)+\int d\tau\, \mathrm{Tr}[J(\tau)\eta(\tau)]}, \tag{72}$$

subject to the condition

$$\int [d\eta]\, \frac{\delta}{\delta J(\tau)} e^{-S_0(Y+\eta)+\int d\tau\, \mathrm{Tr}[J(\tau)\eta(\tau)]} = 0. \tag{73}$$

$\Gamma_1(Y) = \Gamma_{\text{eff}}^1(Y)$ has been calculated for the arbitrary Y satisfying $\delta S_0(Y)/\delta Y = 0$ [3, 25]:

$$\begin{aligned}
\Gamma_1(Y) &= \frac{1}{2}\int d^{10}x_1 \\
&\quad \times \int d^{10}x_2 G_{\text{Fc}}^0(x_1,x_2)\, V_{F(Y)}(x_1)\, V_{F(Y)}(x_2),
\end{aligned} \tag{74}$$

with G_{Fc}^0 being the free supergraviton propagator. Compared with (61), $\Lambda_{F,1}(x_1,x_2) = G_{\text{Fc}}^0(x_1,x_2)$. If the result can be extended to the n-point Green's functions and to the full-loop calculation, there will be

$$\Lambda_F(x_1,\ldots,x_n) = G_{\text{Fc}}(x_1,\ldots,x_n), \tag{75}$$

where G_{Fc} is the connected full Green's function of supergravity. $\Gamma_F^c(x_1,\ldots,x_n)$ and $W(F)$ will then become the n-point vertex function and the effective action of supergravity, respectively.

In quantum field theory, unlike the S-matrix, the effective action is not the observable and thus is not uniquely defined. Different gauges and the parameterization give the different effective actions. $\Gamma(Y)$ and $\Gamma_{\text{eff}}(Y)$ differ from a parameterization transformation. $\Gamma(Y) \neq \Gamma_{\text{eff}}(Y)$. However, to compare the Matrix theory with supergravity, we do need a privileged effective action. On supergravity side, in light-cone gauge, the expected effective action is

$$\begin{aligned}
\Gamma_g[Y(\tau)] &= \sum_n \frac{1}{n!}\int d^{10}x_1 \cdots \int d^{10}x_n G_{\text{Fc}}(x_1,\ldots,x_n) \\
&\quad \times V_{F(Y)}(x_1)\cdots V_{F(Y)}(x_n).
\end{aligned} \tag{76}$$

The effective action defined in (61) is also the Taylor series of $V_{F(Y)}$ and thus could be compared with (76) directly. The standard effective action $\Gamma_{\text{eff}}(Y)$ is expanded as the Taylor series of Y. A careful reorganization is needed to get $V_{F(Y)}$, but it is unclear whether it is always possible to do so. Moreover, under a Legendre transformation, (76) becomes

$$\begin{aligned}
\Gamma_g[Y(\tau)] &= S_{\text{eff}}(F) \\
&\quad + \int d^{10}x F(x)\, V_{F(Y)}(x) \sim S_{\text{eff}}(F) + S_F[Y(\tau)]
\end{aligned} \tag{77}$$

with

$$-\left.\frac{\delta S_{\text{eff}}(F)}{\delta F(x)}\right|_F = V_{F(Y)}(x) = \left.\frac{\delta S_F(Y)}{\delta F(x)}\right|_F. \tag{78}$$

$S_{\text{eff}}(F)$ is the effective action of supergravity. Equation (77) is almost the same as (53). The only problem is whether $W(F) = S_{\text{eff}}(F)$ or not.

2.4. Free Energy of String and the Effective Action of the Background Fields. In string theory, there is a similar story. In [17, 18], it was shown that the effective action of the supergravity could be taken as the renormalized free energy of the strings on background F:

$$S_{\text{eff}}\left[\widetilde{F}(F)\right] = \sum_{m=0}^{\infty} e^{\phi(2-2m)} \int_{M_{2-2m}} [dX]\, e^{-S_F} = W_s(F), \quad (79)$$

where F satisfies the free field equation.

Equation (79) looks consistent with our philosophy: the free energy of the strings/membranes on a given background gives the effective action of the background fields. However, there is a difference: $\widetilde{F}$ is the renormalized field other than the bare field. In fact, in (79), a particular Weyl gauge $g_{ab} = e^{2\sigma}\widehat{g}_{ab}$ is always imposed, and so, $W_s(F,\sigma)$ also has the dependence on σ:

$$\frac{\delta W_s(F,\sigma)}{\delta\sigma} = -\frac{\delta W_s(F,\sigma)}{\delta F}\beta(F). \quad (80)$$

When $\beta(F) = 0$, $W_s(F,\sigma) = W_s(F)$, although $\beta(F) = 0$ is not the necessary condition.

Suppose $S_{\text{eff}}(F)$ is the effective action of the string modes:

$$S_{\text{eff}}(F) = -\frac{1}{2}\int d^D x F\Delta F + S_{\text{int}}(F). \quad (81)$$

Consider F_0 with $\Delta F_0 = 0$. Since $\beta(F_0) \neq 0$, F will evolve along the RG flow; that is, $W_s(F_0,\sigma) = W_s[F(\sigma'),\sigma']$. A special property of F_0 is that it will finally reach an IR fixed point $\widetilde{F}$, $W_s(F_0,\sigma) = W_s(\widetilde{F},\infty)$. In fact, since F_0 satisfies the first-order β equation, the RG flow may bring it to an IR fixed point. Since $\beta(\widetilde{F}) = 0$, $W_s(\widetilde{F},\infty) = W_s(\widetilde{F},\sigma) = W_s(\widetilde{F}) = W_s(F_0,\sigma) = W_s(F_0)$.

$\widetilde{F}$ is calculated from F_0 via

$$\begin{aligned}\widetilde{F} &= F_0 + \Delta^{-1}\frac{\delta S_{\text{int}}\left(\widetilde{F}\right)}{\delta\widetilde{F}} \\ &= F_0(x) + \int d^D y D(x-y,F)\frac{\delta S_{\text{int}}\left(\widetilde{F}\right)}{\delta\widetilde{F}(y)}.\end{aligned} \quad (82)$$

$\delta S_{\text{eff}}(F)/\delta F(x)|_{F=\widetilde{F}} = 0$. Since $\delta S_{\text{eff}}(F)/\delta F^i = \kappa_{ij}\beta(F^j)$, $\beta(\widetilde{F}) = 0$ is equivalent to $\delta S_{\text{eff}}(F)/\delta F(x)|_{F=\widetilde{F}} = 0$ [18]. In [18], it was shown that $W_s(F_0,\sigma) = S_{\text{eff}}(\widetilde{F})$, which means $W_s(\widetilde{F},\sigma) = W_s(\widetilde{F}) = S_{\text{eff}}(\widetilde{F})$. There is a one-to-one correspondence between the solution space of $\delta S_{\text{eff}}(F)/\delta F(x) = 0$ and the solution space of $\Delta F = 0$. So, for all F with $\delta S_{\text{eff}}(F)/\delta F(x) = 0 = \beta(F)$, $W_s(F,\sigma) = S_{\text{eff}}(F)$. In σ-model approach, only for such F, the conformal factor decouples; thus, the calculated string free energy is physical. In this subspace, $W_s(F) = S_{\text{eff}}(F)$.

$W_s(F,\sigma)$ with $\beta(F)\neq 0$ usually depends on σ. However, the σ-dependence drops out for F_0. In fact,

$$\begin{aligned}W_s(F_0,\sigma) &= W_s(F_0) = S_{\text{eff}}\left(\widetilde{F}\right) \\ &= S_{\text{eff}}\left[F_0 + \Delta^{-1}\frac{\delta S_{\text{int}}(F_0)}{\delta F_0} + \cdots\right]\end{aligned} \quad (83)$$

is the S-matrix functional [27]. For the on-shell $p_1,\ldots,p_n$,

$$\begin{aligned}W_s(F_0,\sigma) &= \sum_n \frac{1}{n!}\int d^9 p_1 \cdots \int d^9 p_n S_F^c(p_1,\ldots,p_n) \\ &\quad \times F_0(p_1)\cdots F_0(p_n), \\ F_0(x) &= \int d^9 p F_0(p), \\ S_F^c(p_1,\ldots,p_n) & \\ &= \frac{1}{N_0}\sum_{m=0}^{\infty} e^{\phi(2-2m)} \\ &\quad \times \int_{M_{2-2m}} [dX]\, e^{-S_0} V_F(p_1)\cdots V_F(p_n)\end{aligned} \quad (84)$$

is the connected scattering amplitude. It is only in Polyakov approach can we construct the S-matrix functional in this way since it intrinsically involves some kind of renormalization, which is equivalent to the subtraction of the massless pole exchange contribution [18, 27].

To conclude, for F satisfying $\delta S_{\text{eff}}(F)/\delta F(x) = 0$, $W_s(F,\sigma) = W_s(F) = S_{\text{eff}}(F)$. However, the off-shell extension of the effective action and the free energy are both ambiguous in Sigma-model approach.

Another support for the identification of the string theory free energy on a particular background and the effective action of the background fields comes from AdS/CFT. SYM_4 is a nonperturbative description of the type IIB string theory on $\text{AdS}_5 \times S^5$. It is expected that the Hilbert spaces of both sides are isomorphic to each other. Similar with the Chern-Simons/topological string correspondence, $Z_{\text{SYM}_4} = e^{W_s}$, where W_s is the free energy of the string theory on $\text{AdS}_5 \times S^5$. The question is what will be the string dual of SYM_4 if the background metric of SYM_4 is $\eta_{\mu\nu} + \delta g_{\mu\nu}(x)$ other than $\eta_{\mu\nu}$ for very small $\delta g_{\mu\nu}(x)$. A natural expectation is that such SYM_4 is dual to the type IIB string theory on $\text{AdS}_5 \times S^5$ with a little modification of the background metric $\delta G_{\mu\nu}(x,\sigma)$ that is entirely determined by $\delta g_{\mu\nu}(x)$. Correspondingly, $Z_{\text{SYM}_4}[\delta g_{\mu\nu}(x)] = e^{W_s[\delta G_{\mu\nu}(x,\sigma)]}$, where $W_s[\delta G_{\mu\nu}(x,\sigma)]$ is the free energy of the string theory on $\text{AdS}_5\times S^5$ with the background perturbation $\delta G_{\mu\nu}(x,\sigma)$ being turned on. Note that, for the stringy explanation of the SYM_4 partition function to be possible, the type IIB dual must have the definite background since the string partition function is always defined on a given background. Also, for the state correspondence to be valid, the dual type IIB string theory should have the definite background; otherwise, it is impossible to determine the string spectrum. Now, return to the original topic. Gauge theory calculation gives

$Z_{\text{SYM}_4}[\delta g_{\mu\nu}(x)] = e^{S_{\text{eff}}[\delta G_{\mu\nu}(x,\sigma)]}$, where $S_{\text{eff}}[\delta G_{\mu\nu}(x,\sigma)]$ is the supergravity effective action for the modified background fields, so $S_{\text{eff}}[\delta G_{\mu\nu}(x,\sigma)] = W_s[\delta G_{\mu\nu}(x,\sigma)]$ if $Z_{\text{SYM}_4} = e^{W_s}$ holds.

There is a naive way to interpret $W_s(F) = S_{\text{eff}}(F)$. Suppose the classical supergravity action is $S_c(F)$, from the field theory's point of view:

$$e^{S_{\text{eff}}(F)} = \int_{1\text{PI}} \left[d\widetilde{F}\right] e^{-S_c(F+\widetilde{F})}, \tag{85}$$

where $\widetilde{F}$ is the fluctuation on F or, in other words, supergravitons living on background F. The effective action is the sum of the connected 1PI vacuum-vacuum diagrams of the supergravitons on background F. The elementary propagator and the vertices can be read from $S_c(F+\widetilde{F})$. Now, consider the string theory on background F, we may have

$$e^{S_{\text{eff}}(F)} = Z_s(F) = e^{W_s(F)}, \tag{86}$$

where $W_s(F)$ is the sum of the irreducible vacuum-vacuum string diagrams on background F. Note that, in string diagram, there is nither a concept of 1PI nor 1PR. Also, there is no classical action like $S_c(F+\widetilde{F})$ to determine the basic constitution of the diagram. The integration simply covers all possible string configurations. Equation (86) can be taken as the stringy refined version of (85). In (86), we secretly assumed that the unphysical worldsheet conformal factor is decoupled. In Polyakov approach, this is possible only when F is the solution of $\delta S_{\text{eff}}(F)/\delta F = 0$. The cancelation of the Weyl anomaly gives the e.o.m for the effective action of the supergravity, including the α' corrections, so $S_{\text{eff}}(F)$ should be the effective action with the α' corrections taken into account.

A natural M-theory extension is

$$e^{S_{\text{eff}}(F)} = Z_M(F) = \int [dY]\, e^{-S_F^M(Y)}, \tag{87}$$

where $S_F^M(Y)$ is the supermembrane action on $11d$ supergravity background F. The membrane is already the second quantized object, so the left-hand side of (87) is $e^{S_{\text{eff}}(F)}$ other than $S_{\text{eff}}(F)$. For the generic $11d$ background F, $S_F^M(Y)$ should be covariantm, and, so, the worldvolume metric must be introduced and integrated, making the supermembrane theory nonrenormalizable. For F which is translation invariant along x^-, the light-cone gauge can be imposed. The configurations are then truncated to those with the light-cone momentum p^+. The integration out of such configurations on background F gives the effective action of F with the radius of $x^- \sim 1/p^+$.

3. Matrix Theory on a Particular Vacuum

If $W(R,F)$ in (42) is the effective action of the $11d$ supergravity field $F(x)$ that is translation invariant along x^-, for the on-shell $F(x)$, there will be $\delta W(R,F)/\delta F(x) = -\nu_F(x) = 0$. However, for $F=\overline{F}$, which is obviously on shell, the current densities $\overline{\nu}_{g_{++}}$, $\overline{\nu}_{g_{--}}$, and $\overline{\nu}_{g_{+-}}$ are nonvanishing. (Note that the vertex operators for g_{++} and g_{+-} are quite like the vertex operators for tachyon and dilaton in bosonic string theory, in which, there is also a tadpole in flat spacetime [28].)

$$\begin{aligned} &W(R,F) \\ &= -\int d^{10}x\left[\overline{\nu}_{g_{++}}g_{++}(x) + \overline{\nu}_{g_{--}}g_{--}(x) + \overline{\nu}_{g_{+-}}g_{+-}(x)\right] \\ &\quad + \frac{1}{2}\int d^{10}x_1 \\ &\quad \times \int d^{10}x_2 \Gamma^c_{R,g_{++},g_{--}}(x_1,x_2)\, g_{++}(x_1)\, g_{--}(x_2) \\ &\quad + \frac{1}{2}\int d^{10}x_1 \\ &\quad \times \int d^{10}x_2 \Gamma^c_{R,g_{+-},g_{+-}}(x_1,x_2)\, g_{+-}(x_1)\, g_{+-}(x_2) + \cdots . \end{aligned} \tag{88}$$

With $\alpha,\beta = +,-$, one may try to solve $g_{\alpha\beta}(x)$ from $\delta W(R,F)/\delta g_{\alpha\beta}(x) = 0$, and then take this value other than $g_{+-}(x) = -1$, $g_{--}(x) = g_{++}(x) = 0$ as the background to do the expansion. $\overline{\nu}_{g_{\alpha\beta}}$ are constants, so the generated $g_{\alpha\beta}(x)$ are also constants (although the constant $g_{\alpha\beta}(x)$ does not really solve the equation) and do not represent the substantial change of the background. We will simply neglect these tadpoles. $\overline{\nu}_{g_{++}}$ should be distinguished from $\delta S_{R,F}(Y)/\delta g_{++}(x)|_{Y=0} = (N/R)\delta^{10}(x)$, which is the light-cone momentum density of a supergraviton localized at $(\tau,0,\ldots,0)$ and, of course, will produce the nontrivial $g_{--}(x)$.

In (54), $F=F(R,Y)$ is the functional of Y solved through (55). For the arbitrary F, we may define

$$\begin{aligned} e^{\Omega(R,Y,F)} &= \int \left[d\widetilde{Y}\right] e^{-S_{R,F}(\widetilde{Y}) + S_{R,F}(Y)} \\ &= \int [d\eta]\, e^{-S_{R,F}(Y+\eta) + S_{R,F}(Y)} \\ &= \int [d\eta]\, e^{-S_{R,F}(Y;\eta)}. \end{aligned} \tag{89}$$

$\Omega(R,Y,F) = W(R,F) + S_{R,F}(Y)$ is the action of the source-gravity coupled system and is not necessarily on shell with respect to gravity. If $S_{R,F}(\eta)$ gives the Matrix theory description of M theory on background F, $S_{R,F}(Y;\eta)$ will describe M theory on background F, in presence of the brane Y. If Y satisfies the equations of motion, that is, $\delta S_{R,F}(Y)/\delta Y = 0$,

$$\begin{aligned} S_{R,F}(Y;\eta) &= \int d\tau \frac{\delta S_{R,F}(Y)}{\delta Y(\tau)}\eta(\tau) \\ &\quad + \frac{1}{2}\int d\tau\, d\tau' \frac{\delta^2 S_{R,F}(Y)}{\delta Y(\tau)\,\delta Y(\tau')}\eta(\tau)\,\eta(\tau') + \cdots \\ &= \frac{1}{2}\int d\tau\, d\tau' \frac{\delta^2 S_{R,F}(Y)}{\delta Y(\tau)\,\delta Y(\tau')}\eta(\tau)\,\eta(\tau') + \cdots \end{aligned} \tag{90}$$

$S_{R,F}(Y;\eta)$ starts from the quadratic term. For F satisfying $\delta\Omega(R,Y,F)/\delta F(x) = 0$, $S_{R,F}(Y;\eta)$ then gives a description of

M theory on a background F generated by the brane Y. In this case, $\Omega[R, Y, F(Y)] = \Gamma(Y)$:

$$0 = \int [d\eta]\, e^{-S_{R,F}(Y;\eta)} \frac{\delta S_{R,F}(Y;\eta)}{\delta F(x)} = \left\langle \frac{\delta S_{R,F}(Y;\eta)}{\delta F(x)} \right\rangle. \quad (91)$$

The expectation values of all current densities vanish. There is no tadpole.

The $U(\infty)$ symmetry of Matrix theory is destroyed by Y. In particular, if

$$Y = \operatorname{diag}\left[\underbrace{M_1, \ldots, M_1}_{N_1}, \ldots, \underbrace{M_m, \ldots, M_m}_{N_m}\right], \quad (92)$$

the preserved gauge symmetry is $U(N_1) \times \cdots \times U(N_m)$. Nevertheless, the original $U(\infty)$ symmetry still has its manifestation. For all $u \in U(\infty)$,

$$\eta \longrightarrow u\eta u^{+} + \left(uYu^{+} - Y\right), \qquad \eta \longrightarrow \eta + i[\alpha, \eta] + i[\alpha, Y], \quad (93)$$

so for $u \notin U(N_1) \times \cdots \times U(N_m)$, η transforms like a gauge field.

3.1. Applied to PWMM. In the following, we will focus on a special example: the plane wave matrix model [2]. PWMM is a Matrix theory description of M-theory on $11d$ pp-wave background:

$$\begin{aligned} ds^2 &= -2dx^{+}dx^{-} + \sum_{i=1}^{9} dx^i dx^i \\ &\quad - \left[\sum_{a=1}^{3} \frac{\mu^2}{9}(x^a)^2 + \sum_{a'=4}^{9} \frac{\mu^2}{36}\left(x^{a'}\right)^2\right] dx^{+}dx^{+} \\ F_{123+} &= \mu. \end{aligned} \quad (94)$$

The background preserves 32 supersymmetries, while the rest $11d$ supergravity solutions with 32 supersymmetries are flat spacetime, $\mathrm{AdS}_4 \times S^7$, and $\mathrm{AdS}_7 \times S^4$ [29]. In pp-wave, the dynamics of the M-theory sector with the light-cone momentum $p^{+} = N/R$ is described by the $U(N)$ matrix model:

$$\begin{aligned} S_R^{\mathrm{PW}} &= R \int dx^{+} \\ &\quad \times \operatorname{Tr}\left\{ \frac{1}{4R^2}\left(D_{+}X^i\right)^2 \right. \\ &\quad - \frac{i}{R}\theta D_{+}\theta + \frac{1}{2}\left[X^i, X^j\right]^2 + 2\theta\gamma_i\left[X^i, \theta\right] \\ &\quad - \left(\frac{\mu}{6R}\right)^2 (X^a)^2 - \left(\frac{\mu}{12R}\right)^2 \left(X^{a'}\right)^2 \\ &\quad \left. - \frac{\mu}{6R} i\epsilon_{abc} X^a X^b X^c - \frac{\mu}{8R}\theta\gamma_{123}\theta \right\}, \end{aligned} \quad (95)$$

$N \to \infty$, $R \to \infty$. PWMM also preserves 32 supersymmetries and has the same symmetry group as that of the pp-wave background.

The classical supersymmetric solutions of the action are [2]

$$\begin{aligned} \left[\widehat{X}^i, \widehat{X}^j\right] &= i\frac{\mu}{3R}\epsilon^{ijk}\widehat{X}^k, \quad i, j, k = 1, 2, 3, \\ \widehat{X}^i &= 0, \quad i = 4, \ldots, 9, \\ \dot{\widehat{X}}^i &= 0, \quad i = 1, \ldots, 9. \end{aligned} \quad (96)$$

$\{\widehat{X}^1, \widehat{X}^2, \widehat{X}^3\}$ form the N dimensional representation of $SU(2)$. Suppose $(3R/\mu)\widehat{X}^i = L^i$, $i = 1, 2, 3$, L^i can be decomposed as

$$L^i = \operatorname{diag}\left[\underbrace{L^i_{j_1} \cdots L^i_{j_1}}_{N_1} \underbrace{L^i_{j_2} \cdots L^i_{j_2}}_{N_2} \cdots \underbrace{L^i_{j_t} \cdots L^i_{j_t}}_{N_t}\right], \quad (97)$$

where $L^i_{j_s}$ $(s = 1, \ldots, t)$ stands for the spin j_s irreducible representation of $SU(2)$:

$$(2j_1 + 1)N_1 + (2j_2 + 1)N_2 + \cdots + (2j_t + 1)N_t = N. \quad (98)$$

The solution can be taken as a set of N_s coincident fuzzy spheres with the radius $(\mu/3R)\sqrt{j_s(j_s+1)}$ [2]. The $U(N)$ gauge symmetry is broken into $U(N_1) \times U(N_2) \times \cdots \times U(N_t)$. All fuzzy spheres are concentric. When $j_s \to \infty$, the fuzzy spheres become a set of membrane spheres given by $\sum_{i=1}^{3}(x_s^i)^2 = r_s^2$ with $r_s \propto (2j_s + 1)\mu/6R$. Each spherical membrane has $p^{-} = 0$, $p^{+} = (2j+1)/R$, so they are also called the giant gravitons.

The solution can also be interpreted as the spherical $M5$ branes with a dual assignment of the light-cone momentum [30]. Any partition of N may be represented by a Young diagram whose column lengths are the elements in the partition. In the $M2$ interpretation, such a diagram corresponds to a state with one membrane for each column with the number of boxes in the column being the number of units of momentum. In the dual $M5$ interpretation, it is the rows of the Young diagram that correspond to the individual $M5$, with the row lengths giving the number of units of momentum carried by each $M5$. In both cases, the total light-cone momentum is always N/R. If N_s are finite, they are fuzzy $M5$ branes. When $N_s \to \infty$, the fuzzy $M5$ become the spherical $M5$ wrapping S^5 given by $\sum_{i=4}^{9}(x_r^i)^2 = r_r^2$. In patricular, the trivial vacuum $\widehat{X}^i = 0$ represents a single $M5$ brane.

All of the supersymmetric solutions preserve 16 nonlinearly realized supersymmetries. They are the 1/2 BPS states on pp-wave background. Although the background and the Lagrangian are both maximally supersymmetric, there is no state in matrix model preserving all of the 32 supersymmetries. The reason is that all states have the same nonzero light-cone momentum N/R, which itself would destroy the linearly realized supersymmetries. The situation is different in SYM_4 in which, we do have a vacuum preserving

32 supersymmetries, representing the ground state of the string theory on $AdS_5 \times S^5$. (Brane like) 1/2 BPS states are giant gravitons on $AdS_5 \times S^5$ [31–33].

Similar to the PWMM, "tiny graviton matrix model" (TGMT) is proposed as the nonperturbative description of the type IIB string theory on pp-wave background [34–36]. TGMT can also be taken as the DLCQ of the type IIB string theory on $AdS_5 \times S^5$, capturing the physics seen from the infinite momentum frame (IMF) since, in IMF, $AdS_5 \times S^5$ is viewed as the pp-wave [34–36]. Although TGMT preserves 32 supersymmetries, the vacuum configurations, carrying the definite light-cone momentum, are all 1/2 BPS, matching exactly with the 1/2 BPS states on type IIB pp-wave background, which are giant gravitons (spherical $D3$ branes) and type IIB strings. Lifted to $11d$, the light-cone dimension is replaced by T^2, while the TGMT, which is a discrete regularization of the $D3$ branes in type IIB picture, becomes the regularization of the $M5$ branes [35]. Note that the 4-form field in type IIB pp-wave, when lifted to $11d$, becomes the 6-form field coupling electrically with the $M5$ branes, so it is natural to construct the matrix model via the discrete regularization of $M5$ branes other than the usual $M2$ branes. On type IIB pp-wave, $D3$ branes are $M5$ branes wrapping T^2, while the type IIB strings are membranes wrapping one of S^1. In contrast to the PWMM, in TGMT, the trivial vacuum $X = 0$ represents type IIB string (with $M2$ origin), while the nontrivial vacuum represents $D3$ branes (with $M5$ origin), which is probably because the PWMM and the TGMT are constructed from $M2$ and $M5$, respectively. Recall that, in PWMM, the fuzzy configuration may have the $M2$ and $M5$ dual interpretations [30], it is interesting to figure out whether a similar $F1$-$D3$ dual interpretation also exists for configurations in TGMT.

With $\widehat{Y}$ denoting the supersymmetric vacuum in (96), M theory on pp-wave background in presence of the brane $\widehat{Y}$ is described by the Matrix model with the action

$$S_R^{\mathrm{PW}}\left(\widehat{Y};\eta\right) = S_R^{\mathrm{PW}}\left(\widehat{Y}+\eta\right) - S_R^{\mathrm{PW}}\left(\widehat{Y}\right). \tag{99}$$

Since $(\delta S_R^{\mathrm{PW}}(Y)/\delta Y)|_{Y=\widehat{Y}} = 0$, $S_R^{\mathrm{PW}}(\widehat{Y};\eta)$ starts from the quadratic term.

For simplicity, in the following, we will assume R is absorbed into the fields. If the backreaction of the brane $\widehat{Y}$ on pp-wave background is turned on, the field $F(\widehat{Y})$ will be generated:

$$\begin{aligned} e^{\Gamma^{\mathrm{PW}}(\widehat{Y})} &= \int [d\eta]\, e^{-S_{F(\widehat{Y})}^{\mathrm{PW}}(\eta)+S_{F(\widehat{Y})}^{\mathrm{PW}}(\widehat{Y})} \\ &= e^{W^{\mathrm{PW}}[F(\widehat{Y})]+S_{F(\widehat{Y})}^{\mathrm{PW}}(\widehat{Y})}, \end{aligned} \tag{100}$$

where

$$\left.\frac{\delta S_F^{\mathrm{PW}}(\widehat{Y})}{\delta F(x)}\right|_{F=F(\widehat{Y})} = -\left.\frac{W^{\mathrm{PW}}(F)}{\delta F(x)}\right|_{F=F(\widehat{Y})}. \tag{101}$$

pp-wave with $\widehat{Y}$ added respects the $SU(2 \mid 4)$ symmetry, so the generated $F(\widehat{Y})$ as well as the corresponding $S_{F(\widehat{Y})}^{\mathrm{PW}}(\eta)$ will have the same symmetry.

With the fields $F(\widehat{Y})$ given, in principle, one can write down $S_{F(\widehat{Y})}^{\mathrm{PW}}(\eta)$ explicitly. The classical supersymmetric solutions of $S_{F(\widehat{Y})}^{\mathrm{PW}}(\eta)$ are still (96). To see this, note that the spherical giant gravitons are 1/2 BPS states on pp-wave, so they should not exert force to each other [26], or, in other words, giant gravitons are still stable even if the backreaction of the other giant gravitons on pp-wave is turned on. Indeed, in [37], the giant graviton on the backreacted geometry is analyzed. The stable configuration is still the same as that in the pp-wave case. In [38], the quantum effective action of PWMM around $\widehat{Y}$ was calculated at the one-loop level. $\widehat{Y}$ is also the stationary point of the effective action. We may have $\delta\Gamma^{\mathrm{PW}}(Y)/\delta Y|_{Y=\widehat{Y}} = \delta S_{F(\widehat{Y})}^{\mathrm{PW}}(Y)/\delta Y|_{Y=\widehat{Y}} = 0$. With the pp-wave background replaced by the backreacted geometry, $S_R^{\mathrm{PW}}(\widehat{Y};\eta)$ is modified to

$$S_{F(\widehat{Y})}^{\mathrm{PW}}\left(\widehat{Y};\eta\right) = S_{F(\widehat{Y})}^{\mathrm{PW}}\left(\widehat{Y}+\eta\right) - S_{F(\widehat{Y})}^{\mathrm{PW}}\left(\widehat{Y}\right). \tag{102}$$

$S_{F(\widehat{Y})}^{\mathrm{PW}}(\widehat{Y};\eta)$ still starts from the quadratic term since $\delta S_{F(\widehat{Y})}^{\mathrm{PW}}(Y)/\delta Y|_{Y=\widehat{Y}} = 0$.

Notice that, although any configuration $\widehat{Y}'$ in (96) may be the classical vacuum of $S_{F(\widehat{Y})}^{\mathrm{PW}}(\eta)$, $\widehat{Y}$ is special because the vacuum expectation values of current densities vanish only for $S_{F(\widehat{Y})}^{\mathrm{PW}}(\widehat{Y};\eta)$ but not for the generic $S_{F(\widehat{Y})}^{\mathrm{PW}}(\widehat{Y}';\eta)$.

The $11d$ geometry produced by 1/2 BPS states of PWMM was constructed in [19, 39]. The geometry has a bubble structure containing noncontractible 7 cycles and 4 cycles supporting F_4 and F_7 fluxes. The geometry is smooth without singularity and, then, sourceless. This is an explicit realization of the geometric transition [40]. The backreaction makes the the worldvolume of the M branes shrink and the transverse sphere blow up. As a result, although we start from the source-gravity coupled action, the obtained supergravity solution is smooth and satisfies the sourceless equations too. The brane action as well as the current density is zero on the generated supergravity background.

Return to (100),

$$\begin{gathered} S_{F(\widehat{Y})}^{\mathrm{PW}}\left(\widehat{Y}\right) = 0, \qquad \left.\frac{\delta S_F^{\mathrm{PW}}(\widehat{Y})}{\delta F(x)}\right|_{F=F(\widehat{Y})} = 0, \\ \Gamma^{\mathrm{PW}}\left(\widehat{Y}\right) = W^{\mathrm{PW}}\left[F\left(\widehat{Y}\right)\right], \qquad \left.\frac{\delta W^{\mathrm{PW}}(F)}{\delta F(x)}\right|_{F=F(\widehat{Y})} = 0. \\ e^{W^{\mathrm{PW}}[F(\widehat{Y})]} = \int [d\eta]\, e^{-S_{F(\widehat{Y})}^{\mathrm{PW}}(\widehat{Y};\eta)}. \end{gathered} \tag{103}$$

$\widehat{Y}$ is the momentum eigenstate in x^- direction, so the generated $11d$ background $F(\widehat{Y})$ is translation invariant along x^-. In large r limit, the local structure of the giant gravitons is not important while the asymptotic geometry is just the pp-wave with the perturbation roughly given by [39]

$$\begin{aligned} ds_{11}^2 &= 2dx^+dx^- + g_{--}dx^-dx^- + dx^idx^i \\ g_{--} &= \frac{Q}{r^7}, \end{aligned} \tag{104}$$

which is the field produced by the supergraviton which is static in $9d$ space, carrying the definite light-cone momentum [41]. $Q = 30\pi^3 N/R^2$. (It is not $Q = 30\pi^3 l_p^9 N/R^2$ because l_p is absorbed into R and r.) With the coordinate transformation $x \to g_s^{-1/3} x$, $x^- \to g_s^{2/3} x^-$, (104) becomes

$$\begin{aligned} ds_{11}^2 &= 2 g_s^{1/3} dx^+ dx^- + \frac{g_s^{4/3} Q}{r^7} dx^- dx^- \\ &\quad + g_s^{-2/3} dx^i dx^i, \end{aligned} \tag{105}$$

while the radius of x^- is now $g_s^{-2/3} R$.

Under the x^- reduction, in string frame,

$$\begin{gathered} ds^2 = e^{-2\phi/3} G_{mn} dx^m dx^n + e^{4\phi/3} \left(dx^- + A_m dx^m\right)^2. \\ g_{mn} = e^{-2\phi/3} G_{mn} + e^{4\phi/3} A_m A_n, \qquad g_{--} = e^{4\phi/3}, \\ g_{m-} = e^{4\phi/3} A_m. \end{gathered} \tag{106}$$

In particular, for (105),

$$\begin{gathered} ds_{10}^2 = -\frac{r^{7/2}}{Q^{1/2}} dx^+ dx^+ + \frac{Q^{1/2}}{r^{7/2}} dx^I dx^I, \\ e^{\phi} = g_s \left(\frac{Q}{r^7}\right)^{3/4}, \qquad A_+ = \frac{r^7}{g_s Q} \end{gathered} \tag{107}$$

is the near-horizon geometry of the $D0$ branes [41].

The type IIA solution coming from the x^- reduction of the $11d$ field $F(\widehat{Y})$ was constructed in [19, 37]. When $r \to \infty$, the perturbation, which is the reduction of (104) along x^-, is the near-horizon geometry of N coincident $D0$-branes. The appearance of the near-horizon geometry is because of the null reduction. The reduction of (104) along $x^+ - x^-$ gives the $D0$ brane solution [41]. Different from AdS/CFT, here, no near-horizon limit is taken, and the brane solution itself becomes the near-horizon geometry when reduced along x^-.

3.2. The Gauge Theory Dual from the Matrix Model. According to the previous proposal, the Matrix theory description of the M theory on background (104) in presence of the supergraviton with $p^+ = N/R$ is $S_{R,F}(Y;\eta)$, where F is the field in (104), $Y = 0$ (if F is the field in (105), $S_{R,F} \to (1/g_s)\, S_{R,F}$.) Cosider

$$S_{R,F}(0;\eta) = S_R(\eta) + \int dx^+ \operatorname{Tr}\left(g_{--} T^{--}\right)(\eta). \tag{108}$$

S_R is the BFSS action in (17). On the other hand, according to AdS/CFT, the gauge theory description of the type IIA string theory on background (104) is SYM_1 with the action $S_{R=1}(\eta)$. Since (104) becomes (107) under the x^- reduction, it is desirable to find a limit to make (108) reduce to $S_{R=1}(\eta)$.

Consider the coordinate transformation $x^I \to x^I/R$, $x^- \to R x^-$, under which

$$\begin{aligned} F(x) &= \big(g_{+-}, g_{IJ}, C_{I+-}, C_{IJK}, \psi_I, g_{++}, \\ &\qquad g_{I+}, g_{--}, g_{I-}, C_{IJ+}, C_{IJ-}, \psi_+, \psi_-\big) \\ &\longrightarrow F_*(R,x) = R^q F\left(\frac{x}{R}\right) \\ &= \left(R g_{+-}, \frac{g_{IJ}}{R^2}, C_{I+-}, \frac{C_{IJK}}{R^3}, \frac{\psi_I}{R}, g_{++}, \right. \\ &\qquad \left. \frac{g_{I+}}{R}, R^2 g_{--}, g_{I-}, \frac{C_{IJ+}}{R^2}, \frac{C_{IJ-}}{R}, \psi_+, R\psi_-\right). \end{aligned} \tag{109}$$

Correspondingly, in Matrix theory action $S_{R,F}$, with a field redefinition $X^I \to X^I/R$ and also a $x^- \to R x^-$ rescaling to make the radius of x^- become 1, the background fields appearing in action are just $F_*(R,X)$. $S_{R,F} \to S_{1,F_*(R)}$. With a further rescaling $\theta \to \theta/R^{3/2}$, $S_{R,F} \to S_{1,F^*(R)}/R^3$. In $S_{1,F^*(R)}$, it is

$$\begin{aligned} F^*(R,X) &= R^p F\left(\frac{X}{R}\right) \\ &= \left(g_{+-}, g_{IJ}, C_{I+-}, C_{IJK}, \psi_I, R^2 g_{++}, R g_{I+}, \right. \\ &\qquad \left. \frac{g_{--}}{R^2}, \frac{g_{I-}}{R}, R C_{IJ+}, \frac{C_{IJ-}}{R}, R\psi_+, \frac{\psi_-}{R}\right) \end{aligned} \tag{110}$$

that will appear.

Return to (108). For any R, $S_{R,F}$ is equivalent to $S_{1,F^*(R)}/R^3$, in which $g^*_{--}(R,x) = g_{--}(r/R)/R^2 \sim R^3/r^7$. In $R \to 0$ limit, $g^*_{--}(R,x) \to 0$,

$$\begin{aligned} \lim_{R\to 0} S_{R,F} &\Longleftrightarrow \lim_{R\to 0} \frac{1}{R^3} S_{R=1} \\ &= \lim_{R\to 0} \frac{1}{R^3} \int dx^+ \operatorname{Tr}\left(\frac{1}{2} D_+ X^I D_+ X^I + \frac{1}{4}\left[X^I, X^J\right]^2 \right. \\ &\qquad \left. - i\theta D_+ \theta + \theta \gamma_I \left[X^I, \theta\right]\right). \end{aligned} \tag{111}$$

SYM_1 arrives. Note that, for finite x, $\lim_{R\to 0} x/R = \infty$, so by taking the $R \to 0$ limit, the background field that matters lives at the $r \to \infty$ region, far away from the source.

For PWMM, $R^2 g_{++}(x/R) = g_{++}(x)$, and $R C_{IJ+}(x/R) = C_{IJ+}(x)$, all fields are marginal, so, for the arbitrary R [5],

$$\begin{aligned} S_R^{\mathrm{PW}} &\Longleftrightarrow \frac{1}{R^3} S_{R=1}^{\mathrm{PW}} \\ &= \frac{1}{R^3} \int dx^+ \\ &\qquad \times \operatorname{Tr}\left\{\frac{1}{4}\left(D_+ X^i\right)^2 - i\theta D_+ \theta\right. \end{aligned}$$

$$+\frac{1}{2}\left[X^i, X^j\right]^2 + 2\theta\gamma_i\left[X^i, \theta\right]$$
$$-\left(\frac{\mu}{6}\right)^2 (X^a)^2 - \left(\frac{\mu}{12}\right)^2 \left(X^{a'}\right)^2$$
$$-\frac{\mu}{6} i\epsilon_{abc} X^a X^b X^c - \frac{\mu}{8}\theta\gamma_{123}\theta\Big\}. \quad (112)$$

Now consider the background $F(\widehat{Y})$ coming from the backreaction of the giant graviton $\widehat{Y}$ on pp-wave. Suppose $\widehat{Y}$ is a vacuum in (96) with

$$2j_s + 1 = n + \alpha_s, \quad (113)$$

$$n \longrightarrow \infty, \quad \alpha_s \text{ finite}. \quad (114)$$

The radii of the spherical membranes are $r_s \sim (n + \alpha_s)/R$, $s = 1, \ldots, t$. $F(\widehat{Y})$ reduced along x^- gives the smooth type IIA solution that is constructed in [19]. The generic solution of type IIA supergravity with $SU(2 \mid 4)$ symmetry is characterized by a function $V(\rho, \eta)$ and is given as [19]

$$ds_{10}^2 = \left(\frac{\ddot{V} - 2\dot{V}}{-V''}\right)^{1/2} \times \left\{-4\frac{\ddot{V}}{\ddot{V} - 2\dot{V}} dx^+ dx^+ + \frac{-2V''}{\dot{V}}\left(d\rho^2 + d\eta^2\right) + 4d\Omega_5^2 + 2\frac{V''\dot{V}}{\Delta} d\Omega_2^2\right\},$$
$$e^{4\phi} = \frac{4\left(\ddot{V} - 2\dot{V}\right)^3}{-V''\dot{V}^2\Delta^2},$$
$$C_1 = -\frac{2\dot{V}'\dot{V}}{\ddot{V} - 2\dot{V}} dx^+,$$
$$F_4 = dC_3, \qquad C_3 = -4\frac{\dot{V}^2 V''}{\Delta} dx^+ \wedge d^2\Omega,$$
$$H_3 = dB_2, \qquad B_2 = \left(\frac{\dot{V}\dot{V}'}{\Delta} + \eta\right) d^2\Omega,$$
$$\Delta = \left(\ddot{V} - 2\dot{V}\right)V'' - \left(\dot{V}'\right)^2, \quad (115)$$

where the dot and the prime represent the derivatives with respect to $\log\rho$ and η, respectively. V can be taken as an electrostatic potential for an axially symmetric system with conducting disks and a background potential. ρ is the distance from the center axis, and η is the coordinate in the direction along the center axis. $V(\rho, \eta) = V_b(\rho, \eta) + v_{\overline{\eta}}(\rho, \eta)$, where V_b is the background potential and $v_{\overline{\eta}}$ is determined by a configuration of conducting disks. Each $SU(2 \mid 4)$ symmetric theory is specified by V_b; each vacuum of the theory is specified by a configuration of conducting disks.

From (115), one can also get the uplifted $11d$ solution. For example,

$$ds_{11}^2 = \frac{2^{2/3}\left(\ddot{V} - 2\dot{V}\right)}{\left(-V''\dot{V}^2\Delta^2\right)^{1/3}} dx^- dx^- - \frac{2^{5/3}\dot{V}'\dot{V}}{\left(-V''\dot{V}^2\Delta^2\right)^{1/3}} dx^+ dx^- + \frac{2^{5/3}\dot{V}^{1/3}\left[\left(\dot{V}'\right)^2 - \ddot{V}V''\right]}{\left(-V''\Delta^2\right)^{1/3}} dx^+ dx^+ + \frac{2^{2/3}\left(\dot{V}\Delta\right)^{1/3}}{\left(-V''\right)^{1/3}}\left\{\frac{-V''}{\dot{V}}\left(d\rho^2 + d\eta^2\right) + 2d\Omega_5^2 + \frac{V''\dot{V}}{\Delta} d\Omega_2^2\right\}. \quad (116)$$

For PWMM,

$$V_b^{\text{PW}}(\rho, \eta) = \rho^2\eta - \frac{2}{3}\eta^3. \quad (117)$$

Only the region $\eta \geq 0$ is meaningful. There is an infinitely large conducting disk sitting at $\eta = 0$. Vacuum (97) corresponds to t disks located at $\overline{\eta}_1 = (\pi/2R)(2j_1 + 1)$, $\overline{\eta}_2 = (\pi/2R)(2j_2 + 1), \ldots, \overline{\eta}_t = (\pi/2R)(2j_t + 1)$. The electric charges on these disks are equal to $(\pi^2/8)N_1, (\pi^2/8)N_2, \ldots, (\pi^2/8)N_t$, respectively. $\overline{\eta}_s$ is the radius of the spherical membranes. The correspondence between the spacetime coordinates and the PWMM fields is $(X^1, X^2, X^3) \leftrightarrow (\eta, \Omega_2)$, $(X^4, \ldots, X^9) \leftrightarrow (\rho, \Omega_5)$, $x^+ \leftrightarrow x^+$.

With the given disk configuration, the corresponding potential is

$$V(\rho, \eta) = \rho^2\eta - \frac{2}{3}\eta^3 + v_{\overline{\eta}}(\rho, \eta), \quad (118)$$

which, when plugged into (115) and (116), gives the $10d$ solution $F_{10}(x)$ as well as the $11d$ solution $F(x)$. x stands for the 9 space coordinates since the solution is translation invariant along x^+ and x^-.

Under the coordinate transformation $x \rightarrow x/R$, $x^- \rightarrow Rx^-$, the disks are now located at $\overline{\eta}_s = (\pi/2)(2j_s + 1)$, while the fields will transform as $F(x) \rightarrow F_*(R, x) = R^q F(x/R)$, $F_{10}(x) \rightarrow F_{*10}(R, x)$. As a result, $S_{R,F} \rightarrow S_{1,F_*(R)}$. $F_{*10}(R, x)$ and $F_*(R, x)$ can be obtained by plugging

$$V\left(\frac{\rho}{R}, \frac{\eta}{R}\right) = \frac{1}{R^3}\left(\rho^2\eta - \frac{2}{3}\eta^3\right) + v_{\overline{\eta}/R}\left(\frac{\rho}{R}, \frac{\eta}{R}\right) \quad (119)$$

into (115) and (116). Except for $S_{1,F_*(R)}$, we also have $S_{R,F} \sim S_{1,F^*(R)}/R^3$ with $F^*(R, x) = R^p F(x/R)$ given by (110). Plug

$$R^3 V\left(\frac{\rho}{R}, \frac{\eta}{R}\right) = \rho^2\eta - \frac{2}{3}\eta^3 + R^3 v_{\overline{\eta}/R}\left(\frac{\rho}{R}, \frac{\eta}{R}\right) \quad (120)$$

into (115) and (116); one can get $F_{10}^*(R, x)$ and the corresponding $F^*(R, x)$. In particular, for PWMM, $V = V_b^{\text{PW}}$,

$R^3V(\rho/R,\eta/R) = V(\rho,\eta)$, so $F^*(R,x) = F(x)$ as we have seen before. Both $V(\rho/R,\eta/R)$ and $R^3V(\rho/R,\eta/R)$ satisfy cylindrically symmetric Laplace equation, so $F_*(R,x)$, and $F^*(R,x)$ are also the type IIA supergravity solutions. In fact, $F(x)$, $F_*(R,x)$ and $F^*(R,x)$ are related to each other by the coordinate transformation. $S_{R,F} \Leftrightarrow S_{1,F_*(R)} \Leftrightarrow S_{1,F^*(R)}/R^3$.

The limit of interest is $n \to \infty$, $R \to \infty$, $R^3/n = \alpha^2$ and μ fixed. In this limit, the concentric spherical membranes all have the infinite radii with the difference $(\alpha_i-\alpha_j)/R \to 0$. For the finite ρ, η, $\rho/R \to 0$, $\eta/R \to 0$, we need to consider the behavior of $v_{\overline{\eta}}(\rho,\eta)$ around 0. Since $\overline{\eta}_s/R \to \infty$, near 0, $v_{\overline{\eta}/R}$ can be approximated as the potential generated by dipoles located at $\pm\overline{\eta}_s/R$:

$$R^3 v_{\overline{\eta}/R}\left(\frac{\rho}{R},\frac{\eta}{R}\right) \sim \sum_{s=1}^{t} N_s R^3 \left\{\left[\left(\frac{\overline{\eta}_s-\eta}{R}\right)^2+\left(\frac{\rho}{R}\right)^2\right]^{-1/2} - \left[\left(\frac{\overline{\eta}_s+\eta}{R}\right)^2+\left(\frac{\rho}{R}\right)^2\right]^{-1/2}\right\} \sim \sum_{s=1}^{t}\frac{2R^4N_s\eta}{\overline{\eta}_s^2} \longrightarrow 0. \tag{121}$$

$\lim R^3V(\rho/R,\eta/R) = \rho^2\eta - (2/3)\eta^3$. Indeed, with $R^3V(\rho/R,\eta/R)$ plugged into (116), $n \to \infty$, $R \to \infty$, R^3/n being fixed, the $11d$ solution finally approaches the pp-wave. As a result, $\lim S_{R,F} \Leftrightarrow \lim S_{R=1}^{\mathrm{PW}}/R^3$. Even if the background field is $F(\widehat{Y})$, in the limit taken, we still get PWMM.

On the other hand, plug $R^3V(\rho/R,\eta/R)$ into the reduced $10d$ solution (115); $\ddot{V}-2\dot{V} = R^3(\ddot{v}_{\overline{\eta}/R}-2\dot{v}_{\overline{\eta}/R})$; the $v_{\overline{\eta}/R}$ part now matters. Let $w(\rho^2,\eta) = R^3v_{\overline{\eta}/R}(\rho/R,\eta/R)$:

$$\begin{gathered} ds_{10}^2 = 4\left\{-\left(\frac{\partial_{\rho^2}^2 w}{\eta}\right)^{-1/2} dx^+dx^+ + \left(\frac{\partial_{\rho^2}^2 w}{\eta}\right)^{1/2}\left[d\rho^2+\rho^2 d\Omega_5^2+d\eta^2+\eta^2 d\Omega_2^2\right]\right\}, \\ e^{4\phi} = \left(\frac{\partial_{\rho^2}^2 w}{\eta}\right)^3, \\ C_1 = -2\left(\frac{\partial_{\rho^2}^2 w}{\eta}\right)^{-1} dx^+, \\ F_4 = dC_3, \qquad C_3 = -16\eta^3 dx^+\wedge d^2\Omega, \\ H_3 = dB_2, \qquad B_2 = \left(4\eta^2\partial_{\rho^2}^2 w - \partial_{\rho^2} w + \eta\partial_{\rho^2}\partial_\eta w\right) d^2\Omega. \end{gathered} \tag{122}$$

The $10d$ solution has the dependence on the disk configuration. Equation (122) is $\lim F_{10}^*(R,x)$. With $e^\phi \to R^3e^\phi$, $C_1 \to C_1/R^3$, $C_3 \to C_3/R^3$, we will get $\lim F_{*10}(R,x)$. By taking the $R \to \infty$ limit, the relevant region is $\rho \to 0$, $\eta \to 0$, which is again far from the source. The $11d$ field approaches the pp-wave, but the reduced $10d$ field still depends on the disk configuration. Similarly, for (104), suppose $Q = Nk^7$; then, when $k \to 0$, the $11d$ background is fat, while the associated $10d$ background (107) is still the near-horizon geometry but becomes singular now. Nevertheless, let $u = r/k$, $r \to 0$, u fixed; (107) can be written as

$$\begin{gathered} ds_{10}^2 = k^2\left(-\frac{u^{7/2}}{N^{1/2}}du^+du^+ + \frac{N^{1/2}}{u^{7/2}}du^I du^I\right), \\ e^{4\phi/3} = \frac{N}{u^7}, \qquad A_+ = \frac{u^7}{N}. \end{gathered} \tag{123}$$

This is similar to the limit taken in AdS/CFT [7].

Now, consider the exact form of $S_{1,F_*(R)}$. For the generic $11d$ background $F(x)$, the bosonic part of $S_{1,F_*(R)}$ is

$$\begin{aligned} S_{1,F_*(R)}^b = \int dx^+ \times \mathrm{Tr}\Bigg(g_{*+-}\Bigg\{&\frac{D_+X^I D_+X^J}{2g_{*+-}^2} g_{*IJ} + \frac{D_+X^I}{g_{*+-}}\left(c_*^J + \frac{g_{*+}^J}{g_{*+-}}\right) g_{*IJ} \\ &+ \frac{1}{4}\left[X^I,X^J\right]\left[X^K,X^L\right] g_{*IK}g_{*JL} + \frac{g_{*++}}{2g_{*+-}^2} + \frac{c_{*+}}{g_{*+-}}\Bigg\}\Bigg) + S_{g_{*I-}} + S_{g_{*--}}. \end{aligned} \tag{124}$$

The relation between $F_*(R,x)$ and the x^- reduced $10d$ background $F_{*10}(R,x)$ is

$$\begin{gathered} g_{*mn} = e^{-2\phi_*/3}G_{*mn} + e^{4\phi_*/3}A_{*m}A_{*n}, \\ g_{*--} = e^{4\phi_*/3}, \qquad g_{*m-} = e^{4\phi_*/3}A_{*m}. \end{gathered} \tag{125}$$

In the limit with $n \to \infty$, $R \to \infty$, R^3/n fixed, if $g_{--}^* \to 0$, $g_{I-}^* = 0$, the rest $F_{10}^*(R,x)$ is finite, there will be

$$\begin{gathered} R^3S_{g_{I-}} = 0, \qquad R^3S_{g_{--}} \longrightarrow 0, \\ G_{*++} \longrightarrow -e^{2\phi_*}A_{*+}A_{*+}, \qquad G_{*IJ} = e^{2\phi_*/3}g_{*IJ}, \\ \sqrt{-G_{*++}} \longrightarrow e^{\phi_*}A_{*+}. \end{gathered} \tag{126}$$

With these relations,

$$\begin{aligned}\lim S^b_{1,F_*(R)} = \lim \Bigg\{ \frac{1}{e^{\phi_*}} \int dx^+ \operatorname{Tr} \sqrt{-G_{*++}} \\ \times \bigg(-\frac{1}{2} G_{*IJ} G^{++}_* D_+ X^I D_+ X^J \\ - G_{*IJ} G^{++}_* D_+ X^I g^J_{*+} \\ + \left(-G^{++}_*\right)^{1/2} G_{*IJ} C^J_{*KL} \\ \times D_+ X^I \left[X^K, X^L\right] \\ + \frac{1}{4} G_{*IJ} G_{*KL} \left[X^I, X^K\right]\left[X^J, X^L\right] \\ - \frac{1}{2} e^{2\phi_*/3} G^{++}_* g_{*++} \\ + \left(-G^{++}_*\right)^{1/2} C_{*+JK} \left[X^J, X^K\right] \bigg) \Bigg\} \\ = \lim S^b_{1,F_{*10}(R)}.\end{aligned} \tag{127}$$

For $g_{*++} = \mu_{++} X^I X^J g_{*IJ}$, $g^J_{*+} = \nu_+ X^J$,

$$\begin{aligned} e^{2\phi_*/3} G^{++}_* g_{*++} &= \mu_{++} G^{++}_* G_{*IJ} X^I X^J, \\ G_{*IJ} G^{++}_* D_+ X^I g^J_{*+} &= \nu_+ G_{*IJ} G^{++}_* D_+ X^I X^J. \end{aligned} \tag{128}$$

So, in the limit taken, $S^b_{1,F_*(R)}$ could be identified with $S^b_{1,F_{*10}(R)}$, the action of SYM_1 on the $10d$ background $F_{*10}(R, x)$. The discussion can also be extended to the full $S_{1,F_*(R)}$ with the fermionic part included.

Specified to the background generated by $\widehat{Y}$ on pp-wave, (126) is satisfied for the solution in (122), so $\lim S_{1,F_*(R)} = \lim S^{\mathrm{PW}}_{R=1}/R^3 = \lim S_{1,F_{*10}(R)}$. Indeed, one can verify the validity of (127) by directly plugging (122) in it. In the limit taken, PWMM can be taken as the SYM_1 living on $10d$ background (122). Equation (104) also satisfies the above criteria. Actually,

$$\begin{aligned} \int dx^+ \operatorname{Tr} \bigg(\frac{1}{2} D_+ X^I D_+ X^I + \frac{1}{4} \left[X^I, X^J\right]^2 \\ - i\theta D_+ \theta + \theta \gamma_I \left[X^I, \theta\right] \bigg) \\ = \int dx^+ \\ \times \operatorname{Tr} \Bigg\{ \frac{\sqrt{-G_{++}}}{e^{\phi}} \bigg(-\frac{1}{2} G_{IJ} G^{++} D_+ X^I D_+ X^J \\ + \frac{1}{4} G_{IJ} G_{KL} \left[X^I, X^K\right]\left[X^J, X^L\right] \\ - i e^{\phi} \left(-G^{++}\right)^{1/2} \theta D_+ \theta \\ + e^{\phi} G^{1/2}_{II} \theta \gamma^I \left[X^I, \theta\right] \bigg) \Bigg\}, \end{aligned} \tag{129}$$

where ϕ, G_{++}, and so forth are the near-horizon geometry of the $D0$ branes in (107) with $g_s = 1$.

In the limit with $n \to \infty$, $R \to \infty$, $R^3/n = \alpha^2$ fixed, (122) gives the solution in the region of finite ρ and η. To study the region near the spherical membrane shells with $\rho, \eta - n$ finite, a change of variables $\eta \to \eta + n$ can be made. As is shown in [42], in this limit,

$$\begin{aligned} \frac{1}{R^3} V^{\mathrm{PW}}_b \\ = \frac{1}{R^3} \left[\rho^2 (\eta + n) - \frac{2}{3} (\eta + n)^3\right] \longrightarrow \frac{n}{R^3} \left(\rho^2 - 2\eta^2\right) \\ = \frac{1}{\alpha^2} V^{\mathrm{SYM}_{R\times S^2}}_b, \end{aligned} \tag{130}$$

where $V^{\mathrm{SYM}_{R\times S^2}}_b$ is the background potential for $\mathrm{SYM}_{R\times S^2}$. Alternatively, we may directly plug

$$\begin{aligned} V\left(\frac{\rho}{R}, \frac{\eta + n}{R}\right) \\ = \frac{1}{R^3} \left[\rho^2 (\eta + n) - \frac{2}{3} (\eta + n)^3\right] + v_{\overline{\eta}/R} \left(\frac{\rho}{R}, \frac{\eta + n}{R}\right) \end{aligned} \tag{131}$$

into (116) and (115), taking the limit at the end. Let $v_{\overline{\eta}/R}(\rho/R, (\eta + n)/R) = u(\rho^2, \eta)$; the $11d$ background $F_*(R, x)$ is

$$\begin{aligned} ds^2_{11} = \left(\frac{2}{\alpha^2 n^2 \partial^2_{\rho^2} u}\right)^{2/3} \\ \times \Bigg[\frac{\alpha^4 n \partial^2_{\rho^2} u}{4} dx^- dx^- \\ - 2\alpha^2 n \left(1 + \partial_{\rho^2} \partial_{\eta} u\right) dx^+ dx^- \\ + n^2 \left(4 dx^+ dx^+ - d\Omega^2_2\right) \\ -4 \left(d\eta^2 + d\rho^2 + \rho^2 d\Omega^2_5\right) \Bigg], \end{aligned} \tag{132}$$

which has the topology of $R \times R^7 \times S^2 \times R$. The $10d$ background $F_{*10}(R, x)$ is

$$\begin{aligned} ds^2_{10} = \left(\frac{\partial^2_{\rho^2} u}{n}\right)^{-1/2} \left(-4 dx^+ dx^+ + d\Omega^2_2\right) \\ + 4 \left(\frac{\partial^2_{\rho^2} u}{n}\right)^{1/2} \left(d\eta^2 + d\rho^2 + \rho^2 d\Omega^2_5\right), \\ e^{\phi} = \frac{\alpha^2}{2} \left(\frac{\partial^2_{\rho^2} u}{n}\right)^{1/4}, \end{aligned}$$

$$C_1 = -\frac{2\left(1+\partial_{\rho^2}\partial_\eta u\right)}{\alpha^2\partial^2_{\rho^2}u}dx^+,$$

$$F_4 = dC_3, \qquad C_3 = -\frac{4n}{\alpha^2\partial^2_{\rho^2}u}dx^+\wedge d^2\Omega,$$

$$H_3 = dB_2, \qquad B_2 = \left(-\frac{1+\partial_{\rho^2}\partial_\eta u}{4\partial^2_{\rho^2}u}+\eta\right)d^2\Omega. \tag{133}$$

dx^+dx^+, and $d\Omega_2^2$ now have the same prefactor. Equations (122) and (133) can be taken as the background for $1d$ and $3d$ gauge theories, respectively.

On gauge theory side, to study the fluctuation around the spherical membranes, we should expand $S^{\mathrm{PW}}_{R=1}/R^3$ around $\widehat{Y}$. In [20], it was shown that in the limit of $n\to\infty$, R^3/n fixed, (i) PWMM around a certain vacuum is equivalent to $\mathrm{SYM}_{R\times S^2}$ around each vacuum and (ii) $\mathrm{SYM}_{R\times S^2}$ around a certain vacuum with a periodicity imposed is equivalent to $\mathrm{SYM}_{R\times S^3/Z_k}$ around each vacuum. In particular, $\mathrm{SYM}_{R\times S^3}$ can be realized as the PWMM around a certain vacuum with a periodicity condition imposed.

Concretely, $(n/R^3)\,S^{\mathrm{PW}}_{R=1}(\widehat{Y};\eta) = (n/R^3)S^{\mathrm{PW}}_{R=1}(\widehat{Y}+\eta) = S^{R\times S^2}(\widehat{y}+y)$, where $S^{R\times S^2}$ is the action of the $U(\overline{N})$ $\mathrm{SYM}_{R\times S^2}$ with $\overline{N}=N_1+\cdots+N_t$:

$$\begin{aligned}S^{R\times S^2} = \frac{1}{g^2_{R\times S^2}}\int dx^+\frac{d\Omega_2}{\mu^2}&\times\mathrm{Tr}\Big\{-\frac{1}{4}F_{ab}F^{ab}-\frac{1}{2}D_a\Phi D^a\Phi\\&-\frac{\mu^2}{2}\Phi^2+\mu F_{12}\Phi-\frac{1}{2}D_aX_mD^aX_m\\&-\frac{\mu^2}{8}X_m^2+\frac{1}{4}\left[X_m,X_n\right]^2+\frac{1}{2}\left[\Phi,X_m\right]^2\\&-\frac{i}{2}\overline{\lambda}\Gamma^aD_a\lambda+\frac{i\mu}{8}\overline{\lambda}\Gamma^{123}\lambda\\&-\frac{1}{2}\overline{\lambda}\Gamma^3\left[\Phi,\lambda\right]-\frac{1}{2}\overline{\lambda}\Gamma^m\left[X_m,\lambda\right]\Big\},\end{aligned}\tag{134}$$

$1/\mu$ is the radius of S^2 which is parameterized by (θ,ϕ). $g^2_{R\times S^2}=R^3/\mu^3 n$. $\widehat{y}$ represents the vacuum:

$$\begin{aligned}\widehat{\Phi} &= \frac{\mu}{2}\,\mathrm{diag}\left[\underbrace{\alpha_1,\ldots,\alpha_1}_{N_1},\underbrace{\alpha_2,\ldots,\alpha_2}_{N_2},\ldots,\underbrace{\alpha_t,\ldots,\alpha_t}_{N_t}\right],\\ \widehat{A}_1 &= 0,\\ \widehat{A}_2 &= \begin{cases}\tan\dfrac{\theta}{2}\,\widehat{\Phi} & \text{in region I}\\ -\cot\dfrac{\theta}{2}\,\widehat{\Phi} & \text{in region II.}\end{cases}\end{aligned}\tag{135}$$

Region I and II correspond to $0\le\theta<(\pi/2)+\varepsilon$ and $(\pi/2)-\varepsilon<\theta\le\pi$, respectively. α_s is identified with the α_s in (113). The $U(\overline{N})$ gauge group is spontaneously broken to $U(N_1)\times U(N_2)\times\cdots\times U(N_t)$. Equation (135) corresponds to t disks located at $\overline{\eta}_1=(\pi/2)\alpha_1$, $\overline{\eta}_2=(\pi/2)\alpha_2,\ldots,\overline{\eta}_t=(\pi/2)\alpha_t$ with the electric charges on each equal to $(\pi^2/8)N_1,(\pi^2/8)N_2,\ldots,(\pi^2/8)N_t$. The previous $\overline{\eta}_s=(\pi/2)(n+\alpha_s)$ in PWMM now becomes $\overline{\eta}_s=(\pi/2)\alpha_s$ due to the $\eta\to\eta+n$ redefinition.

The correspondence between the spacetime coordinates and the $\mathrm{SYM}_{R\times S^2}$ fields is $\Phi\leftrightarrow\eta$, $(X^4,\ldots,X^9)\leftrightarrow(\rho,\Omega_5)$, $(x^+,\Omega_2)\leftrightarrow(x^+,\Omega_2)$. Equation (134) is the $\mathrm{SYM}_{R\times S^2}$ on flat background. Plug (133) into (134), similar to (127), for bosonic part, we have

$$\begin{aligned}S_b^{R\times S^2} = \mu^3\int dx^+\frac{d\Omega_2}{\mu^2}&\times\mathrm{Tr}\left\{\frac{\sqrt{-G}}{e^\phi}\right.\\&\times\Big(-\frac{1}{4}G^{ac}G^{bd}F_{ab}F_{cd}\\&-\frac{1}{2}G^{ab}G_{\eta\eta}D_a\Phi D_b\Phi-\frac{\mu^2}{2}G^{\Omega_2}G_{\eta\eta}\Phi^2\\&+\mu(-G)^{-1/2}G_{\eta\eta}^{1/2}F_{12}\Phi\\&-\frac{1}{2}G^{ab}G_{mn}D_aX^mD_bX^n\\&-\frac{\mu^2}{8}G^{\Omega_2}G_{mn}X^mX^n\\&+\frac{1}{4}G_{mp}G_{nq}\left[X^m,X^n\right]\left[X^p,X^q\right]\\&\left.+\frac{1}{2}G_{\eta\eta}G_{mn}\left[\Phi,X^m\right]\left[\Phi,X^n\right]\Big)\right\},\end{aligned}\tag{136}$$

where $G^{ab}\sim G_{mn}\sim G_{\eta\eta}\sim G^{\Omega_2}\sim(\partial^2_{\rho^2}u/n)^{1/2}$, $G=\det G_{ab}\sim -(\partial^2_{\rho^2}u/n)^{-3/2}$, $e^\phi\sim\alpha^2(\partial^2_{\rho^2}u/n)^{1/4}$. The action remains the same, so (134) is also the action of $\mathrm{SYM}_{R\times S^2}$ on background (133). The conclusion holds for fermionic part except for a rescaling of λ.

Indeed, as we will show in Appendix A, such phenomenon is very common. Generically, the action of SYM_{p+1} on flat background is equal to the action of the SYM_{p+1} on the near-horizon geometry of the Dp branes. (For $p\ge 6$, the worldvolume theories of Dp branes do not decouple from the bulk, as is discussed in [7].) For SCFT_3 and SCFT_6 on $M2$ and $M5$, such requirement can even offer some clues for the structure of the field theory.

$u = u(\rho^2, \eta)$ is the function of the radial directions. For the given field configuration $[\Phi, X^4, \ldots, X^9]$, let

$$\begin{aligned}&\left[\widehat{\Phi}, \widehat{X}^4, \ldots, \widehat{X}^9\right]\\ &= \left[\left(\frac{\partial^2_{\rho^2} u}{n}\right)^{1/4} \Phi, \left(\frac{\partial^2_{\rho^2} u}{n}\right)^{1/4} X^4, \ldots, \left(\frac{\partial^2_{\rho^2} u}{n}\right)^{1/4} X^9\right]; \end{aligned} \tag{137}$$

then

$$\begin{aligned} S_b^{R\times S^2} &= \mu^3 \int dx^+ \frac{d\Omega_2}{\mu^2} \\ &\times \operatorname{Tr}\left\{\frac{\sqrt{-G}}{e^{\phi}}\right. \\ &\times \left(-\frac{1}{4} G^{ac} G^{bd} F_{ab} F_{cd}\right. \\ &- \frac{1}{2} G^{ab} D_a \widehat{\Phi} D_b \widehat{\Phi} - \frac{\mu^2}{2} G^{\Omega_2} \widehat{\Phi}^2 \\ &+ \mu (-G)^{-1/2} F_{12} \widehat{\Phi} - \frac{1}{2} G^{ab} D_a \widehat{X}^m D_b \widehat{X}^n \\ &- \frac{\mu^2}{8} G^{\Omega_2} \widehat{X}^m \widehat{X}^n \\ &+ \frac{1}{4}\left[\widehat{X}^m, \widehat{X}^n\right]\left[\widehat{X}^p, \widehat{X}^q\right] \\ &\left.\left.+ \frac{1}{2}\left[\widehat{\Phi}, \widehat{X}^m\right]\left[\widehat{\Phi}, \widehat{X}^n\right]\right)\right\}. \end{aligned} \tag{138}$$

Compared with (134), with $X^m \rightarrow \widehat{X}^m$, $\Phi \rightarrow \widehat{\Phi}$, the $3d$ background fields on $R \times S^2$ will get the radial dependence:

$$\begin{aligned} e^{\phi} &= 1 \longrightarrow e^{\phi} = \frac{\alpha^2}{2}\left(\frac{\partial^2_{\rho^2} u}{n}\right)^{1/4} \left(\rho^2, \eta\right), \\ ds_3^2 &= -4 dx^+ dx^+ + d\Omega_2^2 \\ &\longrightarrow ds_3^2 = \left(\frac{\partial^2_{\rho^2} u}{n}\right)^{-1/2} \left(\rho^2, \eta\right)\left(-4 dx^+ dx^+ + d\Omega_2^2\right). \end{aligned} \tag{139}$$

This is some kind of realization of the holography, on which, we will discuss more in the next section. One special feature here is that the background fields depend on two radial directions ρ and η. Let $r = (\rho^2 + \eta^2)^{1/2}$; ds^2_{10} in (133) can be written as

$$\begin{aligned} ds_{10}^2 &= \left(\frac{\partial^2_{\rho^2} u}{n}\right)^{-1/2} \left(-4 dx^+ dx^+ + d\Omega_2^2\right) \\ &+ 4\left(\frac{\partial^2_{\rho^2} u}{n}\right)^{1/2} dr^2 + 4r^2 \left(\frac{\partial^2_{\rho^2} u}{n}\right)^{1/2} d\Omega_6^2, \end{aligned} \tag{140}$$

with $\mathrm{SYM}_{R\times S^2}$ living at $r = \infty$. However, with the energy E indentified with r, the RG flow cannot give $u(\rho^2, \eta)$, which is not just the function of r. Instead, we will make the $X^m \rightarrow \widehat{X}^m$, $\Phi \rightarrow \widehat{\Phi}$ transformation to recover $\rho\eta$ dependence of the $3d$ background fields in $SYM_{R\times S^2}$.

For PWMM, we have

$$e^{W^{\mathrm{PW}}(F)} = \int [d\eta]\, e^{-S^{\mathrm{PW}}(\widehat{Y};\eta)}. \tag{141}$$

$W^{\mathrm{PW}}(F)$ is the x^- reduction of the $11d$ supergravity effective action of the field generated by the brane $\widehat{Y}$ on pp-wave. In the limit with $n \rightarrow \infty$, $R \rightarrow \infty$, $R^3/n = \alpha^2$ fixed, under the change of variables $\eta \rightarrow \eta + n$,

$$\begin{aligned} S^{\mathrm{PW}}\left(\widehat{Y}; \eta\right) &\longrightarrow S^{R\times S^2}\left(\widehat{y}; y\right) \\ W^{\mathrm{PW}}(F) &\longrightarrow W^{R\times S^2}(F), \end{aligned} \tag{142}$$

where $W^{R\times S^2}(F)$ is the type IIA action for the above-mentioned supergravity solution dual to a vacuum of $\mathrm{SYM}_{R\times S^2}$. Then,

$$e^{W^{R\times S^2}(F)} = \int [dy]\, e^{-S^{R\times S^2}(\widehat{y};\, y)}. \tag{143}$$

As is demonstrated in [19, 20], from $\mathrm{SYM}_{R\times S^2}$ and the corresponding type IIA solution, it is also possible to get the $\mathrm{SYM}_{R\times S^3}$ and the associated type IIB solution. In (135), suppose

$$\begin{aligned} \widehat{\Phi} &= \frac{\mu}{2} \\ &\times \operatorname{diag}\Big[\ldots, \underbrace{s-1, \ldots, s-1}_{M}, \underbrace{s, \ldots, s}_{M}, \underbrace{s+1, \ldots, s+1}_{M}, \ldots\Big], \end{aligned} \tag{144}$$

where s runs from $-\infty$ to ∞. Expanding the action (134) around this vacuum and imposing the condition $y^{(s+1,t+1)} = y^{(s,t)}$ on all of the field fluctuations, one will get [20]

$$\begin{aligned} &S^{R\times S^2}\left(\widehat{y} + y\right)\Big|_{[y^{(s+1,t+1)} = y^{(s,t)}]} \\ &= S^{R\times S^3}(z) \\ &= \frac{1}{g^2_{R\times S^3}} \int dt \frac{d\Omega_3}{(\mu/2)^3} \\ &\quad \times \operatorname{Tr}\left\{-\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} D_a X_m D^a X_m - \frac{1}{12} \widehat{R} X_m^2\right. \\ &\quad - \frac{i}{2} \bar{\lambda} \Gamma^a D_a \lambda - \frac{1}{2} \bar{\lambda} \Gamma^m \left[X_m, \lambda\right] \\ &\quad \left.+ \frac{1}{4}\left[X_m, X_n\right]^2\right\}, \end{aligned} \tag{145}$$

where

$$g^2_{R\times S^3} = \frac{4\pi}{\mu} g^2_{R\times S^2}. \tag{146}$$

$S^{R\times S^3}$ is the action of the $U(M)$ SYM on $R \times S^3$.

This is a special example of Taylor's prescription for the compactification (the T-duality) in matrix models [43]. The new ingredient is the nontrivial gauge field, which makes a nontrivial fibration of S^1 over S^2 rather than a direct product; as a result, it is S^3 other than $S^2 \times S^1$ that is obtained [20].

On gravity side, start from the trivial vacuum of $\mathrm{SYM}_{R\times S^2}$, for which there is only a single disk with the electric charge $(\pi^2/8)M$; compactify the η direction; the disk configuration in covering space will contain the infinite copies of disks with the period $\pi/2$, corresponding to the vacuum (144). The type IIA geometry generated by (144), after a T-duality transformation, becomes the type IIB geometry $\mathrm{AdS}_5 \times S^5$ [19]. On field theory side, since η is compactified, the field fluctuations should respect the periodicity condition $y^{(s+1,t+1)} = y^{(s,t)}$. Taylor's prescription for the compactification also involves the T-duality transformation, making $\mathrm{SYM}_{R\times S^2}$ in type IIA become $S^{R\times S^3}$ in type IIB. Equation (143) turns into

$$e^{W^{R\times S^3}(F)} = \int [dz]\, e^{-S^{R\times S^2}(z)}. \tag{147}$$

For the trivial vacuum of $S^{R\times S^3}$, F represents $\mathrm{AdS}_5 \times S^5$ background.

Finally, the geometry arising from the backreaction of the $D3$ giant gravitons with the definite light-cone momentum on type IIB pp-wave background was also constructed in [39]. It is tempting to find the corresponding gauge theory dual. one attempt is to expand the TGMT [34–36] around the corresponding 1/2 BPS configuration, and then take the certain limit. TGMT is the discrete regularization of the $D3$ branes, so the resulted gauge theory should be a $4d$ gauge theory as is required since the geometry is generated by spherical $D3$ branes on pp-wave.

4. Holography in AdS/CFT

SYM_4 is the gauge dual of the string theory on $\mathrm{AdS}_5 \times S^5$. A natural question is what will be the gauge dual of the string theory if the field perturbation is added to $\mathrm{AdS}_5\times S^5$. For BFSS matrix model, we have matrices X^i, $i = 1, \dots, 9$ representing nine transverse coordinates, so for a scalar field

$$\begin{aligned}\phi\left(t, x^1, \dots, x^9\right) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_9=0}^{\infty} \frac{1}{n_1! \cdots n_9!} & \\ \times \left(\partial_1^{n_1} \cdots \partial_9^{n_9} \phi\right)(t, 0, \dots, 0) & \\ \times \left(x^1\right)^{n_1} \cdots \left(x^9\right)^{n_9}, &\end{aligned} \tag{148}$$

we may get the operator realization

$$\begin{aligned}\Phi(t) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_9=0}^{\infty} \frac{1}{n_1! \cdots n_9!} & \\ \times \left(\partial_1^{n_1} \cdots \partial_9^{n_9} \phi\right)(t, 0, \dots, 0) & \\ \times \mathrm{STr}\left[\left(X^1\right)^{n_1} \cdots \left(X^9\right)^{n_9}\right], &\end{aligned} \tag{149}$$

with x^i replaced by matrix X^i. Adding $\Phi(t)$ to the Lagrangian gives a matrix model on a background with the scalar field $\phi(t, x^1, \dots, x^9)$ turned on. The situation for SYM_4 is a little different. In BFSS model, the original background is flat; all of the fields could be expanded as the Taylor series in Cartesian coordinates. For SYM_4, the original background is $\mathrm{AdS}_5 \times S^5$; fields are expanded in terms of the spherical harmonics on S^5. For scalar h,

$$\begin{aligned}h(x, y) &= \sum_{n=2}^{\infty} h_n(x, |y|)\, Y^n\left(\frac{y}{|y|}\right), \\ Y^n\left(\frac{y}{|y|}\right) &= \frac{C_{a_1\cdots a_n}\left(y^{a_1} \cdots y^{a_n}\right)}{|y|^n},\end{aligned} \tag{150}$$

where $Y^n(y/|y|)$ is a spherical harmonic of rank n. $h_n(x, |y|)$ is a scalar field on AdS_5 transforming in the $(0, n, 0)$ irrep of $SO(6)$. It is necessary to find the operator correspondence of $Y^n(y/|y|)$. In SYM_4, we have $C_{a_1\cdots a_n}\,\mathrm{Tr}(X^{a_1} \cdots X^{a_n})$, which, however, is the operator realization of $C_{a_1\cdots a_n}(y^{a_1} \cdots y^{a_n})$.

Naively, let $R^2 = e^{2\Sigma} = X^m X^m$, $\Sigma = (1/2)\ln(X^m X^m)$; R and Σ may be the operators corresponding to $r = |y| = e^{\sigma}$ and σ, respectively. For the $5d$ field $F(x, r)$, for all $\sigma_0 = \ln r_0$, formally,

$$\begin{aligned}F(x, R) &= f(x, \Sigma) \\ &= f^{(0)}(x, \sigma_0) + f^{(1)}(x, \sigma_0)(\Sigma - \sigma_0 I) \\ &\quad + \frac{1}{2} f^{(2)}(x, \sigma_0)(\Sigma - \sigma_0 I)^2 + \cdots \\ &= f^{(0)}(x, \sigma_0) + f^{(1)}(x, \sigma_0)\left[\frac{1}{2}\ln\left(\frac{X^m X^m}{r_0^2}\right)\right] \\ &\quad + \frac{1}{2} f^{(2)}(x, \sigma_0)\left[\frac{1}{2}\ln\left(\frac{X^m X^m}{r_0^2}\right)\right]^2 + \cdots.\end{aligned} \tag{151}$$

Functions $f^{(n)}(x, \sigma_0)$ are coupling constants of the $4d$ gauge theory living in (x, σ_0). Similar to (28), when the gauge theory moves along σ, $\Sigma \to \Sigma - aI$, $f^{(n)}(x, \sigma_0) \to f^{(n)}(x, \sigma_0 + a)$, Σ could be decomposed into the $U(1)$ part σ and the traceless part $\Sigma_s = \Sigma - \sigma I$:

$$\begin{aligned}\sigma &= \frac{1}{N}\mathrm{Tr}\,\Sigma = \frac{1}{2N}\mathrm{Tr}\,\ln\left(X^m X^m\right) \\ &= \frac{1}{2N}\ln\,\det\left(X^m X^m\right), \\ \Sigma_s &= \frac{1}{2}\ln\left(X_s^m X_s^m\right), \qquad \det\left(X_s^m X_s^m\right) = 1,\end{aligned} \tag{152}$$

$X^m = e^{\sigma} X_s^m$. The configuration $(X^5, \dots, X^{10})$ is equivalent to $(X_s^5, \dots, X_s^{10}, \sigma)$. $\sigma \to \sigma + a$, $X^m \to e^a X^m$.

One may take $C_{a_1\cdots a_n}\operatorname{Tr}(X_s^{a_1}\cdots X_s^{a_n})$ as the operator corresponding to $Y^n(y/|y|)$ and consider the gauge theory with the vertex operator perturbation realized as (in fact, we will choose $\widehat{X}$ instead of X_s with $\det(\widehat{X}^m\widehat{X}^m) = \infty$. $C_{a_1\cdots a_n}\operatorname{Tr}(\widehat{X}^{a_1}\cdots\widehat{X}^{a_n}) \sim Y^n(\widehat{y})$, where $|\widehat{y}| = \infty$)

$$H_n(x,\sigma) = h_n(x,\sigma)\,C_{a_1\cdots a_n}\operatorname{Tr}\left(X_s^{a_1}\cdots X_s^{a_n}\right); \tag{153}$$

$h_n(x,\sigma)$ can be the arbitrary $5d$ function. Gauge theory like this is difficult to approach directly. We still prefer SYM$_4$ with $h_n(x)C_{a_1\cdots a_n}\operatorname{Tr}(X^{a_1}\cdots X^{a_n})$ added, which, after a suitable transition, will become a gauge theory with the operator $h_n(x,\sigma)C_{a_1\cdots a_n}\operatorname{Tr}(\widehat{X}^{a_1}\cdots\widehat{X}^{a_n})$. The $5d$ field $h_n(x,\sigma)$ is not arbitrary anymore but is determined by $h_n(x)$.

This is quite similar to the noncritical string coupling with $2d$ gravity. For critical string with 26 coordinates, any $26d$ background fields can be represented by the vertex operator perturbations on the string worldsheet action. For noncritical string with, for example, 25 coordinates, only the vertex operator perturbations corresponding to $25d$ fields can be constructed. However, if the noncritical string is coupled to the conformal mode of the $2d$ gravity so that the total number of degrees of freedom is 26, after a property transformation with the partition function kept invariant, the theory will become the critical string coupling with the $26d$ background fields. The $26d$ fields are induced from the $25d$ fields with the conformal mode acting as the 26th dimension. For SYM$_4$, the role of the conformal mode is played by σ.

4.1. Noncritical String Coupling with 2d Gravity. Let us have a simple review of the noncritical string [44–49]. Consider the bosonic string living in d dimensional spacetime. The corresponding nonlinear sigma model action is

$$\begin{aligned} S(g,X) = \frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{g}\,\Big[&g^{ab}\partial_a X^\mu\partial_b X^\nu G_{\mu\nu}(X) \\ &+\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu B_{\mu\nu}(X) \\ &+\alpha' R\Phi(X) + T(X) + \cdots\Big], \end{aligned} \tag{154}$$

$\mu,\nu = 1,\ldots,d$. The partition function is defined as

$$\begin{gathered} Z = \int\frac{D_g g D_g X}{\operatorname{vol}(\mathrm{diff}_2)}\;e^{-S(g,X)}. \\ \int\frac{D_g g D_g X}{\operatorname{vol}(\mathrm{diff}_2)}e^{-S(g,X)} = \int\frac{D_{e^\omega g}(e^\omega g)\,D_{e^\omega g}X}{\operatorname{vol}(\mathrm{diff}_2)}e^{-S(e^\omega g,X)}. \end{gathered} \tag{155}$$

The theory is conformal invariant since all g are integrated. The integration over the small g gives the divergence, to cure which a cutoff should be introduced, destroying the conformal invariance. Under the conformal gauge fixing $g = e^\varphi\widehat{g}, \varphi \geq 0$, $\widehat{g}$ is a small metric giving the cut-off scale

$$\frac{D_g g}{\operatorname{vol}(\mathrm{diff}_2)} = \frac{D_g\varphi}{\operatorname{vol}(\mathrm{conf}_2)}e^{-S_L(\varphi;\widehat{g})}, \tag{156}$$

where $S_L(\varphi;\widehat{g})$ is the Liouville action. With $D_g X \rightarrow D_{\widehat{g}}X$, $D_g\varphi \rightarrow D_{\widehat{g}}\varphi$, the partition function becomes

$$\begin{aligned} Z(\widehat{g},0) &= \int_{\varphi=0}^{\infty}\frac{D_g\varphi D_g X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-S(g,X)-S_L(\varphi;\widehat{g})} \\ &= \int_{\varphi=0}^{\infty}\frac{D_{\widehat{g}}\varphi D_{\widehat{g}}X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-\widehat{S}(\varphi,X;\widehat{g})}, \end{aligned} \tag{157}$$

where

$$\begin{aligned} &\widehat{S}(\varphi,X;\widehat{g}) \\ &= \frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{\widehat{g}}\,\Big[\widehat{g}^{ab}\partial_a\varphi\partial_b\varphi \\ &\qquad + \widehat{g}^{ab}\partial_a X^\mu\partial_b X^\nu\widehat{G}_{\mu\nu}(X,\varphi) \\ &\qquad + \epsilon^{ab}\partial_a X^\mu\partial_b X^\nu\widehat{B}_{\mu\nu}(X,\varphi) \\ &\qquad + \alpha'\widehat{R}\widehat{\Phi}(X,\varphi) + \widehat{T}(X,\varphi) + \cdots\Big]. \end{aligned} \tag{158}$$

The original d dimensional background field $F(X)$, after the gravitational dressing, becomes the $d+1$ dimensional field $\widehat{F}(X,\sigma)$. $\widehat{F}(X,0) = F(X)$. One can also make a change of the variables to move the boundary from $\varphi = 0$ to $\varphi = -\infty$:

$$\begin{aligned} Z(\widehat{g},0) &= \int_{\varphi=0}^{\infty}\frac{D_{\widehat{g}}\varphi D_{\widehat{g}}X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-\widehat{S}(\varphi,X;\widehat{g})} \\ &= \int_{\varphi=0}^{\infty}\frac{D_{\widehat{g}}(\varphi-\infty)\,D_{\widehat{g}}X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-\widehat{S}(\varphi-\infty+\infty,X;\widehat{g})} \\ &= \int_{\varphi=-\infty}^{\infty}\frac{D_{\widehat{g}}\varphi D_{\widehat{g}}X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-\widehat{S}(\varphi+\infty,X;\widehat{g})} \\ &= \int_{\varphi=-\infty}^{\infty}\frac{D_{\widehat{g}}\varphi D_{\widehat{g}}X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-\widehat{S}'(\varphi,X;\widehat{g})}. \end{aligned} \tag{159}$$

In $\widehat{S}'$, $\widehat{F}'(\varphi) = F(\varphi+\infty)$.

Since $g = e^{\varphi+\omega}e^{-\omega}\widehat{g}$, for $\omega \geq 0$,

$$\begin{aligned} Z(\widehat{g},0) &= \int_{\varphi=0}^{\infty}\frac{D_{e^{-\omega}\widehat{g}}(\varphi+\omega)\,D_{e^{-\omega}\widehat{g}}X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-\widehat{S}(\varphi+\omega,X;e^{-\omega}\widehat{g})} \\ &= \int_{\varphi=\omega}^{\infty}\frac{D_{e^{-\omega}\widehat{g}}\varphi D_{e^{-\omega}\widehat{g}}X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-\widehat{S}(\varphi,X;e^{-\omega}\widehat{g})} = Z(e^{-\omega}\widehat{g},\omega). \end{aligned} \tag{160}$$

With the cutoff being introduced, the conformal transformation should be accompanied by a change of the boundary:

$$\begin{aligned} Z(e^{-\omega}\widehat{g},0) - Z(\widehat{g},0) &= \int_{\varphi=0}^{\omega}\frac{D_{e^{-\omega}\widehat{g}}\varphi D_{e^{-\omega}\widehat{g}}X}{\operatorname{vol}(\mathrm{conf}_2)}e^{-\widehat{S}(\varphi,X;e^{-\omega}\widehat{g})} \\ &= Z(e^{-\omega}\widehat{g},0) - Z(e^{-\omega}\widehat{g},\omega). \end{aligned} \tag{161}$$

With the cutoff being removed, $Z(\widehat{g}) = Z(e^{-\omega}\widehat{g})$.

$\widehat{S}(\varphi, X; \widehat{g})$ is the action of the bosonic string coupling with the background fields $\widehat{F}(X,\sigma)$. The conformal invariance indicates that $\widehat{F}(X,\sigma)$ should be the on-shell solution of gravity. To arrive at this result, it is important that, in (157), the change of the $2d$ metric is always compensated by the adjustment of the background fields, keeping the form of the sigma model invariant.

4.2. Inducing the Radial Dependent Fields. Now, consider SYM_4 with the action

$$\begin{aligned} I\left[\widehat{g}_{\mu\nu}, X, A, \lambda\right] = & -\frac{1}{g_{YM}^2}\int d^4x\sqrt{-\widehat{g}} \\ & \times \operatorname{Tr}\Bigl(\frac{1}{2}\widehat{g}^{\mu\nu}\widehat{g}^{\lambda\sigma}F_{\mu\lambda}F_{\nu\sigma} \\ & + \widehat{g}^{\mu\nu}D_\mu X^m D_\nu X^m \\ & + \frac{1}{6}\widehat{R}X_m^2 - i\bar{\lambda}\Gamma^\mu D_\mu\lambda \\ & - \bar{\lambda}\Gamma^m\left[X_m,\lambda\right] \\ & - \frac{1}{2}\left[X^m, X^n\right]^2\Bigr). \end{aligned} \tag{162}$$

Select $\widehat{X}$ so that X can be expressed as $X = e^\sigma\widehat{X}$ with $\sigma \in (-\infty, 0]$. $\widehat{X}$ is the infinite matrix representing the maximum X. The partition function is

$$\begin{aligned} Z\left(\widehat{g}_{\mu\nu}, \widehat{X}\right) &= \int^{\widehat{X}} D_{\widehat{g}}XD_{\widehat{g}}AD_{\widehat{g}}\lambda\,\exp\left\{-I\left[\widehat{g}_{\mu\nu}, X, A, \lambda\right]\right\} \\ &= e^{W(\widehat{g}_{\mu\nu},\widehat{X})}. \end{aligned} \tag{163}$$

Due to the Weyl invariance,

$$I\left[\widehat{g}_{\mu\nu}, X, A, \lambda\right] = I\left[e^{2\omega}\widehat{g}_{\mu\nu}, e^{-\omega}X, A, e^{-(3/2)\omega}\lambda\right]. \tag{164}$$

Suppose

$$\begin{aligned} D_{\widehat{g}}XD_{\widehat{g}}AD_{\widehat{g}}\lambda = & D_{e^{2\omega}\widehat{g}}\left(e^{-\omega}X\right)D_{e^{2\omega}\widehat{g}} \\ & \times AD_{e^{2\omega}\widehat{g}}\left(e^{-(3/2)\omega}\lambda\right)e^{S^c[\omega;\widehat{g}]}, \end{aligned} \tag{165}$$

with no counterterm added, SYM_4 is conformal invariant, so at least for finite ω and $\widehat{g}_{\mu\nu}$, $S^c[\omega;\widehat{g}] = 0$. $Z(\widehat{g}_{\mu\nu};\widehat{X}) = Z(e^{-2\omega}\widehat{g}_{\mu\nu}; e^\omega\widehat{X})$. However, this may not be the case when $\omega = \infty$ or when $\widehat{g}_{\mu\nu} \to 0$. $S^c[\omega;\widehat{g}]$ will still be kept for the infinite conformal transformation.

Nevertheless, with $\widehat{g}_{\mu\nu} = \eta_{\mu\nu}$, we always have $S^c[\sigma;\widehat{g}] = 0$. With X parameterized as $X = e^\sigma\widehat{X}$,

$$\begin{aligned} Z\left(\widehat{g}_{\mu\nu};\widehat{X}\right) = & \int_{\sigma=-\infty}^{0} D_{\widehat{g}}\left(e^\sigma\widehat{X}\right)D_{\widehat{g}}AD_{\widehat{g}}\lambda \\ & \times\exp\left\{-I\left[\widehat{g}_{\mu\nu}, e^\sigma\widehat{X}, A, \lambda\right]\right\} \end{aligned} \tag{166}$$

$$\begin{aligned} = & \int_{\sigma=-\infty}^{0} D_{e^{2\sigma}\widehat{g}}\widehat{X}D_{e^{2\sigma}\widehat{g}}AD_{e^{2\sigma}\widehat{g}}\lambda \\ & \times\exp\left\{-I\left[e^{2\sigma}\widehat{g}_{\mu\nu}, \widehat{X}, A, \lambda\right]\right\}. \end{aligned} \tag{167}$$

From (166) to (167), the integration over σ in X is converted to the integration over σ in g. The partition function of SYM_4 on flat background is equal to the partition function of a gauge theory on a curved background. From (167), one may read the background metric:

$$\begin{aligned} ds^2 &= e^{2\sigma}dx_4^2 + r_3^2\left(d\sigma^2 + d\Omega_5^2\right) \\ &= \left(\frac{r}{r_3}\right)^2 dx_4^2 + \left(\frac{r_3}{r}\right)^2\left(dr^2 + r^2 d\Omega_5^2\right). \end{aligned} \tag{168}$$

$r = e^\sigma r_3$. $r \in [0, +\infty)$, $r_3 = +\infty$, and $\sigma \in (-\infty, 0]$. $r_3 = +\infty$ because we take $\widehat{X}$ as the standard matrix. For finite r_3, $\sigma \in (-\infty, +\infty]$. With r_3 being the radius of $\mathrm{AdS}_5 \times S^5$, (167) could be taken as the partition function of a gauge theory on $\mathrm{AdS}_5 \times S^5$.

Equations (166) and (167) have the direct extension to SCFT_3 and SCFT_6 for $M2$ and $M5$. For SCFT_3, X has the weight $1/2$, and the Weyl transformation is $(e^\sigma\widehat{X}, \widehat{g}_{\mu\nu}) \to (\widehat{X}, e^{4\sigma}\widehat{g}_{\mu\nu})$. The induced metric is

$$\begin{aligned} ds^2 &= e^{4\sigma}dx_3^2 + r_2^2\left(d\sigma^2 + d\Omega_7^2\right) \\ &= \left(\frac{r}{r_2}\right)^4 dx_3^2 + \left(\frac{r_2}{r}\right)^2\left(dr^2 + r^2 d\Omega_7^2\right) \end{aligned} \tag{169}$$

with $r = e^\sigma r_2$. For SCFT_6, X has the weight 2, and the Weyl transformation is $(e^\sigma\widehat{X}, \widehat{g}_{\mu\nu}) \to (\widehat{X}, e^\sigma\widehat{g}_{\mu\nu})$. The induced metric is

$$\begin{aligned} ds^2 &= e^{\sigma}dx_6^2 + r_5^2\left(d\sigma^2 + d\Omega_4^2\right) \\ &= \left(\frac{r}{r_5}\right) dx_6^2 + \left(\frac{r_5}{r}\right)^2\left(dr^2 + r^2 d\Omega_4^2\right), \end{aligned} \tag{170}$$

with $r = e^\sigma r_5$. Equations (169) and (170) are the near-horizon geometries of $M2$ and $M5$, respectively.

More generically, as we will show in Appendix A, for SYM_{p+1}, SCFT_3, and SCFT_6 on Dp, $M2$, and $M5$, the action on flat background and the action on the near-horizon geometry of Dp, $M2$, and $M5$ are the same. As a result, for $M2$ and $M5$ with the near-horizon geometry given by (169) and (170), $X \to (r_2/r)X$ and $X \to (r_5/r)X$ are always accompanied by $dx_3^2 \rightarrow (r/r_2)^4 dx_3^2$ and $dx_6^2 \rightarrow (r/r_5) dx_6^2$. For Dp with the near-horizon geometry

$$ds^2 = \left(\frac{r}{r_p}\right)^{(7-p)/2} dx_{p+1}^2 + \left(\frac{r}{r_p}\right)^{-(7-p)/2} \left(dr^2 + r^2 d\Omega_{8-p}^2\right), \\ e^{\Phi} = g_s \left(\frac{r}{r_p}\right)^{(7-p)(p-3)/4}, \tag{171}$$

the $X \rightarrow (r_p/r)^{(7-p)/4} X$ transformation will then make $dx_{p+1}^2 \rightarrow (r/r_p)^{(7-p)/2} dx_{p+1}^2$, $e^{\Phi} = g_s \rightarrow e^{\Phi} = g_s (r/r_p)^{(7-p)(p-3)/4}$. The coupling constant as well as the metric now gets the radial dependence. $((7-p)/2)/((7-p)/2) = 1$, and the weight of X is 1.

In the above situations, the transformations are all made for X. This is not always the case. Consider the $2dN = (4,4)$ SCFT and the $AdS_3 \times S^3 \times T^4$ geometry [50, 51]. The scalars X^i, $i = 1,2,3,4$ are related to the transverse T^4, for which the metric is dx_4^2. There is no radial dependent prefactor for X to absorb, so X has the weight 0. In fact, the chiral primaries are constructed from the fermions Ψ that have the weight 1/2 and are representations of the R-symmetry group $SO(4)$ associated with S^3.

Return to SYM_4, with the operator perturbation added:

$$I\left[g_{\mu\nu 0}, \phi_0, X, A, \lambda\right] \\ = I\left[g_{\mu\nu 0}, X, A, \lambda\right] + \int d^4x \sqrt{-g_{\mu\nu 0}}\, \phi_0(x) O(x), \tag{172}$$

where $O = C_{a_1 \cdots a_n} \mathrm{Tr}(X^{a_1} \cdots X^{a_n})$. $g_{\mu\nu 0}$ is the arbitrary $4d$ metric. The partition function is (here, the path integral measure depends on both $g_{\mu\nu 0}$ and ϕ_0. However, except for $g_{\mu\nu 0}$, no other field will enter into the path integral measure directly. Even though, the transformation of the path integral measure still has the dependence on other external fields. An explicit example is the chiral anomaly in gauge theory. We will discuss the path integral measure in more detail in Appendix B)

$$Z\left(g_{\mu\nu 0}, \phi_0; \widehat{X}\right) = \int^{\widehat{X}} D_{(g_0,\phi_0)} X D_{(g_0,\phi_0)} A D_{(g_0,\phi_0)} \lambda \\ \times \exp\left\{-I\left[g_{\mu\nu 0}, \phi_0, X, A, \lambda\right]\right\}. \\ I\left[g_{\mu\nu 0}, \phi_0, X, A, \lambda\right] \\ = I\left[e^{2\omega} g_{\mu\nu 0}, e^{(n-4)\omega} \phi_0, e^{-\omega} X, A, e^{-(3/2)\omega} \lambda\right]. \tag{173}$$

For finite ω, $g_{\mu\nu 0}$, ϕ_0,

$$Z\left(g_{\mu\nu 0}, \phi_0; \widehat{X}\right) = Z\left(e^{2\omega} g_{\mu\nu 0}, e^{(n-4)\omega} \phi_0; e^{-\omega} \widehat{X}\right). \tag{174}$$

We still want to do a $X \rightarrow \widehat{X}$ transformation:

$$D_{(g_0,\phi_0)}\left(e^{\sigma} \widehat{X}\right) D_{(g_0,\phi_0)} A D_{(g_0,\phi_0)} \lambda \\ \times \exp\left\{-I\left[g_{\mu\nu 0}, \phi_0, e^{\sigma} \widehat{X}, A, \lambda\right]\right\} \\ = D_{(e^{2\sigma} g_0, e^{(n-4)\sigma} \phi_0)} \widehat{X} D_{(e^{2\sigma} g_0, e^{(n-4)\sigma} \phi_0)} \\ \times A D_{(e^{2\sigma} g_0, e^{(n-4)\sigma} \phi_0)}\left(e^{-(3/2)\sigma} \lambda\right) \\ \times \exp\left\{-I\left[e^{2\sigma} g_{\mu\nu 0}, e^{(n-4)\sigma} \phi_0, \widehat{X}, A, e^{-(3/2)\sigma} \lambda\right] \\ + S^c\left(\sigma; g_{\mu\nu 0}, \phi_0\right)\right\}. \tag{175}$$

When $\sigma = -\infty$, $S^c(\sigma; g_{\mu\nu 0}, \phi_0)$ may not be zero. Instead of $e^{2\sigma} g_{\mu\nu 0}$ and $e^{(n-4)\sigma} \phi_0$, we can find the suitably adjusted $g_{\mu\nu}(\sigma)$ and $\phi(\sigma)$ so that

$$D_{(g_0,\phi_0)}\left(e^{\sigma} \widehat{X}\right) D_{(g_0,\phi_0)} A D_{(g_0,\phi_0)} \lambda \\ \times \exp\left\{-I\left[g_{\mu\nu 0}, \phi_0, e^{\sigma} \widehat{X}, A, \lambda\right]\right\} \\ = D_{[g(\sigma),\phi(\sigma)]} \widehat{X} D_{[g(\sigma),\phi(\sigma)]} A D_{[g(\sigma),\phi(\sigma)]} \lambda \\ \times \exp\left\{-I\left[g_{\mu\nu}(\sigma), \phi(\sigma), \widehat{X}, A, \lambda\right]\right\}. \tag{176}$$

In (176), the integration over $\widehat{X}$, A, and λ has already been carried out; both sides only depend on the $4d$ function $\sigma(x)$. Obviously, $g_{\mu\nu 0} = g_{\mu\nu}(0)$, $\phi_0 = \phi(0)$. We also require

$$D_{[g(\sigma_1),\phi(\sigma_1)]}\left(e^{\sigma_2} \widehat{X}\right) D_{[g(\sigma_1),\phi(\sigma_1)]} A D_{[g(\sigma_1),\phi(\sigma_1)]} \lambda \\ \times \exp\left\{-I\left[g_{\mu\nu}(\sigma_1), \phi(\sigma_1), e^{\sigma_2} \widehat{X}, A, \lambda\right]\right\} \\ = D_{[g(\sigma_1+\sigma_2),\phi(\sigma_1+\sigma_2)]} \widehat{X} D_{[g(\sigma_1+\sigma_2),\phi(\sigma_1+\sigma_2)]} \\ \times A D_{[g(\sigma_1+\sigma_2),\phi(\sigma_1+\sigma_2)]} \lambda \\ \times \exp\left\{-I\left[g_{\mu\nu}(\sigma_1+\sigma_2), \phi(\sigma_1+\sigma_2), \widehat{X}, A, \lambda\right]\right\}. \tag{177}$$

The $5d$ functions $g_{\mu\nu}(x,\sigma)$ and $\phi(x,\sigma)$ are induced from the $4d$ fields $g_{\mu\nu 0}(x)$ and $\phi(x)$. For finite σ, $g_{\mu\nu}(\sigma) = e^{2\sigma} g_{\mu\nu 0}$, $\phi(\sigma) = e^{(n-4)\sigma} \phi_0$.

More generically, with the $4d$ background field F_0 turned on, for the given $4d$ function $\sigma(x)$, since $\sigma \le 0$, $e^{\sigma} \widehat{X} \rightarrow \widehat{X}$ is a UV to IR transformation, the effect of which can be cancelled by the adjustment of the coupling constants, keeping the partition function invariant. $e^{\sigma} \widehat{X} \rightarrow \widehat{X}$, $F_0 \rightarrow F(\sigma)$. With all possible operators included, the change of the scale can always be compensated by the change of the background fields, leaving the form of the Lagrangian invariant:

$$
\begin{aligned}
&Z\left(g_{\mu\nu 0}, F_0; \widehat{X}\right) \\
&\quad = \int_{\sigma=-\infty}^{0} D_{[g(\sigma),F(\sigma)]} \widehat{X} D_{[g(\sigma),F(\sigma)]} A D_{[g(\sigma),F(\sigma)]} \lambda \\
&\qquad \times \exp\left\{-I\left[g_{\mu\nu}(\sigma), F(\sigma), \widehat{X}, A, \lambda\right]\right\} \\
&\quad = \widehat{Z}\left(g_{\mu\nu}(\sigma), F(\sigma); \widehat{X}; 0\right).
\end{aligned} \tag{178}
$$

In (178), with F specified to ϕ, the induced action $I[g_{\mu\nu}(\sigma), F(\sigma), \widehat{X}, A, \lambda]$ contains the term $\int d^4x\sqrt{-g(\sigma)}\phi(x,\sigma)\widehat{O}(x)$ with $\widehat{O}(x)$ given by $\widehat{O} = C_{a_1\cdots a_n}\operatorname{Tr}(\widehat{X}^{a_1}\cdots\widehat{X}^{a_n})$. $\widehat{O}(x) \sim C_{a_1\cdots a_n}(\widehat{y}^{a_1}\cdots\widehat{y}^{a_n})$. $|\widehat{y}| = \infty$ since $\widehat{X}$ is the infinite matrix. The corresponding $10d$ field is $\Phi(x,\sigma,\Omega) = |\widehat{y}|^{n-4}\phi(x,\sigma)Y^n(\Omega)$ and $\mathbf{g}_{\mu\nu}(x,\sigma) = |\widehat{y}|^4 g_{\mu\nu}(x,\sigma)$, where $Y^n(\Omega)$ is the spherical harmonic on S^5. Equation (178) is the partition function of the gauge theory on $\mathrm{AdS}_5\times S^5$ with the background fields $\Phi(x,\sigma,\Omega)$ and $\mathbf{g}_{\mu\nu}(x,\sigma)$ turned on.

On gravity side, the asymptotic expansion of the gravity solution $\mathbf{g}(\sigma)$ is [52]

$$
\mathbf{g}(\sigma) = e^{2\sigma}\left[g_0 + e^{-2\sigma}g_2 + e^{-4\sigma}g_4 - 2\sigma e^{-4\sigma}h_4\right] + O\left(e^{-3\sigma}\right), \tag{179}
$$

where $\sigma \in (-\infty,+\infty)$. Let $\sigma \to \sigma + \sigma_\infty$, with $\sigma \in (-\infty, 0]$, $\sigma_\infty = +\infty$, and $g(\sigma) = e^{-2\sigma_\infty}\mathbf{g}(\sigma)$.

$$
\begin{aligned}
g(\sigma) = e^{2\sigma}&\left[g_0 + e^{-2(\sigma+\sigma_\infty)}g_2 + e^{-4(\sigma+\sigma_\infty)}g_4\right. \\
&\left.-2(\sigma+\sigma_\infty)e^{-4(\sigma+\sigma_\infty)}h_4\right] + O\left(e^{-5\sigma_\infty}e^{-3\sigma}\right).
\end{aligned} \tag{180}
$$

For finite σ, $g(\sigma) = e^{2\sigma}g_0$. Similarly, for the solution of the scalar field $\Phi(\sigma)$ [53],

$$
\begin{aligned}
\Phi(\sigma) = e^{(n-4)\sigma}&\left[\phi_0 + e^{-2\sigma}\phi_2 + \cdots + e^{(4-2n)\sigma}\phi_{2n-4}\right. \\
&\left.-2\sigma e^{(4-2n)\sigma}\varphi_{2n-4}\right] + O\left[e^{-(n+1)\sigma}\right]
\end{aligned} \tag{181}
$$

with $\sigma \in (-\infty,+\infty)$. $\phi(\sigma) = e^{(4-n)\sigma_\infty}\Phi(\sigma+\sigma_\infty)$.

$$
\begin{aligned}
\phi(\sigma) = e^{(n-4)\sigma}&\left[\phi_0 + e^{-2(\sigma+\sigma_\infty)}\phi_2 + \cdots\right. \\
&+ e^{(4-2n)(\sigma+\sigma_\infty)}\phi_{2n-4} \\
&\left.-2(\sigma+\sigma_\infty)e^{(4-2n)(\sigma+\sigma_\infty)}\varphi_{2n-4}\right] \\
&+ O\left[e^{(3-2n)\sigma_\infty}e^{-(n+1)\sigma}\right],
\end{aligned} \tag{182}
$$

with $\sigma \in (-\infty, 0]$. When $n > 2$, $\phi(\sigma) = e^{(n-4)\sigma}\phi_0$ for finite σ. For $\mathbf{g}$-Φ coupled system, the subleading terms are determined by both g_0 and ϕ_0. $\Phi(x,\sigma)$ and $\mathbf{g}_{\mu\nu}(x,\sigma)$ are fields on AdS_5 with r_3 finite. It is expected that the $X \to \widehat{X}$ transformation will give $g_{\mu\nu}(x,\sigma)$ and $\phi(x,\sigma)$ in (180) and (182).

From (177),

$$
\begin{aligned}
&D_{[g(\sigma),\phi(\sigma)]}\widehat{X}D_{[g(\sigma),\phi(\sigma)]}AD_{[g(\sigma),\phi(\sigma)]}\lambda \\
&\qquad \times \exp\left\{-I\left[g_{\mu\nu}(\sigma), \phi(\sigma), \widehat{X}, A, \lambda\right]\right\} \\
&\quad = D_{[g(\sigma+\omega),\phi(\sigma+\omega)]}\left(e^{-\omega}\widehat{X}\right)D_{[g(\sigma+\omega),\phi(\sigma+\omega)]} \\
&\qquad \times AD_{[g(\sigma+\omega),\phi(\sigma+\omega)]}\lambda \\
&\qquad \times \exp\left\{-I\left[g_{\mu\nu}(\sigma+\omega), \phi(\sigma+\omega), e^{-\omega}\widehat{X}, A, \lambda\right]\right\}.
\end{aligned} \tag{183}
$$

As a result,

$$
\begin{aligned}
&Z\left(g_{\mu\nu 0}, \phi_0; \widehat{X}\right) \\
&\quad = \widehat{Z}\left(g_{\mu\nu}(\sigma), \phi(\sigma); \widehat{X}; 0\right) \\
&\quad = \int_{\sigma=-\infty}^{0} D_{[g(\sigma+\omega),\phi(\sigma+\omega)]}\left(e^{-\omega}\widehat{X}\right)D_{[g(\sigma+\omega),\phi(\sigma+\omega)]} \\
&\qquad \times AD_{[g(\sigma+\omega),\phi(\sigma+\omega)]}\lambda \\
&\qquad \times \exp\left\{-I\left[g_{\mu\nu}(\sigma+\omega), \phi(\sigma+\omega), e^{-\omega}\widehat{X}, A, \lambda\right]\right\} \\
&\quad = \widehat{Z}\left(g_{\mu\nu}(\sigma+\omega), \phi(\sigma+\omega); e^{-\omega}\widehat{X}; 0\right) \\
&\quad = \int_{\sigma=-\infty}^{\omega} D_{[g(\sigma),\phi(\sigma)]}\left(e^{-\omega}\widehat{X}\right)D_{[g(\sigma),\phi(\sigma)]}AD_{[g(\sigma),\phi(\sigma)]}\lambda \\
&\qquad \times \exp\left\{-I\left[g_{\mu\nu}(\sigma), \phi(\sigma), e^{-\omega}\widehat{X}, A, \lambda\right]\right\} \\
&\quad = \widehat{Z}\left(g_{\mu\nu}(\sigma), \phi(\sigma); e^{-\omega}\widehat{X}; \omega\right).
\end{aligned} \tag{184}
$$

With $\widehat{X} \to e^{-\omega}\widehat{X}$, the induced fields become $g_{\mu\nu}(\sigma+\omega)$ and $\phi(\sigma+\omega)$. Also, since

$$
Z\left(g_{\mu\nu 0}, \phi_0; \widehat{X}\right) = Z\left(e^{2\omega}g_{\mu\nu 0}, e^{(n-4)\omega}\phi_0; e^{-\omega}\widehat{X}\right), \tag{185}
$$

we have

$$
\widehat{Z}\left(g_{\mu\nu}(\sigma), \phi(\sigma); \widehat{X}; 0\right) = \widehat{Z}\left(g^{\omega}_{\mu\nu}(\sigma), \phi^{\omega}(\sigma); e^{-\omega}\widehat{X}; 0\right), \tag{186}
$$

where $g^{\omega}_{\mu\nu}(\sigma)$ and $\phi^{\omega}(\sigma)$ are fields induced from $e^{2\omega}g_{\mu\nu 0}$ and $e^{(n-4)\omega}\phi_0$, respectively. There will be $g^{\omega}_{\mu\nu}(\sigma) = g_{\mu\nu}(\sigma+\omega)$, $\phi^{\omega}(\sigma) = \phi(\sigma+\omega)$. Indeed, in (182) and (180), with ϕ_0 and g_0 replaced by $e^{(n-4)\omega}\phi_0$ and $e^{2\omega}g_0$, one may get $\phi(\sigma+\omega)$ and $g(\sigma+\omega)$. $g_0 \to e^{2\omega}g_0$, $\phi_0 \to e^{(n-4)\omega}\phi_0$, $\sigma \to \sigma+\omega$ is a diffeomorphism transformation of the gravity solution $\mathbf{g}_{\mu\nu}(\sigma)$ and $\Phi(\sigma)$ [54].

Equation (184) is valid for $[g_{\mu\nu}(x,\sigma), \phi(x,\sigma)]$ induced from $[g_{\mu\nu 0}(x), \phi_0(x)]$. For the generic $5d$ functions $[\widetilde{g}_{\mu\nu}(x,\sigma), \widetilde{\phi}(x,\sigma)]$, we only have

$$
\begin{aligned}
&\widehat{Z}\left(\widetilde{g}_{\mu\nu}(x,\sigma), \widetilde{\phi}(x,\sigma); \widehat{X}; 0\right) \\
&\quad = \widehat{Z}\left(\widetilde{g}^{\omega}_{\mu\nu}(x,\sigma), \widetilde{\phi}^{\omega}(x,\sigma); e^{-\omega}\widehat{X}; 0\right)
\end{aligned} \tag{187}
$$

for some $5d$ functions $[\widetilde{g}^{\omega}_{\mu\nu}(x,\sigma),\widetilde{\phi}^{\omega}(x,\sigma)]$. For constant ω, under the scale transformation $x \rightarrow e^{\omega}x$,

$$\begin{aligned}&\widehat{Z}\left(\widetilde{g}_{\mu\nu}(x,\sigma),\widetilde{\phi}(x,\sigma);\widehat{X};0\right)\\&\quad=\widehat{Z}\left(e^{-2\omega}\widetilde{g}^{\omega}_{\mu\nu}(e^{\omega}x,\sigma),\widetilde{\phi}^{\omega}(e^{\omega}x,\sigma);e^{-\omega}\widehat{X};0\right).\end{aligned}\tag{188}$$

In special cases, if $e^{-2\omega}\widetilde{g}^{\omega}_{\mu\nu}(\sigma)=\widetilde{g}_{\mu\nu}(\sigma)$, $\widetilde{\phi}^{\omega}(\sigma)=\widetilde{\phi}(\sigma)$, the background is "conformal". For $[g_{\mu\nu}(x,\sigma),\phi(x,\sigma)]$, this requires $g_{\mu\nu}(\sigma+\omega)=e^{2\omega}g_{\mu\nu}(\sigma)$, $\phi(\sigma+\omega)=\phi(\sigma)$ and thus $g_{\mu\nu}(\sigma)=e^{2\sigma}g_{\mu\nu0}$, $\phi(\sigma)=\phi_0$. ϕ is the marginal field.

For fields which are "conformal",

$$\begin{aligned}&\widehat{Z}\left(\widetilde{g}_{\mu\nu}(x,\sigma),\widetilde{\phi}(x,\sigma);\widehat{X};0\right)\\&\quad=\widehat{Z}\left(\widetilde{g}_{\mu\nu}(e^{\omega}x,\sigma),\widetilde{\phi}(e^{\omega}x,\sigma);e^{-\omega}\widehat{X};0\right),\end{aligned}\tag{189}$$

and so

$$\begin{aligned}&\frac{\delta\widehat{Z}\left(\widetilde{g}_{\mu\nu}(x,\sigma),\widetilde{\phi}(x,\sigma);\widehat{X};0\right)}{\delta\widetilde{\phi}(x_0,\sigma_0)}\\&\quad=\frac{\delta\widehat{Z}\left(\widetilde{g}_{\mu\nu}(e^{\omega}x,\sigma),\widetilde{\phi}(e^{\omega}x,\sigma);e^{-\omega}\widehat{X};0\right)}{\delta\widetilde{\phi}(e^{\omega}x_0,\sigma_0)}.\end{aligned}\tag{190}$$

As a result,

$$\begin{aligned}&\left\langle\sqrt{-\widetilde{g}(x_0,\sigma_0)}\widehat{O}(x_0)\delta(\sigma-\sigma_0)\right\rangle\\&\quad=\int_{\sigma=-\infty}^{0}D_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}\widehat{X}D_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}AD_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}\lambda\\&\qquad\times\exp\left\{-I\left[\widetilde{g}_{\mu\nu}(x,\sigma),\widetilde{\phi}(x,\sigma),\widehat{X},A,\lambda\right]\right\}\\&\qquad\times\sqrt{-\widetilde{g}(x_0,\sigma_0)}\widehat{O}(x_0)\delta(\sigma-\sigma_0)\\&\quad=\left\langle e^{4\omega}\sqrt{-\widetilde{g}(e^{\omega}x_0,\sigma_0)}\widehat{O}^{\omega}(e^{\omega}x_0)\delta(\sigma-\sigma_0)\right\rangle_{\omega}\\&\quad=\int_{\sigma=-\infty}^{0}D_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}\left(e^{-\omega}\widehat{X}\right)D_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}AD_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}\lambda\\&\qquad\times\exp\left\{-I\left[\widetilde{g}_{\mu\nu}(e^{\omega}x,\sigma),\widetilde{\phi}(e^{\omega}x,\sigma),e^{-\omega}\widehat{X},A,\lambda\right]\right\}\\&\qquad\times e^{(4-n)\omega}\sqrt{-\widetilde{g}(e^{\omega}x_0,\sigma_0)}\widehat{O}(e^{\omega}x_0)\delta(\sigma-\sigma_0)\\&\quad=\int_{\sigma=-\infty}^{0}D_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}\widehat{X}D_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}AD_{[\widetilde{g}(\sigma),\widetilde{\phi}(\sigma)]}\lambda\\&\qquad\times\exp\left\{-I\left[\widetilde{g}_{\mu\nu}(x,\sigma),\widetilde{\phi}(x,\sigma),\widehat{X},A,\lambda\right]\right\}\\&\qquad\times e^{(4-n)\omega}\sqrt{-\widetilde{g}(e^{\omega}x_0,\sigma_0)}\widehat{O}(e^{\omega}x_0)\delta(\sigma-\sigma_0)\\&\quad=e^{(4-n)\omega}\left\langle\sqrt{-\widetilde{g}(e^{\omega}x_0,\sigma_0)}\widehat{O}(e^{\omega}x_0)\delta(\sigma-\sigma_0)\right\rangle,\end{aligned}\tag{191}$$

and then

$$\begin{aligned}0&=\left\langle\sqrt{-\widetilde{g}(x_0,\sigma_0)}\widehat{O}(x_0)\delta(\sigma-\sigma_0)\right\rangle\\&=\frac{\delta\widehat{Z}\left(\widetilde{g}_{\mu\nu}(x,\sigma),\widetilde{\phi}(x,\sigma);\widehat{X};0\right)}{\delta\widetilde{\phi}(x_0,\sigma_0)}.\end{aligned}\tag{192}$$

$\widetilde{\phi}(x,\sigma)$ is on shell.

Restricted to $[g_{\mu\nu}(x,\sigma),\phi(x,\sigma)]$, usually, $[e^{-2\omega}g_{\mu\nu}(\sigma+\omega),\phi(\sigma+\omega)]\neq[g_{\mu\nu}(\sigma),\phi(\sigma)]$, $[g_{\mu\nu}(\sigma),\phi(\sigma)]$ is not "conformal". Instead of (191), if we require

$$\begin{aligned}&\left\langle\sqrt{-g(x_0,\sigma_0)}\widehat{O}(x_0)\delta(\sigma-\sigma_0)\right\rangle\\&\quad=\int_{\sigma=-\infty}^{0}D_{[g(\sigma),\phi(\sigma)]}\widehat{X}D_{[g(\sigma),\phi(\sigma)]}AD_{[g(\sigma),\phi(\sigma)]}\lambda\\&\qquad\times\exp\left\{-I\left[g_{\mu\nu}(x,\sigma),\phi(x,\sigma),\widehat{X},A,\lambda\right]\right\}\\&\qquad\times\sqrt{-g(x_0,\sigma_0)}\widehat{O}(x_0)\delta(\sigma-\sigma_0)\\&\quad=\left\langle e^{(4-n)\omega}\sqrt{-g(x_0,\sigma_0)}\widehat{O}(x_0)e^{ip(\sigma+\omega)}\right\rangle_{\omega}\\&\quad=\int_{\sigma=-\infty}^{0}D_{[g(\sigma+\omega),\phi(\sigma+\omega)]}\left(e^{-\omega}\widehat{X}\right)D_{[g(\sigma+\omega),\phi(\sigma+\omega)]}\\&\qquad\times AD_{[g(\sigma+\omega),\phi(\sigma+\omega)]}\lambda\\&\qquad\times\exp\left\{-I\left[g_{\mu\nu}(\sigma+\omega),\phi(\sigma+\omega),e^{-\omega}\widehat{X},A,\lambda\right]\right\}\\&\qquad\times e^{(4-n)\omega}\sqrt{-g(x_0,\sigma_0)}\widehat{O}(x_0)e^{ip(\sigma+\omega)}\\&\quad=\int_{\sigma=-\infty}^{0}D_{[g(\sigma),\phi(\sigma)]}\widehat{X}D_{[g(\sigma),\phi(\sigma)]}AD_{[g(\sigma),\phi(\sigma)]}\lambda\\&\qquad\times\exp\left\{-I\left[g_{\mu\nu}(\sigma),\phi(\sigma),\widehat{X},A,\lambda\right]\right\}\\&\qquad\times e^{(4-n)\omega}\sqrt{-g(x_0,\sigma_0)}\widehat{O}(x_0)e^{ip(\sigma+\omega)}\\&\quad=e^{(ip-n+4)\omega}\left\langle\sqrt{-g(x_0,\sigma_0)}\widehat{O}(x_0)e^{ip\sigma}\right\rangle;\end{aligned}\tag{193}$$

still,

$$\begin{aligned}0&=\left\langle\sqrt{-g(x_0,\sigma_0)}\widehat{O}(x_0)e^{ip\sigma}\right\rangle\\&=\left\langle\sqrt{-g(x_0,\sigma_0)}\widehat{O}(x_0)\delta(\sigma-\sigma_0)\right\rangle\\&=\frac{\delta\widehat{Z}\left(g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right)}{\delta\phi(x_0,\sigma_0)}.\end{aligned}\tag{194}$$

$\phi(x,\sigma)$ is also on shell.

$\phi(x,\sigma)$ is the functional of $\phi_0(x)$. In general,

$$\frac{\delta\widehat{Z}\left(g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right)}{\delta\phi_0(x_0)} = \frac{\delta Z\left(g_{\mu\nu 0}(x),\phi_0(x);\widehat{X}\right)}{\delta\phi_0(x_0)} \neq 0 \tag{195}$$

since nothing could guarantee that the one-point function of the SYM_4 would vanish in presence of the arbitrary source $\phi_0(x)$. The physical field is $\Phi = e^{(n-4)\sigma_\infty}\phi$, so there is also

$$\frac{\delta\widehat{Z}\left(g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right)}{\delta\Phi(x_0,\sigma_0)} = 0. \tag{196}$$

The difference between $I[g_{\mu\nu 0}(x),\phi_0(x),X,A,\lambda]$ and $I[g_{\mu\nu}(x,\sigma),\phi(x,\sigma),\widehat{X},A,\lambda]$ is that X is replaced by $\widehat{X}$, while the field $F_0(x)$ becomes $F(x,\sigma)$. $I[g_{\mu\nu}(x,\sigma),\phi(x,\sigma),\widehat{X},A,\lambda]$ has the natural interpretation as a gauge theory on $AdS_5\times S^5$ with the external field

$$\begin{aligned}\sqrt{-g(x,\sigma)}F(x,\sigma,\Omega) &= \sqrt{-g(x,\sigma)}F(x,\sigma)\,\widehat{Y}^n \\ &= e^{n\sigma_\infty}\sqrt{-g(x,\sigma)}F(x,\sigma)\,Y^n \\ &= \sqrt{-\mathbf{g}(x,\sigma)}\mathbf{F}(x,\sigma)\,Y^n\end{aligned} \tag{197}$$

turned on, where $F(x,\sigma)$ and $g_{\mu\nu}(x,\sigma)$ can be the arbitrary $5d$ functions. $\mathbf{g}_{\mu\nu} = e^{4\sigma_\infty}g_{\mu\nu}$ and $\mathbf{F} = e^{(n-4)\sigma_\infty}F$ are physical fields on AdS_5. According to the previous philosophy,

$$\begin{aligned}&\widehat{Z}\left(g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right) \\ &\quad= \exp\left\{S_g\left[g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right]\right\},\end{aligned} \tag{198}$$

$S_g[g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0]$ is the effective action of the type IIB supergravity on $AdS_5\times S^5$.

For a special subset of fields $\{\phi(x,\sigma)\}$ that can be derived from the $4d$ fields $\phi_0(x)$,

$$\frac{\delta\widehat{Z}\left(g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right)}{\delta\phi(x_0,\sigma_0)} = 0, \tag{199}$$

and then

$$\frac{\delta S_g\left[g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right]}{\delta\Phi(x_0,\sigma_0)} = 0; \tag{200}$$

that is, $\Phi(x,\sigma)$ is the on-shell solution of the supergravity. For the given $\phi_0(x)$, there are two ways to get the induced $\phi(x,\sigma)$: one is through (176) and the other is to find the solution of $\delta S_g/\delta\Phi(x_0,\sigma_0) = 0$, imposing $\lim_{\sigma\to 0}e^{(4-n)\sigma_\infty}\Phi(x,\sigma) = \phi_0(x)$ as the boundary condition.

Finally, we arrive at

$$\begin{aligned}W\left(g_{\mu\nu 0},\phi_0;\widehat{X}\right) &= \ln Z\left(g_{\mu\nu 0},\phi_0;\widehat{X}\right) \\ &= S_g\left[g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right].\end{aligned} \tag{201}$$

Starting from $I[g_{\mu\nu 0}(x),\phi_0(x),X,A,\lambda]$, SYM_4 with the source term added, after a $X \to \widehat{X}$ transformation, we get $I[g_{\mu\nu}(x,\sigma),\phi(x,\sigma),\widehat{X},A,\lambda]$, the action of the gauge theory on $AdS_5\times S^5$, whose free energy may be equal to $S_g[g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0]$. During the transition, $F(x,\sigma)$ are carefully adjusted to make the partition function of $I[g_{\mu\nu 0}(x),\phi_0(x),X,A,\lambda]$ and $I[g_{\mu\nu}(x,\sigma),\phi(x,\sigma),\widehat{X},A,\lambda]$ remain the same, then we arrive at (201), where $F(x,\sigma)$ is induced from $F_0(x)$ and is on shell with respect to S_g.

4.3. Imposing the Cutoff

$$\begin{aligned}&S_g\left[g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};0\right] \\ &\quad= \int_{-\infty}^{0}d\sigma\int d^4x L\left[g_{\mu\nu}(x,\sigma),\phi(x,\sigma)\right],\end{aligned} \tag{202}$$

where L is the Lagrangian of the supergravity. S_g is divergent. One may impose a cutoff $\overline{\sigma}$, $\overline{\sigma} = -\infty$,

$$\begin{aligned}&S_g\left[g_{\mu\nu}(x,\sigma),\phi(x,\sigma);\widehat{X};\overline{\sigma}\right] \\ &\quad= \int_{-\infty}^{\overline{\sigma}}d\sigma\int d^4x L\left[g_{\mu\nu}(x,\sigma),\phi(x,\sigma)\right].\end{aligned} \tag{203}$$

On gauge theory side, the one-loop effective action $W(g_{\mu\nu 0}) = -\ln\det(D)/2$ is also divergent. With a cutoff ϵbeing introduced [55],

$$\begin{aligned}W\left(g_{\mu\nu 0};\epsilon\right) &= \frac{1}{2}\int_\epsilon^\infty\frac{d\rho}{\rho}\operatorname{Tr}\left(e^{-\rho D}\right) \\ &= \int_{-\infty}^{\overline{\sigma}}d\sigma\operatorname{Tr}\left(\exp\left\{-e^{-2(\sigma+\sigma_\infty)}D\right\}\right) \\ &= \int_{-\infty}^{\overline{\sigma}}d\sigma\operatorname{Tr}\left(\exp\left\{-e^{-2\sigma}\mathbf{D}\right\}\right) \\ &= W\left(\mathbf{g}_{\mu\nu}(0);\overline{\sigma}\right),\end{aligned} \tag{204}$$

where $e^{-2(\sigma+\sigma_\infty)} = \rho$, $\mathbf{g}_{\mu\nu}(0) = e^{2\sigma_\infty}g_{\mu\nu 0}$, $\mathbf{D}$ is the Laplace operator with the metric $\mathbf{g}_{\mu\nu}(0)$. Obviously, $W(e^{2\omega}g_{\mu\nu 0},e^{2\omega}\epsilon) = W(g_{\mu\nu 0},\epsilon) = W(e^{2\omega}\mathbf{g}_{\mu\nu}(0);\overline{\sigma}-\omega) = W(\mathbf{g}_{\mu\nu}(0);\overline{\sigma})$. For small but finite ρ [56],

$$\begin{aligned}\operatorname{Tr}\left(e^{-\rho D}\right) &= \sum_{n=0}^{\infty}A_{2n}\rho^{n-2}, \\ A_{2n} &= \int d^4x\sqrt{-g}\,a_{2n}(g),\end{aligned} \tag{205}$$

so

$$\begin{aligned}W\left(g_{\mu\nu 0}\right) &= \lim_{\epsilon\to 0}W\left(g_{\mu\nu 0},\epsilon\right) \\ &= \lim_{\epsilon\to 0}\left[\frac{A_0}{4\epsilon^2}+\frac{A_2}{2\epsilon}-\frac{A_4\ln\epsilon}{2}\right]+W_{\text{ren}}\left(g_{\mu\nu 0}\right).\end{aligned} \tag{206}$$

The renormalized $W_{\text{ren}}(g_{\mu\nu 0})$ is finite. This is the realization of the holographic renormalization [52] on gauge theory side. With $g_{\mu\nu 0} \to e^{2\omega} g_{\mu\nu 0}$, $\epsilon \to e^{2\omega}\epsilon$, [52]

$$W_{\text{ren}}\left(e^{2\omega} g_{\mu\nu 0}\right) - W_{\text{ren}}\left(g_{\mu\nu 0}\right) = \omega A_4. \tag{207}$$

A_4 is the conformal anomaly [57]. On the other hand,

$$\begin{aligned} & W\left(\mathbf{g}_{\mu\nu}(0); \overline{\sigma}\right) \\ &\quad = \frac{\mathbf{A}_0}{4e^{-4\overline{\sigma}}} + \frac{\mathbf{A}_2}{2e^{-2\overline{\sigma}}} + \mathbf{A}_4\overline{\sigma} + W_{\text{nonloc}}\left(\mathbf{g}_{\mu\nu}(0)\right), \end{aligned} \tag{208}$$

so

$$\begin{aligned} W\left(\mathbf{g}_{\mu\nu}(0); 0\right) &= \frac{\mathbf{A}_0}{4} + \frac{\mathbf{A}_2}{2} + W_{\text{nonloc}}\left(\mathbf{g}_{\mu\nu}(0)\right) \\ &= W_{\text{loc}}\left(\mathbf{g}_{\mu\nu}(0)\right) + W_{\text{nonloc}}\left(\mathbf{g}_{\mu\nu}(0)\right), \\ W_{\text{nonloc}}\left(\mathbf{g}_{\mu\nu}(0)\right) &- W_{\text{ren}}\left(g_{\mu\nu 0}\right) = \sigma_\infty A_4. \end{aligned} \tag{209}$$

$W_{\text{loc}}(\mathbf{g}_{\mu\nu}(0)) = \mathbf{A}_0/4 + \mathbf{A}_2/2$ and $W_{\text{nonloc}}(\mathbf{g}_{\mu\nu}(0))$ could be compared with the local part and the nonlocal part of the gravity action in [58]. The nonlocal part contains the logarithmical divergence [59], which is just $\sigma_\infty A_4$.

On gravity side, the corresponding gravity solution has the near boundary expansion

$$G(g_0, \rho) = \frac{g_0}{\rho} + g_2 + \rho\left(g_4 + \ln \rho h_4\right) + O\left(\rho^2\right). \tag{210}$$

At $\rho = \epsilon$, the metric is $G(g_0, \epsilon)$. The on-shell gravity action with the cutoff $\rho = \epsilon$ is

$$S\left(g_{\mu\nu 0}, \epsilon\right) = S\left[G\left(g_0, \epsilon\right)\right] = \int_\epsilon^\infty \frac{d\rho}{2\rho} L(\rho), \tag{211}$$

which is entirely determined by the boundary value $G(g_0, \epsilon)$.

$$\begin{aligned} S\left(g_{\mu\nu 0}\right) &= \lim_{\epsilon\to 0} S\left(g_{\mu\nu 0}, \epsilon\right) \\ &= \lim_{\epsilon\to 0}\left\{\frac{l}{16\pi G_N}\int d^4x\sqrt{-g} \right. \\ &\qquad \times\left[\frac{a_0(g)}{\epsilon^2} + \frac{a_2(g)}{\epsilon} \right. \\ &\qquad\qquad \left.\left. - a_4(g)\ln\epsilon\right]\right\} \\ &\quad + S_{\text{ren}}\left(g_{\mu\nu 0}\right). \end{aligned} \tag{212}$$

The renormalized $S_{\text{ren}}(g_{\mu\nu 0})$ is finite [52]. Since $G(e^{2\omega} g_0, e^{2\omega}\rho) = G(g_0, \rho)$, $G(e^{2\omega} g_0, e^{2\omega}\epsilon) = G(g_0, \epsilon)$, $S(e^{2\omega} g_{\mu\nu 0}, e^{2\omega}\epsilon) = S(g_{\mu\nu 0}, \epsilon)$ [52]

$$\begin{aligned} & S_{\text{ren}}\left(e^{2\omega} g_{\mu\nu 0}\right) - S_{\text{ren}}\left(g_{\mu\nu 0}\right) \\ &\quad = \frac{l}{8\pi G_N}\int d^4x\sqrt{-g}\,\omega a_4(g) = \omega A_4. \end{aligned} \tag{213}$$

In both gauge theory and gravity, subtracting of the infinity introduces the conformal anomaly.

In (204), ϵ is the UV cutoff in gauge theory. It is desirable to find a direct and exact way to impose it. In the following, we will consider a cutoff imposed on X, which, although has some relevance with ϵ, is still not it.

Take $\overline{X} = e^{\overline{\sigma}}\widehat{X}$ other than $\widehat{X}$ as the upper bound of X; the partition function is

$$\begin{aligned} Z\left(g_{\mu\nu 0}; \overline{X}\right) &= \int^{\overline{X}} D_{g_0}XD_{g_0}AD_{g_0}\lambda \exp\left\{-I\left[g_{\mu\nu 0}, X, A, \lambda\right]\right\} \\ &= \widehat{Z}\left(g_{\mu\nu}(\sigma); \widehat{X}; \overline{\sigma}\right). \end{aligned} \tag{214}$$

For finite $g_{\mu\nu 0}$ and ω, $Z(g_{\mu\nu 0}; e^{\omega}\overline{X}) = Z(e^{2\omega} g_{\mu\nu 0}; \overline{X})$. There is also

$$\begin{aligned} Z\left(g_{\mu\nu 0}; e^{\omega}\overline{X}\right) &= \widehat{Z}\left(g_{\mu\nu}(\sigma); \widehat{X}; \overline{\sigma} + \omega\right) \\ &= \widehat{Z}\left(g_{\mu\nu}(\sigma + \omega); \widehat{X}; \overline{\sigma}\right) = Z\left(e^{2\omega} g_{\mu\nu 0}; \overline{X}\right), \end{aligned} \tag{215}$$

since the field induced from $e^{2\omega} g_{\mu\nu 0}$ is $g_{\mu\nu}(\sigma + \omega)$. From $Z(e^{2\omega} g_{\mu\nu 0}; \overline{X})$ and $Z(g_{\mu\nu 0}; e^{\omega}\overline{X})$, the induced fields have the same boundary value $g_{\mu\nu}(\overline{\sigma} + \omega)$.

For the infinitesimal $4d$ function $\delta\sigma(x)$,

$$\begin{aligned} & Z\left(g_{\mu\nu 0}; e^{\delta\sigma}\overline{X}\right) \\ &\quad = \int^{e^{\delta\sigma}\overline{X}} D_{g_0}XD_{g_0}AD_{g_0}\lambda \exp\left\{-I\left[g_{\mu\nu 0}, X, A, \lambda\right]\right\} \\ &\quad = \exp\left\{\int d^4x L^a_{(g_0;\overline{X})}(x)\,\delta\sigma(x)\right\} Z\left(g_{\mu\nu 0}; \overline{X}\right), \end{aligned} \tag{216}$$

where

$$\begin{aligned} L^a_{(g_0;\overline{X})}(x) &= \lim_{\delta\sigma\to 0}\left[\frac{1}{\delta\sigma(x)}\right. \\ &\qquad \times\left(\int^{e^{\delta\sigma(x)}\overline{X}} D_{g_0}XD_{g_0}AD_{g_0}\lambda\right. \\ &\qquad\qquad \times\exp\left\{-I\left[g_{\mu\nu 0}, X, A, \lambda\right]\right\} \\ &\qquad\qquad \left.\left. - Z\left(g_{\mu\nu 0}; \overline{X}\right)\right)\right] \\ &\qquad \times Z^{-1}\left(g_{\mu\nu 0}; \overline{X}\right). \end{aligned} \tag{217}$$

With $S^a(g_0; \overline{X}) = \int d^4x L^a_{(g_0;\overline{X})}(x)$, we may have

$$\begin{aligned} & W\left(g_{\mu\nu 0}\right) \\ &\quad = \int_0^\infty d\sigma S^a\left(g_0; e^{\sigma}\overline{X}\right) + W\left(g_{\mu\nu 0}; \overline{X}\right) \\ &\quad = \int_{-\infty}^\infty d\sigma S^a\left(g_0; e^{\sigma}\overline{X}\right) = \int_0^\infty \frac{d\rho}{\rho} S^a\left(g_0; \frac{\overline{X}}{\rho}\right), \end{aligned} \tag{218}$$

where $\rho = e^{-\sigma}$

$$W\left(g_{\mu\nu 0};\overline{X}\right) = \int_{-\infty}^{0} d\sigma S^{a}\left(g_0; e^{\sigma}\overline{X}\right) = \int_{1}^{\infty} \frac{d\rho}{\rho} S^{a}\left(g_0; \frac{\overline{X}}{\rho}\right) \tag{219}$$

$$W\left(g_{\mu\nu 0}; e^{\delta\sigma}\overline{X}\right) - W\left(g_{\mu\nu 0};\overline{X}\right) = \int d^4x L^{a}_{(g_0;\overline{X})}(x)\,\delta\sigma(x). \tag{220}$$

Also, we may define $L^{b}_{(g_0;\overline{X})}$, with

$$W\left(e^{2\delta\sigma} g_{\mu\nu 0};\overline{X}\right) - W\left(g_{\mu\nu 0};\overline{X}\right) = \int d^4x L^{b}_{(g_0;\overline{X})}(x)\,\delta\sigma(x) \tag{221}$$

if

$$\begin{aligned} &W\left(e^{2\delta\sigma} g_{\mu\nu 0};\overline{X}\right) - W\left(g_{\mu\nu 0}; e^{\delta\sigma}\overline{X}\right) \\ &\quad = \int d^4x L^{c}_{(g_0;\overline{X})}(x)\,\delta\sigma(x), \end{aligned} \tag{222}$$

$$L^{c}_{(g_0;\overline{X})}(x) = L^{b}_{(g_0;\overline{X})}(x) - L^{a}_{(g_0;\overline{X})}(x).$$

For finite $\delta\sigma$ and $g_{\mu\nu 0}$, $L^{c}_{(g_0;\overline{X})}(x) = 0$. For infinite $\delta\sigma$, for example, $W(g_{\mu\nu 0}; e^{\omega}\widehat{X})$ and $W(e^{2\omega} g_{\mu\nu 0};\widehat{X})$ with $\omega = -\infty$,

$$\begin{aligned} W\left(g_{\mu\nu 0}; e^{\omega}\widehat{X}\right) &= \ln \widehat{Z}\left(g_{\mu\nu}(\sigma+\omega);\widehat{X};0\right), \\ W\left(e^{2\omega} g_{\mu\nu 0};\widehat{X}\right) &= \ln \widehat{Z}\left(g^{\omega}_{\mu\nu}(\sigma);\widehat{X};0\right). \end{aligned} \tag{223}$$

$g_{\mu\nu}(\omega) \neq e^{2\omega} g_{\mu\nu 0}$, so $g^{\omega}_{\mu\nu}(\sigma) \neq g_{\mu\nu}(\sigma+\omega)$.

Although (219) and (204) look similar, the two kinds of cutoffs are different. In particular,

$$\begin{aligned} \ln Z\left(g_{\mu\nu 0};\overline{X}\right) &= \ln \widehat{Z}\left(g_{\mu\nu}(\sigma);\widehat{X};\overline{\sigma}\right) \neq S_g\left[g_{\mu\nu}(x,\sigma);\widehat{X};\overline{\sigma}\right] \\ &= \int_{-\infty}^{\overline{\sigma}} d\sigma \int d^4x L\left[g_{\mu\nu}(x,\sigma)\right] \end{aligned} \tag{224}$$

because the $Z(g_{\mu\nu 0};\widehat{X}) - Z(g_{\mu\nu 0};\overline{X})$ part also has the contribution to $S_g[g_{\mu\nu}(x,\sigma);\widehat{X};\overline{\sigma}]$.

Let

$$\begin{aligned} S^{b}\left(g_0;\overline{X}\right) &= \int d^4x L^{b}_{(g_0;\overline{X})}(x), \\ S^{c}\left(g_0;\overline{X}\right) &= \int d^4x L^{c}_{(g_0;\overline{X})}(x); \end{aligned} \tag{225}$$

then, $S^{c}(g_0;\overline{X}) = S^{b}(g_0;\overline{X}) - S^{a}(g_0;\overline{X})$. From (206) or (212), one can see

$$\begin{aligned} \epsilon \frac{\partial}{\partial\epsilon} W\left(g_{\mu\nu 0},\epsilon\right) &= -\frac{A_0}{2\epsilon^2} - \frac{A_2}{2\epsilon} - \frac{A_4}{2} + O(\epsilon) \\ &= -g_{\mu\nu 0}\frac{\partial}{\partial g_{\mu\nu 0}} W\left(g_{\mu\nu 0},\epsilon\right). \end{aligned} \tag{226}$$

For finite $\overline{X}$ and ϵ, (226) cannot be directly identified with S^{a} and S^{b}. However, when $\overline{X} \to \widehat{X}$, $\epsilon \to 0$,

$$S^{b}\left(g_0;\widehat{X}\right) = \lim_{\epsilon\to 0}\left[\frac{A_0}{\epsilon^2} + \frac{A_2}{\epsilon} + A_4\right] = S^{a}\left(g_0;\widehat{X}\right). \tag{227}$$

$A_0/\epsilon^2 + A_2/\epsilon + A_4$ is the conformal anomaly, which, with the infinite part subtracted, becomes A_4.

In the previous subsection, the radial dependent function $F(\sigma)$ is induced via (176). Nevertheless, it can also be induced from $W(g_{\mu\nu 0};X)$, with X being the upper bound of the integration. Take a particular finite matrix X^* as the standard so that the arbitrary configuration of X can be represented by $X = e^{\sigma}X^*$ with $\sigma \in (-\infty,+\infty)$:

$$W\left(g_{\mu\nu 0};X\right) = W\left(g_{\mu\nu 0};e^{\sigma}X^*\right) = \ln Z\left(g_{\mu\nu 0};e^{\sigma}X^*\right). \tag{228}$$

Starting from $\sigma = 0$, with σ increasing, $W(g_{\mu\nu 0};e^{\sigma}X^*)$ will also increase, so $g_{\mu\nu 0}$ should change accordingly to make the partition function invariant. Namely, we have

$$W\left(g_{\mu\nu 0};X^*\right) = W\left(g_{\mu\nu}(\sigma);e^{\sigma}X^*\right) \tag{229}$$

with $g_{\mu\nu}(0) = g_{\mu\nu 0}$

$$\frac{\delta W}{\delta g_{\mu\nu}(\sigma)}\frac{\delta g_{\mu\nu}(\sigma)}{\delta\sigma} + \frac{\delta W}{\delta\sigma} = 0. \tag{230}$$

For finite σ, $g_{\mu\nu}(\sigma) = e^{-2\sigma}g_{\mu\nu 0}$. When $\sigma \to \infty$, $g_{\mu\nu}(\sigma) \to 0$, the simple scaling relation may not be valid. For finite $g_{\mu\nu 0}$ and ω, $W(g_{\mu\nu 0};e^{-\omega}X^*) = W(e^{-2\omega}g_{\mu\nu 0};X^*)$. If

$$\begin{aligned} W\left(e^{-2\omega}g_{\mu\nu 0};X^*\right) &= W\left[g_{\mu\nu}(\omega+\sigma);e^{\sigma}X^*\right] \\ &= W\left(g_{\mu\nu}(\omega);X^*\right), \end{aligned} \tag{231}$$

there will be

$$W\left(g_{\mu\nu 0};e^{-\omega}X^*\right) = W\left[g_{\mu\nu}(\sigma);e^{\sigma}\left(e^{-\omega}X^*\right)\right]. \tag{232}$$

So, in (229), with $g_{\mu\nu}(0) = g_{\mu\nu 0}$ fixed and X^* replaced by $e^{-\omega}X^*$, the induced $g_{\mu\nu}(\sigma)$ remains the same. Also, with X^* fixed and $g_{\mu\nu}(0)$ replaced by $g_{\mu\nu}(\omega)$, the induced field will be $g_{\mu\nu}(\omega+\sigma)$. $g_{\mu\nu}(\sigma)$ is the unique function trajectory that does not depend on the X^* chosen.

We can compare it with the previously mentioned noncritical string coupling with $2d$ gravity:

$$Z\left(F;\widehat{g}\right) = \int_{\varphi=0}^{\infty}\frac{D_{(e^{\varphi}\widehat{g})}\varphi D_{(e^{\varphi}\widehat{g})}X}{\mathrm{vol}\left(\mathrm{conf}_2\right)}e^{-S(F,e^{\varphi}\widehat{g},X)-S_L(\varphi,\widehat{g})}, \tag{233}$$

where $F(X)$ is the d dimensional background field. $\widehat{g}$ is the cutoff metric. With $\widehat{g}$ replaced by $\widehat{g}^{\omega} = e^{\omega}\widehat{g}$,

$$\begin{aligned} Z\left(F;e^{\omega}\widehat{g}\right) &= \int_{\varphi=0}^{\infty}\frac{D_{(e^{\varphi}\widehat{g}^{\omega})}\varphi D_{(e^{\varphi}\widehat{g}^{\omega})}X}{\mathrm{vol}\left(\mathrm{conf}_2\right)}e^{-S(F,e^{\varphi}\widehat{g}^{\omega},X)-S_L(\varphi,\widehat{g}^{\omega})} \\ &= \int_{\varphi=\omega}^{\infty}\frac{D_{(e^{\varphi}\widehat{g})}\varphi D_{(e^{\varphi}\widehat{g})}X}{\mathrm{vol}\left(\mathrm{conf}_2\right)}e^{-S(F,e^{\varphi}\widehat{g},X)-S_L(\varphi,\widehat{g})}. \end{aligned} \tag{234}$$

$Z(F; e^{\omega}\widehat{g}) \neq Z(F;\widehat{g})$. $\widehat{g} \rightarrow e^{\omega}\widehat{g}$ is a conformal transformation, which should be accompanied by $F \rightarrow F(\omega)$ to make the partition function invariant. $Z(F;\widehat{g}) = Z(F(\omega); e^{\omega}\widehat{g})$. $F(\omega)$ is a function trajectory that does not depend on the $\widehat{g}$ chosen. A further $e^{\varphi}\widehat{g}^{\omega} \rightarrow \widehat{g}^{\omega}$ transformation gives

$$Z\left(F\left(\omega\right); e^{\omega}\widehat{g}\right) = \int_{\varphi=0}^{\infty} \frac{D_{\widehat{g}^{\omega}}\varphi D_{\widehat{g}^{\omega}}X}{\operatorname{vol}\left(\operatorname{conf}_2\right)} e^{-\widehat{S}(\varphi, X, F^{\omega}(\varphi), \widehat{g}^{\omega})}, \quad (235)$$

in which $F^{\omega}(\varphi) = F(\varphi + \omega)$ is induced.

Just as (224), $\ln Z(F(\omega); e^{\omega}\widehat{g})$ cannot be identified with the $d+1$ dimensional gravity action with a cutoff. The gravity action interpretation is possible only when $\widehat{g} \rightarrow 0$; that is, the cutoff is removed.

Here, φ is directly related to the worldsheet metric, $g = e^{\varphi}\widehat{g}$. $F(\varphi)$ gives the RG flow of the d dimensional fields. In gauge theory case, σ represents the radial direction, along which all fields, including the metric $g_{\mu\nu}$, will evolve. $F(\sigma)$ gives the radial evolution of the fields, which cannot be directly identified with the RG flow. To discuss the RG flow, we should consider the renormalized $W_{\text{ren}} = S_{\text{ren}}$, for which (see, e.g., [60])

$$\begin{aligned} &\int d^4x\sqrt{g_0}\omega\left(x\right)\left[-2g_0^{\mu\nu}\frac{\delta}{\delta g_0^{\mu\nu}} + \left(\Delta^i - 4 + \widehat{\beta}^i\right)F_0^i\frac{\delta}{\delta F_0^i}\right]W_{\text{ren}} \\ &\quad = \int d^4x\left[\omega\left(x\right)A + \partial_{\mu}\omega\left(x\right)Z^{\mu}\right], \end{aligned} \quad (236)$$

where A represents the local anomalies. For the renormalization scale μ,

$$\begin{aligned} &\mu\frac{\partial W_{\text{ren}}}{\partial\mu} + \int d^4x\sqrt{g_0} \\ &\quad \times\left[2g_0^{\mu\nu}\frac{\delta}{\delta g_0^{\mu\nu}} - \left(\Delta^i - 4\right)F_0^i\frac{\delta}{\delta F_0^i}\right]W_{\text{ren}} = 0, \\ &\mu\frac{\partial W_{\text{ren}}}{\partial\mu} + \int d^4x\sqrt{g_0}\left[\beta^i\frac{\delta W_{\text{ren}}}{\delta F_0^i}\right] = \int d^4xA. \end{aligned} \quad (237)$$

$\widehat{\beta}^i$ is the anomalous dimension. $\beta^i = \widehat{\beta}^i F_0^i$ is the β function.

5. Conclusion

With $N \rightarrow \infty$being fixed and R varying, the free energy of the Matrix theory on a x^{-}-translation invariant $11d$ supergravity background F is the functional of R and F; that is, $W = W(R, F)$. Under the coordinate transformation

$$x^{-} = \alpha x'^{-}, \qquad x^{+} = f\left(x'^{+}\right), \qquad x = g\left(x'\right), \quad (238)$$

with $x = (x^1, \ldots, x^9)$, if $R \rightarrow R'$, $F \rightarrow F'$, $W(R,F) = W(R', F')$. $W(R,F)$ preserves part of the $11d$ diffeomorphism invariance. $W(R, F)$ can be compared with $S_{\text{eff}}(R, F)$, the effective action of the supergravity for the same field F. On field theory side, naively, $S_{\text{eff}}(F)$ is calculated from

$$e^{S_{\text{eff}}(F)} = \int_{1\text{PI}} \left[d\widetilde{F}\right] e^{-S_{\text{cla}}(F+\widetilde{F})} \quad (239)$$

with S_{cla} being the classical action of the $11d$ supergravity. $e^{S_{\text{eff}}(F)}$ can be taken as the sum of the 1PI graphs of the vacuum-vacuum amplitude for supergravitons living on background F.(Of course, (239) is not well-defined.) In M theory, the basic objects are membranes other than particles. $e^{W(F)}$ is the sum of the membrane configurations on background F. In some sense, $W(F)$ gives an M theory refined version of the $S_{\text{eff}}(F)$ in (239).

In Matrix theory, the problem is that we only consider the M theory sector with the definite light-cone momentum p^{+} and the background F that is translation invariant along x^{-}. For the same F but different p^{+}, $W(R, F)$ also has the R or, equivalently, p^{+} dependence. Nevertheless, $W(R,F) = W(F_0, RF_{-}, RF_{--})$, with R being the radius of the x^{-}. In fact, for S_{eff} on the same $11d$ background, there is also $S_{\text{eff}}(R,F) = S_{\text{eff}}(F_0, RF_{-}, RF_{--})$. In $W(p^{+}, F)$, p^{+} is only a parameter encoding the scale of x^{-}, so it is enough to consider the sector with the definite p^{+}. $N \rightarrow \infty$ in Matrix theory is kept fixed in order to maintain the consistency with membrane theory, which is only characterized by one parameter p^{+}.

On field theory side, to describe the supergravity interactions among M theory objects with no light-cone momentum exchange, one may consider $\Gamma_g(Y)$ given by

$$e^{\Gamma_g(Y)} = \int [dF]\, e^{S_{R,F}(Y) - S_{\text{cla}}(R,F)}, \quad (240)$$

where F is the zero mode of the $11d$ supergravity along x^{-}. The integration out of F gives the effective action for the M theory object Y with the supergravity interactions (without transferring the light-cone momentum) all taken into account. Under the Legendre transformation, we may get $\Gamma_g(Y) = S_{\text{eff}}(R,F) + S_{R,F}(Y)$, where S_{eff} is given by (239). $\delta[S_{\text{eff}}(R,F) + S_{R,F}(Y)]/\delta F = 0$. With $S_{\text{eff}}(R,F)$ being replaced by $W(R,F)$, one can define another effective action $\Gamma(Y) = W(R,F) + S_{R,F}(Y)$, which is totally constructed from the Matrix theory with no input like $S_{\text{cla}}(R,F)$ added. $\Gamma(Y)$ is p^{+} independent, which is expected, since the scattering amplitudes with the zero light-cone momentum transfer should not depend on p^{+} due to the Lorentz invariance [61]. When $\delta S_0(Y)/\delta Y = 0$, at the one-loop order, $\Gamma(Y)$ and the standard Matrix theory effective action $\Gamma_{\text{eff}}(Y)$ are the same.

A special property of Matrix theory is that various brane configurations have the natural matrix realization, so it is possible to construct some AdS/CFT type gauge/gravity correspondences from it. In [20], $\text{SYM}_{R\times S^2}$, $\text{SYM}_{R\times S^3/Z_k}$, and $\text{SYM}_{R\times S^3}$ are all obtained by expanding the PWMM around the particular 1/2 BPS states, while the backreaction of the 1/2 BPS states on pp-wave after the x^{-} reduction produces the dual $10d$ geometry [19]. A special type IIA limit is taken in both gauge theory and gravity side. In this limit, the backreacted $11d$ geometry approaches the pp-wave, so the Matrix theory dual is still PWMM. On the other hand, the x^{-} reduced $10d$ geometry (122) is a $D0$-type solution. The action of the PWMM equals the action of the SYM_R on background (122). To study the fluctuations around the 1/2 BPS states, the $\eta \rightarrow \eta + n$ and the $Y \rightarrow Y + \widehat{Y}$ redefinition should be made on gravity and the Matrix theory side. The $10d$ geometry then becomes (133), a $D2$-type solution. Correspondingly,

$S^{\mathrm{PW}}(Y+\widehat{Y})$ becomes the action of $\mathrm{SYM}_{R\times S^2}$, which, with the background (133) plugged in, remains invariant. With the $\mathrm{SYM}_{R\times S^2}$ and the gravity dual at hand, a further T-duality-like transformation gives $\mathrm{SYM}_{R\times S^3}$ and $\mathrm{AdS}_5\times S^5$ [19, 21].

SYM_4 is the nonpertubative definition of the type IIB string theory on $\mathrm{AdS}_5\times S^5$. It is unlikely such gauge/string correspondence will suddenly vanish just because the metric in SYM_4 deviates $\eta_{\mu\nu}$ a little. If the correspondence still exists, the gravity dual will be the type IIB string theory on $\mathrm{AdS}_5\times S^5$ with a little background perturbation turned on. Then a one-to-one correspondence should exist between the $4d$ field $F_0(x)$ in SYM_4 and the $5d$ field $\mathbf{F}(x,\sigma)$ on AdS_5. The natural candidate of $\mathbf{F}(x,\sigma)$ is the gravity solution on AdS_5 with $F_0(x)$ the boundary condition. For the correspondence to be valid, it is necessary to derive $\mathbf{F}(x,\sigma)$ merely from SYM_4. In the simplest situation, when the background metric in gauge theory is the standard $\eta_{\mu\nu}$, the near-horizon geometry of $D3$, $M2$, and $M5$ could be induced from SYM_4, SCFT_3, and SCFT_6 by a $X\to\widehat{X}$ transformation. It is expected that the same method, when applied to SYM_4 with the arbitrary $F_0(x)$ turned on, will give the corresponding $\mathbf{F}(x,\sigma)$. With $X\to\widehat{X}$, $F_0(x)\to F(x,\sigma)$, SYM_4 becomes a gauge theory living in $5d$ background $\mathbf{F}(x,\sigma)$ times the transverse S^5. The free energy can be expressed as the functional of $\mathbf{F}(x,\sigma)$; that is, $W_0(F_0)=W(\mathbf{F})$. $\delta W(\mathbf{F})/\delta\mathbf{F}(x,\sigma)=0$, so if $W(\mathbf{F})$ is the $10d$ gravity action as is in Matrix theory case, $\mathbf{F}(x,\sigma)$ will be the on-shell solution.

Appendices

A. The Action on the Flat Background and the Near-Horizon Geometry

The action of the $(p+1)$-dimensional SYM theory on $R^{9,1}$ background could be written as

$$\begin{aligned} I_0 = -\frac{1}{g_p^2}\int d^{p+1}x\sqrt{-\widehat{g}} \\ \times \mathrm{Tr}\Big(\frac{1}{2}\widehat{g}^{\mu\lambda}\widehat{g}^{\nu\sigma}F_{\mu\nu}F_{\lambda\sigma} + \widehat{g}^{\mu\nu}\widehat{g}_{mn}D_\mu X^m D_\nu X^n \\ - i\overline{\lambda}\Gamma^\mu D_\mu\lambda - \overline{\lambda}\Gamma^m\left[X_m,\lambda\right] \\ -\frac{1}{2}\widehat{g}_{mp}\widehat{g}_{nq}\left[X^m,X^n\right]\left[X^p,X^q\right]\Big), \end{aligned} \tag{A.1}$$

where $\widehat{g}_{\mu\nu}=\eta_{\mu\nu}$, $\widehat{g}_{mn}=\delta_{mn}$, $\widehat{g}=\det\widehat{g}_{\mu\nu}$. The near-horizon geometry of the Dp branes is

$$\begin{aligned} g_{\mu\nu} &= \left(\frac{r}{r_p}\right)^{(7-p)/2}\eta_{\mu\nu}, \qquad g_{mn} = \left(\frac{r}{r_p}\right)^{-(7-p)/2}\delta_{mn}, \\ e^\Phi &= g_s\left(\frac{r}{r_p}\right)^{(7-p)(p-3)/4}, \qquad A_{01\cdots p} = \left(\frac{r}{r_p}\right)^{7-p}\varepsilon_{01\cdots p}. \end{aligned} \tag{A.2}$$

With r replaced by the matrix $R=(X^mX^m)^{1/2}$, the gauge theory on the near-horizon geometry background has the action

$$\begin{aligned} I_F = -\frac{1}{4\pi}\int d^{p+1}x\ \mathrm{Tr}\Big\{\sqrt{-g}e^{-\Phi} \\ \times\Big(\frac{1}{2}g^{\mu\lambda}g^{\nu\sigma}F_{\mu\nu}F_{\lambda\sigma} \\ + g^{\mu\nu}g_{mn}D_\mu X^m D_\nu X^n \\ - i\overline{\lambda}\Gamma^\mu D_\mu\lambda - \overline{\lambda}\Gamma^m\left[X_m,\lambda\right] \\ -\frac{1}{2}g_{mp}g_{nq} \\ \times\left[X^m,X^n\right]\left[X^p,X^q\right]\Big)\Big\}, \end{aligned} \tag{A.3}$$

where $g=\det g_{\mu\nu}$. With $4\pi g_s=1$, except for a rescaling of λ, $I_0=I_F$. The SYM action in $R^{9,1}$ and the SYM action in the near-horizon geometry are the same. Before and after the backreaction, the action remains invariant.

Similarly, the near-horizon geometries of $M2$ and $M5$ branes are

$$\begin{aligned} g_{\mu\nu} &= \left(\frac{r}{r_2}\right)^4\eta_{\mu\nu}, \qquad g_{mn} = \left(\frac{r}{r_2}\right)^{-2}\delta_{mn}, \\ A_{012} &= \left(\frac{r}{r_2}\right)^6\varepsilon_{012}, \\ g_{\mu\nu} &= \left(\frac{r}{r_5}\right)\eta_{\mu\nu}, \qquad g_{mn} = \left(\frac{r}{r_5}\right)^{-2}\delta_{mn}, \\ A_{012} &= \left(\frac{r}{r_5}\right)^3\varepsilon_{012}, \end{aligned} \tag{A.4}$$

respectively. For both $M2$ and $M5$,

$$\sqrt{-g}g^{\mu\nu}g_{mn}D_\mu X^m D_\nu X^n = \sqrt{-\widehat{g}}\widehat{g}^{\mu\nu}\widehat{g}_{mn}D_\mu X^m D_\nu X^n. \tag{A.5}$$

But

$$\begin{aligned} \sqrt{-g}g_{mp}g_{nq}\left[X^m,X^n\right]\left[X^p,X^q\right] \\ \neq \sqrt{-\widehat{g}}\widehat{g}_{mp}\widehat{g}_{nq}\left[X^m,X^n\right]\left[X^p,X^q\right]. \end{aligned} \tag{A.6}$$

Instead, for $M2$, (the 3-algebra was proposed in [62–64] to construct the model for two coincident $M2$ branes)

$$\begin{aligned} \sqrt{-g}g_{lp}g_{mq}g_{nr}\left[X^l,X^m,X^n\right]\left[X^p,X^q,X^r\right] \\ = \sqrt{-\widehat{g}}\widehat{g}_{lp}\widehat{g}_{mq}\widehat{g}_{nr}\left[X^l,X^m,X^n\right]\left[X^p,X^q,X^r\right]. \end{aligned} \tag{A.7}$$

We see another necessity of the sextic potential. For $3d$ SCFT, the dimension of the scalar field is $1/2$, so the potential term should contain six scalars. For $M5$, (3-algebra like this appeared in [65])

$$\begin{aligned} &\sqrt{-g}g_{\mu\nu}g_{mq}g_{nr}\left[C^{\mu},X^{m},X^{n}\right]\left[C^{\nu},X^{q},X^{r}\right] \\ &\quad = \sqrt{-\widehat{g}}\widehat{g}_{\mu\nu}\widehat{g}_{mq}\widehat{g}_{nr}\left[C^{\mu},X^{m},X^{n}\right]\left[C^{\nu},X^{q},X^{r}\right]. \end{aligned} \tag{A.8}$$

The scalar dimension is 2, so, effectively, the potential should contain two-dimension 2 scalars and two-dimension 1 scalars.

B. Path Integral Measure and the External Fields

For simplicity, consider the scalar field with the action

$$\begin{aligned} &I\left[g_{\mu\nu0},\phi_0,X\right] \\ &= \int d^4x\sqrt{-g_0}\,\mathrm{Tr}\left(g_0^{\mu\nu}\partial_\mu X^m\partial_\nu X^m + \frac{1}{6}R_0X_m^2 + \phi_0\widetilde{O}\right), \end{aligned} \tag{B.1}$$

where $\widetilde{O} = C_{a_1\cdots a_n}(X^{a_1}\cdots X^{a_n})$. $I[e^{2\omega}g_{\mu\nu0}, e^{(n-4)\omega}\phi_0, e^{-\omega}X] = I[g_{\mu\nu0},\phi_0,X]$. Following [66], we will define the scalar density field $\overline{X}$ to eliminate the dependence of the path integral measure on $g_{\mu\nu0}$:

$$\overline{X}^m = (-g_0)^{1/4}X^m. \tag{B.2}$$

In terms of $\overline{X}$,

$$\begin{aligned} I\left[g_{\mu\nu0},\phi_0,X\right] &= \overline{I}\left[g_{\mu\nu0},\phi_0,\overline{X}\right] \\ &= \int d^4x\,\mathrm{Tr}\left[g_0^{\mu\nu}\partial_\mu\overline{X}^m\partial_\nu\overline{X}^m \right. \\ &\qquad \left. + \frac{1}{6}R_0\overline{X}_m^2 + (-g_0)^{(2-n)/4}\phi_0\overline{\widetilde{O}}\right] \\ \overline{I}\left[e^{2\omega}g_{\mu\nu0},e^{(n-4)\omega}\phi_0,e^{\omega}\overline{X}\right] &= \overline{I}\left[g_{\mu\nu0},\phi_0,\overline{X}\right]. \end{aligned} \tag{B.3}$$

The partition function is then

$$Z\left(g_{\mu\nu0},\phi_0\right) = \int D\overline{X}\exp\left\{-\overline{I}\left[g_{\mu\nu0},\phi_0,\overline{X}\right]\right\}, \tag{B.4}$$

with no $g_{\mu\nu0}$ entering into the path integral measure.

$$\begin{aligned} &Z\left(e^{2\omega}g_{\mu\nu0},e^{(n-4)\omega}\phi_0\right) \\ &\quad = \int D\overline{X}\exp\left\{-\overline{I}\left[e^{2\omega}g_{\mu\nu0},e^{(n-4)\omega}\phi_0,\overline{X}\right]\right\} \\ &\quad = \int D\overline{X}\exp\left\{-\overline{I}\left[g_{\mu\nu0},\phi_0,e^{-\omega}\overline{X}\right]\right\} \\ &\quad = \int D\left(e^{\omega}\overline{X}'\right)\exp\left\{-\overline{I}\left[g_{\mu\nu0},\phi_0,\overline{X}'\right]\right\}, \end{aligned} \tag{B.5}$$

where $\overline{X}' = e^{-\omega}\overline{X}$. If $D(e^{\omega}\overline{X}') = e^{A_\omega}D\overline{X}'$; then,

$$\ln Z\left(e^{2\omega}g_{\mu\nu0},e^{(n-4)\omega}\phi_0\right) = A_\omega + \ln Z\left(g_{\mu\nu0},\phi_0\right) \tag{B.6}$$

with A_ω being the conformal anomaly. Note that to arrive at (B.6), we potentially assumed

$$\begin{aligned} &\int D\overline{X}\exp\left\{-\overline{I}\left[g_{\mu\nu0},\phi_0,\overline{X}\right]\right\} \\ &\quad = \int D\left(e^{-\omega}\overline{X}\right)\exp\left\{-\overline{I}\left[g_{\mu\nu0},\phi_0,e^{-\omega}\overline{X}\right]\right\}. \end{aligned} \tag{B.7}$$

On gravity side, this is equivalent to assume that $\lim_{\epsilon\to 0}S(g_{\mu\nu0},\phi_0,\epsilon) = \lim_{\epsilon\to 0}S(g_{\mu\nu0},\phi_0,e^{2\omega}\epsilon)$, so

$$A_\omega = \lim_{\epsilon\to 0}S\left(e^{2\omega}g_{\mu\nu0},e^{(n-4)\omega}\phi_0,\epsilon\right) - \lim_{\epsilon\to 0}S\left(g_{\mu\nu0},\phi_0,\epsilon\right). \tag{B.8}$$

A_ω in (B.8) could be compared with (227). From (B.5),

$$\begin{aligned} \left.\frac{\delta A_\omega}{\delta\omega}\right|_{\omega=0} &= \frac{1}{Z}\int D\overline{X}\exp\left\{-\overline{I}\left[g_{\mu\nu0},\phi_0,\overline{X}\right]\right\}\frac{\delta\overline{I}}{\delta\overline{X}}\overline{X} \\ &= \left\langle\frac{\delta\overline{I}}{\delta\overline{X}}\overline{X}\right\rangle, \end{aligned} \tag{B.9}$$

indicating that A_ω will depend on both $g_{\mu\nu0}$ and ϕ_0. Indeed, the direct calculation of the conformal anomaly gives $A_\omega = A_\omega(g_{\mu\nu0},\phi_0)$ composed by the gravity part $(g_{\mu\nu0})$ and the matter part (ϕ_0). Correspondingly, the transformation of the path integral measure will also depend on $g_{\mu\nu0}$ and ϕ_0, although neither of them enters into $D\overline{X}$ explicitly.

Similarly, when the theory is coupled to the external $SU(4)_R$ gauge field A_μ^a, although A_μ^a does not enter into the path integral measure, the Jacobian of the path integral measure under the R-symmetry transformation gives the R-symmetry anomaly which is the function of A_μ^a.

Finally, on gravity side, one can read the R-symmetry anomaly from the type IIB supergravity action directly but should make the regularization of the action first to get the conformal anomaly. Correspondingly, on gauge theory side, the R-symmetry anomaly exists originally, while the conformal anomaly is introduced by regularization.

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References

[1] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, "M theory as a matrix modela conjecture," *Physical Review D*, vol. 55, no. 8, pp. 5112–5128, 1997.

[2] D. Berenstein, J. Maldacena, and H. Nastase, "Strings in flat space and pp waves from $N = 4$ super Yang Mills," *Journal of High Energy Physics*, vol. 2002, no. 4, article 13, 2002.

[3] W. Taylor and M. Van Raamsdonk, "Supergravity currents and linearized interactions for matrix theory configurations with fermionic backgrounds," *Journal of High Energy Physics*, vol. 1999, no. 4, article 013, 1999.

[4] A. Dasgupta, H. Nicolai, and J. Plefka, "Vertex operators for the supermembrane," *The Journal of High Energy Physics*, no. 5, article 7, 28, 2000.

[5] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, "Matrix perturbation theory for M-theory on a pp-wave," *Journal of High Energy Physics*, vol. 2002, no. 5, article 56, 2002.

[6] J. Maldacena, "The large N limit of superconformal field theories and supergravity," *Advances in Theoretical and Mathematical Physics*, vol. 2, no. 2, pp. 231–252, 1998.

[7] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, "Supergravity and the large N limit of theories with sixteen supercharges," *Physical Review D*, vol. 58, no. 4, Article ID 046004, 1998.

[8] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Physics Letters B*, vol. 428, no. 1-2, pp. 105–114, 1998.

[9] E. Witten, "Anti de Sitter space and holography," *Advances in Theoretical and Mathematical Physics*, vol. 2, no. 2, pp. 253–291, 1998.

[10] S. Sethi and L. Susskind, "Rotational invariance in the M(atrix) formulation of type IIB theory," *Physics Letters B*, vol. 400, no. 3-4, pp. 265–268, 1997.

[11] L. Susskind, "T duality in M(atrix) theory and S duality in field theory," http://xxx.lanl.gov/abs/hep-th/9611164.

[12] O. J. Ganor, S. Ramgoolam, and W. Taylor, "Branes, fluxes and duality in M(atrix)-theory," *Nuclear Physics B*, vol. 492, no. 1-2, pp. 191–204, 1997.

[13] B. de Wit, J. Hoppe, and H. Nicolai, "On the quantum mechanics of supermembranes," *Nuclear Physics B*, vol. 305, no. 4, pp. 545–581, 1988.

[14] L. Motl, "Proposals on non-perturbative superstring interactions," http://lanl.arxiv.org/abs/hep-th/9701025.

[15] R. Dijkgraaf, E. Verlinde, and H. Verlinde, "Matrix string theory," *Nuclear Physics B*, vol. 500, no. 1–3, pp. 43–61, 1997.

[16] Y. Sekino and T. Yoneya, "From supermembrane to matrix string," *Nuclear Physics B*, vol. 619, no. 1–3, pp. 22–50, 2001.

[17] E. S. Fradkin and A. A. Tseytlin, "Effective field theory from quantized strings," *Physics Letters B*, vol. 158, no. 4, pp. 316–322, 1985.

[18] A. A. Tseytlin, "Sigma model approach to string theory effective actions with tachyons," *Journal of Mathematical Physics*, vol. 42, no. 7, pp. 2854–2871, 2001.

[19] H. Lin and J. Maldacena, "Fivebranes from gauge theory," *Physical Review D*, vol. 74, no. 8, Article ID 084014, 40 pages, 2006.

[20] G. Ishiki, S. Shimasaki, Y. Takayama, and A. Tsuchiya, "Embedding of theories with $SU(2|4)$ symmetry into the plane wave matrix model," *Journal of High Energy Physics*, vol. 20069, no. 11, article 089, 2006.

[21] T. Ishii, G. Ishiki, S. Shimasaki, and A. Tsuchiya, "$N = 4$ super Yang-Mills theory from the plane wave matrix model," *Physical Review D*, vol. 78, no. 10, Article ID 106001, 21 pages, 2008.

[22] B. deWit, K. Peeters, and J. Plefka, "Superspace geometry for supermembrane backgrounds," *Nuclear Physics B*, vol. 532, Article ID 980320, pp. 99–123, 1998.

[23] W. Taylor and M. Van Raamsdonk, "Multiple D0-branes in weakly curved backgrounds," *Nuclear Physics B*, vol. 558, no. 1-2, pp. 63–95, 1999.

[24] W. Taylor and M. Van Raamsdonk, "Multiple Dp-branes in weak background fields," *Nuclear Physics B*, vol. 573, no. 3, pp. 703–734, 2000.

[25] D. Kabat and W. Taylor, "Linearized supergravity from Matrix theory," *Physics Letters B*, vol. 426, no. 3-4, pp. 297–305, 1998.

[26] A. A. Tseytlin, "'No-force' condition and BPS combinations of p-branes in 1/2 and $(2, 0)$ dimensions," *Nuclear Physics B*, vol. 487, no. 1-2, pp. 141–154, 1997.

[27] A. Jevicki and C. Lee, "S-matrix generating functional and effective action," *Physical Review D*, vol. 37, no. 6, pp. 1485–1491, 1988.

[28] E. S. Fradkin and A. A. Tseytlin, "Quantum string theory effective action," *Nuclear Physics B*, vol. 261, no. 1, pp. 1–27, 1985.

[29] J. Figueroa-O'Farrill and G. Papadopoulos, "Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory," *Journal of High Energy Physics*, vol. 2001, no. 08, article 036, 2001.

[30] J. Maldacena, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, "Tranverse fivebranes in Matrix theory," *Journal of High Energy Physics*, vol. 2003, no. 1, article 038, 2003.

[31] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler, "Giant gravitons in conformal field theory," *Journal of High Energy Physics*, vol. 2002, no. 4, article 34, 2002.

[32] A. Hashimoto, S. Hirano, and N. Itzhaki, "Large branes in AdS and their field theory dual," *The Journal of High Energy Physics*, vol. 2000, no. 8, article 51, 2000.

[33] J. McGreevy, L. Susskind, and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," *The Journal of High Energy Physics*, vol. 2000, no. 6, article 8, 2000.

[34] M. M. Sheikh-Jabbari, "Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture," *Journal of High Energy Physics*, vol. 2004, no. 9, article 17, 2004.

[35] M. M. Sheikh-Jabbari and M. Torabian, "Classification of all 1/2 BPS solutions of the tiny graviton matrix theory," *Journal of High Energy Physics*, vol. 2005, no. 4, article 001, 2005.

[36] M. Ali-Akbari, M. M. Sheikh-Jabbari, and M. Torabian, "Tiny graviton matrix theory/SYM correspondence: analysis of BPS states," *Physical Review D*, vol. 74, no. 6, Article ID 066005, 2006.

[37] H. Lin, "The supergravity dual of the BMN matrix model," *Journal of High Energy Physics*, vol. 2004, no. 12, article 001, 2004.

[38] K. Sugiyama and K. Yoshida, "Giant graviton and quantum stability in the Matrix model on a pp-wave background," *Physical Review D*, vol. 66, no. 8, Article ID 085022, 9 pages, 2002.

[39] H. Lin, O. Lunin, and J. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," *Journal of High Energy Physics*, vol. 2004, no. 10, article 025, 2004.

[40] F. Cachazo, K. A. Intriligator, and C. Vafa, "A large N duality via a geometric transition," *Nuclear Physics B*, vol. 603, Article ID 010306, 3 pages, 2001.

[41] K. Becker, M. Becker, J. Polchinski, and A. Tseytlin, "Higher order graviton scattering in M(atrix) theory," *Physical Review D*, vol. 56, no. 6, pp. R3174–R3178, 1997.

[42] H. Ling, A. R. Mohazab, H.-H. Shieh, G. van Anders, and M. Van Raamsdonk, "Little string theory from a double-scaled matrix model," *Journal of High Energy Physics*, vol. 2006, no. 10, article 018, 2006.

[43] W. Taylor, "D-brane field theory on compact spaces," *Physics Letters B*, vol. 394, no. 3-4, pp. 283–287, 1997.

[44] I. Antoniadis, C. Bachas, J. Ellis, and D. V. Nanopoulos, "Cosmological string theories and discrete inflation," *Physics Letters B*, vol. 211, no. 4, pp. 393–399, 1988.

[45] I. Antoniadis, C. Bachas, J. Ellis, and D. V. Nanopoulos, "An expanding universe in string theory," *Nuclear Physics B*, vol. 328, no. 1, pp. 117–139, 1989.

[46] I. Antoniadis, C. Bachas, J. Ellis, and D. V. Nanopoulos, "Comments on cosmological string solutions," *Physics Letters B*, vol. 257, no. 3-4, pp. 278–284, 1991.

[47] J. Distler and H. Kawai, "Conformal field theory and 2D quantum gravity," *Nuclear Physics B*, vol. 321, no. 2, pp. 509–527, 1989.

[48] J. Polchinski, "A two-dimensional model for quantum gravity," *Nuclear Physics B*, vol. 324, no. 1, pp. 123–140, 1989.

[49] A. Dhar and S. R. Wadia, "Noncritical strings, RG flows and holography," *Nuclear Physics B*, vol. 590, no. 1-2, pp. 261–272, 2000.

[50] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Physics Letters B*, vol. 379, no. 1–4, pp. 99–104, 1996.

[51] J. R. David, G. Mandal, and S. R. Wadia, "Microscopic formulation of black holes in string theory," *Physics Reports*, vol. 369, no. 6, pp. 549–686, 2002.

[52] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," *The Journal of High Energy Physics*, no. 7, article 23, 1998.

[53] S. de Haro, K. Skenderis, and S. N. Solodukhin, "Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence," *Communications in Mathematical Physics*, vol. 217, no. 3, pp. 595–622, 2001.

[54] C. Imbimbo, A. Schwimmer, S. Theisen, and S. Yankielowicz, "Diffeomorphisms and holographic anomalies," *Classical and Quantum Gravity*, vol. 17, no. 5, pp. 1129–1138, 2000.

[55] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, vol. 7 of *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, UK, 1982.

[56] B. S. DeWitt, *Dynamical Theory of Groups and Fields*, Gordon and Breach Science Publishers, New York, NY, USA, 1965.

[57] M. J. Duff, "Twenty years of the Weyl anomaly," *Classical and Quantum Gravity*, vol. 11, no. 6, pp. 1387–1403, 1994.

[58] J. De Boer, E. Verlinde, and H. Verlinde, "On the holographic renormalization group," *Journal of High Energy Physics*, vol. 4, no. 8, article 003, pp. 1–15, 2000.

[59] M. Fukuma, S. Matsuura, and T. Sakai, "A note on the Weyl anomaly in the holographic renormalization group," *Progress of Theoretical Physics*, vol. 104, no. 5, pp. 1089–1108, 2000.

[60] J. Erdmenger, "Field-theoretical interpretation of the holographic renormalization group," *Physical Review D*, vol. 64, no. 8, Article ID 085012, 12 pages, 2001.

[61] T. Banks, "Tasi lectures on matrix theory," http://arxiv.org/abs/hep-th/9911068.

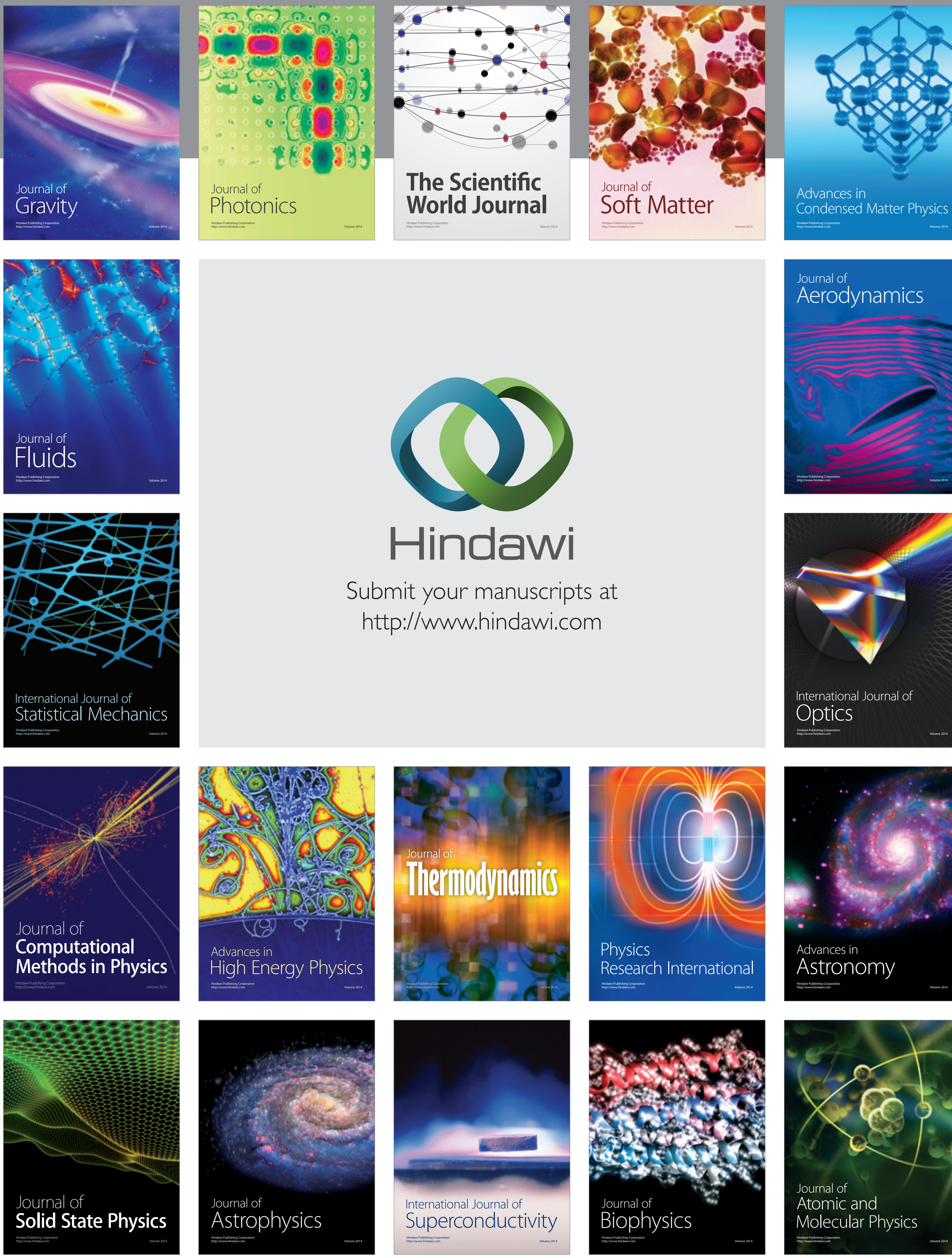
[62] J. Bagger and N. Lambert, "Modeling multiple M2-branes," *Physical Review D*, vol. 75, no. 4, Article ID 045020, 7 pages, 2007.

[63] J. Bagger and N. Lambert, "Gauge symmetry and supersymmetry of multiple M2-branes," *Physical Review D*, vol. 77, no. 6, Article ID 065008, 6 pages, 2008.

[64] A. Gustavsson, "Algebraic structures on parallel M2-branes," *Nuclear Physics B*, vol. 811, no. 1-2, pp. 66–76, 2009.

[65] N. Lambert and C. Papageorgakis, "Nonabelian (2,0) tensor multiplets and 3-algebras," *Journal of High Energy Physics*, vol. 2010, no. 8, article 083, 2010.

[66] K. Fujikawa, "Comment on chiral and conformal anomalies," *Physical Review Letters*, vol. 44, no. 26, pp. 1733–1736, 1980.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