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# BRS Operators and Covariant Derivatives in Loop Space for p-Branes coupled to Yang-Mills

J. A. Dixon

Center for Theoretical Physics
Physics Department
Texas A & M University
College Station, Texas 77843, USA

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ABSTRACT

Canonical forms are given for the nilpotent BRS operator $\delta$ and the covariant 'loop space' derivative $\mathcal{D}_\mu$ for the p-brane fields for all odd p. The defining characteristic of $\mathcal{D}_\mu$ is that it is a functional derivative operator which generalizes the ordinary functional derivative and also commutes with $\delta$. Methods of construction for the canonical forms are discussed.

# 1 Introduction

The existence of two alternative formulations of 10-dimensional supergravity-Yang-Mills field theory, suggests that there may exist[1] a fundamental heterotic super-5-brane theory that is dual [2][3] to the heterotic string (1-brane). These two p-brane (p=5 and p=1) theories should couple to the two dual versions (with antisymmetric tensors $B_6$ and $B_2$ respectively) of the supergravity-Yang-Mills theories. The action for the heterotic superstring coupled to this supergravity-Yang-Mills background is known (See [4] for the formulation closest to our present needs). The bosonic part of the corresponding action for the 5-brane coupled to Yang-Mills has been discussed in [5], but this has not yet been made supersymmetric. On the other hand, the actions for supersymmetric p-branes that are not coupled to Yang-Mills are known[6].

One way to get the equations of motion of the $B_2$ theory is to derive it from light-like integrability applied to the covariant derivatives $\mathcal{D}_\mu(\sigma)$ in loop superspace[4]. Hence one might be able to make some progress in finding the heterotic supersymmetric 5-brane by examining the analog problem for the 5-brane. However, the present discussion is restricted to bosonic fields only.

In this paper, the formulation of the BRS operator and the covariant derivative for the coupling of Yang-Mills to the p-branes for odd p are discussed in a general way. The operators for even p will no doubt involve products of the odd p structures as happens in [5]. In general, this formulation involves complicated polynomials in the Yang-Mills field $A_\mu^a$ and a composite field $K_i^a$. In this paper no new examples of these polynomials are given, but the way they arise is explained and it is proved that they always exist. Various methods for their construction are discussed. A summary of the results is given in the conclusion. The notation here follows that of [7].

# 2 Covariant Functional Derivatives

How should one define a covariant derivative? What properties should it have? One way to find a covariant derivative is to express the velocity in terms of the Hamiltonian variables, and then quantize the theory. For example in electromagnetism, one gets the covariant derivative:

$$\mathcal{D}_\mu = \dot{X}_\mu = p_\mu + A_\mu = i \left[ \frac{\partial}{\partial X^\mu} - i A_\mu \right] \tag{1}$$

Another way is to define the covariant derivative to be a generalization of the ordinary derivative which commutes with the BRS operator $\delta$. These two ways are probably equivalent, and both are used for the string in [4].

A covariant derivative should have several properties, and these can best be enumerated by considering a simple example in field theory, which also turns out to be relevant to our problem below. Suppose that $\phi^i$ is a scalar field coupled to Yang-Mills theory. The covariant derivative can be written in the following way:

$$D^i_{\mu j} \phi^j = \partial_\mu \phi^i + A^a_\mu T^{ai}{}_j \phi^j \tag{2}$$

where the matrices $T^a$ satisfy:

$$[T^a, T^b] = f^{abc} T^c \tag{3}$$

The covariant derivative can be obtained by defining a functional derivative operator as follows:

$$\mathcal{D}_\mu = \int d^D x \left\{ D^i_{\mu j} \phi^j(x) \frac{\delta}{\delta \phi^i(x)} \right\} \tag{4}$$

Using this expression for $\mathcal{D}_\mu$ one finds that:

$$[\mathcal{D}_\mu \phi]^i = D^i_{\mu j} \phi^j \tag{5}$$

Now note the following features of this functional derivative form of the covariant derivative:

1. $\mathcal{D}_\mu$ is a functional derivative operator which is integrated over spacetime and summed over all indices except the one uncontracted spacetime index $\mu$. The advantage in writing $\mathcal{D}_\mu$ in this way is that it makes $\mathcal{D}_\mu$ into a derivation on the algebra of polynomial functionals of the field $\phi^i$, so that the covariant derivative behaves much like an ordinary derivative. When acting on invariant local polynomials in $\phi^i$, like say $d_{ijk}\phi^i\phi^j\phi^k$, where $d_{ijk}$ is an invariant tensor, $\mathcal{D}_\mu$ yields the same result as the ordinary derivative.

2. In this functional derivative form, one can verify that $\mathcal{D}_\mu$ commutes with the BRS operator:

$$[\mathcal{D}_\mu, \delta] = 0 \tag{6}$$

where the anticommuting BRS operator $\delta$ is:

$$\delta = \int d^D x \left\{ D^{ab}_\mu \omega^b \frac{\delta}{\delta A^a_\mu(x)} - \frac{1}{2} f^{abc} \omega^b \omega^c \frac{\delta}{\delta \omega^a(x)} - T^{ai}_{\ \ j} \phi^j \omega^a \frac{\delta}{\delta \phi^i(x)} \right\} \tag{7}$$

Note that

(a) $\delta$ is an anti-derivation (an anticommuting derivation),

(b) $\delta$ is a functional derivative operator integrated over space-time and summed over all indices,

(c) $\delta$ is nilpotent:

$$\delta^2 = 0. \tag{8}$$

3. Because of the above properties, the (multiple) commutators:

$$\mathcal{D}_{\mu_2 \mu_1} = [\mathcal{D}_{\mu_2}, \mathcal{D}_{\mu_1}] \tag{9}$$

$$\mathcal{D}_{\mu_k \cdots \mu_1} = [\mathcal{D}_{\mu_k}, \mathcal{D}_{\mu_{k-1} \cdots \mu_1}]; \quad (k \geq 3) \tag{10}$$

also satisfy

$$[\mathcal{D}_{\mu_k \cdots \mu_1}, \delta] = 0 \tag{11}$$

For example, in this case, one gets:

$$\mathcal{D}_{\mu\nu} = \int d^D x \left\{ -F^a_{\mu\nu} T^{ai}_{\ \ j} \phi^j \frac{\delta}{\delta \phi^i(x)} \right\} \tag{12}$$

Here the Yang-Mills curvature is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\mu A^a_\nu + f^{abc} A^b_\mu A^c_\nu \tag{13}$$

4. In this functional form, one can add covariant derivatives for different fields, and this will generally change the value of these commutators above. This is particularly true if one adds 'covariant derivatives' for gauge fields. For example, one could take:

$$\mathcal{D}_{\mu\,\mathrm{new}} = \int d^D x \left\{ D^i_{\mu j}\phi^j(x)\frac{\delta}{\delta\phi^i(x)} + \frac{1}{2}F^a_{\mu\nu}\frac{\delta}{\delta A^a_\nu} \right\} \tag{14}$$

which still satisfies (6), but then quite a different result for $\mathcal{D}_{\mu\nu}$, is obtained, namely:

$$\mathcal{D}_{\mu\nu\,\mathrm{new}} = -\frac{3}{4}\int d^D x\; D^{ab}_\rho F^b_{\mu\nu}\frac{\delta}{\delta A^a_\rho} \tag{15}$$

Only the gauge part of the commutator remains–the $\frac{\delta}{\delta\phi}$ part is gone.

# 3 Covariant Derivative for String

For the reasons above, we will look for a Yang-Mills gauge covariant derivative for a p-brane that is a functional derivative operator $\mathcal{D}_\mu$ that commutes with the nilpotent BRS functional derivative operator $\delta$:

$$[\delta, \mathcal{D}_\mu] = 0 \tag{16}$$

In [7], the functional derivative form of the BRS operators $\delta$ for the string and the 3-brane was discussed, and those results will be used here. The results for the string were based on the analysis in [4]. For the string, the result in [7] was that the BRS operator is:

$$\delta = \delta_{\mathrm{fields}} + \delta_{\mathrm{string}} \tag{17}$$

and

$$\delta_{\text{fields}} = \int d^D x \left\{ D^{ab}_\mu \omega^b \frac{\delta}{\delta A^a_\mu(x)} - \frac{1}{2} f^{abc} \omega^b \omega^c \frac{\delta}{\delta \omega^a(x)} \right.$$
$$+ [\partial_{[\mu} \Lambda_{\nu]} + n \partial_{[\mu} \omega^a A^a_{\nu]}] \frac{\delta}{\delta B_{\mu\nu}(x)} + [-\partial_\mu B + n \omega^a \partial_\mu \omega^a] \frac{\delta}{\delta \Lambda_\mu(x)}$$
$$\left. + \frac{1}{6} n f^{abc} \omega^a \omega^b \omega^c \frac{\delta}{\delta B} \right\} \tag{18}$$

$$\delta_{\text{string}} = \prod_{\sigma' \nu, m} \int dX^\nu(\sigma') dy^m(\sigma') \left\{ \left( \int d\sigma \left[ -\omega^a T_a(\sigma) + \Lambda_\mu \frac{dX^\mu}{d\sigma} \right] \Phi \right) \frac{\delta}{\delta \Phi} \right\} \tag{19}$$

Here $\Phi$ is the string field, which is to be considered an arbitrary functional of the functions $X^\mu(\sigma)$ and $y^m(\sigma)$ in the same sense that in field theory the scalar fields $\phi^i$ are arbitrary functions of $x$. $X^\mu(\sigma)$ and $y^m(\sigma)$ are to be considered a set of D and Dim(G) arbitrary functions of the string parameter $\sigma$, respectively. The functional $\Phi$ has no explicit $\sigma$ dependence since it is integrated over many copies of the $\sigma$ variables–one for each variable $X$ or $y$ in it. One could write it in the form (ignoring the $y$ variables):

$$\Phi[X] = \sum_n \prod_{i=1}^n \int d\sigma_i \Phi_{\mu_1 \cdots \mu_n}(\sigma_1 \cdots \sigma_n) X^{\mu_1}(\sigma_1) \cdots X^{\mu_n}(\sigma_n). \tag{20}$$

Thus the integrated $\Lambda$ term above is simply a factor which multiplies $\Phi$, whereas the $T^a$ term above is a functional derivative operator which operates on the $y$ variables in $\Phi$.

For the covariant derivative on the string functional $\Phi$, we want an operator that commutes with the above $\delta$. A form which satisfies this equation for the string is given by:

$$\mathcal{D}_{\text{string}} = \prod_{\sigma' \rho, m} \int dX^{\rho}(\sigma') dy^{m}(\sigma') \left[ \int d\sigma V^{\mu}[X(\sigma)] \mathcal{D}_{\mu}(\sigma) \Phi \right] \frac{\delta}{\delta \Phi}$$

$$= \prod_{\sigma' \rho, m} \int dX^{\rho}(\sigma') dy^{m}(\sigma')$$

$$\left( \int d\sigma V^{\nu}[X(\sigma)] \left[ \frac{\delta}{\delta X^{\nu}(\sigma)} + A^{a}_{\nu} T_{a}(\sigma) + [n A^{a}_{\nu} A^{a}_{\lambda} - 2 B_{\nu\lambda}] \frac{dX^{\lambda}}{d\sigma} \right] \Phi \right) \frac{\delta}{\delta \Phi} \quad (21)$$

Here $V^{\nu}[X(\sigma)]$ can be an arbitrary function of $X(\sigma)$ and it could have more indices if desired. It cannot be a function of $y(\sigma)$ however. This operator $\mathcal{D}_{\mu\,\text{string}}(\sigma)$ was given (in a different form) in [4], where it was deduced both from the requirement that $[\delta, \mathcal{D}_{\mu\,\text{string}}(\sigma)] = 0$ and from the heterotic Green-Schwarz superstring action using canonical quantization as in (1) above. Using the Kac-Moody algebra obeyed by the $T^{a}$ operators:

$$[T^{a}(\sigma), T^{b}(\sigma')] = f^{abc} T^{c}(\sigma) \delta(\sigma - \sigma') + 2n \delta^{ab} \frac{d}{d\sigma} \delta(\sigma - \sigma'), \quad (22)$$

it is straightforward to show that:

$$[\delta, \mathcal{D}_{\mu}(\sigma)] = [\delta_{\text{string}}, \mathcal{D}_{\mu\,\text{string}}(\sigma)] + [\delta_{\text{fields}}, \mathcal{D}_{\mu\,\text{string}}(\sigma)] = 0 \quad (23)$$

# 4 BRS operators for p-branes

In [7], the BRS operator for the 3-brane was discussed in detail. Here form notation will be used, since it is more convenient. The operator $\delta$ should however be thought of as a functional derivative operator as in (18) and (19). The generalization of the results in [7] to the case of arbitrary p is:

$$\delta \Phi = \int \left[ -\omega^{a} T^{a}(\sigma) d^{p}\sigma + n C^{1}_{p} + \Lambda^{1}_{p} \right] \Phi \quad (24)$$

$$\delta A^a = -d\omega^a - f^{abc} A^b \omega^c \tag{25}$$

$$\delta \omega^a = -\frac{1}{2} f^{abc} \omega^b \omega^c \tag{26}$$

$$\delta B^0_{p+1} = n I^1_{p+1} + d\Lambda^1_p \tag{27}$$

$$\delta \Lambda^1_p = n I^2_p + dB^2_{p-1} \tag{28}$$

$$\cdots$$

$$\delta B^{p+1}_0 = n I^{p+2}_0 \tag{29}$$

Here $T^a$ generates Yang-Mills transformations on $K^a_i[y(\sigma)]$:

$$\delta K^a(\sigma') = \left[ \int \omega^b(\sigma) T^b(\sigma) d^p\sigma, K^a(\sigma') \right] = -d\omega^a(\sigma') - f^{abc} K^b(\sigma') \omega^c(\sigma') \tag{30}$$

Here it is assumed that the expressions depend only on the parameters indicated below:

$$C^1_p(\sigma) = C^1_p\Big( A[X(\sigma)], \omega[X(\sigma)], K[y(\sigma)] \Big) \tag{31}$$

$$I^i_{p+2-i}[A,\omega](\sigma) = I^i_{p+2-i}\Big( A[X(\sigma)], \omega[X(\sigma)] \Big) \tag{32}$$

$$I^i_{p+2-i}[K,\omega](\sigma) = I^i_{p+2-i}\Big( K[y(\sigma)], \omega[X(\sigma)] \Big) \tag{33}$$

$$T^a(\sigma) = T^a[y(\sigma), \frac{\delta}{\delta y(\sigma)}] \tag{34}$$

Note that on the right hand sides of equations (24-29), only $T^a(\sigma)$ is an operator–the rest of the quantities are just numbers, once field values are assigned. The expressions $I^i_{p+2-i}$ appear in the descent equations of Yang-Mills theory:

$$\delta I^i_{p+2-i} = dI^{i+1}_{p+1-i} \quad i = 0, 2 \cdots p+1. \tag{35}$$

$$\delta I_0^{p+2} = 0 \tag{36}$$

Following the analysis in [7], we will assume that the $y$-dependent operators $T^a$ satisfy the following algebra:

$$\int d^p\sigma \omega^a(\sigma) T^a(\sigma) \int d^p\sigma' \omega^b(\sigma') T^b(\sigma') =$$
$$\int \left[\frac{1}{2} f^{abc} \omega^a(\sigma) \omega^b(\sigma) T^c(\sigma) d^p\sigma + n I_p^2(K,\omega)\right] \tag{37}$$

Note that here $I_p^2$ is a function of $K$ and not $A$, in accord with the fact that $T^a$ depends only on the $y$ variables, as does $K$. In a Hamiltonian treatment, these variables $K$ should arise from the action given in [5], and can identified with the spatial parts of the $K_i^a$ defined there. The above operator (24-29) satisfies $\delta^2 = 0$ if:

$$\delta C_p^1(A,K,\omega) = I_p^2(A,\omega) - I_p^2(K,\omega) + dC_{p-1}^2(A,K,\omega) \tag{38}$$

where the argument $\sigma$ is understood everywhere. Here $C_{p-1}^2(A,K,\omega)$ is a new function which arises because nilpotence requires only that the integral of this expression be zero.

But equation (38) is just a descent from the equation that was solved in [5]. The general form of these descent equations is:

$$\delta C_{p+2-i}^{i-1}(A,K,\omega) = I_{p+2-i}^i(K,\omega)$$
$$- I_{p+2-i}^i(A,\omega) + dC_{p+1-i}^i(A,K,\omega); \quad (i \geq 1) \tag{39}$$

In the above, the operator $\delta$ performs Yang-Mills transformations on $A$ and $K$ and can be written in the form of equations (75–77) below. In [5], a formula was given for $C^0_{p+1}(A,K)$, which satisfies this equation for $i=1$. In that paper, we ignored the term $C^1_p(A,K,\omega)$ since we were only interested in the action there. Evidently one can obtain $C^1_p$ from the results in [5]. Note that in (39), the expression $dC^i_{p+1-i}(A,K,\omega)$ generally will not be zero because acting on (39) with $\delta$ gives:

$$0 = dI^{i+1}_{p+1-i}(K,\omega) - dI^{i+1}_{p+1-i}(A,\omega) - d\delta C^i_{p+1-i}(A,K,\omega);\ \ (i \geq 1) \tag{40}$$

and since the first two terms are not zero, and do not cancel in general, the third term is not zero. When the descent arrives at terms which are not dependent on the gauge fields, but only on the ghost $\omega$, the terms will become zero.

Better methods, probably using the 'Russian Formula'[9], to find $C^1_p(A,K,\omega)$ are doubtless available, but this paper is concerned only with the existence of $C^1_p$, which follows from the above.

The two simplest examples of $C^1_p$ are, for the string,

$$C^1_1 = 0 \tag{41}$$

and for the 3-brane (see [7]),

$$C^1_3 = d^{abc}K^aA^bd\omega^c \tag{42}$$

It is easy to verify equation (38) for these two cases, using

$$I^2_3(A,\omega) = d^{abc}A^ad\omega^bd\omega^c \tag{43}$$

To go beyond the 3-brane requires the general form of $I_p^2(K,\omega)$ for p odd. It will be shown below that this can be written in the form:

$$\int_\sigma I_p^2(K,\omega) = \int_\sigma \omega^a d\omega^b M_{p-1}^{(ab)}[K] \tag{44}$$

For $p=1$ we have

$$M_0^{ab} = \delta^{ab}. \tag{45}$$

and for $p \geq 3$:

$$M_{p-1}^{ab} = dN_{p-2}^{ab} \tag{46}$$

So it is true for all odd $p$ that:

$$dM_{p-1}^{ab} = 0 \tag{47}$$

This can be seen from the following expressions, which, for $p \geq 3$, were derived in [8]:

$$I_1^2 = \omega^a d\omega^a \tag{48}$$

$$\begin{aligned} I_{p+2-i}^i(K,\omega) = {} & \frac{(p+3)(p+1)\cdots(p+3-2i)}{2^{i+1}i!} \\ & \int_0^1 dt(1-t)^i P[(d\omega)^i, K, (tdK+t^2K^2)^{\frac{p+1-2i}{2}}] \\ = {} & d\omega^{a_1}\cdots d\omega^{a_i} N^{a_1\cdots a_i}[K];\ (p \text{ odd});\ (p\geq 3);\ (i \leq \frac{p+1}{2}) \end{aligned} \tag{49}$$

where $N^{a_1\cdots a_i}[K]$ is defined by the above and the notation $P$ is defined by:

$$P[H_1, H_2, (H_3)^n] \equiv d^{(aba_1\cdots a_n)} H_1^a H_2^b H_3^{a_1}\cdots H_3^{a_n} \tag{50}$$

for any Lie-algebra valued forms $H_j$. Here

$$(K^2)^a \equiv \frac{1}{2} f^{abc} K^b K^c \tag{51}$$

Removing the $\omega$ fields by functional differentiation and replacing $N$ by its dual yields the form:

$$\left[T^a(\sigma), T^b(\sigma')\right] = f^{abc}T^c(\sigma)\delta^p(\sigma-\sigma') + n\partial_i\tilde{N}^{(ab)[ij]}(\sigma)\frac{\partial}{\partial\sigma'^j}\delta^p(\sigma-\sigma') \quad (52)$$

where

$$N_{p-2}^{(ab)} = N_{i_1\ldots i_{p-2}}^{(ab)}d\sigma^{i_1}\cdots d\sigma^{i_{p-2}} \quad (53)$$

$$\tilde{N}^{(ab)[i_1i_2]}(\sigma) = \epsilon^{i_1\ldots i_p}N_{i_3\ldots i_p}^{(ab)}(\sigma) \quad (54)$$

for $p \geq 3$. This is the general form of the Mickelsson-Faddeev algebra for arbitrary odd p.

## 4.1 Covariant Derivatives for p-branes

Now we want to generalize the covariant derivative from the string case to the case of the p-brane for general odd p. We continue to think of the operator in the form (21), but shall use a simpler notation by applying the operator to the p-brane functional $\Phi$ and removing the integration over $\sigma$.

The following is the canonical form:

$$\mathcal{D}_\mu(\sigma)\Phi = \left\{\frac{\delta}{\delta X^\mu(\sigma)} + A_\mu^a(\sigma)T_a(\sigma) + nJ_{\mu p}^0(\sigma) - (p+1)B_{\mu p}(\sigma)\right\}\Phi \quad (55)$$

where the new function $J$ is a function of the fields:

$$J_{\mu p}^0(\sigma) = J_{\mu p}^0\left[A[X(\sigma)], K[y(\sigma)]\right] \quad (56)$$

and has the form:

$$J_{\mu p}^0 = J_{\mu\mu_1\cdots\mu_p}^0\Pi^{\mu_1\cdots\mu_p} \quad (57)$$

as does $B_{\mu p}$:

$$B_{\mu p} = B_{\mu\mu_1\cdots\mu_p}\Pi^{\mu_1\cdots\mu_p} \tag{58}$$

Covariance of this covariant derivative requires that it satisfy

$$[\delta, \mathcal{D}_\mu(\sigma)] = 0 \tag{59}$$

which implies that:

$$\begin{aligned}\left[\delta, J^0_{\mu p}[A,K](\sigma)\right] &= -(p+1)I^1_{\mu p}[A,\omega](\sigma)\\ -\left[\left(\frac{\delta}{\delta X^\mu(\sigma)} + A^a_\mu(\sigma)T_a(\sigma)\right), \int_{\sigma'} C^1_p(\sigma')\right]&\\ + A^a_\mu(\sigma)d\omega^b(\sigma)M^{(ab)}_{p-1}(\sigma) &\equiv J^1_{\mu p}[A,K,\omega](\sigma)\end{aligned} \tag{60}$$

where we have used equations (24,44,55) and have defined a new quantity $J^1_{\mu p}[A,K,\omega](\sigma)$. This is an equation which determines $J^0_{\mu p}[A,K]$ in terms of $I^1_p[A,\omega]$, $C^1_p[A,K,\omega]$ and $M^{(ab)}_{p-1}[K]$. Why should a $J^0_{\mu p}[A,K]$ exist which satisfies this equation? Note that (59) implies the relation:

$$\left[\delta, J^1_{\mu p}[A,K,\omega]\right] = 0 \tag{61}$$

It is easy to verify that the $\Lambda$ dependence disappears from $J^1_{\mu p}$, so that indeed $J^1_{\mu p}$ depends only on the variables $A$, $K$ and $\omega$. Then it is also evident that in equation (61), only the transformations of $A$, $K$ and $\omega$ are relevant. These transformations are summarized and used in the equations (75-77) in the Appendix below. However note that $J^1_{\mu p}[A,K,\omega]$ actually arises from $\delta$ transformations including (27) and (24), so that it is not at all obvious that a $J^0_{\mu p}[A, K]$ which is only a function of $A$, $K$ and $\omega$ and which satisfies (60) should exist–all we know so far is that some function of a larger set of variables with a larger transformation rule exists which gives rise to $J^1_{\mu p}[A, K, \omega]$.

In appendix A it is shown that the local unintegrated cohomology of the BRS operator restricted to the fields $A, K, \omega$ is trivial in the ghost charge one sector. It then follows that $J^0_{\mu p}$ exists with the desired properties.

It is instructive to see how this works for the string:

$$M_1^{(ab)} = \delta^{ab} \tag{62}$$

We already know from (21) the form of $J^0_{\mu 1}$ for the string:

$$J^0_{\mu 1} = A^a_\mu A^a_\nu \frac{\partial X^\nu}{\partial \sigma} \tag{63}$$

and it is easy to verify that (60) holds for this case. The form of $J^0_{\mu 3}$ for the 3-brane is much more complicated, and it will not be evaluated explicitly here. It is a form of the general structure:

$$J_{\mu 3} = c_1 A^a_\mu dA^b A^c d^{abc} + c_2 A^a_\mu dK^b A^c d^{abc} + \cdots \tag{64}$$

invoving $A$, $K$, $d$ and invariant tensors. Here $c_i$ are numerical coefficients and there are about a dozen of them. There are at least four ways to construct the polynomials $J$ and $C$ discussed here:

1. Brute force–choose the most general possible solution with arbitrary coefficients and then fix the coefficients by algebra using (38) and (60).

2. Cartan Homotopy operators-find a solution to the homotopy equations as outlined in [9] and [8], as was done in [5], and then integrate over t.

3. Construct the covariant derivative using the Hamiltonian methods given in [4] with the Lagrangian given in [5].

4. Fock space homotopy–use the methods of[10].

All of these are quite labor intensive. For many purposes it is probably sufficient to know that the expressions exist and satisfy the relevant equations–frequently a long expression conveys less useful information than the knowledge that it satisfies certain defining equations. If actual construction is needed, the second method is probably the best, and usually the answer should probably be left in integral form like (49), since that form preserves much of the information.

# 5 Commutators of the Covariant Derivatives

We know that

$$\mathcal{D}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] \tag{65}$$

commutes with the BRS operator, and therefore should represent some sort of curvature. We find for the string, setting $V^\mu(\sigma)$ in (21) equal to $\delta^\mu_\nu$, that:

$$\mathcal{D}_{\mu\nu\,\text{string}} = \prod_{\sigma'\nu,m} \int dX^\nu(\sigma') dy^m(\sigma')$$

$$\left\{ \left( \int d\sigma \left[ F^a_{\mu\nu} T_a(\sigma) - 6[Q[A]_{\mu\nu\lambda} + \partial_{[\mu} B_{\nu\lambda]}] \frac{dX^\lambda}{d\sigma} \right] \Phi \right) \frac{\delta}{\delta\Phi} \right\} \tag{66}$$

where $Q[A]_{\mu\nu\lambda}$ is some unfamiliar function of $A$. This is not what one might naively expect, since it does not involve the invariant

$$H_3^0 = dB_2^0 + I_3^0 \tag{67}$$

However we can add a term to the covariant derivative:

$$\mathcal{D}_{\mu\ \mathrm{new}} = \mathcal{D}_{\mu\ \mathrm{old}} - \frac{1}{2}\int\ d^D x\ \left\{ F^a_{\mu\nu}\frac{\delta}{\delta A^a_\nu} \right\} \tag{68}$$

as was done in equation (14). Here by $\mathcal{D}_{\mu\ \mathrm{old}}$ we mean (55). Using (68) we do get a result close to the naively expected one:

$$\mathcal{D}_{\mu\nu\ \mathrm{new}}\Phi = -6\int d\sigma H^0_{\mu\nu\lambda}\frac{dX^\lambda}{d\sigma}\Phi \tag{69}$$

Since the central extended $T^a$ term is no longer present here, a zero commutator with $\delta$ demands that the familiar invariant (67) must appear.

The same trick will work also for the p-branes, but there the invariants obtained by commutation may also depend on $K$, since $\mathcal{D}_\mu$ depends on $K$.

# 6 Conclusion

This paper has introduced two new polynomial functions of $A$ and $K$ that appear in general when one couples p-branes (with odd p) to Yang-Mills theory. The polynomial $C^1_p[A,\omega,K]$ is needed to define the nilpotent BRS operator $\delta$ for the general p-brane. It is simply related (by descent equations) to the polynomial $C^0_{p+1}[A,K]$ that appears in the WZW part of the p-brane action in [5]. The polynomial $J^0_{\mu p}[A,K]$ is needed in the covariant derivative

$\mathcal{D}_\mu(\sigma)$ to ensure that $[\delta, \mathcal{D}_\mu(\sigma)] = 0$, and a proof that it can always be constructed has been given for all odd p.

In summary, for all odd $p \geq 1$, the BRS transformation of the p-brane field $\Phi$ takes the form

$$\delta\Phi = \int \left[ -\omega^a T^a(\sigma) d^p\sigma + n C_p^1[A, K](\sigma) + \Lambda_p^1(\sigma) \right] \Phi \tag{70}$$

where the p-form $C_p^1(A, K, \omega)$ satisfies the relation:

$$\delta C_p^1(A, K, \omega) = I_p^2(K, \omega) - I_p^2(A, \omega) + dC_{p-1}^2(A, K, \omega) \tag{71}$$

The general Mickelsson-Faddeev algebra has been discussed. It is determined by the ghost charge two term in the descent equations:

$$\int_\sigma I_p^2(K, \omega)(\sigma) = \int_\sigma \left( \omega^a d\omega^b M_{p-1}^{(ab)}[K] \right) = \int_\sigma \left( d\omega^a d\omega^b N_{p-2}^{(ab)}[K] \right) \tag{72}$$

where the latter ($N$) form is valid for $p \geq 3$. Then the covariant derivative takes the form:

$$\mathcal{D}_\mu(\sigma)\Phi = \left\{ \frac{\delta}{\delta X^\mu(\sigma)} + A_\mu^a(\sigma) T_a(\sigma) + n J_{\mu p}^0(\sigma) - (p+1) B_{\mu p}(\sigma) \right\} \Phi \tag{73}$$

where the function $J_{\mu p}^0$ is determined by:

$$[\delta, J_{\mu p}^0(\sigma)] = J_{\mu p}^1(\sigma) \tag{74}$$

with $J_{\mu p}^1$ a calculable function determined by $C_p^1(A, K, \omega)$ and $I_p^2(K, \omega)(\sigma)$.

In conclusion, the bosonic part of the loop space BRS operator and the loop space covariant derivative of the string generalize to the p-branes in a fairly simple way, as did the bosonic part of the action coupled to background Yang-Mills fields. It remains to be seen whether fermions with heterosis and $\kappa$ symmetry can also be incorporated.

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# 7 Appendix A: Construction of $J_p^0$.

**Theorem**

*The local unintegrated BRS cohomology of the nilpotent BRS operator defined by:*

$$\delta A_\mu^a = D(A)_\mu^{ab}\omega^b = \partial_\mu \omega^a + f^{abc} A_\mu^b \omega^c \tag{75}$$

$$\delta K_i^a(y) = D(K)_i^{ab}\omega^b = \partial_i X^\mu \partial_\mu \omega^a + f^{abc} K_i^b \omega^c \tag{76}$$

$$\delta \omega^a(x) = -\frac{1}{2} f^{abc} \omega^b \omega^c \tag{77}$$

*consists of sums of products of functions of the form*

$$I[\partial_\rho, A_\mu^a, K_i^b, \partial_j X^\nu] T^g(\omega), \tag{78}$$

*where the terms are separately invariant:*

$$\delta I[\partial_\rho, A_\mu^a, K_i^b, \partial_j X^\nu] = 0 \tag{79}$$

$$\delta T^g(\omega) = 0 \tag{80}$$

Here the superscript $g$ is the ghost number of the corresponding expression. In other words all solutions of

$$\delta P^g = 0 \tag{81}$$

can be expressed in the form

$$P^g = \delta P^{g-1}$$

$$+ I[\partial_\rho, A^a_\mu, K^b_i, \partial_j X^\nu] T^g(\omega), \tag{82}$$

where $P^g$ are unintegrated local polynomials that depend only on the fields $A, K, \omega$ and the derivative operator. Now for a semisimple group, the polynomials $T^g(\omega)$ exist only for $g = 3, 5, 7...$ and not for $g = 1$, which is all we need for present purposes. In particular, since there are (for semisimple groups) no $T^g$ for $g = 1$, it follows that every polynomial $J^1$ of ghost charge one which satisfies:

$$\delta J^1[A^a_\mu, K^a_i] = 0 \tag{83}$$

can be written in the form:

$$J^1 = \delta J^0[A^a_\mu, K^a_i] \tag{84}$$

where $J^0[A^a_\mu, K^a_i]$ is another local polynomial. This is the result used in the text to show that a polynomial $J^0_{\mu p}$ satisfying (60) exists for all odd p.

**Sketch of Proof:**

This result is easily proved along the lines given in [10] using the following field redefinitions:

$$\phi^a_i = \Pi^\mu_i A^a_\mu - K^a_i \tag{85}$$

which give rise to transformations like (7) in the introduction. Note that since here we do not have an integration over spacetime, we do not need to consider the full operator with an exterior derivative term $\delta + d$ here but only the part $\delta$. We then use a spectral sequence based on the counting operator:

$$N = N(A) + N(\omega) + N(\phi) \tag{86}$$

This means that

$$\delta_0 = \int d^D x \partial_\mu \omega^a \frac{\delta}{\delta A^a_\mu} \tag{87}$$

So the space $E_1$ is of the form:

$$E_1 = E_1[A', \phi, \omega] \tag{88}$$

where it is understood that only undifferentiated $\omega$ appears in this expression. Next

$$d_2 = \Pi_1 \omega^a \left[ T^a - \frac{1}{2} Y^a \right] \Pi_1 \tag{89}$$

Then the theorem is a direct consequence of the arguments in [10], and the demonstration is in fact much easier here than in that case because there is no exterior derivative here to give rise to exceptional vectors which require the consideration of $d_r$ for $r \geq 2$. The spectral sequence collapses at $E_2 = \ker \Delta_1$ and the result is as stated.

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