

## The Complete Spectrum of the $W_N$ String

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### ABSTRACT

We obtain the complete physical spectrum of the  $W_N$  string, for arbitrary  $N$ . The  $W_N$  constraints freeze  $N - 2$  coordinates, while the remaining coordinates appear in the currents only *via* their energy-momentum tensor. The spectrum is then effectively described by a set of ordinary Virasoro-like string theories, but with a non-critical value for the central charge and a discrete set of non-standard values for the spin-2 intercepts. In particular, the physical spectrum of the  $W_N$  string includes the usual massless states of the Virasoro string. By looking at the norms of low-lying states, we find strong indications that all the  $W_N$  strings are unitary.

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## 1. Introduction

String theory is two-dimensional gravity coupled to a critical matter system that includes free scalar fields which are interpreted as coordinates on the target spacetime. The gauge-fixed string action has a Virasoro symmetry, which provides a powerful organising principle for describing the physical spectrum and interactions of the theory. The Virasoro algebra is the simplest example of a more general class of infinite-dimensional symmetry algebras, known generically as  $W$  algebras, which can be characterised by the fact that they contain higher-spin currents as well as the spin-2 energy-momentum tensor  $T(z)$  which generates the Virasoro subalgebra. One may then construct generalisations of two-dimensional gravity, by gauging matter systems with  $W$  symmetries. If the matter systems are critical, and include free scalars, then one obtains  $W$ -string theories. Owing to the non-linearity of  $W$  algebras, the construction of these  $W$ -string theories seems to be much more complicated than that of the usual Virasoro string. Some progress has been made recently in constructing specific examples, principally for the case of  $W_3$  [1,2,3].

$W_3$  is the first example in the infinite sequence of non-linear  $W_N$  algebras, which are generated by primary currents of spins  $3, \dots, N$ , together with the energy-momentum tensor  $T(z)$ . In this paper, we study the general features of  $W_N$  strings for arbitrary  $N$ . Realisations of  $W_N$  in terms of  $(N - 1)$  free scalars  $\varphi_2, \dots, \varphi_N$  can be obtained from the Miura transformation for  $su(N)$  [4]. These realisations can be generalised by observing that the scalar  $\varphi_2$  enters the currents only *via* its energy-momentum tensor (with background charge), which may then be replaced by an arbitrary energy-momentum tensor with the same central charge [5,2]. If the new energy-momentum tensor is chosen to comprise  $D$  free scalar fields  $X^\mu$  together with the original scalar  $\varphi_2$ , then these  $(D+1)$  scalars, together with  $\varphi_3, \dots, \varphi_N$ , form the coordinates on the target spacetime. The  $W_N$  constraints, however, “freeze” the momentum components of  $\varphi_3, \dots, \varphi_N$  for physical states to certain specific values. This implies that only the scalars  $\varphi_2$  and  $X^\mu$  are physically-observable coordinates, thus describing a  $(D+1)$ -dimensional spacetime. Furthermore, as we shall show, higher-level physical states in the theory can only involve excitations in the (unfrozen)  $(\varphi_2, X^\mu)$  directions. This is because states with excitations in the (frozen)  $(\varphi_3, \dots, \varphi_N)$  directions turn out to have momentum components in these directions that are incompatible with momentum conservation, and thus all such states have zero norm and are to be set equal to zero.

Because the physically-observable coordinates  $\varphi_2$  and  $X^\mu$  enter the  $W_N$  currents only through their energy-momentum tensor  $T^{\text{eff}}$ , the above considerations imply that the  $W_N$ -string theories closely resemble ordinary string theory. The central charge  $c^{\text{eff}}$  of  $T^{\text{eff}}$  is related to the total central charge  $c_N$  of the  $W_N$  realisation. Requiring that  $c_N$  take the critical value

$$c_N^* = 2(N - 1)(2N^2 + 2N + 1) \tag{1.1}$$

implies that  $c^{\text{eff}}$  for  $W_N$  should take the value

$$c^{\text{eff}} = 25 + \frac{6}{N(N+1)}. \quad (1.2)$$

This suggests that there may be a connection with a Virasoro minimal model [2], in the sense that  $c^{\text{eff}}$  may be rewritten as

$$c^{\text{eff}} = 26 - \left(1 - \frac{6}{N(N+1)}\right), \quad (1.3)$$

where 26 is the critical central charge of the usual Virasoro string and the term between brackets is precisely the central charge of the unitary  $(N, N+1)$  Virasoro minimal model.

In this paper, we show how the tachyonic physical states of the  $W_N$  string may be classified and generated by the action of the Weyl group of  $su(N)$ . This enables us to explain some findings of [2], where it was noticed in the special cases of  $W_3$  and  $W_4$  that certain ‘‘diagonal’’ states in the Kac tables of the corresponding minimal models arose at this tachyonic level. Our results for the tachyonic states apply to any  $W_N$ -string theory, revealing a connection not only with the diagonal states of the  $(N, N+1)$  Virasoro minimal model, but also with the diagonal states of certain  $W_M$  minimal models, for all  $M < N$ . We also find a connection between higher-level states of the  $W_N$  string and certain ‘‘off-diagonal’’ entries in the Kac table of the relevant minimal model. However, these particular higher-level states involve excitations in the frozen directions  $(\varphi_3, \dots, \varphi_N)$ , and, as we mentioned earlier, they therefore have zero norm.

Because the higher-level states with excitations in these frozen directions have zero norm, it follows that the only physical states in the spectrum of the  $W_N$  string are the tachyons described above, and those higher-level states that involve excitations *only* in the unfrozen directions. The consequence of this is that the spectrum of physical states of the  $W_N$  string is essentially given by the spectrum for Virasoro strings with effective central charge (1.2), and a discrete set of effective intercepts  $L_0^{\text{eff}}$ . We shall show that these values of  $L_0^{\text{eff}}$  all satisfy the unitarity bounds arising from level-1 and level-2 physical states. These results provide a strong indication of the unitarity of the  $W_N$  string.

The paper is organised as follows. In the next section we review how scalar realisations of  $W_N$  may be obtained by using the Miura transformation of  $su(N)$ , and we derive an explicit formula for  $W_N$  currents in terms of  $W_{N-1}$  currents and one extra scalar field (called  $\varphi_N$  in our notation). Using these results, we prove the relation between the central charges of the  $W_N$  and  $W_{N-1}$  algebras that was conjectured in [5], and which leads to the relation between (1.1) and (1.2). In section 3, we give the critical central charge for the  $W_N$  string, and determine the physical-state conditions. In particular, this involves calculating the higher-spin intercepts, which, *a priori*, one does not know without detailed knowledge of the BRST operator for the  $W_N$  gauge theory. In [2], it was proposed that a particular

tachyonic operator called the “cosmological-constant operator,” for which the momentum is a certain multiple of the background-charge vector of the  $(N - 1)$ -scalar Miura realisation, is always a physical operator. This “cosmological solution” enables one to determine the values of the higher-spin intercepts, for which we give general formulae. In section 4, we prove that the solutions to the physical-state conditions for the tachyons of the  $W_N$  string are generated by the action of the Weyl group of  $su(N)$  on the cosmological solution. We give a general proof for arbitrary  $W_N$  that this spectrum of states is associated with the “diagonal” highest-weight states for the  $(N, N + 1)$  Virasoro minimal model. In section 5, we consider higher-level physical states, and obtain additional “off-diagonal” entries of the Kac table. We also discuss the no-ghost theorem for  $W_N$  strings, and argue in particular that the higher-level states with excitations in the frozen directions have zero norm (these include all those associated with the off-diagonal entries of the Kac table). The remaining physical states have non-negative norm. In section 6, we return to the assumption of the existence of the “cosmological-constant operator” as a physical operator. We give strong arguments supporting this assumption, by demonstrating that if the values of the higher-spin intercepts are different from those determined by this particular physical operator, then one gets a non-unitary theory. Finally, in section 7, we present our conclusions and discuss some open problems.

## 2. The Miura Transformation and the $W_N \rightarrow W_{N-1}$ Reduction.

A realisation of the  $W_N$  algebra in terms of  $(N - 1)$  free scalars  $\vec{\varphi}^{(N)} \equiv (\varphi_2, \dots, \varphi_N)$  is given by the Miura transformation for  $su(N)$  [4]

$$\prod_{k=1}^N \left( \alpha_0 \partial + \vec{h}_k^{(N)} \cdot (\partial \vec{\varphi}^{(N)}) \right) = \sum_{\ell=0}^N W_\ell^{(N)} (\alpha_0 \partial)^{N-\ell}, \quad (2.1)$$

where the  $\vec{h}_k^{(N)}$  are  $(N - 1)$ -component vectors satisfying

$$\begin{aligned} \vec{h}_i^{(N)} \cdot \vec{h}_j^{(N)} &= \delta_{ij} - \frac{1}{N}, \\ \sum_{i=1}^N \vec{h}_i^{(N)} &= 0. \end{aligned} \quad (2.2)$$

It follows immediately from (2.1) that  $W_0^{(N)} = 1$  and  $W_1^{(N)} = 0$ . The quantities  $W_\ell^{(N)}$  with  $2 \leq \ell \leq N$  are spin- $\ell$  currents that generate the  $W_N$  algebra. They are not primary with respect to the energy-momentum tensor  $W_2^{(N)}$ , but can be made so for  $\ell \geq 3$  by adding derivatives and composites of lower-spin currents. Since this is not essential for our purposes, we shall not take the trouble to do so. By convention, we shall always order products such

as the one in (2.1) in decreasing order of  $k$ , *i.e.* the largest- $k$  factor sits at the left. Normal ordering of the quantum operators will always be understood. The fields  $\varphi_i$  satisfy the operator-product expansions

$$\varphi_i(z)\varphi_j(w) \sim -\delta_{ij} \log(z-w). \quad (2.3)$$

The  $\vec{h}_i^{(N)}$  for  $1 \leq i \leq N-1$  are the weights of the  $\mathbf{N}$  representation of  $su(N)$ , and  $\vec{h}_N^{(N)}$  is defined by the second equation in (2.2). The simple roots  $\vec{e}_i^{(N)}$  of  $su(N)$  are given in terms of these weights by

$$\vec{e}_i^{(N)} = \vec{h}_i^{(N)} - \vec{h}_{i+1}^{(N)}, \quad 1 \leq i \leq N-1. \quad (2.4)$$

For later purposes we also introduce the Weyl vector  $\vec{\rho}^{(N)}$ , given by

$$\vec{\rho}^{(N)} = \sum_{j=1}^{N-1} (N-j)\vec{h}_j^{(N)} = \frac{1}{2} \sum_{j=1}^{N-1} j(N-j)\vec{e}_j^{(N)}. \quad (2.5)$$

A convenient choice of representation for the weights  $\vec{h}_i^{(N)}$  is given by [2]

$$\begin{aligned} \vec{h}_1^{(N)} &= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \dots, \frac{1}{\sqrt{N(N-1)}} \right), \\ \vec{h}_p^{(N)} &= \left( \underbrace{0, \dots, 0}_{p-2}, -\frac{p-1}{\sqrt{p(p-1)}}, \frac{1}{\sqrt{p(p+1)}}, \dots, \frac{1}{\sqrt{N(N-1)}} \right), \end{aligned} \quad (2.6)$$

where  $p$  runs from 2 to  $N$ .

This choice of vectors  $\vec{h}_i^{(N)}$  for  $W_N$  has the nice property that

$$\vec{h}_i^{(N)} = \left( \vec{h}_i^{(N-1)}, \frac{1}{\sqrt{N(N-1)}} \right) \quad (2.7)$$

for  $1 \leq i \leq N-1$ . In other words, the first  $(N-2)$  components of the first  $(N-1)$  vectors are precisely the  $\vec{h}_i^{(N-1)}$  vectors for  $W_{N-1}$ . This enables one to re-express the  $W_N$  currents in terms of  $W_{N-1}$  currents together with the scalar field  $\varphi_N$  which is the last component of the  $\vec{\varphi}^{(N)}$  fields of the  $W_N$  realisation. To see this, we begin by writing the left-hand side of (2.1), using (2.7) and (2.6), as

$$\left( \alpha_0 \partial - (N-1)(\partial \phi_N) \right) \prod_{\ell=1}^{N-1} \left( \alpha_0 \partial + \vec{h}_\ell^{(N-1)} \cdot (\partial \vec{\varphi}^{(N-1)}) + (\partial \phi_N) \right), \quad (2.8)$$

where we have defined

$$\phi_N \equiv \frac{1}{\sqrt{N(N-1)}} \varphi_N. \quad (2.9)$$

The  $\prod_{\ell=1}^{N-1}$  factors in (2.8) may be rewritten as

$$e^{-\phi_N/\alpha_0} \prod_{\ell=1}^{N-1} \left( \alpha_0 \partial + \vec{h}_\ell^{(N-1)} \cdot (\partial \vec{\varphi}^{(N-1)}) \right) e^{\phi_N/\alpha_0}. \quad (2.10)$$

Using the Miura transformation (2.1) for  $W_{N-1}$ , this can be written, using Leibniz's rule, as

$$\sum_{m=0}^{N-1} \sum_{q=0}^{N-m-1} \binom{N-m-1}{q} W_m^{(N-1)} P_q(\phi_N) (\alpha_0 \partial)^{N-m-q-1}, \quad (2.11)$$

where we have defined  $P_q(\phi_N)$ , which is a differential polynomial in  $\partial\phi_N$ , by

$$P_q(\phi_N) \equiv e^{-\phi_N/\alpha_0} \left( (\alpha_0 \partial)^q e^{\phi_N/\alpha_0} \right). \quad (2.12)$$

Note that  $P_q(\phi_N)$  satisfies the recursion relation

$$P_q(\phi_N) = \alpha_0 \partial P_{q-1}(\phi_N) + \partial \phi_N P_{q-1}(\phi_N). \quad (2.13)$$

Rewriting the double sum in (2.11), substituting back into (2.8), and equating powers of  $(\alpha_0 \partial)$  in (2.1), we obtain the explicit relation

$$W_k^{(N)} = \sum_{q=0}^k \binom{N+q-k}{q} \left[ \frac{N-k}{N+q-k} W_{k-q}^{(N-1)} P_q(\phi_N) + \alpha_0 \partial \left( W_{k-q-1}^{(N-1)} P_q(\phi_N) \right) - (N-1) (\partial \phi_N) W_{k-q-1}^{(N-1)} P_q(\phi_N) \right]. \quad (2.14)$$

All products of operators are understood to be normal ordered with respect to the basic scalar fields  $\varphi_i$ . Currents  $W_m^{(N-1)}$  with  $m < 0$  are defined to be zero. (A formula equivalent to (2.10) was also derived in [2].)

Equation (2.14) gives a realisation of the  $W_N$  currents in terms of those for  $W_{N-1}$ , together with an additional scalar field  $\varphi_N$ . Applying this recursively leads to a realisation of the  $W_N$  algebra in terms of  $\varphi_2$ , which appears only *via* its energy-momentum tensor, and  $(N-2)$  additional scalar fields  $(\varphi_3, \dots, \varphi_N)$ . Since  $\varphi_2$  commutes with the other scalars, its energy-momentum tensor may be replaced by an arbitrary one that commutes with  $(\varphi_3, \dots, \varphi_N)$  and that has the same central charge.\*

Using equation (2.14) for  $k=2$ , we may relate the energy-momentum tensors for  $W_N$  and  $W_{N-1}$  as follows:

$$W_2^{(N)} = W_2^{(N-1)} - \frac{1}{2} (\partial \varphi_N)^2 + \frac{1}{2} \sqrt{N(N-1)} \alpha_0 \partial^2 \varphi_N. \quad (2.15)$$

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\* Note that this allows one in particular to realise  $W_N$  on any affine Lie algebra  $g$  with  $\text{rank}(g) \geq N-2$ , generalising the results of [6]. The energy-momentum tensor is then realised on the coset  $g/h$ , where  $h$  is an  $(N-2)$ -dimensional Abelian subalgebra of  $g$ , and the currents  $(\partial \varphi_3, \dots, \partial \varphi_N)$  are taken to be the generators of this subalgebra.

Applying this recursively, we obtain

$$\begin{aligned} W_2^{(N)} &= \sum_{j=2}^N \left( -\frac{1}{2}(\partial\varphi_j)^2 + \frac{1}{2}\sqrt{j(j-1)}\alpha_0\partial^2\varphi_j \right) \\ &= -\frac{1}{2}\partial\vec{\varphi}^{(N)} \cdot \partial\vec{\varphi}^{(N)} + \alpha_0\vec{\rho}^{(N)} \cdot \partial^2\vec{\varphi}^{(N)}, \end{aligned} \quad (2.16)$$

which therefore generates the Virasoro algebra with central charge

$$c_N = (N-1)\left(1 + N(N+1)\alpha_0^2\right), \quad (2.17)$$

in agreement with [4]. The recursion relation

$$c_N = -2 + \frac{N+1}{N-2}c_{N-1} \quad (2.18)$$

conjectured in [5] follows straightforwardly from (2.17), and the fact that the background-charge parameter  $\alpha_0$  for the  $W_{N-1}$  Miura transformation in (2.10) is identical to that for the  $W_N$  transformation (2.1).

The scalar  $\varphi_2$  appears in (2.16) (and indeed, as already mentioned, in all the currents in (2.14)) *via* its energy-momentum tensor

$$T(\varphi_2) = -\frac{1}{2}(\partial\varphi_2)^2 + \frac{1}{\sqrt{2}}\alpha_0\partial^2\varphi_2. \quad (2.19)$$

This generates the Virasoro algebra with central charge  $c = 1 + 6\alpha_0^2$ . The contribution from  $\varphi_2$  can thus be replaced by an arbitrary energy-momentum tensor with this central charge, *i.e.*

$$c = \frac{6c_N}{N(N^2-1)} + \frac{2(N-2)(N+3)}{N(N+1)} - \left(1 - \frac{6}{N(N+1)}\right). \quad (2.20)$$

### 3. Physical-state Conditions

We now turn to the consideration of the physical spectrum of the  $W_N$  string. In the remaining sections of the paper, when there is no possibility of confusion, we shall suppress the label  $N$  that we have been using to indicate that the quantities are associated with the  $W_N$  algebra.

Physical states  $|\text{phys}\rangle$  of the  $W_N$  string are defined by the conditions

$$(W_s)_m |\text{phys}\rangle = 0, \quad m \geq 1, \quad (3.1a)$$

$$(W_s)_0 |\text{phys}\rangle = \omega_s |\text{phys}\rangle, \quad (3.1b)$$

where the Laurent modes  $(W_s)_m$  of the spin- $s$  current  $W_s$  for the  $W_N$  algebra are defined by  $W_s(z) = \sum_m (W_s)_m z^{-m-s}$ . The constants  $\omega_s$  are the intercepts for the zero modes of the spin- $s$  currents. In principle, they can be determined by requiring that the nilpotent BRST operator for the algebra annihilate the physical vacuum (including ghosts). In practice, however, the construction of the BRST operator for the  $W_N$  algebra is very complicated, and has been given only for the case of  $N = 3$  [7]. We shall show later in this section how the intercepts may be determined by simpler methods. Note, incidentally, that it is sufficient to impose (3.1b) for all  $s$ , together with (3.1a) for  $s = 2$  and  $m = 1$  and  $m = 2$ , since the rest of the constraints in (3.1a) then follow from the commutation relations of the  $W_N$  algebra.

The requirement of nilpotency of the BRST operator determines the central charge of the  $W_N$ -string theory. Even though the  $W_N$  algebra is non-linear, the spin-2 current generates a linear subalgebra. Thus the total central charge, which must be zero for nilpotence, is simply the sum of those for the matter and ghost sectors. The ghosts for the spin- $s$  current contribute  $-2(6s^2 - 6s + 1)$  to the ghostly central charge, and so the critical central charge  $c_N^*$  for the matter sector is given by

$$\begin{aligned} c_N^* &= 2 \sum_{s=2}^N (6s^2 - 6s + 1) \\ &= 2(N-1)(2N^2 + 2N + 1). \end{aligned} \tag{3.2}$$

From (2.17), we see that the background-charge parameter  $\alpha_0$  is then given by its critical value  $\alpha_0^*$ , namely

$$(\alpha_0^*)^2 = \frac{(2N+1)^2}{N(N+1)}. \tag{3.3}$$

From now on, we shall always assume that the central charge and  $\alpha_0$  take their critical values.

The necessity of using BRST methods to determine the intercepts  $\omega_s$  in (3.1b) can be avoided if one knows a specific example of an operator that creates a physical state, since then one can simply act on it with  $(W_s)_0$  and read off the values of the intercepts. In [2], such an operator, called the ‘‘cosmological-constant operator’’ was proposed. Specifically, for  $W_N$ , it is a tachyonic operator of the form

$$V_{\text{cosmo}} = e^{\lambda \vec{\rho} \cdot \vec{\varphi}}, \tag{3.4}$$

where  $\vec{\rho}$  is the Weyl vector given in (2.5), and  $\lambda$  is a certain constant to be determined. For  $W_3$ , since one knows the values of the intercepts from the BRST construction in [7], one can explicitly verify that such a physical operator exists. In [2], it was argued from classical correspondence-principle considerations that such a physical operator should occur for all higher  $W_N$  algebras too. In section 6, we shall present a stronger argument that supports this proposal. For now, we shall proceed on the assumption that a physical operator of the

form (3.4) indeed exists. It then remains to determine the value of the constant  $\lambda$ . This can be done by using an independent argument that enables us to calculate the spin-2 intercept (see for example [2,8]). Since  $T^{\text{tot}} \equiv T^{\text{mat}} + T^{\text{ghost}}$  annihilates  $|\text{phys}\rangle \otimes |\text{vac}\rangle_{\text{ghost}}$ , and the spin-2 subalgebra of  $W_N$  is linear, we may read off the spin-2 intercept as the negative of the intercept for the spin-2 ghost current acting on  $|\text{vac}\rangle_{\text{ghost}}$ . In other words, the BRST charge is required, as usual, to annihilate  $|\text{phys}\rangle \otimes |\text{vac}\rangle_{\text{ghost}}$ . The ghost vacuum is defined by

$$|\text{vac}\rangle_{\text{ghost}} \equiv \prod_{s=2}^N \prod_{m=1}^{s-1} (c_s)_m |0\rangle, \quad (3.5)$$

where  $|0\rangle$  is the  $SL(2, C)$ -invariant vacuum, and  $(c_s)_m$  are the Laurent modes of the usual ghost field for the spin- $s$  current. Thus we find that the spin-2 intercept is given by [2]

$$\omega_2 = \sum_{s=2}^N \sum_{m=1}^{s-1} m = \frac{1}{6}N(N^2 - 1). \quad (3.6)$$

From (2.16) and (3.4) it follows that  $\lambda$  is given by

$$\lambda = \left(1 \pm \frac{1}{2N+1}\right) \alpha_0^*. \quad (3.7)$$

The two values for  $\lambda$  in (3.7) are related by a reflection symmetry, as we shall explain in section 4. Without loss of generality, we shall take the  $+$  sign in (3.7), and refer to the corresponding operator (3.4) as the ‘‘cosmological solution.’’

We are now in a position to compute the intercepts for the  $W_N$  string. To do this, we first compute the eigenvalues of the zero modes of the  $W_N$  currents acting on arbitrary tachyonic states, and then substitute the cosmological solution defined in (3.4) and (3.7) into these eigenvalues. Thus consider an arbitrary tachyonic operator

$$V_{\vec{\beta}} = e^{\vec{\beta} \cdot \vec{\varphi}}. \quad (3.8)$$

The highest-order pole of the operator-product expansion  $W_s(z)V_{\vec{\beta}}(w)$  is of order  $s$ , implying that the tachyonic state, obtained from (3.8), satisfies (3.1a). The eigenvalue of this state under the action of  $(W_s)_0$  can be read off from this highest-order pole. Since (3.8) satisfies

$$\partial\varphi_j(z)V_{\vec{\beta}}(0) \sim -\frac{\beta_j V_{\vec{\beta}}(0)}{z}, \quad (3.9)$$

this pole can be obtained simply by replacing  $\partial\vec{\varphi}$  by  $-\vec{\beta}/z$  in formula (2.14) [4]. Let us define the functions  $U_s^{(n)}(z)$  for  $2 \leq n \leq N$  and  $0 \leq s \leq n$  by  $U_0^{(n)}(z) = 1$ ,  $U_1^{(n)}(z) = 0$ ,

$U_{t<0}^{(n)}(z) = 0$  and the recursion relation

$$U_s^{(n)}(z) = \sum_{q=0}^s \binom{n+q-s}{q} \left[ \frac{n-s}{n+q-s} U_{s-q}^{(n-1)}(z) P_q(\zeta_n/z) \right. \\ \left. + \alpha_0 \partial \left( U_{s-q-1}^{(n-1)}(z) P_q(\zeta_n/z) \right) - (n-1) \frac{\zeta_n}{z} U_{s-q-1}^{(n-1)}(z) P_q(\zeta_n/z) \right], \quad (3.10)$$

where we have introduced

$$\zeta_n = -\frac{\beta_n}{\sqrt{n(n-1)}}. \quad (3.11)$$

The eigenvalues  $v_s(\vec{\beta})$  of  $V_{\vec{\beta}}$  under  $(W_s)_0$  for the  $W_N$  string are then given by

$$v_s(\vec{\beta}) = U_s^{(N)}(z) \Big|_{z=1}. \quad (3.12)$$

The intercepts  $\omega_s$  for the  $W_N$ -string theory can be obtained by substituting the cosmological solution (3.4), with  $\lambda$  given by (3.7), into (3.12) and (3.10), and solving the recursion relations. Even though we have not been able to find a closed-form expression for all the  $\omega_s$  as a function of  $N$  and  $s$ , we have found that the intercepts for any  $W_N$ -string theory for some low spins  $s$  can be written as

$$\begin{aligned} \omega_2 &= \frac{1}{6}(N+1)N(N-1), \\ \omega_3 &= -\frac{1}{6}(N+1)N(N-1)(N-2)\alpha_0^*, \\ \omega_4 &= \frac{1}{360}(N-1)(N-2)(N-3)(5N^3 + 228N^2 + 223N + 54), \\ \omega_5 &= -\frac{1}{180}(N-1)(N-2)(N-3)(N-4)(5N^3 + 108N^2 + 103N + 24)\alpha_0^*, \\ \omega_6 &= \frac{1}{45360N(N+1)}(N-1)(N-2)(N-3)(N-4)(N-5) \\ &\quad \times (35N^6 + 7238N^5 + 110728N^4 + 201646N^3 + 14378N^2 + 45666N + 5400). \end{aligned} \quad (3.13)$$

The expression for  $\omega_2$  is just (3.6). We shall show in section 6 how  $\omega_3$  may be calculated in general. For  $s \geq 4$ , we have arrived at the above expressions for  $\omega_s$  by polynomial fitting from specific results for small values of  $N$ ; we have then verified these formulae for all algebras up to and including  $W_{20}$ .

An important remark is in order here. The intercepts computed above are the ones for the  $W_N$  currents as defined by the Miura transformation (2.1). The intercepts for the primary  $W_N$  currents can be obtained from the ones presented here. As an example, the primary spin-3 current (for any  $N$ ) is given by  $W_3 - \frac{1}{2}(N-2)\alpha_0^*\partial W_2$  and hence has intercept  $\omega_3 + (N-2)\alpha_0^*\omega_2 = 0$ , which agrees with the known result for  $N = 3$  [7]. In general the intercepts for the primary currents are much more complicated than the ones above. The determination of the physical states is a basis-independent question and thus there is

no advantage in working in the more cumbersome “primary basis.” To avoid unnecessary complication we shall therefore continue to use the more natural “Miura basis.”

#### 4. The Physical Spectrum for Tachyonic States

Having determined the intercepts for the  $W_N$ -string theory in the previous section, we are now in a position to discuss the physical spectrum of this theory. In this section we shall only consider tachyonic states, relegating higher-level states to section 5.

##### 4.1 The Weyl Reflection Symmetry of Tachyonic States

The eigenvalues  $v_s(\vec{\beta})$  for the zero modes of  $W_s$  acting on the tachyonic state (3.8) can also be obtained directly from the Miura transformation (2.1), by making use of (3.9). One finds by letting the differential operators act on  $z^j$ , for  $0 \leq j \leq N - 2$ , that [4]

$$\sum_{s=0}^j \frac{j!}{(j-s)!} (\alpha_0^*)^s v_{N-s}(\vec{\beta}) = \prod_{k=1}^N \left[ \alpha_0^*(j+1-k) - \vec{h}_k \cdot \vec{\beta} \right]. \quad (4.1)$$

Shifting  $\vec{\beta}$ , so that

$$\vec{\beta} = \vec{\gamma} + \alpha_0^* \vec{\rho}, \quad (4.2)$$

leads to

$$\sum_{s=0}^j \frac{j!}{(j-s)!} (\alpha_0^*)^s v_{N-s}(\vec{\beta}) = \prod_{k=1}^N \left[ \frac{1}{2} \alpha_0^*(2j+1-N) - \vec{h}_k \cdot \vec{\gamma} \right]. \quad (4.3)$$

From this identity one can see that the eigenvalues  $v_s(\vec{\beta})$ , which are now polynomials in the shifted momentum  $\vec{\gamma}$ , are invariant under  $su(N)$  Weyl reflections of  $\vec{\gamma}$ . To show this, we first note that Weyl reflections of the simple roots of  $su(N)$ , as given in (2.4), are defined by

$$S_{\vec{e}_i}(\vec{e}_j) \equiv \vec{e}_j - (\vec{e}_j \cdot \vec{e}_i) \vec{e}_i. \quad (4.4)$$

This implies that the action of  $S_{\vec{e}_i}$  on  $\vec{h}_k$  interchanges  $\vec{h}_i$  with  $\vec{h}_{i+1}$  whilst leaving all the other  $\vec{h}_k$  fixed. From the invariance of the scalar product, it follows that a Weyl reflection of  $\vec{\gamma}$

$$\vec{\gamma} \longrightarrow S_{\vec{e}_i}(\vec{\gamma}) = \vec{\gamma} - (\vec{\gamma} \cdot \vec{e}_i) \vec{e}_i, \quad (4.5)$$

for any simple root  $\vec{e}_i$  of  $su(N)$ , merely permutes the ordering of the factors in the right-hand side of (4.3). Since the simple roots generate the entire Weyl group, we conclude that indeed the polynomials  $v_s$  are invariant under all Weyl reflections of the shifted momentum  $\vec{\gamma}$ .

## 4.2 The Tachyon Spectrum of $W_N$ -String Theory

To determine the tachyonic states in the  $W_N$ -string spectrum, one has to solve the physical-state conditions (3.1b), *i.e.*

$$v_s(\vec{\beta}) = \omega_s, \quad 2 \leq s \leq N, \quad (4.6)$$

where the intercepts  $\omega_s$  are given by  $v_s(\lambda\vec{\rho})$ , according to the discussion of the previous section. Since  $v_s(\vec{\beta})$  is a polynomial of degree  $s$  in  $\vec{\beta}$ , it follows that the set of equations (4.6) will have  $N!$  solutions. Because, as we have seen, the Weyl group of  $su(N)$  acting on the shifted momentum  $\vec{\gamma}$  leaves (4.6) invariant, we learn that it maps solutions of (4.6) into solutions. In fact, we know one solution of (4.6), *viz.* the cosmological solution, which is given by (3.4) and (3.7); this is a solution by construction. Since the Weyl vector  $\vec{\rho}$  is not a fixed point of the Weyl group, and the dimension of the Weyl group of  $su(N)$  is  $N!$ , we therefore conclude that we obtain *all* the tachyonic physical states from the cosmological solution by the action of the Weyl group on it.

An explicit procedure for writing down the  $N!$  elements of the Weyl group of  $su(N)$  can be described as follows. Defining  $S_i \equiv S_{\vec{e}_i}$ , these elements can be obtained by taking the product of one entry from each column of

$$\begin{pmatrix} 1 \\ S_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ S_2 \\ S_2 S_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ S_3 \\ S_3 S_2 \\ S_3 S_2 S_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ S_{N-1} \\ S_{N-1} S_{N-2} \\ \vdots \\ S_{N-1} S_{N-2} \cdots S_1 \end{pmatrix}, \quad (4.7)$$

giving  $N!$  (inequivalent) choices in all. Applying these to the shifted momentum

$$\vec{\gamma}^{\text{cosmo}} = \frac{\alpha_0^*}{2N+1} \vec{\rho} \quad (4.8)$$

of the cosmological solution fills out all the  $N!$  tachyonic physical states of the  $W_N$  string. Note that included amongst these  $N!$  solutions generated by (4.7) is one that corresponds to taking the  $-$  sign instead of the  $+$  sign in (3.7). It is obtained by choosing the Weyl reflection generated by the product of the bottom entries in each column of (4.7), and corresponds to the reflection  $\vec{\rho} \rightarrow -\vec{\rho}$ .

## 4.3 The Target Spacetime of $W_N$ Strings

In the previous subsection, we showed how the set of  $N!$  tachyonic physical states of the  $W_N$  string are generated by the Weyl group acting on the cosmological solution given by (3.4) and (3.7). In the discussions so far, we have considered an  $(N-1)$ -dimensional

target space, with coordinates  $(\varphi_2, \varphi_3, \dots, \varphi_N)$ . Since the physical-state conditions imply that the momentum components  $\beta_j$  can take only specific, discrete values (for example, the  $N!$  tachyon solutions), there is no sensible notion of a physical spacetime yet. To obtain a physical-spacetime interpretation, we can carry out the procedure described in section 2, of replacing the energy-momentum tensor of the  $\varphi_2$  scalar by an arbitrary energy-momentum tensor with the same central charge, which is obtained by substituting the critical value  $c_N^*$  for  $W_N$ , given by (3.2), into (2.20), leading to the expression given in (1.2).

We shall take this energy-momentum tensor to be that for  $\varphi_2$  plus  $D$  additional free scalar fields  $X^\mu$ , one of which will be chosen to be timelike, and the rest spacelike. Thus we have

$$T^{\text{eff}} = -\frac{1}{2}(\partial\varphi_2)^2 + Q\partial^2\varphi_2 - \frac{1}{2}\eta_{\mu\nu}\partial X^\mu\partial X^\nu. \quad (4.9)$$

The background charge  $Q$  must be chosen so that  $T^{\text{eff}}$  has central charge given by (1.2), and so

$$\begin{aligned} Q^2 &= \frac{1}{12}\left(6(\alpha_0^*)^2 - D\right) \\ &= \frac{1}{12}\left(24 + \frac{6}{N(N+1)} - D\right). \end{aligned} \quad (4.10)$$

Note that  $Q$  is non-zero for all  $W_N$  strings (with  $N \geq 3$ ), regardless of the number  $D$  of additional scalars  $X^\mu$ . It is for this reason that we choose to separate the coordinates into  $\varphi_2$ , which carries the background charge, and the remaining  $X^\mu$ , which have no background charge.

When one realises the  $W_N$  algebra with just the  $(N-1)$  scalars  $\vec{\varphi}$ , all  $(N-1)$  components of the momentum  $\vec{\beta}$  are ‘‘frozen’’ by the physical-state conditions to specific discrete sets of values  $\vec{\beta}^{\text{froz}}$ , such as those of the  $N!$  tachyonic solutions which we are considering in this section. The effect of introducing extra scalar fields  $X^\mu$  is that the momentum components  $(\beta_3, \beta_4, \dots, \beta_N)$  continue to be frozen to exactly the same sets of values  $(\beta_3^{\text{froz}}, \beta_4^{\text{froz}}, \dots, \beta_N^{\text{froz}})$ , whilst the momentum components  $(\beta_2, \beta_\mu)$  satisfy

$$L_0^{\text{eff}} = -\frac{1}{2}\beta_2^2 + Q\beta_2 - \frac{1}{2}\beta_\mu\beta^\mu, \quad (4.11)$$

where

$$L_0^{\text{eff}} \equiv -\frac{1}{2}(\beta_2^{\text{froz}})^2 + \frac{1}{\sqrt{2}}\alpha_0^*\beta_2^{\text{froz}}. \quad (4.12)$$

The fact that the momentum components  $(\beta_3, \beta_4, \dots, \beta_N)$  remain unchanged, and the remaining  $\beta$ 's satisfy (4.11), is a consequence of the special way in which the new coordinates  $X^\mu$  are introduced into the theory, as a modification of the original energy-momentum tensor for  $\varphi_2$ . Thus  $\beta_2^{\text{froz}}$  appeared in the physical-state conditions only through the combination  $L_0^{\text{eff}}$  defined by (4.12), and so after adding the extra coordinates  $L_0^{\text{eff}}$  remains unchanged.

The conclusion of the above discussion is that we can effectively view the tachyonic spectrum of the  $W_N$  string as being composed of sets of Virasoro-type physical states, all

with the same central charge  $c^{\text{eff}}$  given by (1.2), but with different intercepts  $L_0^{\text{eff}}$  given by substituting the discrete  $\beta_2^{\text{froz}}$  values into (4.12). In section 5, we shall show that this in fact holds for the spectrum of higher-level states also.

#### 4.4 $W_N$ Strings and Minimal Models

In [2], it was noticed that the tachyonic physical states for the  $W_3$  and  $W_4$  strings display a numerological connection with the Virasoro minimal models with central charges  $c = 1/2$  and  $c = 7/10$  respectively. In this subsection, we shall generalise and explain this numerological connection.

As already mentioned in the introduction, one can rewrite  $c^{\text{eff}}$  in the suggestive form (1.3), which consists of a term equal to the critical central charge for the usual Virasoro string minus the central charge of a unitary minimal model. This is indicative of a possible connection between the  $W_N$  string and the  $(N, N + 1)$  Virasoro minimal model with central charge

$$c(N) = 1 - \frac{6}{N(N+1)}. \quad (4.13)$$

The suggestion is strengthened by the fact that if one rewrites  $L_0^{\text{eff}}$  as

$$L_0^{\text{eff}} = 1 - L_0^{\text{min}}, \quad (4.14)$$

where 1 is the value for the intercept of the critical Virasoro string, then  $L_0^{\text{min}}$  is precisely the dimension of a primary field of the minimal model with central charge given by (4.13). The values of  $L_0^{\text{eff}}$  corresponding to all the tachyonic physical states can be obtained from (4.12). Substituting these values into (4.14) yields all the ‘‘diagonal’’ entries of the Kac table of the relevant minimal model, as we now show.

Since the values of  $L_0^{\text{eff}}$  can be determined by  $\beta_2^{\text{froz}}$ , we need only compute these latter values. They are easily obtained by acting with the Weyl group on the cosmological solution. It turns out that for the  $W_N$  string the shifted momentum components  $\gamma_2^{\text{froz}}$  can take the following values

$$\gamma_2^{\text{froz}} = \pm \frac{\alpha_0^* k}{\sqrt{2}(2N+1)}, \quad (4.15)$$

where  $k$  is an integer satisfying

$$1 \leq k \leq N - 1. \quad (4.16)$$

One can easily see this from the Weyl reflections

$$\begin{pmatrix} 1 \\ S_1 \end{pmatrix} \otimes S_2 S_3 \cdots S_k(\vec{\rho}) = \vec{\rho} - \sum_{j=(2,1)}^k (k-j+1)\vec{e}_j, \quad (4.17)$$

where the first lower bound in the summation of the right-hand side corresponds to the upper entry in left-hand side, and the second bound to the lower entry. From (4.2), (4.12), (4.14) and (4.15), we therefore find the following values for  $L_0^{\min}$ :

$$L_0^{\min} = \frac{k^2 - 1}{4N(N + 1)}, \quad (4.18)$$

where the integer  $k$  lies in the interval given by (4.16). The dimensions of the primary fields of the Virasoro minimal model with central charge given by (4.13) are

$$\Delta_{(r,s)} = \frac{[(N + 1)r - Ns]^2 - 1}{4N(N + 1)}, \quad (4.19)$$

where  $r$  and  $s$  are integers lying in the ranges  $1 \leq r \leq N - 1$  and  $1 \leq s \leq N$  respectively. Thus we see that the weights in (4.18) precisely correspond to the  $s = r = k$  entries in the Kac table of the minimal model, *i.e.* the “diagonal” ones. In section 5, we shall show how other entries of the Kac table arise from higher-level physical states.

We have exhibited  $2(N - 1)$  Weyl reflections (4.17) which generate the  $2(N - 1)$  distinct values of  $\beta_2^{\text{froze}}$  of the tachyonic states of the  $W_N$  string. Since there are  $N!$  such states, it follows that in general  $\beta_2^{\text{froze}}$  is degenerate. Indeed, we find that  $(N - k)(N - 2)!$  different tachyonic states have identical shifted-momentum component  $\gamma_2^{\text{froze}}$  for each  $k$  and each choice of sign in (4.15). Thus for each allowed  $k$ , the value of  $L_0^{\min} = \Delta_{(k,k)}$  in (4.18) occurs with degeneracy  $2(N - k)(N - 2)!$ .

The relation between  $W_N$  strings and the dimensions of the primary fields of Virasoro minimal models that we have just described is, in fact, a special case of a more general association that can be made between  $W_N$  strings and the dimensions for minimal models of  $W_M$  algebras, with  $M < N$ . Instead of replacing just  $\varphi_2$  by a new energy-momentum tensor, one could as well replace  $(\varphi_2, \dots, \varphi_M)$  by new  $W_M$  currents. These now have to satisfy the  $W_M$  algebra with central charge

$$c_M^{\text{eff}} = c_M^* - c_M^{\min}(N), \quad (4.20)$$

where  $c_M^*$  is the critical central charge for  $W_M$  (generalising the 26 of (1.3)) and the remainder,

$$c_M^{\min}(N) = (M - 1) \left[ 1 - \frac{M(M + 1)}{N(N + 1)} \right], \quad (4.21)$$

corresponds to the central charge of the relevant  $W_M$  minimal model. By analogy with (4.14), we write

$$L_0^{\text{eff}}(\beta_2, \beta_3, \dots, \beta_M) = \frac{1}{6}M(M^2 - 1) - L_0^{\min} M, \quad (4.22)$$

where  $L_0^{\text{eff}}(\beta_2, \beta_3, \dots, \beta_M)$  is the contribution to the spin-2 intercept of the  $W_N$  string from the scalars realising  $W_M$ , and is fixed by the physical-state conditions. By substituting

all possible values of the frozen-momentum components  $(\beta_2^{\text{froz}}, \beta_3^{\text{froz}}, \dots, \beta_M^{\text{froz}})$ , obtained by acting with (4.7) on the cosmological solution (4.8), into (4.22), one then finds that  $L_0^{\text{min } M}$  takes values in the Kac table of the corresponding minimal model of  $W_M$ . For example, if  $M = N - 1$ , one finds the following values for  $L_0^{\text{min } M}$ :

$$L_0^{\text{min } M} = \frac{k(N - k - 1)}{2(N^2 - 1)}, \quad k = 0, 1, \dots, \left[ \frac{N - 1}{2} \right], \quad (4.23)$$

where  $[x]$  denotes the integer part of  $x$ . The values given in (4.23) are equal to the ‘‘diagonal’’ entries of the Kac table\* of the  $W_{N-1}$  minimal model with central charge given by  $c_{N-1}^{\text{min}}(N)$  as defined in (4.21). The connection between  $W_N$  strings and  $W_M$  minimal models is, however, not so clear as in the case of Virasoro minimal models, since there does not seem to be any way of defining effective intercepts for the higher-spin currents analogous to  $L_0^{\text{eff}}$ .

## 5. Higher-level Physical States and the No-ghost Theorem

Having discussed the tachyonic states in the previous section, we shall now turn our attention to higher-level states. If one considers cases where the excitations occur purely in the unfrozen directions  $\varphi_2$  and  $X^\mu$ , the analysis is similar to that for ordinary string theory, and can be carried out for arbitrary  $W_N$  strings, at arbitrary level number. For excitations involving the frozen directions  $(\varphi_3, \dots, \varphi_N)$ , the analysis is much more complicated. By looking at special cases for the  $W_3$ ,  $W_4$ , and  $W_5$  strings, a general pattern seems to emerge, indicating that these states have frozen-momentum components that are incompatible with momentum conservation in their two-point functions, and thus that they have zero norm. Consequently, the physical spectrum of higher-level states comprises excitations only in the unfrozen directions. For these we show, by looking at level-1 and level-2 states, that they have non-negative norms.

### 5.1 The Higher-level Physical Spectrum of $W_N$ Strings

The most general level-1 state in the  $(N - 1)$ -scalar realisation of the  $W_N$  algebra is given by

$$V_{(\vec{\xi}, \vec{\beta})}(z) = \vec{\xi} \cdot \partial \vec{\varphi} e^{\vec{\beta} \cdot \vec{\varphi}}, \quad (5.1)$$

where  $\vec{\xi} \equiv (\xi_2, \xi_3, \dots, \xi_N)$  is a polarisation vector. In addition to the zero-mode constraints (3.1b), there is one other non-trivial constraint coming from (3.1a):

$$(W_2)_1 V_{(\vec{\xi}, \vec{\beta})}(0) |0\rangle = 0. \quad (5.2)$$

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\* By diagonal, we mean the dimensions with  $\ell_i = \ell'_i$  in the notation of [4,9].

In the tachyonic case the physical-state conditions can be treated in a general way, since these states are automatically eigenstates of the zero modes  $(W_s)_0$  of the  $W_N$  currents, and since their eigenvalues can be obtained by algebraic means; see (3.12). For level-1 states, the physical-state conditions are much more complicated: level-1 states are not automatically eigenstates of  $(W_s)_0$ ; they have to satisfy the additional condition (5.2); and finally, one has to compute non-trivial operator-product expansions in order to find the explicit form of the constraints. For these reasons, we have not found a treatment as general as the one for the tachyonic case. However we shall present a general pattern for certain level-1 states, based on a complete analysis for  $W_3$ ,  $W_4$  and  $W_5$ , suggesting a generalisation to  $W_N$  for arbitrary  $N$ . Moreover, we shall also give the results of our complete analysis of the level-2 physical states for the  $W_3$  and  $W_4$  cases.

Let us consider splitting the scalars  $(\varphi_2, \dots, \varphi_N)$  into two sets:  $\vec{\varphi}_b \equiv (\varphi_2, \dots, \varphi_s)$  and  $\vec{\varphi}_\# \equiv (\varphi_{s+1}, \dots, \varphi_N)$ , where  $s$  is a fixed integer between 2 and  $N - 1$ . Consider a general physical operator of the form

$$P(\vec{\varphi}_b, \vec{\varphi}_\#) = \tilde{P}(\vec{\varphi}_b) e^{\vec{\beta}_\# \cdot \vec{\varphi}_\#}, \quad (5.3)$$

where  $\tilde{P}(\vec{\varphi}_b)$  is of the form

$$\tilde{P}(\vec{\varphi}_b) = \tilde{R}(\vec{\varphi}_b) e^{\vec{\beta}_b \cdot \vec{\varphi}_b}, \quad (5.4)$$

with  $\tilde{R}(\vec{\varphi}_b)$  a given differential polynomial in  $(\partial\varphi_2, \dots, \partial\varphi_s)$ . This means that the corresponding state has excitations only in the  $(\varphi_2, \dots, \varphi_s)$  directions. One can then show that the values to which the momentum components  $(\beta_{s+1}, \dots, \beta_N)$  of the operator (5.3) are frozen by the physical-state conditions are independent of the detailed structure of  $\tilde{P}(\vec{\varphi}_b)$  and are therefore equal to the corresponding frozen-momentum components of the tachyonic physical states.<sup>‡</sup>

An important result that follows from this property is that higher-level physical states of the  $W_N$ -string theory (for all levels) defined by (5.3) with  $s = 2$  will have the momentum components  $(\beta_3, \dots, \beta_N)$  frozen identically to their tachyonic values. Thus these physical states are, just like the tachyonic physical states, related to the “diagonal” entries in the Kac table of the  $(N, N + 1)$  Virasoro minimal model. The physical-state conditions for states which are not of this form are much more complicated and we have not been able to solve them in general. However, the complete set of level-1 physical states for  $W_3$ ,  $W_4$  and  $W_5$

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<sup>‡</sup> This follows from the fact that the  $W_N$  currents can be written as linear combinations of  $W_2^{(N-1)}, \dots, W_s^{(N-1)}$  (which do not depend on  $\vec{\varphi}_\#$ ) with coefficients which are differential polynomials in  $\partial\vec{\varphi}_\#$ . The detailed structure of  $\tilde{P}(\vec{\varphi}_b)$  then enters the physical-state conditions only through the operator-product expansions between  $W_j^{(N-1)}$ , for  $j = 2, 3, \dots, s$ , and  $\tilde{P}(\vec{\varphi}_b)$ , which can be rewritten, using (2.14), as operator-product expansions between  $W_j^{(N)}$  currents and the physical operator  $P(\vec{\varphi}_b, \vec{\varphi}_\#)$ . One can then replace  $W_j^{(N)}$  by its intercept value  $\omega_j^{(N)}$ , which is independent of the structure of  $\tilde{P}(\vec{\varphi}_b)$ .

suggests that the higher-level states for any  $W_N$ -string theory will recover the remaining entries of the Kac table.

Let us first consider level-1 physical states of the form (5.1) for the  $W_3$ -string theory. As the case  $\vec{\xi} = (\xi_2, 0)$  has already been discussed, we shall assume that  $\xi_3 \neq 0$ . We need only give the results for the momentum component  $\beta_2$ , since these are sufficient to establish the connection with the corresponding minimal model. Solving the physical-state conditions in this case leads to six possible values of  $\beta_2$ . In terms of the shifted-momentum component  $\gamma_2$  they are given by (4.15), where now  $k \in \{1, 2, 5\}$ . Using equation (4.14), the  $k = 1$  and  $k = 2$  values correspond to the diagonal entries 0 and  $1/16$  of the Kac table of the Virasoro minimal model with  $c = 1/2$ , whereas  $k = 5$  gives the remaining dimension of this minimal model, namely  $\Delta_{(2,1)} = 1/2$ . Thus, by considering level-1 physical states of the  $W_3$  string one finds all the dimensions of the  $c = 1/2$  Virasoro minimal model.

Consider now the  $W_4$  string. Suppose first that the polarisation components  $\xi_3$  and  $\xi_4$  are both non-zero. In that case, the shifted-momentum component  $\gamma_2$  is again given by (4.15), where now  $k \in \{1, 2, 3, 6, 7\}$ . The values  $k = 1$ ,  $k = 2$  and  $k = 3$  lead to the diagonal entries of the  $c = 7/10$  Virasoro minimal models, whereas  $k = 6$  and  $k = 7$  lead to the off-diagonal dimensions  $\Delta_{(2,1)} = 7/16$  and  $\Delta_{(3,2)} = 3/5$  respectively. Only one dimension of this minimal model has not yet been found, namely  $\Delta_{(3,3)} = 3/2$ . The cases where  $\xi_3 = 0$  or  $\xi_4 = 0$  do not lead to anything new.

The  $W_5$  case is again very similar. The values of  $k$  which do not lead to diagonal entries of the  $c = 4/5$  Virasoro minimal model are now 7, 8 and 9. They lead to the off-diagonal entries  $\Delta_{(2,1)} = 2/5$ ,  $\Delta_{(3,2)} = 21/40$  and  $\Delta_{(4,3)} = 2/3$  respectively.

The pattern that seems to emerge for the level-1 physical states of the general  $W_N$  string should now be clear. The new values of  $k$  in (4.15) and (4.18) are given by

$$k \in \{N + 2, N + 3, \dots, 2N - 1\} \quad (5.5)$$

and lead to the off-diagonal dimensions  $\Delta_{(r,r-1)}$ , with  $r = 2, 3, \dots, N - 1$ , of the  $(N, N + 1)$  Virasoro minimal model.

The analysis of the spectrum of level-1 physical states for the  $W_N$  string thus gives more evidence for the connection between the  $W_N$  string and the  $(N, N + 1)$  Virasoro minimal model. For  $W_3$  this level-1 spectrum exhausts all the dimensions of the relevant minimal model. For the other  $W_N$ -string theories we expect the remaining off-diagonal dimensions to appear from physical states at sufficiently high level.

We have, in fact, constructed all the level-2 physical states of the  $W_3$  and  $W_4$  strings. All these states are, once again, related to the relevant minimal model. For  $W_4$ , however, there is still no physical state at this level which corresponds to the dimension  $3/2$  primary field of the  $c = 7/10$  Virasoro minimal model. We expect, nevertheless, that this state will emerge from the higher-level physical spectrum.

In subsection 4.4 we generalised the connection between  $W_N$  strings and Virasoro minimal models in the tachyonic case to a connection with  $W_M$  minimal models. This generalisation seems to hold for the higher-level physical spectrum as well, where off-diagonal dimensions of the  $W_M$  minimal models appear. We have checked this explicitly for the connection between the  $W_4$ -string theory and the  $W_3$  minimal model with  $c = 4/5$ . In this case the two off-diagonal entries,  $2/3$  and  $2/5$ , of the corresponding Kac table, as well as the diagonal ones,  $0$  and  $1/15$ , emerge in the level-1 and level-2 physical spectrum of the  $W_4$  string.

## 5.2 The No-ghost Theorem for $W_N$ Strings

Having analysed the physical spectrum of the  $W_N$  string, we are now going to use these results to discuss the no-ghost theorem for  $W_N$ -string theories. As explained in subsection 4.3, the target spacetime of the  $W_N$  string only acquires a physical interpretation if the energy-momentum tensor for  $\varphi_2$  is replaced by a new energy-momentum tensor (4.9) with central charge  $c^{\text{eff}}$  given in (1.2). We shall first show that after adding extra coordinates  $X^\mu$ , all higher-level physical states that involve excitations only in the unfrozen directions, *i.e.* that are of the form (5.3) with  $s = 2$  (and  $\varphi_2$  replaced by the set  $\{\varphi_2, X^\mu\}$ ), have positive semi-definite norm. Next, we shall argue, based on some explicit examples, that all the physical states that are not of this form have zero norm and hence do not describe physical degrees of freedom. This absence of negative-norm states at low-lying levels is usually a good indication of the ghost freedom of the theory.

Higher-level physical operators of the form

$$R(\varphi_2, X^\mu) e^{\vec{\beta} \cdot \vec{\varphi} + \beta_\mu X^\mu} \quad (5.6)$$

have, as we have explained in the previous subsection, the same set of frozen values for the momentum components  $(\beta_3^{\text{froz}}, \dots, \beta_N^{\text{froz}})$ , regardless of the explicit form of  $R(\varphi_2, X^\mu)$ , *i.e.* of their level. This implies that all the states of the form (5.6) with given values of  $(\beta_3^{\text{froz}}, \dots, \beta_N^{\text{froz}})$  have the same value of  $L_0^{\text{eff}}$ :

$$\begin{aligned} L_0^{\text{eff}} &\equiv \frac{1}{6}N(N^2 - 1) - \sum_{j=3}^N \left[ -\frac{1}{2}(\beta_j^{\text{froz}})^2 + \frac{1}{\sqrt{j(j-1)}} \alpha_0^* \beta_j^{\text{froz}} \right] \\ &= n - \frac{1}{2}\beta_2^2 + Q\beta_2 - \frac{1}{2}\beta_\mu\beta^\mu, \end{aligned} \quad (5.7)$$

where  $n$  is the level number of the states (5.6), and the background charge  $Q$  is given by (4.10). The computation of the norm of such physical states is therefore analogous to that for physical states in Virasoro-string theory, but with central charge  $c^{\text{eff}}$  and intercept  $L_0^{\text{eff}}$ , given in (1.2) and (5.7). One can thus use the well-known method of deriving unitarity

bounds for the intercept  $a$  in Virasoro-string theory with a given central charge  $c$ . For level-1 physical states this bound is independent of the value of the central charge and is given by  $a \leq 1$ . For level 2, the bounds depend on the value of the central charge: they are given by

$$a \leq \frac{37 - c - \sqrt{(c-1)(c-25)}}{16} \quad \text{or} \quad a \geq \frac{37 - c + \sqrt{(c-1)(c-25)}}{16}. \quad (5.8)$$

In string theory, these bounds are sufficient to establish the unitarity of the theory at all levels. Combining these bounds for the central charge  $c^{\text{eff}}$  given by (1.2) requires the intercept to satisfy

$$a \leq \frac{3(N-1)}{4N} \quad \text{or} \quad \frac{3(N+2)}{4(N+1)} \leq a \leq 1. \quad (5.9)$$

The values of  $L_0^{\text{eff}}$  given in (5.7) can be most easily obtained by using the tachyonic physical states. Substituting (4.18) into (4.14) gives the values

$$L_0^{\text{eff}} = \frac{(2N+1)^2 - k^2}{4N(N+1)}, \quad k = 1, 2, \dots, N-1. \quad (5.10)$$

Each value of  $L_0^{\text{eff}}$  in (5.10) corresponds to the value of the intercept for a Virasoro-type string with central charge (1.2). One can easily see that each such intercept satisfies the unitarity bounds (5.9). We therefore conclude that all the physical states of the form (5.6) have positive semi-definite norm. This demonstration that physical states having excitations only in the unfrozen directions have non-negative norm concludes the first part of the no-ghost theorem.

The above discussion is not valid any more for states that are not of the form (5.6). However, we shall argue, based on momentum-conservation considerations, that all these states have vanishing norm. The point is that higher-level physical states are always of the form  $|\text{phys}\rangle = R|p\rangle$ , where  $|p\rangle$  is a tachyon-like state and  $R$  is a differential polynomial in the free scalars with polarisation tensors as coefficients. The norms of such states are

$$\langle \text{phys} | \text{phys} \rangle = \mathcal{N}(R) \langle p | p \rangle, \quad (5.11)$$

where  $\mathcal{N}(R)$  is a function of the scalar products of the polarisation tensors. The physical state  $|\text{phys}\rangle$  is a null state if  $\langle p | p \rangle = 0$ , which occurs when momentum conservation cannot be satisfied. Under these circumstances, the sign of  $\mathcal{N}(R)$  is immaterial.

The momentum-conservation law for tachyonic states appears as a delta function in their two-point function. For a  $W_N$ -string theory with effective energy-momentum tensor given in equation (4.9) this is expressed by

$$\left\langle e^{\vec{\beta}' \cdot \vec{\varphi} + \beta'_\mu X^\mu} e^{\vec{\beta} \cdot \vec{\varphi} + \beta_\mu X^\mu} \right\rangle \propto \delta(\beta'_2 + \beta_2 - 2Q) \prod_\mu \delta(\beta'_\mu + \beta_\mu) \prod_{j=3}^N \delta(\beta'_j + \beta_j - 2\alpha_0^* \rho_j), \quad (5.12)$$

where  $\rho_j = \sqrt{j(j-1)}/2$  is the  $j$ -th component of the Weyl vector. Since the addition of extra coordinates allows  $\beta_2$  and  $\beta_\mu$  to take continuous values, while leaving  $(\beta_3, \dots, \beta_N)$  frozen to the same discrete values, it may happen that the momentum-conservation law cannot be satisfied in the  $(\varphi_3, \dots, \varphi_N)$  directions. In this case the delta function is zero in these directions, implying that the two-point function (5.12) vanishes.

Let us first consider the tachyonic states. Since, in order to discuss their physical spectrum, it is not essential to introduce extra coordinates  $X^\mu$ , we shall restrict ourselves to the  $(N-1)$ -scalar realisation. Suppose that  $\vec{\beta}_+ = \alpha_0^* \vec{\rho} + \vec{\gamma}$  is a solution of the physical-state conditions for the tachyon case (which means that  $\vec{\gamma}$  can be obtained from a Weyl reflection (4.7) on  $\vec{\gamma}^{\text{cosmo}}$  given in (4.8)). It then follows that  $\vec{\beta}_- = \alpha_0^* \vec{\rho} - \vec{\gamma}$  is also a solution of the physical-state conditions. Since  $\vec{\beta}_+ + \vec{\beta}_- = 2\alpha_0^* \vec{\rho}$ , we see from (5.12) that the momenta  $\vec{\beta}_+$  and  $\vec{\beta}_-$  are conjugate, implying that the corresponding two-point function (5.12) is not zero. Thus, tachyonic states have positive norm. Since physical operators of the form (5.6) have their momentum components  $(\beta_3, \dots, \beta_N)$  frozen identically to the tachyonic values, the same argument teaches us that these states are, in general, not null states.

The conclusion for higher-level physical states that have excitations in frozen directions is different. In this case, the frozen momenta  $\vec{\beta}^{\text{froz}}$  do not seem to appear in conjugate pairs, and therefore the momentum-conservation law cannot be satisfied. Any two-point function between such states is thus zero, implying that all these states are null. Although we do not have a general proof of this statement, we have checked it for all level-1 states of the  $W_3$ ,  $W_4$  and  $W_5$  strings and for all level-2 states of the  $W_3$  and  $W_4$  strings. Indeed we find in all these examples that the component  $\gamma_s$  ( $s \geq 3$ ) of the shifted momentum  $\vec{\gamma}$  is always positive, where  $s$  is defined just above equation (5.3). The condition for having a conjugate-momentum pair  $\vec{\beta}$  and  $\vec{\beta}'$  is, as we have seen above, that their shifted momenta  $\vec{\gamma}$  and  $\vec{\gamma}'$  should satisfy  $\vec{\gamma} = -\vec{\gamma}'$ . Since  $\gamma_s$  is always positive for the states in question, such conjugate pairs do not occur. It seems reasonable to expect that this pattern will hold in general. Proving that all states involving excitations in the frozen directions have zero norm, and thus do not contribute to the physical spectrum, establishes the second part of the no-ghost theorem for  $W_N$  strings.

Summarising this discussion, we conclude that the complete physical spectrum of the  $W_N$  string is given by the tachyonic states discussed in section 4, together with all higher-level physical states that have excitations only in the unfrozen directions.

## 6. Uniqueness of the Higher-spin Intercepts

In section 3 we computed the intercepts for the  $W_N$  string under the assumption that a particular tachyonic operator, namely the cosmological operator (3.4), is physical. After determining the constant  $\lambda$  in (3.4) by using the known spin-2 intercept (3.6), the intercepts for all higher-spin currents followed.

In [2], a classical correspondence-principle argument for the existence of the cosmological solution as a physical operator was given. Of course a rigorous proof of the existence of this solution would require a full BRST analysis in order to determine the actual values of the intercepts, to verify that they were the same as those obtained from the cosmological solution. Such an analysis is extremely difficult and has only been performed for the  $W_3$  algebra, confirming the existence of the cosmological solution in this case.

In this section we shall analyse this question from another point of view, and present very strong evidence in favour of the existence of the cosmological solution as a physical operator. In fact we shall argue that the assumption that the  $W_N$  string is a unitary theory implies that the intercepts must be *precisely* those obtained from the cosmological solution.

The demonstration proceeds in two stages. The first stage consists of requiring that all tachyonic physical states should occur in conjugate pairs, such that they can have non-zero norms consistent with the momentum-conservation conditions for frozen directions discussed in subsection 5.2. This requirement, which does not involve the assumption of unitarity, is very natural from the point of view of the string theory. It has the consequence of uniquely determining the spin-3 intercept for any  $W_N$  string. For the  $W_3$  string, this is therefore sufficient. It also places curiously-stringent constraints on the intercepts for the higher-spin currents. The tightness of the bounds on these intercepts, which is inessential to our further arguments, is nevertheless very intriguing. The second stage of our demonstration consists of demanding unitarity of the physical spectrum. This has the effect of pinning down the values of the higher-spin intercepts precisely, to those given by the cosmological solution.

From the discussion in subsection 5.2, it follows immediately that if the tachyonic physical states are to have non-zero norm, then their momenta  $(\beta_2, \dots, \beta_N)$  must occur in conjugate pairs. In terms of the shifted momentum, this means that if  $\vec{\gamma}$  is a solution of the physical-state conditions, then  $-\vec{\gamma}$  must be also. In other words if, without loss of generality, we write  $\vec{\gamma}$  as

$$\vec{\gamma} \equiv (\gamma_2, \gamma_3, \gamma_4, \dots, \gamma_N) = x \left( \frac{1}{\sqrt{2}}, \frac{t_3}{\sqrt{6}}, \frac{t_4}{\sqrt{12}}, \dots, \frac{t_N}{\sqrt{N(N-1)}} \right) \alpha_0^*, \quad (6.1)$$

then the physical-state conditions for tachyonic states must all be *even* functions of  $x$ .

For the case of  $W_3$ , substituting (6.1) into the physical-state conditions (3.1b) for tachyonic states gives

$$\begin{aligned}\omega_2 &= (\alpha_0^*)^2 \left[ 1 - \frac{1}{12}(t_3^2 + 3)x^2 \right], \\ \omega_3 &= -\alpha_0^* \omega_2 + \frac{1}{108}(\alpha_0^*)^3 t_3 (t_3^2 - 9)x^3,\end{aligned}\tag{6.2}$$

where  $\alpha_0^* = 7/(2\sqrt{3})$ . Thus the requirement that these equations be even in  $x$  immediately implies that  $\omega_3 = -\alpha_0^* \omega_2$ . Using the general result (3.6) for the spin-2 intercept, we therefore find  $\omega_3 = -4\alpha_0^*$  for the  $W_3$  string. This agrees with the result obtained previously from the cosmological solution (which itself agrees with the result from the BRST analysis for  $W_3$  [7], as explained earlier). The absence of the  $x^3$  term in (6.2) requires that  $t_3 = 0, \pm 3$ . It is easy to check that these values correspond to the three conjugate pairs of frozen momenta for the tachyons of the  $W_3$  string, obtained by the action of the Weyl group on the cosmological solution.

Turning now to the case of the  $W_N$  string, we first note that the above discussion generalises straightforwardly to give the spin-3 intercept of equation (3.13). For the higher-spin currents the argument becomes more subtle. We shall now illustrate this with a few examples. Let us first consider the case of the  $W_4$  string. Substituting (6.1) into the physical-state conditions (3.1b) for tachyons gives

$$\begin{aligned}\omega_2 &= (\alpha_0^*)^2 \left[ \frac{5}{2} - \frac{1}{24}(2t_3^2 + t_4^2 + 6)x^2 \right], \\ \omega_3 &= -2\alpha_0^* \omega_2 + \frac{1}{216}(\alpha_0^*)^3 (2t_3 + t_4)(t_3 - t_4 + 3)(t_3 - t_4 - 3)x^3, \\ \omega_4 &= -\frac{9}{16}(\alpha_0^*)^4 - \frac{3}{4}(\alpha_0^*)^2 \omega_2 - \frac{3}{2}\alpha_0^* \omega_3 + \frac{1}{6912}(\alpha_0^*)^4 t_4 (4t_3 - t_4)(t_4 + 2t_3 - 6)(t_4 + 2t_3 + 6)x^4,\end{aligned}\tag{6.3}$$

where  $\alpha_0^* = 9/(2\sqrt{5})$ . From the single condition needed to make these equations of even order in  $x$ , it follows that there are three 1-parameter families of solutions at this stage. Using the value for the spin-2 intercept given by (3.6) we find that  $\omega_3$  is then uniquely determined, as mentioned above, to be  $-20\alpha_0^*$ , but that  $\omega_4$  is a function of the free parameter, say  $t_3$ , for each of the three families. The range of  $\omega_4$  is the same for each family, and spans a remarkably small interval:

$$81.89859375 = \frac{524151}{6400} \leq \omega_4 \leq \frac{524176}{6400} = 81.9025.\tag{6.4}$$

The cosmological solution corresponds to  $\omega_4 = 819/10$ . We have no explanation, beyond the superficial one, for why the allowed range in (6.4) is so small. It seems however that this is a generic feature of  $W_N$  strings, which we have also observed for  $W_5$  and  $W_6$ , suggesting a deeper structure that has still to be elucidated.

We have just seen that requiring that the tachyonic physical states of the  $W_4$  string arise in conjugate pairs determines the spin-3 intercept uniquely, and restricts the spin-4 intercept to lie in the interval given in (6.4). We shall now consider the consequences of unitarity.

We discussed this in subsection 5.2 for the case where the intercepts were assumed to take the values determined by the cosmological solution. We saw in particular that, under this assumption, the physical states of the  $W_N$ -string theory of the form (5.6) have a set of  $L_0^{\text{eff}}$  values given in (5.7) that lie in the interval  $3(N+2)/(4(N+1)) \leq L_0^{\text{eff}} \leq 1$  of equation (5.9), including states that saturate the lower and the upper bounds. As we shall now show, if the  $\omega_4$  intercept for  $W_4$  is not given by its cosmological value, then some states will violate these unitarity bounds.

The values of  $L_0^{\text{eff}}$  for these states are given by substituting their (frozen)  $\beta_2$  values into (5.7) (there are no extra  $X^\mu$  coordinates in our present discussion). In terms of the shifted momentum given by (6.1), we therefore have

$$L_0^{\text{eff}} = \frac{81}{80}(1 - x^2) \quad (6.5)$$

for the  $W_4$  case. Since we already know the value of  $\omega_2$  from (3.6), the first equation in (6.3) may be used to express the free parameter  $t_3$  in terms of  $x^2$ , and hence, using (6.5), in terms of  $L_0^{\text{eff}}$ . Thus for each family we may express the spin-4 intercept  $\omega_4$  as a function of  $L_0^{\text{eff}}$ . Two of the three families give the same result,

$$\omega_4(L_0^{\text{eff}}) = \omega_4^{\text{cosmo}} - 4(1 - L_0^{\text{eff}})\left(L_0^{\text{eff}} - \frac{77}{80}\right), \quad (6.6)$$

and the third family gives

$$\omega_4(L_0^{\text{eff}}) = \omega_4^{\text{cosmo}} + (1 - L_0^{\text{eff}})\left(L_0^{\text{eff}} - \frac{9}{10}\right). \quad (6.7)$$

In these equations  $\omega_4^{\text{cosmo}}$  denotes the cosmological value,  $819/10$ , for the spin-4 intercept for  $W_4$ . Equation (6.6) shows that there is a state that has  $L_0^{\text{eff}} = 1$  when  $\omega_4 = \omega_4^{\text{cosmo}}$  and has  $L_0^{\text{eff}} > 1$  when  $\omega_4$  is *larger* than  $\omega_4^{\text{cosmo}}$ . On the other hand, equation (6.7) shows that there is another state that has  $L_0^{\text{eff}} = 1$  when  $\omega_4 = \omega_4^{\text{cosmo}}$  and has  $L_0^{\text{eff}} > 1$  when  $\omega_4$  is *smaller* than  $\omega_4^{\text{cosmo}}$ . What is happening is that the degeneracy of the set of physical states at a given  $L_0^{\text{eff}}$  value described in subsection 4.4 is (partially) lifted when  $\omega_4$  is not equal to its cosmological value  $\omega_4^{\text{cosmo}}$ . For some such states,  $L_0^{\text{eff}}$  increases with *increasing*  $\omega_4$ , while for other states  $L_0^{\text{eff}}$  increases with *decreasing*  $\omega_4$ . Therefore by considering the full set of originally-degenerate  $L_0^{\text{eff}} = 1$  states, we see that unitarity will be violated for any value of  $\omega_4$  except the cosmological value.<sup>†</sup>

We have also checked completely that the same conclusions hold in the case of the  $W_5$  string. Since the general pattern is very similar to the  $W_4$  case, we shall be very brief about

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<sup>†</sup> In fact the requirement of unitarity by itself is sufficient to determine *all* the intercepts, including the spin-2 intercept which we calculated in section 3 by a different method. One simply uses the same procedure that we have described for  $\omega_4$  to express  $\omega_2$  as a function of  $L_0^{\text{eff}}$ , and sees that the unitarity bounds are violated unless  $\omega_2$  is given by (3.6).

it. The intercepts of the spin-4 and spin-5 currents again depend on one free parameter and their ranges are, once more, remarkably small:

$$\begin{aligned}
499.59555 \dots &= \frac{1798544}{3600} \leq \omega_4 \leq \frac{1798569}{3600} = 499.6025, \\
-515.205 &= -\frac{927369}{1800} \leq \frac{\omega_5}{\alpha_0^*} \leq -\frac{927344}{1800} = -515.19111 \dots .
\end{aligned}
\tag{6.8}$$

The unitarity bounds (5.9) for  $N = 5$  are violated except when the intercepts take the values determined by the cosmological solution.

If  $N > 5$ , then the situation is more complicated. In the case of  $W_6$ , for example, the higher-spin intercepts now depend on two free parameters and lie again in extremely small ranges. Although we have not analysed the general case in detail, we expect that only the cosmological solution will give a unitary theory.

## 7. Conclusions

In this paper we have studied the physical spectrum of  $W_N$ -string theories. Starting from the Miura transformation for  $su(N)$ , we derived an explicit formula giving the currents of  $W_N$  in terms of those of  $W_{N-1}$ , together with one extra free scalar. Applying this recursively leads to realisations of  $W_N$  in terms of  $(N-2)$  free scalar fields  $(\varphi_3, \dots, \varphi_N)$  and an arbitrary energy-momentum tensor. By taking this energy-momentum tensor to be realised in terms of additional free scalar fields, one has the starting point for a  $W_N$ -string theory. In order to study the physical-state conditions for this  $W_N$  string, we used a method based on a unitarity argument to determine the intercepts of the  $W_N$  currents, avoiding the necessity of performing the complete BRST analysis of the  $W_N$  gauge theory. This allowed us to derive explicit formulae for these intercepts.

Using these values of the intercepts, we gave a construction of all the tachyonic physical states for the general  $W_N$  string. All these states can be obtained by acting on a particular physical state, the cosmological solution, with the Weyl group of  $su(N)$ . We also constructed all the level-1 and level-2 physical states in some specific examples. Our results indicate that the physical spectrum of the  $W_N$  string bears a strong resemblance to ordinary Virasoro string theory, but with a non-critical value of the central charge (1.2) and a discrete set of intercepts, given by (4.14) and (4.18), which includes 1, together with other values. These values for the central charge and the intercepts are very suggestive of a connection between  $W_N$  strings and minimal models. The physical states of the  $W_N$  string can, in a certain sense, be viewed as “gravitational dressings” of the primary fields of the corresponding minimal model [2]. The precise nature of this connection is, however, still a mystery.

Our finding that unitarity requires the values of the intercepts for the  $W_N$  currents to be those given by the cosmological solution is one of the crucial results of this paper. We studied

the no-ghost theorem by analysing level-1 and level-2 states. These, and indeed as we have seen *all* higher-level states, can be divided into two categories, namely those that comprise excitations *only* in the (unfrozen)  $(\varphi_2, X^\mu)$  directions, and the remaining ones which include excitations in the (frozen)  $(\varphi_3, \dots, \varphi_N)$  directions. The first category describes the higher-level states of the effective Virasoro-string theory alluded to above; it is the unitarity of these states that determines the values of the  $W_N$  intercepts. The second category, as we showed in various examples, describes higher-level physical states that *all* have zero norm, since they do not occur in conjugate pairs, implying that the two-point functions of any two of these states vanishes by momentum conservation. Since only the first category of higher-level states contributes to the physical spectrum of the  $W_N$  string, the derivation of the *entire* physical spectrum of the  $W_N$  string reduces to finding the physical spectra of a set of Virasoro-type string theories, with the non-standard values of the central charge and intercepts given above. Thus we have found the complete spectrum of the  $W_N$  string.

We have shown that  $W_N$ -string theory reduces to effective Virasoro-string theories with  $(D + 1)$  coordinates  $X^\mu$  and  $\varphi_2$ . For  $N \geq 3$ , the coordinate  $\varphi_2$  has a non-vanishing background charge  $Q$ , given in (4.10). Owing to this background charge, the theory does not have  $(D + 1)$ -dimensional target-space Poincaré invariance. Amongst other things, this makes the definition of a  $(D + 1)$ -dimensional mass ambiguous. This problem resolves itself by taking  $D$  to be greater than 24, since then the background charge becomes imaginary. As discussed in [3] for the  $W_3$  string, this implies that the  $\varphi_2$  coordinate has to live on a circle, since the functional integral becomes periodic in  $\varphi_2$  with period  $\pi/|Q|$ . Therefore its momentum component  $\beta_2$  is quantised, and is given by

$$\beta_2 = 2mi|Q|, \quad m \in \mathbf{Z}. \quad (7.1)$$

(Recall that in our conventions, the momenta are imaginary.) Because  $\varphi_2$  is then compactified, the remaining coordinates  $X^\mu$  describe a  $D$ -dimensional Minkowski spacetime *à la* Kaluza-Klein. Since this spacetime *is* Poincaré invariant, the definition of  $D$ -dimensional mass is now unambiguous and given by  $\mathcal{M}^2 = \beta_\mu \beta^\mu$ . Using (5.7), (4.14), (4.10), (4.18) and (7.1) this can be rewritten for the  $W_N$  string as

$$\mathcal{M}^2 = -2 + 2n + \frac{k^2 - 1}{2N(N + 1)} + \frac{1}{3}m(m - 1) \left[ D - 24 - \frac{6}{N(N + 1)} \right], \quad (7.2)$$

where  $n$  is the level number of the physical state,  $m$  is the Kaluza-Klein mode number and  $k$  is an integer labelling the diagonal entries of the Kac table of the relevant minimal model, satisfying  $1 \leq k \leq N - 1$ . In particular we see from this that the  $k = 1$  case with  $m = 0$  or  $m = 1$  has precisely the mass spectrum of ordinary Virasoro-string theory and includes, therefore, a massless vector at level  $n = 1$ . Curiously, some sporadic cases develop extra massless physical states, occurring at level  $n = 0$ . These massless tachyons arise at  $D = 25$ ,  $m = -2, 3$ ,  $k = 5$  for  $W_N$  strings with  $N \geq 6$ , and at  $D = 27$ ,  $m = -1, 2$ ,  $k = 3$  for  $N \geq 4$ .

The issue of the existence of massless states in the spectrum of the  $W_N$  string had already been raised some time ago. Since the spin-2 intercept (3.6) for the  $W_N$  string increases with  $N$ , naive expectations might suggest that the  $(\text{mass})^2$  of physical states would correspondingly be decreased, opening the possibility of having massless higher-spin states. In fact, as seen in [3] and this paper, the effect of having momentum components frozen by the  $W_N$  constraints is that the relevant quantity that determines the masses of the physical states is  $L_0^{\text{eff}}$  rather than the spin-2 intercept (3.6). Since  $L_0^{\text{eff}} \leq 1$ , it follows that the values of  $(\text{mass})^2$  are either the same as those for the usual Virasoro string (when  $L_0^{\text{eff}} = 1$ ) or shifted *upwards*. The original version of [3] mistakenly discarded the physical states of the  $W_3$  string with  $L_0^{\text{eff}} = 1$ , by imposing an over-stringent requirement of hermiticity of the  $W_3$  currents. This led to the erroneous conclusion that the  $W_3$  string did not contain massless states at all. (Further discussion of this issue may be found in [10], and the revised version of [3].) In fact, as we have just seen in the previous paragraph,  $W_N$  strings do contain massless states. In particular, the massless level-1 states will give rise to a massless graviton in the case of a closed  $W_N$  string. However, neither the open nor the closed  $W_N$  string has massless states with spins higher than 2.

The next step in the understanding of the  $W_N$  string is to construct an interacting theory.

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## NOTE ADDED

After this paper was completed, we encountered a paper that has some overlap with our work [12]. It discusses the relation between the tachyonic spectrum of the  $W_N$  string and the diagonal states of the corresponding minimal model, although the rôle of the Weyl group as the organising symmetry is not found.

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