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On Sibling and Exceptional W Strings

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ABSTRACT

We discuss the physical spectrum for W strings based on the algebras B_n, D_n, E_6, E_7 and E_8. For a simply-laced W string, we find a connection with the $(h, h+1)$ unitary Virasoro minimal model, where h is the dual Coxeter number of the underlying Lie algebra. For the W string based on B_n, we find a connection with the $(2h, 2h+2)$ unitary $N = 1$ super-Virasoro minimal model.

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1. Introduction

W strings have received a considerable amount of attention recently. They are described by two-dimensional conformal field theories with non-linear local symmetry algebras, namely W algebras, which are higher-spin extensions of the Virasoro algebra. It has been known for some time that every simple Lie algebra is the progenitor of an associated W algebra, with the Virasoro algebra arising from $su(2)$. Even though simple Lie algebras can be viewed as "coupled" $su(2)$ subalgebras, one cannot understand Lie algebras by studying $su(2)$ alone. It seems that W strings can similarly be viewed as "coupled" Virasoro strings, so, in the same vein, one should not expect the full richness of string theories to be uncovered by studying the Virasoro string only.

The results obtained hitherto have been restricted to the W_N string, whose symmetry algebra, $W_N = WA_{N-1}$, is derived from the Lie algebra $su(N) \equiv A_{N-1}$ [1,2,3,4,5]. In particular, it has been shown that the physical spectrum of the W_N string is effectively given by that of ordinary Virasoro strings, but with a non-standard central charge $c^{\rm eff} = 26 - c(N)$, and a set of non-standard values for the spin-2 intercepts $L_0^{\rm eff} = 1 - \Delta_{(r,r)}$, where $c(N) = 1 - 6/\big(N(N+1)\big)$ is the central charge of the $(N, N+1)$ unitary Virasoro minimal model and $\Delta_{(r,r)}$ are the values of the diagonal entries of the weights $\Delta_{(r,s)}$ appearing in its Kac table [5].

In this paper, we generalise these results to W strings whose underlying W algebras are derived from any of the simply-laced Lie algebras $A_n \equiv su(n+1)$, $D_n \equiv so(2n)$, E_6, E_7 and E_8. These string theories all turn out to be connected with $(h, h+1)$ unitary Virasoro minimal models, where h is the dual Coxeter number of the corresponding Lie algebra. In addition we show that for the non-simply-laced Lie algebras $B_n \equiv so(2n+1)$, the corresponding W strings are connected with $(2h, 2h+2)$ unitary $N = 1$ super-Virasoro minimal models. A similar analysis seems not to be possible for the $C_n \equiv sp(2n)$, F_4 and G_2 algebras.

For any W algebra based on a rank-n Lie algebra, there exists a realisation in terms of n free scalars. All the W algebras that we shall consider have the remarkable property that one of these scalars appears in the W currents only *via* its energy-momentum tensor. Consequently this energy-momentum tensor may be replaced by an arbitrary one, $T^{\rm eff}$, with the same central charge.* The physical-state conditions from the W currents have the effect of determining the intercept $L_0^{\rm eff}$ of $T^{\rm eff}$, and of "freezing" the remaining $(n-1)$ scalars of the realisation of the W algebra. By choosing $T^{\rm eff}$ to be the energy-momentum tensor for an independent set of free scalars X^μ, one then has a starting point for a W-string theory. These scalars X^μ acquire the interpretation of being the coordinates of the target spacetime. The physical spectrum of the W string is thus determined by the values of $L_0^{\rm eff}$,

* In the case of the WB_n algebra considered here, one also needs a free fermion, in order to realise the algebra. This, and one of the free scalars, appear only *via* their super energy-momentum tensor.

which follow from the W constraints. Since these values are independent of the explicit form of $T^{\rm eff}$, it is sufficient for us to choose the simplest realisation, *i.e.* to revert to the original n-scalar realisation of the W algebra, in order to calculate them. In the rest of this paper, even though we shall usually work with the n-scalar realisation, it is to be understood that our results will acquire a string-theoretic interpretation when the extra X^μ coordinates are introduced.

2. Simply-laced Algebras

2.1 The General Structure of Simply-laced W Strings

The discussion of W strings in the case of simply-laced Lie algebras can be given in a rather general way. In these cases the energy-momentum tensor for the corresponding W algebra is given by

$$T(z) = -\tfrac{1}{2}\partial\vec{\varphi}\cdot\partial\vec{\varphi} + \alpha_0\vec{\rho}\cdot\partial^2\vec{\varphi}\ , \tag{2.1}$$

where $\vec{\varphi} \equiv (\varphi_1,\ \varphi_2,\ldots,\ \varphi_n)$ are free scalar fields, and $\vec{\rho}$ is the Weyl vector (*i.e.* half the sum of the positive roots) of the underlying Lie algebra g, which has rank n. The central charge of this realisation is

$$c = n + 12\,\alpha_0^2\,\vec{\rho}^{\,2}\ . \tag{2.2}$$

Using the Freudenthal strange formula $\vec{\rho}^{\,2} = \frac{1}{12}h\,\dim(g)$, this can be rewritten as

$$c = n + \alpha_0^2\,h\,\dim(g)\ , \tag{2.3}$$

where h is the dual Coxeter number‡ and $\dim(g)$ is the dimension of the Lie algebra g, which, for simply-laced algebras, is given by

$$\dim(g) = n(h+1)\ . \tag{2.4}$$

In these expressions the standard normalisation, in which the simple roots have $(\text{length})^2 = 2$, is being used.

Anomaly freedom of the W-string theory requires that the central charge take its critical value, which is determined by the condition that it cancel the contribution from the ghosts in the BRST quantisation procedure. For every current, with spin s, generating the W algebra, there is a contribution $-2(6s^2-6s+1)$ to the ghostly central charge, implying that the critical central charge is given by

$$c^* = 2\sum_{\{s\}}\big(6s^2-6s+1\big)\ . \tag{2.5}$$

‡ For convenience, we recall that the dual Coxeter numbers for the simple Lie algebras are as follows: $A_n,\ n+1$; $B_n,\ 2n-1$; $C_n,\ n+1$; $D_n,\ 2n-2$; $E_6,\ 12$; $E_7,\ 18$; $E_8,\ 30$; $F_4,\ 9$; and $G_2,\ 4$.

For the W algebras derived from the simply-laced Lie algebras, the set $\{s\}$ of spins of the generating currents runs over the orders of the independent Casimir operators [6,7,8]. These are

$$\begin{aligned} A_n &: \quad 2,\ 3,\ 4,\ \ldots,\ n+1 \\ D_n &: \quad 2,\ 4,\ 6,\ \ldots,\ 2n-4,\ 2n-2;\ n \\ E_6 &: \quad 2,\ 5,\ 6,\ 8,\ 9,\ 12 \\ E_7 &: \quad 2,\ 6,\ 8,\ 10,\ 12,\ 14,\ 18 \\ E_8 &: \quad 2,\ 8,\ 12,\ 14,\ 18,\ 20,\ 24,\ 30\ . \end{aligned} \tag{2.6}$$

This leads to a general formula for the critical central charge [6]

$$c^* = 2n(2h^2 + 2h + 1)\ . \tag{2.7}$$

Using (2.3) and (2.4), this gives the following expression for the background-charge parameter α_0^*:

$$(\alpha_0^*)^2 = \frac{(2h+1)^2}{h(h+1)}\ . \tag{2.8}$$

From now on, we shall always take the critical values for the central charge and the background charge parameter.

It was shown in [2,5] that the WA_n algebra can be realised in terms of the currents of the WA_{n-1} algebra and one extra free scalar field (see also [9], where this was first observed for WA_2, and conjectured for WA_n). We shall show in subsection 2.2 that the WD_n algebra may similarly be realised in terms of the currents of the WD_{n-1} algebra and one extra free scalar field, and we shall give evidence in subsection 2.3 for a similar reduction procedure for the exceptional algebras. Applying the above mentioned reduction recursively, one can then realise these W algebras in terms of an arbitrary energy-momentum tensor $T^{\rm eff}$ and $(n-1)$ free scalar fields. As explained in [1,2,3,5], the W constraints have the effect of "freezing" the $(n-1)$ free scalars.* Thus if $T^{\rm eff}$ is taken to be the energy-momentum tensor for a new set of free scalar fields X^μ, one is left with an effective Virasoro-like string theory, where the X^μ fields have the interpretation of being target-spacetime coordinates. This effective string theory has a non-standard value for the central charge $c^{\rm eff}$, and the W constraints imply that the spin-2 intercept $L_0^{\rm eff}$ of $T^{\rm eff}$ takes values from a finite set. As we shall see, the central charge $c^{\rm eff}$ is always given by

$$c^{\rm eff} = 1 + 6(\alpha_0^*)^2\ . \tag{2.9}$$

* Identifying the frozen coordinates is not always straightforward. Indeed, for a general choice of basis of the simple roots of the underlying Lie algebra, the unfrozen coordinate turns out to be a non-trivial linear combination of the scalars $(\varphi_1, \ldots, \varphi_n)$.

The relation between these W strings and Virasoro minimal models emerges by substituting the critical value α_0^* for the background-charge parameter given in (2.8) into (2.9), leading to

$$c^{\rm eff} = 26 - \Big[1 - \frac{6}{h(h+1)}\Big] \ . \tag{2.10}$$

Here, 26 is the critical central charge for the Virasoro string and the remainder is precisely the central charge of the $(h, h+1)$ unitary Virasoro minimal model. This connection with minimal models is further strengthened by the fact that if one rewrites the effective intercepts $L_0^{\rm eff}$ as

$$L_0^{\rm eff} = 1 - L_0^{\rm min} \ , \tag{2.11}$$

where 1 is the value for the intercept of the critical Virasoro string, then $L_0^{\rm min}$ takes values precisely from a subset of the dimensions of the primary fields of the $(h, h+1)$ minimal model. This has been shown in [4,5] for the WA_n strings, and we shall show it below for the WD_n strings.

We shall focus for now on the tachyonic physical spectrum of these simply-laced W strings. Here "tachyonic" signifies states at level 0, obtained by acting on the $SL(2,C)$ vacuum with operators of the form

$$V_{\vec\beta} = e^{\vec\beta\cdot\vec\varphi} \ . \tag{2.12}$$

(Recall that since the additional scalar fields X^μ appear in the W currents only through their energy-momentum tensor, it suffices, when discussing tachyonic states, to consider just one such extra scalar, which is in fact the one that automatically remains in the reduction process that we described earlier.) The physical-state conditions for tachyonic states are given by

$$\big(W_s\big)_0 |{\rm phys}\rangle = \omega_s |{\rm phys}\rangle \ , \tag{2.13}$$

where $\big(W_s\big)_0$ denotes the zero Laurent mode of the spin-s current $W_s(z)$, and ω_s its intercept. $W_2(z)$ is the energy-momentum tensor $T(z)$, given in (2.1).

These intercepts should in principle be determined by requiring that the nilpotent BRST operator for the algebra annihilate the physical vacuum. In practice, however, the construction of the BRST operator is very complicated and has only been performed for the $W_3 = WA_2$ algebra [10]. The intercepts can, however, be determined if one knows a particular physical state, since one can then simply read them off by acting with $\big(W_s\big)_0$ on that state. In fact such a physical state has been proposed for the WA_n algebra. It is obtained by acting on the $SL(2,C)$ vacuum with the "cosmological-constant operator"

$$V_{\rm cosmo} = e^{\lambda\alpha_0^*\vec\rho\cdot\vec\varphi} \ , \tag{2.14}$$

where $\vec\rho$ is the Weyl vector for A_n and λ is a certain constant to be determined. Since one knows the values of the intercepts for the WA_2 algebra, one can explicitly check the existence of such a physical state in this case. Classical-correspondence considerations presented in [2] already suggested that such a physical state indeed exists for all the WA_n algebras. Stronger evidence towards this assumption was given in [5], where it was shown that, if the WA_n-string theory is to be unitary, then this "cosmological solution" is necessarily contained in its physical spectrum. The simplicity of the form of the cosmological operator (2.14) leads us to conjecture its existence for *all* the simply-laced W algebras. We shall thus assume that such a state is contained in the physical spectrum of the simply-laced W strings.

The value of the constant λ in the cosmological operator (2.14) can be computed by an independent argument that enables one to determine the spin-2 intercept ω_2 from the structure of the ghost vacuum. The total energy-momentum tensor $T^{\rm tot} \equiv T^{\rm mat} + T^{\rm ghost}$ generates a linear algebra, and consequently, since the BRST quantisation procedure requires $T^{\rm tot}$ to annihilate $\big|{\rm phys}\big\rangle \otimes \big|{\rm vac}\big\rangle_{\rm ghost}$, one can read off the spin-2 intercept ω_2 as the negative of the intercept for $T^{\rm ghost}$ on the ghost vacuum, which is defined by

$$\big|{\rm vac}\big\rangle_{\rm ghost} \equiv \prod_{\{s\}} \prod_{m=1}^{s-1} \big(c_s\big)_m \big|0\big\rangle, \tag{2.15}$$

where $\big|0\big\rangle$ is the $SL(2,C)$ vacuum and $\big(c_s\big)_m$ are the Laurent modes of the usual ghost field for the spin-s current. The spin-2 intercept is therefore given by

$$\omega_2 = \tfrac{1}{2}\sum_{\{s\}} s(s-1) = \tfrac{1}{6} n h(h+1)\ . \tag{2.16}$$

Note that using the Freudenthal strange formula and (2.4), one finds the important property

$$\omega_2 = 2\,\vec{\rho}^{\,2}\ . \tag{2.17}$$

The value of λ can now be determined by requiring that the cosmological operator (2.14) be a primary field of dimension ω_2 with respect to the energy-momentum tensor (2.1). This gives a quadratic equation for λ. As will be clear later, the two solutions of this equations are related by a Weyl reflection, and one can thus, without loss of generality, take one of the solutions, *e.g.*

$$\lambda = \frac{2(h+1)}{2h+1}\ , \tag{2.18}$$

and refer to this solution as the cosmological solution.

The complete tachyonic physical spectrum of these simply-laced W strings then follows from this cosmological solution. To see this, one first recalls [7,8,11] that the eigenvalues $v_s(\vec{\beta})$ of a state created by any tachyonic operator (2.12) under the action of $\big(W_s\big)_0$ are invariant under the action of the Weyl group $\mathcal{W}$ of the underlying Lie algebra on the shifted momentum $\vec{\gamma}$, which is defined by

$$\vec{\beta} = \alpha_0^*(\vec{\rho} + \vec{\gamma})\ . \tag{2.19}$$

Since the physical-state conditions for tachyons are given by $v_s(\vec{\beta}) = \omega_s$, it follows that the action of Weyl group on the shifted momentum $\vec{\gamma}$ maps solutions of these physical-state conditions into solutions. In fact, we know one solution of the tachyonic physical-state conditions, *viz.* the cosmological solution; this is a solution by construction. Weyl reflecting the shifted momentum of this solution,

$$\vec{\gamma}^{\rm cosmo} = \frac{1}{2h+1}\,\vec{\rho}\ , \tag{2.20}$$

thus leads to new physical states of the corresponding W string theory. Since the Weyl vector is not a fixed point of the Weyl group, one can therefore construct $\dim(\mathcal{W})$ different tachyonic physical states. On the other hand $v_s(\vec{\beta})$ is a polynomial of degree s in $\vec{\beta}$, and thus also in the shifted momentum $\vec{\gamma}$, and so it follows that the tachyonic physical-state conditions will have $\left(\prod_{\{s\}} s\right)$ different solutions. Remarkably, this expression, which is the product of the orders of the independent Casimir operators, is precisely the dimension of the Weyl group! (Note that this is true for *any* simple Lie algebra.) We therefore conclude that the action of the Weyl group on the shifted momentum (2.19) of the cosmological solution indeed generates the entire tachyonic physical spectrum.

Having found the complete tachyonic physical spectrum of these W strings, one can now compute the effective spin-2 intercepts $L_0^{\rm eff}$. To do this one has to identify the unfrozen coordinate. This requires knowledge of the explicit form of the realisation of the W algebra. It is therefore necessary to continue the discussion of these W strings case by case.

2.2 W Strings for Classical Simply-laced Algebras

Since the WA_n string has been treated in detail in [5], we shall just summarise the results. In this particular case, the complete physical spectrum (including *all* higher-level states) has been obtained. Physical states with excitations in the frozen directions have zero norm and thus decouple from the theory. The remaining physical states have positive semi-definite norm and their effective intercepts, at all higher levels, take the same set of $L_0^{\rm eff}$ values as those for the tachyonic states. Using (2.11), this leads to the following values of $L_0^{\rm min}$:

$$L_0^{\rm min} = \frac{k^2-1}{4h(h+1)}\ , \qquad k = 1, 2, \ldots, h-1\ . \tag{2.21}$$

These are precisely the diagonal entries $\Delta_{(k,k)}$ of the Kac table of the $(h, h+1)$ unitary Virasoro minimal model, whose dimensions are

$$\Delta_{(r,s)} = \frac{[(h+1)r - hs]^2 - 1}{4h(h+1)}\ , \tag{2.22}$$

with $1 \le r \le h-1$ and $1 \le s \le h$.

Let us now turn our attention to the WD_n case. The WD_n algebra is generated by currents $W_{2k}^{(n)}(z)$ of spin $s = 2k$, with $k = 1, \ldots, n-1$, and a current $U^{(n)}(z)$ of spin n. Since $D_2 \cong A_1 \times A_1$ is not simple, we shall restrict ourselves to $n \geq 3$. A realisation of WD_n in terms of n free scalar fields $(\varphi_1, \ldots, \varphi_n)$ has been given by Fateev and Lukyanov in [7,8]. The spin-n current is given by the Miura-type transformation

$$U^{(n)}(z) = (\alpha_0 \partial - \partial\varphi_n)(\alpha_0\partial - \partial\varphi_{n-1}) \cdots (\alpha_0\partial - \partial\varphi_2)\partial\varphi_1 \ , \tag{2.23}$$

where, as usual, normal ordering is assumed. The $W_{2k}^{(n)}$ currents can then be read off from the operator-product expansion $U^{(n)}(z)U^{(n)}(w)$:

$$U^{(n)}(z)U^{(n)}(w) \sim \frac{a_n}{(z-w)^{2n}} + \sum_{k=1}^{n-1} \frac{a_{n-k}}{(z-w)^{2n-2k}} \big(W_{2k}^{(n)}(z) + W_{2k}^{(n)}(w)\big) \ . \tag{2.24}$$

In this expression a_k are constants, given by $a_k = (-)^{k+1}\prod_{j=1}^{k-1}[1 + 2j(2j+1)\alpha_0^2]$, which fix the normalisations of the higher-spin currents to their standard forms.

Note that the way in which the WD_n currents are obtained is rather different from the procedure that one uses for the WA_n algebra. In that case, all the currents are obtained directly from a Miura transformation. In fact in general for any simple Lie algebra, a classical W algebra can be obtained from a Miura transformation [12]. In the case of the WA_n algebra, one can straightforwardly take the classical currents and interpret them as currents for the full quantum WA_n algebra, by simply replacing products of fields by normal-ordered products. A realisation of the classical WD_n can be similarly obtained from a Miura transformation, involving, however, a pseudo-differential operator [7,8]. In view of this it is simpler to obtain the currents that generate the full quantum WD_n algebra from the operator-product expansion of $U^{(n)}$ with itself [7,8], according to (2.24). In our discussion, however, we shall never need to know the explicit form of the $W_{2k}^{(n)}$ currents for $k \geq 2$. The only information that we need about these higher-spin currents is that, acting on tachyonic operators (2.12), they give shifted-momentum polynomials that are invariant under the action of the Weyl group, as we discussed earlier.

From (2.23) and (2.24) it follows that the energy-momentum tensor $T = W_2^{(n)}$ is given by (2.1), with

$$\vec{\rho} = (0,\ 1,\ 2,\ \ldots,\ n-1)\ . \tag{2.25}$$

This is indeed the Weyl vector of D_n, with simple roots given by

$$\begin{aligned} \vec{e}_i &= \vec{\sigma}_{n-i} - \vec{\sigma}_{n+1-i}, \qquad i = 1, 2, \ldots, n-1 \\ \vec{e}_n &= \vec{\sigma}_1 + \vec{\sigma}_2 \ , \end{aligned} \tag{2.26}$$

with $\{\vec{\sigma}_i\}$ $(i = 1, \ldots, n)$ the canonical orthonormal basis $\vec{\sigma}_i \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$ for R^n, where the 1 occurs in the i-th entry.

The reduction procedure mentioned earlier can now easily been proven since $U^{(n)}$ can be rewritten as

$$U^{(n)} = \alpha_0 \partial U^{(n-1)} - \partial\varphi_n U^{(n-1)} \ , \tag{2.27}$$

where $U^{(n-1)}$ is the spin-$(n-1)$ current of WD_{n-1}. It then follows from (2.24) that the $W_{2k}^{(n)}$ currents can, likewise, be written in terms of the WD_{n-1} currents together with one extra scalar field φ_n.

Applying this reduction recursively, one can thus express the currents of the WD_n algebra in terms of the currents of WD_3 together with $(n-3)$ scalar fields $(\varphi_4, \ldots, \varphi_n)$ (recall that we only consider the case $n \geq 3$). Since $D_3 \cong A_3$, the WD_3 algebra is isomorphic to WA_3. We can then use the recursion procedure proven in [2,5] to realise WA_3 in terms of an arbitrary effective energy-momentum tensor T^{eff} together with 2 scalar fields. We can summarise this recursion procedure schematically as

$$WD_n \to WD_{n-1} \to \cdots \to WD_3 \cong WA_3 \to WA_2 \to WA_1 = \text{Virasoro} \ . \tag{2.28}$$

Since the WA_n algebras have already been treated, it is convenient to rewrite the realisation of WD_3 in the basis in which WA_3 has been studied. This can be obtained by the orthogonal transformation

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \widetilde{\varphi}_1 \\ \widetilde{\varphi}_2 \\ \widetilde{\varphi}_3 \end{pmatrix} \ . \tag{2.29}$$

Here $(\varphi_1, \varphi_2, \varphi_3)$ are the scalar fields of the WD_3 realisation as defined in (2.23), and $(\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{\varphi}_3)$ are the scalars of the WA_3 realisation obtained from the Miura transformation as given in [5]. In this basis the unfrozen coordinate is $\widetilde{\varphi}_1 = -(\varphi_2 - \varphi_3)/\sqrt{2}$. The effective energy-momentum tensor in the case of the WD_n algebra is thus

$$T^{\text{eff}} \equiv -\tfrac{1}{2}\partial\widetilde{\varphi}_1 \partial\widetilde{\varphi}_1 + \tfrac{1}{\sqrt{2}}\alpha_0^* \partial^2 \widetilde{\varphi}_1 = -\tfrac{1}{4}\big(\partial\varphi_2 - \partial\varphi_3\big)^2 - \tfrac{1}{2}\alpha_0^*\big(\partial^2\varphi_2 - \partial^2\varphi_3\big) \ , \tag{2.30}$$

and generates a Virasoro algebra with central charge c^{eff} given by (2.9). The effective intercept L_0^{eff} for the WD_n string is then given by

$$L_0^{\text{eff}} = -\tfrac{1}{4}(\beta_2 - \beta_3)^2 - \tfrac{1}{2}\alpha_0^*(\beta_2 - \beta_3) \ . \tag{2.31}$$

Using equations (2.8), (2.11) and (2.19) it follows that

$$L_0^{\text{min}} = \frac{\big[(2h+1)(\gamma_2 - \gamma_3)\big]^2 - 1}{4h(h+1)} \ , \tag{2.32}$$

where γ_2 and γ_3 are shifted-momentum components of tachyonic physical states. Note that substituting for the cosmological solution, *i.e.* $\gamma_2^{\text{cosmo}} = 1/(2h+1)$ and $\gamma_3^{\text{cosmo}} = 2/(2h+1)$, one finds $L_0^{\text{min}} = 0$ which is the identity operator of the $(h, h+1)$ Virasoro minimal model.

As mentioned in the previous subsection, the complete tachyonic physical spectrum of the WD_n string can be obtained from the action of the Weyl group of D_n on the shifted-momentum $\vec{\gamma}^{\rm cosmo}$ given in (2.20) with $\vec{\rho}$ given by (2.25). The Weyl group of D_n acts on $\vec{\rho}$ by permuting its components and/or changing their signs [13]. The shifted momenta for all the tachyonic physical states are thus given by

$$\vec{\gamma} = \frac{1}{2h+1}\Big(0,\ 1,\ 2,\ \ldots,\ n-1\Big)\ , \tag{2.33}$$

together with all possible permutations of the components and all possible combinations of sign changes, making $2^{n-1}n!$ states in all. Hence writing $\gamma_2 - \gamma_3 = k/(2h+1)$, we find that the absolute value of k can take all integer values in the range $1,\ 2,\ \ldots,\ 2n-3$. Since $h = 2n-2$ for D_n, it follows that $L_0^{\rm min}$ takes precisely the values given in (2.21). We thus conclude that the tachyonic physical states of the WD_n string are indeed related to the diagonal entries of the Kac table of the $(h, h+1)$ unitary Virasoro minimal model.

2.3 W Strings for Exceptional Simply-laced Algebras

Since a Miura-type realisation for the WE_6, WE_7 and WE_8 algebras has not yet been constructed, it seems that *a priori* the corresponding string theories cannot at present be discussed. However, the structure of the WA_n and the WD_n strings suggests a possible generalisation to these cases. The main reason why the classical simply-laced W strings are related to Virasoro minimal models seems to be equation (2.17), which expresses the spin-2 intercept ω_2 in terms of the Weyl vector of the underlying Lie algebra. Since this equation also holds for the exceptional simply-laced W strings, the entire discussion of subsection 2.1 can be performed for these cases as well. The only missing link is a recursion relation which enables one to identify the unfrozen coordinate and prove that the central charge $c^{\rm eff}$ of the effective energy-momentum tensor $T^{\rm eff}$ is given by (2.9).

It is, however, not too difficult to see how such a recursion procedure might work. A possible reduction follows from noting that $E_5 \cong D_5$, which leads schematically to

$$WE_8 \rightarrow WE_7 \rightarrow WE_6 \rightarrow WE_5 \cong WD_5 \rightarrow WD_4 \rightarrow WD_3 \cong WA_3 \rightarrow WA_2 \rightarrow \text{Virasoro}\ . \tag{2.34}$$

Alternatively, since $E_4 \cong A_4$ one can consider

$$WE_8 \rightarrow WE_7 \rightarrow WE_6 \rightarrow WE_5 \rightarrow WE_4 \cong WA_4 \rightarrow WA_3 \rightarrow \cdots \rightarrow \text{Virasoro}\ . \tag{2.35}$$

In both of these schemes, the reduction is assumed to be proceeding *via* the sequence of canonical embeddings $E_8 \supset E_7 \supset E_6 \supset \cdots$. (By "canonical," we mean an embedding described by deleting a vertex in the Dynkin diagram.) *A priori*, this is not the only route that the reduction might follow; for example, one has an alternative sequence of canonical embeddings $E_p \supset A_{p-1} \supset \cdots \supset A_1$. However, it should be emphasised that one does not have a free choice in deciding which reduction scheme is to be selected. Rather, this is dictated by the detailed form of the Miura transformation. For example, we have seen in the previous subsection that the reduction process for WD_n singles out φ_n, *i.e.* the field that sits at the uttermost left of the product of factors in the Miura transformation (2.23), corresponding to the canonical embedding $D_n \supset D_{n-1}$. Even though there are other canonical embedding sequences, such as $D_n \supset A_{n-1} \supset A_{n-2} \supset \cdots \supset A_1$, the Miura transformation imposes the specific reduction scheme (2.28). Thus for the exceptional WE_p algebras, one has to know the detailed form of the Miura transformation in order to settle the issue of which reduction scheme is selected.

No matter what reduction scheme is actually selected by the Miura transformation, it is natural to expect that the relation that holds between WA_n and WD_n strings and minimal models should hold also in the case of the exceptional simply-laced W strings. The tachyonic physical spectrum of the WE_6, WE_7 and WE_8 strings would then be related to the diagonal primary fields of the $(12,13)$, $(18,19)$ and $(30,31)$ unitary Virasoro minimal models respectively.

3. Non-simply-laced Algebras

The orders of the independent Casimir operators for the non-simply-laced simple Lie algebras are

$$\begin{aligned} B_n &: \quad 2,\ 4,\ 6,\ \ldots,\ 2n \\ C_n &: \quad 2,\ 4,\ 6,\ \ldots,\ 2n \\ F_4 &: \quad 2,\ 6,\ 8,\ 12 \\ G_2 &: \quad 2,\ 6\ . \end{aligned} \tag{3.1}$$

For WB_n a Miura-type realisation has been given in [7]. In addition to bosonic currents having the spins listed for B_n in (3.1), there is also a fermionic current with spin $(n+\frac{1}{2})$. The first non-trivial example, WB_2, was constructed explicitly in [14].

The fermionic spin-$(n+\frac{1}{2})$ current $Q^{(n)}(z)$ of WB_n plays an analogous rôle to the bosonic spin-n current $U^{(n)}(z)$ in the WD_n algebra. It is given by the Miura-type transformation

$$Q^{(n)}(z) = (\alpha_0\partial - \partial\varphi_n)\,(\alpha_0\partial - \partial\varphi_{n-1})\,\cdots\,(\alpha_0\partial - \partial\varphi_1)\,\psi\ , \tag{3.2}$$

where ψ is a free fermion field. The bosonic currents $W^{(n)}_{2k}$ can be read off from the operator-product expansion $Q^{(n)}(z)Q^{(n)}(w)$:

$$Q^{(n)}(z)Q^{(n)}(w) \sim \frac{b_n}{(z-w)^{2n+1}} + \sum_{k=1}^{n} \frac{b_{n-k}}{(z-w)^{2n+1-2k}}\big(W^{(n)}_{2k}(z) + W^{(n)}_{2k}(w)\big)\ . \tag{3.3}$$

In this expression b_k are constants, given by $b_k = \prod_{j=1}^{k}[1 + 2j(2j-1)\alpha_0^2]$. We shall only need to know the explicit form of the spin-2 current $T(z) = W_2^{(n)}$. For the higher-spin currents, it suffices for our purposes to observe that the eigenvalues of all the currents acting on tachyonic operators are invariant under the action of the Weyl group of B_n.

The energy-momentum tensor, which can be read from (3.3),

$$T(z) = -\tfrac{1}{2}\partial\vec{\varphi}\cdot\partial\vec{\varphi} + \alpha_0\vec{\rho}\cdot\partial^2\vec{\varphi} + \tfrac{1}{2}\partial\psi\,\psi\ , \tag{3.4}$$

has central charge

$$c = (n+\tfrac{1}{2})\big(1 + 2n(2n-1)\alpha_0^2\big)\ . \tag{3.5}$$

In (3.4) $\vec{\rho}$ is the Weyl vector of the B_n algebra, which takes the form

$$\vec{\rho} = \big(\tfrac{1}{2},\ \tfrac{3}{2},\ \ldots,\ n-\tfrac{1}{2}\big)\ , \tag{3.6}$$

with the simple roots being given by

$$\begin{aligned} \vec{e}_i &= \vec{\sigma}_{n-i} - \vec{\sigma}_{n+1-i}, \qquad i = 1,2,\ldots,n-1 \\ \vec{e}_n &= \vec{\sigma}_1\ . \end{aligned} \tag{3.7}$$

Note that $\vec{\rho}^{\,2} = \frac{1}{12}n(4n^2-1)$, which we have used in writing (3.5).

From (3.2), we see that

$$Q^{(n)} = \alpha_0\partial Q^{(n-1)} - \partial\varphi_n Q^{(n-1)}\ , \tag{3.8}$$

which enables us to express the currents of WB_n in terms of those of WB_{n-1} together with the extra scalar φ_n. Applying this procedure recursively, we may therefore realise WB_n in terms of the currents of WB_1 together with $(n-1)$ scalars $(\varphi_2,\ldots,\varphi_n)$. The WB_1 algebra, which is generated by currents of spins $\frac{3}{2}$ and 2, is isomorphic to the $N=1$ super-Virasoro algebra. Thus replacing these currents, initially realised in terms of the scalar φ_1 and the fermion ψ, by an arbitrary realisation with the same central charge $c^{\rm eff} = \frac{3}{2} + 3\alpha_0^2$, we find that the WB_n string is effectively reduced to $N=1$ worldsheet super-Virasoro strings. Schematically, this can be summarised by the diagram

$$WB_n \to WB_{n-1} \to\ \cdots\ \to WB_2 \to WB_1 = \text{super-Virasoro}\ . \tag{3.9}$$

The critical central charge for the WB_n string, given by (2.5) with an appropriate sign change for the contribution from the ghosts for the spin-$(n+\frac{1}{2})$ fermionic current, is

$$c^* = (2n+1)(8n^2-4n+1)\ . \tag{3.10}$$

This leads to the critical value of the background-charge parameter

$$(\alpha_0^*)^2 = \frac{(4n-1)^2}{2n(2n-1)}\ . \tag{3.11}$$

Using these results, one finds that the central charge of the effective $N=1$ superstring is

$$c^{\rm eff} = 15 - \frac{3}{2}\Big[1 - \frac{8}{4n(4n-2)}\Big]\ . \tag{3.12}$$

Here 15 is the critical central charge for the $N=1$ superstring, and the remainder is the central charge of the $(4n-2, 4n)$ unitary $N=1$ super-Virasoro minimal model. Following the discussion for the simply-laced case, we shall now compute the effective spin-2 intercepts and rewrite them as

$$L_0^{\rm eff} = \tfrac{1}{2} - L_0^{\rm min}\ , \tag{3.13}$$

where $\frac{1}{2}$ is the spin-2 intercept for the critical $N=1$ superstring. We shall see that $L_0^{\rm min}$ takes values from the diagonal entries of the corresponding $(4n-2, 4n)$ super minimal model.

As in the simply-laced case, we shall assume that the cosmological operator $e^{\lambda\alpha_0^*\vec\rho\cdot\vec\varphi}$ creates a physical state when acting on the $SL(2,C)$ vacuum.‡ The constant λ can be determined from the knowledge of the intercept of the spin-2 current. The latter can be derived by augmenting the ghost-vacuum discussion of subsection 2.1 to include the contribution of the ghosts for fermionic currents. As explained in [15], the total contribution to the spin-2 intercept for a theory with both bosonic and fermionic currents is given by

$$\omega_2 = \tfrac{1}{2}\sum_{\substack{\{s\}\\ \rm bosonic}} s(s-1) - \tfrac{1}{2}\sum_{\substack{\{s\}\\ \rm fermionic}} (s-\tfrac{1}{2})^2\ . \tag{3.14}$$

For the case of the WB_n algebra, we therefore find

$$\omega_2 = \tfrac{1}{6} n(4n^2-1)\ . \tag{3.15}$$

Solving the resulting quadratic equation for λ, and without loss of generality choosing just one of its roots to define the cosmological solution, we find

$$\lambda = \frac{4n}{4n-1}\ . \tag{3.16}$$

The physical-state conditions for tachyonic states are invariant under the action of the Weyl group of B_n on the shifted momentum as defined in (2.19). The dimension of the Weyl group is $2^n n!$, which is precisely equal to the product of the spins of the bosonic currents in

‡ We shall consider only the Neveu-Schwarz sector here, for which the fermionic current, since it has half-integer modes, imposes a physical-state condition that is automatically satisfied for tachyonic operators.

the WB_n algebra. Since the physical-state condition from a spin-s current is a polynomial of degree s in the shifted momentum, it follows that $2^n n!$ is also the number of tachyonic physical states. Thus, since the Weyl vector is not a fixed point of the Weyl group, we obtain the entire tachyonic spectrum by Weyl-reflecting the cosmological solution.

Since the reduction from WB_n to WB_1 freezes the fields $(\varphi_2, \ldots, \varphi_n)$, it follows that the energy-momentum tensor for the effective super-Virasoro string theory is given by

$$T^{\rm eff} = -\tfrac{1}{2}\big(\partial\varphi_1\big)^2 + \tfrac{1}{2}\alpha_0^*\partial^2\varphi_1 + \tfrac{1}{2}\partial\psi\,\psi\ . \tag{3.17}$$

Thus by acting on tachyonic operators of the form (2.12), we find that $L_0^{\rm eff}$ is given by

$$L_0^{\rm eff} = -\tfrac{1}{2}\beta_1^2 + \tfrac{1}{2}\alpha_0^*\beta_1 = \tfrac{1}{8}(\alpha_0^*)^2\big(1 - 4\gamma_1^2\big)\ , \tag{3.18}$$

where the second equality is formulated in terms of the shifted-momentum component γ_1. To find the allowed values of γ_1, one has to act with the Weyl group on the cosmological solution. This has the effect of permuting the components, and changing their signs in all possible ways [13], giving $2^n n!$ possibilities. Consequently γ_1 can take the values

$$\gamma_1 = \pm\frac{2k-1}{2(4n-1)}, \qquad k = 1, 2, \ldots, n\ . \tag{3.19}$$

Using (3.18), we find that $L_0^{\rm min}$, defined by (3.13), takes the values

$$L_0^{\rm min} = \frac{(2k-1)^2 - 1}{16n(2n-1)}\ . \tag{3.20}$$

The general result for the dimensions $\Delta_{(r,s)}$ of the primary fields in the Neveu-Schwarz sector of the $N = 1$ super-Virasoro $(m, m+2)$ minimal model is

$$\Delta_{(r,s)} = \frac{[(m+2)r - ms]^2 - 4}{8m(m+2)}\ , \tag{3.21}$$

where $1 \le r \le m-1$ and $1 \le s \le m+1$, with $(r-s)$ an even integer. Comparing with (3.20), we conclude that the tachyonic spectrum of the WB_n string is related to the $r = s = 2k-1$ diagonal entries of the Kac table of the $(4n-2, 4n)$ unitary super-Virasoro minimal model, for $k = 1, 2, \ldots, n$. Unlike the simply-laced case, we therefore find only a subset of the diagonal entries at this tachyonic level. It may be that the remaining diagonal entries arise from higher-level states, or from the Ramond sector.

There exists a different W algebra based on B_n. It can be obtained from a Hamiltonian reduction of B_n, and is generated by bosonic currents of spins $2, 4, \ldots, 2n$ only [12]. The spin-2 intercept for the corresponding string theory can be easily found to be given by $\omega_2 = \frac{1}{6}n(n+1)(4n-1)$, which is not equal to $2\vec{\rho}^{\,2}$, where $\vec{\rho}$ is given by (3.6). For the WB_n

string discussed in detail above, which has an additional fermionic spin-$(n+\frac{1}{2})$ current, this important relation (2.17) *is* satisfied.* It seems that this relation between the spin-2 intercept and the Weyl vector is crucial in establishing the connection between W strings and minimal models.

In the cases of C_n, F_4 and G_2, equation (2.17) is not satisfied. One can however, as for B_n, add additional currents in the cases of C_n and F_4 so that (2.17) is satisfied. For C_n, this can be done by adding a spin-$(n+1)$ current, but then the resulting field content is precisely that of the WD_{n+1} algebra. For F_4, adding a fermionic current of spin $\frac{17}{2}$ leads to a spin-2 intercept satisfying (2.17). To show the possible connection with a minimal model, however, one needs an explicit realisation of the corresponding W algebra.

4. Conclusion and Discussions

In this paper, we have studied some properties of W strings based on sibling and exceptional simple Lie algebras. We have found that many of the features encountered previously for WA_n strings [5] carry over to these other examples, notably the connection between the tachyonic spectrum and minimal models.

In the case of the WA_n string, it was shown that the only higher-level states that contribute to the physical spectrum are those that involve excitations exclusively in the unfrozen directions [5]. Moreover these higher-level physical states have a set of effective spin-2 intercepts identical to the corresponding set for the tachyonic level. Consequently the physical states at level ℓ have $(\text{mass})^2$ given by 2ℓ plus the $(\text{mass})^2$ of the corresponding tachyonic physical states. However, this notion of mass is rather problematical, owing to the presence of background charges; see refs. [3,5] for further discussion of this point. In view of our findings it is natural to expect that higher-level physical states will again arise only in the unfrozen directions for the W strings discussed in this paper. Strong evidence for the unitarity of WA_n strings was presented in [5]; it is likely that the W strings considered here are unitary also.

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* Recall that the $(\text{length})^2$ of the long simple roots is normalised to 2.

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