CERN-TH.5231/89
CPTH-A890.0389
MAD/TH/88-23
CPT-TAMU-83/89

AN EXPANDING UNIVERSE IN STRING THEORY

I. Antoniadis and C. Bachas
Centre de Physique Théorique, Ecole Polytechnique
91128 Palaiseau, France

John Ellis
CERN - Geneva

and

D.V. Nanopoulos
Physics Department, Texas A&M University
College Station, Texas, U.S.A.

ABSTRACT

We present solutions of the bosonic, heterotic and type II string theories whose space-time manifold is a linearly expanding homogeneous and isotropic Universe. These solutions are obtained by giving a background charge to the time co-ordinate on the world-sheet. We find the spectrum, demonstrate positivity of the Hilbert space up to the second excited level, and construct modular-invariant partition functions. The central charge of the transverse excitations is a free parameter that controls the asymptotic density of states; the critical dimension and gauge group can in particular be made arbitrary large. We show how to construct dual, factorizable, energy-conserving amplitudes in this background, discuss their interpretation and comment on the initial singularity and flatness problems in the light of our results.

CERN-TH.5231/89
CPTH-A890.0389
MAD/TH/88-23
CPT-TAMU-83/89
November 1988

1. INTRODUCTION

Nowadays the radius of the Universe is much larger than the Planck length, or any other fundamental physical scale, and its curvature is correspondingly very small. Therefore it is normally thought to be a good approximation in particle physics to neglect the expansion of the universe, and treat the four large space-time dimensions as infinite and flat. This is why strings have been mostly studied up to now in four- or higher-dimensional Minkowski space-time. The role of a non-trivial gravitational background has received very little attention, even in attempts to discuss the physics of the early universe. This is somewhat paradoxical, because strings have been mainly advocated as consistent quantum theories of gravity [1]: it is therefore precisely in non-trivial gravitational settings that one should expect our field theory intuition to fail completely , and string theory to teach us something new and interesting. The purpose of this paper is to study string propagation in a simple, time- dependent cosmological background, which illustrates how many properties such as the critical dimension, gauge group and behaviour of couplings are radically modified, and gives us some new insights into the fundamental cosmological problems of the initial singularity and flatness.

This simple cosmological background [2,3] is obtained by giving a linear time-dependence to the dilaton field. From the world-sheet point of view this corresponds to putting an (imaginary) charge at infinity for the time- coordinate. Conformal invariance is not destroyed provided one arranges for the total central charge to remain unchanged. The "physical metric" , with correctly normalized Einstein action , describes a linearly expanding D-dimensional Robertson-Walker universe. In the particular case D=4 one may furthermore curve space into a 3-sphere (thus respecting isotropy and homogeneity), by giving a linear time dependence to the axion. The corresponding conformal theory is a Wess-Zumino-Witten model [4] on the SO(3) group manifold. In fact in the absence of dynamic "matter" these **exact** solutions are unique in the following sense [3]: any other homogeneous, isotropic solution of the one-loop β-function equations with cosmological term approaches them asymptotically at large times.

Although a period of linear expansion could have occured at some earlier epoch, it is not within the scope of this paper to propose a realistic cosmological scenario. We will rather use these simple models as playgrounds, in which to address a host of interesting questions about strings in an expanding universe: is their ultraviolet behaviour modified? How do anomalies cancel ? How is the coupling affected by the time-dependence of the dilaton? What happens to space-time supersymmetry [5]? Is the expansion rate quantized? And how do strings avoid the initial singularity? The sometimes surprising answers we will find to such questions, could very well be relevant in other cosmological solutions .

In section 2 we start by describing the simple cosmological solutions we will study. Their basic parameter is the background charge $2iQ$ of the time-coordinate. From the geometric point of view it gives the expansion rate in conformal time. It is related to the central charge of transverse excitations through the conformal-anomaly cancellation condition :

$$c_{transv} = 24 + 12Q^2 \qquad\qquad (12 + 12Q^2) \tag{1.1}$$

for the bosonic (supersymmetric) string. This implies in particular that the critical dimension and maximal heterotic gauge group are modified by the expansion of the universe, so that these solutions are not classical backgrounds in 26 (or 10) dimensional space-time.

In section 3 we study the free bosonic string in this background. Besides the change in the dimension, the effect of the background can be summarized by a constant negative shift in the spectrum of the flat wave-operator. We show that positivity up to the second massive level and modular invariance give no additional constraints on Q, other than the anomaly cancellation condition (1.1) , and do not restrict the range of allowed energies. This is to be contrasted with the Coulomb gas representation of minimal models, or string theory on the 3-sphere, where unitarity and modular invariance restrict both the allowed central charge and the zero mode spectrum. On the other hand, condition (1.1) could by itself give an "external" quantization of the expansion rate Q [3], due to the fact that the central charge of internal compactified coordinates cannot be varied continuously.

In section 4 we turn to heterotic and type-II supersymmetric strings. Simple counting of degrees of freedom shows that space-time supersymmetry is lost when the background is switched on. Fermions in this background still obey the flat Dirac equation without a mass shift. We show how to construct anomaly-free chiral theories in 10+16n dimensions, with a rank 16+8n gauge group. We also show how for an appropriate expansion rate, the type II string can have a free spectrum identical to that of the bosonic string. Finally we argue that in an expanding universe type II strings could contain the gauge and matter content of the standard model.

In section 5 we study interactions. We show how to define dual , factorizable , energy-conserving amplitudes . Technically this requires the use of screening operators [6], and a restriction of the values of Q to a discrete but **dense** set in the real line . By calculating the 3-point amplitude in closed form we can, however, show that it does not depend on the precise way of screening , and argue that it should have an analytic continuation to all values of Q. Properly intepreting these amplitudes is a subtle issue, because the background is not asymptotically flat. It seems however that string theory chooses to respect a symmetry under combined time-translations and dilaton shifts, which furthermore allows it to avoid the problem of the initial singularity.

The linearly-expanding string solutions are probably unstable, as suggested among other things by the presence of one-loop divergences. Their eventual fate is discussed in the concluding and speculative section 6. One possibility is a series of transitions towards smaller and finally vanishing cosmological term, and hence also expansion rate and 3-space curvature. Whether and how this happens is one of the most interesting questions raised by our work.

2. THE COSMOLOGICAL SOLUTIONS

The action for a bosonic string moving in some arbitrary graviton, dilaton and antisymmetric tensor field backgrounds is [7]:

$$S = \int \frac{d^2\xi}{4\pi}(\sqrt{\gamma}G_{\mu\nu}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu + B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu + \frac{\sqrt{\gamma}}{2}\Phi R^{(2)}) \qquad (2.1)$$

where we have set the Regge slope $\alpha' = 1$, $\gamma^{\alpha\beta}$ is the metric on the world-sheet, and

$$\int R^{(2)}\sqrt{\gamma} = 2(1-g) - N_h$$

is the Euler characteristic, with g the genus of the surface and N_h the number of marked points or holes . For constant dilaton $e^{\frac{\Phi}{2}}$ is the string coupling constant; indeed one gets a power of $e^{\frac{\Phi}{2}}$ for each external leg and e^{Φ} for each loop, as is expected in a theory with only cubic couplings. The classical string equations of motion are the conditions for the vanishing of the β-functions [7]. They follow from an action principle; when written in terms of the σ-model backgrounds, this action is multiplied by an overall $e^{-\Phi}$ factor. The physical space-time metric, with a correctly normalized Einstein action, is:

$$g_{\mu\nu} = e^{-\frac{2\Phi}{D-2}} G_{\mu\nu} \tag{2.2}$$

Kinetic energies of scalar fields are automatically normalized in the physical metric. All this is well known and standard. We turn now to a description of the homogeneous and isotropic cosmological solutions we will study.

Conformal theories. The simplest time-dependent consistent background has [2,3]

$$G_{\mu\nu} = \eta_{\mu\nu} \qquad ; \qquad B_{\mu\nu} = 0 \tag{2.3a}$$

and a linearly growing dilaton

$$\Phi = -2QX^0 \tag{2.3b}$$

This is an exact solution corresponding to a conformal theory with energy-momentum tensor :

$$T_{zz} = -\frac{1}{2}\partial_z X^\mu \partial_z X_\mu - Q\partial_z^2 X^0 \tag{2.4}$$

where our convention for the metric signature is $(-++....+)$. $T_{\bar{z}\bar{z}}$ is given by the same expression with $z \to \bar{z}$. Using free field contractions one can easily show that T_{zz} closes a Virasoro algebra with central charge

$$c = D - 12Q^2 \tag{2.5}$$

where D is the dimension of space-time. To cancel the Weyl anomaly we should demand that c=26 , which implies that the critical dimension is bigger than 26. Note that once the Weyl anomaly has been cancelled, the use of a free field contraction for X^0 is consistent, since on the cylinder one may completely fix the gauge $\gamma_{\alpha\beta} = \delta_{\alpha\beta}$, so that $R^{(2)} = 0$ and the linear term drops from the σ-model action.

Energy-momentum tensors of the type (2.4) have been extensively used in the literature, particularly in the Coulomb-gas representation of minimal models [6]. A Coulomb gas is described by a positive metric boson ϕ with energy-momentum tensor :

$$T_{zz} = -\frac{1}{2}\partial_z\phi\partial_z\phi - iq\partial_z^2\phi \tag{2.6}$$

where $2q$ is a charge placed at infinity. This energy-momentum tensor can be obtained from (2.4) by an imaginary rotation of the time coordinate: $X^0 \to iX^0$. Alternatively it corresponds to an **imaginary** dilaton background linear in some **space** direction. The central charge is again $1-12q^2$ since the change of sign in the free-field propagator is compensated by the i in the dilaton background. Despite this similarity , the CFT of the time coordinate differs from minimal models in at least two crucial respects: (i) As opposed to ϕ, the time cannot be compactified; its corresponding zero-mode spectrum is therefore necessarily continuous rather than discrete, and (ii) Unitarity is required only after having imposed the Virasoro gauge conditions; as a result there are no extra constraints either on the "charge" $2iQ$ or on the zero mode spectrum. We will demonstrate this in section 3 .

The reason we cannot compactify the time coordinate, is that the σ-model action (2.1) with a **real** dilaton background (2.3b) has no symmetry under time translations . For a purely imaginary dilaton $Q \to iq$ on the other hand, e^{-S} would be invariant under the finite shift $X^0 \to X^0 + \frac{2\pi}{q}n$ with $n \in Z$, since the Euler number is an integer. One is thus allowed to compactify, which is precisely the case with minimal models. A comment is also in order concerning the solutions with a real dilaton linear in some space-coordinate (say $\Phi = 2QX^1$) , for which $c = D + 12Q^2$ and the critical dimension is less than 26 . These models will not ineterest us, as they violate isotropy in some necessarily non-compact space .*

Let us go back now to the time-dependent solution (2.3) . To keep things simple we have considered (D-1) flat space-coordinates. Some of these can of course be made compact; more generally we may add any "internal" unitary conformal model to the conformal theory (2.4). Denoting by d the number of uncompactified, or large, space-time dimensions, we have :

$$c = d - 12Q^2 + c_I \tag{2.7}$$

with c_I the internal central charge. Note that c_I and $12Q^2$ need not any more be integer. Note also that the separation of "internal" from "space-time" coordinates makes sense only if the radii of the former are sufficiently small, i.e. if the spectrum of conformal weights of the internal model has splittings of order say one.

Restricting our attention to $d = 4$, we see that there is one further modification of the "space-time" part of the CFT, which respects space-isotropy and homogeneity. This consists of replacing the three free space coordinates by a level-k Wess-Zumino-Witten model [4] on the SU(2) (or SO(3), depending on one's choice of topology) group manifold. The σ-model backgrounds G_{ij} and $H_{ijl} \equiv 3D_{[i}B_{jl]}$ are the appropriately normalized [4] metric and volume element on the 3-sphere. The central charge now reads:

$$c = 4 - 12Q^2 - \frac{6}{k+2} + c_I \tag{2.8}$$

---

* Liouville models [8] also have an energy-momentum tensor of the Coulomb-gas type, although their world-sheet action principle is quite different from the one considered here.The connection, if any, is not clear to us. After completion of this work we learned that J.Polchinski is investigating this question.

Again this separation really makes sense only when the sphere-radius is hierarchically larger than the Planck length, i.e. $k \gg 1$. This might in fact be hard to satisfy , because on the one hand c should be 26, and on the other all presently known candidates for the internal unitary model have central charges that fall in discrete series, which converge to integers or half-integers from below . Thus the space-curvature could be forced to be of order one, in which case the separation in (2.8) should be taken with a grain of salt.

Space-Time Interpretation. The physical space-time metric corresponding to the solution (2.3) is

$$g_{\mu\nu} = e^{\frac{4QX^0}{D-2}} \eta_{\mu\nu} \tag{2.9}$$

After a time redefinition

$$t = \frac{D-2}{2Q} e^{\frac{2Q}{D-2}X^0} \tag{2.10}$$

and a rescaling of space-variables one finds :

$$(ds)^2 = -(dt)^2 + t^2 dX^i dX_i \tag{2.11a}$$

while the dilaton in these variables reads :

$$\Phi = (2-D) log \frac{2Qt}{D-2} \tag{2.11b}$$

The metric (2.11a) describes a linearly expanding (or contracting, depending on the sign of Q) D-dimensional Robertson- Walker universe, with vanishing space-curvature.

More generally we can replace D by d, the number of "large" space-time dimensions. In the particular case $d = 4$, since we can curve space into a 3-sphere, the metric takes the more general form:

$$(ds)^2 = -(dt)^2 + t^2 \left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \qquad (2.12a)$$

while the axion field b, defined through a duality transformation : $H_{\lambda\mu\nu} = e^{2\phi}\epsilon_{\lambda\mu\nu\rho}D^{\rho}b$, acquires a linear time- dependence:

$$b = 2Q^2\sqrt{\kappa}t \qquad (2.12b)$$

Here $\kappa = \frac{1}{2Q^2k}$ is the 3-space curvature. Since the Kac-Moody anomaly k is a positive integer, κ must be non-negative; this follows also from reality of the axion background. For the non-allowed value $\kappa = -1$, the metric (2.12a) is a reparametrization of flat space, known as the Milne universe *. For any other value, the Ricci scalar curvature $R = \frac{6(1+\kappa)}{t^2}$ does not vanish and has an initial singularity at $t = 0$.

Finally, let us for completeness note that for $Q = 0$ the universe does not expand, but space can still be curved into a 3-sphere. The metric:

$$(ds)^2 = -(dX^0)^2 + [\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)] \qquad (2.13a)$$

describes a static Einstein universe, the dilaton is constant: $\Phi = \Phi_0$, while the axion grows again linearly in time:

$$b = 2e^{-\Phi_0}\sqrt{\kappa}X^0 \qquad (2.13b)$$

From the field-theoretic point of view, there are two, somewhat complimentary, ways of thinking about these non-trivial string solutions . The first is to note

* In ref. [3] we have referred to all these linearly expanding solutions as Milne universes; this might have caused some confusion.

that the "central charge deficit" $\delta c = c_I - 26 + d$ adds a tree-level cosmological term $\int \sqrt{g} e^{\Phi}$ to the effective action of the d-dimensional graviton, dilaton and axion fields . Roughly speaking , δc could be thought of as the value of the matter field potential at some saddle point or local minimum (see figure 1). It is precisely this cosmological term that causes space to expand and/or acquire a non-vanishing curvature.

A second way of thinking about these solutions is to note that both the axion and dilaton fields are Goldstone bosons of classical string symmetries *: they enter in the string Lagrangian only through their derivatives or, in the case of the dilaton, an overall scale factor. Giving them a **linear time dependence** affects all other fields by at most shifting their (space-time independent) minima. The only exception is the graviton, which couples to energy density, and whose equations of motion are thus drastically modified. This way of thinking about the solutions (2.11-13) suggests , among other things, that they are somehow unique: indeed, it can be shown [3] that any other solution of the one-loop β-function equations, approaches them asymptotically at large times.

Before concluding this section let us again stress the one, genuinely stringy phenomenon which a field theoretic language cannot accomodate. This is the fact that a non-trivial expansion rate modifies the critical dimension which becomes a free (hopefully dynamical?) parameter, as is fitting for a theory of everything. To be more precise we should say that even for static backgrounds, the space-time dimension is not a well-defined notion in string theory ; there is however a universal parameter, which controls the asymptotic density of states . As we will however see in the upcoming section, even this now becomes a variable, related to the expansion rate Q.

---

* We thank C.Kounnas for a very illuminating discussion on this point .

3. SPECTRUM OF FREE BOSONIC STRING

We turn now to the study of the free bosonic string in the linearly-expanding universe. In the absence of string interactions this background is , as we will see, amazingly simple: its effect can be entirely summarized by a constant shift in the eigenvalues of the flat box operator, together with an overall rescaling of plane-waves, that keeps them normalized at all times. The rest of the analysis goes through almost exactly as in the conventional 26-dimensional string model.

A. The Wave Equation. Consider for simplicity the solution (2.2) with D flat uncompactified space-time dimensions . The Virasoro generators, i.e. the moments of the energy-momentum tensor (2.4) are :

$$L_N = \frac{1}{2}\sum_n \eta_{\mu\nu} X^{\mu}_{N-n} X^{\nu}_{n} + iQ(N+1)X^0_N \tag{3.1}$$

where X^{μ}_{l} are the moments of the operators $i\partial X^{\mu}$. These satisfy canonical commutation relations :

$$[X^{\mu}_{m}, X^{\nu}_{n}] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{3.2}$$

The hermiticity conditions are:

$$X^{\mu\dagger}_{n} = X^{\mu}_{-n} + 2iQ\eta^{\mu 0}\delta_{n,0} \tag{3.3}$$

ensuring that the Virasoro generators obey $L^{\dagger}_{N} = L_{-N}$ as usual. Conditions (3.3) imply that the zeroth component of momentum has a fixed imaginary part :

$$p^0 = E + iQ \tag{3.4}$$

with E real . We will sometimes refer to E in the sequel as the energy .

The lowest-lying state is the tachyon :

$$|p\rangle = e^{-ip^\mu X_\mu(0)}|0\rangle$$

annihilated by all lowering operators X_n^μ $(n > 0)$. It satisfies the mass-shell condition :

$$\frac{1}{2}p^\mu p_\mu + iQp^0 = -\frac{1}{2}(E^2 + Q^2 - \vec{p}^2) = 1 \tag{3.5}$$

This is easy to understand, since the quadratic piece of the Lagrangian of a scalar field in the background (2.3) reads :

$$\mathcal{L}_{scalar} = e^{2QX^0}(-\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2) \tag{3.6a}$$

Thus the rescaled field $\tilde{\phi} = e^{QX^0}\phi$ obeys the free wave equation in **flat** space with "shifted mass" $m^2 - Q^2$. As a result, ϕ is a superposition of plane waves in conformal coordinates, times the rescaling factor e^{-QX^0} , which takes care of the expansion of the universe. This factor precisely kills the growing modes in the propagator of "tachyonic" rescaled fields, in the forward light- cone. Thus the "tachyonic" mass shifts need not, a priori, signal classical instabilities.

Let us in passing note that the scalar-field Lagrangian (3.6a), can be rewritten in the background metric (2.9) as follows:

$$\mathcal{L}_{scalar} = \sqrt{g}(-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{(D-2)}{4(D-1)}\frac{m^2}{Q^2}R\phi^2) \tag{3.6b}$$

where we have here used the fact that the scalar curvature is: $R = 4Q^2\frac{D-1}{D-2}e^{-\frac{4Q}{D-2}X^0}$ for this background. The absence of an explicit mass term is a consequence of global Weyl invariance. This is crucial since otherwise the wave equation would not reduce to the one in flat space-time. Note also that minimally and conformally coupled fields correspond to $\frac{m^2}{Q^2} = 0$ and 1 respectively.

The above arguments can be extended to all other bosonic modes, including the graviton and dilaton, all of which obtain the same "mass-shift" $-Q^2$

after appropriate rescaling. In the case of the graviton and dilaton the field-theory calculation is somewhat more involved: the effective Lagrangian written in terms of the σ-model fields is :

$$\mathcal{L}_{grav+dil} = \sqrt{G}e^{-\Phi}(R(G) + (D\Phi)^2 - 4Q^2)$$

and the rescaled quantum fields are defined by

$$\Phi = -2QX^0 + e^{-QX^0}\tilde{\Phi}; \qquad G_{\mu\nu} = \eta_{\mu\nu} + e^{-QX^0}\tilde{h}_{\mu\nu}$$

B. Positivity of the Hilbert Space. For a string moving on a compact group manifold it is known [9,4] that unitarity puts severe restrictions both on the central charge, and on the zero-mode spectrum, i.e. the allowed highest- weight group representations. More recently, this has been also argued to be the case for string theory on the SU(1,1) manifold [10], which has a 3- dimensional cosmological interpretation. Finally, unitarity is known to restrict both the background charge and the conformal weights in minimal models [11]. It is thus natural to ask, whether any similar restrictions on Q and E exist for the background (2.3).

There is good reason to believe that the answer to this question is "no". Indeed, one can define a manifestly positive-definite light-cone gauge [2]: $X^+ = P^+\tau$, because the Virasoro generators (3.1) are still linear in X^-, which can therefore be expressed in terms of the transverse components, by explicitly solving the gauge conditions. As long as there are no anomalies, there should be no obstruction to choosing such a gauge. We will now explicitly verify this argument, by checking positivity up to the second excited level, directly in the covariant gauge.

As usual we will restrict ourselves to the left-moving sector. The most general state at the first excited level is

$$\varsigma_\mu X^\mu_{-1}|p\rangle$$

where the L_0 and L_1 Virasoro gauge conditions require:

$$p^*_\mu \varsigma^\mu = 0 \tag{3.7a}$$

$$p^*_\mu p^\mu = 0 \tag{3.7b}$$

Here ς^μ is the polarization vector and the complex conjugate momentum is $p^*_\mu = (E - iQ, \vec{p})$. Using equations (3.7) we can write the norm of this state as follows:

$$\varsigma^*_\mu \varsigma^\mu = \sum_{i,j=1}^{D-1} \varsigma^{i*}(\delta_{i,j} - \frac{p_i p_j}{E^2 + Q^2})\varsigma^j$$

Thus one has as usual one longitudinal null state and D-2 positive norm transverse ones.

As in the old dual model the non-trivial constraint comes from the requirement of positive norm at the second excited level. The general state is:

$$|\varsigma\epsilon\rangle = (\varsigma_{\mu\nu} X^\mu_{-1} X^\nu_{-1} + \epsilon_\mu X^\mu_{-2})|p\rangle$$

There are now three non-trivial Virasoro gauge conditions which read:

$$2\epsilon_\mu p^\mu + \varsigma^\mu_\mu - 6iQ\epsilon_0 = 0 \tag{3.8a}$$

$$2\varsigma_{\mu\nu} p^{\mu *} + 2\epsilon_\nu = 0 \tag{3.8b}$$

$$-\frac{1}{2} p^*_\mu p^\mu = 1 \tag{3.8c}$$

Let us use the SO(D-1) invariance under space-rotations to bring the space- momentum along the direction $i = 1 : \vec{p} = (p, 0....0)$. Solving eq.(3.8b) for ϵ_ν we may now express the norm as a sum of three pieces:

$$\begin{aligned}\langle\varsigma\epsilon|\varsigma\epsilon\rangle &= -2\varsigma^*_{\mu\nu}\varsigma^{\mu\nu} + 2\epsilon^*_\mu\epsilon^\mu = \\ &= \left\{2\sum_{i\neq j=2}^{D-1}|\varsigma_{ij}|^2\right\} \quad + \sum_{i=2}^{D-1}\left\{-4|\varsigma_{0i}|^2 + 4|\varsigma_{1i}|^2 + 2|\varsigma_{0i}(E-iQ) + \varsigma_{1i}p|^2|\right\} \\ &+ \left\{2|\varsigma_{00}|^2 + 2|\varsigma_{11}|^2 - 4|\varsigma_{01}|^2 - 2|\varsigma_{00}(E-iQ) + \varsigma_{01}p|^2\right. \\ &\left.+2|\varsigma_{01}(E-iQ) + \varsigma_{11}p|^2 + 2\sum_{i=2}^{D-1}|\varsigma_{ii}|^2\right\}\end{aligned} \tag{3.9}$$

There is still one constraint, eq. (3.8a), which can be written as follows:

$$\sum_{i=2}^{D-1}\varsigma_{ii} = (1 + 2E^2 - 4Q^2 - 6iQE)\varsigma_{00} + p(4E - 6iQ)\varsigma_{01} + (2p^2 - 1)\varsigma_{11} \tag{3.10}$$

and which relates only the components of the polarization tensor that appear in the third bracket of the expression for the norm.

The first piece of the norm, depending on ς_{ij} for $i \neq j$ is manifestly positive. The second piece is in matrix notation:

$$\sum_{i=2}^{D-1}(\varsigma^*_{0i}\varsigma^*_{1i})\begin{pmatrix}-4 + 2(E^2 + Q^2) & 2p(E - iQ) \\ 2p(E + iQ) & 4 + 2p^2\end{pmatrix}\begin{pmatrix}\varsigma_{0i} \\ \varsigma_{1i}\end{pmatrix} \tag{3.11}$$

Using the mass shell condition (3.8c) one sees that the norm-matrix has one positive and one zero eigenvalue for every i, in other words there are D-2 positive and as many null states. We finally consider the third piece of the norm (3.9). By symmetry its minimum is at $\varsigma_{ii} = \varsigma$ for all i. Solving for ς from condition (3.10) we may rewrite this third piece as $2vMv\dagger$ where:

$$v = (\varsigma_{00}, \varsigma_{01}, \varsigma_{11})$$

and the hermitean matrix M is :

$$M = \begin{pmatrix} 1-E^2-Q^2+\frac{1}{D-2}A^*A & M_{21}^* & M_{31}^* \\ -p(E+iQ)+\frac{2p}{D-2}(2E-3iQ)A & \frac{4p^2}{D-2}(4E^2+9Q^2 & M_{32}^* \\ \frac{1}{D-2}(2p^2-1)A & \frac{2p}{D-2}(2E+3iQ)(2P^2-1)+p(E+iQ) & p^2+1+\frac{1}{D-2}(2p^2-1)^2 \end{pmatrix}$$

with $A = 1 + 2E^2 - 4Q^2 + 6iQE$. We have calculated the eigenvalues of this matrix using REDUCE, with the result:

$$\lambda_1 = 0 \tag{3.12a}$$

$$\lambda_2\lambda_3 = \frac{1}{D-2}(26 - D + 12Q^2)[3E^4 - 6E^2(1 - Q^2) + 3Q^4 - 6Q^2 + 1] \tag{3.12b}$$

$$\lambda_2 + \lambda_3 = \frac{1}{D-2}[24E^4 + (80Q^2 - 48)E^2 + 56Q^4 - 100Q^2 + 26] \tag{3.12c}$$

There is thus always one null state. Considering the other two eigenvalues, we first note that the quantities inside the brackets in eqs.(3.12b,c) are always positive for $E^2 \geq 2 - Q^2$ i.e. in the physical region of real momenta. Consequently both eigenvalues are non-negative if and only if

$$D < 26 + 12Q^2 = D_{critical} \tag{3.13}$$

Precisely at the critical dimension one of them vanishes signalling the appearance of one extra null state.

To summarize , let us count the total number of linearly independent positive norm states: it is equal to the number of components of a symmetric two-index tensor $\varsigma_{\mu\nu}$ minus one (because of the constraint (3.8a)), minus the D-2 null states (3.11), minus the null state corresponding to (3.12a), minus one more at the critical dimension, i.e.:

$$(\#level2states) = \begin{cases} \frac{D(D+1)}{2} - D, & \text{if } D < D_{cr} \\ \frac{D(D+1)}{2} - D - 1 & \text{if } D = D_{cr} \end{cases} \tag{3.14}$$

Precisely at D_{cr} one gets the same number of states as in the manifestly unitary "light-cone gauge", where $\varsigma_{\mu\nu}$ and ϵ_μ have $\frac{(D-2)(D-1)}{2} + (D-2)$ transverse components. All this is of course very similar to the usual analysis of the 26-dimensional bosonic string. Furthermore, positivity is clearly not affected if some flat space dimensions are replaced by other internal unitary models. We have verified this explicitly in the case of one extra free fermion.

C. Partition Function and Modular Invariance . Assuming no restriction on the energy E at all higher excited levels, we may write the partition function for the D-dimensional expanding universe as follows:

$$Z_D^{bosonic} = \int dE d\vec{p}\, (q\bar{q})^{-\frac{c}{24} - \frac{1}{2}(E^2+Q^2-\vec{p}^2)} \sum_{oscillators} q^N \bar{q}^{\bar{N}} Z_{ghost} \tag{3.15 }$$

where $q = e^{2i\pi\tau}$ and N and $\bar{N}$ are the left and right frequencies . Setting $c = D - 12Q^2$ one finds easily:

$$Z_D^{bosonic} = (2Im\tau)^{\frac{2-D}{2}} (\eta\bar{\eta})^{2-D} \tag{3.16 }$$

with $\eta = q^{-1/24} \prod_{n>0}(1-q^n)$ the Dedekind function. This is manifestly modular invariant, even though holomorphic factorization is lost at $D - 2 \neq 0(mod24)$.

The partition function (3.16) gives a number of states in agreement with the counting in the light-cone gauge. Since the dimension of space-time is bigger than 26, the asymptotic density of states is not the same as in the usual dual model:

$$\rho(m) \sim_{m\to\infty} exp(2\pi\sqrt{\frac{D-2}{2}}m) = exp(2\pi\sqrt{2+Q^2}m) \tag{3.17}$$

This change can be attributed to the "shift" in the tachyon position. It raises the issue of whether Q (which is related to the field-theory cosmological constant) is an external parameter of string theory or reflects a choice of vacuum.

Finally it is easy to modify the partition function (3.16) to take into account compactification or the addition of internal unitary conformal models. In general one has:

$$Z^{bosonic} = Z_d^{bosonic} Z_{anything} \tag{3.18}$$

where d are the uncompactified dimensions and the extra factor is the modular-invariant partition function of any combination of unitary models, with central charge c_I . This is the partition function of a string theory in a linearly-expanding universe with expansion rate fixed by the anomaly condition $26 = d - 12Q^2 + c_I$.

4. TYPE II AND HETEROTIC STRINGS

It is straightforward to extend the cosmological solution (2.3a,b) to heterotic and type II superstrings . The energy-momentum tensor and supercharge of the underlying conformal theory are:

$$T_B = -\frac{1}{2}\partial X_\mu \partial X^\mu + Q\partial^2 X^0 - \frac{1}{2}\psi_\mu \partial\psi^\mu \tag{4.1a}$$

$$T_F = -\psi_\mu \partial X^\mu + 2Q\partial\psi^0 \tag{4.1b}$$

and the anomaly cancels when

$$\hat{c}_I + d - 8Q^2 = 10 \tag{4.2}$$

where $\hat{c}_I$ corresponds to some additional internal superconformal system. In the heterotic string the cancellation in the non-supersymmetric sector is, as before :

$$c_I + d - 12Q^2 = 26 \tag{4.3}$$

The lowest lying possible fermionic excitations obey a flat massless wave equation, like conformally coupled scalars . To see why consider for somplicity the case $\hat{c}_I = 0$. The mass-shell condition for a lowest-lying Ramond state , taking into account the subtraction of the $d-2$ transverse fermionic zero modes , is :

$$\frac{1}{2}(E^2 + Q^2 - \vec{p}^2) = -\frac{1}{2} + \frac{d-2}{16} \tag{4.4}$$

Using the anomaly condition (4.2) we find $E^2 - \vec{p}^2 = 0$ in this case, as claimed . Alternatively, this follows from the the supercharge (4.1b), whose moments are

$$G_N = i\sum_n \psi^\mu_{N-n} X_{\mu,n} - 2Q(N+\frac{1}{2})\psi^0_N$$

When acting on a highest weight state the zeroth moment becomes

$$G_0 = -i(\gamma_0 E - \vec{\gamma}\vec{p})$$

which is precisely the massless Dirac operator in flat space-time. The addition of an internal superconformal system does not change the argument: one simply replaces d by $d+\hat{c}$ in eq. (4.4) . From the field-theory point of view the quadratic piece of the Lagrangian of a fermion in the background (2.3) is:

$$\mathcal{L}_{ferm} = e^{2QX^0}(\bar{\psi}\overleftrightarrow{\partial}_\mu \gamma^\mu \psi + m\bar{\psi}\psi)$$

The rescaled field $\tilde{\psi} = e^{QX^0}\psi$ thus obeys the free Dirac equation in flat space-time **without a mass shift**. Note, however, that in field-theoretic language it is hard to accomodate the fact that the dimension of the spinor varies also with Q.

A. Type II. Due to the requirement of world-sheet supersymmetry the analysis of modular invariance is somewhat more involved than in the case of the bosonic string, but can be easily done , for instance in the fermionic language, along the lines of references [12] . We will only discuss here a few simple examples. Consider first the type II superstring in D uncompactified dimensions and without any internal superconformal system. The only modular invariant partition functions with correct spin-statistics are :

$$Z_D^{typeII} = \frac{1}{2}\left[(\frac{\Theta_2\bar{\Theta}_2}{\eta\bar{\eta}})^{D/2-1} + (\frac{\Theta_3\bar{\Theta}_3}{\eta\bar{\eta}})^{D/2-1} + (\frac{\Theta_4\bar{\Theta}_4}{\eta\bar{\eta}})^{D/2-1}\right] Z_D^{bosonic} \qquad (4.5)$$

with $Z_D^{bosonic}$ given by eq. (3.14b), and for $D = 10 + 16m$

$$Z_D^{typeII,chiral} = \frac{1}{4}\left|-(\frac{\Theta_2}{\eta})^{D/2-1} + (\frac{\Theta_3}{\eta})^{D/2-1} - (\frac{\Theta_4}{\eta})^{D/2-1}\right|^2 Z_D^{bosonic} \qquad (4.6)$$

Theory (4.5) has only bosonic excitations. Note that for D=18 it is identical to the partition function of the usual 26-dimensional bosonic string, compactified on an appropriate 8-dimensional torus. This might shed some light on the question of the equivalence of the bosonic and supersymmetric strings: In contrast to earlier efforts [13] we here obtain identical spectra without any truncation. To further pursue this issue, however, one must also take into account interactions.

Theory (4.6) can have chiral fermions. Such partition functions were already considered in references [14] and shown to be free of field theory anomalies. At that time they were thought to correspond to string theories with a non- vanishing conformal anomaly; as we see here this need not be the case if one allows a non-trivial dilaton background. Clearly the partition function (4.6) does not vanish for $D > 10$, since supersymmetry is lost [5] as can be seen by a simple counting of the degrees of freedom.

The number of modular invariant theories is much bigger, if one compactifies some coordinates and/or adds other internal superconformal systems. Unlike the bosonic string one can however no more factorize the partition function as in eq. (3.18) , because Ramond and Neveu-Schwarz sectors are coupled together. It is very easy to construct specific examples but we will refrain fron doing so. What we want to stress is the following potentially important fact: that from the point of view of 4 dimensions the expansion of the universe allows $\hat{c}_I > 6$. It may thus be possible to evade the arguments of reference [15] and construct a type II theory with the gauge and matter content of the standard model. Although our specific solution may not be realistic, it illustrates how time-dependent backgrounds could modify such arguments in string theory.

B.Heterotic. The analysis is also easily extended to heterotic string theories. In the simple case of an integer critical dimension $D = 10 + 8Q^2$ one may satisfy the condition (4.3) by adding

$$N_f = 32 + 8Q^2$$

free fermions in the non-supersymmetric sector. One possible modular-invariant partition function is:

$$Z_D^{heterotic} = \frac{1}{2}\left[-(\frac{\Theta_2}{\eta})^4(\frac{\Theta_2\bar{\Theta}_2}{\eta\bar{\eta}})^{4Q^2} + (\frac{\Theta_3}{\eta})^4(\frac{\Theta_3\bar{\Theta}_3}{\eta\bar{\eta}})^{4Q^2} - (\frac{\Theta_4}{\eta})^4(\frac{\Theta_4\bar{\Theta}_4}{\eta\bar{\eta}})^{4Q^2}\right] Z_G Z_D^{bosonic} \tag{4.7}$$

where $G = E_8 \times E_8$ or $G = SO(32)$, and Z_G is the level-1 partition function of the group G . If furthermore $Q^2 = 2m$, one may also construct a factorizable partition function:

$$Z_D^{het,chiral} = \frac{1}{2}\left[-(\frac{\Theta_2}{\eta})^{4+8m} + (\frac{\Theta_3}{\eta})^{4+8m} - (\frac{\Theta_4}{\eta})^{4+8m}\right] Z_G Z_D^{bosonic} \tag{4.8}$$

where now $G = SO(32+16m), E_8 \times E_8 \times .... \times E_8$ or any other rank-(16+8m) group with an even selfdual root lattice. One effect of the expansion of the universe is thus again to enlarge the allowed gauge groups. Note that the action of the unrescaled gauge bosons

$$\mathcal{L}_{gauge} = e^{2QX^0} TrF^2$$

is invariant under the usual gauge transformations:

$$\delta A_\mu^a = \partial_\mu \Lambda^a + f^{abc} A_\mu^b \Lambda^c$$

The rescaled bosons: $\tilde{A}_\mu^a = e^{QX^0} A_\mu^a$, on the other hand obtain like all other bosonic fields a "tachyonic mass shift" $-Q^2$; their field strength and gauge-transformation law are appropriately modified to take care of the time-dependent coupling constant . Finally, a large number of modular-invariant theories can again be obtained by compactification and/or addition of internal unitary (super in the left sector) conformal models.

5. INTERACTIONS

The only quantities one may calculate in string theory are correlation functions of vertex operators for the emission of on-shell particles, i.e. particles satisfying the free equations of motion. In a flat background these are of course S-matrix elements. In the case at hand, however, space-time is not asymptotically flat, and it is not a priori clear how an S-matrix should be defined. Here we will take a more pragmatic point of view: we will describe the unique sensible prescription for calculating finite string amplitudes in this background. As we will see, this prescription amounts to leaving a symmetry under combined time-translations and dilaton shifts unbroken; we have no argument for why string theory should (or should not) respect such a symmetry. A better understanding of this issue would, we believe, yield important insight into the dynamics of the dilaton and the structure of the string vacuum.

Consider first a free bosonic field in the background (2.3) , and assume $m^2 \geq Q^2$. As discussed in section 3 this can be expanded in terms of plane waves:

$$\phi(x) = e^{-QX^0} \int_k \left(a_k e^{-i\omega_k X^0 + i\vec{k}\vec{X}} + a_k^\dagger e^{i\omega_k X^0 + i\vec{k}\vec{X}} \right) \tag{5.1}$$

where the frequency, or (real) energy is: $\omega_k = \sqrt{\vec{k}^2 + m^2 - Q^2}$. The invariant inner product in this background is:

$$(\phi_1, \phi_2) \sim \int d^{D-1}x \; a^{D-1} \phi_1^* \overleftrightarrow{\partial}_t \phi_2 \sim e^{2QX^0} \int d^{D-1}x \; \phi_1^* \overleftrightarrow{\partial}_0 \phi_2 \tag{5.2}$$

where t is the Robertson-Walker time, a the scale factor, and we have used equation (2.10) . With this inner product the plane-waves in (5.1) are orthonormalized **at all times** . The factor e^{-QX^0} is the dilution of plane-waves due to the expansion of the universe. We can quantize the free field ϕ by requiring the usual commutation relations for $a_k, a_k^\dagger$. In and out vacua are defined as the states annihilated by all negative-energy operators a_k . Clearly, both the space-momentum in comoving coordinates, and the the energy are conserved by the time-evolution which does not, in particular, mix positive and negative frequencies; consequently there is neither **scattering**, nor **particle creation** in this background. To be sure, this conclusion need not hold for "tachyonic" excitations ($m^2 < Q^2$), for which frequencies can become purely imaginary.

We turn now to the harder issue of string interactions. Our guiding principle will be **duality** and **factorizability**; the former follows from conformal symmetry which we should therefore make sure to respect . Consider inserting N vertex operators V_i:

$$\int V(z,\bar{z}) = \int : Polynomial(\partial_z X^\mu, \partial_{\bar{z}} X^\nu) e^{-ip^\mu X_\mu} : \tag{5.3}$$

on the sphere. Since each momentum carries a fixed imaginary part, eq.(3.4), there is a net charge iNQ which must be cancelled to make the integration over the time-zero mode finite. Part of it is counterbalanced by the background charge $-2iQ$ placed at infinity [6].The existence of this background charge is a consequence of SL(2,C) invariance , which requires that

$$_{out}\langle 0|e^{-2QX^0}|0\rangle_{in} = 1 \tag{5.4}$$

where e^{-2QX^0} is an operator of conformal weight zero, that is identified with the identity . This background charge plays precisely the role of the e^{2QX^0} factor in the covariant definition of the inner product , eq.(5.2), in field theory.

For more than two external legs there is, however, still a left over charge $i(N-2)Q$ which would make all amplitudes diverge. One possible attitude is to say that this is normal. Indeed, assuming $\phi(x)$ approaches asymptotically the free field (5.1), and using the covariant inner product (5.2), it is straightforward to reduce S-matrix elements to amputated on-shell Green's functions of rescaled fields. These diverge if the coupling varies exponentially with time , which would be the end of the story. There is however an alternative: In Coulomb gas models one cancells the excess charge , while respecting conformal invariance, by inserting dimension-(1,1) screening operators [6]

$$V_{\pm} =: e^{i\alpha_{\pm} X^0} : \tag{5.5a}$$

$$\alpha_{\pm} = i(Q \pm \sqrt{2+Q^2}) \tag{5.5b}$$

integrated over the world-sheet. Let us for the moment take this as a prescription, and see what we get.

Consider first the 3-point amplitude. To cancel the excess charge we must demand that:

$$n\alpha_{+} + m\alpha_{-} = -iQ \tag{5.6}$$

for some non-negative integers n and m ($n \leq m$), which implies that:

$$Q^2 = \frac{2(n-m)^2}{(2n+1)(2m+1)} \tag{5.7}$$

Assuming for simplicity that the external particles are tachyons, we have:

$$A^{(3)} \sim \langle \prod_{i=1}^{3} \int d^2 z_i : e^{-ip_i^{\mu} X_{\mu}(z_i)} : \prod_{k=1}^{n} \int d^2 u_k : e^{i\alpha_{+} X^0(u_k)} : \prod_{j=1}^{m} \int d^2 v_j : e^{i\alpha_{-} X^0(v_j)} : \rangle \tag{5.8}$$

The zero-mode integration forces real energy-momentum to be conserved:

$$E_1 + E_2 + E_3 = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$$

Furthermore, using SL(2,C) invariance, we can fix $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. Dropping the infinite volume of the group, and doing the free field contractions we find:

$$A^{(3)} \sim \prod_{k=1}^{n} \int d^2u_k \prod_{j=1}^{m} \int d^2v_j |u_k|^{-2\alpha_+ p_1^0} |1-u_k|^{-2\alpha_+ p_2^0} |v_j|^{-2\alpha_- p_1^0} |1-v_j|^{-2\alpha_- p_2^0}$$
$$\prod_{k>k'} |u_k - u_{k'}|^{-2\alpha_+^2} \prod_{j>j'} |v_j - v_{j'}|^{-2\alpha_-^2} \prod_{k,j} |u_k - v_j|^{-4} \tag{5.9}$$

These integrations have been performed explicitly by Dotsenko and Fateev [6] . After some straightforward manipulations, the result can be written as :

$$A^{(3)} = N(n,m) f(E_1) f(E_2) f(E_3) \tag{5.10a}$$

where N is an energy-independent normalization, and:

$$f(E) = \prod_{k=1}^{n} \frac{\Gamma(-m - \alpha_+ E - k\alpha_+^2/2)}{\Gamma(1+m+\alpha_+ E + k\alpha_+^2/2)} \prod_{j=1}^{m} \frac{\Gamma(-\alpha_- E - j\alpha_-^2/2)}{\Gamma(1+\alpha_- E + j\alpha_-^2/2)} \tag{5.10b}$$

Up to an overall polarization factor, the same result holds for the 3-point function of transverse gauge bosons.

Two questions arise immediately, concerning the amplitude (5.10). First, if one replaces n and m by :

$$n\prime = \frac{1}{2}(p(2n+1)-1); \qquad m\prime = \frac{1}{2}(p(2m+1)-1) \tag{5.11}$$

with p any odd positive integer , the value of the background charge Q , eq. (5.7) does not change. On the other hand, $A^{(3)}$ seems to depend explicitly on n and m, i.e. on the precise way of screening. Fortunately, this is not the case *: using the fact that $\alpha_+ = i\sqrt{\frac{2(2m+1)}{(2n+1)}}$, $\alpha_+\alpha_- = 2$, and the Gauss-Legendre multiplication formula:

$$\prod_{r=0}^{m-1} \Gamma(z + \frac{r}{m}) = (2\pi)^{\frac{1}{2}(m-1)} m^{\frac{1}{2}-mz} \Gamma(mz) \tag{5.12}$$

one can show that $f(E)/f(0)$ is invariant under the change (5.11). Thus with an appropriate choice of the normalization N , the amplitude is only a function of energy and the background charge Q, as it should.

The second question concerns the quantization condition (5.7): it looks as if the amplitude can only be defined for a **discrete**, though **dense**, set of values of Q. We believe that this is an artifact of our prescription, and that $A^{(3)}$ should have an analytic continuation to the entire real line. Although we cannot exhibit this explicitly, we have checked it in the large E limit, where one indeed finds a smooth Q-dependence:

$$f(E) \sim |E|^{-Q^2/2} \tag{5.13}$$

A few more remarks are in order concerning the 3-point amplitude: It is, of course, energy-dependent (except when $m = n$, i.e. $Q = 0$) as one expects in a time-varying background. It is everywhere analytic, with isolated poles off the real axis.It falls off as a power law, eq.(5.13), at high energies, which shows again that the ultraviolet properties of the string theory are modified by the expansion of the universe. Finally, in the field-theory limit where the Regge slope $\alpha\prime$ vanishes, i.e. E

* Note that in the bosonic string the screening operators (5.5) are vertex operators for the emission of tachyons at zero space-momentum. Thinking of (5.10) as a $3+n+m$-amplitude is, however, we believe misleading, both because it depends on n and m only through Q, but also because in supersymmetric strings the old tachyon can be projected out of the spectrum.

and Q are both small, $f(E)$ goes to a constant , independent of the dimensionless ratio E/Q . This is contrary to the naive expectation that the string coupling constant should vary with time, and brings us back to the issue of interpreting the screening-operator prescription for computing amplitudes.

In order to understand what is going on, it is instructive to consider first the case of the static Einstein universe , eq. (2.13) . One may wonder why we call this background "static", when the axion field b grows linearly in time. The answer is that, if we neglect non-perturbative effects, b is the Goldstone boson of a broken symmetry: in other words there is exact symmetry under axion shifts: $b \rightarrow b + constant$, which means that b enters in the Lagrangian only through its derivatives . As a result a constant b-gradient breaks only Lorentz, but not time-translation invariance, and energy is at least perturbatively conserved. This is of course obvious from the world-sheet point of view , since the corresponding conformal theory is a WZW SU(2) model and a **free** time coordinate without background charge.

Unlike the axion, the dilaton is the Goldstone boson of only a classical symmetry: this is because a shift $\Phi \rightarrow \Phi + constant$, rescales the entire Lagrangian or, equivalently, the Planck constant. On the other hand , adding a constant Φ_0 to the background (2.3) amounts simply to a shift of the origin of conformal time. Suppose now that the "correct vacuum" was a superposition of states with different Φ_0 : then time-translation invariance would be formally restaured, i.e. Green's functions of rescaled fields would be time-independent, and scattering amplitudes would conserve real energy, as we indeed find with our prescription. Furthermore the theory does not suffer from an initial singularity. Note that for $Q = 0$ such a superposition would be inconsistent, since it would imply loss of cluster decomposition and factorization *, but it is not clear how and whether this argument

* We thank D.Gross for raising this point

applies when $Q \neq 0$.

Consider finally higher N-point amplitudes. For the values (5.7) of Q we can screen the excess charge by inserting $(N-2)n$ and $(N-2)m$ operators V_+ and V_- respectively. To keep things simple let us take $n = 0$, $m = 1$, so that $Q^2 = \frac{2}{3}$ and a single screening operator suffices to define $A^{(3)}$. For $A^{(4)}$ one needs two screening charges $\alpha_- = -iQ$. Injecting momenta p_1, p_2, p_3 and p_4 at the points $z_1 = z$,$z_2 = 1$,$z_3 = 0$ and $z_4 = \infty$ one finds:

$$A^{(4)} \sim \int d^2z d^2v_1 d^2v_2 |z-1|^{2p_1p_2} |z|^{2p_1p_3} |z-v_1|^{2iQp_1^0} |z-v_2|^{2iQp_1^0} |v_1|^{2iQp_3^0} \\ \times |v_2|^{2iQp_3^0} |1-v_1|^{2iQp_2^0} |1-v_2|^{2iQp_2^0} |v_1-v_2|^{2Q^2} \tag{5.14}$$

The integrand is SL(2,C) invariant if all p_i are put on mass-shell, and total real momentum is conserved : $\sum_1^4 p_i^\mu = -4iQ\eta^{\mu 0}$. This implies **duality** since $z \to 1-z$ and $v_i \to 1-v_i$ interchanges p_2 and p_3 (i.e. sends s to t), while $z \to \frac{1}{z}$ and $v_i \to \frac{1}{v_i}$ interchanges p_3 and p_4 (i.e. sends t to u) . To find the pole structure let us consider sending z and v_1 to ∞ ; this corresponds to factorizing two 3-point amplitudes as shown in fig.2 . Poles will appear when the exponent of the overall scale factor is a non-negative even integer, i.e. :

$$4 + 2p_1p_2 + 2p_1p_3 + 4iQp_1^0 + 2iQp_3^0 + 2iQp_2^0 + 2Q^2 = 2N \tag{5.15a}$$

with $N \geq 0$. Using the value of the background charge $Q^2 = \frac{2}{3}$ we can rewrite this as

$$\frac{1}{2}(E_1+E_4)^2 - \frac{1}{2}(\vec{p}_1+\vec{p}_4)^2 = -1 - \frac{1}{2}Q^2 + N \tag{5.15b}$$

which is precisely the mass spectrum of the free string. Note that to get the right pole structure, it was crucial to be able to separate two correctly screened 3-point functions, as shown in fig.2. In general a weaker condition than (5.6) suffices for screening 4- or higher-point amplitudes: for instance for $Q^2 = \frac{1}{4}$, we can write down an SL(2,C) invariant 4-point amplitude, using a single screening charge. This amplitude would be dual, but would neither factorize nor have the correct pole structure.

6.CONCLUDING REMARKS

The methods of the previous section can be used to write down higher-loop scattering amplitudes. The background charge is now $2(g-1)iQ$, with g the genus of the surface ; the remaining excess charge can be cancelled by screening, so that real energy is conserved. Due to the tachyonic mass shifts of bosonic fields, these amplitudes would, however, diverge even for type II or heterotic strings. For instance the one-loop vacuum-to-vacuum amplitude diverges, since partition functions grow at least like $e^{Q^2 Im\tau}$ at large $Im\tau$, for all models .

Assuming these divergences are a signal of instability, the question that arises is how an era of linear expansion would come to an end. If one takes seriously the field- theoretic interpretation of δc as the value of the "matter-field potential" at one of its saddle points, then the universe should make transitions to smaller and smaller values of δc , and hence also expansion rate , and 3-space curvature. These transitions could occur by "rolling down a hill" if the internal conformal theory has relevant operators, in accordance with Zamolodchikov's c-theorem [16]; in string-theory-language this would happen if there were tachyons, even before taking into account the $-Q^2$ mass shifts. Otherwise transitions could occur through tunelling. Furthermore, as we already argued in ref.[3], the allowed values of δc could be discrete and bounded away from zero, since all known unitary CFTs have central charges that accumulate to integers or half-integers from below . Thus after a finite number of transitions one should reach flat 4d Minkowski space-time. Unfortunately current string technology does not allow us to address in a more quantitative way this very interesting question.

Aknowledgements. We thank J.-L.Gervais, D.Gross, J.Iliopoulos, C.Kounnas, K.S.Narain, B.Rostand and T.Tomaras for discussions. We also thank D.Hansel for help with the REDUCE program.

REFERENCES

[1] For a review see M.B.Green, J.H.Schwarz and E.Witten, Superstring Theory, Cambridge University Press, 1987 .

[2] R.C.Myers, Phys.Lett. **B199** (1987) 371 .

[3] I.Antoniadis, C.Bachas, J.Ellis and D.V.Nanopoulos, Phys.Lett. **B211** (1988) 371 .

[4] E.Witten, Comm.Math.Phys. **92** (1984) 451 ; V.Knizhnik and A.B.Zamolodchikov, Nucl.Phys. **B247** (1984) 83 ; D.Gepner and E.Witten, Nucl.Phys. **B278** (1986) 493.

[5] This was also discussed recently by S.P.de Alwis, J.Polchinski and R.Schimmrigk, UTTG-23-88 preprint (1988).

[6] B.L.Feigin and D.B.Fuchs, Moscow preprint (1983) ; V.Dotsenko and V.Fateev, Nucl.Phys. **B240** [**FS12**] (1984) 312 and **B251** [**FS13**] (1985) 691 .

[7] E.S.Fradkin and A.A.Tseytlin, Phys.Lett. **B158** (1985) 315; C.G.Callan, D.Friedan, E.Martinec and M.Perry, Nucl.Phys. **B262** (1985) 593; A.Sen, Phys.Rev. **D32** (1985) 2102 and Phys.Rev.Lett. **55** (1985) 1846 .

[8] For a recent review see J.-L.Gervais, in Proceedings in String Theory, P.Di Vecchia and J.L.Petersen Eds. , World Scientific 1988 and references therein.

[9] P.Goddard and D.Olive, Int.J. of Mod.Phys. **A1** (1986) 303 .

[10] J.Balog, L.O'Raifeartaigh, P.Forgacs and A.Wipf, MPI-PAE/PTh 66/88 preprint (October 1988) .

[11] D.Friedan, Z.Qiu and S.Shenker, Phys.Rev.Lett. **52** (1984) 1575 .

[12] H.Kawai, D.C.Lewellen and S.H.H.Tye, Phys.Rev.Lett. **57** (1986) 1832 and Nucl.Phys. **B288** (1987)1 ; I.Antoniadis, C.Bachas and C.Kounnas, Nucl.Phys. **B289** (1987) 87 ; I.Antoniadis and C.Bachas , Nucl.Phys. **B298** (1988) 586 .

[13] P.G.D.Freund, Phys.Rev.Lett. **151B** (1985) 387 ; A.Casher, F.Englert, H.Nicolai and A.Taormina, Phys.Lett. **B662** (1985) 121 ; F.Englert, H.Nicolai and A.Schellekens, Nucl.Phys. **B274** (1986) 315 .

[14] A.Schellekens and N.Warner, Phys.Lett. **B177** (1986) 317 .

[15] L.Dixon, V.Kaplunovsky and C.Vafa, Nucl.Phys. **B294** (1987) 43.

[16] A.B.Zamolodchikov, JETP Lett. **43** (1986) 731 .

FIGURE CAPTIONS

Fig.1 : A matter field potential that could give rise to flat (A) and linearly expanding (B) universes in field theory. Note however that this picture may be misleading in string theory, because the matter-field content is not the same at points A and B .

Fig.2: The integration region giving the pole-structure of $A^{(4)}$ in the u channel, as discussed in the text. Had we used a single screening operator, $A^{(4)}$ would not have factorized into two 3-point amplitudes.

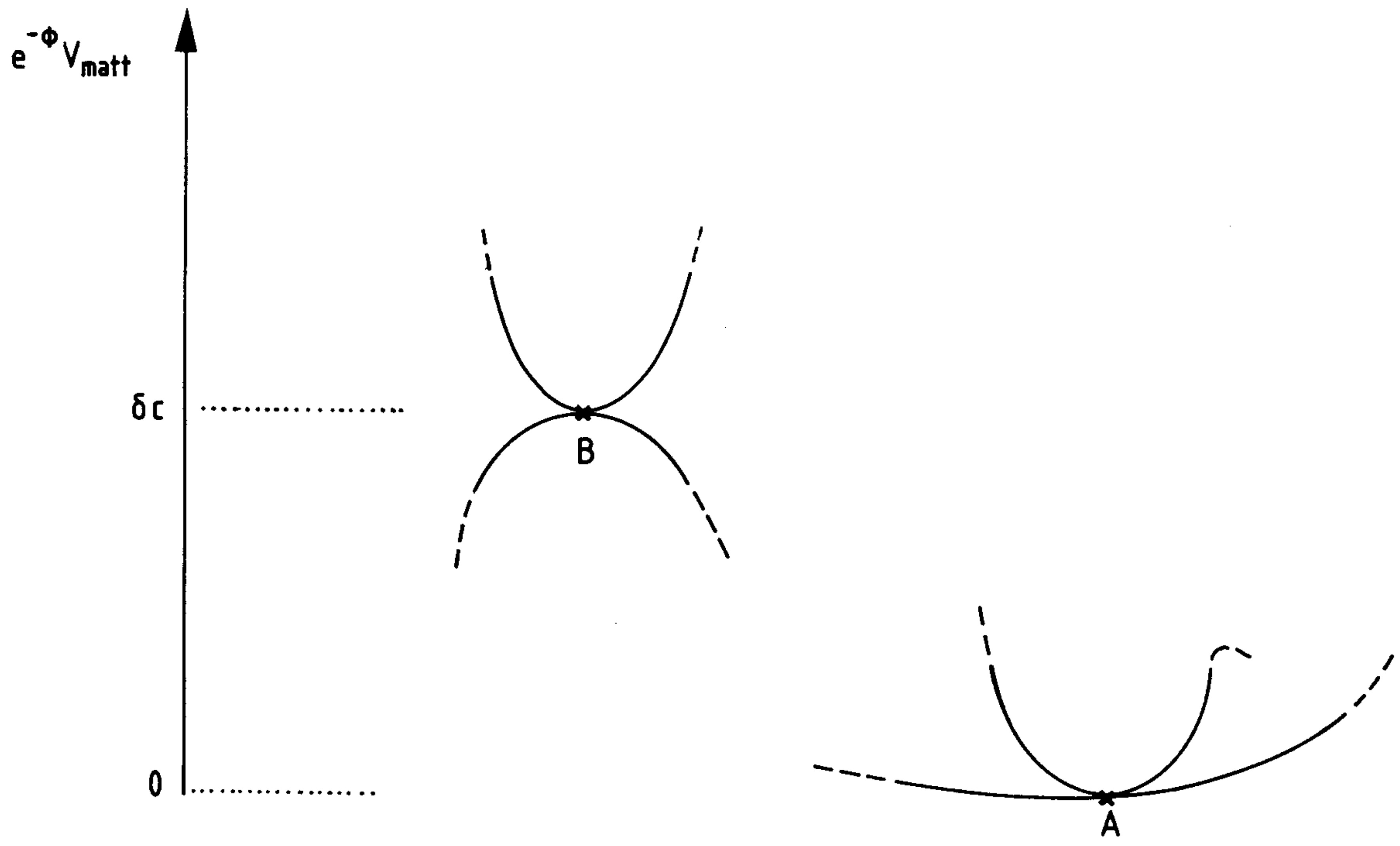

FIGURE 1

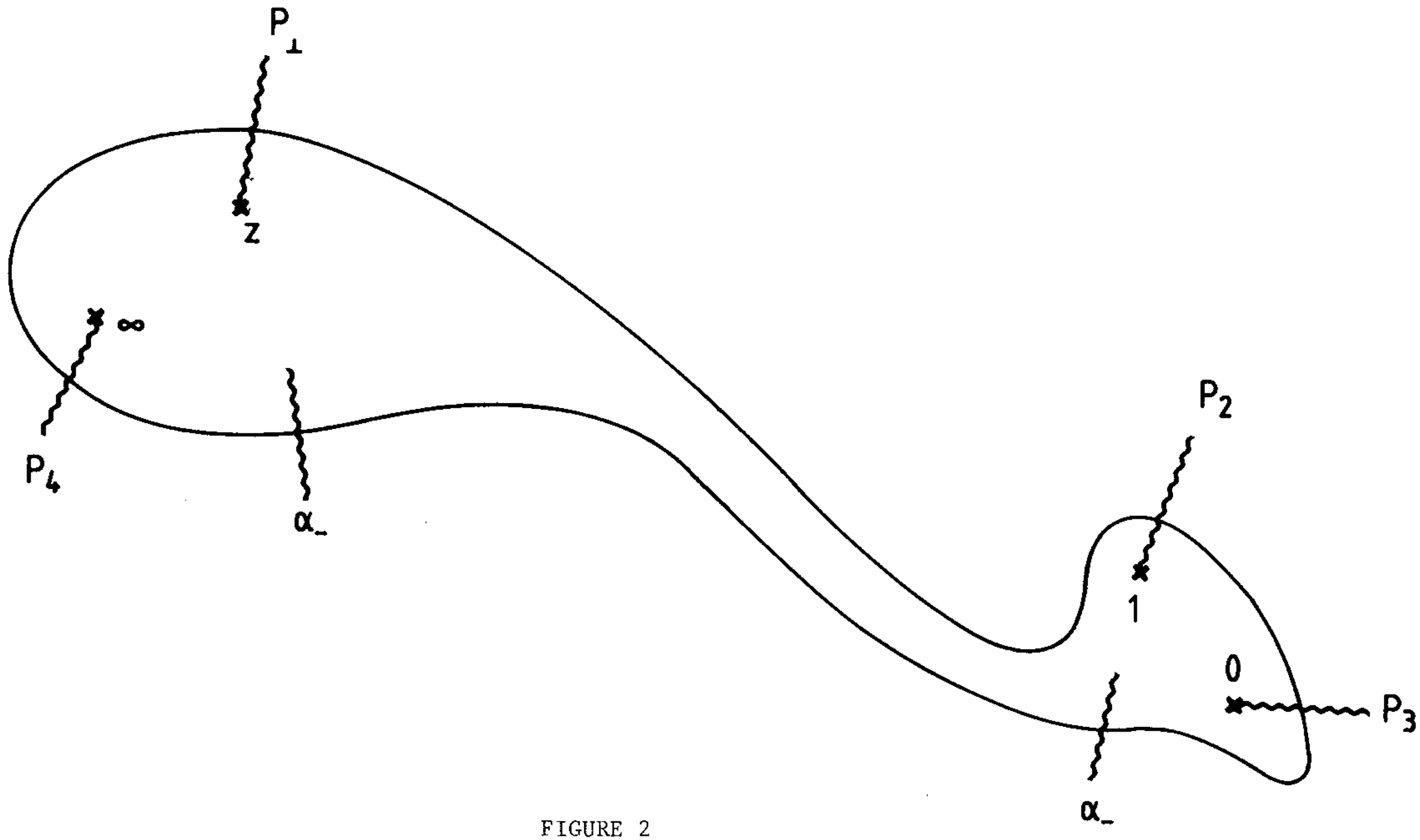

FIGURE 2