HOMOGENIZATION OF NONLINEAR RANDOM PARABOLIC OPERATORS

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Abstract. In the paper we consider the homogenization of nonlinear random parabolic operators. Depending on the ratio between time and spatial scales different homogenization regimes are studied and the homogenization procedure is carried out. The parameter dependent auxiliary problem is investigated and used in the construction of the homogenized operator.

1. Introduction. Let $Q_0 \in R^n$ be a bounded open set with Lipshitz boundary and $Q = (0,T) \times Q_0$. On Q we consider nonlinear evolution operators

$$L_\epsilon u = D_t u - div(a(\frac{x}{\epsilon^\beta}, \frac{t}{\epsilon^\alpha}, u, D_x u)) + a_0(\frac{x}{\epsilon^\beta}, \frac{t}{\epsilon^\alpha}, u, D_x u).$$

It is assumed that the temporal and spatial heterogeneities have random homogeneous nature which will be described more precisely later. We are interested in the asymptotic behavior of L_ϵ as $\epsilon \to 0$. G-convergence theory for parabolic operators guarantees that the limiting operator L^* belongs to the same class of parabolic operators. G-convergence of nonlinear parabolic operators has been studied in [7, 9]. To find the form of L^* some assumptions on the nature of spatial and temporal heterogeneities of a and a_0 need to be imposed. In the periodic setting the homogenization of nonlinear parabolic equations is carried out in [7]. Using two-scale convergence the homogenization of nonlinear parabolic equations for some values of α and β is investigated in [6]. In [1], time homogenization of random nonlinear abstract parabolic equations has been studied. The homogenization of linear parabolic operators with almost periodic and random coefficients has been studied in [11, 10].

In this paper we consider the homogenization of nonlinear parabolic equations when the fluxes, a and a_0 are random homogeneous fields with respect to temporal and spatial variables. We show that the homogenized operator has the form

$$L^* u = D_t u - div(a^*(x,t,u,D_x u)) + a_0^*(x,t,u,D_x u),$$

where the calculation of a^* and a_0^* depends on the ratio between α and β. Our homogenization results are of statistical nature, i.e., homogenization takes place for almost all realizations. As in the case of linear operators depending on the ratio between α and β different regimes are considered: self-similar case ($\alpha = 2\beta$); non self-similar case ($\alpha < 2\beta$); non self-similar case ($\alpha > 2\beta$); spatial case ($\alpha = 0$); temporal case ($\beta = 0$). These regimes yield different asymptotic behavior of L_ϵ which is determined by the solution of auxiliary problem. The auxiliary problem contains a parameter, which is characterized by the ratio between α and β. Depending on the ratio between α and β the solution of the auxiliary problem has different nature that determines the homogenized operator. As in [10] the solution of the auxiliary problem does not have independent meaning and we employ near solutions extensively.

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The main idea in carrying out the homogenization procedure is as follows. First we construct a solution for the parabolic equation by rescaling the solution of corresponding auxiliary problem. In this way the parameter involved in the auxiliary problem is set in terms of some power of ϵ. Next we study the convergence of the solutions or near solutions of the auxiliary problem as $\epsilon \to 0$. Further the results on the convergence of arbitrary solutions for G-converging sequence of operators allow us to calculate the homogenized operator based on a particular solution. This technique has been employed for periodic case in [7].

Since we consider random operators, our main result, Theorem 4.1 (below), is of statistical nature. It says that homogenization takes place almost surely. As in the case of nonlinear elliptic operators, one can deduce from this statistical result the individual homogenization theorem for almost periodic nonlinear parabolic operators. One needs only to consider almost periodic functions as realizations of appropriate random fields (in this case Ω is the Bohr compactification of R^{n+1}) and follow the proof of Theorem 3.3.1 [7]. We would like to mention that in the linear case there is more general individual homogenization theorem [10] that holds when Ω is a compact topological space and the dynamical system T is strictly ergodic. A nonlinear counterpart of this result is still an open problem even in the case of monotone elliptic operators.

Our motivation for considering homogenization of nonlinear parabolic equations comes from the applications arisen in flow in porous media for both saturated and unsaturated media, though one encounter nonlinear parabolic equations in many different applications. Due to uncertainties and general nature of the heterogeneities in subsurface flows one no longer can assume periodicity. We employ the results of the present work for the development and analysis of efficient numerical homogenization schemes in our subsequent paper [3]. In the porous media applications one is often interested in the gradients of the solutions. In [3] we construct numerical correctors for the solution of nonlinear parabolic equations. The auxiliary problem proposed in this work play a central role in the calculation of numerical correctors. These correctors further allow us to obtain the convergence of our numerical schemes for the gradients of the solutions. We would like to note that the homogenization results obtained in this work are important in addressing the robustness of the numerical homogenization schemes for more realistic porous media applications.

Finally we would like to note that the homogenization results and the analysis presented in this paper avoids many details involved in [10] since we study neither the individual homogenization nor the correctors. Moreover, the homogenization procedure presented in the paper differs from the one in [10]. In our paper we carry out the homogenization using the solution of an auxiliary problem and the theorem on G-convergence of arbitrary solutions.

The paper is organized as follows. In the next section we present some basic facts that are used later in the analysis. Section 3 is devoted to the auxiliary problem. In the following section we present the homogenization results.

2. Preliminaries. Let (Ω, Σ, μ) be a probability space and let $L^p(\Omega)$ denote the space of all p-integrable functions. Consider $(n+1)$-parametric dynamical system on Ω, $T(z) : \Omega \to \Omega$, $z = (x, t) \in R^{n+1}$ ($t \in R$, $x \in R^n$) that satisfies the following conditions: 1) $T(0) = I$, and $T(x + y) = T(x)T(y)$; 2) $T(z) : \Omega \to \Omega$ preserves the measure μ on Ω; 3) For any measurable function $f(\omega)$ on Ω, the function $f(T(z)\omega)$ defined on $R^{n+1} \times \Omega$ is also measurable.

$U(z)f(\omega) = f(T(z)\omega)$ defines a $(n+1)$-parameter group of isometries in the space of $L_p(\Omega)$. $U(z)$ is strongly continuous. Further we assume that the dynamical system T is ergodic, i.e., any measurable T-invariant function on Ω is constant. Denote by $\langle\cdot\rangle$ the mean value over Ω,

$$\langle f\rangle = \int_\Omega f(\omega)d\mu(\omega), \quad \langle u,v\rangle = \int (u,v)d\mu(\omega).$$

Throughout the paper C denotes a generic constant, $\|\cdot\|_{p,Q}$ denotes $L^p(Q)$ as well as $L^p(Q)^n$ norms and q is defined by $1/p+1/q=1$. The notation a.e. is often omitted.

For further analysis we will need Birkoff Ergodic Theorem. Denote

$$M\{f\} = \lim_{s\to\infty} \frac{1}{s^{n+1}|K|}\int_{K_s} f(z)dz,$$

where $K\subset R^{n+1}$, $|K|\neq 0$, and $K_s=\{z\in R^{n+1}: s^{-1}z\in K\}$. Let $f(\frac{x}{\epsilon})$ be bounded in $L^p_{loc}(R^{n+1})$, $1\le p<\infty$. Then f has mean value $M\{f\}$ if and only if $f(x/\epsilon)\to M\{f\}$ weakly in $L^p_{loc}(R^{n+1})$ as $\epsilon\to 0$ [4].

THEOREM 2.1. *(**Birkhoff Ergodic Theorem**)*
Let $f\in L^p(\Omega)$, $1\le p<\infty$. Then

$$\langle f\rangle = M\{f(T(z)\omega)\} \text{ a.e. on } \Omega.$$

Consider the equation on $Q=[0,T]\times Q_0$

$$\begin{aligned} D_t u_\epsilon = div\ a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon) - a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon) + f \ \text{ in } Q_0 \\ u_\epsilon = 0 \text{ on } \partial Q_0 \\ u_\epsilon(t=0)=0. \end{aligned} \tag{2.1}$$

We assume that $a(\omega,\eta,\xi)$ and $a_0(\omega,\eta,\xi)$, $\eta\in R$ and $\xi\in R^n$ are Caratheodory functions satisfying the following inequalities

- for any (η,ξ)
$$|a(\omega,\eta,\xi)|^{p'} + |a_0(\omega,\eta,\xi)|^{p'} \le c_0(|\eta|^p+|\xi|^p)+c(\omega), \text{ a.e. on } \Omega, \tag{2.2}$$
where $p>1$, $c_0>0$ and $c(\omega)$ belongs to $L^1(\Omega)$.
- for any (η,ξ) and $(\eta,\xi^{'})$
$$(a(\omega,\eta,\xi)-a(\omega,\eta,\xi^{'}),\xi-\xi^{'}) \ge C|\xi-\xi^{'}|^p, \text{ a.e. on } \Omega. \tag{2.3}$$
- for any (η,ξ)
$$(a(\omega,\eta,\xi),\xi)+a_0(\omega,\eta,\xi)\eta \ge C|\xi|^p - C_1 \text{ a.e. on } \Omega. \tag{2.4}$$
- for any $\chi=(\eta,\xi)$ and $\chi'=(\eta^{'},\xi^{'})$
$$\begin{aligned} |a(\omega,\eta,\xi)-a(\omega,\eta^{'},\xi^{'})|^{p'} + |a_0(\omega,\eta,\xi)-a_0(\omega,\eta^{'},\xi^{'})|^{p'} \le \\ k[(h(\omega)+|\chi|^p+|\chi'|^p)\nu(|\xi-\xi'|)+(h(\omega)+|\chi|^p+|\chi'|^p)^{1-s/p}|\xi-\xi'|^s], \text{ a.e. on } \Omega, \end{aligned} \tag{2.5}$$
where $k>0$, $0<s<\min(p,p')$, $\nu(r)$ is continuity modulus (i.e., a nondecreasing continuous function on $[0,+\infty)$ such that $\nu(0)=0$, $\nu(r)>0$ if $r>0$, and $\nu(r)=1$ if $r>1$), and $h\in L^1(\Omega)$.

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$$p > \frac{2n}{n+2}$$

Next we briefly review G-convergence results for non-monotone operators [7] that will be used in the analysis (G-convergence for monotone operators has been studied in [2]). Consider $Q = [0,T] \times Q_0$ and introduce

$$V_0 = L^p(0,T,W_0^{1,p}(Q_0)), \quad \overline{V} = L^p(0,T,W^{1,p}(Q_0)), \quad W = \{u \in L^p(0,T,W_0^{1,p}(Q_0)), D_t u \in L^q(0,T,W^{-1,q}(Q_0))\},$$
$$\overline{W} = \{u \in \overline{V}, D_t u \in L^q(0,T,W^{-1,q}(Q_0))\}, \quad W_0 = \{u \in W, u(0) = 0\}. \tag{2.6}$$

Consider a sequence of general parabolic operators L_k, $L_k u = D_t u - div(a_k(x,t,u,D_x u)) + a_{0,k}(x,t,u,D_x u)$ and $Lu = D_t u - div(a(x,t,u,D_x u)) + a_0(x,t,u,D_x u)$. We assume that L_k and L satisfy (2.2)-(2.5). Next we briefly mention the definition for G-convergence for sequence of operators L_k to L. For more details we refer to [7]. Based on L_k and L we define the sequence of operators $L_k^1(u,v) = D_t u - div(a_k(x,t,v,D_x u))$, $L^1(u,v) = D_t u - div(a(x,t,v,D_x u))$ and the fluxes

$$\Gamma^k(u,v) = a_k(t,x,v,D_x u), \quad \Gamma_0^k(u,v) = a_{0,k}(t,x,v,D_x u)$$
$$\Gamma(u,v) = a(t,x,v,D_x u), \quad \Gamma_0(u,v) = a_0(t,x,v,D_x u).$$

Given $v \in V_0$, $L_k^1(u,v)$ and $L^1(u,v)$ are strictly monotone parabolic operators. Therefore, for any $v \in V_0$ and $f \in W'$ there exist unique solutions $u_k \in W_0$ and $u \in W_0$ of $L_k^1(u_k,v) = f$ and $L^1(u,v) = f$ [8]. Without loss of generality we assume $k \to \infty$.

Definition (G-convergence) A sequence L_k G-converges to L if for any $v \in V_0$ and $f \in L^q(0,T,W^{-1,q}(Q_0))$ we have

$$u_k \to u$$

as $k \to \infty$ weakly in W_0, and

$$\Gamma^k(u_k,v) \to \Gamma(u,v)$$
$$\Gamma_0^k(u_k,v) \to \Gamma_0(u,v)$$

as $k \to \infty$ weakly in $L^q(Q)^n$ and $L^q(Q)$ respectively.

Remark. We would like to note that in [7] (where to our best knowledge G-convergence for this class of operators is first introduced) the author calls G-convergent sequence defined as above "strongly G-convergent sequence".

Next we mention the theorem on the convergence of arbitrary solutions for G-convergent sequence of operators [7] that will be used in our analysis.

Theorem 2.2. *Assume L_k G-converges to L, $u_k \in \overline{W}$, $f_k, f \in L^q(0,T,W^{-1,q}(Q_0))$, $L_k u_k = f_k$, $u_k \to u$ weakly in $\overline{W}$, and $f_k \to f$ strongly in W_0'. Then $Lu = f$, and*

$$a_k(x,t,u_k,D_x u_k) \to a(x,t,u,D_x u),$$
$$a_{0,k}(x,t,u_k,D_x u_k) \to a_0(x,t,u,D_x u)$$

as $k \to \infty$ weakly in $L^q(Q)^n$ and $L^q(Q)$ respectively.

Following to [10] we define spaces similar to $L^p(W^{1,p})$ on Ω in the following way. Denote by $\partial_{full} = (\partial_1, \cdots, \partial_{n+1})$ the collection of generators of the group $U(z)$.

There is a dense subspace $S \subset L^p(\Omega)$ that is contained in the domains of all operators $\partial^\alpha_{full} = \partial_1^{\alpha_1} \cdots \partial_{n+1}^{\alpha_{n+1}}$, $\alpha \in Z_+^{n+1}$. Next we introduce smoothing operators J^δ. Set $K(z) \in C_0^\infty(R^{n+1})$ be a non-negative even function such that

$$\int_{R^{n+1}} K(z)dz = 1,$$

and $K^\delta(z) = \delta^{-(n+1)}K(z/\delta)$. Define the operator J^δ as follows

$$J^\delta f(\omega) = \int_{R^{n+1}} K^\delta(z) f(T(z)\omega)dz.$$

J_δ is a bounded operator in the space of $L^p(\Omega)$ whose norm is not greater than 1. For a generic realization of f we have

$$J^\delta f(T(z)\omega) = \int_{R^{n+1}} K^\delta(z - z_1) f(T(z_1)\omega)dz_1.$$

The latter shows that a generic realization of $J^\delta f$ belongs to $C^\infty(R^{n+1})$. Thus, for $f \in L^p(\Omega)$ the function $J_\delta f$ belongs to the domain $D(\partial^\alpha, L^p(\Omega))$ for any $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_{n+1}^{\alpha_{n+1}}$. More discussion of J^δ can be found in [7]. The following lemma is important for the analysis (see [7], page 139).

Lemma 2.3. *For any $f \in L^p(\Omega)$*

$$\lim_{\delta \to 0} \|J^\delta f - f\|_{L^p(\Omega)} = 0.$$

Clearly, $J^\delta L^p \subset S$, $\delta > 0$.

Further denote by $\mathcal{V} = \mathcal{V}^p$ the completion of S with respect to the semi-norm

$$\|f\|_{\mathcal{V}} = \left(\sum_{i=1}^n \|\partial_i f\|^p_{L_p(\Omega)} \right)^{1/p}.$$

Note that the completion with respect to a seminorm "cuts off" the kernel of the semi-norm. The operator $\partial = (\partial_1, \ldots, \partial_n) : \mathcal{V} \to L^p(\Omega)^n$ is an isometric embedding. Moreover, the space $\mathcal{V}$ is reflexive with dual denoted by $\mathcal{V}'$. By duality we define the operator $\mathbf{div} : L^q(\Omega)^n \to \mathcal{V}'$, where

$$\langle \mathbf{div} u, w \rangle = -\langle u, \partial w \rangle, \ \forall u \in L^q(\Omega)^n, \ w \in \mathcal{V}^q. \tag{2.7}$$

We note that the elements of $\mathcal{V}$ in general do not have independent meaning and contains fields that are not spatially homogeneous. Note that [10, 7] (page 138 in [7]) the operators ∂_i may be viewed as derivatives along trajectories of the dynamical system $T(z)$

$$(\partial_i f)(T(z)\omega) = \frac{\partial}{\partial z_i} f(T(z)\omega) \tag{2.8}$$

for a.e. $\omega \in \Omega$ and $f \in D(\partial_i, L^p(\Omega))$. For our further analysis we introduce

$$T_1(t) = T(0, \ldots, 0, t), \quad T_2(x) = T(x_1, \ldots, x_n, 0). \tag{2.9}$$

Denote by

$$M_t\{f\} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(T_1(\tau)\omega)d\tau, \quad M_x\{f\} = \lim_{|K| \to \infty} \frac{1}{|K|} \int_K f(T_2(y)\omega)dy.$$

Next we introduce the differentiation with respect to time ∂_{n+1}. Define an unbounded operator σ from $\mathcal{V}$ into $\mathcal{V}'$ as follows. We say that $v \in \mathcal{V}$ belongs to $D(\sigma)$ if there exists $f \in \mathcal{V}'$ such that

$$\langle v, \partial_{n+1}\phi\rangle = -\langle f, \phi\rangle, \ \forall \phi \in S$$

and set $\sigma v = f$. It is easily seen that $\sigma\phi = \partial_{n+1}\phi$, $\phi \in S$. Moreover, σ commutes with operators J^δ. Therefore, σ is a closed linear operator from $\mathcal{V}$ to $\mathcal{V}'$. Let σ^+ be the adjoint operator (acting from $\mathcal{V}$ to $\mathcal{V}'$). Then

$$\sigma^+ = -\sigma,$$

i.e., σ is a skew symmetric operator. For further analysis we denote $\mathcal{W} = D(\sigma)$. Clearly, $\mathcal{W} = \mathcal{D}(\sigma)$ is dense in $\mathcal{V}$. Because $\langle \sigma u, u\rangle \geq 0$, $\forall u \in \mathcal{D}(\sigma)$, and $\langle \sigma^+ u, u\rangle \geq 0$, $\forall u \in \mathcal{D}(\sigma^+)$, $\sigma : \mathcal{W} \to \mathcal{V}^{'}$ is a maximal monotone operator [5].

3. Auxiliary problem. In this section we will study an auxiliary problem and near solutions for it. Consider the auxiliary problem

$$\mu\sigma w^\mu - \textbf{div}\ a(\omega, \eta, \xi + \partial w^\mu) = 0. \tag{3.1}$$

Define the operator A from $\mathcal{V}$ to $\mathcal{V}^{'}$ as

$$\langle Au, v\rangle = \langle a(\omega, \eta, \xi + \partial u), \partial v\rangle. \tag{3.2}$$

It can be easily verified that A is strongly monotone, i.e., $\langle Au - Av, u - v\rangle \geq C\|u - v\|_{\mathcal{V}}^p$, continuous, and coercive operator from $\mathcal{V}$ to $\mathcal{V}^{'}$. Since σ is maximal monotone whose domain $\mathcal{W}$ is dense in $\mathcal{V}$ it follows from [5] that the solution of (3.1) in $\mathcal{W}$ exists. Uniqueness follows from the fact that $(\sigma u, u) = 0$ and A is strongly monotone. Thus we have the following lemma.

Lemma 3.1. *Equation (3.1) has a unique solution, $w^\mu \in \mathcal{W}$, and*

$$\|w^\mu\|_{\mathcal{W}} \leq C. \tag{3.3}$$

For the analysis we need to consider near solutions for which w (the solution of (3.1)) is approximated with the functions that have smooth realizations.

For each element $v \in \mathcal{W}$ we define its near smooth element as follows. Since S is dense in $\mathcal{W}$ we can approximate $v \in \mathcal{W}$ with the elements $v_k \in S$, $v_k \to v$ in $\mathcal{W}$. Consider

$$J^\delta v_k = \int_{R^{n+1}} K^\delta(z - z_1) v_k(T(z_1)\omega) dz_1.$$

Clearly,

$$\|J^\delta v_k\|_{\mathcal{V}} \leq C\|v_k\|_{\mathcal{V}},$$

where C is independent of k. Since σ commutes with J^δ we have

$$\|J^\delta \sigma v_k\|_{\mathcal{V}'} \leq C\|\sigma v_k\|_{\mathcal{V}'}.$$

Consequently, $\|v_k^\delta\|_{\mathcal{W}} \leq C\|v_k\|_{\mathcal{W}}$, where $v_k^\delta = J^\delta v_k$. Hence, we can continue J^δ to $\mathcal{W}$, and denote by $v^\delta = J^\delta v$. In order v^δ to be a near solution of (3.3), one needs

$$\|w - v^\delta\|_{\mathcal{V}} \to 0, \quad \|\sigma v^\delta + Av^\delta\|_{\mathcal{V}'} \to 0$$

as $\delta \to 0$. The first limit is true due to the approximation property. The second limit is true due to the fact that J^{δ} commutes with σ, and A is continuous from $\mathcal{V}$ to $\mathcal{V}^{'}$.

Near solution for the auxiliary equation (3.1) has the form

$$\mu\sigma w_{\delta}^{\mu} + A w_{\delta}^{\mu} = \mathbf{div}\rho_{\delta}, \tag{3.4}$$

where **div** is defined by (2.7) and

$$\lim_{\delta\to 0}\langle|\rho_{\delta}|^{p}\rangle = 0. \tag{3.5}$$

The right hand side of (3.4) can be written as $\mathbf{div}\rho_{\delta}$ because it is an element of $\mathcal{V}^{'}$ (see [10]). The auxiliary equation on a typical realization has a form

$$\int_{R^{n+1}}(\mu D_{\tau}w_{\delta}^{\mu}(T(z)\omega)\psi(z)+(a(T(z)\omega,\eta,\xi+D_y w_{\delta}^{\mu}),D_y\psi(z)))dz = \int_{R^{n+1}}(\rho_{\delta}(T(z)\omega),D_y\psi(z))dz,\ \forall\psi\in C_0^{\infty}(R^{n+1}), \tag{3.6}$$

where $z=(y,\tau)\in R^{n+1}$ and $\rho_{\delta}\to 0$ in $L^{p}(\Omega)$ as $\delta\to 0$. By Ergodic Theorem

$$\int_{K}|\rho_{\delta}(T(z_{\epsilon})\omega)|^{p}dz \to |K|\langle|\rho_{\delta}(\omega)|^{p}\rangle \tag{3.7}$$

as $\epsilon\to 0$ for any $\delta>0$, where $z_{\epsilon}=(x/\epsilon^{\alpha},t/\epsilon^{\beta})\in R^{n+1}$. Furthermore the right hand side of (3.7) converges to zero as $\delta\to 0$ for each $\epsilon>0$.

The following lemma is needed for our analysis.

LEMMA 3.2. *Assume $\rho_{\delta}\in L^{p}(\Omega)$ and $\langle|\rho|^{p}\rangle < s(\delta)$, where $s(\delta)\to 0$ as $\delta\to 0$. Then for any sequence $\delta\to 0$ there exists a sequence $\epsilon_0(\delta)$, such that $\epsilon_0(\delta)\to 0$ as $\delta\to 0$, and for any $Q\subset R^{n+1}$*

$$\int_{Q}|\rho_{\delta}(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega)|^{p}dxdt < s(\delta),\ \forall\epsilon<\epsilon_0(\delta),$$

where $s(\delta)\to 0$ as $\delta\to 0$.

Proof.

Introduce $Q_{\epsilon}=\{(x,t)|(\epsilon^{\beta}x,\epsilon^{\alpha}t)\in Q\}$. Then

$$\int_{Q}|\rho_{\delta}(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega)|^{p}dxdt = \epsilon^{n\beta+\alpha}\int_{Q_{\epsilon}}|\rho_{\delta}(T(y,\tau)\omega)|^{p}dyd\tau \to meas(Q)\langle|\rho_{\delta}|^{p}\rangle < s(\delta),$$

as $\epsilon\to 0$. Here we have used Birkhoff Ergodic Theorem. From here it follows that there exists a sequence $\epsilon_0(\delta)$ such that for all $\epsilon<\epsilon_0(\delta)$, $\int_Q|\rho_{\delta}(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega)|^{p}dxdt < s(\delta)$, where $s(\delta)\to 0$ as $\delta\to 0$.

Q.E.D.

Throughout the paper $s(\delta)$ denotes a generic sequence that converges to zero as $\delta\to 0$.

4. Homogenization. The homogenization of parabolic equation depends on the relation between time and spatial scales [10, 7]. In particular we consider

$$L_{\epsilon}u = D_t u - div(a(x/\epsilon^{\beta},t/\epsilon^{\alpha},\omega,u,D_x u)) + a_0(x/\epsilon^{\beta},t/\epsilon^{\alpha},\omega,u,D_x u) \tag{4.1}$$

where

$$a(y,\tau,\omega,\eta,\xi)=a(T(z)\omega,\eta,\xi),\quad a_0(y,\tau,\omega,\eta,\xi)=a_0(T(z)\omega,\eta,\xi),\quad z=(y,\tau)\in R^{n+1}$$

Depending on α and β we distinguish the following cases:

- Self-similar case ($\alpha = 2\beta$)
- Non self-similar case ($\alpha < 2\beta$)
- Non self-similar case ($\alpha > 2\beta$)
- Spatial case ($\alpha = 0$)
- Temporal case ($\beta = 0$)

The homogenization for each case is presented next. The main idea of this procedure is as follows. First we construct a solution for the parabolic equation by rescaling the solution of corresponding auxiliary problem (3.1). After the rescaling μ in (3.1) may depend on ϵ. Further, we study the convergence of the solution of auxiliary problem as $\epsilon \to 0$. Then employing the results on the convergence of arbitrary solutions for G-convergent sequence of operators we calculate the homogenized fluxes a^* and a_0^* using a constructed solution. This technique has been employed for periodic case in [7].

Our main result is the following.

THEOREM 4.1.

L_ϵ G-converges to L^*, where L^* is given by

$$L^*u = D_t u - div(a^*(\omega, x, t, u, D_x u)) + a_0^*(\omega, x, t, u, D_x u). \tag{4.2}$$

a^* and a_0^* are defined as follows.

- *For self-similar case ($\alpha = 2\beta$),*

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle,$$
$$a_0{}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle,$$

  where $w_{\eta,\xi} = w^{\mu=1} \in \mathcal{W}$ is the unique solution of

$$\sigma w^{\mu=1} - \mathbf{div}\ a(\omega, \eta, \xi + \partial w^{\mu=1}) = 0. \tag{4.3}$$

- *For non self-similar case ($\alpha < 2\beta$),*

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle,$$
$$a_0{}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle,$$

  where $w_{\eta,\xi} = w^0 \in \mathcal{V}$ is the unique solution of

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial w^0) = 0. \tag{4.4}$$

- *For non self-similar case ($\alpha > 2\beta$),*

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle,$$
$$a_0{}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle,$$

  where $w_{\eta,\xi} = w^\infty \in \mathcal{V}_s$ is the unique solution of

$$-\mathbf{div}\ \overline{a}(\omega, \eta, \xi + \partial w^\infty) = 0. \tag{4.5}$$

  $\overline{a}$ and $\mathcal{V}_s$ is defined in section 4.2.2.

- *For spatial case ($\alpha = 0$),*

$$a(\omega, \eta, \xi) = M_x\{a(T_2(x)\omega, \eta, \xi + \partial w_{\eta,\xi}(T_2(x)\omega))\},$$
$$a_0(\omega, \eta, \xi) = M_x\{a_0(T_2(x)\omega, \eta, \xi + \partial w_{\eta,\xi}(T_2(x)\omega))\},$$

  where $w_{\eta,\xi} = w_x \in \mathcal{V}$

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial w_x) = 0. \tag{4.6}$$

- *For temporal case ($\beta = 0$), the homogenized fluxes are defined by*

$$\begin{aligned} a^*(\omega,\eta,\xi) &= P_1 a(\omega,\eta,\xi), \\ a_0^*(\omega,\eta,\xi) &= P_1 a_0(\omega,\eta,\xi), \end{aligned} \tag{4.7}$$

where P_1 is defined in (4.29).

The theorem on the convergence of arbitrary solutions (Theorem 2.2) for G-convergent sequence of operators allows us not to restrict ourselves to a particular boundary or initial conditions. In particular, from Theorem 2.2 and Theorem 4.1 we have

THEOREM 4.2. *Let $u_\epsilon \in \overline{W}$ be a solution of $L_\epsilon u_\epsilon = f$, $f \in L^q(0,T,W^{-1,q}(Q_0))$, such that $\|u_\epsilon\|_{\overline{W}}$ is bounded. Then u_ϵ converges to u as $\epsilon \to 0$ weakly in $\overline{W}$ (up to a subsequence) where u is a solution of $L^* u = f$, and L^* is defined in (4.2).*

Remark. We note that the ergodicity assumption is not essential for the proof of the theorem. One can carry out the proof for non-ergodic case essentially in the same manner as that for the ergodic case. The homogenized operators for non-ergodic case will be invariant functions with respect to $T(z)$.

Remark. Note that in the case of spatial and temporal homogenization the homogenized operator depends on ω. If the operator is random in time variable one can apply the results of [1]. However, the results of [1] do not imply convergence of the fluxes.

Remark. Under additional regularity assumption on a and a_0 with respect to the time variable, G-convergence results follow from the G-convergence of elliptic parts of parabolic operators. However, this additional assumption is too restrictive and not well suited to the case of random operators.

Remark. In the analysis, for simplicity we assume (2.4), though the homogenization results can be obtained under a weaker assumption,

$$(a(\omega,\eta,\xi) - a(\omega,\eta,\xi^{'}), \xi - \xi^{'}) \geq C(1 + |\xi|^p + |\xi^{'}|^p)^{p-\beta} |\xi - \xi'|^\beta, \text{ a.e. on } \Omega.$$

4.1. Self-similar case ($\alpha = 2$, $\beta = 1$). Take $\mu = 1$ in (3.1), consider near solutions $w_\delta^{\mu=1}$, and set

$$w_{\epsilon,\delta}^{\mu=1}(x,t,\omega) = \epsilon w_\delta^{\mu=1}(T(x/\epsilon, t/\epsilon^2)\omega). \tag{4.8}$$

$w_{\epsilon,\delta}^{\mu}$ satisfies in R^{n+1} for a.e. ω

$$D_t w_{\epsilon,\delta}^{\mu=1} - div(a(T(x/\epsilon, t/\epsilon^2)\omega, \eta, \xi + D_x w_{\epsilon,\delta}^{\mu=1})) = div_x \rho_\delta, \tag{4.9}$$

where $\langle |\rho_\delta|^p \rangle \to s(\delta)$, where $s(\delta) \to 0$ as $\delta \to 0$.

LEMMA 4.3. *For every $\delta > 0$ $w_{\epsilon,\delta}^{\mu=1} \to 0$ weakly in $\overline{W}$ as $\epsilon \to 0$.*

Proof. Using the fact that $w_\delta^{\mu=1} \in \mathcal{W}$ and (4.8) we obtain that $\|w_{\epsilon,\delta}^{\mu=1}\|_{L^p(0,T,W^{1,p}(Q_0))} \leq C$ and $w_{\epsilon,\delta}^{\mu=1} \to 0$ in $L^p(0,T,L^p(Q_0))$ as $\epsilon \to 0$ for every $\delta > 0$ (cf.[10]). Next we will show that

$$\|D_t w_{\epsilon,\delta}^{\mu=1}\|_{L^q(0,T,W^{-1,q}(Q_0))} \leq C.$$

From equation (4.9) we obtain that

$$\|D_t w_{\epsilon,\delta}^{\mu=1}\|_{L^q(0,T,W^{-1,q}(Q_0))} \leq C \|w_{\epsilon,\delta}^{\mu=1}\|_{L^p(0,T,W^{1,p}(Q_0))} + C\|\rho_\delta\|_{p,Q} \leq C.$$

Consequently, $w_{\epsilon,\delta}^{\mu=1}$ is bounded in $\overline{W}$. Since this family is weakly compact in $\overline{W}$ and converges to zero in $L^p(0,T,L^p(Q_0))$, it converges to zero weakly in $\overline{W}$.

Q.E.D.

Define

$$L^*u = D_t u - div(a^*(u, D_x u)) + a_0^*(u, D_x u), \tag{4.10}$$

where

$$\begin{aligned} a^*(\eta,\xi) &= \langle a(\omega,\eta,\xi+\partial w^{\mu=1})\rangle \\ a_0^*(\eta,\xi) &= \langle a_0(\omega,\eta,\xi+\partial w^{\mu=1})\rangle. \end{aligned}$$

THEOREM 4.4. *If $\alpha = 2\beta$ then L_ϵ G-converges to L^* defined by (4.10) for a.e. $\omega \in \Omega$.*

Proof.

The equation for $w_{\epsilon,\delta}^{\mu=1}$ (4.9) can be written as

$$D_t w_{\epsilon,\delta}^{\mu=1} - div(a(T(x/\epsilon,t/\epsilon^2)\omega,\eta,\xi+D_x w_{\epsilon,\delta}^{\mu=1}) + a_0(T(x/\epsilon,t/\epsilon^2)\omega,\eta,\xi+D_x w_{\epsilon,\delta}^{\mu=1}) = h_{\epsilon,\delta} + div_x\rho_\delta,$$

where $h_{\epsilon,\delta} = a_0(T(x/\epsilon,t/\epsilon^2)\omega,\eta,\xi+D_x w_{\epsilon,\delta}^{\mu=1})$.

Next we choose two sequences $\delta\to 0$ and $\epsilon(\delta)\to 0$ such that $w_{\epsilon(\delta),\delta}^{\mu=1}\to 0$ weakly in $\overline{W}$, and $\rho_\delta \to 0$ in $L^q(Q)^n$ as $k\to\infty$. This is possible for any sequence $\delta\to 0$ because of Lemma 4.3 and Lemma 3.2. Consider a generic sequence of $\delta_k \to 0$ as $k\to\infty$ and corresponding $\epsilon_k = \epsilon(\delta_k)$. Then $w_k^{\mu=1} = w_{\epsilon_k,\delta_k}^{\mu=1} \to 0$ weakly in $\overline{W}$, and $\rho_k = \rho_{\delta_k} \to 0$ in $L^q(Q)^n$ as $k\to\infty$. Consider for each $\omega\in\Omega$

$$L_k u = D_t u - div(a(T(x/\epsilon_k,t/\epsilon_k^2)\omega,\eta,\xi+D_x u) + a_0(T(x/\epsilon_k,t/\epsilon_k^2)\omega,\eta,\xi+D_x u).$$

It is known [7] that L_k G-converges to $\tilde{L}$ (up to a subsequence),

$$\tilde{L}u = D_t u - div(\tilde{a}(\omega,t,x,\eta,\xi+D_x u)) + \tilde{a}_0(\omega,t,x,\eta,\xi+D_x u).$$

Moreover, for a.e. ω

$$\begin{aligned} a(T(x/\epsilon_k,t/\epsilon_k^2)\omega,\eta,\xi+D_x w_k^{\mu=1}) &\to \tilde{a}(\omega,t,x,\eta,\xi) \\ a_0(T(x/\epsilon_k,t/\epsilon_k^2)\omega,\eta,\xi+D_x w_k^{\mu=1}) &\to \tilde{a}_0(\omega,t,x,\eta,\xi), \end{aligned} \tag{4.11}$$

as $k\to\infty$ weakly in $L^q(Q)^n$ and $L^q(Q)$. Our goal is to find the form of $\tilde{a}$ and $\tilde{a}_0$. Note that $h_k = h_{\epsilon_k,\delta_k}$ is bounded in $L^q(Q)$ and $h_k \to h = \tilde{a}_0(\omega,t,x,\eta,\xi)$ as $k\to\infty$ weakly in $L^q(Q)$. Thus from the convergence of arbitrary solutions for G-convergent sequence of operators we also have $u=0$ is a solution of $\tilde{L}u = h$. On the other hand using Ergodic Theorem

$$\begin{aligned} a(T(x/\epsilon_k,t/\epsilon_k^2)\omega,\eta,\xi+D_x w_k^{\mu=1}) &\to \langle a(\omega,\eta,\xi+\partial w^{\mu=1})\rangle \\ a_0(T(x/\epsilon_k,t/\epsilon_k^2)\omega,\eta,\xi+D_x w_k^{\mu=1}) &\to \langle a_0(\omega,\eta,\xi+\partial w^{\mu=1})\rangle, \end{aligned} \tag{4.12}$$

as $k\to\infty$ weakly in $L^q(Q)^n$ and $L^q(Q)$ because $a(T(x/\epsilon_k,t/\epsilon_k^2)\omega,\eta,\xi+D_x w_{\epsilon_k,\delta_k}^{\mu=1})$ and $a_0(T(x/\epsilon_k,t/\epsilon_k^2)\omega,\eta,\xi+D_x w_{\epsilon,\delta}^{\mu=1})$ are homogeneous fields. Comparing (4.11) and (4.12) we obtain

$$\begin{aligned} \tilde{a}(\omega,t,x,\eta,\xi) &= \langle a(\omega,\eta,\xi+\partial w^{\mu=1})\rangle \\ \tilde{a}_0(\omega,t,x,\eta,\xi) &= \langle a_0(\omega,\eta,\xi+\partial w^{\mu=1})\rangle. \end{aligned}$$

Finally it can be easily verified that $u=0$ is the solution of $\tilde{L}u = h$.

Q.E.D.

4.2. Non self-similar cases.

4.2.1. Case $\alpha < 2\beta$, $\beta = 1$. To construct the homogenized operator for this case we will set $\mu = \epsilon^{2-\alpha}$ in (3.1) and need to study the limit of w^μ as $\mu \to 0$.

LEMMA 4.5.
$w^\mu \to w^0$ as $\mu \to 0$ in $\mathcal{V}$, where $w^0 \in \mathcal{V}$ is the unique solution of

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial w^0) = 0. \tag{4.13}$$

Proof.
The uniqueness of the solution of (4.13) can be shown in the same as that of (3.1) using the fact that A (see (3.2)) is strongly monotone operator from $\mathcal{V}$ to $\mathcal{V}'$.

To show the convergence we follow [7]. Define $w^{0,k} \in \mathcal{W}$ such that $w^{0,k} \to w^0$ in $\mathcal{V}$. Such sequence exists since $\mathcal{W}$ is dense in $\mathcal{V}$. Then

$$\|w^\mu - w^{0,k}\|_{\mathcal{V}}^p \leq C\langle Aw^\mu - Aw^{0,k}, w^\mu - w^{0,k}\rangle = C\langle \mu\sigma(w^\mu - w^{0,k}) + Aw^\mu - Aw^{0,k}, w^\mu - w^{0,k}\rangle \leq$$
$$C\langle -\mu\sigma w^{0,k} - Aw^{0,k}, w^\mu - w^{0,k}\rangle \leq C(\|\mu\sigma w^{0,k}\|_{\mathcal{V}'} + \|Aw^{0,k}\|_{\mathcal{V}'})\|w^\mu - w^{0,k}\|_{\mathcal{V}}$$

Here we have used the fact that $\mu\sigma w^\mu - Aw^\mu = 0$. Next using the fact that $Aw_0 = 0$ we have

$$\|w^\mu - w^{0,k}\|_{\mathcal{V}} \leq C(\|\mu\sigma w^{0,k}\|_{\mathcal{V}'} + \|Aw^0 - Aw^{0,k}\|_{\mathcal{V}'}).$$

Thus,

$$\|w^\mu - w^0\|_{\mathcal{V}} \leq \|w^\mu - w^{0,k}\|_{\mathcal{V}} + \|w^{0,k} - w^0\|_{\mathcal{V}} \leq C(\mu\|\sigma w^{0,k}\|_{\mathcal{V}'} + \|Aw^0 - Aw^{0,k}\|_{\mathcal{V}'}) + \|w^{0,k} - w^0\|_{\mathcal{V}}. \tag{4.14}$$

Next for any $\delta > 0$ we can choose k sufficiently large such that $\|w^{0,k} - w^0\|_{\mathcal{V}} < \delta$ and $\|Aw^0 - Aw^{0,k}\|_{\mathcal{V}'} < \delta$. The latter is possible since A is continuous from $\mathcal{V}$ to $\mathcal{V}'$. Next choosing μ sufficiently small we have $\mu\|\sigma w^{0,k}\|_{\mathcal{V}'} < \delta$, and hence from (4.14)

$$\|w^\mu - w^0\|_{\mathcal{V}} < C\delta.$$

Q.E.D.

Define

$$L^* u = D_t u - div(a^*(u, D_x u)) + a_0^*(u, D_x u), \tag{4.15}$$

where

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w^0)\rangle$$
$$a_0^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w^0)\rangle,$$

w^0 is the solution of (4.13).

THEOREM 4.6. *If $\alpha < 2\beta$, $\beta = 1$ then L_ϵ G-converges to L^* defined by (4.15).*

Proof.
Set $\mu = \epsilon^{2-\alpha}$ in (3.1), consider near solutions of (3.1), w_δ^μ, and set

$$w_{\epsilon,\delta} = \epsilon w_\delta^\mu(T(x/\epsilon, t/\epsilon^\alpha)\omega).$$

Furthermore, set

$$w_{\epsilon,\delta}^0 = \epsilon w_\delta^0(T(x/\epsilon, t/\epsilon^\alpha)\omega),$$

where $w^0_{\epsilon,\delta}$ are near solutions of (4.13). Then $w_{\epsilon,\delta}$ satisfies in R^{n+1} for a.e. ω

$$D_t w^{\mu}_{\epsilon,\delta} - div(a(T(x/\epsilon, t/\epsilon^2)\omega, \eta, \xi + D_x w^{\mu}_{\epsilon,\delta})) = div_x \rho_\delta, \tag{4.16}$$

where $\langle|\rho_\delta|^p\rangle < s(\delta)$, where $s(\delta) \to 0$ as $\delta \to 0$. As in the proof of Theorem 4.4 we choose two sequences $\delta \to 0$ and $\epsilon(\delta) \to 0$ such that $w_{\epsilon(\delta),\delta} \to 0$ weakly in $\overline{W}$, and $\rho_\delta \to 0$ in $L^q(Q)^n$ as $k \to \infty$. This is possible for any sequence $\delta \to 0$ because of Lemma 4.3 and Lemma 3.2. Consider a generic sequence of $\delta_k \to 0$ as $k \to \infty$ and corresponding $\epsilon_k = \epsilon(\delta_k)$. Then $w_k = w_{\epsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$, and $\rho_k = \rho_{\delta_k} \to 0$ in $L^q(Q)$ as $k \to \infty$.

As in the proof of Theorem 4.4 using the convergence of arbitrary solutions for G-convergent sequence of operators we have that for a.e. ω

$$\begin{aligned} a(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w_k) &\to \tilde{a}(\omega, t, x, \eta, \xi) \\ a_0(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w_k) &\to \tilde{a}_0(\omega, t, x, \eta, \xi) \end{aligned} \tag{4.17}$$

weakly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. Set $w^0_k = w^0_{\epsilon_k,\delta_k}$. Using Ergodic Theorem we have

$$\begin{aligned} a(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w^0_k) &\to \langle a(\omega, \eta, \xi + \partial w^0)\rangle \\ a_0(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w^0_k) &\to \langle a_0(\omega, \eta, \xi + \partial w^0)\rangle \end{aligned} \tag{4.18}$$

weakly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$ because $a(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w^0_k)$ and $a_0(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w^0_k)$ are homogeneous fields.

Since $\|w^\mu - w^0\|_{\mathcal{V}} \to 0$ as $\mu \to 0$ we can obtain $\|D_x w_k - D_x w^0_k\|_{p,Q} \to 0$ as $k \to \infty$. The latter can be shown using triangular inequality, $\|D_x w_k - D_x w^0_k\|_{p,Q} \le \|D_x w_k - D_x w^\mu\|_{p,Q} + \|D_x w^\mu - D_x w^0\|_{p,Q} + \|D_x w^0 - D_x w^0_k\|_{p,Q}$ and Lemma 3.2. Indeed, $\|D_x w_k - D_x w^\mu\|_{p,Q}$ and $\|D_x w^0 - D_x w^0_k\|_{p,Q}$ can be estimated using Lemma 3.2 since they represent the error associated with near solutions, and $\langle|\partial w^\mu_\delta - \partial w^\mu|^p\rangle < s(\delta)$, and $\langle|\partial w^0_\delta - \partial w^0|^p\rangle < s(\delta)$, where $s(\delta) \to 0$ as $\delta \to 0$. Thus there exists a subsequence of $\delta_k \to 0$ (as $k \to \infty$) such that $\|D_x w_k - D_x w^\mu\|_{p,Q}$ and $\|D_x w^0 - D_x w^0_k\|_{p,Q}$ converge to zero as $k \to \infty$. The term $\|D_x w^\mu - D_x w^0\|_{p,Q}$ converges to zero as $\mu \to 0$ or $\epsilon \to 0$ because of Lemma 4.5. Thus,

$$\begin{aligned} a(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w_k) - a(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w^0_k) &\to 0 \\ a_0(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w_k) - a_0(T(x/\epsilon_k, t/\epsilon_k^\alpha)\omega, \eta, \xi + D_x w^0_k) &\to 0 \end{aligned} \tag{4.19}$$

in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. Combining (4.17), (4.18) and (4.19) we see that L^* defines the homogenized operator.

Q.E.D.

4.2.2. Case $\alpha > 2\beta$, $\beta = 1$. For the analysis of this case we will need to consider asymptotic behavior of w^μ as $\mu \to \infty$. This requires the average of $a(\omega, \eta, \xi)$ over the time variable which will be defined next. $a(T(z)\omega, \eta, \xi)$, $z = (x, t) \in R^{n+1}$ can be considered as a continuous function from $L^p_{loc}(R^{n+1})$ to $L^p(\Omega)$ for each $\eta \in R$ and $\xi \in R^n$. Consider an average of a over t variable at fixed $x = 0$,

$$\overline{a}(\omega, \eta, \xi) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a(T_1(t)\omega, \eta, \xi)dt = M_t\{a(\omega, \eta, \xi)\}$$

The function $\overline{a}(\omega, \eta, \xi)$ is defined on $L^p(\Omega)$ for each $\eta \in R$ and $\xi \in R^n$.

Next we introduce the space $\mathcal{V}_s$. Consider the subset of S consisting of functions

$$f(\omega) = M_t\{f(T_1(t)\omega\}.$$

Denote by $\mathcal{V}_s$ the completion of this set with respect to the norm

$$\|f\| = (\sum_{i=1}^{n} \|\partial_i f\|^p_{L_p(\Omega)})^{1/p}.$$

Define the operator $(\overline{A}u, v) = (\overline{a}(\omega, \eta, \xi + \partial u), \partial v)$. $\overline{A} : \mathcal{V}_s \to \mathcal{V}_s^{'}$ is bounded, continuous and strongly monotone. Indeed, for any $u, v \in \mathcal{V}_s$

$$\langle \overline{A}u - \overline{A}v, u - v\rangle = \langle Au - Av, u - v\rangle \geq C\langle|\partial(u - v)|^p\rangle.$$

This implies the existence and uniqueness of $w^\infty \in \mathcal{V}_s$ which is the solution of

$$-\mathbf{div}(\overline{a}(\omega, \eta, \xi + \partial w^\infty)) = 0. \tag{4.20}$$

For further analysis we denote near solutions of (4.20) by w^∞_δ. Then

LEMMA 4.7. $\lim_{\mu\to\infty} \|w^\mu - w^\infty\|_{\mathcal{V}} = 0$

Proof.

Set

$$w^\mu_\delta = w^\infty + \frac{1}{\mu} v_\delta,$$

where v_δ will be defined later. Note that δ does not indicate near solutions here. We will show that w^μ_δ approximates w^μ for large μ. We have

$$\mu\sigma w^\mu_\delta + Aw^\mu_\delta = \sigma v_\delta + f_1 + f_{2,\delta},$$

where A is defined as previously by $Au = \mathbf{div}\ a(\omega, \eta, \xi + \partial u)$ and

$$f_1 = \mathbf{div}\ a(\omega, \eta, \xi + \partial w^\infty),$$
$$f_2^\delta = \mathbf{div}(a(\omega, \eta, \xi + \partial w^\mu_\delta) - a(\omega, \eta, \xi + \partial w^\infty)).$$

Note that for any $\phi \in \mathcal{V}_s$

$$(f_1, \phi) = \int_\Omega a(\omega, \eta, \xi + \partial w^\infty)\partial_{full}\phi d\mu(\omega) =$$
$$\int_\Omega \overline{a}(\omega, \eta, \xi + \partial w^\infty)\partial\phi d\mu(\omega) = 0.$$

Consider σ as a closed operator from $\mathcal{V}$ to $\mathcal{V}^{'}$. The kernel of σ is $\mathcal{V}_s$. Using the fact that the range of σ is dense in the orthogonal complement of $ker(\sigma^+)$, and the fact that $\sigma^+ = -\sigma$ (σ^+ is the adjoint of σ) we have that there exist $v_\delta \in \mathcal{W}$, $g_\delta \in \mathcal{V}^{'}$ such that

$$f_1 = -\sigma v_\delta + g_\delta$$

and $\|g_\delta\|_{\mathcal{V}'} \leq s(\delta)$, where $s(\delta) \to 0$ as $\delta \to 0$. This is the way we define v_δ. Then

$$C\|w^\mu_\delta - w^\mu\|^p_{\mathcal{V}} \leq (\mu\sigma(w^\mu_\delta - w^\mu) + Aw^\mu_\delta - Aw^\mu, w^\mu_\delta - w^\mu) =$$
$$(\mu\sigma w^\mu_\delta + Aw^\mu_\delta, w^\mu_\delta - w^\mu) = (\sigma v_\delta + f_1 + f_{2,\delta}, w^\mu_\delta - w^\mu) =$$
$$(g_\delta + f_{2,\delta}, w^\mu_\delta - w^\mu) \leq (\|g_\delta\|_{\mathcal{V}'} + \|f_{2,\delta}\|_{\mathcal{V}'})\|w^\mu_\delta - w^\mu\|_{\mathcal{V}}.$$

This implies

$$\|w_\delta^\mu - w^\mu\|_{\mathcal{V}} \leq C(\|g_\delta\|_{\mathcal{V}'} + \|f_{2,\delta}\|_{\mathcal{V}'})^{1/(p-1)}.$$

On the other hand using Holder continuity of a we have

$$\|f_{2,\delta}\|_{\mathcal{V}'} \leq C\|w_\delta^\mu - w^\infty\|_{\mathcal{V}} = C\mu^{-1}\|v_\delta\|_{\mathcal{V}}.$$

Consequently,

$$\|w_\delta^\mu - w^\mu\|_{\mathcal{V}} \leq C(s(\delta) + \mu^{-1}\|v_\delta\|_{\mathcal{V}})^{1/(p-1)}.$$

Furthermore,

$$\|w^\mu - w^\infty\|_{\mathcal{V}} \leq \|w_\delta^\mu - w^\infty\|_{\mathcal{V}} + \|w^\mu - w_\delta^\mu\|_{\mathcal{V}} \leq \mu^{-1}\|v_\delta\|_{\mathcal{V}} + \|w^\mu - w_\delta^\mu\|_{\mathcal{V}}.$$

Thus, for any $\zeta > 0$ we can choose δ sufficiently small such that for all $\mu > \mu_0$ we have $\|w^\mu - w^\infty\|_{\mathcal{V}} < \zeta$.

Q.E.D.

Define

$$L^* u = D_t u - div(a^*(u, D_x u)) + a_0^*(u, D_x u), \tag{4.21}$$

where

$$\begin{aligned} a^*(\eta,\xi) &= \langle a(\omega,\eta,\xi+\partial w^\infty)\rangle, \\ a_0^*(\eta,\xi) &= \langle a_0(\omega,\eta,\xi+\partial w^\infty)\rangle, \end{aligned} \tag{4.22}$$

w^∞ is the solution of (4.20).

Theorem 4.8. *If $\alpha > 2\beta$, $\beta = 1$ then L_ϵ G-converges to L^* defined by (4.21) for a.e. $\omega \in \Omega$.*

Proof.

Set $\mu = \epsilon^{2-\alpha} \to \infty$ as $\epsilon \to 0$ and

$$w_{\epsilon,\delta} = \epsilon w_\delta^\mu(T(x/\epsilon, t/\epsilon^\alpha)\omega), \quad w_{\epsilon,\delta}^\infty = \epsilon w_\delta^\infty(T(x/\epsilon, t/\epsilon^\alpha)\omega),$$

where w_δ^μ is near solutions of (3.1) and w_δ^∞ is near solutions of (4.20). Then $w_{\epsilon,\delta}$ satisfies in R^{n+1} for a.e. ω

$$D_t w_{\epsilon,\delta} - div(a(T(x/\epsilon, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_{\epsilon,\delta}) = div_x \rho_\delta,$$

where $\langle|\rho_\delta|^p\rangle \to 0$ as $\delta \to 0$. As in the proof of Theorem 4.4 we choose two sequences $\delta \to 0$ and $\epsilon(\delta) \to 0$ such that $w_{\epsilon(\delta),\delta} \to 0$ weakly in $\overline{W}$, and $\rho_\delta \to 0$ in $L^q(Q)^n$ as $k \to \infty$. This is possible for any sequence $\delta \to 0$ because of Lemma 4.3 and Lemma 3.2. Consider a generic sequence of $\delta_k \to 0$ as $k \to \infty$ and corresponding $\epsilon_k = \epsilon(\delta_k)$. Then $w_k = w_{\epsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$, and $\rho_k = \rho_{\delta_k} \to 0$ in $L^q(Q)^n$ as $k \to \infty$. Using the convergence of arbitrary solutions for G-convergent sequence of operators as in the proof of Theorem (4.4) we obtain that for a.e. ω

$$\begin{aligned} a(T(x/\epsilon_k, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_k) &\to \tilde{a}(\omega, t, x, \eta, \xi) \\ a_0(T(x/\epsilon_k, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_k) &\to \tilde{a}_0(\omega, t, x, \eta, \xi) \end{aligned} \tag{4.23}$$

weakly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. On the other hand using Ergodic Theorem we have

$$\begin{aligned} a(T(x/\epsilon_k, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_k^\infty) &\to \langle a(\omega, \eta, \xi + \partial w^\infty)\rangle \\ a_0(T(x/\epsilon_k, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_k^\infty) &\to \langle a_0(\omega, \eta, \xi + \partial w^\infty)\rangle \end{aligned} \tag{4.24}$$

weakly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. Since $\lim_{\mu\to\infty} \|w^\mu - w^\infty\|_{\mathcal{V}} = 0$ as in the proof of Theorem 4.6 we obtain that $\|D_x w_k - D_x w_k^\infty\|_{p,Q} \to 0$ as $k \to \infty$. Consequently,

$$\begin{aligned} a(T(x/\epsilon_k, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_k) - a(T(x/\epsilon_k, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_k^\infty) &\to 0 \\ a_0(T(x/\epsilon_k, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_k) - a_0(T(x/\epsilon_k, t/\epsilon^\alpha)\omega, \eta, \xi + D_x w_k^\infty) &\to 0 \end{aligned} \tag{4.25}$$

in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. Combining (4.23), (4.24) and (4.25) we see that L^* defines the homogenized operator.

Q.E.D.

4.3. Spatial homogenization ($\alpha = 0$, $\beta = 1$). Consider

$$\begin{aligned} a(T_1(t)\omega, \eta, \xi) &= M_x\{a(T_2(x)\omega, \eta, \xi + \partial w_x(T_2(x)\omega))\} \\ a_0(T_1(t)\omega, \eta, \xi) &= M_x\{a_0(T_2(x)\omega, \eta, \xi + \partial w_x(T_2(x)\omega))\} \end{aligned}$$

where T_1 is defined in (2.9) and $w_x = w^0 \in \mathcal{V}$ (see (4.13)) satisfies

$$-\mathbf{div}\; a(\omega, \eta, \xi + \partial w_x) = 0. \tag{4.26}$$

The existence and uniqueness of this equation is discussed previously. Next we show that the homogenized operator has the form

$$L^*(\omega)u = D_t u - div(a(T_1(t)\omega, u, D_x u)) + a_0(T_1(t)\omega, u, D_x u). \tag{4.27}$$

THEOREM 4.9. *If $\alpha = 0$, $\beta = 1$ then L_ϵ G-converges to L^* defined by (4.27) for a.e. $\omega \in \Omega$.*

Proof.

Set $\mu = \epsilon^2$ and

$$w_{\epsilon,\delta} = \epsilon w_\delta^\mu(T(x/\epsilon, t)\omega),$$

where w_δ^μ is near solutions of (3.1). Then $w_{\epsilon,\delta}$ satisfies in R^{n+1} for a.e. ω

$$D_t w_{\epsilon,\delta} - div(a(T(x/\epsilon, t)\omega, \eta, \xi + D_x w_{\epsilon,\delta}) = div_x \rho_\delta,$$

where $\langle|\rho_\delta|^p\rangle \to 0$ as $\delta \to 0$. As in the proof of Theorem 4.4 we choose two sequences $\delta \to 0$ and $\epsilon(\delta) \to 0$ such that $w_{\epsilon(\delta),\delta} \to 0$ weakly in $\overline{W}$, and $\rho_\delta \to 0$ in $L^q(Q)^n$ as $k \to \infty$. This is possible for any sequence $\delta \to 0$ because of Lemma 4.3 and Lemma 3.2. Consider a generic sequence of $\delta_k \to 0$ as $k \to \infty$ and corresponding $\epsilon_k = \epsilon(\delta_k)$. Then $w_k = w_{\epsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$, and $\rho_k = \rho_{\delta_k} \to 0$ in $L^q(Q)$ as $k \to \infty$. Using the convergence of arbitrary solutions for G-convergent sequence of operators as in the proof of Theorem 4.4 we obtain that for a.e. ω

$$\begin{aligned} a(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_k) &\to \tilde{a}(\omega, t, x, \eta, \xi), \\ a_0(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_k) &\to \tilde{a}_0(\omega, t, x, \eta, \xi) \end{aligned}$$

weakly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. Set $w_{x,k} = w_x(T(t, x/\epsilon_k)\omega)$. Using Ergodic Theorem and the argument as in [10] (page 228)

$$a(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_{x,k}) = a(T_1(t)T_2(x/\epsilon_k)\omega, \eta, \xi + D_x w_{x,k}) \to M_x\{a(T_1(t)T_2(x)\omega, \eta, \xi + \partial w_x)\rangle\}$$
$$a_0(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_{x,k}) = a_0(T_1(t)T_2(x/\epsilon_k)\omega, \eta, \xi + D_x w_{x,k}) \to M_x\{a_0(T_1(t)T_2(x)\omega, \eta, \xi + \partial w_x)\rangle\}$$

weakly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$ because $a(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_{x,k})$ and $a_0(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_{x,k})$ are homogeneous fields.

Since $\|w^\mu - w_x\|_{\mathcal{V}} \to 0$ as $\mu \to 0$ as in the Theorem 4.6 we obtain $\|D_x w_k - D_x w_{x,k}\|_{p,Q} \to 0$ as $k \to \infty$. Consequently,

$$\begin{aligned} a(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_k) - a(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_{x,k}) \to 0 \\ a_0(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_k) - a_0(T(x/\epsilon_k, t)\omega, \eta, \xi + D_x w_{x,k}) \to 0 \end{aligned} \tag{4.28}$$

in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. Thus,

$$\begin{aligned} \tilde{a}(x, t, \eta, \xi) &= \langle a(\omega, \eta, \xi + \partial w_x)\rangle \\ \tilde{a}_0(x, t, \eta, \xi) &= \langle a_0(\omega, \eta, \xi + \partial w_x)\rangle. \end{aligned}$$

Q.E.D.

4.4. Time homogenization ($\beta = 0$, $\alpha = 1$). Following to [10] introduce the orthogonal projection operator $P_1 f$

$$P_1 f = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(T_1(\tau)\omega) d\tau. \tag{4.29}$$

Consider

$$\begin{aligned} a^*(\omega, \eta, \xi) &= P_1 a(\omega, \eta, \xi) \\ a_0^*(\omega, \eta, \xi) &= P_1 a_0(\omega, \eta, \xi). \end{aligned} \tag{4.30}$$

Next we will show that the homogenized operator is given by

$$L^*(\omega)u = D_t u - a^*(\omega, u, Du) - a_0^*(\omega, u, Du). \tag{4.31}$$

THEOREM 4.10. *If $\alpha = 0$, $\beta = 1$ then L_ϵ G-converges to L^* defined by (4.31).*
Proof.
Consider

$$F = P_1 a(\omega, \eta, \xi) - a(\omega, \eta, \xi), \quad f = \mathbf{div} F,$$

where **div** is defined by (2.7). Since $(f, \phi) = \langle F, \partial\phi\rangle = 0$ for any $\phi \in \mathcal{V}_s$ we have as in the proof of Lemma 4.7 that there exist $w_\zeta \in \mathcal{W}$, $g_\zeta \in \mathcal{V}'$ such that

$$f = -\sigma w_\zeta + g_\zeta,$$

$\|g_\zeta\|_{\mathcal{V}'} \le s(\zeta)$, where $s(\zeta) \to 0$ as $\zeta \to 0$.

We will employ the theorem on the convergence of arbitrary solution for w. Since $w \in \mathcal{W}$ we need near solutions. Set

$$w_{\epsilon,\delta,\zeta} = \epsilon w_{\delta,\zeta}(T(x, t/\epsilon)\omega),$$

where $w_{\delta,\zeta}$ is an approximation of w_ζ that has smooth realizations, and $\|w_{\delta,\zeta}-w_\zeta\|_{\mathcal{V}} \leq s(\delta)$, $s(\delta) \to 0$ as $\delta \to 0$. It can be easily shown that $w_{\epsilon,\delta,\zeta} \to 0$ as $\epsilon \to 0$ weakly in $\overline{W}$ and strongly in $\overline{V}$ for any $\delta > 0$ and $\zeta > 0$. This follows from the following,

$$\|w_{\delta,\zeta}(T(x,t/\epsilon)\omega)\|^p_{\overline{V}} = \int_0^t \int_{Q_0} |D_x w_{\delta,\zeta}(T(x,\tau/\epsilon)\omega)|^p dx d\tau = \epsilon \int_0^{t/\epsilon} \int_{Q_0} |D_x w_{\delta,\zeta}(T(x,\tau)\omega)|^p dx d\tau \to \|w_{\delta,\zeta}\|_{\mathcal{V}},$$

$$\|\epsilon D_t w_{\delta,\zeta}(T(x,t/\epsilon)\omega)\|^q_{\overline{V}'} = \int_0^t \|\epsilon D_\tau w_{\delta,\zeta}(T(x,\tau/\epsilon)\omega)\|^q_{W^{-1,q}(Q_0)} d\tau = \epsilon \int_0^{t/\epsilon} \|D_\tau w_{\delta,\zeta}(T(x,\tau)\omega)\|^q_{W^{-1,q}(Q_0)} d\tau \to \|w_{\delta,\zeta}\|_{\mathcal{V}'}.$$

Thus that $w_{\epsilon,\delta,\zeta}$ is compact in $\overline{W}$ and converges to zero in $\overline{V}$. Next we show that for any sequence $\zeta \to 0$ and $\delta \to 0$ there exists a sequence $\epsilon = \epsilon(\delta,\zeta) \to$ such that $w_{\epsilon,\delta,\zeta} \to 0$ weakly in $\overline{W}$ and strongly in $\overline{V}$ as well as $g_{\epsilon,\zeta} = g_\zeta(T(x,t/\epsilon)\omega) \to 0$ in $\overline{V}'$. Clearly this holds for $w_{\epsilon,\delta,\zeta}$ since it converges for any $\delta > 0$ and $\zeta > 0$. To show this for $g_{\epsilon,\zeta}$ we follow the Lemma 3.2.

$$\|g_{\epsilon,\zeta}\|_{\overline{V}'} = \int_0^t \|D_x g_\zeta(T(x,\tau/\epsilon)\omega)\|^q_{q,Q_0} d\tau = \epsilon \int_0^{t/\epsilon} \|D_x g_\zeta(T(x,\tau/\epsilon)\omega)\|^q_{q,Q_0} d\tau \to \|g_\zeta\|_{\mathcal{V}'} < s(\zeta)$$

as $\epsilon \to 0$. Here $s(\zeta) \to 0$ as $\zeta \to 0$. Consequently, for a given sequence of $\zeta \to 0$ there exists a sequence $\epsilon(\zeta) \to 0$ such that $g_{\epsilon,\zeta} \to 0$ in $\overline{V}'$. Next we choose a generic sequence $\delta_k \to 0$, $\zeta_k \to 0$ and $\epsilon_k(\delta_k,\zeta_k) \to 0$ such that $w_{\epsilon,\delta,\zeta} \to 0$ weakly in $\overline{W}$ and strongly in $\overline{V}$ as well as $g_{\epsilon,\zeta} = g_\zeta(T(x,t/\epsilon)\omega) \to 0$ in $\overline{V}'$. Then w_k satisfies in R^{n+1} for a.e. $\omega \in \Omega$

$$D_t w_k - div(a(T(x,t/\epsilon_k)\omega,\eta,\xi+D_x w_k)) + a_0(T(x,t/\epsilon_k)\omega,\eta,\xi+D_x w_k) = g_k - div(a^*(T_2(x)\omega,\eta,\xi)) + \phi_k + \psi_k,$$

where

$$\phi_k = -div(a(T(x,t/\epsilon_k)\omega,\eta,\xi+D_x w_k) - a(T(x,t/\epsilon_k)\omega,\eta,\xi)), \quad \psi_k = a_0(T(x,t/\epsilon_k)\omega,\eta,\xi+D_x w_k).$$

Because of Holder continuity of a, (2.5), we obtain that $\phi_k \to 0$ in $\overline{V}'$. Similarly $\psi_k \to \psi = a_0(T_2(x)\omega,\eta,\xi)$ weakly in $L^q(Q)$ for a.e. ω. Using the theorem on the convergence of arbitrary solutions we have that for a.e. ω L_k G-converges to $\tilde{L}$, $\tilde{L}u = D_t u - div(\tilde{a}(\omega,x,t,\eta,\xi)) + \tilde{a}_0(\omega,x,t,\eta,\xi)$. Here

$$a(T(x,t/\epsilon_k)\omega,\eta,\xi+D_x w_k) \to \tilde{a}(\omega,x,t,\eta,\xi)$$
$$a_0(T(x,t/\epsilon_k)\omega,\eta,\xi+D_x w_k) \to \tilde{a}_0(\omega,x,t,\eta,\xi)$$

weakly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$ respectively, and $u = 0$ is a solution of $\tilde{L}u = -div(a^*(T_2(x)\omega,\eta,\xi)) + a_0(T_2(x)\omega,\eta,\xi)$. On the other hand using Ergodic Theorem and the argument as in [10] (page 228) we obtain

$$a(T(x,t/\epsilon_k)\omega,\eta,\xi) \to a^*(T_2(x)\omega,\eta,\xi)$$
$$a_0(T(x,t/\epsilon_k)\omega,\eta,\xi) \to a_0^*(T_2(x)\omega,\eta,\xi)$$

weakly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. Here a^* and a_0^* are defined by (4.30).

Because of $w_k \to 0$ strongly in $\overline{V}$

$$a(T(x,t/\epsilon_k)\omega,\eta,\xi+D_x w_k) - a(T(x,t/\epsilon_k)\omega,\eta,\xi) \to 0$$
$$a_0(T(x,t/\epsilon_k)\omega,\eta,\xi+D_x w_k) - a_0(T(x,t/\epsilon_k)\omega,\eta,\xi) \to 0$$

strongly in $L^q(Q)^n$ and $L^q(Q)$ as $k \to \infty$. Thus, $\tilde{a} = a^*$ and $\tilde{a}_0 = a_0^*$.
Q.E.D.

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