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NUMERICAL HOMOGENIZATION OF NONLINEAR RANDOM PARABOLIC OPERATORS*

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Abstract. In this paper we study the numerical homogenization of nonlinear random parabolic equations. This procedure is developed within a finite element framework. A careful choice of multiscale finite element bases and the global formulation of the problem on the coarse grid allow us to prove the convergence of the numerical method to the homogenized solution of the equation. The relation of the proposed numerical homogenization procedure to multiscale finite element methods is discussed. Within our numerical procedure one is able to approximate the gradients of the solutions. To show this feature of our method we develop numerical correctors that contain two scales, the numerical and the physical. Finally, we would like to note that our numerical homogenization procedure can be used for the general type of heterogeneities.

Key words. homogenization, multiscale, upscaling, random, nonlinear, parabolic, finite element

AMS subject classification. 65N99

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1. Introduction.

In this paper we consider numerical homogenization techniques for nonlinear parabolic equations:

$$D_t u_\epsilon - \operatorname{div}(a_\epsilon(x,t,u_\epsilon,D_x u_\epsilon)) + a_{0,\epsilon}(x,t,u_\epsilon,D_x u_\epsilon) = f, \tag{1.1}$$

where ϵ is a small scale. Our motivation in considering (1.1) mostly stems from the applications of flow in porous media (multiphase flow in saturated porous media, flow in unsaturated porous media), though many applications of nonlinear parabolic equations of these kinds occur in transport problems. Many problems in subsurface modeling have a multiscale nature where the heterogeneities associated with the media are no longer periodic. Furthermore, the level of detail and uncertainty incorporated into geologic characterization of subsurfaces typically exceeds the capabilities of traditional flow simulators by a wide margin. For this reason, some type of upscaling, or numerical homogenization, of the detailed geologic model must be performed before the model can be used for flow simulation. The numerical homogenization is, in general, nontrivial because heterogeneities at all scales have a significant effect, and these must be captured in the coarsened subsurface description.

Our main goal in the paper is to propose and analyze a numerical homogenization procedure that is applicable to heterogeneities of general nature. The analysis of the numerical method employs previous results on G-convergence [19] as well as homogenization [7] of nonlinear parabolic equations. It was shown that a solution u_ϵ converges to u (up to a subsequence) in an appropriate sense, where u is a solution of

$$D_t u - \operatorname{div}(a^*(x,t,u,D_x u)) + a_0^*(x,t,u,D_x u) = f. \tag{1.2}$$

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In [7] the homogenized fluxes a^* and a_0^* are computed under the assumption that the heterogeneities are strictly stationary random fields with respect to both space and time. The numerical homogenization procedure for (1.1) should account for the functional dependence of the macroscopic quantities on the solution and its gradients.

The numerical homogenization procedure proposed in the paper uses general finite element procedure and solves local problems that are further coupled in the global formulation. The local problems are formulated in the domains (spatial and time) with carefully selected boundary and initial conditions. The size of the local domains is much larger than that of heterogeneities. Moreover, with a careful choice of local problems we guarantee the uniqueness of the solutions of these local problems. Because of a careful choice of local problems, as well as the formulation of the discrete problem, we obtain the convergence of the solutions under some assumptions. The formulation of the local problems can be simplified, depending on the relation between temporal and spatial scales. This is discussed in the end of section 3. Our numerical procedure, as we show in the paper, shares some common elements with recently developed multiscale finite element methods [11], where the local information is incorporated into base functions that are further coupled in the global formulation.

The numerical homogenization procedure yields a coarse scale solution that converges to a solution of the homogenized equation (1.2). To capture the oscillations of the solution the corrector results are needed. To our best knowledge the correctors for nonlinear random parabolic equation are not known. In the second part of the paper we construct the correctors which are used to show the convergence of gradients of the solutions for our numerical procedure. The constructed correctors use two scales, the physical scale and the numerical scale, the latter being much larger than the former. The convergence for the corrector is obtained. These results show us a way to obtain numerically the fine scale features of the solution using the solutions of the local problems computed previously. We would like to note that the computation of the oscillation of solutions is important for the application to flow in porous media and other transport problems.

In the paper we consider some numerical examples. One of the examples is related to a heterogeneous convection diffusion equation. Assuming that velocity is a zero-mean divergence-free field that has a homogeneous stream function we obtain the homogenized equation that contains “extra diffusion” (known as enhanced diffusion). The latter is due to the effects of the convection at smaller scales. We would like to note that this problem for linear convection has been of great interest [10, 15]. The application of the numerical homogenization procedure to Richards equation is also considered.

The paper is organized as follows. In the next section we present some basic facts that are used later in the analysis. Section 3 is devoted to the numerical homogenization method and its analysis. In the following section the corrector results are derived. Finally, in section 5 we present numerical results. Conclusions are drawn in the last section.

2. Preliminaries. Let (Ω, Σ, μ) be a probability space and $L^p(\Omega)$ denote the space of all p-integrable functions. Consider a $(d+1)$-dimensional dynamical system on Ω, $T(z) : \Omega \to \Omega$, $z = (x,t) \in R^{d+1}$ ($t \in R$, $x \in R^d$), that satisfies the following conditions: (1) $T(0) = I$, and $T(x+y) = T(x)T(y)$; (2) $T(z) : \Omega \to \Omega$ preserve the measure μ on Ω; (3) for any measurable function $f(\omega)$ on Ω, the function $f(T(z)\omega)$ defined on $R^{d+1} \times \Omega$ is also measurable.

$U(z)f(\omega) = f(T(z)\omega)$ defines a $(d+1)$-parameter group of isometries in the space of $L^p(\Omega)$. $U(z)$ is strongly continuous. Further, we assume that the dynamical system T is ergodic; i.e., any measurable T-invariant function on Ω is constant. Denote by $\langle\cdot\rangle$ the mean value over Ω,

$$\langle f\rangle = \int_\Omega f(\omega)d\mu(\omega), \quad \langle u,v\rangle = \int (u,v)d\mu(\omega).$$

For further analysis we will need the Birkhoff ergodic theorem. Denote

$$M\{f\} = \lim_{s\to\infty} \frac{1}{s^{d+1}|K|}\int_{K_s} f(z)dz,$$

where $K \subset R^{d+1}$, $|K| \neq 0$, and $K_s = \{z \in R^{d+1} : s^{-1}z \in K\}$. Let $f(\frac{z}{\epsilon})$ be bounded in $L^p_{loc}(R^{d+1})$, $1 \le p < \infty$. Then f has mean value $M\{f\}$ if and only if $f(z/\epsilon) \to M\{f\}$ weakly in $L^p_{loc}(R^{d+1})$ as $\epsilon \to 0$ [19, p. 134]. The Birkhoff ergodic theorem states that if $f \in L^p(\Omega)$, $1 \le p < \infty$, then

$$\langle f\rangle = M\{f(T(z)\omega)\} \text{ a.e. on } \Omega.$$

Define $Q = Q_0 \times [0,T]$, where $Q_0 \subset R^d$ is a domain with Lipschitz boundaries, and introduce

$$\begin{aligned} &V_0 = L^p(0,T,W_0^{1,p}(Q_0)), \quad \overline{V} = L^p(0,T,W^{1,p}(Q_0)),\\ &W = \{u \in V_0, D_t u \in L^q(0,T,W^{-1,q}(Q_0))\},\\ &\overline{W} = \{u \in \overline{V}, D_t u \in L^q(0,T,W^{-1,q}(Q_0))\}, \quad W_0 = \{u \in W, u(0) = 0\}. \end{aligned} \tag{2.1}$$

For further analysis X' will denote the dual of the space X. Let $u_\epsilon \in W_0$ be a solution of

(2.2)
$$D_t u_\epsilon = \operatorname{div} a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon) - a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon) + f \text{ in } Q,$$

and denote $L_\epsilon u = D_t u - \operatorname{div} a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u, D_x u) + a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u, D_x u)$.

We assume that $a(\omega,\eta,\xi)$ and $a_0(\omega,\eta,\xi)$, $\eta \in R$, and $\xi \in R^d$ are Caratheodory functions satisfying the following inequalities:

- for any (η,ξ)
$$|a(\omega,\eta,\xi)| + |a_0(\omega,\eta,\xi)| \le C(1 + |\eta|^{p-1} + |\xi|^{p-1}) \text{ a.e. on } \Omega; \tag{2.3}$$
- for any (η,ξ_1) and (η,ξ_2)
$$(a(\omega,\eta,\xi_1) - a(\omega,\eta,\xi_2), \xi_1 - \xi_2) \ge C|\xi_1 - \xi_2|^p \text{ a.e. on } \Omega; \tag{2.4}$$
- for any (η,ξ)
$$(a(\omega,\eta,\xi),\xi) + a_0(\omega,\eta,\xi)\eta \ge C|\xi|^p - C_1 \text{ a.e. on } \Omega; \tag{2.5}$$
- for any $\chi_1 = (\eta_1,\xi_1)$ and $\chi_2 = (\eta_2,\xi_2)$
$$\begin{aligned} &|a(\omega,\eta_1,\xi_1) - a(\omega,\eta_2,\xi_2)| + |a_0(\omega,\eta_1,\xi_1) - a_0(\omega,\eta_2,\xi_2)|\\ &\le C(1 + |\chi_1|^{p-1} + |\chi_2|^{p-1})\nu(|\xi_1 - \xi_2|)\\ &\qquad + C(1 + |\chi_1|^{p-1-s} + |\chi_2|^{p-1-s})|\xi_1 - \xi_2|^s \text{ a.e. on } \Omega, \end{aligned} \tag{2.6}$$

  where $0 < s < \min(p-1,1)$, and $\nu(r)$ is a continuity modulus (i.e., a nondecreasing continuous function on $[0,+\infty)$ such that $\nu(0) = 0$, $\nu(r) > 0$ if $r > 0$, and $\nu(r) = 1$ if $r > 1$);

- $$p \geq 2.$$

For further analysis we define q by $\frac{1}{p} + \frac{1}{q} = 1$. Note that various other coercivity conditions can be also imposed instead of (2.5).

Next, we briefly mention the definition for G-convergence for the sequence of nonlinear parabolic operators following [19, p. 176]. Consider a sequence of general parabolic operators L_ϵ, $L_\epsilon u = D_t u - \mathrm{div}(a_\epsilon(x,t,u,D_x u)) + a_{0,\epsilon}(x,t,u,D_x u)$ and $Lu = D_t u - \mathrm{div}(a^*(x,t,u,D_x u)) + a_0^*(x,t,u,D_x u)$. We assume that L_ϵ and L satisfy (2.3)–(2.6) with $a(\omega,\eta,\xi)$, $a_0(\omega,\eta,\xi)$ replaced by $a_\epsilon(x,t,\eta,\xi)$, $a_{0,\epsilon}(x,t,\eta,\xi)$ as well as $a^*(x,t,\eta,\xi)$, $a_0^*(x,t,\eta,\xi)$. Based on L_ϵ and L we define the sequence of operators $L_\epsilon^1(u,v) = D_t u - \mathrm{div}(a_\epsilon(x,t,v,D_x u))$, $L^1(u,v) = D_t u - \mathrm{div}(a^*(x,t,v,D_x u))$ and the fluxes

$$\Gamma^\epsilon(u,v) = a_\epsilon(x,t,v,D_x u), \quad \Gamma_0^\epsilon(u,v) = a_{0,\epsilon}(t,x,v,D_x u),$$
$$\Gamma(u,v) = a^*(x,t,v,D_x u), \quad \Gamma_0(u,v) = a_0^*(t,x,v,D_x u).$$

Given $v \in V_0$, $L_\epsilon^1(u,v)$ and $L^1(u,v)$ are strictly monotone parabolic operators [19, p. 176]. Therefore, for any $v \in V_0$ and $f \in W^{'}$ there exist unique solutions $u_\epsilon \in W_0$ and $u \in W_0$ of $L_\epsilon^1(u_\epsilon, v) = f$ and $L^1(u,v) = f$ [20].

Definition (G-convergence). *A sequence L_ϵ G-converges to L if for any $v \in V_0$ and $f \in L^q(0,T,W^{-1,q}(Q_0))$ we have*

$$u_\epsilon \rightarrow u$$

weakly in W_0 as $\epsilon \rightarrow 0$ and

$$\Gamma^\epsilon(u_\epsilon, v) \rightarrow \Gamma(u,v),$$
$$\Gamma_0^\epsilon(u_\epsilon, v) \rightarrow \Gamma_0(u,v)$$

weakly in $L^q(Q)^d$ and $L^q(Q)$, respectively, as $\epsilon \rightarrow 0$.

Remark 2.1. *We would like to note that in* [19] *(where to our best knowledge G-convergence for this class of operators is first introduced) the author calls the G-convergent sequence defined as above the "strongly G-convergent sequence." The theorem on the convergence of arbitrary solutions for the G-convergent sequence of operators* [19] *that will be used in our analysis follows.*

Theorem 2.1. *Assume L_ϵ G-converges to L, $u_\epsilon \in \overline{W}$, $f_\epsilon, f \in L^q(0,T,W^{-1,q}(Q_0))$, $L_\epsilon u_\epsilon = f_\epsilon$, $u_\epsilon \rightarrow u$ weakly in $\overline{W}$, and $f_\epsilon \rightarrow f$ strongly in $W_0^{'}$. Then $Lu = f$, and*

$$a_\epsilon(x,t,u_\epsilon,D_x u_\epsilon) \rightarrow a^*(x,t,u,D_x u),$$
$$a_{0,\epsilon}(x,t,u_\epsilon,D_x u_\epsilon) \rightarrow a_0^*(x,t,u,D_x u)$$

weakly in $L^q(Q)^d$ and $L^q(Q)$, respectively, as $\epsilon \rightarrow 0$.

To formulate the auxiliary problem for the homogenization we need the following preliminaries. Following to [23] we define spaces similar to $\overline{V}$ on Ω in the following way. Denote by $\partial_{full} = (\partial_1, \ldots, \partial_{d+1})$ the collection of generators of the group $U(z)$. There is a dense subspace $S \subset L^p(\Omega)$ that contains in the domains of all operators $\partial_{full}^\alpha = \partial_1^{\alpha_1} \cdots \partial_{d+1}^{\alpha_{d+1}}$, $\alpha \in Z_+^{d+1}$.

Further, denote by $\mathcal{V}$ the completion of S with respect to the seminorm

$$\|f\| = \left(\sum_{i=1}^{d} \|\partial_i f\|_{L^p(\Omega)}^p \right)^{1/p}.$$

The operator $\partial : \mathcal{V} \to L^p(\Omega)^n$ is an isometric embedding, $\partial = (\partial_1, \dots, \partial_d)$. In particular, the space $\mathcal{V}$ is reflexive with its dual denoted by $\mathcal{V}'$. By duality the operators $\mathbf{div} : L^q(\Omega)^n \to \mathcal{V}'$, where $\langle \mathbf{div}\ u, v\rangle = -\langle u, \partial v\rangle$. We note that $\mathcal{V}$, in general, contains fields that are not spatially homogeneous. Note that in [23, 19] the operators ∂_i may be viewed as derivatives along trajectories of the dynamical system $T(z)$:

$$(\partial_i f)(T(z)\omega) = \frac{\partial}{\partial z_i} f(T(z)\omega) \tag{2.7}$$

for a.e. $\omega \in \Omega$ and $f \in D(\partial_i, L^p(\Omega))$. For further analysis we introduce

$$T_1(t) = T(0, \dots, 0, t), \quad T_2(x) = T(x_1, \dots, x_d, 0). \tag{2.8}$$

We denote

(2.9)
$$M_t\{f\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(T(0,\tau)\omega)d\tau, \quad M_x\{f\} = \lim_{|K|\to\infty} \frac{1}{|K|} \int_K f(T(y,0)\omega)dy.$$

We note that the average of a,

$$\overline{a}(\omega, \eta, \xi) = M_t\{a(\omega, \eta, \xi)\}, \tag{2.10}$$

is defined on $L^p(\Omega)$ for each $\eta \in R$ and $\xi \in R^d$. Consider the subset of S consisting of functions

$$f(\omega) = M_t\{f\}.$$

Denote by $\mathcal{V}_s$ the completion of this set with respect to the norm

$$\|f\| = \left(\sum_{i=1}^d \|\partial_i f\|^p_{L^p(\Omega)} \right)^{1/p}.$$

To formulate an auxiliary problem we introduce the differentiation with respect to time ∂_{d+1}. Define an unbounded operator σ from $\mathcal{V}$ into $\mathcal{V}'$ as follows. We say that $v \in \mathcal{V}$ belongs to $D(\sigma)$ if there exists $f \in \mathcal{V}'$ such that

$$\langle v, \partial_{d+1}\phi\rangle = -\langle f, \phi\rangle \quad \forall \phi \in S, \tag{2.11}$$

and we set $\sigma v = f$. It is easily seen that $\sigma\phi = \partial_{d+1}\phi$, $\phi \in S$. Therefore, σ is a closed linear operator from $\mathcal{V}$ to $\mathcal{V}'$. Let σ^+ be the adjoint operator (acting from $\mathcal{V}$ to $\mathcal{V}'$). Then

$$\sigma^+ = -\sigma;$$

i.e., σ is a skew-symmetric operator. In analogy with (2.1) denote $\mathcal{W} = D(\sigma)$. Clearly, $\mathcal{W} = \mathcal{D}(\sigma)$ is dense in $\mathcal{V}$, and σ is a maximal monotone operator [7].

Consider the auxiliary problem

$$\mu\sigma N^\mu_{\eta,\xi} - \mathbf{div}\ a(\omega, \eta, \xi + \partial N^\mu_{\eta,\xi}) = 0. \tag{2.12}$$

Define the operator A from $\mathcal{V}$ to $\mathcal{V}'$ as

$$\langle Au, v\rangle = \langle a(\omega, \eta, \xi + \partial u), \partial v\rangle.$$

It can be easily verified that A is a strongly monotone, continuous, and coercive operator from $\mathcal{V}$ to $\mathcal{V}'$. Since σ is maximal monotone it follows from [14] that the solution of (2.12) in $\mathcal{W}$ exists. Uniqueness follows from the fact that $(\sigma u, u) = 0$ and A is strongly monotone. Thus we have the following lemma [7].

LEMMA 2.2. *Equation* (2.12) *has a unique solution,* $N^\mu_{\eta,\xi} \in \mathcal{W}$, *and*

$$\|N^\mu_{\eta,\xi}\|_{\mathcal{V}} \le C. \tag{2.13}$$

The homogenization of nonlinear parabolic equations depends on the ratio between α and β and is presented in [7]. The following cases are distinguished: (1) self-similar case ($\alpha = 2\beta$); (2) non–self-similar case ($\alpha < 2\beta$); (3) non–self-similar case ($\alpha > 2\beta$); (4) spatial case ($\alpha = 0$); (5) temporal case ($\beta = 0$).

THEOREM 2.3. L_ϵ *G-converges to* L^*, *where* L^* *is given by*

$$L^* u = D_t u - \operatorname{div}(a^*(\omega, x, t, u, D_x u)) + a_0^*(\omega, x, t, u, D_x u). \tag{2.14}$$

a^* *and* a_0^* *are defined as follows:*

- *For self-similar case* ($\alpha = 2\beta$),

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$
$$a_0{}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$

where $N_{\eta,\xi} = N^{\mu=1} \in \mathcal{W}$ *is the unique solution of*

$$\sigma N^{\mu=1} - \mathbf{div}\ a(\omega, \eta, \xi + \partial N^{\mu=1}) = 0. \tag{2.15}$$

- *For non–self-similar case* ($\alpha < 2\beta$),

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$
$$a_0{}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$

where $N_{\eta,\xi} = N^0 \in \mathcal{V}$ *is the unique solution of*

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial N^0) = 0. \tag{2.16}$$

- *For non–self-similar case* ($\alpha > 2\beta$),

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$
$$a_0{}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$

where $N_{\eta,\xi} = N^\infty \in \mathcal{V}_s$ *is the unique solution of*

$$-\mathbf{div}\ \overline{a}(\omega, \eta, \xi + \partial N^\infty) = 0. \tag{2.17}$$

$\overline{a}$ *is defined in* (2.10).

- *For spatial case* ($\alpha = 0$),

$$a(T_1(t)\omega, \eta, \xi) = M_x\{a(T_2(x)\omega, \eta, \xi + \partial N_{\eta,\xi}(T_2(x)\omega))\},$$
$$a_0(T_1(t)\omega, \eta, \xi) = M_x\{a_0(T_2(x)\omega, \eta, \xi + \partial N_{\eta,\xi}(T_2(x)\omega))\},$$

where $N_{\eta,\xi} = N_x \in \mathcal{V}$ *is the unique solution of*

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial N_x) = 0. \tag{2.18}$$

- *For temporal case* ($\beta = 0$)*, the homogenized fluxes are defined by*

$$a^*(\omega, \eta, \xi) = M_t\{a(\omega, \eta, \xi)\},\\ a_0^*(\omega, \eta, \xi) = M_t\{a_0(\omega, \eta, \xi)\}, \tag{2.19}$$

where M_t *is defined in* (2.9).

For the temporal case one can also define $N_{\eta,\xi}$ in the following way (see the proof of Theorem 4.8 in [7]). Define $F = a(\omega, \eta, \xi) - M_t a(\omega, \eta, \xi)$, and $f = \mathbf{div}\ F$. Then it can be shown that there exists N such that

$$f = -\sigma N + g, \tag{2.20}$$

where $\|g\|_{\mathcal{V}'} \leq \delta$, for arbitrary small δ. The latter follows from the fact that the range of σ is dense in the orthogonal complement of the kernel of σ, and f belongs to the kernel of σ. The proof of this theorem extensively uses near solutions of (2.12) since $N^{\mu}_{\eta,\xi}$ is no longer a homogeneous random field. Near solutions will be needed later on in this paper, though we will not discuss them here.

The theorem on the convergence of arbitrary solutions (Theorem 2.1) for the G-convergent sequence of operators allows us not to restrict ourselves to a particular boundary or initial conditions. In particular, from Theorems 2.1 and 2.3 we have the following.

THEOREM 2.4. *Let* $u_\epsilon \in \overline{W}$ *be a solution of* $L_\epsilon u_\epsilon = f$, $f \in L^q(0, T, W^{-1,q}(Q_0))$, *such that* $\|u_\epsilon\|_{\overline{W}}$ *is bounded. Then* u_ϵ *converges to* u *as* $\epsilon \to 0$ *weakly in* $\overline{W}$ *(up to a subsequence), where* u *is a solution of* $L^* u = f$, *and* L^* *is defined in* (2.14).

REMARK 2.2. *We note that the ergodicity assumption is not essential for the proof of the theorem. One can carry out the proof for the nonergodic case essentially in the same manner as that for the ergodic case. The homogenized operators for the nonergodic case will be invariant functions with respect to* $T(z)$.

For the sake of the simplicity of our further analysis we will assume that the homogenized operator does not depend on x or t. This corresponds to self-similar and non–self-similar cases. For the spatial homogenization case the homogenized operator does not depend on x or t if the fluxes are independent of time. Similarly, for the time homogenization case fluxes should be independent of space. The analysis of the numerical homogenization procedure can be carried out for general heterogeneities using the techniques of G-convergence theory.

3. Numerical computation of the homogenized solution.

3.1. Numerical homogenization method. Consider $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T$ and $\max(t_i - t_{i-1}) = h_t$. Denote $Q_i^{x,t} = [t_i, t_{i+1}] \times Q_0$, $\overline{V_i} = L^p(t_i, t_{i+1}, W^{1,p}(Q_0))$, and $\overline{W_i} = \{u \in \overline{V_i},\ D_t u \in L^q(t_i, t_{i+1}, W^{-1,q}(Q_0))\}$. Throughout the paper $\|\cdot\|_{p,Q}$ denotes the $L^p(Q)$-norm.

The computation of the homogenized solution will be performed for a.e. ω. For this reason we omit everywhere the notation "a.e. ω." To solve the homogenized equation, $u \in W_0$,

$$D_t u - \operatorname{div}(a^*(u, D_x u)) + a_0^*(u, D_x u) = f(x), \tag{3.1}$$

we employ the standard finite element method. Introduce a finite dimensional space over the standard triangular or tetrahedral partitions K of $Q_0 = \bigcup K$,

(3.2)
$$S^h = \{v_h \in C^0(\overline{Q_0}) : \text{the restriction } v_h \text{ is linear for each element } K \text{ and } v_h = 0 \text{ on } \partial Q_0\},$$

$diam(K) \le Ch_x$. Consider $u_h(t) \in S^h$ such that

$$\int_{Q_0} D_t u_h v_h dx + A^*(u_h, v_h) = \int_{Q_0} f v_h dx \quad \forall v \in S^h,$$

where

$$A^*(u, v) = \int_{Q_0} ((a^*(u, D_x u), D_x v) + a_0^*(u, D_x u) v) dx.$$

The main idea of the numerical homogenization technique is to approximate $A^*(u_h, v_h)$ using the solution of the local problems. Denote by $\phi_i^0(x)$ a basis in S^h consisting of piecewise linear functions such that $\phi_i^0(x_j) = \delta_{ij}$ (δ_{ij} is the Kronecker symbol), and x_j are the nodal points of the finite element partition. Consider $u_h = \sum_{i=1}^N \theta_i(t)\phi_i^0(x)$, where $\theta_i^{n+1} = \theta_i(t = t_{n+1})$ are defined by

(3.3)

$$\sum_i (\theta_i^{n+1} - \theta_i^n) \int_{Q_0} \phi_i^0(x)\phi_j^0(x) dx + \int_{t_n}^{t_{n+1}} \int_{Q_0} ((a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon), D_x \phi_j^0) + a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon)\phi_j^0) dx dt = \int_{t_n}^{t_{n+1}} \int_{Q_0} f \phi_j^0 dx dt.$$

Here $l^\theta = \sum_i \theta_i^{n+1}\phi_i^0(x)$ (see also the remark that follows), and v_ϵ is the solution of the local problem and computed as

$$D_t v_\epsilon = \operatorname{div} a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon) \text{ in } K \times [t_n, t_{n+1}], \tag{3.4}$$

where $v_\epsilon = l^\theta$ on ∂K, $v_\epsilon(x, t = t_n) = l^\theta$, and

$$\eta^{l^\theta} = \frac{1}{|K|} \int_K l^\theta dx.$$

For further analysis θ and ζ denote discrete vectors defined at the nodal points, and $l^\theta(x) \in S^h$ and $l^\zeta(x) \in S^h$ are the functions that linearly interpolate these vectors into Q_0, e.g., $l^\theta = \sum_{i=1}^N \theta_i \phi_i^0(x)$.

Remark 3.1. *Note that numerical homogenization procedure* (3.3) *can be performed both in explicit and implicit manners. For the explicit implementation* $l^\theta = \sum \theta_i^n \phi_i^0(x)$ *and for the implicit one* $l^\theta = \sum \theta_i^{n+1}\phi_i^0(x)$.

Equation (3.3) defines our numerical homogenization procedure. Note that this method couples the local information that is obtained by solving (3.4) in the global formulation of the problem via (3.3). The choice of the local problems, (3.4), as well as the global formulation (3.3) are carefully selected for the robustness of the numerical method.

Introduce the discrete operator $A^{h,\epsilon}$ as follows:

$$(A^{h,\epsilon}\theta, \zeta) = \int_{t_n}^{t_{n+1}} \int_{Q_0} ((a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon), D_x l^\zeta) + a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon) l^\zeta) dx dt, \tag{3.5}$$

where $l^\zeta = \sum_i \zeta_i \phi_i^0(x)$, and v_ϵ is the solution of the local problem (3.4). The numerical homogenization procedure introduced above has the following discrete representation:

$$M(\theta^{n+1} - \theta^n) + A^{h,\epsilon}(\theta^{n+1}) = b, \tag{3.6}$$

where $M_{ij} = \int_{Q_0} \phi_i^0(x)\phi_j^0(x)dx$ is a mass matrix, $A^{h,\epsilon}$ is defined by (3.5), $b_i = \int_{t_n}^{t_{n+1}} \int_{Q_0} f\phi_i^0 dxdt$. Equation (3.6) is solved using Newton's method or its variations. For the explicit formulation of the numerical homogenization procedure $A^{h,\epsilon}(\theta^{n+1})$ is replaced by $A^{h,\epsilon}(\theta^n)$ in (3.6).

REMARK 3.2. *Note that the solution of* (3.4) *exists, is unique, and guarantees the operator* $A^{h,\epsilon}$ *is single valued.*

Our goal is to show the following.

THEOREM 3.1. $u_h = \sum_i \theta_i(t)\phi_i^0(x)$ *converges to* u*, a solution of the homogenized equation* (3.1) *in* $V_0 = L^p(0,T,W_0^{1,p}(Q_0))$ *as* $\lim_{h\to 0}\lim_{\epsilon\to 0}$ *under additional not restrictive assumptions that will be discussed later.*

REMARK 3.3. *The proof of the theorem uses the convergence of the solutions and the fluxes, and, consequently, it is applicable for the case of general heterogeneities that uses G-convergence theory. Since the G-convergence of the operators occurs up to a subsequence the numerical solution converges to a solution of a homogenized equation (up to a subsequence of* ϵ*).*

REMARK 3.4. *Note that one can compute the effective fluxes* $a^*(x,t,\eta,\xi)$ *and* $a_0^*(x,t,\eta,\xi)$ *for each* η *and* ξ *and coarse block using the solutions of the local problems similar to* (3.4)*. This procedure may not be efficient because one does not always know a priori the range of* η *and* ξ*. In this respect, the numerical homogenization procedure solves the local problems selectively.*

3.2. The numerical homogenization method and multiscale finite element methods. The numerical homogenization procedure presented in the previous section can be formulated within the framework of *multiscale finite element methods* (MsFEM) [11]. To do this we will first formulate MsFEM in a slightly different manner from that presented in [11] for *the linear problem.* Consider a standard finite dimensional S^h space over a coarse triangulation of Q_0, (3.2), and define $E^{MsFEM} : S^h \to V_\epsilon^h$ in the following way. For each $u_h \in S^h$ there is a corresponding element $u_{h,\epsilon}$ in V_ϵ^h that is defined by

$$D_t u_{h,\epsilon} - \text{div}(a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega) D_x u_{h,\epsilon}) = 0 \text{ in } K \times [t_n, t_{n+1}], \tag{3.7}$$

with boundary condition $u_{h,\epsilon} = u_h$ on ∂K, and $u_{h,\epsilon}(t = t_n) = u_h$. For the linear equations E^{MsFEM} is a linear operator, and the obtained multiscale space, V_ϵ^h, is a linear space on $Q_0 \times [t_n, t_{n+1}]$. Moreover, the basis in the space V_ϵ^h can be obtained by mapping the basis functions of S^h. For *the nonlinear parabolic equations* considered in this paper the operator E^{MsFEM} is constructed similar to (3.7) using the local problems; i.e., for each $u_h \in S^h$ there is a corresponding element $u_{h,\epsilon}$ in V_ϵ^h that is defined by

$$D_t u_{h,\epsilon} - \text{div}(a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta, D_x u_{h,\epsilon})) = 0 \text{ in } K \times [t_n, t_{n+1}], \tag{3.8}$$

with boundary condition $u_{h,\epsilon} = u_h$ on ∂K, and $u_{h,\epsilon}(t = t_n) = u_h$. Here $\eta = \frac{1}{|K|}\int_K u_h dx$. Note that E^{MsFEM} is a nonlinear operator and V_ϵ^h is no longer a linear space.

The following method that can be derived from general multiscale finite element framework is equivalent to our numerical homogenization procedure. Find $u_h(t) \in S^h$ such that

$$\int_{t_n}^{t_{n+1}} \int_{Q_0} D_t u_h v_h dxdt + A(u_h, v_h) = \int_{t_n}^{t_{n+1}} \int_{Q_0} f v_h dxdt \quad \forall v_h \in S^h,$$

where

$$A(u_h, w_h) = \sum_K \int_{t_n}^{t_{n+1}} \int_K ((a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{u_h}, D_x u_{h,\epsilon}), D_x w_h) \\ + a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{u_h}, D_x u_{h,\epsilon}) w_h) dx dt,$$

where $u_{h,\epsilon}$ is the solution of the local problem (3.8), $\eta^{u_h} = \frac{1}{|K|}\int_K u_h dx$, and u_h is known at $t = t_n$.

We would like to note that the operator E^{MsFEM} can be constructed using larger domains, as it is done in MsFEM with oversampling [11]. This way one reduces the effects of the boundary conditions and initial conditions. In particular, for the temporal oversampling it is only sufficient to start the computations before t_n and end them at t_{n+1}. Consequently, the oversampling domain for $K \times [t_n, t_{n+1}]$ consists of $[\tilde{t}_n, t_{n+1}] \times S$, where $\tilde{t_n} < t_n$ and $K \subset S$. More precise formulation and detailed numerical studies of the oversampling technique for nonlinear equations are currently under investigation. Further, we would like to note that oscillatory initial conditions can be imposed (without using oversampling techniques) based on the solution of the elliptic part of the local problems (3.8). These initial conditions at $t = t_n$ are the solutions of

$$-\text{div}(a(T(x/\epsilon^\beta, t_n/\epsilon^\alpha)\omega, \eta, D_x u_{h,\epsilon})) = 0 \text{ in } K, \tag{3.9}$$

or

$$-\text{div}(\overline{a}(T(x/\epsilon^\beta)\omega, \eta, D_x u_{h,\epsilon})) = 0 \text{ in } K, \tag{3.10}$$

where $\overline{a}(T(x/\epsilon^\beta)\omega, \eta, \xi) = \frac{1}{t_{n+1}-t_n}\int_{t_n}^{t_{n+1}} a(T(x/\epsilon^\beta, \tau/\epsilon^\alpha)\omega, \eta, \xi) d\tau$ and $u_{h,\epsilon} = u_h$ on ∂K. The latter can become efficient, depending on the interplay between the temporal and spatial scales. This issue is discussed below.

Note that in the case of periodic media the local problems can be solved in a single period in order to construct $A(u_h, v_h)$. This technique, which localizes the computation, is similar to the recently proposed method [6]. In general, one can solve the local problems in a domain different from K (an element) to calculate $A(u_h, v_h)$, and our analysis is applicable to these cases. Note that the numerical advantages of our approach over the fine scale simulation is similar to that of MsFEM. In particular, for each Newton's iteration a linear system of equations on *a coarse grid* is solved.

3.2.1. Special cases. For some special cases the operator E^{MsFEM} introduced in the previous section can be simplified.

1. *Linear separable case.* Let $u_\epsilon \in W_0$ be a solution of

$$D_t u_\epsilon = \text{div}(a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon) D_x u_\epsilon) + f \quad \text{in } Q,$$

where a has the form $a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta) = a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega) k(\eta)$. In this case V_ϵ^h is the same as that for the linear case.

2. *Various time and spatial scale heterogeneities.* Consider

$$D_t u_\epsilon = \text{div}(a(T(x/\epsilon^\beta)\omega, t, u_\epsilon, D_x u_\epsilon)) + f \quad \text{in } Q,$$

and assume a to be sufficiently smooth with respect to t. In this case the homogenized operator can be constructed using the parameter dependent elliptic equation

$$-\text{div}(a(T(x/\epsilon^\beta)\omega, t, u_\epsilon, D_x u_\epsilon)) = f \quad \text{in } Q_0.$$

The local problems for this case can be constructed by solving, instead of (3.4), $-\text{div}(\overline{a}(T(x/\epsilon^\beta)\omega, \eta^{l^\theta}, D_x v_\epsilon)) = f$, where

$$\overline{a}(T(x/\epsilon^\beta)\omega, \eta, \xi) = \frac{1}{t_{n+1} - t_n} \int_{t_n}^{t_{n+1}} a(T(x/\epsilon^\beta)\omega, t, \eta, \xi) dt.$$

This way we can avoid solving local time-dependent problems.

In general, one can avoid solving the local parabolic problems if the ratio between α and β is known and solve instead a simplified equation. For example, if $\alpha < 2\beta$ one can solve instead of (3.4) the local problem $-\text{div}(a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega), \eta^{l^\theta}, D_x v_\epsilon)) = 0$; if $\alpha > 2\beta$ one can solve instead of (3.4) the local problem $-\text{div}(\overline{a}(T(x/\epsilon^\beta)\omega), \eta^{l^\theta}, D_x v_\epsilon))$ $= 0$, where $\overline{a}$ is an average over time (see (2.10)), while if $\alpha = 2\beta$ we need to solve the parabolic equation in $K \times [t_n, t_{n+1}]$, (3.4).

We would like to note that, in general, one can use (3.9) or (3.10) as oscillatory initial conditions, and these initial conditions can be efficient for some cases. For example, for $\alpha > 2\beta$ with initial conditions given by (3.10) the solutions of the local problems (3.8) can be computed easily since they are approximated by (3.10). Moreover, one can expect better accuracy with (3.10) for the case $\alpha > 2\beta$ because this initial condition is more compatible with the local heterogeneities compared to the artificial linear initial conditions (cf. (3.8)). The comparison of various oscillatory initial conditions, including the ones obtained by the oversampling method, is a subject of future studies.

3.3. Proof of Theorem 3.1. The proof of the theorem will be carried out in the following manner. First, we will show the existence of the discrete solution. Second, the convergence of the discrete solution to a solution of the homogenized equation will be demonstrated. For our analysis we will use zero trace functions $v_\epsilon^b = v_\epsilon - l^\theta$ (cf. (3.4)), which satisfies

$$D_t v_\epsilon^b = \text{div}\, a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, \xi + D_x v^b) \text{ in } K, \tag{3.11}$$

where ξ is constant

$$\xi = D_x l^\theta,$$

$v^b = 0$ on ∂K, and $v^b(x, t = t_n) = 0$. ξ will denote the gradient of l^θ in further analysis. Define the norm of $\|\theta\|$ (finite dimensional) by

$$\|\theta\| = \left(\sum_K \int_K (|l^\theta|^p + |D_x l^\theta|^p) dx \right)^{1/p}.$$

This norm is equivalent to $(\sum_K \int_K (|\eta^{l^\theta}|^p + |\xi_K|^p) dx)^{1/p}$ or any other norm in the corresponding finite dimensional space. Moreover, because of $\|\theta\| = \|l^\theta\|_{W^{1,p}(Q_0)} \le C\|D_x l^\theta\|_{p,Q_0}$, $\|\theta\|$ is majorized by $(\sum_K \int_K |\xi_K|^p dx)^{1/p}$.

LEMMA 3.2. *$A^{h,\epsilon}$ is coercive for sufficiently small h_x, i.e.,*

$$(A^{h,\epsilon}\theta, \theta) \ge C \int_{t_n}^{t_{n+1}} \|\theta\|^p dt - C_0. \tag{3.12}$$

Proof.

(3.13)

$$\begin{aligned}(A^{h,\epsilon}\theta,\theta) &= \sum_K \int_{t_n}^{t_{n+1}} \int_K ((a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon), D_x l^\theta) \\ &\qquad + a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon) l^\theta) dxdt \\ &= \sum_K \int_{t_n}^{t_{n+1}} \int_K ((a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon), D_x l^\theta) \\ &\qquad + a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon) \eta^{l^\theta}) dxdt \\ &\quad + \sum_K \int_{t_n}^{t_{n+1}} \int_K a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon)(l^\theta - \eta^{l^\theta}) dxdt =: I_1 + I_2,\end{aligned}$$

where I_1 and I_2 denote the first and second term on the right-hand side that involve $\sum_K$. For the first term we have

(3.14)

$$\begin{aligned}I_1 &= \sum_K \int_{t_n}^{t_{n+1}} \int_K ((a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x l^\theta + D_x v_\epsilon^b), D_x l^\theta + D_x v_\epsilon^b) \\ &\qquad + a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x l^\theta + D_x v_\epsilon^b) \eta^{l^\theta}) dxdt \\ &\quad - \sum_K \int_{t_n}^{t_{n+1}} \int_K (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x \eta^{l^\theta} + D_x v_\epsilon^b), D_x v_\epsilon^b) dxdt \\ &\geq C \sum_K \int_{t_n}^{t_{n+1}} \int_K |D_x l^\theta + D_x v_\epsilon^b|^p dxdt + \frac{1}{2} \sum_K \int_K |D_x v_\epsilon^b(t = t_{n+1})|^2 dx - C_0 \\ &\geq C \sum_K \int_{t_n}^{t_{n+1}} \int_K |D_x v_\epsilon|^p dxdt - C_0,\end{aligned}$$

where v_ϵ^b is defined by (3.11).

Using the trace inequality (see, e.g., [13]) $\|u\|_{p,\partial K} \leq C \|D_x u\|_{p,K}$ we can obtain the lower bound for (3.14). Denote K_1 to be rescaled K such that $diam(K_1) = O(1)$, $y = x/h_x$, $v_\epsilon^1 = v_\epsilon(y h_x)$. Then

(3.15)

$$\begin{aligned}&\int_{t_n}^{t_{n+1}} \int_K |D_x v_\epsilon|^p dxdt = C \frac{h_x^d}{h_x^p} \int_{t_n}^{t_{n+1}} \int_{K_1} |D_y v_\epsilon^1|^p dydt \geq C \frac{h_x^d}{h_x^p} \int_{t_n}^{t_{n+1}} \int_{\partial K_1} |v_\epsilon^1|^p dS_y dt \\ &= C h_x^d \int_{t_n}^{t_{n+1}} \int_{\partial K_1} |l^\theta|^p dS_y dt.\end{aligned}$$

l^θ can be written as $l^\theta = \xi \cdot (x - x_0) + \eta^{l^\theta}$, where $\xi = D_x l^\theta$ and x_0 is chosen such that $\frac{1}{|K|} \int_K l^\theta dx = \eta^{l^\theta}$. Then we have

$$\begin{aligned}\sum_K \int_{t_n}^{t_{n+1}} \int_K |D_x v_\epsilon|^p dxdt &\geq C \sum_K h^d \int_{t_n}^{t_{n+1}} \int_{\partial K_1} |\xi \cdot (x - x_0) + \eta^{l^\theta}|^p dl dt \\ &= C \sum_K h^d \int_{t_n}^{t_{n+1}} \|\theta\|^p dt = C \int_{t_n}^{t_{n+1}} \|\theta\|^p dt.\end{aligned} \tag{3.16}$$

The latter can be shown using the equivalence of the norm in finite dimensional space. Indeed, $\int_{t_n}^{t_{n+1}}\int_{\partial K_1}|\xi\cdot(x-x_0)+\eta^{l^\theta}|^p dl$ defines a norm in the finite dimensional space of (ξ,η). Since all norms are equivalent in finite dimensional space we obtain (3.16).

For the second term, I_2, on the right-hand side of (3.13) we have

$$\begin{aligned}|I_2| &\le Ch_x\sum_K\int_{t_n}^{t_{n+1}}\int_K a_0(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,\eta^{l^\theta},D_xv_\epsilon)|D_xl^\theta|dxdt\\ &\le Ch_x\sum_K\int_{t_n}^{t_{n+1}}\int_K(|\eta^{l^\theta}|^p+|\xi|^p)dxdt\le Ch_x\int_{t_n}^{t_{n+1}}\|\theta\|^p dt.\end{aligned}\tag{3.17}$$

Combining (3.13), (3.14), and (3.17) we obtain

$$(A^{h,\epsilon}\theta,\theta)\ge(C-C_1h_x)\int_{t_n}^{t_{n+1}}\|\theta\|^p dt-C_0.\qquad\square$$

Lemma 3.3. *$A^{h,\epsilon}$ is uniformly continuous in any compact set of θ's. Moreover, for any θ_1 and θ_2 in a compact set,*

$$\|A^{h,\epsilon}\theta_1-A^{h,\epsilon}\theta_2\|^p\le C\left(\sum_K\int_{t_n}^{t_{n+1}}\int_K(|D_xl^{\theta_1}-D_xl^{\theta_2}|^p+\nu(|\eta^{l^{\theta_1}}-\eta^{l^{\theta_2}}|))dxdt\right)^{1/p}.$$

Proof. Clearly,

$$\begin{aligned}&\|A^{h,\epsilon}\theta_1-A^{h,\epsilon}\theta_2\|\\ &\le\sum_K\left|\int_{t_n}^{t_{n+1}}\int_K(a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,\eta^{l^{\theta_1}},D_xv_1)-a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,\eta^{l^{\theta_2}},D_xv_2))dxdt\right|\\ &\quad+\sum_K\left|\int_{t_n}^{t_{n+1}}\int_K(a_0(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,\eta^{l^{\theta_1}},D_xv_1)-a_0(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,\eta^{l^{\theta_2}},D_xv_2))dxdt\right|,\end{aligned}\tag{3.18}$$

where $D_tv_i=\operatorname{div}(a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,\eta^{l_i^\theta},D_xv_i))$ in $K\times[t_n,t_{n+1}]$, $v_i=l^{\theta_i}$ on ∂K, and $v_i(t=t_n)=l^{\theta_i}$ $(i=1,2)$. It can be easily shown that $\int_{t_n}^{t_{n+1}}\int_K|D_xv_i|^pdxdt\le C\int_{t_n}^{t_{n+1}}\int_K|D_xl^{\theta_i}|^pdxdt$. Thus, $\sum_K\int_{t_n}^{t_{n+1}}\int_K|D_xv_i|^pdxdt\le C$.

For the first term on the right-hand side of (3.18) we have

$$\begin{aligned}&\left|\sum_K\int_{t_n}^{t_{n+1}}\int_K(a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,\eta^{l^{\theta_1}},D_xv_1)-a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,\eta^{l^{\theta_2}},D_xv_2))dxdt\right|\\ &\le C\sum_K\int_{t_n}^{t_{n+1}}\int_K(1+|\eta^{l^{\theta_1}}|^{p-1}+|\eta^{l^{\theta_2}}|^{p-1}|D_xv_1|^{p-1}+|D_xv_2|^{p-1})\nu(|\eta^{l^{\theta_1}}-\eta^{l^{\theta_2}}|)dxdt\\ &\quad+C\sum_K\int_{t_n}^{t_{n+1}}\int_K(1+|\eta^{l^{\theta_1}}|^{p-1-s}+|\eta^{l^{\theta_2}}|^{p-1-s}+|D_xv_1|^{p-1-s}+|D_xv_2|^{p-1-s})|D_xv_1-D_xv_2|^sdxdt\\ &\le C\left(\sum_K\int_{t_n}^{t_{n+1}}\int_K\nu(|\eta^{l^{\theta_1}}-\eta^{l^{\theta_2}}|)^pdxdt\right)^{1/p}+C\left(\sum_K\int_{t_n}^{t_{n+1}}\int_K(1+|\eta^{l^{\theta_1}}|^{\frac{p(p-1-s)}{(p-s)}}+|\eta^{l^{\theta_2}}|^{\frac{p(p-1-s)}{(p-s)}}\right.\\ &\quad\left.+|D_xv_1|^{\frac{p(p-1-s)}{(p-s)}}+|D_xv_2|^{\frac{p(p-1-s)}{(p-s)}})dxdt\right)^{\frac{(p-s)}{p}}\left(\sum_K\int_{t_n}^{t_{n+1}}\int_K|D_xv_1-D_xv_2|^pdxdt\right)^{1/p}\\ &\le C\left(\sum_K\int_{t_n}^{t_{n+1}}\int_K\nu(|\eta^{l^{\theta_1}}-\eta^{l^{\theta_2}}|)dxdt\right)^{1/p}+C\left(\sum_K\int_{t_n}^{t_{n+1}}\int_K|D_xv_1-D_xv_2|^pdxdt\right)^{1/p}.\end{aligned}\tag{3.19}$$

Here we have used $l^{\theta_i} = \eta^{l^{\theta_i}} + \xi_i(x - x_0)$ $(i = 1, 2)$, Cauchy and Holder inequalities, along with the facts that $D_x v_1$ and $D_x v_2$ are bounded in $(L^p(t_n, t_{n+1}, Q_0))^d$ and $\nu(r)^p$ is still a continuity modulus. The estimate for the second term on the right-hand side of (3.19) can be derived as follows:

(3.20)
$$\begin{aligned}
&\sum_K \int_{t_n}^{t_{n+1}} \int_K |D_x v_1 - D_x v_2|^p dxdt \\
&\le C \sum_K \int_{t_n}^{t_{n+1}} \int_K (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_1}}, D_x v_1) - a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_1}}, D_x v_2), D_x v_1 - D_x v_2) dxdt \\
&\le C \sum_K \int_{t_n}^{t_{n+1}} \int_K (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_1}}, D_x v_1) - a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_2}}, D_x v_2), D_x v_1 - D_x v_2) dxdt \\
&\quad + C \sum_K \int_{t_n}^{t_{n+1}} \int_K (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_2}}, D_x v_2) - a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_1}}, D_x v_2), D_x v_1 - D_x v_2) dxdt \\
&\le C \sum_K \int_{t_n}^{t_{n+1}} \int_K (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_1}}, D_x v_1) - a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_2}}, D_x v_2), \\
&\qquad\qquad D_x l^{\theta_1} + D_x v_1^b - (D_x l^{\theta_2} + D_x v_2^b)) dxdt \\
&\quad + \frac{C}{\delta_1} \left(\sum_K \int_{t_n}^{t_{n+1}} \int_K \nu(|\eta^{l^{\theta_1}} - \eta^{l^{\theta_2}}|)^p dxdt \right)^{1/p} + C\delta_1 \sum_K \int_{t_n}^{t_{n+1}} \int_K |D_x v_1 - D_x v_2|^p dxdt.
\end{aligned}$$

Here we have used Cauchy and Holder inequalities, along with the facts that $D_x v_1$ and $D_x v_2$ are bounded in $(L^p(t_n, t_{n+1}, Q_0))^d$. With an appropriate choice of δ_1 we have

(3.21)
$$\begin{aligned}
&\sum_K \int_{t_n}^{t_{n+1}} \int_K |D_x v_1 - D_x v_2|^p dxdt \\
&\le C \sum_K \int_{t_n}^{t_{n+1}} \int_K (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_1}}, D_x v_1) - a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_2}}, D_x v_2), \\
&\qquad\qquad D_x l^{\theta_1} - D_x l^{\theta_2}) dxdt \\
&\quad - \frac{1}{2} C \sum_K \int_{t_n}^{t_{n+1}} \int_K D_t |v_1^b - v_2^b|^2 dxdt + C \left(\sum_K \int_{t_n}^{t_{n+1}} \int_K \nu(|\eta^{l^{\theta_1}} - \eta^{l^{\theta_2}}|) dxdt \right)^{1/p} \\
&\le C \left(\sum_K \int_{t_n}^{t_{n+1}} \int_K |D_x l^{\theta_1} - D_x l^{\theta_2}|^p dxdt \right)^{1/p} + C \left(\sum_K \int_{t_n}^{t_{n+1}} \int_K \nu(|\eta^{l^{\theta_1}} - \eta^{l^{\theta_2}}|) dxdt \right)^{1/p}.
\end{aligned}$$

Here we have used Cauchy and Holder inequalities, along with the facts that $D_x v_1$ and $D_x v_2$ are bounded in $L^p(t_n, t_{n+1}, Q_0)$, $D_t v_i = \mathrm{div}(a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^{\theta_i}}, D_x v_i))$ $(i = 1, 2)$. The second term on the right-hand side of (3.18) can be estimated in an analogous manner. ∎

From Lemmas 3.2 and 3.3 it follows that (3.3) has solutions which are uniformly bounded with respect to ϵ for any h. Next, we take the limit as $\epsilon \to 0$ in (3.3) and show the following lemma.

Lemma 3.4.

$$\lim_{\epsilon \to 0} A^{h,\epsilon} \zeta = A^h \zeta,$$

for any vector ζ, *where* A^h *is defined as*

$$(A^h \theta, \zeta) = \int_{t_n}^{t_{n+1}} \int_{Q_0} ((a^*(\eta^{l^\theta}, D_x l^\theta), D_x l^\zeta) + a_0^*(\eta^{l^\theta}, D_x l^\theta) l^\zeta) dxdt.$$

Proof. Using G-convergence results [19] for arbitrary solutions we have that v_ϵ converges to v_0 in $\overline{W_n}$, where v_0 is the solution of

$$D_t v_0 = \operatorname{div} a^*(\eta^{l^\theta}, D_x v_0) \text{ in } K \times [t_n, t_{n+1}],$$

and $v_0 = l^\theta$ on ∂K, $v_0(x, t = t_n) = l^\theta$. The solution of this equation is $v_0 = l^\theta$. Consequently, using Theorem 2.1 on the convergence of arbitrary solutions for the G-convergent sequence of operators we have

$$a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon) \to a^*(\eta^{l^\theta}, D_x l^\theta),$$
$$a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta^{l^\theta}, D_x v_\epsilon) \to a_0^*(\eta^{l^\theta}, D_x l^\theta)$$

as $\epsilon \to 0$ weakly in $(L^q(t_n, t_{n+1}, Q_0))^d$ and $L^q(t_n, t_{n+1}, Q_0)$, respectively. Next, taking into account (3.5), we get the desired result. □

Note that since $A^{h,\epsilon}$ is uniformly continuous (see Lemma 3.3) the convergence results of Lemma 3.4 hold uniformly in any compact set of ζ's (finite dimensional). Thus taking the limit as $\epsilon \to 0$ of (3.3) yields

(3.22)
$$\sum_i (\theta_i^{n+1} - \theta_i^n) \int_{Q_0} \phi_i^0(x)\phi_j^0(x)dx$$
$$+ \int_{t_n}^{t_{n+1}} \int_{Q_0} ((a^*(\eta^{l^\theta}, D_x l^\theta), D_x\phi_j^0) + a_0^*(\eta^{l^\theta}, D_x l^\theta)\phi_j^0) dxdt = \int_{t_n}^{t_{n+1}} \int_{Q_0} f\phi_j^0 dxdt.$$

Next, we will show that the solution of (3.22) converges to the solution of the homogenized equation. Note that (3.22) is not a standard discretization of the homogenized equation on S^h, where we have $a^*(l^\theta, D_x l^\theta)$ and $a_0^*(l^\theta, D_x l^\theta)$ instead of $a^*(\eta^{l^\theta}, D_x l^\theta)$ and $a_0^*(\eta^{l^\theta}, D_x l^\theta)$. Equation (3.22) is more tractable for computational purposes because the quadrature step can be easily implemented. We rewrite (3.22) as

$$\begin{aligned} &\sum_i \frac{\theta_i^{n+1} - \theta_i^n}{h_t} \int_{Q_0} \phi_i^0(x)\phi_j^0(x)dx \\ &+ \int_{Q_0} ((a^*(\eta^{l^\theta}, D_x l^\theta), D_x\phi_j^0) + a_0^*(\eta^{l^\theta}, D_x l^\theta)\phi_j^0) dx = \int_{Q_0} f\phi_j^0 dx. \end{aligned} \tag{3.23}$$

For simplicity in (3.23) we have assumed that $f = f(x)$.

For each $u_h(t), v_h(t) \in S^h$ such that $u_h(t), v_h(t) \in C(0, T, S^h)$, denote

$$(A^h u_h, v_h) = \int_0^T \int_{Q_0} ((a^*(\eta^{u_h}, D_x u_h), D_x v_h) + a_0^*(\eta^{u_h}, D_x u_h) v_h) dxdt.$$

For further analysis we will use u_h instead of l^θ to denote discrete solutions, $u_h \in S^h$, because we will be studying the continuum limit of the discrete quantities, i.e., the limit as $h \to 0$. Then (3.23) can be written as

$$\frac{1}{h_t}(u_h(t) - u_h(t - h_t)) + A^h u_h = f_h,$$

where f_h is the orthogonal projection of f onto S^h, i.e., $f_h \in S^h$, such that $(f_h, v_h) = (f, v_h)$.

LEMMA 3.5. *A^h is coercive for sufficiently small h_x, i.e.,*

$$(A^h u_h, u_h) \geq C\|u_h\|_{V_0} - C_0.$$

Proof.

$$\begin{aligned}
(A^h u_h, u_h) &= \sum_K \int_0^T \int_K (a^*(\eta^{u_h}, D_x u_h), D_x u_h) dxdt + \sum_K \int_0^T \int_K a_0^*(\eta^{u_h}, D_x u_h) u_h dxdt \\
&= \sum_K \int_0^T \int_K (a^*(\eta^{u_h}, D_x u_h), D_x u_h) dxdt + \sum_K \int_0^T \int_K a_0^*(\eta^{u_h}, D_x u_h) \eta^{u_h} dxdt \\
&\quad + \sum_K \int_0^T \int_K a_0^*(\eta^{u_h}, D_x u_h)(u_h - \eta^{u_h}) dxdt \\
&\geq C \sum_K \int_0^T \int_K |D_x u_h|^p dxdt - C_0 - \left| \sum_K \int_0^T \int_K a_0^*(\eta^{u_h}, D_x u_h)(u_h - \eta^{u_h}) dx \right| \\
&\geq C \sum_K \int_K |D_x u_h|^p dxdt - C_0 - C_2 h_x \sum_K \int_0^T \int_K |D_x u_h|^p dxdt \\
&= (C - C_2 h_x) \sum_K \int_0^T \int_K |D_x u_h|^p dxdt - C_0. \qquad \square
\end{aligned}$$

Next, we show that $A^h(\theta)$ converges to $A(\theta)$ uniformly in V_0' for any uniformly bounded set in V_0, where A is defined by $(A(u_h), v_h) = \sum_K \int_0^T \int_K ((a^*(u_h, D_x u_h), D_x v_h) + a_0^*(u_h, D_x u_h) v_h) dxdt$.

LEMMA 3.6.

$$(A^h(u_h) - A(u_h), v_h) \to 0$$

for any uniformly bounded family of u_h and compact family of v_h in V_0.

Proof. Consider

$$\begin{aligned}
(A^h(u_h) - A(u_h), v_h) = \sum_K \int_0^T \int_K &((a^*(\eta^{u_h}, D_x u_h) - a^*(u_h, D_x u_h), D_x v_h) \\
&+ (a_0^*(\eta^{u_h}, D_x u_h) - a_0^*(u_h, D_x u_h) v_h)) dxdt.
\end{aligned}$$

Using the estimates for a^* we have

(3.24)

$$\begin{aligned}
&\left| \sum_K \int_0^T \int_K (a^*(\eta^{u_h}, D_x u_h) - a^*(u_h, D_x u_h), D_x v_h) dxdt \right| \\
&\leq C \sum_K \int_0^T \int_K (1 + |\eta^{u_h}|^{p-1} + |D_x u_h|^{p-1} + |u_h|^{p-1}) \nu(|\eta^{u_h} - u_h|) |D_x v_h| dxdt \\
&\leq \left(\sum_K \int_0^T \int_K (1 + |u_h|^p + |D_x u_h|^p) dxdt \right)^{1/q} \left(\int_K |D_x v_h|^p \nu(|\eta^{u_h} - u_h|)^p dxdt \right)^{1/p} \\
&\leq \left(\int_Q (1 + |u_h|^p + |D_x u_h|^p) dxdt \right)^{1/q} \left(\int_Q |D_x v_h|^p \nu(|\eta^{u_h} - u_h|)^p dxdt \right)^{1/p} \\
&\leq (C + \|u_h\|_{V_0}^p)^{1/q} \left(\int_Q |D_x v_h|^p \nu(h|D_x u_h|)^p dxdt \right)^{1/p}.
\end{aligned}$$

Here we have used $\int_K |\eta^{u_h}|^p dx \leq \int_K |u_h|^p dx$ (by Jensen inequality) and $|u_h - \eta^{u_h}| \leq Ch|D_x u_h|$. Because of Lemma 4.3 we obtain that the right-hand side of (3.24) converges to zero for any uniformly bounded family of $u_h \in V_0$ and compact family $v_h \in V_0$ as $h \to 0$. The estimate for a_0 can be obtained in a similar way:

$$\left|\sum_K \int_0^T \int_K (a_0^*(\eta^{u_h}, D_x u_h) - a_0^*(u_h, D_x u_h), D_x v_h) dx dt\right| \leq (C + \|u_h\|_{V_0}^p)^{1/q} \left(\int_Q |v_h|^p \nu(h|D_x u_h|)^p dx dt\right)^{1/p}. \tag{3.25}$$

Note that the right-hand side of (3.25) converges to zero for any uniformly bounded family of $u_h \in V_0$ and $v_h \in V_0$. □

Next, we will show that u_h converges to the solution of the homogenized equation weakly in V_0. Our proof will follow the same lines as the Bardos–Brezis theorem (see [20, p. 128]). The difference in our case is that we do not have the original operator but have its uniform approximation. To simplify the presentation we denote

$$[u, v] = \int_0^T \int_{Q_0} uv dx dt. \tag{3.26}$$

$[A_h u_h, v_h] = (A_h u_h, v_h)$ is assumed. Consider

$$J_h u_h + A_h(u_h) = f_h, \tag{3.27}$$

where $J_h u_h = \frac{1}{h_t}(u_h(t) - u_h(t - h_t))$. Denote the corresponding generator by J. Here $u_h = u_h(t) \in V_0$ is considered as a function with values in $W_0^{1,p}(Q_0)$. It can be easily shown that the solution of the discrete equation exists. Taking the value of (3.27) at u_h and noting $[J_h u_h, u_h] \geq 0$ (see [20]) we obtain that

$$[A_h(u_h), u_h] \leq [f_h, u_h].$$

Consequently, u_h is bounded in V_0; thus $A(u_h)$ is bounded in $V_0^{'}$, from where it follows that $u_h \to u$ and $A_h u_h \to g$ weakly in V_0 and $V_0^{'}$, respectively, as $h \to 0$. Next, for each v in $D(J^+)$, where J^+ denotes the adjoint of L, we choose a sequence v_h such that $v_h \to v$ in V_0 and $J_h^+ v_h \to J^+ v$ in $V_0^{'}$. Consider (3.27) at v_h,

$$[f_h - A_h(u_h), v_h] = [J_h u_h, v_h], \tag{3.28}$$

or

$$[f_h - A_h(u_h), v_h] = [u_h, J_h^+ v_h].$$

Taking the limit as $h \to 0$ we obtain $[f - g, v] = [u, J^+ v]$, for any $v \in D(J^+)$, where $[g, v] = \lim_{h\to 0}[A_h u_h, v_h]$. From here, making use of the resolvents of J (as it is done in [20]) we have in $V_0^{'}$

$$Ju + g = f,$$

$u \in D(J)$. It remains to show that $g = A(u)$, where u is a weak limit of u_h. Again for any v choosing a sequence $v_h \to v$ in V_0 we have

$$[g, v] = \lim_{h\to 0}[A_h(u_h), v_h] = \lim_{h\to 0}[A_h(u_h) - A(u_h), v_h] + \lim_{h\to 0}[A(u_h), v_h] = \lim_{h\to 0}[A(u_h), v_h]. \tag{3.29}$$

Thus $A(u_h) \to g$ weakly in V_0'. To show $g = A(u)$ it remains to show

$$\lim_{h\to 0}[A(u_h) - g, u_h] = 0. \tag{3.30}$$

From here, using the fact that the operator A is type M [20], we will obtain $A(u) = g$; thus u is a solution of the homogenized equation. Moreover, since our differential operators are also type S_+ (see [21]) we obtain that u_h strongly converges to u, a solution of the homogenized equation. This completes the proof of the fact that u_h converges strongly to u, a solution of the homogenized equation, in V_0 as $h \to 0$.

For (3.30) to hold, additional conditions are needed which will be discussed next. These are the conditions required for Theorem 3.1 to hold. We will discuss various conditions that can be used in different situations. Note that (3.30) can be written as

$$\lim_{h\to 0}[A(u_h) - A_h(u_h), u_h] = 0.$$

The left-hand side can be written as

$$\begin{aligned}[A(u_h) - A_h(u_h), u_h] = &\int_0^T \int_{Q_0} (a^*(\eta^{u_h}, D_x u_h) - a^*(u_h, D_x u_h), D_x u_h) dxdt \\ &+ \int_0^T \int_{Q_0} (a_0^*(\eta^{u_h}, D_x u_h) - a_0^*(u_h, D_x u_h)) u_h dxdt.\end{aligned} \tag{3.31}$$

It can be easily shown that the second term converges to zero as $h \to 0$. Indeed, taking into account that u_h is uniformly bounded in V_0:

$$\begin{aligned}&\left|\int_0^T \int_{Q_0} (a_0^*(\eta^{u_h}, D_x u_h) - a_0^*(u_h, D_x u_h)) u_h dxdt\right| \\ &\quad \le \int_0^T \int_{Q_0} (1 + |D_x u_h|^{p-1} + |u_h|^{p-1}) \nu(|\eta^{u_h} - u_h|) u_h dxdt \\ &\quad \le C \int_0^T \int_{Q_0} (1 + |D_x u_h|^{p-\alpha}) \nu(|\eta^{u_h} - u_h|) dxdt,\end{aligned}$$

where $\alpha > 0$. By Lemma 4.3 the right-hand side converges to zero since $D_x u_h$ is bounded in $(L^p(Q))^d$. The first term on the right-hand side of (3.31) does not converge to zero in general. Indeed, for this term using (2.6) we have

$$\begin{aligned}&\left|\int_0^T \int_{Q_0} (a^*(\eta^{u_h}, D_x u_h) - a^*(u_h, D_x u_h), D_x u_h) dxdt\right| \\ &\qquad \le C \int_0^T \int_{Q_0} \nu(|\eta^{u_h} - u_h|)(1 + |D_x u_h|^p) dxdt.\end{aligned} \tag{3.32}$$

The right-hand side does not necessarily converge to zero unless $D_x u_h$ is uniformly bounded in $(L^{p+\alpha}(Q))^d$ or under assumptions different from (2.6). It is not difficult to construct a function whose L^p-norm is of order one over a finite number of elements K, and $\nu(|\eta^{u_h} - u_h|)$ is also of order one in these elements. Next, we will discuss assumptions that allow us to state that (3.32) converges to zero, and, consequently, (3.30) holds.

First, we note that if we use instead of (2.6)

$$|a(\omega, \eta, \xi) - a(\omega, \eta', \xi)| \le C(1 + |\eta|^{p-1} + |\eta'|^{p-1} + |\xi|^{p-1-r})|\eta - \eta'|^r \tag{3.33}$$

($0 < r < 1$) for a, then the right-hand side of (3.32) converges to zero. This condition is used in the homogenization of parabolic operators in previous findings (see, e.g., [21, 17]). It can be easily checked that if we have (3.33) for higher order terms (i.e., a) and (2.6) for lower order terms (i.e., a_0), then all our previous calculations are valid; moreover, (3.32) converges to zero, which implies that $g = Au$. Indeed, in this case

$$
\begin{aligned}
&\left| \int_0^T \int_{Q_0} (a^*(\eta^{u_h}, D_x u_h) - a^*(u_h, D_x u_h), D_x u_h) dx dt \right| \\
&\leq C \int_0^T \int_{Q_0} |\eta^{u_h} - u_h|^r (1 + |D_x u_h|^{p-r}) dx dt \leq C h_x^r \int_0^T \int_{Q_0} |D_x u_h|^p dx dt,
\end{aligned} \tag{3.34}
$$

where in the last step we have used $|\eta^{u_h} - u_h| \leq C h_x |D_x u_h|$. Clearly, the right-hand side of (3.34) converges to zero for any uniformly bounded family of u_h in V_0. Under the following condition, $\int_{Q_0} |a^*(\eta_1(x), \xi(x)) - a^*(\eta_2(x), \xi(x))|^q dx \leq C \int_{Q_0} \|\eta_1 - \eta_2\|_{p,Q_0} \cdot (1 + |\xi(x)|^p) dx$, which is more general (than (3.33)), one can also show (3.30) (cf. (3.32)). Another case of (3.34) converging to zero is when the elliptic part of our parabolic operator is strongly monotone. The analysis for the strongly monotone parabolic operators is different from the one presented here (cf. [9]), and one can use directly the monotonicity condition to show the convergence of the numerical solution. Moreover, in the periodic case the explicit convergence rate in terms of ϵ and h can be obtained. Note that for the strongly monotone random operators we actually do not need to study the limit as $h \to 0$ as we did in the above analysis because in the limit $\epsilon \to 0$ standard finite element discretization of the homogenized equation will be obtained.

Another condition under which (3.32) converges to zero is that $D_x u_h$ is uniformly bounded in $(L^{p+\alpha}(Q))^d$ for some $\alpha > 0$. One can assume additional not restrictive regularity assumptions [16] for input data and obtain Meyers-type estimates, $\|D_x u\|_{p+\alpha,Q} \leq C$, for the homogenized solutions. In this case it is reasonable also to assume that the discrete solutions are uniformly bounded in $(L^{p+\alpha}(Q))^d$. Meyers-type estimates for approximate solutions of linear elliptic problems have been previously obtained in [2]. We have obtained results on Meyers-type estimates for our approximate solutions in the case $p = 2$ [8]. The results can be generalized to parabolic equations. We are currently studying the generalizations of these results to arbitrary p. One can formulate some other conditions which will allow us to show that (3.27) converges to zero (for example, $|\frac{\partial}{\partial \eta} a| \leq C$ (see [18])), or another condition that can be practical for computational purposes is that the homogenized solution is in C^α, $\alpha > 0$. The latter can also be obtained from the Sobolev imbedding theorem for sufficiently large p.

REMARK 3.5. *We would like to note that the additional condition required for Theorem* 3.1 *to hold is that the gradient of the numerical solution, $D_x u_h$, is in $(L^{p+\alpha}(Q))^d$ for some $\alpha > 0$. This condition can be replaced by other conditions that were discussed above.*

The above analysis can be carried out for general heterogeneities using G-convergence theory. To show it we can use instead of (3.3)

$$
\begin{aligned}
\int_{Q_0} (u_h(t) - u_h(t - h_t)) w_h dx + \int_{t-h_t}^t \int_{Q_0} &((a_\epsilon(x, t, \eta^{u_h}, D_x v_\epsilon), D_x w_h) \\
&+ a_{0,\epsilon}(x, t, \eta^{u_h}, D_x v_\epsilon) w_h) dx dt = \int_{t-h_t}^t \int_{Q_0} f w_h dx dt,
\end{aligned}
$$

where w_h is an arbitrary element of S^h, and v_ϵ is the solution of an appropriate local problem. Further, taking a limit as $\epsilon \to 0$ in the same way as we did before one can obtain an equation similar to (3.22),

$$\int_{Q_0} (u_h(t) - u_h(t-h_t)) w_h dx + \int_{t-h_t}^{t} \int_{Q_0} ((a^*(x,t,\eta^{u_h}, D_x v_\epsilon), D_x w_h) \\ + a_0^*(x,t,\eta^{u_h}, D_x v_\epsilon) w_h) dx dt = \int_{t-h_t}^{t} \int_{Q_0} f w_h dx dt.$$

The further analysis can be carried out along the same lines as we did above, assuming additionally that a^* and a_0^* are Holder continuous with respect to spatial and temporal variables (cf. [9]). We would like to note that in the case of the general G-convergent sequence of operators the convergence is up to a subsequence; i.e., the numerical solution will converge to a solution of a homogenized equation (up to a subsequence of ϵ) as it was formulated in Theorem 3.1.

REMARK 3.6. *To construct an approximation that strongly converges to* an oscillatory solution *in V_0 norm given homogenized solution or its strong approximation in V_0 we need corrector results that will be described in section* 4.

4. Numerical correctors and the approximations of the gradients. Define M_h in the following way:

$$M_h \phi(x,t) = \sum_i 1_{Q^i} \frac{1}{|Q^i|} \int_{Q^i} \phi(y,\tau) dy d\tau, \tag{4.1}$$

where $\bigcup Q_0^i = Q_0$ and $\bigcup T^i = [0,T]$, $Q^i = Q_0^i \times T^i$, Q_0^i and T^i have empty intersections, respectively, and the maximum diameter of Q_0^i and T^i are h_x and h_t, respectively, and $h = (h_x, h_t)$. Without loss of generality we assume Q_0^i are arbitrary domains with Lipschitz boundaries (in particular the triangular partition of Q_0; cf. (3.2)). Note that for any $\phi \in L^p(Q)$

$$M_h \phi \to \phi \text{ in } L^p(Q) \text{ as } h \to 0. \tag{4.2}$$

Further, denote

$$P(T(y,\tau)\omega, \eta, \xi) = \xi + w_{\eta,\xi}(T(y,\tau)\omega), \tag{4.3}$$

where $w_{\eta,\xi} = \partial N$ and N is the solution of (2.15), (2.16), (2.17), (2.18), or (2.20) depending on the ratio between α and β. Note that the realizations of N can be defined using near solutions (see [7] for details).

One of our main results is the following.

THEOREM 4.1. *Let u_ϵ and u be solutions of* (2.2) *and* (3.1)*, respectively, and P is defined by* (4.3) *in each Q_i. Then*

$$\lim_{h\to 0} \lim_{\epsilon \to 0} \int_Q |P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, M_h D_x u) - D_x u_\epsilon|^p dx dt \to 0,$$

μ-a.e.

We will omit μ-a.e. notation in further analysis. To make the expressions in the proof more concise we introduce the notation

$$\mathcal{P}_\epsilon = P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, M_h D_x u).$$

Theorem 4.1 indicates that the gradient of solutions can be approximated by $P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, M_h D_x u)$. This quantity can be computed based on $M_h D_x u$ and $M_h u$ i.e., the gradient of the coarse solution in each coarse block as we will show later. For the proof of Theorem 4.1 we need the following lemma.

LEMMA 4.2. *For every $\eta \in R$ and $\xi \in R^d$*

$$\|P(\omega, \eta, \xi)\|_{p,\Omega}^p \leq C(1 + |\eta|^p + |\xi|^p).$$

Proof.

$$\begin{aligned}
\|P\|_{p,\Omega}^p &\leq C \int_\Omega (a(\omega, \eta, P) - a(\omega, \eta, 0), P) d\mu(\omega) \\
&\leq C \left| \int_\Omega (a(\omega, \eta, P), P) d\mu(\omega) \right| + \left| \int_\Omega (a(\omega, \eta, 0), P) d\mu(\omega) \right| \\
&\leq \left| \int_\Omega (a(\omega, \eta, P), \xi) d\mu(\omega) \right| + (1 + |\eta|^{p-1}) \left| \int_\Omega P d\mu(\omega) \right| \\
&\leq C\delta_1 \|P\|_{p,\Omega}^p + C\delta_1^{-1/(p-1)} |\eta|^p + C \int_\Omega (1 + |\eta| + |P|)^{p-1} |\xi| d\mu(\omega) \\
&\leq C\delta_2 \|P\|_{p,\Omega}^p + C\delta_2^{-1/(p-1)} (1 + |\xi|^p) + C + C\delta_1 (|\eta|^p + \|P\|_{p,\Omega}^p) + C\delta_1^{-1/(p-1)} |\eta|^p.
\end{aligned}$$

With an appropriate choice of δ_1 and δ_2 we obtain the desired result. ∎

From here it follows that $P(T(y, \tau)\omega, \eta, \xi) \in L^p_{loc}(R^d)$ for a.e. ω and for each $\eta \in R$, $\xi \in R^d$.

The next lemma will also be used in the proof.

LEMMA 4.3. *If $u_k \to 0$ in $L^r(Q)$ $(1 < r < \infty)$ as $k \to \infty$, then*

$$\int_Q \nu(u_k) |v_k|^p dxdt \to 0, \text{ as } k \to \infty$$

for all v_k either (1) *compact in $L^p(Q)$ or* (2) *bounded in $L^{p+\alpha}(Q)$, $\alpha > 0$. Here $\nu(r)$ is continuity modulus defined previously (see* (2.6)*), and $1 < p < \infty$.*

Proof. Since u_k converges in L^r it converges in measure. Consequently, for any $\delta > 0$ there exists Q_δ and k_0 such that $meas(Q \setminus Q_\delta) < \delta$ and $\nu(u_k) < \delta$ in Q_δ for $k > k_0$. Thus

$$\begin{aligned}
\int_Q \nu(u_k) |v_k|^p dxdt &= \int_{Q_\delta} \nu(u_k) |v_k|^p dxdt + \int_{Q \setminus Q_\delta} \nu(u_k) |v_k|^p dxdt \\
&\leq C\delta + C \int_{Q \setminus Q_\delta} |v_k|^p dxdt.
\end{aligned} \tag{4.4}$$

Next, we use the fact that if (1) or (2) satisfies, then the set v_k has equiabsolute continuous norm [12] (i.e., for any $\epsilon > 0$ there exists $\zeta > 0$ such that $meas(Q_\zeta) < \zeta$ implies $\|P_{Q_\zeta} v_k\|_p < \epsilon$, where $P_D f = \{f(x)$, if $x \in D$; 0 otherwise$\}$). Consequently, the second term on the right-hand side of (4.4) converges to zero as $\delta \to 0$. ∎

The proof of the theorem will be based on the following estimate:

$$\begin{aligned}
&\int_Q |\mathcal{P}_\epsilon - D_x u_\epsilon|^p dxdt \qquad (4.5) \\
&\leq C \int_Q (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, \mathcal{P}_\epsilon) - a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), \mathcal{P}_\epsilon - D_x u_\epsilon) dxdt
\end{aligned}$$

$$\leq C\left|\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon) - a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), \mathcal{P}_\epsilon - D_x u_\epsilon)dxdt\right|$$
$$+ C\left|\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, u_\epsilon, \mathcal{P}_\epsilon) - a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon), \mathcal{P}_\epsilon - D_x u_\epsilon)dxdt\right|$$
$$=: I_1 + I_2,$$

where I_1 and I_2 are the first and second terms that involve absolute value. We write the first term on the right-hand side as follows:

(4.6)
$$\begin{aligned}
I_1 &= C\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon) - a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), \mathcal{P}_\epsilon - D_x u_\epsilon)dxdt \\
&= C\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon), \mathcal{P}_\epsilon)dxdt \\
&\quad - C\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon), D_x u_\epsilon)dxdt \\
&\quad - C\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), \mathcal{P}_\epsilon)dxdt \\
&\quad + C\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), D_x u_\epsilon)dxdt.
\end{aligned}$$

We will investigate the right-hand side of (4.6) in the limit as $\epsilon \to 0$. For the first term of the right-hand side of (4.6) we have the following.

LEMMA 4.4.

$$\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon), \mathcal{P}_\epsilon)dxdt \to \int_Q (a^*(M_h u, M_h D_x u), M_h D_x u)dxdt$$

as $\epsilon \to 0$.

Proof.

$$\begin{aligned}
&\int_Q (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon), \mathcal{P}_\epsilon)dxdt \\
&= \sum_i \int_{Q_i} (a(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega, \eta_i, \xi_i + w_{\eta_i,\xi_i}(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega)), \\
&\qquad\qquad \xi_i + w_{\eta_i,\xi_i}(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega))dxdt \\
&\to \sum_i \int_{Q_i} 1_{Q_i}\langle (a(\omega,\eta_i,\xi_i + w_{\eta_i,\xi_i}), \xi_i + w_{\eta_i,\xi_i})\rangle dxdt \\
&= \sum_i \int_{Q_i} 1_{Q_i}(a^*(\eta_i,\xi_i),\xi_i)dxdt + \sum_i \int_{Q_i} 1_{Q_i}\langle (a(\omega,\eta_i,\xi_i + w_{\eta_i,\xi_i}), w_{\eta_i,\xi_i})\rangle dxdt
\end{aligned}$$

as $\epsilon \to 0$. Here we have used the Birkhoff ergodic theorem. The last term is zero because

$$\langle (a(\omega,\eta_i,\xi_i + w_{\eta_i,\xi_i}), w_{\eta_i,\xi_i})\rangle = \langle (a(\omega,\eta_i,\xi_i + w_{\eta_i,\xi_i}), \partial N_{\eta_i,\xi_i})\rangle = -\langle \sigma N_{\eta,\xi}, N_{\eta,\xi}\rangle = 0,$$

where σ, the time derivative in abstract space, is defined in (2.11). The latter is because σ is the skew-symmetric operator.

Finally, we note that the limit can be written as

$$\sum_i \int_{Q_i} 1_{Q_i}(a^*(\eta_i, \xi_i), \xi_i) dxdt = \int_Q (a^*(M_h u, M_h D_x u), M_h D_x u) dxdt. \qquad \square$$

For the second term of the right-hand side of (4.6) we have the following.

Lemma 4.5.

$$\int_Q (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon), D_x u_\epsilon) dxdt \to \int_Q (a^*(M_h u, M_h D_x u), D_x u) dxdt$$

as $\epsilon \to 0$.

Proof.

$$\begin{aligned} &\int_Q (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon), D_x u_\epsilon) dxdt \\ &= \sum_i \int_{Q_i} (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i)), D_x u_\epsilon) dxdt. \end{aligned}$$

$D_x u_\epsilon$ is bounded in $(L^p(Q))^d$ for a.e. ω. In order to show that $a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i))$ is bounded in $(L^r(Q_i))^d$, where $r > q$, we will use a Meyers-type theorem [5, 1, 22]. Using the fact that $P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i))) = \partial N$, where N satisfies either of (2.15), (2.16), (2.17), (2.18), one can use near solutions for N (as we did in [7]) and obtain Meyers-type estimates following [22],

$$\|P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i)\|_{p+\eta,Q} \leq C,$$

where C is independent of ω and depends only on operator constants. From here using bounds for $a(T_y\omega, \eta, \xi)$ we obtain that $a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i))$ is bounded in $(L^r(Q))^d$, where $r > q$ for a.e. ω. Consequently, $(a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i + w_{\eta_i,\xi_i}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega)), D_x u_\epsilon)$ is bounded in $(L^\sigma(Q_i))^d$, $\sigma > 1$, for every η_i and ξ_i. Thus it contains a subsequence that weakly converges to g_i for any i and a.e. ω. Using compensated compactness argument (see Theorem 2.1 of [22]) and near solutions [7] we can obtain that as $\epsilon \to 0$ $g_i = (a^*(\eta_i, \xi_i), D_x u)$. The latter is because $D_x u_\epsilon$ weakly converges to $D_x u$ in $(L^p(Q))^d$ for a.e. ω and $a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i))$ weakly converges to $a^*(\eta_i, \xi_i)$ in $(L^r(Q))^d$. The fact that $D_x u_\epsilon$ weakly converges to $D_x u$ for a.e. ω follows from general G-convergence results [19], and the weak convergence of $a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i))$ is a consequence of the Birkhoff ergodic theorem. Consequently,

$$\begin{aligned} &\sum_i \int_{Q_i} (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i)), D_x u_\epsilon) dxdt \\ &\to \sum_i \int_{Q_i} (a^*(\eta_i, \xi_i), D_x u) dxdt = \int_Q (a^*(M_h u, M_h D_x u), D_x u) dxdt. \qquad \square \end{aligned}$$

For the third term of the right-hand side of (4.6) we have the following.

Lemma 4.6.

$$\int_Q (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), \mathcal{P}_\epsilon) dxdt \to \int_Q (a^*(u, D_x u), M_h D_x u) dxdt$$

as $\epsilon \to 0$.

Proof.

$$\int_Q (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), \mathcal{P}_\epsilon) dxdt$$
$$= \sum_i \int_{Q_i} (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i)) dxdt.$$

Since $|a(\omega,\eta,\xi)| \le C(1+|\eta|^{p-1}+|\xi|^{p-1})$ in $L^q(Q)$ for a.e. ω, $P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i)$ converges to ξ_i in $(L^p(Q))^d$ and bounded in $(L^{p+\eta}(Q))^d$ ($\eta > 0$), $a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon)$ weakly converges to $a^*(u, Du)$ in $(L^q(Q))^d$ (using the theorem on the convergence of arbitrary solutions for G-convergent sequence of operators), similar to the analysis of the second term we obtain that

$$\sum_i \int_{Q_i} (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, \eta_i, \xi_i)) dxdt$$
$$\to \sum_i \int_{Q_i} (a^*(u, D_x u), \xi_i) dxdt = \int_Q (a^*(u, D_x u), M_h D_x u) dxdt. \qquad \square$$

For the fourth term of the right-hand side of (4.6) we have the following.

LEMMA 4.7.

$$\int_Q (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), D_x u_\epsilon) dxdt \to \int_Q (a^*(u, D_x u), D_x u) dxdt$$

as $\epsilon \to 0$.

Proof.

$$\int_Q (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), D_x u_\epsilon) dxdt$$
$$= -\int_Q (\operatorname{div}(a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon)), u_\epsilon) dxdt$$
$$= -\int_Q (D_t u_\epsilon + a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon) - f) u_\epsilon dxdt$$
$$\to -\int_Q (D_t u + a_0^*(u, D_x u) - f) u dxdt = \int_Q (a^*(u, D_x u), D_x u) dxdt.$$

Here we have used the fact that $a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon) \to a_0^*(t, \omega, u, D_x u)$ weakly in $L^q(Q)$ for a.e. ω. The latter follows from the convergence of arbitrary solutions of the G-convergent sequence of operators, Theorem 2.1. $\square$

For the second term, I_2, on the right-hand side of (4.5) we have

(4.7)
$$|I_2| \le C \left| \int_Q (a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, \mathcal{P}_\epsilon) - a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon), \mathcal{P}_\epsilon - D_x u_\epsilon) dxdt \right|$$
$$\le \frac{C}{\delta_1} \left| \int_Q |a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, \mathcal{P}_\epsilon) - a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, \mathcal{P}_\epsilon)|^q dxdt \right|$$
$$+ C\delta_1 \int_Q |\mathcal{P}_\epsilon - D_x u_\epsilon|^p dxdt$$
$$\le \frac{C}{\delta_1} \sum_i \int_{Q_i} \nu(|u_\epsilon - \eta_i|)^q (1 + |\xi_i|^p) dxdt + \frac{C}{\delta_1} \sum_i \int_{Q_i} \nu(|u_\epsilon - \eta_i|)^q (1 + |w_{\eta_i,\xi_i}|^p) dxdt$$
$$+ C\delta_1 \int_Q |\mathcal{P}_\epsilon - D_x u_\epsilon|^p dxdt,$$

where $\nu(r)$ is a continuity modulus defined earlier (see (2.6)). The first term on the right-hand side converges to $\int_Q \nu(|u - M_h u|)^q(1 + |M_h D_x u|^p)dxdt$ by Lemma 4.3. For the second term using Meyers-type estimates (cf. Lemma 4.5) we obtain that w_{η_i,ξ_i} is bounded in $(L^{p+\alpha}(Q))^d$, $\alpha > 0$. Thus using Lemma 4.3 we have that the second term for each i converges to $\int_{Q_i} \nu(|u - \eta_i|)^q(1 + \langle |w_{\eta_i,\xi_i}|^p\rangle)dxdt$, which is not greater than $\int_{Q_i} \nu(|u - \eta_i|)^q(1 + |\eta_i|^p + |\xi_i|^p)dxdt$. Summing this over all i we get $\int_Q \nu(|u - M_h u|)^q(1 + |M_h u|^p + |M_h Du|^p)dxdt$. Thus (4.7) is not greater than

$$\int_Q \nu(|u - M_h u|)^q(1 + |M_h u|^p + |M_h Du|^p)dxdt + C\delta_1 \int_Q |\mathcal{P}_\epsilon - D_x u_\epsilon|^p dxdt.$$

Combining all the estimates for I_1 and I_2 (with an appropriate δ_1 in (4.7)) we have

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_Q |\mathcal{P}_\epsilon - D_x u_\epsilon|^p dxdt \\
&\le C\Bigg(\int_Q (a^*(M_h u, M_h D_x u), M_h D_x u)dxdt - \int_Q (a^*(M_h u, M_h D_x u), D_x u)dxdt \\
&\qquad - \int_Q (a^*(u, D_x u), M_h D_x u)dxdt + \int_Q (a^*(u, D_x u), D_x u)dxdt \\
&\qquad + C\int_Q \nu(|u - M_h u|)^q(1 + |M_h u|^p + |M_h D_x u|^p)dxdt\Bigg).
\end{aligned} \tag{4.8}$$

Next, it is not difficult to show that the right-hand side of (4.8) approaches to zero as $h \to 0$. For this reason we use

$$\begin{aligned}
&\int_Q (a^*(M_h u, M_h D_x u), M_h D_x u)dxdt - \int_Q (a^*(M_h u, M_h D_x u), D_x u)dxdt \\
&\qquad - \int_Q (a^*(u, D_x u), M_h D_x u)dxdt + \int_Q (a^*(u, D_x u), D_x u)dxdt \\
&= \int_Q (a^*(u, D_x u) - a^*(M_h u, M_h D_x u), D_x u - M_h D_x u)dxdt
\end{aligned}$$

and write the right-hand side of (4.8) as

$$\begin{aligned}
&\int_Q (a^*(u, D_x u) - a^*(M_h u, M_h D_x u), D_x u - M_h D_x u)dxdt \\
&+ \int_Q \nu(|u - M_h u|)^q(1 + |M_h u|^p + |M_h D_x u|^p)dxdt.
\end{aligned} \tag{4.9}$$

Next, using the estimate $|a^*(\eta_1, \xi_1) - a^*(\eta_2, \xi_2)| \le C(1 + |\eta_1|^{p-1} + |\eta_2|^{p-1} + |\xi_1|^{p-1} + |\xi_2|^{p-1})\nu(|\eta_1 - \eta_2|) + C(1 + |\eta_1|^{p-1-\tilde{s}} + |\eta_2|^{p-1-\tilde{s}} + |\xi_1|^{p-1-\tilde{s}} + |\xi_2|^{p-1-\tilde{s}})|\xi_1 - \xi_2|^{\tilde{s}}$ (see [19]) we obtain that the right-hand side of (4.9) converges to zero as $h \to 0$. Indeed, for the first term we have

$$\begin{aligned}
&\int_Q (a^*(u, D_x u) - a^*(M_h u, M_h D_x u), D_x u - M_h D_x u)dxdt \\
&\le C\int_Q (1 + |u|^{p-1} + |D_x u|^{p-1} + |M_h u|^{p-1} + |M_h D_x u|^{p-1})\nu(|u - M_h u|)|D_x u - M_h D_x u|dxdt \\
&\quad + C\int_Q (1 + |u|^{p-1-\tilde{s}} + |D_x u|^{p-1-\tilde{s}} + |M_h u|^{p-1-\tilde{s}} + |M_h D_x u|^{p-1-\tilde{s}})|D_x u - M_h D_x u|^{\tilde{s}} dxdt.
\end{aligned}$$

Using the Holder inequality it can be easily shown that the second term converges to zero as $h \to 0$. Using Lemma 4.3 we easily obtain that the first term also converges to zero since $M_h D_x u$ is compact in $(L^p(Q))^d$. Similarly, using Lemma 4.3 and the fact that $M_h D_x u$ is compact in $(L^p(Q))^d$ we obtain that the second term on the right-hand side of (4.9) converges to zero.

Next, we use the corrector results and show that our numerical homogenization solution approximates $D_x u_\epsilon$ in the L^p-norm.

THEOREM 4.8.

$$\lim_{h\to 0}\lim_{\epsilon\to 0}\|D_x(u_{\epsilon,h}-u_\epsilon)\|_{p,Q}=0,$$

where u_ϵ is a solution of (2.2) *and $u_{\epsilon,h}=E^{MsFEM}u_h$ is defined by* (3.8) *(or* (3.4) *in each K).*

Proof. Because of Theorem 4.1 we need to show only that

$$\lim_{h\to 0}\lim_{\epsilon\to 0}\|D_x u_{\epsilon,h}-P(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,M_h u,M_h D_x u)\|_{p,Q}=0.$$

Note that

$$\lim_{\epsilon\to 0}\|D_x u_{\epsilon,h}-P(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,M_h u_h,M_h D_x u_h)\|_{p,K\times[t_n,t_{n+1}]}=0. \tag{4.10}$$

Equation (4.10) is due to the fact that $D_t u_h-\mathrm{div}(a^*(\eta^{u_h},D_x u_h))=0$ in $K\times[t_n,t_{n+1}]$; i.e., the homogenized solution for $u_{\epsilon,h}$ is u_h. Consequently,

$$\lim_{\epsilon\to 0}\|D_x u_{\epsilon,h}-P(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,M_h u_h,M_h D_x u_h)\|_{p,Q}=0.$$

It remains to show that

$$\lim_{h\to 0}\lim_{\epsilon\to 0}\|P(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,M_h u_h,M_h D_x u_h)-P(T(x/\epsilon^\beta,t/\epsilon^\alpha)\omega,M_h u,M_h D_x u)\|_{p,Q}=0.$$

To show the latter we need an estimate for $\int_\Omega |P(\omega,\eta_1,\xi_1)-P(\omega,\eta_2,\xi_2)|^p d\mu(\omega)$. Denote $P_1=P(\omega,\eta_1,\xi_1)$ and $P_2=P(\omega,\eta_2,\xi_2)$. Then

$$\begin{aligned}
C\int_\Omega |P_1-P_2|^p d\mu(\omega) &\le \int_\Omega (a(\omega,\eta_1,P_1)-a(\omega,\eta_1,P_2),P_1-P_2)d\mu(\omega)\\
&= \int_\Omega (a(\omega,\eta_1,P_1)-a(\omega,\eta_2,P_2),P_1-P_2)d\mu(\omega)\\
&\quad+\int_\Omega (a(\omega,\eta_2,P_2)-a(\omega,\eta_1,P_2),P_1-P_2))d\mu(\omega)\\
&\le \int_\Omega (a(\omega,\eta_1,P_1)-a(\omega,\eta_2,P_2),\xi_1-\xi_2)d\mu(\omega)\\
&\quad+\int_\Omega (a(\omega,\eta_2,P_2)-a(\omega,\eta_1,P_2),P_1-P_2))d\mu(\omega).
\end{aligned}$$

From here we can easily obtain

$$\begin{aligned}
\int_\Omega |P_1-P_2|^p d\mu(\omega) &\le C(a^*(\eta_1,\xi_1)-a^*(\eta_2,\xi_2),\xi_1-\xi_2)\\
&\quad+C\int_\Omega (1+|\eta_1|^p+|\eta_2|^p+|P_2|^p)\nu(|\eta_1-\eta_2)d\mu(\omega).
\end{aligned} \tag{4.11}$$

Thus

(4.12)

$$\lim_{h\to 0}\lim_{\epsilon\to 0}\|P(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega, M_h u_h, M_h D_x u_h) - P(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega, M_h u, M_h D_x u)\|_{p,Q}$$
$$\leq \lim_{h\to 0}\int_Q (a^*(M_h u_h, M_h D_x u_h) - a^*(M_h u, M_h D_x u), M_h D_x u_h - M_h D_x u) dxdt$$
$$+ C\lim_{h\to 0}\int_Q (1 + |M_h u_h| + |M_h u| + |M_h D_x u|)^p \nu(|M_h u_h - M_h u|) dxdt.$$

Similar to the analysis of the right-hand side of (4.8) it can be easily verified that the right-hand side of (4.12) converges to zero. □

5. Numerical examples. Consider the following convection-diffusion equation in two dimensions:

$$D_t u_\epsilon - \frac{1}{\epsilon} v(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega)\cdot D_x F(u_\epsilon) - d\Delta_{xx} u_\epsilon = f, \tag{5.1}$$

where $\operatorname{div}_x v = 0$. Assuming that homogeneous stream function $\mathcal{H}(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega)$

$$\mathcal{H} = \begin{pmatrix} 0 & H(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega) \\ -H((x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega) & 0 \end{pmatrix}$$

exists such that $\operatorname{div}_x \mathcal{H} = v$ we obtain

$$D_t u_\epsilon + \operatorname{div}_x(-d\delta_{ij} D_x u_\epsilon + \mathcal{H}(T(x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega) D_x F(u_\epsilon)) = f.$$

The latter is equivalent to

$$D_t u_\epsilon - \operatorname{div}_x(a((x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega, u_\epsilon) D_x u_\epsilon) = f,$$

where

$$a = \begin{pmatrix} -d & H((x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega)F'(u) \\ -H((x/\epsilon^{\beta},t/\epsilon^{\alpha})\omega)F'(u) & -d \end{pmatrix}.$$

We assume that a satisfies the assumptions imposed in previous sections.

Next, we apply the homogenization theorem presented earlier to this example and consider the case $\alpha > 0$, $\beta > 0$. From homogenization theory [7] it follows that u_ϵ converges to u, which satisfies

$$D_t u = \operatorname{div}_x(a^*(u) D_x u),$$

where $a^*_{ij}(\eta) = \langle a(\omega,\eta)(\xi + \partial w_\eta)\rangle$ and $w_\eta = \partial N_\eta$. Here N_η is the solution of an auxiliary problem whose formulation depends on the ratio between α and β. In all the cases, w_η is a linear function with respect to ξ; thus it can be represented as $w^i_\eta = W^{ij}_\eta \xi_i$. Substituting this expression for w_η in the homogenized formula we have

$$a^*_{ij}(\eta) = -d\delta_{ij} + \langle H_{ik} F'(\eta) W^{kj}_\eta \rangle.$$

The second term on the right-hand side, $a^c_{ij} = \langle H_{ik} F'(\eta) W^{kj}_\eta\rangle$, represents enhanced diffusion due to nonlinear heterogeneous convection.

We consider a simple application of our approach in the following way. At each time step the average of u_ϵ, $\frac{1}{Q_0}\int_{Q_0} u_\epsilon dx$, is computed. Based on this average we solve

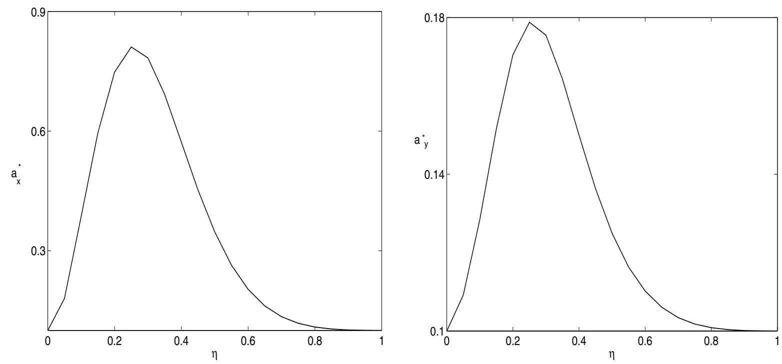

FIG. 5.1. *Enhanced diffusion for horizontal and vertical directions,* $H = 0.5(\sin(t/\epsilon) + \sin(t\sqrt(2)/\epsilon))(\sin(2\pi y/\epsilon) + \sin(2\sqrt(2)\pi y/\epsilon))$.

the local problem (3.4) and compute the enhanced diffusion which is further used to solve the global problem. Further, we will compare our results with the average of the fine scale results. The results where the enhanced diffusion is neglected will also be presented. These tests will demonstrate the importance of the enhanced diffusion. In all the examples below $x = (x_1, x_2)$, and we denote $x = x_1$ and $y = x_2$. All the computations are performed using the standard finite element method on triangular meshes.

First, we present the total diffusivity as a function of η (i.e., average of the solution) for various heterogeneous velocity fields given by the stream functions $H = 0.5(\sin(t/\epsilon^\alpha) + \sin(t\sqrt(2)/\epsilon^\alpha))(\sin(2\pi y/\epsilon) + \sin(2\sqrt(2)\pi y/\epsilon))$. We take $\epsilon = 0.1$ and $d = 0.1$ (molecular diffusion) and vary α, $\alpha = 1, 2$. The flux function is chosen to be the Buckley–Leverett function $F(u) = u^2/(u^2 + 0.2(1-u)^2))$ motivated by porous media flows. The approximation of the enhanced diffusion is computed by solving (5.1) in a unit square.

Next, a set of numerical examples are designed to compare the solutions of the original (fine scale equation) with the solutions of the equations obtained using numerical homogenization with and without enhanced diffusion. We consider (5.1) in a unit square domain with the boundary and initial conditions as follows. $u_\epsilon = 1$ at the inlet ($x_1 = 0$), $u_\epsilon = 0$ at the outlet ($x_1 = 1$), there are no flow boundary conditions on the lateral sides $x_2 = 0$ and $x_2 = 1$, and, initially, $u_\epsilon = 0$; thus flow from left to right will occur.

Our first set of numerical tests use layered flow $H = 0.5(\sin(t/\epsilon)+\sin(t\sqrt(2)/\epsilon)) \times (\sin(2\pi y/\epsilon)+\sin(2\sqrt(2)\pi y/\epsilon))$, where $\epsilon = 0.1$. In Figure 5.1 we plot the total diffusion. Note "the molecular diffusion" is $d = 0.1$. The left plot of this figure represents the total diffusivity in the horizontal direction (along the layers), and the right plot represents the total diffusivity in the vertical direction. Clearly, the diffusion enhances somewhat dramatically in the horizontal direction, that is, along the convection. As we see for $\eta \approx 0.4$ there is an 8 fold increase in the diffusion. Moreover, since $F'(0) = F'(1) = 0$ there is no enhancement if $\eta = 0$ or $\eta = 1$ (this corresponds to pure phases). In Figure 5.2 we compare the averaged solution of the original equation with the one computed using our approach. The averages are taken differently on the left and the right figures. On the left figure of Figure 5.2 we have plotted the average solution as a function of time, i.e., $\frac{1}{Q_0}\int_{Q_0} u(x,t)dx$. Here and below the solid line designates the

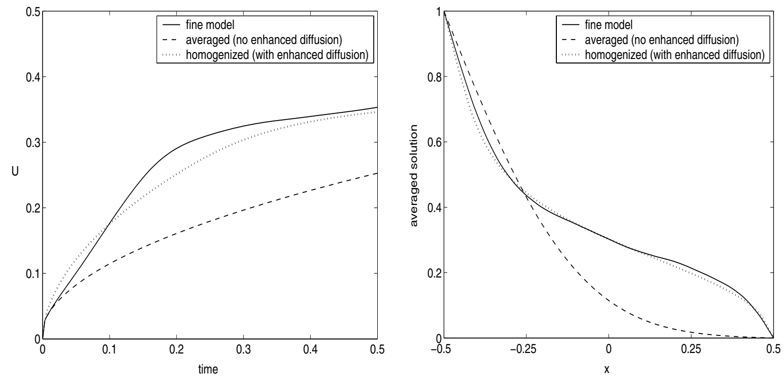

FIG. 5.2. *Left figure: The solutions are averaged over the whole spatial domain. Right figure: The solutions are averaged in the vertical direction (across heterogeneities).* $H = 0.5(\sin(t/\epsilon) + \sin(t\sqrt(2)/\epsilon))(\sin(2\pi y/\epsilon) + \sin(2\sqrt(2)\pi y/\epsilon))$.

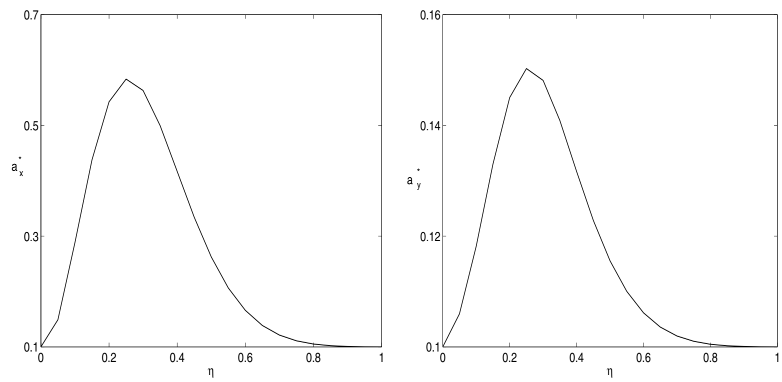

FIG. 5.3. *Enhanced diffusion for horizontal and vertical directions,* $H = 0.5(\sin(t/\epsilon^2) + \sin(t\sqrt(2)/\epsilon^2))(\sin(2\pi y/\epsilon) + \sin(2\sqrt(2)\pi y/\epsilon))$.

fine scale model corresponding to the original equation, and the dotted line designates the solution obtained using our numerical homogenization technique. To illustrate the importance of the enhanced diffusion we also include the solution where the enhanced diffusion is neglected (i.e., the solution of $u_t = d\Delta u$). This solution is designated with the dashed line. On the right figure of Figure 5.2 we have plotted the solution averaged across the heterogeneities (vertical direction) at the time instant $t = 0.5$. Both figures clearly demonstrate the importance of the enhanced diffusion and the robustness of our approach.

For the next set of numerical tests we change only the time scale by assuming $\alpha = 2$. Thus, $H = 0.5(\sin(t/\epsilon^\alpha) + \sin(t\sqrt(2)/\epsilon^\alpha))(\sin(2\pi y/\epsilon) + \sin(2\sqrt(2)\pi y/\epsilon))$, where $\epsilon = 0.1$. In Figure 5.3 we plot the enhanced diffusion. As in the previous case we observe somewhat large enhancement in the horizontal direction. In Figure 5.4 we compare the averaged solutions as we did for the previous example. The results indicate the importance of enhanced diffusion as well as the robustness of our approach.

Next, we present an example where the stream function is a realization of the random field with Gaussian distribution with respect to the spatial variables, $H =$

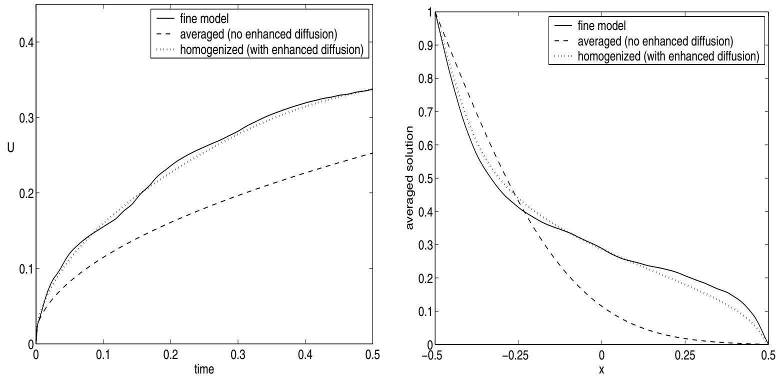

FIG. 5.4. *Left figure: The solutions are averaged over the whole spatial domain. Right figure: The solutions are averaged in the vertical direction (across heterogeneities).* $H = 0.5(\sin(t/\epsilon^2) + \sin(t\sqrt{(2)}/\epsilon^2))(\sin(2\pi y/\epsilon) + \sin(2\sqrt{(2)}\pi y/\epsilon))$.

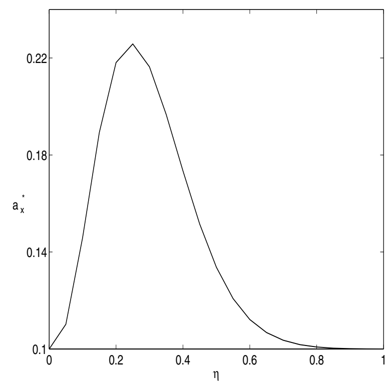

FIG. 5.5. *Enhanced diffusion for horizontal and vertical directions,* $H = 0.5((\sin(t/\epsilon^2) + \sin(t\sqrt{(2)}/\epsilon^2))k(x, y)$, *where* k *is a realization of the random Gaussian field that has correlation length* $l_x = l_y = 0.1$*, mean zero, and variance* 0.5.

$0.5((\sin(t/\epsilon^2)+\sin(t\sqrt{(2)}/\epsilon^2))k(x, y)$, where k is a realization of the random Gaussian field that has correlation length $l_x = l_y = 0.1$, mean zero, and variance 0.5. To generate a realization of the random field with prescribed correlation lengths we use GSLIB [4]. $d = 0.1$ and $F(u) = u^2/(u^2 + 0.2(1-u)^2))$ are used in (5.1). In Figure 5.5 we plot the total diffusivity. As we can see, the enhancement of the diffusion can be up to 2.3 times. Since the stream field is isotropic the total diffusivity in the vertical direction is the same. In Figure 5.6 we compare the averaged solution of the original equation with the one computed using our approach. The averages are taken differently on the left and the right figures as it is done previously. We have observed similar accuracy for other realizations of this random field. These results again demonstrate the importance of enhanced diffusion and the robustness of our approach.

Finally, we consider an application of the numerical homogenization procedure to Richards equation, $D_t u_\epsilon = \operatorname{div}(a_\epsilon(x, u_\epsilon) D_x u_\epsilon)$, where $a_\epsilon(x, \eta) = k_\epsilon(x)/(1+\eta)^{\alpha_\epsilon(x)}$. $k_\epsilon(x) = \exp(\beta_\epsilon(x))$ is chosen such that $\beta_\epsilon(x)$ is a realization of a random field with the exponential variogram [4], the correlation lengths $l_x = 0.3$, $l_y = 0.02$, and the variance

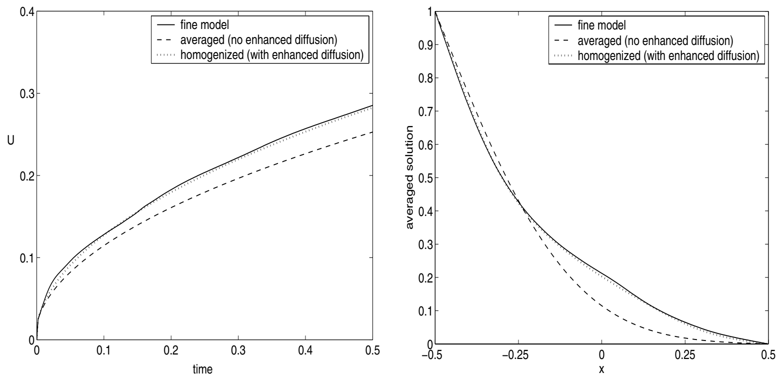

FIG. 5.6. *Left figure: The solutions are averaged over the whole spatial domain. Right figure: The solutions are averaged in the vertical direction.* $H = 0.5(\sin(t/\epsilon^2)+\sin(t\sqrt(2)/\epsilon^2))k(x,y)$*, where k is a realization of the random Gaussian field that has correlation length* $l_x = l_y = 0.1$*, mean zero, and variance* 0.5.

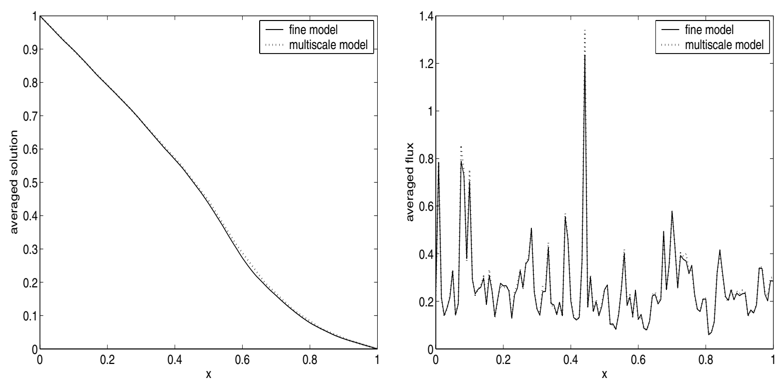

FIG. 5.7. *Left figure: The solutions are averaged in the vertical direction. Right figure: The fluxes are averaged in the vertical direction.*

$\sigma = 1$. $\alpha_\epsilon(x)$ is chosen such that $\alpha_\epsilon(x) = k_\epsilon(x) + \text{const}$ with the spatial average of 2. In Figure 5.7 we compare the solutions (u_ϵ) and the fluxes $(-a_\epsilon(x, u_\epsilon)D_x u_\epsilon)$ corresponding to this equation with boundary and initial conditions given as previously at the time $t = 2$. The solid line designates the fine scale model results computed on the 120×120 grid, and the dotted line designates the coarse scale results computed using the numerical homogenization procedure on the 12×12 coarse grid. Since a_ϵ is independent of t the local problems are chosen to be elliptic, as we discussed before. These results demonstrate the robustness of our approach for anistropic fields where h and ϵ are nearly the same. Currently, we are studying the applications of the oversampling technique to the numerical homogenization procedure.

6. Concluding remarks. In the paper we proposed and studied the convergence of the numerical homogenization scheme for nonlinear parabolic equations. The convergence of the scheme is obtained in the limit $\lim_{h\to 0}\lim_{\epsilon\to 0}$ (see Theorem 3.1). The proof of Theorem 3.1 can be extended to the case of general heterogeneities that uses G-convergence theory. In fact the proof holds when a^* and a_0^* do not depend on spatial and temporal variables. In the periodic case the convergence of the numerical homogenization method can be shown in the limit $\epsilon/h \to 0$ (and $\epsilon \to 0$ if an exact period is used for the local problem). The case of general heterogeneities may involve all possible scales $\alpha(\epsilon)$ such that $\alpha(\epsilon) \to 0$ as $\epsilon \to 0$, and, consequently, our convergence result in Theorem 3.1 cannot be improved. We believe for the homogeneous random case that one can show the convergence of the numerical homogenization procedure in the limit $\epsilon/h \to 0$, and this is currently under investigation.

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