GL-TR-90-0095

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17 April 1990 | Reprint

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PE 62714E
PR 8A10 TA DA WU AH

Contract F19628-88-K-0042

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GL-TR-90-0095

Contract Manager: James Lewkowicz/LWH

Submitted to SIAM Journal of Numerical Analysis

Approved for public release; distribution unlimited

In this paper, a new transformation referred to as the $G_n^{(m)}$-transform is introduced for the purpose of increasing the rate of convergence of a sequence to its limit. In particular, if the sequence is a sequence of partial integrals, $F(x)$, then $G_n^{(m)}[F(x)]$ is shown to converge super fast to $F(\infty)$ under very general conditions. The $G_n^{(m)}$-transform is shown to be closely related to Levin and Sidi's D-transform and the B_n-transform introduced by Gray, et al. Several examples are given.

Convergence acceleration **Tail probability**
$G_n^{(m)}$-transform

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Unclassified	**Unclassified**	**Unclassified**	**SAR**

A General Method for Approximating Improper Integrals

by

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Abstract. In this paper a new transformation referred to as the $G_n^{(m)}$-transform is introduced for the purpose of increasing the rate of convergence of a sequence to its limit. In particular if the sequence is a sequence of partial integrals, $F(x)$, then $G_n^{(m)}[F(x)]$ is shown to converge super fast to $F(\infty)$ under very general conditions. The $G_n^{(m)}$-transform is shown to be closely related to Levin and Sidi's D-transform and the B_n-transform introduced by Gray, et al. Several examples are given.

Key words. convergence acceleration, $G_n^{(m)}$-transform, tail probability

AMS(MOS) subject classifications. 65B05, 65D30, 62E99

1. Introduction

In [2] Gray, Atchinson and McWilliams introduced the G_m-transform. They showed the transform to be of value in evaluating improper integrals, especially in the case of integrals in which the integrand decayed like an exponential function.

The results of Gray, et al were significantly extended by Levin and Sidi [6] to a much more general case, which no longer required the exponential behavior of the integrand in the limit to be

This research was partially supported by DARPA/AFGL Contract No. F19628-88-K-0042.

effective. This was accomplished by introducing what they referred to as the D-transformation. Given a sequence of values of a function, the D-transformation is a very effective tool for increasing the rate of convergence of the sequence to its limit in most cases. In fact, to date, the D-transformation (or its discrete counterpart, the d-transformation) is probably the most effective general method for increasing the rate of convergence of a sequence available. For some cases its effectiveness has been further improved recently by a slight extension by Gray and Wang [4].

Since the original paper by Levin and Sidi a number of papers have been written on the Levin and Sidi transformation and Ford and Sidi [1] have shown that the generalized Richardson extrapolation procedure (GREP) can be used to calculate the D- and d-transformations. One aspect of the D-transformation that has not been investigated very closely is the behavior of the transformation as the step size between an equally spaced sequence of points, where the functional values are taken, approaches zero. In the particular case where the sequence is a sequence of improper integral approximations and the integrand is given at equally spaced points, i.e., at $x_0 + kh$, h fixed and $k = 0$, 1, 2, . . . , N, this phenomonon was investigated, [2], for the G_m-transform. In particular it was demonstrated that in the case of the G_m-transformation, it is often the case that the approximation improved as $h \to 0$. Investigating this property then led to a new transformation referred to as the B_m-transformation [3]. To be specific let

$$F(x) = \int_a^x f(u)\, du \quad , \tag{1}$$

where $F(x) \to S < \infty$ as $x \to \infty$. Then $\lim_{h \to 0} G_m \left[F(x+kh)\right] = B_m[(F(x)]$. The B_m-transformation was then shown to be of value in its own right, particularly in the area of producing functional approximations to tail probabilities (see [3]). The fundamental difference between G_m and B_m-transformations is that the former requires values of the function being transformed at several points while the B_m requires values of the function and its derivatives at a single point. In the case of tail probabilities this distinction allows one to replace the problem of integration completely by differentiation.

In this paper we introduce a new transformation, referred to as the $G_n^{(m)}$-transformation, which is in fact the limit of the Levin and Sidi D-transformation when applied to a function evaluated at equally spaced points. Thus the $G_n^{(m)}$-transformation can be viewed as the extension of the B_m-transformation analogous to the Levin-Sidi generalization of the G_m-transformation. In fact, as we shall see, $B_m = G_1^{(m)}$. The Levin-Sidi generalization has been clearly shown to be a powerful tool for evaluating improper integrals when the integrand is given at a sequence of points; it should be expected, therefore that the $G_n^{(m)}$-transformation will be useful when the integrand and its derivations are given at a single point. This is shown to be the cae in both theory and practice. In particular, under very general conditions, $G_n^{(m)}[F(x)]$ is shown to converge to S faster than F(x) at a rate proportional to x^{-n}. Several examples are given.

2. Background and Definitions

Let F(x) and S be defined as in equation (1) and let

$$\epsilon(x) = S - F(x) = \int_x^\infty f(u)\,du. \tag{2}$$

Further let

$$U_k(x) = x^{\ell_k} \sum_{i=1}^{\infty} \alpha_{k,i}\, x^{-i+1}\ , \tag{3}$$

where $\alpha_{k,1} \neq 0$ and ℓ_k is an integer with $\ell_k \leq k$. Also suppose m is the smallest integer such that $\epsilon(x)$ satisfies the differential equation

$$U_m(x)\, y^{(m)} + U_{m-1}(x)\, y^{(m-1)} + \ldots + U_1(x)\, y' - y = 0, \tag{4}$$

for some set of U_k's. For motivational purposes suppose

$$\alpha_{1,i} = \alpha_{2,i} \cdots = \alpha_{m,i} = 0$$

when $i > n$. Then

$$\sum_{i=1}^{n} \alpha_{m,i} \; x^{\ell_m - i+1} \, \epsilon^{(m)}(x) + \ldots + \sum_{i=1}^{n} \alpha_{1,i} \, x^{\ell_1 - i+1} \, \epsilon'(x) - \epsilon(x) = 0 \; .$$

Thus

$$F(x) = S + \sum_{i=1}^{n} \alpha_{1,i} \, x^{\ell_1 - i+1} \, f(x) + \ldots + \sum_{i=1}^{n} \alpha_{m,i} \, x^{\ell_m - i+1} \, f^{(m-1)}(x) \tag{5}$$

and

$$f^{(k)}(x) = \sum_{i=1}^{n} \alpha_{1,i} \left\{ x^{\ell_1 - i+1} \, f(x) \right\}^{(k+1)} + \ldots + \sum_{i=1}^{n} \alpha_{m,i} \left\{ x^{\ell_m - i+1} \, f^{(m-1)}(x) \right\}^{(k+1)} , \tag{6}$$

$k = 0, 1, \ldots, N-1, \quad N = mn$. Now let

$$H_{k+1}(z_j \, ; \, a_{i,j}) = \begin{vmatrix} z_1 & z_2 & \cdots & z_{k+1} \\ a_{11} & a_{12} & \cdots & a_{1,k+1} \\ \vdots & \vdots & & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k+1} \end{vmatrix} \tag{7}$$

and define

$$G\left[F(x), f(x), \ldots, f^{(N-1)}(x) \, ; \, a_{i,j}(x) \right] = \frac{H_{N+1}(z_j \, ; \, a_{i,j})}{H_{N+1}(c_j \, ; \, a_{i,j})} \; , \tag{8}$$

where $c_1 = 1$, $c_2 = c_3 = \ldots = c_{N+1} = 0$, $z_j = F^{(j-1)}(x)$, and for $j = 1, 2, \ldots, mn+1$,

$$a_{i,j}(x) = \begin{cases} \left[x^{\ell_1 - i+1} f(x)\right]^{(j-1)}, & i = 1, \ldots, n, \\ \left[x^{\ell_2 - i+n+1} f'(x)\right]^{(j-1)}, & i = n+1, \ldots, 2n, \\ \left[x^{\ell_m - i+(m-1)n+1} f^{(m-1)}(x)\right]^{(j-1)}, & i = (m-1)n+1, \ldots, mn\ . \end{cases}$$

From equations (5) and (6) it follows from Kramer's rule, if $H_{N+1}(c_j\ ;\ a_{i,j}) \neq 0$, that

$$G\left[F(x), f(x), \ldots, f^{(N-1)}(x);\ a_{i,j}(x)\right] = S = \int_a^\infty f(u)du \quad . \tag{9}$$

Now suppose f(x) satisfies (4), i.e.,

$$f(x) = \sum_{k=1}^{m} U_k(x)\, f^{(k)}(x)\ , \tag{10}$$

where $U_k(x)$ is defined by (3) and we no longer assume that the series in (3) is of finite length but rather that it converges for all $x \geq a$. Suppose further that

$$\lim_{x \to \infty} U_k^{(i-1)}(x)\, f^{(k-i)}(x) = 0, \tag{11}$$

for k=i, i+1, . . . m, i=1, . . . , m. Then Levin and Sidi [6] have shown that there exists a set of $\alpha^*_{k,i}$ such that

$$U^*_m(x)\, \epsilon^{(m)}(x) + U^*_{m-1}(x)\, \epsilon^{(m-1)}(x) + \ldots + U^*_1(x)\, \epsilon'(x) - \epsilon(x) = r_n(x)\ , \tag{12}$$

where

$$U_k^*(x) = \sum_{i=1}^{n} \alpha_{k,i}^* \, x^{\ell_k - i + 1} \tag{13}$$

and

$$r_n(x) = \sum_{k=1}^{m} f^{(k-1)}(x) \sum_{i=n+1}^{\infty} \alpha_{k,i}^* \, x^{\ell_k - i+1} = O\left(x^{-n} \, \epsilon(x)\right), \tag{14}$$

since $\epsilon(x) \sim \sum_{k=1}^{m} \alpha_{k,1}^* \, x^{\ell_k} f^{(k-1)}(x)$. The notation "$\sim$" implies that if $p(x) \sim q(x)$ then $p(x)/q(x) \to 1$, as $x \to \infty$. From (12) we have

$$F(x) = S + \sum_{k=1}^{m} U_k^*(x) \, f^{(k-1)}(x) + r_n(x), \tag{15}$$

and therefore for $j = 1, 2, \ldots, mn$, we have

$$f^{(j-1)}(x) = \sum_{i=1}^{n} \alpha_{1,i}^* \left(x^{\ell_1 - i+1} f(x)\right)^{(j)} + \ldots + \sum_{i=1}^{n} \alpha_{m,i}^* \left(x^{\ell_m - i+1} f^{(m-1)}(x)\right)^{(j)} + r_n^{(j)}(x)\,. \tag{16}$$

From these observations and equation (9) it is clear that as x and n become large $G[F(x), f(x), \ldots, f^{(mn-1)}(x); a_{ij}(x)]$ approaches S and one would think it would approach S more rapidly than F(x). This leads us to the following definition, which is essentially just an economy of notation over our previous remarks.

Definition 1. Suppose for $x \geq a$, that $f^{(mn+m-1)}(x)$ exists, and $H_{mn+1}\left(c_j\,,\, a_{ij}(x)\right) \neq 0$, then we define the $G_n^{(m)}$-transformation by

$$G_n^{(m)}\left[F(x); a_{i,j}\right] = G\left[F(x), f(x), \ldots, f^{(mn-1)}(x); a_{i,j}(x)\right]. \tag{17}$$

The $G_n^{(m)}$-transform defined by (17) is in fact the limiting case of the D-transformation which we spoke of in the introduction. In fact, had we generated the system of equations defined by equation (15) by incrementing x, i.e., using F(x+kh), instead of differentiating, we would have been led to the

D-transformation of Levin and Sidi. This was shown in [6]. To obtain $G_n^{(m)}$ directly as a limit of the D-transformation one begins with the D-transform and takes the limit as $h \to 0$ after several row and column manipulations. The procedure is exactly analogous to the method used in [2] to obtain the B_m-transform as the limit of the G_m-transform. It follows by definition that $G_1^{(m)} \equiv B_m$.

3. Major Results

In this section we prove rigorously the fundamental properties of the $G_n^{(m)}$-transformation suggested by our previous discussion. The first theorem needs no proof and is really just a restatement of our previous remark but we choose to state it formally for emphasis.

Theorem 1. If $\epsilon(x)$ satisfies equation (4) with $\alpha_{1,i} = \alpha_{2i} = \ldots = \alpha_{m,i} = 0$ when $i>n$, and $H_{mn+1}(c_j; a_{i,j}) \neq 0$, then for $x \geq a$

$$G_n^{(m)}\left[F(x); a_{i,j}\right] \equiv \int_a^\infty f(u)du \quad . \tag{18}$$

The value of a transformation such as $G_n^{(m)}$ does not lie in a theorem such as Theorem 1, even though such theorems are of interest. However it is the next theorem which justifies our interest in the $G_n^{(m)}$-transformation.

Theorem 2. Suppose that $G_n^{(m)}\left[F(x); a_{i,j}\right]$ is defined, and that for all $x \geq a$, $f(x)$ satisfies equation (4), and that $m > 0$ is the smallest integer for which this is true. Suppose further that $F(x) \to S < \infty$ as $x \to \infty$, and that equation (11) holds. Then

$$\frac{S - G_n^{(m)}\left[F(x); a_{i,j}\right]}{S - F(x)} = O\left(x^{-n}\right) . \tag{19}$$

Note that not only does (19) imply that $G_n^{(m)}[F(x); a_{i,j}]$ converge to S, but that it does so more rapidly at a rate proportional to x^{-n} as $x \to \infty$.

Proof. Due to the length and detail required to prove the results we shall divide the proof into two parts. First, we will show the results in some detail for m=1. Following that we will only sketch the proof for m > 1.

Case I: m = 1.

We first show that the denominator in (8)

$$H_{n+1}\left(c_j; a_{i,j}\right) \sim \frac{A_n f^n(x)}{x^{(n-1)n}}, \tag{20}$$

for some $A_n \neq 0$. Let $P_n(x) = H_{n+1}(c_j; a_{i,j})$ and $g(x) = x^{\ell_1} f(x)$, denoted simply by g. Then

$$P_n(x) = \begin{vmatrix} a_{12}(x) & a_{13}(x) & \cdots & a_{1,n+1}(x) \\ a_{22}(x) & a_{23}(x) & \cdots & a_{2,n+1}(x) \\ \vdots & \vdots & & \vdots \\ a_{n,2}(x) & a_{n,3}(x) & \cdots & a_{n,n+1}(x) \end{vmatrix}$$

$$
= \begin{vmatrix}
g' & g'' & \cdots & g^{(n)} \\[2ex]
\frac{1}{x}g' + \left(\frac{1}{x}\right)'g & \frac{1}{x}g'' + 2\left(\frac{1}{x}\right)'g' + \left(\frac{1}{x}\right)''g & \cdots & \sum_{j=0}^{n}\binom{n}{j}\left(\frac{1}{x}\right)^{(j)}g^{(n-j)} \\[2ex]
\vdots & \vdots & & \vdots \\[2ex]
\frac{1}{x^{n-1}}g' + \left(\frac{1}{x^{n-1}}\right)'g & \frac{1}{x^{n-1}}g'' + 2\left(\frac{1}{x^{n-1}}\right)''g' + \left(\frac{1}{x^{n-1}}\right)''g & \cdots & \sum_{j=0}^{n}\binom{n}{j}\left(\frac{1}{x^{n-1}}\right)^{(j)}g^{(n-j)}
\end{vmatrix} .
$$

By elementary row and column operations we obtain

$$
P_n(x) = g^{n-1} \begin{vmatrix}
h_1 & h_2 & \cdots & h_n \\[1ex]
\left(\frac{1}{x}\right)' & \left(\frac{1}{x}\right)'' & \cdots & \left(\frac{1}{x}\right)^{(n)} \\[1ex]
\vdots & \vdots & & \vdots \\[1ex]
\left(\frac{1}{x^{n-1}}\right)' & \left(\frac{1}{x^{n-1}}\right)'' & \cdots & \left(\frac{1}{x^{n-1}}\right)^{(n)}
\end{vmatrix} , \tag{21}
$$

where

$$
\begin{aligned}
h_1 &= g', \\
h_2 &= g'' - 2g'h_1/g , \\
&\vdots \\
h_n &= g^{(n)} - \sum_{j=1}^{n-1}\binom{n}{j} g^{(n-j)} h_j/g.
\end{aligned}
$$

From (16) we note that $f(x) = \alpha_{11}^{*} x^{\ell_1} f'(x)$, which implies that

$$f^{(j)}(x) \sim d_j\, x^{-j\ell_1} f(x), \tag{22}$$

and

$$g^{(j)} \sim d_j'\, x^{-j\ell_1} g\,, \tag{23}$$

for some constants d_j and d_j', $j = 1, 2, \ldots$. Note that $d_1 = 1/\alpha_{11}^*$, where $\alpha_{11}^* \neq 0$ by assumption . It then follows that

$$h_j \sim c_j f(x)/x^{(j-1)\ell_1}\,, \tag{24}$$

for some constants c_j, $j = 1, 2, \ldots, n$.

(i). If $\ell_1 \leq 0$, then it is easily derived from (16) that

$$d_j = d_j' = d_1^j\,,\ j = 1, 2, \ldots .$$

Then

$$c_j = (-1)^j d_1^j\,, j = 1, \ldots, n\,.$$

Therefore

$$P_n(x) \sim g^{n-1} h_n \begin{vmatrix} \left(\frac{1}{x}\right)' & \cdots & \left(\frac{1}{x}\right)^{(n-1)} \\ \vdots & \cdots & \vdots \\ \left(\frac{1}{x^{n-1}}\right)' & \cdots & \left(\frac{1}{x^{n-1}}\right)^{(n-1)} \end{vmatrix}$$

$$= g^{n-1} h_n \frac{b_n}{x^{(n-1)n}} \sim g^{n-1} \frac{c_n f(x)}{x^{(n-1)\ell_1}} \frac{b_n}{x^{(n-1)n}} = A_n \frac{f^n(x)}{x^{(n-1)n}}\,, \tag{25}$$

where $A_n = c_n b_n$ and b_n is a constant. It is seen from elementary theory of differential equations that $b_n \neq 0$, which implies that $A_n \neq 0$.

(ii). Now we suppose that $\ell_1 = 1$. Then one can show that

$$d_j' = (d_1+1)\, d_1(d_1-1) \ldots (d_1-j+2),\ j = 1, 2, \ldots,$$

and thus

$$c_j = (-1)^{j-1}(d_1+1)\,(d_1+2) \ldots (d_1+j),\ j = 1, \ldots, n\,.$$

We then have

$$P_n(x) \sim g^{n-1} f(x) \begin{vmatrix} c_1 & \frac{c_2}{x} & \cdots & \frac{c_n}{x^{n-1}} \\ \left(\frac{1}{x}\right)' & \left(\frac{1}{x}\right)'' & \cdots & \left(\frac{1}{x}\right)^{(n)} \\ \vdots & \vdots & & \vdots \\ \left(\frac{1}{x^{n-1}}\right)' & \left(\frac{1}{x^{n-1}}\right)'' & \cdots & \left(\frac{1}{x^{n-1}}\right)^{(n)} \end{vmatrix}$$

$$= A_n\, g^{n-1} f(x) \frac{1}{x^{(n-1)(n+1)}} = A_n \frac{f^n(x)}{x^{(n-1)n}}\,,$$

where $A_n = b_n \prod_{i=1}^{n} (d_1-i+2)$ for some $b_n \neq 0$, and thus $A_n \neq 0$ since it is easily shown that $-1<\alpha_{11}^*<0$. We have therefore proved (20).

We now consider the numerator of $S-G_n^{(1)}[F(x);\, a_{i,j}]$. Denoting the numerator by $Q_n(x)$, we have

$$Q_n(x) = \begin{vmatrix} r_n(x) & r_n'(x) & \cdots & r_n^{(n)}(x) \\ a_{11}(x) & a_{12}(x) & \cdots & a_{1,n+1}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2,n+1}(x) \\ \vdots & \vdots & & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n+1}(x) \end{vmatrix}$$

$$
= \begin{vmatrix} v_0 & v_1 & \cdots & v_n \\ h_0 & h_1 & \cdots & h_n \\ 0 & \left(\frac{1}{x}\right)' g & \cdots & \left(\frac{1}{x}\right)^{(n)} g \\ \vdots & \vdots & & \vdots \\ 0 & \left(\frac{1}{x^{n-1}}\right)' g & \cdots & \left(\frac{1}{x^{n-1}}\right)^{(n)} g \end{vmatrix} ,
$$

where $h_0 = g$, $v_0 = r_n(x)$ and v_j's correspond to h_j's when g is replaced by $r_n(x)$. From (14) and (22) we know that

$$
r_n^{(j)} = O\left(x^{-n-j\ell_1}\,\epsilon(x)\right), \quad j = 1, \ldots, n .
$$

Thus, by (23),

$$
v_0 = r_n(x) = O\left(x^{-n}\epsilon(x)\right),
$$

$$
v_1 = r_n'(x) = O\left(x^{-n-\ell_1}\,\epsilon(x)\right),
$$

$$
v_2 = r_n''(x) - 2g'v_1/g = O\left(x^{-n-2\ell_1}\,\epsilon(x)\right) ,
$$

$$
\vdots
$$

$$
v_n = r_n^{(n)}(x) - \sum_{j=1}^{n-1}\binom{n}{j} g^{(n-j)}\, v_j/g = O\left(x^{-n-n\ell_1}\epsilon(x)\right).
$$

We then have

$$Q_n(x) = v_0 P_n(x) - v_1\, g^{n-1} \begin{vmatrix} h_0 & h_2 & \cdots & h_n \\ 0 & \left(\frac{1}{x}\right)'' & \cdots & \left(\frac{1}{x}\right)^{(n)} \\ \vdots & \vdots & & \vdots \\ 0 & \left(\frac{1}{x^{n-1}}\right)'' & \cdots & \left(\frac{1}{x^{n-1}}\right)^{(n)} \end{vmatrix} +$$

$$+ \ldots + (-1)^n v_n\, g^{n-1} \begin{vmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ 0 & \left(\frac{1}{x}\right)' & \cdots & \left(\frac{1}{x}\right)^{(n-1)} \\ \vdots & \vdots & & \vdots \\ 0 & \left(\frac{1}{x^{n-1}}\right)' & \cdots & \left(\frac{1}{x^{n-1}}\right)^{(n-1)} \end{vmatrix}$$

$$= O\Big(x^{-n}\, \epsilon(x)\Big)\, P_n(x) + O\left(v_1\, g^{n-1} h_0\, \frac{1}{x^{(n-1)n+n-1}}\right)$$

$$+ O\left(v_2\, g^{n-1}\, h_0\, \frac{1}{x^{(n-1)n+n-2}}\right) + \ldots + O\left(v_n\, g^{n-1} h_0\, \frac{1}{x^{(n-1)n}}\right)$$

$$= O\Big(x^{-n}\epsilon(x)\Big) P_n(x) + O\left(x^{-n+(n-1)(\ell_1-1)}\epsilon(x) P_n(x)\right)$$

$$+ O\left(x^{-n+(n-2)(\ell_1 - 1)} \epsilon(x) P_n(x)\right) + \ldots + O\left(x^{-n}\, \epsilon(x) P_n(x)\right)$$

$$= O\left(x^{-n} \epsilon(x)\right) P_n(x) \,. \tag{26}$$

Hence by (25) and (26), we obtain

$$S - G_n^{(1)}\left[F(x);\, a_{i,j}\right] = Q_n(x) \,/\, P_n(x) = O\left(x^{-n} \epsilon(x)\right)$$

and

$$\frac{S - G_n^{(1)}[F(x);\, a_{i,j}]}{S - F(x)} = O\left(x^{-n}\right), \quad \text{as } x \to \infty \,,$$

which completes the proof for m=1.

Case II: $m \geq 2$.

Because the proof is quite lengthy and its scheme is similar to Case I, we only sketch it. From (10), we can asymptotically express $f^{(m)}(x)$ in terms of $f(x)$, $f'(x)$, . . . , $f^{(m-1)}(x)$ as follows:

$$f^{(m)}(x) \sim \sum_{k=1}^{m-1} \beta_{k,0}\, x^{r_{k,0}}\, f^{(k)}(x) \,, \tag{27}$$

where $\beta_{k,0}$ are nonzero constants, $r_{k,0} = \ell_k - \ell_m$, $\ell_0 = 0$. Because the relative error of $f^{(m)}$ to the right hand side of (27) is a linear combination of negative powers, we have the following lemma whose proof is straightforward.

<u>Lemma 1</u>. For $q = 1, 2, \ldots$,

$$f^{(m+q)}(x) \sim \sum_{k=0}^{m-1} \beta_{k,q}\, x^{r_{k,q}} f^{(k)}(x) \,, \tag{28}$$

where $\beta_{k,q}$ and $r_{k,q}$ are constants, and $r_{k,q} \geq r_{k,q-1} - 1$.

From equation (2), $f^{(m+q)}(x)$ can be asymptotically expressed in terms of f(x), f′(x), ..., $f^{(m-1)}(x)$. Moreover, one can show that there exists an integer $i_0 \leq 1$ such that $f^{(q+1)}(x)$ and $x^{i_0} f^{(q)}(x)$ have the same order when $x \rightarrow \infty$, for q = 1, 2, ... To be specific, suppose m=2. Let $g_k(x) = x^{\ell_k} f^{(k-1)}(x)$, and denote it by g_k, k=1, 2 . Then $g_k^{(j+1)}$ and $x^{i_0} g_k^{(j)}$, have the same order, for j = 1, 2, ... Using the same notation as Case I, we have the denominator

$$P_n(x) = \begin{vmatrix} g_1' & g_1'' & \cdots & g_1^{(2n)} \\ a_{22}(x) & a_{23}(x) & \cdots & a_{2,2n+1}(x) \\ \vdots & \vdots & & \vdots \\ a_{n,2}(x) & a_{n,3}(x) & \cdots & a_{n,2n+1}(x) \\ g_2' & g_2'' & \cdots & g_2^{(2n)} \\ a_{n+2,2} & a_{n+2,3}(x) & \cdots & a_{n+2,2n+1}(x) \\ \vdots & \vdots & & \vdots \\ a_{2n,2}(x) & a_{2n,3}(x) & \cdots & a_{2n,2n+1}(x) \end{vmatrix} .$$

There are $(2n)!/(n!)^2$ minors of order n obtained from the first n rows. Multiply each of these minors by its complementary minor and by a proper sign factor. The sum of all such products is $P_n(x)$ by the

Lapalace expansion. We can prove that each product is at most of order $f^{n-1}(x)\left(f'(x)\right)^{n+1}$ $x^{-n(n-1+(n+1)i_0-\ell_1-\ell_2)}$. In fact, ignoring the sign factor, the first such product is

$$
\begin{vmatrix}
g_1' & \cdots & g_1^{(n)} \\
a_{22}(x) & \cdots & a_{2,n+1}(x) \\
\vdots & & \vdots \\
a_{n,2}(x) & \cdots & a_{n,n+1}(x)
\end{vmatrix}
\begin{vmatrix}
g_2^{(n+1)} & \cdots & g_2^{(2n)} \\
a_{n+2,n+2}(x) & \cdots & a_{n+2,2n+1}(x) \\
\vdots & & \vdots \\
a_{2n,n+2}(x) & \cdots & a_{2n,2n+1}(x)
\end{vmatrix}
$$

$$
\sim b_0 \left(\frac{g_1'}{x^{(n-1)i_0}} \, g_1^{n-1} \, \frac{1}{x^{(n-1)n/2}} \right) \left(\frac{g_2^{(n)}}{x^{ni_0}} \left(g_2^{(n)}\right)^n \frac{1}{x^{(n-1)n/2}} \right)
$$

$$
\sim b_1 \, f^{n-1}(x)\left(f'(x)\right)^{n+1} x^{-n(n-1+(n+1)i_0-\ell_1-\ell_2)},
$$

where b_0 and b_1 are constants, by employing the technique used in the proof of case I. Thus, omitting the detailed tedious arguments, we have

$$
P_n(x) \sim A_n \, f^{n-1}(x) \left(f'(x)\right)^{n+1} x^{-n(n-1+(n+1)d-\ell_1-\ell_2)},
$$

where A_n is a nonzero constant. Similarly to the proof in case I, we can also show that

$$
Q_n(x) = O\left(x^{-n}\epsilon(x)\right) P_n(x).
$$

The case of m=2 is then proved. The proof for the case of m>2 is directly analogous to the case m=2. It is however extremely laborious and we do not include it here.

Corollary 1. Assume the general conditions in Theorem 2. If $\max_{1\le k\le m} | \alpha^*_{k,n+1} | \neq 0$, then

$$\frac{S-G^{(m)}_{n+1}\left[F(x);\, a_{i,j}\right]}{S-G^{(m)}_{n}\left[F(x);\, a_{i,j}\right]} = O\left(\frac{1}{x}\right), \qquad n = 1, 2, \ldots \,. \tag{29}$$

In other words $G^{(m)}_{n+1}$ converges to S faster than $G^{(m)}_{n}$ at a rate proportional to 1/x.

The proof is straightforward from the proof of Theorem 1.

4. Applications

In Section 3 we have established the fast convergence rate of $G^{(m)}_{n}$-transformation. We now consider some examples to show the numerical accuracy of the transformation.

Example 1. We know that

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha\, e^{-x}\, dx\,, \qquad \alpha > -1\,.$$

The integrand $f(x) = x^\alpha e^{-x}$ satisfies

$$f(x) = \frac{x}{\alpha-x}\, f'(x)\,. \tag{30}$$

Equation (30) implies that m=1 and ℓ_1=0. Rewriting (30) as

$$xf'(x) = (\alpha-x)\, f(x)$$

and taking the r-th derivative on both sides leads to

$$f^{(r+1)}(x) = \left(\frac{\alpha-r}{x} - 1\right) f^{(r)}(x) - \frac{r}{x} f^{(r-1)}(x) . \tag{31}$$

Using (31), the $G_n^{(1)}$-transform can be carried out for any n. To be specific, let $\alpha = \frac{5}{2}$. Then $\Gamma(\alpha+1) = \frac{15}{8}\sqrt{\pi} = 3.3233509704478 \ldots$. Table 1 lists the comparison of the $G_n^{(1)}$-transform and Levin and Sidi's $D_n^{(1)}$-transform. Adopting their notations the $D_n^{(1)}$-transform in Table 1 uses the information of F(x) and f(x) at x=1, 2, ... , n+1 while the $G_n^{(1)}$ uses F(x) at x = n+1 and the derivatives at the same point. The $G_n^{(1)}$-transform apparently gives better approximations in this case.

<u>Example</u> <u>2</u>. In the second example we follow Example 4.4 in [6]. The goal is to approximate the integral

$$\int_0^\infty \frac{\log x}{1+x^2}\, dx = 1.460362116753119547\ldots .$$

The integrand f(x) satisfies the differential equation

$$f = -\frac{5x^2+4x+1}{2(2x+1)} f' - \frac{(x^2+1)(x+1)}{2(2x+1)} f'' .$$

Therefore m=2, $\ell_1 = 1$, $\ell_2 = 2$. Rewriting the above equation as

$$(4x+2)f(x) = -(5x^2+4x+1)f'(x) - (x^3+x^2+x+1)\, f''(x)$$

and taking r-th derivative on both sides, we obtain, after simplification,

$$f^{(r+2)}(x) = -\Big\{\Big[(3r+5)x^2 + (2r+4)\, x+r+1\Big]\, f^{(r+1)}(x) + \Big[\Big(3r^2+7r+4\Big)\, x+r^2+3r+2\Big]\, f^{(r)}(x)$$

$$+ r(r+1)^2 f^{(r-1)}(x)\Big\} \Big/ \left(x^3+x^2+x+1\right).$$

Thus all the derivations can be calculated iteratively.

Table 2 compares the $G_n^{(2)}$-transform with the $D_n^{(2)}$-transform, where $D_n^{(2)}$ use $F(x)$, $f(x)$ and $f'(x)$ at $x = x_j = e^{0.2(j-1)}$, $j=1, \ldots, 2n+1$ and $G_n^{(2)}$ uses $F(x)$, $f(x)$, $f'(x)$, ... , $f^{(2n+1)}(x)$ at $x = x_{2n+1}$, the largest value of x_j's. Again $G_n^{(2)}$ is far better than $D_n^{(2)}$. Note that the order of the determinants involved in the calculations is $2n+1$.

An important application of the $G_n^{(m)}$-transform is to approximate tail probabilities. In this case, calculations of the transform need not require <u>any</u> integration. The transformation is therefore useful for determining approximation functions for tail probabilities. Now suppose that $f(x)$ is a probability density function (PDF) and we wish to approximate the tail probability

$$P(a) = \int_a^\infty f(u)du .$$

If we let $a=x$ in (1) then $F(x) = 0$, so that it is clear that $G_n^{(m)}$ in (17) depends only on $f(x)$, $f'(x)$, ... , $f^{(mn+m-1)}(x)$, producing an approximation function for $P(a)$. We use the notation $G_n^{(m)}[f(x)]$ to indicate this specific case. This method has been shown in [5] to be highly accurate even with small values of order n ($n=1, 2, 3$) in approximating tail probabilities. In such lower order transformations, we can usually express the resulting approximations in very simple forms so that the calculations of the determinants are not needed and a hand calculator is sufficient to compute the approximations.

<u>Example 3</u>. To give a specific example we now consider an application first given in [5]. That is, $G_n^{(m)}$-transform to the Pearson family which include such well known PDF's as the normal, χ^2, student t, F and others (see [5]). For the Pearson family of PDFs

$$f(x) = \frac{b_0 + b_1 x + b_2 x^2}{x-a} f'(x) .$$

Clearly, m=1. The tail probabilities of all these distributions may therefore be well approximated by $G_n^{(1)}[f(x); a_{ij}(x)]$. For those distributions with finite support, the method can still be employed simply by transformating the data to a distribution with infinite support.

For larger values of m and n the form of the $G_n^{(m)}$-transform given by equation (17) is a convenient one, and one which is easy to calculate on a micro-computer. However for m=1 and $n \leq 3$ the transform can be simplified to the extent that it is easily computed on a hand calculator. Specifically we have

$$G_1^{(1)}[f(x)] = \frac{-x\, f^2(x)}{x\, f'(x) + \ell_1 f(x)} ,$$

$$G_2^{(1)}[f(x)] = \frac{x\, f^2(x)\, A(x)}{x^2 B(x) - \ell_1(\ell_1-1)f^2(x) - xf'(x)A(x)} ,$$

and

$$G_3^{(1)}[f(x)] = \frac{xf^2(x)\, C(x)}{x^3 D(x) + 3(\ell_1-2)x^2 f(x)\left[B(x) - \left(f'(x)\right)^2\right] + (\ell_1-1)(\ell_2-2)f^2(x)E(x)} ,$$

where $A(x) = xf'(x) + 2(\ell_1-1)f(x)$,

$B(x) = f(x)\, f''(x) - \left(f'(x)\right)^2$,

$C(x) = 3(\ell_1-2)\, f(x)\left[xf'(x) + (\ell_1-1)f(x)\right] - x^2\left[2B(x) - \left(f'(x)\right)^2\right]$,

$D(x) = -f^2(x)\, f^{(3)}(x) + 6f'(x)\, B(x)$,

$E(x) = -3xf'(x) - \ell_1 f(x)$.

To demonstrate the precision of these approximations, we consider the standard normal case. Finding a proper functional approximation to the tail of a normal distribution is a problem which has a rich history in statistics and many approximations have been proposed. In some applications one requires a tail probability that is accurate in the extreme tails as well as at the usual nominal significance levels. This problems has recently been addressed by Hawkes [7] who employed an ad hoc approach to obtain a specialized approximation which is accurate in the very extreme tails. If f is a standard normal density function, i.e.,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

then

$$f(x) = -\frac{1}{x} f'(x)$$

and clearly $\ell_1 = -1$. Furthermore

$$f^{(r)}(x) = -xf^{(r-1)}(x) + (1-k)f^{(r-2)}(x), \; r \geq 2,$$

so that $G_n^{(1)}[f(x)]$ is easy to calculate for any n. Moreover it is easily derived from the definition that

$$G_1^{(1)}[f(x)] = \frac{x}{x^2+1} f(x) \; ,$$

$$G_2^{(1)}[f(x)] = \frac{x(x^2+4)}{(x^2+1)(x^2+4)-2} f(x) \; ,$$

and

$$G_3^{(1)}[f(x)] = \frac{x(x^2+2)(x^2+9)}{x^2(x^2+3)(x^2+9)+6} f(x) \; .$$

Table 3 compares $G_1^{(1)}$, $G_2^{(1)}$ and $G_3^{(1)}$ with the approximations Q_{L2} and Q_{H2} given in [7]; see [7] for details. Note that $G_2^{(1)}[f(x)]$ is essentially as good as the best of Q_{L2} and Q_{H2} and $G_3^{(1)}[f(x)]$ is better.

In fact, these new approximations are exact to many more significant digits than those shown in Table 3 in the extreme tails. For example, when x=8, the absolute errors of $G_2^{(1)}$ and $G_3^{(1)}$ are 3.4 X 10^{-21} and 2.5 X 10^{-23} respectively while those of Q_{L2} and Q_{H2} are 7.8 X 10^{-19} and 2.5 X 10^{-19} respectively. Notice that all $G_1^{(1)}[f(x)]$, $G_2^{(1)}[f(x)]$ and $G_3^{(1)}[f(x)]$ are a rational function times the density function. This is true for all other distributions considered in [5].

5. Concluding Remarks

In this paper we have extended the B_n-transformation in manner exactly analogous to the Levin-Sidi generalization of the G_n-transformation. As in the case of the Levin-Sidi results, the consequences of such an extension are substantial, as was demonstrated by Theorem 2 and the examples. It should be noted that the examples given are monotonic for large x. This is certainly not necessary for the $G_n^{(m)}$-transform to be effective. However as might be expected from our motivation of the $G_n^{(m)}$-transform, these are the types of functions for which the $G_n^{(m)}$-transform would be expected to perform the best. For example, the $G_n^{(m)}$-transform can be used to accelerate the convergence of the integral $\int_0^\infty \frac{\sin t}{t}\,dt$ but is not as effective as Levin and Sidi's $D_n^{(2)}$-transform.

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Table 1

Errors of the Approximations $G_n^{(1)}$ and $D_n^{(1)}$ to

$$\Gamma\left(\frac{7}{2}\right) = \int_0^\infty x^{5/2}\, e^{-x}\, dx = 3.32335097044784227. \ldots$$

n	$\mid G_n^{(1)}[F(n+1); a_{i,j}] - \Gamma(7/2) \mid$	$\mid D_n^{(1)}[F(1); 1] - \Gamma(7/2) \mid$
4	$1.7X10^{-3}$	$5.7X10^{-2}$
6	$1.7X10^{-6}$	$2.6X10^{-4}$
8	$3.1X10^{-9}$	$2.4X10^{-6}$
10	$7.3X10^{-12}$	$2.8X10^{-8}$
12	$1.9X10^{-14}$	$3.6X10^{-10}$
14	$2.9X10^{-16}$	$5.1X10^{-12}$

Table 2

Errors of the Approximations $G_n^{(2)}$ and $D_n^{(2)}$ to

$$S = \int_0^\infty \frac{\log x}{1+x^2}\, dx = 1.460362116753119547. . . .$$

n	$\lvert G_n^{(2)}[F(x_{2n+1}); a_{i,j}] - S \rvert$	$\lvert D_n^{(2)}[F(x_j)] - S \rvert$
2	$8.1\text{X}10^{-3}$	$3.2\text{X}10^{-1}$
3	$7.6\text{X}10^{-4}$	$8.2\text{X}10^{-3}$
4	$3.6\text{X}10^{-5}$	$5.0\text{X}10^{-4}$
5	$5.0\text{X}10^{-7}$	$3.9\text{X}10^{-4}$
6	$3.6\text{X}10^{-11}$	$6.3\text{X}10^{-5}$
7	$1.7\text{X}10^{-12}$	$2.9\text{X}10^{-6}$
8	$1.1\text{X}10^{-15}$	$3.4\text{X}10^{-8}$

Table 3. Approximations to the upper tail of the normal distribution.

x	true tail	Q_{L2}	Q_{H2}	$G_1^{(1)}$	$G_2^{(1)}$	$G_3^{(1)}$
1.2	.11507	.11737	.11402	.09550	.11244	.11504
1.6	.05480	.05589	.05465	.04985	.06452	.05486
2.0	.02275	.02314	.02274	.02160	.02273	.02276
2.4	.008198	.008312	.008197	.007951	.008199	.008200
3.0	.001350	.001364	.001350	.001330	.001350	.001350
4.5	3.398×10^{-6}	3.414×10^{-6}	3.396×10^{-6}	3.385×10^{-6}	3.398×10^{-6}	3.398×10^{-6}
6.0	9.866×10^{-10}	9.891×10^{-10}	9.861×10^{-10}	9.853×10^{-10}	9.866×10^{-10}	9.866×10^{-10}
8.0	6.221×10^{-16}	6.229×10^{-16}	6.218×10^{-16}	6.218×10^{-16}	6.221×10^{-16}	6.221×10^{-16}
10.0	7.620×10^{-24}	7.625×10^{-24}	7.618×10^{-24}	7.618×10^{-24}	7.620×10^{-24}	7.620×10^{-24}
14.0	7.794×10^{-45}	7.796×10^{-45}	7.793×10^{-45}	7.793×10^{-45}	7.794×10^{-45}	7.794×10^{-45}
18.0	9.741×10^{-73}	9.742×10^{-73}	9.741×10^{-73}	9.741×10^{-73}	9.741×10^{-73}	9.741×10^{-73}