Superconductivity near a quantum-critical point: The special role of the first Matsubara frequency

Yuxuan Wang,[1] Artem Abanov,[2] Boris L. Altshuler,[3] Emil A. Yuzbashyan,[4] and Andrey V. Chubukov[5]

[1]*Department of Physics and Institute for Condensed Matter Theory,*
University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA
[2]*Department of Physics. Texas A&M University, College Station, USA*
[3]*Department of Physics. Columbia University, New York, USA*
[4]*Center for Materials Theory, Department of Physics and Astronomy,*
Rutgers University, Piscataway, NJ 08854, USA
[5]*School of Physics and Astronomy and William I. Fine Theoretical Physics Institute,*
University of Minnesota, Minneapolis, MN 55455, USA
(Dated: October 19, 2016)

Near a quantum-critical point in a metal strong fermion-fermion interaction mediated by a soft collective boson gives rise to incoherent, non-Fermi liquid behavior. It also often gives rise to superconductivity which masks the non-Fermi liquid behavior. We analyze the interplay between the tendency to pairing and fermionic incoherence for a set of quantum-critical models with effective dynamical interaction between low-energy fermions. We argue that superconducting T_c is non-zero even for strong incoherence and/or weak interaction due to the fact that the self-energy from dynamic critical fluctuations vanishes for the two lowest fermionic Matsubara frequencies $\omega_m = \pm\pi T$. We obtain the analytic formula for T_c which reproduces well earlier numerical results for the electron-phonon model at vanishing Debye frequency.

Introduction. The interplay between superconductivity and non-Fermi liquid behavior in metals is one of most fascinating issues in the modern physics of correlated electron systems[1–18] A generic metallic system in $D > 1$ is a Fermi liquid with coherent quasiparticles at low energies. This coherence is destroyed if the system is brought to a quantum-critical point (QCP), beyond which it develops an electronic order in spin or charge channel. At a QCP fluctuations of the order parameter become massless. In $D \leq 3$, the four-fermion interaction, mediated by these massless fluctuations, destroys fermionic coherence at $T = 0$, either at specific hot points on the Fermi surface[4,15,19,20], if the order has a finite momentum, or everywhere on the Fermi surface, if the order develops with $q = 0$ (Ref. 21). The same massless fluctuations, however, also mediate the pairing interaction, and if this interaction has an attractive angular component the system can develop a superconducting instability at a finite T, before a QCP is reached. A dome of superconductivity above a QCP prevents a non-Fermi liquid, QC behavior from extending down to the lowest energies.

The existence of superconductivity near a QCP is not guaranteed, however, because strong fermionic self-energy acts against pairing. There are two effects from the self-energy. First, at $T \neq 0$ the self-energy from static (thermal) fluctuations acts as an impurity and may cause pair-breaking. This is crucial for spin-triplet superconductivity, for which thermal self-energy acts as a magnetic impurity[22], but not for spin-singlet superconductivity, for which it acts as a non-magnetic impurity and its singular contribution cancels out by Anderson theorem[23]. In this paper we consider spin-singlet pairing and neglect the contribution from thermal fluctuations. Second, already at $T = 0$ the self-energy produces strong upturn mass renormalization and shrinks the range of a

coherent fermionic behavior. Both these effects are detrimental to superconductivity.

The pairing amplitude and the self-energy come from the same underlying interaction mediated by a soft boson, hence the two are generally of the same order. Zero-temperature studies of specific models in $D = 2$ and in $D = 3 - \epsilon$ have shown[3–5,8–10,14] that superconductivity does develop at a QCP, however these studies also hinted[2–4] that the pairing at a QCP is a threshold problem and may disappear if the self-energy gets enhanced compared to the pairing amplitude. A recent study[16] made this explicit by extending a model in $D = 3 - \epsilon$ to large N in such a way that the self-energy gets enhanced, while the pairing amplitude remains intact. The authors of Ref. 16 performed $T = 0$ analysis and argued that there exists a critical N above which the pairing does not develop because decoherence, caused by strong self-energy, wins over the tendency to pairing due to an attraction.

In this communication we and analyze the same pairing problem, but at a non-zero T. Our result is different from Ref. 16 and earlier work by some of us (Ref. 3) – we argue that superconducting T_c is finite at arbitrary N. The reason is that the competition between the self-energy and the pairing interaction at a finite temperature is qualitatively different from that at $T = 0$. Namely, at a finite T the Matsubara self-energy $\Sigma(\omega_n)$ is a discrete variable, defined at a set of $\omega_n = \pi T(2n + 1)$. It still large for all $n \neq 0, -1$, but at the two lowest Matsubara frequencies $\omega_n = \pm\pi T$ it vanishes if we neglect the contribution from static bosonic fluctuations.[24] At the same time, the pairing interaction $\chi(\Omega_m)$, also taken without the static part (i.e., at bosonic $\Omega_m = 2\pi T m, m \neq 0$) is not reduced at $\Omega_m = \pi T - (-\pi T) = 2\pi T$ compared to $\chi(\Omega_m)$ at other Ω_m. As a result, the pairing interac-

tion between fermions with $\omega_n = \pm\pi T$ is strong, while the competing contribution from the self-energy is absent. Although this holds only for the two Matsubara frequencies, we show that this is sufficient to render T_c finite. Moreover, T_c is not small and has a power-law dependence of the coupling constant, which is stronger than than the logarithmical divergence in BCS theory, although the latter is obtained by summing up an infinite set of Matsubara points.

In broader terms, we argue against the commonly used procedure[3,4,8–10,14,16] to obtain T_c at a QCP by computing the pairing susceptibility $\chi_{pp}(\omega)$ at $T = 0$, associating the superconducting region with the range of N where $\chi_{pp}(\omega)$ becomes negative below some ω^*, and identifying T_c with $O(\omega^*)$. We argue that T_c has to be determined from the actual calculations at a finite T, and T_c generally does not scale with ω^*, except for special cases like models in $D = 3 - \epsilon$ and $N = O(1)$.

To be specific, our conclusion holds for a set of QC models with dynamical interaction between fermions, for which the Eliashberg approximation[25] is valid. Within this approximation, the momentum integration in the gap equation can be carried out explicitly, and the analysis of superconductivity reduces to a set of equations for the frequency dependent pairing vertex $\Phi(\omega_m)$ and fermionic self-energy $\Sigma(\omega_m)$, both originating from the effective, momentum-averaged interaction $\chi(\omega_m - \omega'_m)$. We consider a generic case of $\chi(\Omega_m) = (g/|\Omega_m|)^\gamma$, where g is the effective fermion-boson coupling. We list specific examples of different γ below. In particular, $\gamma = 2$ corresponds to much studied strong coupling limit of electron-phonon interaction[1,29,31,32]. We argue that T_c is non-zero for any γ, even if the self-energy is enhanced after a proper extension of the model to large N, as in[16]. Moreover, at large N, $T_c \approx [g/(2\pi)]/N^{1/\gamma} \approx 0.16g/N^{1/\gamma}$ is fully determined by the two lowest Matsubara frequencies. At $N = 1$ this formula yields $T_c \approx 0.16g$. This value is very close to $T_c \approx 0.18g$ obtained numerically for $\gamma = 2$ (Refs. 28 and 29). which implies that T_c for QC electron-phonon problem is predominantly determined by just the two lowest Matsubara frequencies.

The model. We consider a system of fermions at the boundary between a Fermi liquid state and a state with a long-range order in either spin or charge channel (ferromagnetism, nematic order, spin/charge-density-wave, etc). At a QCP, the propagator of a soft boson becomes massless and mediates singular interaction between fermions. Like we said, we treat this interaction as attractive in at least one pairing channel. This is true for QCP towards density-wave instabilities[30], but we caution that this is not always the case – e.g., for fermions at the half-filled lowest Landau level, long range current-current interaction mediated by gapless gauge fluctuations is repulsive in all channels[14].

We assume, following earlier work[3,4,7,10,13–16,26,27] that bosons can be treated as slow modes compared to fermions, i.e., the Eliashberg approximation is valid. Within this approiximation one can explicitly integrate

over the momentum component perpendicular to the Fermi surface reduce the integral equations for the self-energy Σ and the pairing vertex Φ to the set for $\Sigma(\mathbf{k}_F, \omega_m)$ and $\Phi(\mathbf{k}_F, \omega_m)$ on the Fermi surface. We will be interested in the solution for T_c, hence we set $\Phi(\mathbf{k}_F, \omega_m)$ to be infinitesimally small and approximate $\Sigma(\mathbf{k}_F, \omega_m)$ by its normal state value. We make one additional approximation – assume that the dependence of $\Phi(\mathbf{k}_F, \omega_m)$ on ω_m and on on the momentum direction along the Fermi surface can be factorized, i.e. that $\Phi(\mathbf{k}_F, \omega_m) = f_\Phi(\mathbf{k}_F)\Phi(\omega_m)$, where f_Φ has the symmetry of the corresponding superconducting state[4,18], and neglect the momentum dependence of $\Sigma(\mathbf{k}_F, \omega_m)$. Under this approximation, the integration over momentum component along the Fermi surface can be done explicitly[4,15], and the set of equations for T_c reduces to the integral equation for $\Phi(\omega_m)$ and the equation for the normal state self-energy $\Sigma(\omega_m)$:

$$\Phi(\omega_m) = \frac{g^\gamma}{N}\pi T \sum_{m'\neq m} \frac{\Phi(\omega_{m'})}{|\omega_{m'} + \Sigma(\omega_{m'})|} \; \frac{1}{|\omega_m - \omega_{m'}|^\gamma},$$

$$\Sigma(\omega_m) = g^\gamma \pi T \sum_{m'\neq m} \frac{\text{sign}(\omega_{m'})}{|\omega_m - \omega_{m'}|^\gamma}, \tag{1}$$

where we incorporated the overall factors from the integration over momentum into g. Like we said, we neglect the terms with $m = m'$ in Eq. (1) because for spin-singlet pairing such terms cancel out between $\Phi(\omega_m)$ and $\Sigma(\omega_m)$ We discuss this in more detail in Ref. 33. The overall factor $1/N$ is the result of extending the model to an SU(N) global symmetry which involves both fermions and bosons[16]. We treat N as a parameter. Our goal is to understand whether there is a critical N above which $T_c = 0$, i.e. the normal state extends down to $T = 0$.

Models described by Eq. (1) include a model for color superconductivity[5] ($\gamma = 0_+$, $\chi(\Omega_m) \propto \log|\omega_m|$), models for spin- and charge-mediated pairing in $D = 3 - \epsilon$ dimension[10,14,16] ($\gamma = O(\epsilon) \ll 1$), a 2D pairing model[35] with interaction peaked at $2k_F$ ($\gamma = 1/4$), 2D models for pairing at a nematic/Ising-ferromagnetic QCP[2,17,22] ($\gamma = 1/3$), a 2D hot-spot model for pairing at the (π, π) SDW QCP[3,4,36,37] and at a 2D CDW QCP[32,38], 2D models for pairing by undamped fermions ($\gamma = 1$), the strong coupling limit of phonon-mediated superconductivity[1,28,29], and models with parameter-dependent γ (Refs. 8 and 9).

The argument for the threshold. To set the stage for our analysis, we briefly display the argument for the existence of a threshold in N for T_c. The argument is based on the analysis of the pairing susceptibility at $T = 0$ for $0 < \gamma < 1$ (Refs. 3 and 16). At $T = 0$ the self-energy has a non-Fermi liquid form: $\Sigma(\omega_m) = |\omega_m|^{1-\gamma}\omega_0^\gamma \text{sign}(\omega_m)$, where $\omega_0 = g[2/(1 - \gamma)]^{1/\gamma}$. Substituting this $\Sigma(\omega_m)$ into the equation for $\Phi(\omega)$ and adding up a bare Φ_0, one can compute the $T = 0$ pairing susceptibility $\chi_{pp}(\omega_m) = \Phi(\Omega_m)/\Phi_0$ at $\omega_m < \omega_0$ order by order in $1/N$. The building block for series for $\chi_{pp}(\omega_m)$ is $\int d\omega_{m'} 1/(|\omega_{m'} - \omega_m|^\gamma |\omega_{m'}|^{1-\gamma})$, where the first term
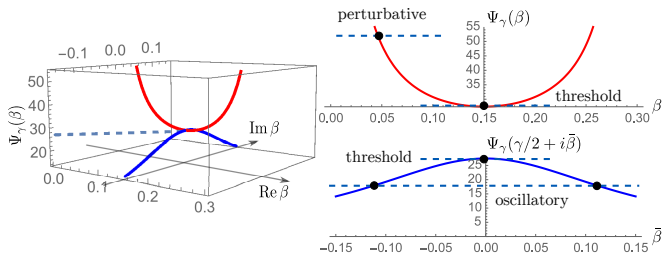
FIG. 1. Left: The plot of the function $\Psi_\gamma(\beta)$ for $\gamma = 0.3$. Right: the solution of the equation $((1-\gamma)/2N)\Psi_\gamma(\beta) = 1$. For large N, β is real, as in perturbation theory (red line), i.e., superconductivity does not develop. For $N < N_{cr}$, $\beta = \gamma/2 \pm i\bar{\beta}$ is complex (blue line). For a complex β, $\Phi(\omega_m)$ is oscillatory in frequency, implying that T_c is finite.

comes from the interaction and the second from the self-energy. The integrand scales as $1/|\omega_{m'}|$ at $\omega_{m'} > \omega_m$, hence the series for $\chi_{pp}(\omega_m)$ are logarithmical. At $N \gg 1$ the coupling is weak and one can just sum up the series of leading logarithms, like in BCS theory. However, this analogy does not go further because in our case, each logarithm is cut by ω_m rather than by T and the summation of the logarithms yields $\chi_{pp}(\omega_m) = 1 + \alpha \log \frac{\omega_0}{|\omega_m|} + \frac{\alpha^2}{2}\left(\log \frac{\omega_0}{|\omega_m|}\right)^2 + \frac{\alpha^3}{6}\left(\log \frac{\omega_0}{|\omega_m|}\right)^3 + ... = (\omega_0/|\omega_m|)^\alpha$, where $\alpha = (1-\gamma)/N$. This susceptibility is positive i.e. the summation of the leading logarithms does not give rise to pairing.

This line of reasoning is developed further by solving for the susceptibility beyond the logarithmical approximation. The $1/\omega_{m'}$ scaling of the kernel suggests a power-law form $\Phi(\omega_m) \propto (\omega_0/|\omega_m|)^\beta$ at $\omega_m < \omega_0$. Substituting this into (1) and evaluating the integrals, we obtain an equation on β of the form $(\alpha/2)\Psi_\gamma(\beta) = 1$, where $\Psi_\gamma(\beta) = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} + \Gamma(1-\gamma)\left(\frac{\Gamma(\beta)}{\Gamma(1-\gamma+\beta)} + \frac{\Gamma(\gamma-\beta)}{\Gamma(1-\beta)}\right)$. We plot $\Psi_\gamma(\beta)$ in Fig. 1. Solving for β as a function of α and γ and choosing the branch which gives $\beta \approx \alpha$ at small α, consistent with logarithmical perturbation theory, we find that β increases with α, reaches the value $\gamma/2$ at a critical $\alpha_{cr} = (1-\gamma)/N_{cr}$, and at larger α (i.e., smaller N) becomes complex: $\beta = \gamma/2 \pm i\bar{\beta}$, where $\bar{\beta} \propto (\alpha - \alpha_{cr})^{1/2} \sim (N_{cr} - N)^{1/2}$. As the consequence, χ_{pp} becomes an oscillating function of ω_m: $\chi_{pp}(\omega_m) \propto (\omega_0/|\omega_m|^{\gamma/2})\cos(\bar{\beta}\log(\omega_0/|\omega_m|) + \psi_0)$, where ψ_0 is an arbitrary phase. Oscillations of the pairing susceptibility cannot be obtained within a perturbation theory and their presence was interpreted as the sign that the system has already underwent a pairing instability at some finite T_c. To obtain T_c, earlier works used $T = 0$ form of $\chi_{pp}(\omega_m)$ and identified T_c with the largest ω_m at which $\chi_{pp}(\omega_m)$ first becomes negative. At $\alpha \geq \alpha_{cr}$, when β is small, this yields[3,16,39] $T_c \sim \omega_0 e^{-a/(N_{cr}-N)^{1/2}}$, where $a = O(1)$.

Finite T analysis. We now perform the actual analysis at a finite T and argue that it yields a result different from the one at $T = 0$. Namely, we argue that

T_c is non-zero for any N and only tends to zero when N tends to infinity. We show that this result originates from the vanishing of the self-energy at Matsubara frequencies $\omega_m = \pm\pi T$. The special role of the lowest Matsubara frequencies cannot be detected in the $T = 0$ analysis in which Matsubara frequency is a continuous variable.

Vanishing of the self-energy $\Sigma(\omega_m = \pm\pi T)$ can be readily seen from Eq. (1). We have $\Sigma(\pi T) = [g/(2\pi T)]^\gamma \pi T \sum_{m' \neq 0} \text{sign}(2m'+1)/|m'|^\gamma$, and the sums over positive and negative m' cancel each other. The same holds for $\omega_m = -\pi T$. For any other $m \geq 1$, $\Sigma(\omega_m > 0) \sim \omega_m(g/(2\pi T))^\gamma \gg \omega_m$, i.e at low T the self-energy at $|\omega_m| \neq \pi T$ well exceeds the bare ω_m term in the fermionic propagator. Note in passing that the vanishing of $\Sigma(\omega_m = \pm\pi T)$ in our analysis does not actually imply that at this frequency a fermion is a free quasiparticle, because we eliminated from $\Sigma(\omega_m)$ the contribution from static critical fluctuations (the $m = m'$ term in in Eq. (1)). Such contribution is irrelevant for the pairing, but it is parametrically larger than T near a QCP, hence the full self-energy has a non-Fermi liquid form even at $\omega_m = \pm\pi T$.

To make our point about T_c, we consider large N and small T. Neglecting ω_m compared to the self-energy for all m except $m = 0$ and $m = -1$, using the symmetry conditions $\Phi(\omega_m) \equiv \Phi_m = \Phi_{-m-1}$ and $\Sigma(\omega_m) \equiv \Sigma_m = -\Sigma_{-m-1}$, and introducing $\bar{\Phi}_m \equiv \Phi_m/(\pi T K_T)$, $\bar{\Sigma}_m \equiv \Sigma_m/(\pi T K_T)$, where $K_T = [g/(2\pi T)]^\gamma \gg 1$, we re-write the gap equation in (1) as a set of coupled equations for $\bar{\Phi}_{m=0,-1}$ and $\bar{\Phi}_{m>0}$:

$$\bar{\Phi}_0 = \frac{K_T}{N}\bar{\Phi}_{-1} + \frac{1}{N}\sum_{m>0}\frac{\bar{\Phi}_m}{\bar{\Sigma}_m}\left[\frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma}\right]$$

$$\bar{\Phi}_{m>0} = \frac{K_T}{N}\left[\frac{\bar{\Phi}_0}{m^\gamma} + \frac{\bar{\Phi}_{-1}}{(m+1)^\gamma}\right]$$

$$+\frac{1}{N}\sum_{m'>0,m'\neq m}\frac{\bar{\Phi}_m}{\bar{\Sigma}_m}\left[\frac{1}{|m-m'|^\gamma} + \frac{1}{(m+m'+1)^\gamma}\right] \quad (2)$$

We distinguish $\bar{\Phi}_0$ and $\bar{\Phi}_{-1}$ in (2) only for illustrative purposes. In fact, the two are equal, $\bar{\Phi}_0 = \bar{\Phi}_{-1}$.

At vanishing $1/N$ Eq. (2) has a solution at $K_T = N$, i.e. at $T = T_c = (g/2\pi)/N^{1/\gamma}$. Indeed, the first equation in (2) is satisfied, while the second one determines $\bar{\Phi}_m$ for all $m > 0$ in terms of $\bar{\Phi}_0$: $\bar{\Phi}_{m>0} = \bar{\Phi}_0\left[\frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma}\right]$. Plugging this $\bar{\Phi}_{m>0}$ into the first equation in (2), we obtain T_c with $1/N$ correction (see Ref. 33 for details)

$$T_c \approx \frac{g}{2\pi}\frac{1}{N^{1/\gamma}}\left(1 + \frac{\delta_\gamma}{N\gamma}\right), \quad (3)$$

where $\delta_\gamma = \sum_{m>0}[1/m^\gamma + 1/(m+1)^\gamma]^2/\bar{\Sigma}_m$ is a number of order one. We see that T_c is non-zero for *any* N, i.e., no matter how strong is the self-energy at Matsubara frequencies ω_m with $m \neq 0, -1$. We also see that superconducting T_c is predominantly determined by the two lowest Matsubara frequencies, for which the pairing
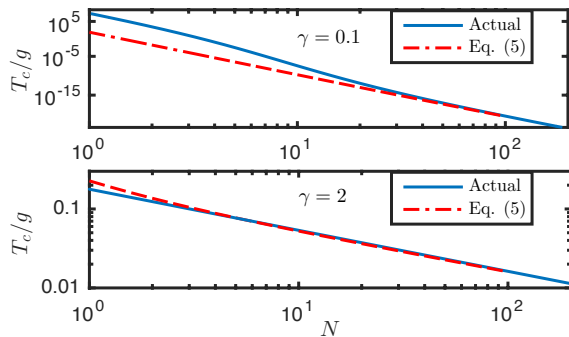
FIG. 2. Superconducting T_c, obtained by solving the gap Eq. (2) numerically (labeled as "actual"), vs. the analytical result from Eq. (3). Upper panel: $\gamma = 0.1$, lower panel: $\gamma = 2$. In both cases the analytical T_c perfectly matches the numerical one at large N. For $N = O(1)$, the numerical solution yields much larger T_c than Eq. (3) for $\gamma = 0.1$, but for $\gamma = 2$ numerical and analytical results remain close even for $N = 1$.

interaction is strong, but the self-energy vanishes. This new understanding is very different from the previous one that superconductivity at a QCP originated from the pairing of incoherent fermions at $T = 0$.

The value of T_c. In Fig. 2 we show T_c given by Eq. (3), together with the numerical solution of the gap equation. We see that at large N the actual solution and the one from Eq. (3) agree quite well, as expected. The agreement does not extend to $N \sim 1$ at small γ, but gets progressively better for larger γ, for which T_c is predominately determined by the first two Matsubara frequencies even for $N = 1$, i.e. $T_c \approx g/(2\pi)$. Other Matsubara frequencies account only for a small correction to $T_c = g/(2\pi)$. To verify this, we computed the leading correction in $1/\gamma$ for an arbitrary N and obtained $T_c = \frac{g}{2\pi}(s/N)^{1/\gamma}$ where $s = s(N)$ is determined from $J_{3/2+N/s}(1/s)/J_{1/2+N/s}(1/s) = s - 1$, where J is a Bessel function (see Ref.[33] for detail) At $N = 1, s = 1.1843$, at $N \gg 1$, $s = 1 + 1/(2N)$, in agreement with Eq. (3) (In Eq. (3), $\delta_\gamma \to 1/2$ at $\gamma \to \infty$). For the strong coupling limit of electron-phonon superconductivity ($\gamma = 2$, $N = 1$), $T_c \approx 0.17g$, which is very close to $0.18g$, obtained in extensive numerical studies[28,29] on a large mesh of Matsubata frequencies. This has been noticed in Ref. 28 but not related to the absence of the self-energy at $\omega_m = \pm \pi T$.

For completeness, we also computed T_c at small γ and $N = O(1)$. In this regime $T_c \gg \omega_0$ (see Fig. 2) and the self-energy is again irrelevant, but now simply because at $T = T_c$, $\omega_m \gg \Sigma(\omega_m)$ for all m. Neglecting $\Sigma(\omega_{m'})$ in Eq. (1), we obtain (see Ref.[33] for details)

$$T_c \sim \omega_0(\gamma N)^{-1/\gamma} \sim \frac{g}{2\pi N^{1/\gamma}}e^{\log(b/\gamma)/\gamma} \gg \frac{g}{2\pi N^{1/\gamma}}, \quad (4)$$

where $b = O(1)$. A similar result for the pairing scale has been obtained in Refs. [14,40] using RG procedure. Note in passing that the divergence of T_c at $\gamma \to 0$ is the consequence of the fact that in this limit the effective

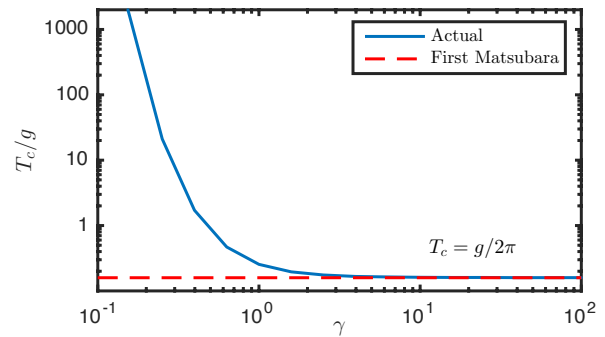FIG. 3. The numerical result for T_c at $N = 1$ as a function of γ. At small γ, T_c is determined by all Matsubara frequencies and increases exponentially with decreasing γ (see the text). At $\gamma > 1$ it rapidly approaches $T_c = g/2\pi$, which we obtained analytically from the two lowest Matsubara frequencies.

interaction $\chi(\Omega_m) = (g/|\Omega_m|)^\gamma$ tends to a constant, while there is no upper cutoff in the theory. If we add a cutoff, we indeed obtain that T_c saturates.

In Fig. 3 we plot T_c at $N = 1$ obtained numerically from the Eliashberg equation (1). We see that at $\gamma > 1$, T_c rapidly approaches $g/2\pi$ – the result which we obtained analytically from the two lowest Matsubara frequencies. We emphasize that at both small and large γ the fermionic self-energy is irrelevant for T_c. At $\gamma \sim 1$, it does affect the value of T_c, but is not crucial in the sense that a comparable T_c is obtained without including the self-energy.

Conclusion. In this paper we computed superconducting T_c for a set of quantum-critical models with Eliashberg-type effective dynamical interaction between low-energy fermions. We found that superconductivity always develops above a quantum-critical point, no matter what is the interplay between the pairing interaction and fermionic incoherence at $T = 0$. We argued that the proper calculation of T_c should be done directly at a finite temperature, and T_c is non-zero due to the fact that at a finite T the self-energy vanishes at the two lowest fermionic Matsubara frequencies $\omega_m = \pm \pi T$. This implies that fermionic incoherence at a QCP is not an obstacle for superconductivity. We caution, however, that this is true for the Eliashberg T_c, which does not include fluctuations of the pairing gap. The analysis of the gap fluctuations requires separate consideration.

[1] R. Combescot, Phys. Rev. B **51**, 11625 (1995)

[2] N. E. Bonesteel, I. A. McDonald, and C. Nayak Phys. Rev. Lett. 77, 3009 (1996).

[3] Ar. Abanov, A. V. Chubukov, and A.M. Finkelstein, Europhys. Lett. **54**, 488 (2001).

[4] Ar. Abanov, A.V. Chubukov, and J. Schmalian, Adv. Phys. **52**, 119 (2003); Ar. Abanov and A..V. Chubukov, Phys. Rev. Lett., **83**, 1652 (1999).

[5] D. T. Son, Phys. Rev. D **59**, 094019 (1999); A. Chubukov, and J. Schmalian, Phys. Rev. B. **72**, 174520 (2005).

[6] S.-S. Lee, Phys. Rev. B **80**, 165102 (2009); D. Dalidovich and S.-S. Lee, Phys. Rev. B 88, 245106 (2013).

[7] M. A. Metlitski and S. Sachdev, Phys. Rev. B, **82**, 075127 (2010); *ibid* **82**, 075128 (2010).

[8] S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, Phys. Rev. B **80**, 155129 (2009); E. G. Moon, and S. Sachdev, Phys. Rev. B **80**, 035117 (2009).

[9] E.G. Moon and A. V. Chubukov, J. Low Temp. Phys. 161, 263 (2010).

[10] D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Phys. Rev. B 82, 045121 (2010).

[11] see R. Mahajan, D. M. Ramirez, S. Kachru, and S. Raghu, Phys. Rev. B 88, 115116 (2013); A. L. Fitzpatrick, S. Kachru, J. Kaplan, and S. Raghu, Phys. Rev. B 88, 125116 (2013); A. L. Fitzpatrick, S. Kachru, J. Kaplan, and S. Raghu, Phys. Rev. B 89, 165114 (2014); G. Torroba and H. Wang, Phys.Rev. B 90, 165144 (2014); A. L. Fitzpatrick, G. Torroba, and H. Wang, Phys. Rev. B 91, 195135 (2015) and references therein.

[12] E. Fradkin, S. A. Kivelson, M. J. Lawler, J. P. Eisenstein, and A. P. Mackenzie, arXiv:0910.4166.

[13] P. Monthoux, D. Pines, and G. Lonzarich, Nature 450, 1177 (2007); D.J. Scalapino, Rev. Mod. Phys. 84, 1383 (2012)

[14] M. A. Metlitski, D. F. Mross, S. Sachdev, and T. Senthil, Phys. Rev. B 91, 115111 (2015).

[15] H. Meier, C. Pépin, M. Einenkel and K.B. Efetov, Phys. Rev. B **89**, 195115 (2014) and references therein; K. B. Efetov Phys. Rev. B **91**, 045110 (2015).

[16] S. Raghu, G. Torroba, and H. Wang, Phys. Rev. B 92, 205104 (2015).

[17] S. Lederer, Y. Schattner, E. Berg, and S.A. Kivelson, Phys. Rev. Lett. 114, 097001 (2015).

[18] Jin Mo Bok *et al* Sci. Adv., **2**, E1501329, 2016.

[19] S. A. Hartnoll, D. M. Hofman, M. A. Metlitski, and S. Sachdev Phys. Rev. B 84, 125115 (2011); S. A. Hartnoll, R. Mahajan, M. Punk, and S. Sachdev Phys. Rev. B 89, 155130 (2014); A. V. Chubukov, D. L. Maslov, and V. I. Yudson, Phys. Rev. B 89, 155126 (2014).

[20] S. A. Maier and P. Strack Phys. Rev. B 93, 165114 (2016); S. P. Ridgway and C. A. Hooley Phys. Rev. Lett. 114, 226404 (2015).

[21] There is a large body of literature on QCP with $q = 0$. For recent works, see S. Sur and S.-S. Lee Phys. Rev. B 91, 125136 (2015); M. Punk Phys. Rev. B 91, 115131 (2015) and references therein.

[22] Z. Wang, W. Mao, and K. Bedell, Phys. Rev. Lett., **87**, 257001 (2001). R. Roussev and A.J. Millis, Phys. Rev. B **63**, 140504 (2001); A. V. Chubukov, A. M. Finkelstein, R. Haslinger, and D. K. Morr, Phys. Rev. Lett. **90**, 077002 (2003).

[23] A. J. Millis, S. Sachdev, and C. M. Varma, Phys. Rev. B 37, 4975 (1988); Ar. Abanov, A.V. Chubukov, and M. Norman, Phys. Rev. B 78, 220507 (2008).

[24] A. V. Chubukov and D. L. Maslov Phys. Rev. B 86, 155136 (2012); D. L. Maslov and A. V. Chubukov Phys. Rev. B 86, 155137 (2012).

[25] G.M. Eliashberg, Sov. Phys. JETP, **11**, 696 (1960)

[26] A.B. Migdal, Sov. Phys. JETP **7**, 996 (1958).

[27] R. Haslinger and A. V. Chubukov Phys. Rev. B 68, 214508 (2003).

[28] P.B. Allen and R.C. Dynes, Phys. Rev. B **12**, 905 (1975).

[29] G. Bergmann and D. Rainer, Z. Phys. **59**, 263 (1973); F. Marsiglio and J.P. Carbotte, Phys. Rev. B **43**, 5355 (1991); A.E. Karakozov, E.G. Maksimov, and A.A. Mikhailovsky, Solid State Coimm., **79**, 329 (1991).

[30] see S. Maiti and A. V. Chubukov in "Lectures on the Physics of Strongly Correlated Systems", Avella and Manchini eds., AIP Pblishing, Melville, NY (2013) ; in "Novel Superconductors", Bennemann and Ketterson eds., Oxford University Press (2014) Vietri sul Mare (Salerno), Italy.

[31] F. Marsiglio and J.P. Carbotte, "Electron-Phonon Superconductivity", in "The Physics of Conventional and Unconventional Superconductors", Bennemann and Ketterson eds., Springer-Verlag, (2006) and references therein.

[32] C. Castellani, C. Di Castro, and M. Grilli, Phys. Rev. Lett. **75**, 4650 (1995); A. Perali, C. Castellani, C. Di Castro, and M. Grilli, Phys. Rev. B 54, 16216 (1996); S. Andergassen, S. Caprara, C. Di Castro, and M. Grilli, Phys. Rev. Lett. 87, 056401 (2001).

[33] See Supplemental Material [url], which includes Refs. 7, 11, 14–16, 23, and 34.

[34] K. B. Efetov, H. Meier and C. Pépin, Nat. Phys. **9** 442, (2013).

[35] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B 52, 5563 (1995); D. Bergeron, D. Chowdhury, M. Punk, S. Sachdev, and A.-M. S. Tremblay Phys. Rev. B 86, 155123 (2012); Y. Wang and A. V. Chubukov Phys. Rev. B 88, 024516 (2013).

[36] A. J. Millis, Phys. Rev. B 45, 13047 (1992).

[37] Y. Wang and A. V. Chubukov, Phys. Rev. Lett. 110, 127001 (2013); A. V. Chubukov and P. Wölfle, Phys. Rev. B 89, 045108 (2014).

[38] D. Chowdhury and S. Sachdev, Phys. Rev. B 90, 134516 (2014), *ibid* 245136 (2014); Y. Wang and A. V. Chubukov Phys. Rev. B 92, 125108 (2015).

[39] D. V. Khveshchenko and W. F. Shively, Phys. Rev. B 73, 115104 (2006); A. V. Chubukov and A. M. Tsvelik, Phys. Rev. B 76, 100509 (2007).

[40] M. Metlitski, private communication.

Supplemental Material

I. CANCELLATION OF THE SINGULAR SELF-ACTION $m = m'$ TERM IN EQ. (1)

In the right hand side of Eq. (1) from the main text we have excluded the $m = m'$ terms for both the self-energy and the pairing vertex. We argued that this procedure is legitimate because such terms come from thermal fluctuations, which act as non-magnetic impurity. In an s-wave superconductor, the $m = m'$ contributions to the self energy and the pairing vertex cancel out exactly, by Anderson's theorem. Here we show that singular contributions with $m = m'$ cancel out even if the pairing channel is different from an s-wave.

We first emphasize that the $m = m'$ contribution has to be analyzed (and eliminated) within the original theory with $N = 1$ before extending it to large N. Otherwise, there will be unphysical divergencies. The elimination of the $m = m'$ term before large N extension is legitimate because the goal of taking N to be large is to enhance self-energy from quantum fluctuations [the one which leads to $\Sigma(\omega_m) \propto \omega_m^\alpha$ with $\alpha < 1$]. The $m = m'$ term is not relevant at $T = 0$, and its singular contribution comes from thermal fluctuations. The extension to large N, which we used, is tailored to analyze quantum, but not thermal fluctuations.

We next remind the reader how the cancellation occurs for s-wave pairing. For this, we consider the coupling via Einstein phonons (the case $\gamma = 2$ in our notations). The Eliashberg equations are given by

$$\Sigma(\omega_m) = \pi T \sum_{m'} \chi(\omega_m - \omega'_m) \operatorname{sgn}(\omega'_m),$$

$$\Phi(\omega_m) = \pi T \sum_{m'} \frac{\chi(\omega_m - \omega'_m)}{|\omega'_m + \Sigma(\omega'_m)|} \Phi(\omega'_m), \tag{S5}$$

where

$$\chi(\omega_m - \omega'_m) = \frac{g^2}{|\omega_m - \omega'_m|^2 + \omega_E^2}, \tag{S6}$$

is the propagator of an Einstein phonon. In the limit $\omega_E \to 0$, the $m = m'$ terms in both equations diverge. We keep ω_E finite at intermediate steps and set it to zero only at the end of calculations.

To see the cancellation of $m = m'$ terms, we introduce the gap function

$$\Delta(\omega_m) \equiv \frac{\Phi(\omega_m)}{1 + \Sigma(\omega_m)/\omega_m}, \tag{S7}$$

and re-express the Eliashberg equation for $\Phi(\omega_m)$ in (S5) as

$$\Delta(\omega_m) = T \sum_{m'} \chi(\omega_m - \omega'_m) \left[\frac{\Delta(\omega'_m)}{\omega'_m} - \frac{\Delta(\omega_m)}{\omega_m} \right]$$

$$T \sum_{m' \neq m} \chi(\omega_m - \omega'_m) \left[\frac{\Delta(\omega'_m)}{\omega'_m} - \frac{\Delta(\omega_m)}{\omega_m} \right]$$

$$+ T \chi(0) \left[\frac{\Delta(\omega'_m)}{\omega'_m} - \frac{\Delta(\omega_m)}{\omega_m} \right]_{m = m'} \tag{S8}$$

We see that the term with $m = m'$ vanishes, as long as ω_E is non-zero. Eliminating this term and re-introducing

$$\tilde{\Sigma}(\omega_m) = \pi T \sum_{m' \neq m} \chi(\omega_m - \omega'_m) \operatorname{sgn}(\omega'_m) \tag{S9}$$

and

$$\tilde{\Phi}(\omega_m) = \Delta(\omega_m) \left(1 + \frac{\tilde{\Sigma}(\omega_m)}{\omega_m} \right), \tag{S10}$$

we obtain the same set of Eliashberg equations as Eq. (S5), but with $m' \neq m$ in the sum over Matsubara frequencies. Taking now the $\omega_E \to 0$ limit, we obtain Eq. (1) in the main text for $\gamma = 2$. [In the main text we reverted to the
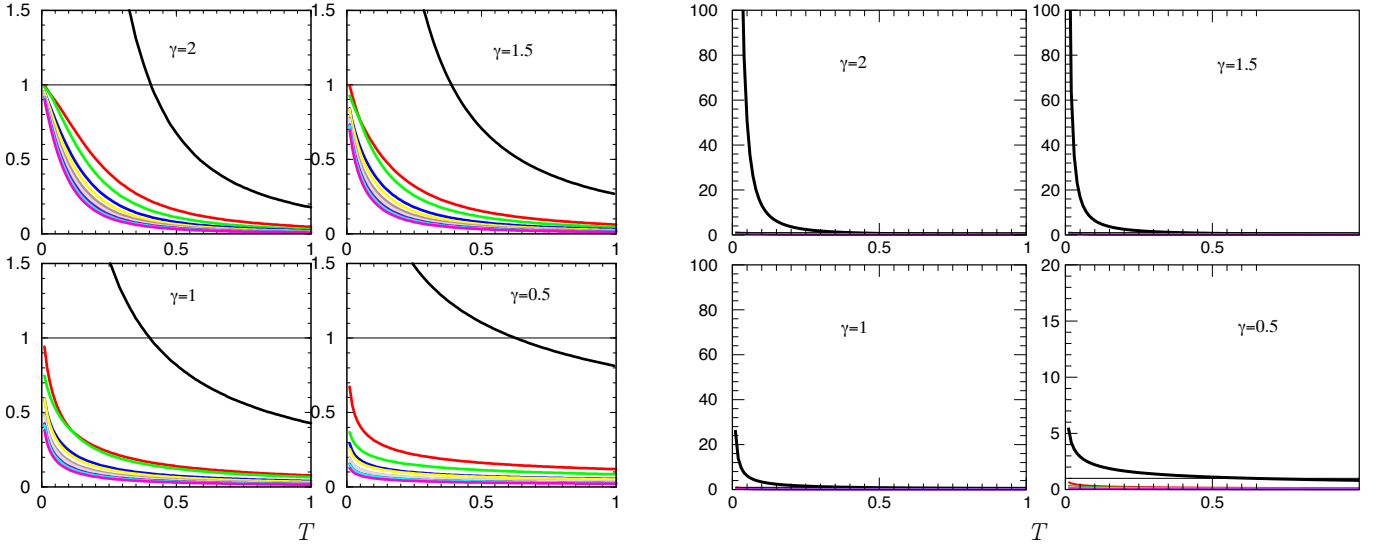
FIG. S4. The first 10 eigenvalues of the equation for $\Phi(\omega_m)$ for different γ. The existence of the threshold in N would imply that the largest eigenvalue saturates at a finite value at $T \to 0$. We see instead that it keeps increasing with decreasing T. We verified that the largest eigenvalue λ follows our result: $\lambda \propto 1/T^\gamma$. Upper and lower panels show the same behavior in different vertical ranges.

un-tilded notation, $\tilde{\Sigma}(\omega_m) \to \Sigma(\omega_m)$ and $\tilde{\Phi}(\omega_m) \to \Phi(\omega_m)$. These variables, however, should not be confused with $\Sigma(\omega_m)$ and $\Phi(\omega_m)$ in Eq. (S5).

We now show that the cancellation of the singular terms holds even when the pairing is not an s-wave. For this, we move one step back and consider the pairing mediated by a critical collective mode with a momentum-dependent propagator $\chi(k_\parallel - k'_\parallel, \omega_m - \omega'_m)$ between fermions at the Fermi surface ($|\mathbf{k}| = |\mathbf{k}'| = k_\mathbf{F}$).

For definiteness, we assume that the susceptibility is peaked at zero transferred momentum and set

$$\chi(\mathbf{k} - \mathbf{k}', \omega_m - \omega'_m) \propto \frac{1}{(\mathbf{k} - \mathbf{k}')^2 + |\omega_m - \omega'_m|^{2\gamma} + \xi^{-2}} \tag{S11}$$

At the critical point, $\xi^{-1} = 0$, and 1D integration over $k - k'$ yields $\int dx \chi(x, \Omega_m) \propto 1/|\Omega_m|^\gamma$. Other forms of χ, which yield the same frequency dependence of the "local" susceptibility can also be used – the end result of the analysis of $m = m'$ term will be the same.

Integrating in the fermionic propagators over the momenta transverse to the Fermi surface, we obtain the set of Eliashberg equations in the form

$$\Sigma(k_\parallel, \omega_m) = T \sum_{m'} \int \frac{dk'_\parallel}{4\pi v_F} \chi(k_\parallel - k'_\parallel, \omega_m - \omega'_m) \operatorname{sgn}(\omega'_m),$$

$$\Phi(k_\parallel, \omega_m) = T \sum_{m'} \int \frac{dk'_\parallel}{4\pi v_F} \frac{\chi(k_\parallel - k'_\parallel, \omega_m - \omega'_m)}{|\omega'_m + \Sigma(k'_\parallel, \omega'_m)|} \Phi(k'_\parallel, \omega'_m), \tag{S12}$$

where both $\mathbf{k}$ and $\mathbf{k}'$ are on the Fermi surface. In both equations in (S12), the integrals over $k'_\parallel$ in the term with $m = m'$ are singular at the quantum critical point $\xi^{-1} = 0$. To regularize the divergence, we again keep ξ finite at intermediate steps and set it to infinity only at the end of calculations.

Like before, we introduce the superconducting gap function as

$$\Delta(k_\parallel, \omega_m) \equiv \frac{\Phi(k_\parallel, \omega_m)}{1 + \Sigma(k_\parallel, \omega_m)/\omega_m}, \tag{S13}$$

and reexpress the equation for Φ in (S12) as

$$
\begin{aligned}
\Delta(k_\|, \omega_m) =& T \sum_{m'} \int \frac{dk'_\|}{4\pi} \chi(k_\| - k'_\|, \omega_m - \omega'_m) \left[\frac{\Delta(k'_\|, \omega'_m)}{|\omega'_m|} - \frac{\Delta(k_\|, \omega_m)}{|\omega_m|} \right] \\
=& T \sum_{m' \neq m} \int \frac{dk'_\|}{4\pi} \chi(k_\| - k'_\|, \omega_m - \omega'_m) \left[\frac{\Delta(k'_\|, \omega'_m)}{|\omega'_m|} - \frac{\Delta(k_\|, \omega_m)}{|\omega_m|} \right] \\
&+ T \int \frac{dk'_\|}{4\pi} \chi(k_\| - k'_\|) \frac{\Delta(k'_\|, \omega_m) - \Delta(k_\|, \omega_m)}{|\omega_m|}
\end{aligned}
\tag{S14}
$$

The terms with $\Delta(k'_\|, \omega_m)$ and $\Delta(k_\|, \omega_m)$ in the last line in Eq. (S14), when taken separately, diverge at $\xi \to \infty$. However, the singular terms obviously cancel out in the full expression. This has been noted in Ref. S2. The remainder is non-singular, but is also non-zero for a k-dependent gap function, and we label it as

$$
\Delta(k_\|, \omega_m) \beta(k_\|, \omega_m) \equiv T \int \frac{dk'_\|}{4\pi} \chi(k_\| - k'_\|) \frac{\Delta(k'_\|, \omega_m) - \Delta(k_\|, \omega_m)}{|\omega_m|}.
\tag{S15}
$$

Introducing then

$$
\tilde{\Sigma}(k_\|, \omega_m) = T \sum_{m' \neq m} \int \frac{dk'_\|}{4\pi v_F} \chi(k_\| - k'_\|, \omega_m - \omega'_m) \operatorname{sgn}(\omega'_m),
\tag{S16}
$$

and

$$
\tilde{\Phi}(k_\|, \omega_m) = \Delta(k_\|, \omega_m) \left(1 + \frac{\tilde{\Sigma}(k_\|, \omega_m)}{\omega_m} \right),
\tag{S17}
$$

we re-express Eliashberg equations as

$$
\begin{aligned}
\tilde{\Sigma}(k_\|, \omega_m) =& \tilde{\Sigma}(\omega_m) = T \sum_{m' \neq m} \int \frac{dk'_\|}{4\pi v_F} \chi(k_\| - k'_\|, \omega_m - \omega'_m) \operatorname{sgn}(\omega'_m), \\
\tilde{\Phi}(k_\|, \omega_m) \left(1 - \frac{\beta(k_\|, \omega_m)}{1 + \tilde{\Sigma}(k_\|, \omega_m)/\omega_m} \right) =& T \sum_{m' \neq m} \int \frac{dk'_\|}{4\pi v_F} \frac{\chi(k_\| - k'_\|, \omega_m - \omega'_m)}{|\omega'_m + \Sigma(k'_\|, \omega'_m)|} \tilde{\Phi}(k'_\|, \omega'_m),
\end{aligned}
\tag{S18}
$$

In a general case, this expression cannot be reduced to integral equation in frequency only, because β term cannot be absorbed into the coupling constant, and also because the integration over k' along the Fermi surface cannot be carried out explicitly if Δ is momentum-dependent. This last complication is often by-passed by approximating the momentum dependence of Δ by one of lattice harmonics. Then momentum integral can be done explicitly.

An additional complication arises when $\chi(k, \omega_m)$ is peaked at finite $\mathbf{k}$ [$\mathbf{k} = (\pi, \pi)$ for antiferromagnetic QCP]. Then integration over the component of $\mathbf{k}'$ transverse to the Fermi surface necessarily introduces an additional term into the bosonic propagator for $\mathbf{k} = \mathbf{k}' = k_F$, when $\mathbf{k}_F$ is away from a hot spot. As a result, the momentum integration in Eq. (S18) cannot be carried out explicitly even if one approximates the gap functions in hot regions as constants and neglect the dependence of the self-energy $\tilde{\Sigma}(k'_\|, \omega_m)$ on $k'_\|$. For example, if we take antiferromagnetic spin-fermion model with

$$
\chi(k, \Omega_m) = \frac{\bar{g}}{k^2 + \gamma|\Omega_m| + \xi^{-2}},
\tag{S19}
$$

expand the dispersion near the hot spot $\mathbf{k}_h$ as $\epsilon_k = v_x \delta k_x + v_y \delta k_y$ and the one near the hot spot at $\mathbf{k}_h + (\pi, \pi)$ as $\epsilon_{k+(\pi,\pi)} = v_x \delta k_x - v_y \delta k_y$, integrate over momenta $\mathbf{k}'$ in the gap equation transverse to the Fermi surface and approximate the bosonic propagator by its value for particles at the Fermi surface, we obtain effective

$$
\chi(k_\|, k'_\|, \omega_m) = \frac{\bar{g}}{(k'_\| - k_\|)^2 + [4v_y^2/(v_x^2 + v_y^2)]k_\|^2 + \gamma|\Omega|_m + \xi^{-2}}.
\tag{S20}
$$

Without the additional $k_\|^2$ term one could explicitly integrate over $k_\| - k'_\|$ and obtain Eq. (1) from the main text with $g = \bar{g}^2/16\gamma$. In the presence of the $k_\|^2$ term, the kernel is no longer a function of $k'_\| - k_\|$, and one has to solve integral equation in both momentum and frequency.

This additional complication, however, does not introduce qualitative changes as additional terms do not change the scaling form of the gap equation at relevant frequencies (i.e., typical $k'_\parallel \sim k_\parallel \sim \gamma \Omega_m$). Furthermore, the strength of the additional term in Eq. (S20) for the antiferromagnetic case and the strength of β term, which is the "leftover" from $m = m'$ term in the gap equation for non-s-wave gap, depend on the ratio of v_y/v_x (Ref. S1) and can be made small by varying the geometry of the Fermi surface. For this reason, we neglected the complications due to the "leftover" term $m = m'$ and due to the presence of additional $k_\parallel^2$ in $\chi(k, k', \Omega_m)$ in the hot spot case (and to momentum dependence of the self-energy along the Fermi surface), integrated explicitly over $k'_\parallel - k_\parallel$, and analyzed the integral equation for $\Phi(\omega_m)$ in the frequency domain.

II. THE FAILURE OF THE ARGUMENT FOR THE THRESHOLD

We present numerical and analytical arguments for the failure of the argument for the existence of the threshold in N for superconductivity at the quantum-critical point.

The numerical argument is the following. If the threshold exists, then at the threshold value $N = N_{cr}$, the linearized gap equation has a solution at $T = 0$. One can verify this by solving the gap equation as an eigenvalue/eigenfunction equation, with N playing the role of the eigenvalue. The existence of $N = N_{cr}$ would imply that the largest eigenvalue of the gap equation tends to a finite value N_{cr} at $T \to 0$.

In Fig. S4 we present the solution of the gap equation for different γ on a mesh of 2000×2000 Matsubara frequencies. We clearly see that for all γ, one eigenvalue gradually increases as T decreases and shows no tendency of saturation. We have explicitly verified that at small T, this eigenvalue grows as $1/T^\gamma$, precisely as we found in the main text.

There are several analytical arguments as well. They all are valid for $\gamma < 1$, when there is a finite frequency ω_0, above which the self-energy $\Sigma(\omega_m) = (\omega_m)^{1-\gamma}\omega_0^\gamma$ becomes smaller than the bare frequency. The analysis of the gap equation at $T = 0$ for $\gamma > 1$ is more involved as it requires a proper regularization of divergencies. There is no such problem in finite T analysis because the self-energy is finite at $T > 0$ for all γ.

The goal of analytical reasoning is to show that the oscillatory solution with $\chi_{pp}(\omega_m) \propto (\omega_0/|\omega_m|)^{\gamma/2 \pm i\beta}$, which, we remind, is valid for $\omega_m \ll \omega_0$, cannot be matched with the solution at $\omega_m \gg \omega_0$. One argument, which requires some lengthy calculations and will be presented in a separate publication, is based on the fact that, by power counting, the corrections to the oscillatory solutions should hold in powers of $(\omega_m/\omega_0)^\gamma$. These corrections come from keeping the bare ω term in the fermionic propagator along with the fermionic self-energy $\Sigma(\omega_m)$. If the prefactors for the series were finite, the corrections would be determined by internal ω'_m of order ω_m, i.e. the theory would remain local. The locality is essential as it is assumed when the pairing problem is reformulated in terms of a differential RG equation rather than the integral equation, as in Refs. S3–S5. However, the calculation shows that the prefactors for these corrections are ultra-violent divergent within the perturbation theory, which implies that they are determined by internal frequencies ω'_m of order ω_0 rather than ω_m. This implies that beyond leading order in ω, the theory becomes non-local, and the exact behavior even at the smallest frequencies is determined by frequencies of order ω_0, where the bare ω term in the fermionic propagator is no longer small compared to $\Sigma(\omega_m)$.

Another analytical argument against the existence of the threshold is the following. Consider again the equation for $\Phi(\omega)$ at $T = 0$ (Eq. (1) from the main text), and rescale frequencies by g. We then have

$$\Phi(\Omega) = \frac{1-\gamma}{2N} \int_{-\infty}^{\infty} \frac{\Phi(\omega)d\omega}{|\omega - \Omega|^\gamma |\omega|^{1-\gamma}} \frac{1}{1 + (1-\gamma)|\omega|^\gamma} \tag{S21}$$

Let's define the oscillatory solutions $\Phi(\omega_m) \propto |\omega_m|^{-\gamma/2 \pm i\bar{\beta}}$ as $\Phi_\pm(\Omega)$. These $\Phi_\pm(\Omega)$ satisfy

$$\Phi_\pm(\Omega) = \frac{1-\gamma}{2N} \int_{-\infty}^{\infty} \frac{\Phi(\omega)d\omega}{|\omega - \Omega|^\gamma |\omega|^{1-\gamma}} \tag{S22}$$

Let's multiply Eq. (S21) by $\frac{\Phi_\pm(\Omega)}{|\Omega|^{1-\gamma}}$ and integrate it over Ω. We obtain

$$\int_{-\infty}^{\infty} \frac{\Phi(\Omega)\Phi_\pm(\Omega)d\Omega}{|\Omega|^{1-\gamma}} = \frac{1-\gamma}{2N} \int_{-\infty}^{\infty} \frac{\Phi(\omega)d\omega}{|\omega|^{1-\gamma}(1 + (1-\gamma)|\omega|^\gamma)} \int_{-\infty}^{\infty} \frac{\Phi(\Omega)d\Omega}{|\omega - \Omega|^\gamma |\Omega|^{1-\gamma}} = \int_{-\infty}^{\infty} \frac{\Phi(\omega)\Phi_\pm(\omega)d\omega}{|\omega|^{1-\gamma}} \frac{1}{1 + (1-\gamma)|\omega|^\gamma} \tag{S23}$$

We see that if Eq. (S21) has a solution $\Phi(\omega)$, then it must satisfy

$$\int_{-\infty}^{\infty} \frac{\Phi(\omega)G_1(\omega)d\omega}{|\omega|^{1-\gamma}(1 + (1-\gamma)|\omega|^\gamma)} = 0, \tag{S24}$$

where

$$G_1(\omega) = |\omega|^\gamma \Phi_\pm(\omega) = |\omega|^{\gamma/2 \pm i\bar{\beta}} \tag{S25}$$

Now multiply Eq. (S21) by $\frac{G_1(\Omega)}{|\Omega|^{1-\gamma}(1+(1-\gamma)|\Omega|^\gamma)}$ and integrate over Ω to obtain

$$0 = \int_{-\infty}^\infty \frac{\Phi(\omega)d\omega}{|\omega|^{1-\gamma}} \frac{1}{1+(1-\gamma)|\omega|^\gamma} \int_{-\infty}^\infty \frac{d\Omega}{|\Omega-\omega|^\gamma} \frac{G_1(\Omega)}{|\Omega|^{1-\gamma}(1+(1-\gamma)|\Omega|^\gamma)} = \int_{-\infty}^\infty \frac{\Phi(\omega)G_2(\omega)d\omega}{|\omega|^{1-\gamma}(1+(1-\gamma)|\omega|^\gamma)}, \tag{S26}$$

where

$$G_2(\omega) = \int_{-\infty}^\infty \frac{d\Omega}{|\Omega-\omega|^\gamma} \frac{G_1(\Omega)}{|\Omega|^{1-\gamma}(1+(1-\gamma)|\Omega|^\gamma)}, \tag{S27}$$

and so on. We will get then the series of functions

$$G_n(\omega) = \int_{-\infty}^\infty \frac{d\Omega}{|\Omega-\omega|^\gamma} \frac{G_{n-1}(\Omega)}{|\Omega|^{1-\gamma}(1+(1-\gamma)|\Omega|^\gamma)}, \tag{S28}$$

such that for any n,

$$\int_{-\infty}^\infty \frac{\Phi(\omega)G_n(\omega)d\omega}{|\omega|^{1-\gamma}(1+(1-\gamma)|\omega|^\gamma)} = 0. \tag{S29}$$

At small frequencies, the oscillatory solution is $\Phi(\omega_m) = |A|\left(e^{i\phi}\Phi_+(\omega_m) + e^{-i\phi}\Phi_-(\omega_m)\right)$ This solution has a single free parameter – a phase ϕ. One parameter cannot satisfy infinite set of equation, unless all G_n are multiples of G_1. One can easily check explicitly that they are not. This implies that the oscillatory solution is actually not the solution of the actual equation for $\Phi(\omega_m)$, even at the smallest frequencies.

III. T_c AT LARGE N

We begin with Eq. (2) of the main text

$$\bar{\Phi}_0 = \frac{K_T}{N}\bar{\Phi}_{-1} + \frac{1}{N}\sum_{m>0} \frac{\bar{\Phi}_m}{\bar{\Sigma}_m}\left[\frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma}\right],$$

$$\bar{\Phi}_{m>0} = \frac{K_T}{N}\left[\frac{\bar{\Phi}_0}{m^\gamma} + \frac{\bar{\Phi}_{-1}}{(m+1)^\gamma}\right] + \frac{1}{N}\sum_{m'>0,m'\neq m} \frac{\bar{\Phi}_m}{\bar{\Sigma}_m}\left[\frac{1}{|m-m'|^\gamma} + \frac{1}{(m+m'+1)^\gamma}\right], \tag{S30}$$

where we remind $\bar{\Phi}_m = \Phi_m/(\pi T K_T)$, $K_T = [g/(2\pi T)]^\gamma$, and

$$\bar{\Sigma}_m = \sum_{m'\neq m} \frac{\operatorname{sgn}(2m'+1)}{|m-m'|^\gamma}. \tag{S31}$$

At the leading order in $1/N$, only the first term in the r.h.s. of the equation for $\bar{\Phi}_0$ matters, and using $\bar{\Phi}_0 \equiv \bar{\Phi}_{-1}$, we immediately obtain that T_c is the solution of $K_T/N = 1$, i.e., $T_c \approx g/(2\pi N^{1/\gamma})$.

To obtain the next order correction to T_c, we rewrite the first equation of (S30) as

$$\frac{K_T}{N} = 1 - \frac{1}{N}\sum_{m>0} \frac{\tilde{\Phi}_m}{\bar{\Sigma}_m}\left[\frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma}\right], \tag{S32}$$

where $\tilde{\Phi}_m \equiv \bar{\Phi}_m/\bar{\Phi}_0$, and its leading order value is, from second equation of (S30),

$$\tilde{\Phi}_m = \left[\frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma}\right]. \tag{S33}$$

Plugging (S33) into (S32) we obtain

$$\frac{K_T}{N} = 1 - \frac{1}{N}\sum_{m>0} \frac{1}{\bar{\Sigma}_m}\left[\frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma}\right]^2 \equiv 1 - \frac{\delta_\gamma}{N}. \tag{S34}$$

Solving this equation for T_c, we obtain at a large N

$$T_c \approx \frac{g}{2\pi} \frac{1}{N^{1/\gamma}} \left(1 + \frac{\delta_\gamma}{N\gamma}\right), \tag{S35}$$

where $\delta_\gamma = \sum_{m>0} [1/m^\gamma + 1/(m+1)^\gamma]^2 / \bar{\Sigma}_m$. This is Eq. (3) from the main text. At $\gamma \to \infty$, only the first term in the sum contributes to δ_γ. Using the fact that $\bar{\Sigma}_1 = 2$ [see Eq. (S31)], we obtain $\lim_{\gamma\to\infty} \delta_\gamma = 1/2$.

IV. T_c AT LARGE γ

We again depart from Eq. (1) in the main text

$$\Phi(\omega_m) = \frac{1}{N} g^\gamma \pi T \sum_{m' \neq m} \frac{\Phi(\omega_{m'})}{|\omega_{m'} + \Sigma(\omega_{m'})|} \frac{1}{|\omega_m - \omega_{m'}|^\gamma}$$

$$\Sigma(\omega_m) = g^\gamma \pi T \sum_{m' \neq m} \frac{\text{sign}(\omega_{m'})}{|\omega_m - \omega_{m'}|^\gamma} \tag{S36}$$

In the limit $\gamma \to \infty$ we neglect all terms in the sums except the ones with $m' = m \pm 1$. Then after the change of notations

$$\phi_m = \frac{\Phi(\omega_{m'})}{|\omega_{m'} + \Sigma(\omega_{m'})|}, \tag{S37}$$

we get

$$\phi_m |\omega_m + \Sigma_m| = \frac{1}{N} \pi K_T T (\phi_{m-1} + \phi_{m+1}),$$

$$\Sigma_m = \begin{cases} 0, & m = 0, -1; \\ 2\pi K_T T, & m > 0; \end{cases} \tag{S38}$$

where we used the notation introduced in the paper $K_T = \left(\frac{\bar{g}}{2\pi T}\right)^\gamma$.

Treating $m = 0$ and $m > 0$ separately we find

$$\phi_1 = \phi_0 (s - 1), \qquad \phi_{m+1} = \phi_m (s(2m+1) + 2N) - \phi_{m-1}, \tag{S39}$$

where $s = N/K_T$. We see, that for a given ϕ_0, this equation yields all ϕ_m for $m > 1$. In addition, we need to satisfy the first relation in (S39) and the condition $\phi_{m\to\infty} \to 0$.

The recurrence relation (S39) can be solved by the method of generating functions. We also recognize in (S39) the recurrence relation for the Bessel functions $zZ_{\nu-1}(z) + zZ_{\nu+1} = 2\nu Z_\nu(z)$, where $Z_\nu(z)$ is any of Bessel functions.

The Bessel function, which tends to 0 as $m \to \infty$, is $J_\nu(z)$, so

$$\phi_m = a J_{m+1/2+N/s}(1/s), \tag{S40}$$

where a is an arbitrary constant. We now need to satisfy the first relation of (S39), i.e.

$$a J_{3/2+N/s}(1/s) = (s-1) a J_{1/2+N/s}(1/s). \tag{S41}$$

Thus

$$\frac{J_{3/2+N/s}(1/s)}{J_{1/2+N/s}(1/s)} = s - 1. \tag{S42}$$

This equation defines s (or K_T) as a function of N.

Solving Eq. (S42) numerically for $N = 1$ we obtain

$$s(N = 1) = 1.1843, \tag{S43}$$

as we stated in the main text. At large N, using the asymptotic expansion of the Bessel function for large order at fixed argument,

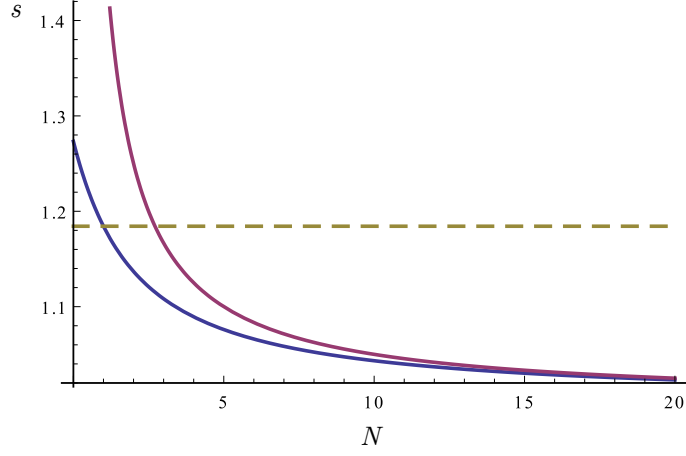$$J_\nu(z) = \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu,$$

FIG. S5. s vs. N given by Eq. (S5) – blue line, $1 + \frac{1}{2N}$ – red line, $s(N = 1)$ – yellow dashed line.

we find

$$s \approx 1 + \frac{1}{2N}. \tag{S44}$$

The full function $s(N)$ is shown in Fig. S5.

V. T_c AT SMALL γ

Here we show the details of our computation of T_c at small γ and $N = O(1)$. Fig. 3 from the main text shows that in this regime the actual T_c is much larger than the contribution of only the lowest Matsubara frequencies, i.e. it is determined by ω_m with $m > 0$. This figure also shows that $T_c \gg \omega_0$, where $\omega_0 = g[2/(1 - \gamma)]^{1/\gamma}$ is the scale below which the self-energy $\Sigma(\omega_m) = (\omega_m)^{1-\gamma}\omega_0^\gamma$ is larger than ω_m. Then, the self-energy is irrelevant simply because at $T = T_c$, $\omega_m \gg \Sigma(\omega_m)$ for all m. Neglecting $\Sigma(\omega_{m'})$ in Eq. (S36), replacing the summation over m by the integration over the frequency (which is justified because typical $m' \gg 1$), introducing new variable $x = (\omega_0/\omega)^\gamma$ and assuming that $\Phi(x)$ is a smooth function, we reduce integral equation (S36) to a differential equation

$$\Phi''(x) + \frac{\Phi(x)}{x\gamma N} = 0, \tag{S45}$$

with the boundary condition $\Phi(x \ll 1) \propto x$ (i.e. $\Phi(\omega_m) \propto 1/|\omega_m|^\gamma$ at $\omega_m \gg \omega_0$). The solution of this equation is

$$\Phi(x) \propto \sqrt{x} J_1 \left(2\sqrt{x/\gamma N}\right), \tag{S46}$$

where J_1 is a Bessel function of the first kind. At small argument $J_1(y) \propto y$, at large y $J_1(y)$ oscillates. Because $x \sim \gamma N$ is the only scale in $\Phi(x)$, it is natural to identify the corresponding frequency $\omega_m = \omega_0/x^{1/\gamma}$ with T_c. This yields

$$T_c \sim \omega_0(\gamma N)^{-1/\gamma} \sim \frac{g}{2\pi N^{1/\gamma}} e^{\log(b/\gamma)/\gamma} \gg \frac{g}{2\pi N^{1/\gamma}}, \tag{S47}$$

where $b = O(1)$.

[S1] Ar. Abanov, A.V. Chubukov, and J. Schmalian, Adv. Phys. **52**, 119 (2003); M. A. Metlitski and S. Sachdev, Phys. Rev. B, **82**, 075127 (2010); **82**, 075128 (2010); K. B. Efetov, H. Meier and C. Pépin, Nat. Phys. **9** 442, (2013).
[S2] A. J. Millis, S. Sachdev, and C. M. Varma, Phys. Rev. B 37, 4975 (1988); Ar. Abanov, A.V. Chubukov, and M. Norman, Phys. Rev. B 78, 220507 (2008).

[S3] R. Mahajan, D. M. Ramirez, S. Kachru, and S. Raghu, Phys. Rev. B 88, 115116 (2013); A. L. Fitzpatrick, S. Kachru, J. Kaplan, and S. Raghu, Phys. Rev. B 88, 125116 (2013); A. L. Fitzpatrick, S. Kachru, J. Kaplan, and S. Raghu, Phys. Rev. B 89, 165114 (2014); G. Torroba and H. Wang, Phys.Rev. B 90, 165144 (2014); A. L. Fitzpatrick, G. Torroba, and H. Wang, Phys.Rev. B 91, 195135 (2015).

[S4] M. A. Metlitski, D. F. Mross, S. Sachdev, and T. Senthil, Phys. Rev. B 91, 115111 (2015).

[S5] S. Raghu, G. Torroba, and H. Wang, Phys. Rev. B 92, 205104 (2015).