

# Origin of Quark-Lepton Flavor in SO(10) with Type II Seesaw

Bhaskar Dutta<sup>1</sup>, Yukihiro Mimura<sup>2</sup> and R.N. Mohapatra<sup>2</sup>

<sup>1</sup>*Department of Physics, Texas A&M University,  
College Station, TX 77843-4242, USA*

<sup>2</sup>*Maryland Center for Fundamental Physics and Department of Physics,  
University of Maryland, College Park, MD, 20742, USA*

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## Abstract

Diverse mass and mixing patterns between the quarks and leptons makes it challenging to construct a simple grand unified theory of flavor. We show that SO(10) SUSY GUTs with type II seesaw mechanism giving neutrino masses provide a natural framework for addressing this issue. A simple ansatz that the dominant Yukawa matrix (the  $\mathbf{10}$ -Higgs coupling to matter) has rank one, appears to simultaneously explain both the large lepton mixings as well as the observed quark flavor hierarchy in these models. A testable prediction of this ansatz is the neutrino mixing,  $U_{e3}$ , which should be observable in planned long baseline experiments.

## I. INTRODUCTION

Understanding the origin of the hierarchical pattern of quark masses and mixings has long been recognized as a challenge for physics beyond the standard model [1]. The discovery of neutrino masses and mixings with totally different flavor pattern than quarks (i.e.  $\theta_{23}^l \sim 45^\circ$  and  $\theta_{12}^l \simeq 35^\circ$  as against  $\theta_{23}^q \sim 2.5^\circ$  and  $\theta_{12}^q \sim 13^\circ$ ) has added more mystery to the flavor problem. In generic bottom-up pictures where quarks and leptons are treated as different species of particles with no particular relation between them, this problem is not so serious since one can simply focus on each sector separately, as is often done for neutrinos [2]. However, in grand unified theories where the quarks and leptons unify at a very high scale, one would naively expect that their masses and mixings would exhibit a similar pattern. The fact that they are so different may be hint of some really new exciting underlying physics. In this note we address this question in the context of supersymmetric SO(10) models with renormalizable Yukawa couplings being responsible for fermion masses.

We show that in SO(10) models with **10**, **126** plus possibly another **10** or **120** Higgs fields where fermion masses are generated by renormalizable Yukawa couplings [3] only and where type II seesaw is responsible for neutrino masses [4], there is a natural way to have a unified understanding of both large lepton mixings and small quark ones. The basic idea is to require that one of the **10** Yukawa couplings is the dominant one contributing to up, down and charged lepton masses and has rank one with other smaller couplings providing neutrino masses as well as most of the quark lepton flavor hierarchy. Rank one plus small corrections as a way to unravel fermion flavor in D-brane models was discussed in [5]. We find that SO(10) models with type II seesaw [3, 4, 6, 7] are ideally suited for such an ansatz. A specific form of the rank one matrix can lead to tri-bimaximal mixing with corrections dictated by the quark flavor pattern.

This paper is organized as follows: in sec. II, we review the mass formulae in SO(10) models with renormalizable couplings; in sec. III, we summarize our basic strategy for understanding the quark lepton flavor in a unified manner, discuss the rank one ansatz and apply it both the two generation case (IIIA) and three generation cases (IIIB). In sec. IIIC, we present realistic three generation models and outline their predictions. Sec. IV is devoted to some specific conjectures for the rank one matrix which could emerge from discrete symmetry models with specific discussion on the correction to tri-bimaximal mixings. Sec. V is devoted to a possible way to obtain the rank one ansatz in SO(10) models and in sec. VI, we present our conclusions.

## II. OVERVIEW OF RENORMALIZABLE SUSY SO(10) MODELS FOR FERMION MASSES

The basic idea in this class of models is to consider SUSY SO(10) theory with Higgs fields that give fermion masses to be in  $\mathbf{10}$  (denoted by  $H$ ) and  $\mathbf{126} + \overline{\mathbf{126}}$  (denoted by  $\Delta$  and  $\overline{\Delta}$ ) plus either an extra  $\mathbf{10}$  ( $H'$ ) or  $\mathbf{120}$  ( $\Sigma$ ). The GUT symmetry is broken by  $\mathbf{210} + \mathbf{54} + \mathbf{126} + \overline{\mathbf{126}}$  [8]. The Yukawa superpotential of this model is:

$$W_Y = h \psi \psi H + f \psi \psi \overline{\Delta} + h' \psi \psi (\Sigma \text{ or } H'), \quad (1)$$

where the symbol  $\psi$  stands for the  $\mathbf{16}$  dimensional representation of SO(10) that represents the matter fields. The coupling matrices  $h$  and  $f$  are symmetric, and  $h'$  is symmetric or anti-symmetric depending on whether we adopt  $H'$  or  $\Sigma$ . The representations  $H$ ,  $H'$  and  $\Delta$  have two standard model (SM) doublets in each of them whereas  $\Sigma$  has four such doublets. The general way to understand so many SM doublets is that at the GUT scale  $M_U$ , once the GUT and the  $B - L$  symmetry are broken, one linear combination of the up-type doublets and one of down-type ones remain almost massless whereas the remaining ones acquire GUT scale masses just like the color triplet and other non-MSSM multiplets. The electroweak symmetry is broken after the light MSSM doublets (to be called  $H_{u,d}$ ) acquire vacuum expectation values (vevs) and they then generate the fermion masses. The resulting mass formulae for different fermion masses are given by:

$$\begin{aligned} Y_u &= h + r_2 f + r_3 h', \\ Y_d &= r_1 (h + f + h'), \\ Y_e &= r_1 (h - 3f + c_e h'), \\ Y_{\nu D} &= h - 3r_2 f + c_\nu h', \end{aligned} \quad (2)$$

where  $Y_a$  are mass matrices divided by the electro-weak vev  $v_{wk}$  and  $r_i$  and  $c_{e,\nu}$  are the mixing parameters which relate the  $H_{u,d}$  to the doublets in the various GUT multiplets. More precisely, the matrices  $h$ ,  $f$  and  $h'$  in  $Y_a$  are multiplied by the Higgs mixings. The precise definitions of the couplings and the Higgs mixings are given in ref.[7]. When  $H'$  is adopted for the  $h'$  coupling,  $c_e = 1$  and  $c_\nu = r_3$ . In generic SO(10) models of this type, the neutrino mass formula has a type I [9] and a type II [10] contributions:

$$\mathcal{M}_\nu = f v_L - M_D \frac{1}{f v_R} M_D^t, \quad (3)$$

where  $v_L$  is the vev of the  $B - L = 2$  triplet in the  $\mathbf{126}$  Higgs field and is given by  $v_L \simeq \frac{\lambda \mu v_{wk}^2}{M_{\Delta L}^2}$ . Note that in general, the two contributions to neutrino mass depend on two different parameters and it is easy to have symmetry breaking pattern in SO(10) [11] where the first

contribution (the type II term) dominates over the type I term. The neutrino mass formula then becomes

$$\mathcal{M}_\nu = f v_L. \quad (4)$$

Note that  $f$  is the same coupling matrix that appears in the charged fermion masses in Eq. (2), up to factors from the Higgs mixings and the Clebsch-Gordan coefficients. The equations (2) and (4) are the key equations in our unified approach to address the flavor problem.

The main hypothesis of our approach is that the fermion mass formula of Eq. (2) are dominated by the matrix  $h$  with the contributions of  $f$  and  $h'$  being small perturbations. In the limit of  $f, h' \rightarrow 0$ , the quark and lepton mixings vanish as do the neutrino masses. We will show below that this simple hypothesis combined with Eq. (4) can simultaneously explain large lepton mixings while keeping the quark mixings being proportional to  $|f|/|h|$  and hence small. We will subsequently assume that the matrix  $h$  has rank one in which case the mass hierarchy can also be explained in a natural manner.

### III. EXPLAINING QUARK-LEPTON FLAVOR HIERARCHIES

The quark and lepton mixing matrices are given by the product of diagonalizing unitary matrices for quark and lepton mass matrices as follows: denoting the diagonalizing matrices of  $M_u, M_d$  by  $V_u$  and  $V_d$  respectively (e.g.,  $V_u M_u M_u^\dagger V_u^\dagger = \text{diag.}(m_u^2, m_c^2, m_t^2)$  and similarly for the down quark mass matrix), the CKM (Cabibbo-Kobayashi-Maskawa) quark mixing matrix is given as  $V_{\text{CKM}} = V_u V_d^\dagger$ . The PMNS (Pontecorvo-Maki-Nakagawa-Sakata) lepton mixing matrix is given as  $U_{\text{PMNS}} = (V_e V_\nu^\dagger)^*$  in the similar notation (e.g.,  $V_\nu \mathcal{M}_\nu V_\nu^\dagger = \text{diag.}(m_1, m_2, m_3)$ ).

In general, when two matrices with random  $O(1)$  elements are considered, the mixing angles of the relative diagonalizing unitary matrices are all  $O(1)$  in radian, while the eigenvalues can have a hierarchy of  $O(0.1)$ . In such an anarchical scenario, the neutrino masses and mixings can be explained (except for the CHOOZ bound of 13 neutrino mixing): the neutrino mixings are generically  $O(1)$  and there is a little hierarchy for the neutrino mass squared difference ratio  $\Delta m_{12}^2/\Delta m_{23}^2$  [12]. On the other hand, since the quark mixings are all smaller than  $O(1)$  and the masses of quarks and charged leptons are very much hierarchical, anarchic mass matrices in general provide no explanation of these observations. Besides, it appears that the mass ratios and CKM mixings have several correlations among them. It is therefore to be expected that the quark and lepton matrices instead of being independent anarchic matrices must have some relations among them and an underlying theory leading

to these relations. In this paper we find that SO(10) with type II seesaw could be such a theory.

When the fermion masses are given by the Eqs. (2) and (4), several possible outcomes are obtained by simple assumptions. To understand these possible outcomes from the Eqs.(2) and (4), let us first ignore  $h'$ . We then have the following possibilities:

*Assumption 1:*

Take  $h, f$  are general rank 3 matrices, and  $f$  is small. This is the case analyzed to fit observed experimental data and to obtain predictions from the minimality of the number of parameters in various papers [4]. Here, we list the properties resulting from the smallness of  $f$  without resorting to any numerical fit.

- The CKM mixings are small, due to the fact that there is an approximate up-down symmetry and  $V_{\text{CKM}} = V_u V_d^\dagger$  [13].
- Bottom-tau unification up to  $O(f/h)$ .
- The 3 neutrino mixings are generically of  $O(1)$  since  $h$  and  $f$  are unrelated matrices. The type II seesaw dominance of the neutrino mass is crucial for the generic largeness of the neutrino mixings.

Thus it is interesting that without any special assumption, the gross features of fermion mixings can be reproduced. This does not, however, throw any light on the mass hierarchies among quarks and leptons, though one can fit the experimental results by the choices of parameters (even in the type I seesaw) [4, 14, 15]. Since we use the experimental data as an input, these scenarios do not provide a fundamental understanding of either the mass hierarchy for quarks and charged leptons, or why the 13 neutrino mixing ( $U_{e3}$ ) is less than  $O(1)$ . Similar situation holds for models where  $h'$  is added [16, 17].

*Assumption 2:*

Let us next consider the specific case when  $h$  is a rank 1 matrix [7], and  $f$  is a rank 3 matrix with eigenvalues of  $f$  being hierarchical ( $f_1, f_2 \ll f_3$ ) and small compared to the elements of  $h$ . As we noted in ref.[7], this choice helps to suppress proton decay in SUSY SO(10) models without invoking huge cancellations among the colored Higgsino exchange amplitudes. In this case we will show in the next two subsections that the following results follow:

1. CKM mixings are small.
2. Approximate bottom-tau unification occurs.

3.  $\frac{m_c}{m_t} : \frac{m_s}{m_b} : \frac{m_\mu}{m_\tau} \simeq r_2 : 1 : -3$ .
4. The quark mixing are related as  $V_{cb} \sim m_s/m_b + e^{i\sigma} m_c/m_t$  (where  $\sigma$  is a phase) and  $V_{ub} \sim V_{cb} f_2/f_3$ .
5. Atmospheric and solar neutrino mixings are generically large, but 13 mixing is  $\sim f_2/f_3$ .

All these predictions are in qualitative agreement with observations. The advantage of the rank one assumption is that it naturally explains the mass hierarchies among quarks and leptons in addition to large lepton mixings. We emphasize that these features are obtained from the rank 1 assumption above without using any numerical inputs, and our claim here is not based on the scenario of the numerical predictions from a fit in which the minimality of parameters plays a key role.

Before we do a full demonstration of these results in the context of a three generation model, let us illustrate the first four points in the context of a two generation model.

### A. A two generation illustration

In this subsection, we apply our rank one hypothesis to the second and the third generation. We will confirm the results 1-4 mentioned above. The starting point is the mass relation from Eqs. (2) and (4) where we ignore the  $h'$  contribution. Using our assumption, we have  $h = (\sin \theta \quad \cos \theta)^t (\sin \theta \quad \cos \theta) h_3$  and  $f = \text{diag}(f_2, f_3)$  (without loss of generality, we can parameterize  $f$  to be diagonal). The parameter  $\theta$  is of  $O(1)$  in general. We now have ten parameters ( $\theta$ ,  $h_3$  and  $r_1$  as real parameters,  $f_2$ ,  $f_3$ , and  $r_2$  as complex parameters, and  $v_L$  for neutrino mass scale) describing ten observables of all lepton and quark mixings and masses.

One can easily obtain  $r_1 m_t \simeq m_b \tan \beta \simeq m_\tau \tan \beta$  at the leading order neglecting  $O(f_3/h_3)$  correction, where  $\tan \beta$  is a ratio of vevs of  $H_u$  and  $H_d$ . Therefore,  $r_1$  corresponds to the freedom of  $\tan \beta$ ,  $r_1 \sim \tan \beta/50$ .

When  $f_2 \ll f_3 \ll h_3$ , we obtain

$$\frac{m_c}{m_t} \simeq r_2 \frac{f_3}{h_3} \sin^2 \theta, \quad \frac{m_s}{m_b} \simeq \frac{f_3}{h_3} \sin^2 \theta, \quad \frac{m_\mu}{m_\tau} \simeq -3 \frac{f_3}{h_3} \sin^2 \theta. \quad (5)$$

Because  $m_c/m_t \ll m_s/m_b$ ,  $r_2$  is small, i.e.  $r_2 \simeq m_c/m_t/(m_s/m_b)$ .

To proceed further, we first diagonalize the charged fermion mass matrices to zeroth order in  $f_2, f_3 \rightarrow 0$ . The matrix diagonalizing this is given by

$$U_0 = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (6)$$

since  $U_0(\sin \theta \ \cos \theta)^t = (0 \ 1)^t$ . Let us now see how the small quark mixings arise despite large mixings in  $U_0$ . Because  $U_0 Y_d U_0^t$  has a small off-diagonal element,  $r_1(f_2 - f_3) \sin \theta \cos \theta$ , the down-type quark mass matrix is diagonalized by  $V_d = \tilde{V}_d U_0$ , where  $\tilde{V}_d$  is close to unit matrix whose off-diagonal element is  $\simeq f_3/h_3 \sin \theta \cos \theta$ . The up-type quark mass matrix is diagonalized by  $V_u = \tilde{V}_u U_0$ , where the off-diagonal element of  $\tilde{V}_u$  is  $\simeq r_2 f_3/h_3 \sin \theta \cos \theta$ . The quark mixing matrix is then given by  $V_{\text{CKM}} = V_u V_d^\dagger = \tilde{V}_u \tilde{V}_d^\dagger$ , in this product  $U_0$  cancels out leaving small mixings between the two generations i.e. small  $V_{cb} \simeq (1 - r_2) f_3/h_3 \sin \theta \cos \theta \simeq (m_s/m_b + e^{i\sigma} m_c/m_t) \cot \theta$ , where  $\sigma$  is a phase of  $r_2$ .

Coming now to lepton mixings, suppose that the charged lepton mass matrix is diagonalized by  $V_\ell$ , then it can be written as  $V_\ell = \tilde{V}_e U_0$ , where  $\tilde{V}_e$  is close to a unit matrix similarly to the quark sector, and is roughly equal to  $V_{\text{CKM}}^\dagger$ . Since the neutrino mass matrix is already diagonal as a parameterization, the PMNS matrix is given by the charged lepton mixings so that  $U_{\text{PMNS}} = V_\ell^* \simeq V_{\text{CKM}}^t U_L^0$ . This leads to a large lepton mixing as desired. In the two generation case,  $\theta$  describes approximately (up to small corrections of order  $V_{cb}$ ) the ‘‘atmospheric mixing angle’’. Since this was an input into our rank one ansatz, we can choose to be large to explain the observations.

If  $f_2, f_3$  and  $r_2$  are assumed to be real, there are six real parameters in this model. In this case,  $m_b, m_s, \theta_{\text{atm}}$  and  $m_2/m_3$  can be written as a function of  $m_c, m_t, m_\mu, m_\tau$  and  $V_{cb}$  for example. Even if  $f_2, f_3$  and  $r_2$  are all complex, we have the following approximate relation at the grand unified scale:

$$\begin{aligned} \frac{m_s}{m_b} &= V_{cb} \tan \theta \left( 1 + O\left(\frac{f_3}{h_3}\right) \right), \\ m_b &= m_\tau \left( 1 + O\left(\frac{f_3}{h_3}\right) \right), \\ m_s &= -\frac{1}{3} m_\mu \left( 1 + O\left(\frac{f_3}{h_3}\right) \right). \end{aligned} \tag{7}$$

Since the relations are satisfied under the assumption of approximate rank 1 property irrespective of the counting of freedom, they are stable even in the case of three generation model. Indeed, the predictivity from the minimality of the parameter is related to the  $O(f_3/h_3)$  corrections, and the minimality does not play a crucial role in the approximate relations from the rank 1 assumption.

It is known that there is a solution that the large atmospheric mixing is obtained even if the smallness of  $f$  is not assumed a priori. In the scenario, the  $b$ - $\tau$  mass convergence as well as the other experimental inputs predict the neutrino mixing as an output [4]. Our main goal in this section is not to give numerical predictions but rather to show how one can get qualitatively expected hierarchical pattern for masses and mixings. Later on we of course study the detailed numerical predictions. As it turns out there is a fine-tuned solution to

fit the experimental data even if  $f_3$  is comparable to  $h_3$ . however such a fine-tuned solution is sensitive to the numerical inputs, and therefore the numerical predictions in this case may be unstable under a possible higher order correction. In our case where the quark and lepton mass hierarchy is predicted by the rank 1 assumption, they are stable under radiative corrections.

While the qualitative predictions are in the expected range, we note that the approximate relation  $\tan \theta_{\text{atm}} \simeq (m_s/m_b)/V_{cb}$  is not very good agreement with the current observation, and small  $h'$  will be invoked to obtain the best fit of the experimental data. We emphasize that our final solutions do not use any fine tuned cancellations, and thus are stable even if we add small corrections to fit the numerical experimental data.

## B. Three generation case

The fermion mass equations for this case are those in Eq. (2) with all coupling matrices being  $3 \times 3$ . The assumption that  $h$  has rank one means that we can write it as

$$h = \begin{pmatrix} c \\ b \\ a \end{pmatrix} \begin{pmatrix} c & b & a \end{pmatrix}, \quad (8)$$

$$f = \text{diag}(f_1, f_2, f_3) \quad (f_{1,2} \ll f_3). \quad (9)$$

Again, we can parameterize  $f$  to be diagonal and  $a, b, c$  to be real without loss of generality. At first, we ignore  $h'$ . In order to analyze the detailed consequences of this assumption, we go to the basis where  $h$  is diagonal. This is achieved by the matrix:

$$U_0 = \begin{pmatrix} \cos \theta_s & \sin \theta_s & 0 \\ -\cos \theta_a \sin \theta_s & \cos \theta_a \cos \theta_s & -\sin \theta_a \\ -\sin \theta_a \sin \theta_s & \sin \theta_a \cos \theta_s & \cos \theta_a \end{pmatrix}, \quad (10)$$

where  $\tan \theta_s = -c/b$  and  $\tan \theta_a = \sqrt{b^2 + c^2}/a$  with

$$U_0 h U_0^t = \text{diag}(0, 0, h_3), \quad (11)$$

where  $h_3 = a^2 + b^2 + c^2$ . It is interesting to note that in the diagonalization matrix there is an ambiguity resulting from the residual SU(2) flavor symmetry in  $h$  (i.e. one of three mixing angles is not fixed at this stage). We choose the unitary matrix  $U_0$  to be an approximate leading order diagonalization matrix of  $Y_u, Y_d$ , and  $Y_e$  as in the previous subsection ( $V_u = \tilde{V}_u U_0, V_d = \tilde{V}_d U_0$ , and  $V_e = \tilde{V}_e U_0$  where  $\tilde{V}_u, \tilde{V}_d, \tilde{V}_e$  are close to a unit matrix). Then, once



the  $f$  contribution is included, the afore mentioned SU(2) flavor symmetry is broken and the ambiguity in mixing angles alluded to above is removed. We wish to note that in our original parameterization of  $U_0$ , we chose the 13 element to be zero since even after including the  $f$ -contribution, the 13 element goes to zero in the limit  $f_{1,2}/f_3 \rightarrow 0$  (which is the limit where  $Y_{u,d,e}$  is rank 2). We have not used prejudices from neutrino experiment.

By the same argument as in the case of two generations,  $U_0$  is cancelled out in the CKM mixing matrix and the quark mixings are small. The PMNS matrix is given by

$$U_{\text{PMNS}} = \tilde{V}_e^* U_0, \quad (12)$$

and since the off-diagonal elements of  $\tilde{V}_\ell$  are small (being related to quark mixings), neglecting the 23 and 13 quark mixings, we get for the solar and atmospheric mixing angles [5]

$$\begin{aligned} \theta_{\text{atm}} &\simeq \theta_a, \\ \theta_\odot &\simeq \theta_s \pm \theta_{13} \cot \theta_a \cos \alpha, \end{aligned} \quad (13)$$

where  $\alpha$  is defined as the diagonal phase matrix  $\text{diag}(1, e^{i\alpha}, e^{i\beta})$  needed to diagonalize the charged lepton mass matrix [18].

We also get a formula for  $U_{e3}$  as follows:

$$U_{e3} = (\tilde{V}_e)_{12} \sin \theta_a. \quad (14)$$

To proceed with the rest of the masses and mixings, let us define the matrices in the  $U_0$  rotation:  $\tilde{Y}_a \equiv U_0 Y_a U_0^t$ ,  $\tilde{f} \equiv U_0 f U_0^t$ , and so on. In this notation,  $\tilde{V}_a$  is a diagonalization matrix of  $\tilde{Y}_a$ . Because  $\tilde{f}_{23} = (f_2 - f_3) \sin \theta_a \cos \theta_a$  and  $\tilde{f}_{13} = (f_2 - f_1) \sin \theta_a \sin \theta_s \cos \theta_s$ , one can obtain

$$V_{ub} \simeq V_{cb} \frac{f_2 - f_1}{f_3} \frac{\sin \theta_s \cos \theta_s}{\cos \theta_a}. \quad (15)$$

Neglecting  $O(f_3/h_3)$  and  $O(f_{1,2}/h_3)$  corrections,  $\tilde{V}_a$  can be approximately as:

$$\tilde{V} = \begin{pmatrix} \cos \check{\theta} & -\sin \check{\theta} & 0 \\ \sin \check{\theta} & \cos \check{\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (16)$$

where

$$\sin \check{\theta} \simeq \frac{f_2 - f_1}{f_3} \frac{\cos \theta_a \sin \theta_s \cos \theta_s}{\sin^2 \theta_a}, \quad (17)$$

and thus

$$U_{e3} \simeq \frac{f_2 - f_1}{f_3} \cot \theta_a \sin \theta_s \cos \theta_s. \quad (18)$$

Thus we have obtained all the features listed before. Due to the generic largeness of the relative mixing angles of the unrelated matrices, solar and atmospheric neutrino mixing angles are of  $O(1)$  generically. On the other hand, 13 mixing is not in the category of the generic largeness since it is related to the ratio of eigenvalues of  $f$ . The eigenvalue ratio is also related to  $V_{ub}/V_{cb}$  implying that the 13 mixing angle has to be small in our approach. It is important to note that we do not assume a particular flavor texture such like hierarchical pattern in one matrix to obtain the feature. The key property to obtain the features for the neutrino mixings is that the correction to the rank one charged lepton mass matrix and the type II seesaw term are unified (or more roughly, simultaneously diagonalized), as a result of SO(10) unification.

### C. Realistic model with $h'$

The discussion above gives the qualitative consequences of the rank one property, and the experimental inputs are not used to obtain the features. The discussion below will address the issue of the experimental data for the first generation. Actually, we have not listed the first generation masses and  $V_{us}$  before. In fact, if  $h' = 0$ , one obtains the following relation among the fermion masses:

$$\frac{m_u}{m_t} : \frac{m_d}{m_b} : \frac{m_e}{m_\tau} \simeq r_2(1 + r_2 X) : 1 + X : -3(1 - 3X), \quad (19)$$

where  $X = f_1 f_2 f_3 / (a^2 f_1 f_2 + b^2 f_1 f_3 + c^2 f_2 f_3)$ . When one fits down quark and electron masses (e.g.,  $X = 0.35$ ), the up quark mass is clearly too large since  $r_2 \simeq m_c/m_t / (m_s/m_b) \sim 0.1$ . As a result, one of the first generation masses cannot be fitted. Besides, since  $\tilde{V}$  in Eq.(16) is common for up- and down-type quarks,  $V_{us}$  becomes too small compared to observations, since  $V_{us} \simeq V_{cb}V_{ub}$ . Therefore, one needs non-vanishing contribution from  $h'$  to obtain realistic masses for the first generation and  $V_{us}$  under the rank 1 assumption.

As is well-known, the empirical relation  $V_{us} \simeq \sqrt{m_d/m_s}$  is obtained when  $(\tilde{Y}_d)_{11} \rightarrow 0$  and  $(\tilde{Y}_d)_{12} \simeq (\tilde{Y}_d)_{21}$ . Therefore we choose  $\tilde{f}_{11} \rightarrow 0$ . When  $(\tilde{Y}_d)_{11}, (\tilde{Y}_e)_{11} \rightarrow 0$  is assumed, the choice of  $(\tilde{Y}_e)_{12}(\tilde{Y}_e)_{21} \sim (\tilde{Y}_d)_{12}(\tilde{Y}_d)_{21}$  satisfies the Georgi-Jarskog (GJ) relation ( $m_e m_\mu m_\tau \sim m_d m_s m_b$ ) for the down-type quarks and charged lepton masses. The up quark mass can be fit by using the freedom of  $r_3$ . As a result, we have the following two solutions typically to fit the first generation masses and  $V_{us}$  in a simple manner.

*Case A:*  $\tilde{f}_{11} \simeq 0$  and  $|\tilde{f}_{12} + \tilde{h}'_{12}| \simeq |-3\tilde{f}_{12} + \tilde{h}'_{12}|$ . The smallness of up quark mass is realized by a cancellation in  $(\tilde{Y}_u)_{12} = r_2 \tilde{f}_{12} + r_3 \tilde{h}'_{12}$ .

In this case,  $h'$  has to be symmetric which we can obtain by employing an extra **10** Higgs

field. For example,  $\tilde{h}'_{12} \simeq \tilde{f}_{12}$  is the simplest solution, giving

$$U_{e3} \simeq \frac{1}{3}V_{us} \sin \theta_a, \quad (20)$$

$$V_{us} \simeq 2 \sin \tilde{\theta} \simeq 2 \frac{f_2 \cos \theta_a}{f_3 \sin^2 \theta_a} \tan \theta_s. \quad (21)$$

where  $\tilde{\theta}$  is given in Eq.(17) and used a relation  $f_1 \simeq -f_2 \tan^2 \theta_s$  from  $\tilde{f}_{11} \simeq 0$ . Assuming that the corrections from the other elements of  $\tilde{h}'$  (e.g.,  $\tilde{h}'_{13,23}$ ) are small, we have an approximate relation

$$\frac{V_{ub}}{V_{cb}} \sim \frac{1}{2}V_{us} \tan^2 \theta_a, \quad (22)$$

which is in good agreement with the experiment.

*Case B:* In this case we have  $\tilde{f}_{11}\tilde{f}_{22} - \tilde{f}_{12}^2 \simeq 0$ . Then, 11 and 12 elements of  $(\tilde{V}\tilde{f}\tilde{V}^t)$  are zero, where  $\tilde{V}$  in Eq.(16) is an approximate diagonalization matrix in the limit  $h' \rightarrow 0$ . The 12 element of  $\tilde{V}\tilde{h}'\tilde{V}^t$  produces the Cabibbo angle. The GJ relation is manifest when  $|c_e| = 1$  and  $(\tilde{V}\tilde{h}'\tilde{V}^t)_{11} \simeq 0$ . The up quark mass is fitted by the smallness of  $r_3$ ,  $m_u/m_c \simeq r_3^2/r_2 m_d/m_s$ .

In this case,  $h'$  can be either symmetric or anti-symmetric. Since 11 element vanishes automatically, anti-symmetric coupling from **120** Higgs field is a better choice. As is noted,  $\tilde{V}$  contributes to  $U_{e3}$ , but it does not contribute to the Cabibbo angle. As a result, we obtain from Eq.(14)

$$|U_{e3}| \simeq \sin \theta_a \left| \sin \tilde{\theta} + e^{i\gamma} c_e \frac{1}{3} V_{us} \right| \simeq \left| \frac{f_2}{f_3} \tan \theta_s \cot \theta_a + e^{i\gamma} c_e \frac{1}{3} V_{us} \sin \theta_a \right|, \quad (23)$$

where  $\gamma$  is a relative phase between  $\tilde{f}_{12}$  and  $\tilde{h}'_{12}$  roughly, and we have used a relation  $f_1 \simeq -f_2 \tan^2 \theta_s$ . Since  $V_{us}$  is generated purely from  $h'$ , it is not directly correlated to  $V_{ub}/V_{cb}$  contrary to the case A.

As we have noted, to fit  $V_{cb}$ ,  $m_s/m_b$  and  $\theta_{\text{atm}}$  very well, one needs a correction in  $\tilde{h}'_{23}$ . However, the correction does not affect the approximate expressions for  $U_{e3}$  very much.

It is interesting that the  $U_{e3}$  is related to the mass ratio of neutrino in both cases. Since the GJ relation and the empirical relation of  $V_{us}$  are not exact relations, there can be a shift from them in a numerical fit analysis.

Here we assumed  $(\tilde{Y}_a)_{11} \rightarrow 0$  to satisfy the GJ relation and the empirical relation of  $V_{us}$  in a simple manner. When one introduces other parameters especially for symmetric  $h'$ , there will be an accidental fine-tuned solution for the relations in a general fit for  $(\tilde{Y}_a)_{11} \neq 0$ . Actually, when  $h'$  is symmetric ( $c_e = 1$ ) and  $r_3 = 1$ , it results in a minimal model in which only one **10** and **126** Higgs fields couple to fermions with  $h$  being rank 3. In the minimal model for the fermion sector, it is known that there is a fine-tuned solution to fit fermion

masses and mixings [15] unless the minimality of the Higgs potential is taken into account. In this case, when the first generation masses and  $V_{us}$  are tuned, there is no freedom to adjust  $U_{e3}$  and thus the approximate relation in the previous subsection holds,  $U_{e3} \sim f_2/f_3$ . When  $r_3 \neq 1$  (but  $r_3 \simeq 1$ ),  $U_{e3}$  can be tuned to be any value (including zero) since first generation masses and  $V_{us}$  can be fitted even if  $(\tilde{Y}_e)_{12} = 0$ . (When  $h$  is rank 1 and  $h'$  is anti-symmetric, there is no such fine-tuned solution. When  $h$  is rank 3, the fine-tuning fit for  $U_{e3} = 0$  is allowed [16].) Therefore, the assumption  $(\tilde{Y}_a)_{11} \rightarrow 0$  to satisfy GJ relation in a simple manner is crucial to keep the  $U_{e3}$  prediction. Actually, when  $(\tilde{Y}_a)_{11} \rightarrow 0$  is assumed, the fine-tune solutions are removed, and the  $U_{e3}$  is predicted as we have noted, irrespective of the number of parameters. We also note that the assumption  $(\tilde{Y}_a)_{11} \rightarrow 0$  is preferable to suppress nucleon decay amplitudes naturally.

The case with the assumption that  $h$  (rank 1) and  $f$  are real and anti-symmetric  $h'$  is pure imaginary (in which case, the charged fermion mass matrices are hermitian) is in fact the model discussed in [7]. For this case, cancellation cannot happen between  $f$  and  $h'$  and thus the numerical fit does not shift very much from the above expression. In the numerical fit, it predicts  $|U_{e3}| = 0.08 - 0.12$  in the case where  $|c_e| = 1$  (and the GJ relation is manifest). Under the hermiticity assumption, one obtains  $e^{i\gamma} = \pm i$  and it is consistent with the above expression. In this case, since  $(\tilde{Y}_e)_{22}$  is real, the PMNS phase is roughly same as the phase in the expression in Eq.(23) and thus

$$\tan \delta_{\text{PMNS}} \simeq \frac{1}{3} \frac{c_e V_{us}}{\sin \tilde{\theta}}, \quad (24)$$

and we obtain  $\delta_{\text{PMNS}} \simeq \pm 30^\circ$  or  $180 \pm 30^\circ$  using the experimental inputs.

For the case of  $|c_e| \neq 1$ , one can also fit the experimental data and the prediction is  $|U_{e3}| = 0.05 - 0.14$ . Since the cancellation is not allowed between  $\tilde{f}_{13}$  and  $\tilde{h}'_{13}$ , the experimental data of  $V_{ub}$  cuts the upper region of experimentally allowed mass squared ratio difference, and then gives an upper bound of  $U_{e3}$ .

#### IV. TRI-BIMAXIMAL ANSATZ

In the previous section, we incorporated large lepton mixings but their values were inputs into the theory. In this section, we consider special cases where the dominant part of the lepton mixing is in the tri-bimaximal form [19]. This would require special form for the rank one matrix  $h$ . We envisage that the rank one form for  $h$  as well as the matrix forms for  $f$  and  $h'$  come from some vacuum alignment of flavon fields, e.g., [20, 21].

In the triplet flavon models, the  $3 \times 3$  matrix can be expanded by the tensor products of the flavon fields when there are three independent flavons. The three flavon fields can be

expressed (without loss of generality, by making unitary transformations) as

$$\phi_1 = (0, 0, 1), \quad \phi_2 = (0, a, b), \quad \phi_3 = (c, d, e). \quad (25)$$

In general, there is no reason for the flavon vevs to be hierarchical, and the large neutrino mixings can originate from  $a \sim b$ ,  $c \sim d \sim e$  [22]. The experimental result from the neutrino oscillation seems to imply a special alignment of flavon vevs rather than the generic largeness, namely [20],

$$\phi_1 = (0, 0, 1), \quad \phi_2 = (0, -1, 1)/\sqrt{2}, \quad \phi_3 = (1, 1, 1)/\sqrt{3}. \quad (26)$$

The vacuum alignment can be obtained by imposing a discrete flavor symmetry [21], leading to tri-bimaximal neutrino mixings.

It is worth pointing out at the beginning that the aligned flavon fields can be written in several ways by choice of coordinates (or by making unitary transformations) and the final results are independent of the coordinate choice.

It is interesting to note that the aligned flavon vevs correspond to a link of a hexahedron, a diagonal line of a lateral surface, a diagonal line of a regular hexahedron, respectively. The interpretation becomes clear when the flavon fields are expanded in terms of the following orthogonal axes of coordinates (called hexahedral coordinate)

$$x_1 = (1, 0, 0), \quad x_2 = (0, 1, 0), \quad x_3 = (0, 0, 1), \quad (27)$$

which correspond to the three lateral links of the regular hexahedron. The hexahedral coordinate is convenient to describe the  $Z_4$  rotation around the surface-diagonal axes of the hexahedron. In fact, the regular hexahedron has different coordinates to describe the symmetry of the shape. One can consider a coordinate system which proves convenient to describe the  $Z_3$  rotation around the vertex-diagonal axes of the hexahedron,

$$x'_1 = (2, -1, -1)/\sqrt{6}, \quad x'_2 = (1, 1, 1)/\sqrt{3}, \quad x'_3 = (0, -1, 1)/\sqrt{2}. \quad (28)$$

The axes of the coordinates  $x'_1$  and  $x'_3$  are on the regular triangle which is formed by three of the hexahedron's vertices, and  $x'_2$  is perpendicular to the triangle. We call this tetrahedral coordinate.

The unitary matrix for the coordinate transformation from hexahedral (unprimed) to tetrahedral (primed) coordinate is the tri-bimaximal (TB) matrix i.e.  $x' = xU_{\text{TB}}^t$ , where

$$U_{\text{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \quad (29)$$

Therefore, in general (irrespective of the rank one assumption), if the charged-lepton mass matrix is (nearly) diagonal in the hexahedral coordinate and the neutrino mass matrix is

(nearly) diagonal in the tetrahedral coordinate, then the neutrino mixing matrix is (nearly) tri-bimaximal and given by:

$$U_{\text{MNSP}} = V_e U_{\text{TB}} V_\nu^\dagger, \quad (30)$$

where  $V_e$  is a diagonalizing matrix of  $Y_e$  in the hexahedral coordinate, and  $V_\nu$  is a diagonalizing matrix of  $\mathcal{M}_\nu$  in the tetrahedral coordinate.

Suppose that the vacuum alignment in Eq.(26) is given in the hexahedral coordinate as we have noted. Then, those flavons in the tetrahedral coordinate are given as ( $\phi'_i = \phi_i U_{\text{TB}}$ )

$$\phi'_1 = (-1, \sqrt{2}, \sqrt{3})/\sqrt{6}, \quad \phi'_2 = (0, 0, 1), \quad \phi'_3 = (0, 1, 0). \quad (31)$$

Therefore, from the discussion in the previous section, one can easily check that the nearly tri-bimaximal neutrino mixings are obtained when  $h$  is rank one formed by  $\phi_1$  (irrespective of the choice of the coordinate), and  $f$  is formed by  $\phi_2$  and  $\phi_3$ . We define  $\phi_4$  which is obtained an outer product of  $\phi_2$  and  $\phi_3$ , i.e.  $\phi_4 = \phi_3 \times \phi_2$ . In the tetrahedral coordinate,  $\phi'_4 = (1, 0, 0)$ .

As we have mentioned, the Yukawa matrices can be expressed in terms of the tensor products of the flavon fields. The symmetric matrices can be formed by six bases. Since we set 11 element to be zero in the hexahedral coordinate, we define the following matrices as the bases to form the linear space of symmetric matrices:

$$Y_1 = \phi_1^t \phi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (32)$$

$$Y_2 = 2\phi_2^t \phi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad (33)$$

$$Y_3 = 2(\phi_3^t \phi_3 - \frac{1}{2}\phi_4^t \phi_4) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad (34)$$

$$Y_4 = \sqrt{6}(\phi_2^t \phi_3 + \phi_3^t \phi_2) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad (35)$$

$$Y_5 = \sqrt{3}(\phi_2^t \phi_4 + \phi_4^t \phi_2) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad (36)$$

where the elements of the matrices are presented in the hexahedral coordinate.

In the following, we will consider two models: (I)  $V_\nu$  is a unit matrix ( $f$  is diagonal in the tetrahedral coordinate), (II)  $V_\nu$  is close to a unit matrix.

### A. Model I : $V_\nu = \mathbf{1}$

The Model I can have both case A and case B solutions as described in the previous section. The case A solutions are, however, more natural in this model. In order to obtain such a solution, we employ additional **10** Higgs to obtain a correction matrix  $h'$ , and  $h'$  is a symmetric matrix.

We arrange the  $h, f, h'$  couplings as follows:

$$h = h_3 Y_1, \quad (37)$$

$$f = h_3 \epsilon (Y_2 + \lambda Y_3), \quad (38)$$

$$h' = h_3 \epsilon \lambda \rho Y_4 \quad (\text{or } h_3 \epsilon \lambda \rho Y_5). \quad (39)$$

Then, since the ratio of eigenvalues of  $f$  is  $1 : \lambda : -\lambda/2$ , we obtain  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2 = \frac{3}{4} \lambda^2$ . In the parameterization in the previous section,  $U_0$  in Eq.(10) is the tri-bimaximal matrix because  $a : b : c = \sqrt{3} : \sqrt{2} : -1$ .

The fermion Yukawa matrices are

$$Y_u = h + r_2 f + r_3 h' \quad (40)$$

$$= h_3 \begin{pmatrix} 0 & \epsilon \lambda (r_2 - r_3 \rho) & \epsilon \lambda (r_2 + r_3 \rho) \\ \epsilon \lambda (r_2 - r_3 \rho) & r_2 \epsilon (1 + \frac{\lambda}{2}) + r_3 x \epsilon \lambda \rho & -\epsilon r_2 (1 - \frac{\lambda}{2}) \\ \epsilon \lambda (r_2 + r_3 \rho) & -\epsilon r_2 (1 - \frac{\lambda}{2}) & 1 + r_2 \epsilon (1 + \frac{\lambda}{2}) - r_3 x \epsilon \lambda \rho \end{pmatrix}, \quad (41)$$

$$Y_d = r_1 (h + f + h') \quad (42)$$

$$= r_1 h_3 \begin{pmatrix} 0 & \epsilon \lambda (1 - \rho) & \epsilon \lambda (1 + \rho) \\ \epsilon \lambda (1 - \rho) & \epsilon (1 + \frac{\lambda}{2}) + x \epsilon \lambda \rho & -\epsilon (1 - \frac{\lambda}{2}) \\ \epsilon \lambda (1 + \rho) & -\epsilon (1 - \frac{\lambda}{2}) & 1 + \epsilon (1 + \frac{\lambda}{2}) - x \epsilon \lambda \rho \end{pmatrix}, \quad (43)$$

$$Y_e = r_1 (h - 3f + h') \quad (44)$$

$$= r_1 h_3 \begin{pmatrix} 0 & \epsilon \lambda (-3 - \rho) & \epsilon \lambda (-3 + \rho) \\ \epsilon \lambda (-3 - \rho) & -3\epsilon (1 + \frac{\lambda}{2}) + x \epsilon \lambda \rho & 3\epsilon (1 - \frac{\lambda}{2}) \\ \epsilon \lambda (-3 + \rho) & 3\epsilon (1 - \frac{\lambda}{2}) & 1 - 3\epsilon (1 + \frac{\lambda}{2}) - x \epsilon \lambda \rho \end{pmatrix}, \quad (45)$$

where  $x = -2$  when  $h' \propto Y_4$ , and  $x = 1$  when  $h' \propto Y_5$ .

For the numerical fits, the parameter  $\epsilon$  is given by  $\epsilon \sim m_s / m_b \sim V_{cb}$ , and the parameter  $\lambda$  is given by  $\lambda(1 - \rho) \sim V_{us}$ .

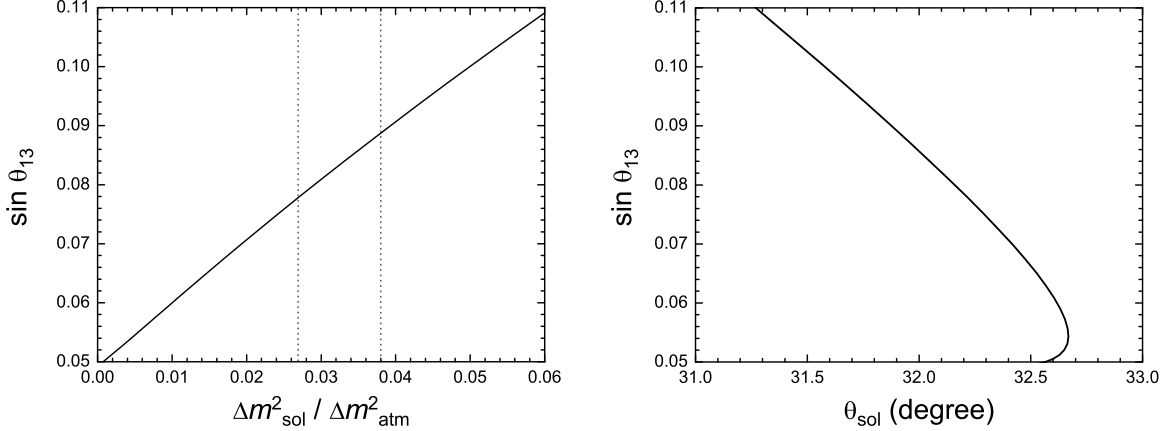


FIG. 1:  $U_{e3}$  is shown as a function of  $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2$  (left) and  $\theta_s$ (right) for Model I.

When  $\rho \simeq -1$ , then Goergi-Jarskog relation is satisfied naturally. At that time,  $(Y_d)_{13} \simeq 0$ . This is interesting since the empirical relation  $|V_{td}| \simeq V_{us}V_{cb}$  is satisfied simultaneously when  $r_2, r_3$  are small (up-type quark masses are more hierarchical rather than down-type quark masses).

The parameter  $r_2$  is fixed as  $|r_2| \sim m_c/m_t/(m_s/m_b)$ . The up quark mass can be made small by a choice  $r_3 \sim r_2/\rho$ .

Cabibbo angle,  $U_{e3}$  and the ratio of mass squared differences  $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2$  are all correlated by the parameter  $\lambda$ . The naive approximate relation is  $U_{e3} \simeq V_{us}/(3\sqrt{2})$  as we have derived in the previous section.

In Fig. 1, we plot  $U_{e3}$  as a function of the mass squared difference ratio. In the plot, we fit  $m_e/m_\tau$  and  $m_\mu/m_\tau$  using  $\rho$  and  $\epsilon$  (which are assumed to be real in the plot). Then,  $U_{e3}$  is calculated as a function of  $\lambda$ . (The mixing angles do not depend on  $h_3$ ). We note that a correction from  $h'$  is needed to fit  $V_{cb}$ , e.g.,  $\Delta h' \propto \phi_1^t \phi_2 + \phi_2^t \phi_1$ . As we have mentioned, such correction do not modify the  $U_{e3}$  very much. In Fig. 1, we also show the plot of  $U_{e3}$  as a function of  $\theta_{\text{sol}}$ . Using the experimental constraint on  $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2$  and the other input from quark masses and mixings, we find that  $U_{e3}$  is predicted to be 0.07-0.08. We note that the smaller side of experimental range of  $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2$  is preferred from the numerical fit, which obeys from the naive relation  $\lambda(1 - \rho) \sim V_{us}$ . The solar mixing angle  $\theta_{\text{sol}}$  is found to be  $\sim 32^\circ$  from the plot which is obeyed by the approximation Eq.(13) when the PMNS phase is 0 (or  $\pi$ ), which is resulting from the assumption where  $\rho$  and  $\epsilon$  are real. The atmospheric mixing angle  $\theta_{\text{atm}}$  is  $45^\circ$  up to  $\pm 2\text{-}3^\circ$  correction from  $V_{cb}$  for Model I irrespective of case A or case B solutions.

The current allowed range for the neutrino parameters at  $2\sigma$  level are as follows [23]:  $\theta_{\text{atm}} = 37^\circ - 51^\circ$ ,  $\theta_{\text{sol}} = 31.8^\circ - 36.4^\circ$  and  $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2 = 0.027 - 0.038$ .



## B. Model II: $V_\nu \neq 1$

The Model II can have both case A and case B solutions as well. In this model, the case B solutions are more natural. It is possible that the  $f$  coupling is not completely diagonal in the tetrahedral coordinate. Using the available freedom, we choose the 12 element of  $f$  in the hexahedral coordinate to obtain the case B solution. Since the 12 elements of  $Y_3 + Y_4$  and  $Y_3 + Y_5$  are zero, one can consider the choice:

$$h = h_3 Y_1, \quad (46)$$

$$f = h_3 \epsilon (Y_2 + \lambda (Y_3 + Y_4)), \quad (47)$$

$$h' = h_3 \epsilon \lambda \rho Y_4. \quad (48)$$

The matrix  $Y_4$  can be replaced with  $Y_5$ . One can also choose  $h'$  to be antisymmetric, e.g.,  $h' \propto \phi_2^t \phi_3 - \phi_3^t \phi_2$ .

In the tetrahedral coordinate, the  $f$  coupling is written when  $f \propto Y_2 + \lambda(Y_3 + Y_4)$  (case B1) as

$$f^{\text{tetra}} \propto \begin{pmatrix} -\frac{1}{2}\lambda & 0 & 0 \\ 0 & \lambda & \sqrt{\frac{3}{2}}\lambda \\ 0 & \sqrt{\frac{3}{2}}\lambda & 1 \end{pmatrix}, \quad (49)$$

and if the notation in the previous section is used, we obtain  $a : b : c = (\sqrt{3} \cos \psi + \sqrt{2} \sin \psi) : (\sqrt{2} \cos \psi - \sqrt{3} \sin \psi) : -1$  where  $\tan 2\psi = \sqrt{6}\lambda/(1 - \lambda)$ . The mass squared ratio is  $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2 = 3/4\lambda^2(1 - 4\lambda + O(\lambda^2))$ . When we use  $f \propto Y_2 + \lambda(Y_3 + Y_5)$  (case B2),

$$f^{\text{tetra}} \propto \begin{pmatrix} -\frac{1}{2}\lambda & 0 & \frac{\sqrt{3}}{2}\lambda \\ 0 & \lambda & 0 \\ \frac{\sqrt{3}}{2}\lambda & 0 & 1 \end{pmatrix}, \quad (50)$$

we obtain  $a : b : c = (\sqrt{3} \cos \psi' - \sin \psi') : \sqrt{2} : (-\cos \psi' - \sqrt{3} \sin \psi')$  where  $\tan 2\psi' = \sqrt{3}\lambda/(1 + \lambda/2)$ . The mass squared ratio is  $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2 = 3/4\lambda^2(1 - \lambda + O(\lambda^2))$ .

We note that if there is a  $O(\lambda)$  correction in the 12 element in the tetrahedral coordinate, it modifies the  $\theta_s$  angle largely, and it separates from the nearly tri-bimaximal mixing, and thus we do not use the choice.

As we have obtained in the previous section,  $U_{e3}$  prediction is

$$|U_{e3}| \sim \left| \sqrt{\frac{1}{2} \frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2}} + e^{i\gamma} \frac{1}{3\sqrt{2}} V_{us} \right|. \quad (51)$$

Clearly, if the parameters  $\rho, \lambda, \epsilon$  are all real, then,  $\gamma = 0$  or  $\pi$ , and the maximal and minimal values of  $U_{e3}$  are obtained. At that time, there is no phase in the PMNS mixing matrix. (The Kobayashi-Maskawa phase can be obtained from a phase of  $r_2$  and/or  $r_3$ .)

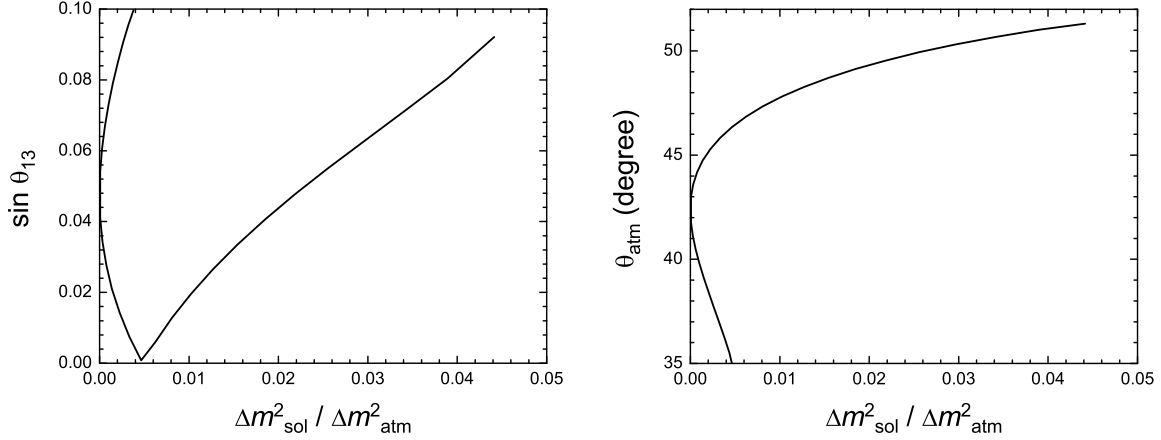


FIG. 2:  $\sin \theta_{13}$  is shown as a function of  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$  (left) and  $\theta_{\text{atm}}$  is shown as a function of  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$  (right) for Model II-case B1 (described in the text).

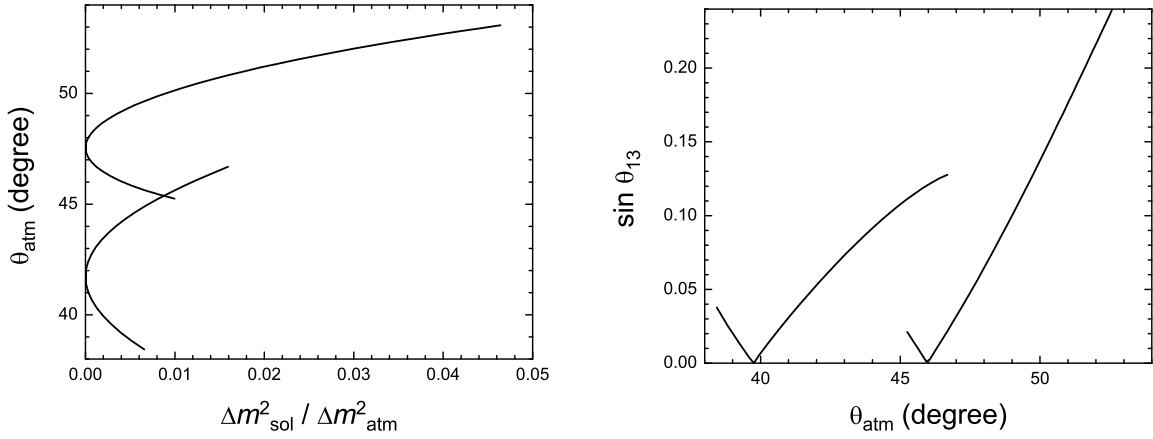


FIG. 3:  $\theta_{\text{atm}}$  is shown as a function of  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$  (left) and  $\sin \theta_{13}$  is shown as a function of  $\theta_{\text{atm}}$  for Model II-case B2 (described in the text).

In Fig.2 (case B1, Eq.(49)) and Fig.3 (case B2, Eq.(50)), we plot  $U_{e3}$  when  $\rho, \lambda, \epsilon$  are real to find the lower and upper limits. Due to the off-diagonal elements of  $f$  in the tetrahedral coordinate, the atmospheric angle shifts from  $45^\circ$ , and the shift is correlated to  $U_{e3}$ , unlike the case of Model I.

In case B1, using the experimental constraint on  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$ , we find that  $U_{e3}$  is predicted to be 0.05-0.08. This solution corresponds to the sign choice  $e^{i\gamma} = -1$  in Eq.(51). We also plot atmospheric mixing angle in Fig.2 (right). It is interesting to note that for the case B1,  $\lambda$  should be negative for  $|\lambda| \sim 0.1 - 0.3$  to fit mass squared difference ratio since  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2 = 3/4\lambda^2(1 - 4\lambda + O(\lambda^2))$ . As a result, the direction of the shift is determined to fit experimental values, i.e.  $\theta_{\text{atm}} > 45^\circ$ .

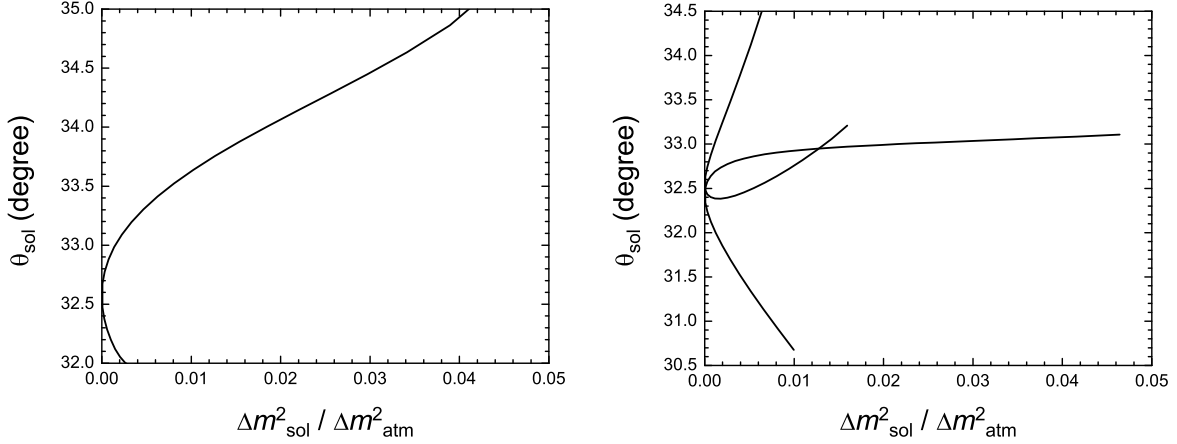


FIG. 4:  $\theta_{\text{sol}}$  is shown as a function of  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$  for Model II-case B1 (left), case B2 (right).

In case B2, the mass squared ratio can be fitted for both signatures of  $\lambda$  (we find two branches in both graphs in Fig.3). The plot in Fig.3 is shown constraining  $\tan^2 \theta_{\text{sol}} > 0.35$ , and one can find that  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$  becomes too small for one of the branch to fit solar mixing angle, and  $\theta_{\text{atm}} > 45^\circ$  is favored in this case as well. Larger  $U_{e3}$  values ( $> 0.15$ ) is preferred once we include the experimental limit on  $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$ . The solution corresponds to the sign choice  $e^{i\gamma} = +1$  in Eq.(51).

As we discussed above that both case A and case B solutions can be obtained in Models I and II. The interesting question is how can we distinguish these two models. For example, the predictions of  $U_{e3}$  (as shown in Figs 1,2 and 3) distinguishes between the cases A and B. Actually, if  $U_{e3}$  is just below the current CHOOZ bound, the case B solution with  $e^{i\gamma} = 1$  is preferred. Since the case A is a more natural solution for Model I and case B is a more natural solution for Model II, one may weakly conclude that this prediction distinguishes between Models I and II. But a stronger way to distinguish these models would be to use the predictions of  $\theta_{\text{atm}}$ . In Model I,  $\theta_{\text{atm}}$  is fixed to be  $45^\circ$  up to  $\pm 2 - 3^\circ$  corrections from  $V_{cb}$ , where as Model II prefers  $\theta_{\text{atm}} > 45^\circ$  (Figs 2 and 3) in the experimentally allowed region. This difference is directly due to the rigorousness of  $\mu - \tau$  symmetry in the  $f$  coupling from the tri-bimaximal ansatz ( $V_\nu = 1$  (Model I)) where the  $f$  coupling has  $Z_2 \times Z_2$  symmetry. In Model I, the deviation from the maximal angle is related to  $V_{cb}$ , while in Model II, the deviation is related to  $U_{e3}$ . The current best fit value of the atmospheric mixing angle ( $\theta_{\text{atm}}^{\text{best fit}} = 43^\circ$  [23]) is nearly the maximal mixing, and it implies the Model I. However, the error is still large. The accurate deviation from the maximal angle will be obtained in future three generation fit of the neutrino oscillations [24], and it will give us an important test for the tri-bimaximal ansatz.

The predictions for  $\theta_{\text{sol}}$  are similar with a small margin in these models, since the devia-

tions from the tri-bimaximal angle ( $\theta_s = 35.3^\circ$ ) are related to  $U_{e3}$  in all cases. In Model I,  $\theta_{\text{sol}}$  is predicted to be  $\sim 32^\circ$ , where as, in Model II (Fig. 4),  $\theta_{\text{sol}}$  is predicted to be  $\sim 34^\circ$  (case B1),  $\sim 33^\circ$  (case B2), once we include all the experimental bounds. The PMNS phase is assumed to be 0 or  $\pi$  in the plot, and the general phase fit will change the predictions of the angle, especially for Model II.

## V. DERIVATION OF RANK ONE ANSATZ

The rank one Yukawa coupling with **10** Higgs field generates the features of flavor hierarchy, and rank 1 matrices can often appear in various ways (flavor symmetry, discrete symmetry, and string models). In this section, we give an SO(10) model, where the rank one ansatz used in our discussion of flavor emerges from a discrete symmetry.

When the direct couplings of chiral fermions with a Higgs field are forbidden by a symmetry, and the effective Yukawa couplings are generated by propagating vector-like matter fields, the rank of the effective Yukawa matrix depends on the number of the vector-like fields. Actually, when there are only one pair of vector-like matter fields as a flavor singlet, the effective Yukawa matrix is rank 1.

The model we assume has one extra vector-like pair of matter fields with mass slightly above the GUT scale (denoted by  $\psi_V \equiv \mathbf{16}_V \oplus \bar{\psi}_V \equiv \overline{\mathbf{16}}_V$ ) and three gauge singlet fields  $Y_a$ . We add a  $Z_4$  discrete symmetry to the model under which the fields  $\psi_a \rightarrow i\psi_a$ , and  $Y_a \rightarrow -iY_a$ . The **10**-Higgs field  $H$  is invariant under this symmetry. The gauge invariant Yukawa superpotential under this assumption is given by

$$W = \psi_V H \lambda \psi_V + M_V \psi_V \bar{\psi}_V + \bar{\psi}_V \sum_a Y_a \psi_a. \quad (52)$$

When we give vevs  $\langle Y_a \rangle \neq 0$ ,  $\psi_V$  and  $\psi_a$  are mixed. The heavy vector-like fields,  $\bar{\psi}_V$  and a linear combination of  $\psi_V$  and  $\psi_a$  (i.e.  $M_V \psi_V + \sum_a Y_a \psi_a$ ), and the effective operator below its scale and at the GUT scale is given by:

$$\mathcal{L}_{eff} = \frac{\lambda}{M_V^2 + \sum_a Y_a^2} \left[ \sum_a Y_a \psi_a \right] H \left[ \sum_b Y_b \psi_b \right]. \quad (53)$$

This gives rise to a rank one  $h$  coupling. We note that it does not contradict the  $O(1)$  top Yukawa coupling, when  $M_V^2 \sim \sum_a Y_a^2$  (or  $M_V^2 < \sum_a Y_a^2$ ).

If we let the  $\overline{\mathbf{126}}$  Higgs field transform like  $-1$  under  $Z_4$ , it can induce the  $f$  coupling with rank three.

Another way to get mass matrix patterns of types in sec. IV is to assume that there are three component flavon fields (denoted by the dimensionless field  $\phi_i \equiv \frac{\Phi}{M}$ ) which are representations of some internal flavor group and constrain their couplings to fermions by

other symmetries. As an example, let us choose four flavon fields which transform as follows under a  $Z_4 \times Z_2 \times Z_2$  group:  $\phi_1(-i, +, +)$ ;  $\phi_2(1, -, +)$ ;  $\phi_3(1, +, -)$ ;  $\phi_4(1, -, -)$ ;  $\psi(i, +, +)$ ;  $\bar{\Delta}(-1, +, +)$ ;  $H(1, +, +)$ ;  $H'(-1, -, -)$ . The invariant Yukawa coupling under these symmetries is:

$$L_Y = \phi_1^t \phi_1 \psi \psi H + \phi_2^t \phi_2 \psi \psi \bar{\Delta} + \phi_3^t \phi_3 \psi \psi \bar{\Delta} + \phi_4^t \phi_4 \psi \psi \bar{\Delta} + \phi_2^t \phi_3 \psi \psi H' + h.c. \quad (54)$$

In general the vevs of the  $\phi_i$  fields align as in Eq. (25) which can then lead to our type of rank one models. With suitable discrete symmetries, e.g.,  $\Delta(27)$  [21], the flavon vevs can align as in Eq. (26), leading to models of the type considered here (Model I). When  $Z_2$  group is chosen instead of  $Z_2 \times Z_2$  group such that  $\phi_1(+)$ ;  $\phi_2(-)$ ;  $\phi_3(-)$ ;  $\phi_4(+)$ ;  $\psi(+)$ ;  $\bar{\Delta}(+)$ ;  $H(+)$ ;  $H'(+)$ ,  $\phi_2^t \phi_3$  term is allowed in the  $\psi \psi \bar{\Delta}$  coupling and Model II (B1) can be considered. More details on the flavon vev alignment with discrete symmetries and implications for rank one models is currently under investigation.

## VI. CONCLUSION

In conclusion, we have shown how a simple ansatz for the dominant Yukawa coupling matrix in renormalizable SO(10) models can lead to a unified understanding of the diverse quark and lepton flavor hierarchies. We suggest this as a possible way to address the challenge of a unified description of quark-lepton flavor. We have not attempted in this note to derive our ansatz from any specific discrete symmetries, although we show a guideline to obtain a rank one form from a higher than GUT scale theory. This may be the next step towards a complete theory of flavor.

Within our rank one hypothesis, we have considered two classes of models which are nearly tri-bimaximal and point out that both a measurement of  $\theta_{13}$  and the atmospheric mixing angle  $\theta_{\text{atm}}$  can distinguish between these models. In these models, the natural minimal value of  $\theta_{13}$  is around 0.05, whereas the maximal value can be larger than 0.15. This range of  $\theta_{13}$  can be probed in the upcoming experiments. The current estimate of  $\theta_{13}$  using the  $1.5\sigma$  excess of events in MINOS  $\nu_\mu - \nu_e$  appearance channel [25] and all other experimental data is  $\sin^2 \theta_{13} \simeq 0.02 \pm 0.01$  ( $1\sigma$ ) [26].

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