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ON THE ACCURACY OF THE FINITE VOLUME ELEMENT METHOD BASED ON PIECEWISE LINEAR POLYNOMIALS*

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Dedicated to Professor John R. Cannon on the occasion of his 65th birthday

Abstract. We present a general error estimation framework for a finite volume element (FVE) method based on linear polynomials for solving second-order elliptic boundary value problems. This framework treats the FVE method as a perturbation of the Galerkin finite element method and reveals that regularities in both the exact solution and the source term can affect the accuracy of FVE methods. In particular, the error estimates and counterexamples in this paper will confirm that the FVE method cannot have the standard $O(h^2)$ convergence rate in the L^2 norm when the source term has the minimum regularity, only being in L^2, even if the exact solution is in H^2.

Key words. elliptic, finite volume, error estimates, counterexamples

AMS subject classifications. 65N10, 65N15, 65N35

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1. Introduction.

In this paper, we consider the accuracy of finite volume element (FVE) methods for the following elliptic boundary value problem: Find $u = u(x)$ such that

$$-\nabla \cdot (A\nabla u) = f, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega, \tag{1.1}$$

where Ω is a bounded convex polygon in R^2 with boundary $\partial\Omega$, $A = \{a_{i,j}(x)\}$ is a 2×2 symmetric and uniformly positive definite matrix in Ω, and the source term $f = f(x)$ has enough regularity so that this boundary value problem has a unique solution in a certain Sobolev space.

Finite volume (FV) methods have a long history as a class of important numerical tools for solving differential equations. In the early literature [26, 27] they were investigated as the so-called integral finite difference methods, and most of the results were given in one-dimensional cases. FV methods also have been termed as box schemes, generalized finite difference schemes, and integral-type schemes [20]. Generally speaking, FV methods are numerical techniques that lie somewhere between finite difference and finite element methods; they have a flexibility similar to that of finite element methods for handling complicated solution domain geometries and boundary conditions; and they have a simplicity for implementation comparable to finite difference methods with triangulations of a simple structure. More important, numerical solutions generated by FV methods usually have certain conservation features that

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are desirable in many applications. However, the analysis of FV methods lags far behind that of finite element and finite difference methods. Readers are referred to [3, 6, 17, 21, 22, 25] for some recent developments.

The FVE method considered in this paper is a variation of the FV method, which also can be considered as a Petrov–Galerkin finite element method. Much has been published on the accuracy of FVE methods using conforming linear finite elements. Some early work published in the 1950s and 1960s can be found in [26, 27]. Later, the authors of [20] and their colleagues obtained optimal-order H^1 error estimates and superconvergence in a discrete H^1 norm. They also obtained L^2 error estimates of the following form:

$$||u - u_h||_0 \leq Ch^2||u||_{W^{3,p}(\Omega)}, \qquad p > 1,$$

where u and u_h comprise the solution of (1.1) and its FVE solution, respectively. Note that the order in this estimate is optimal, but its regularity requirement on the exact solution seems to be too high compared with that for finite element methods having an optimal-order convergence rate when the exact solution is in $W^{2,p}(\Omega)$ or $H^2(\Omega)$. Optimal-order H^1 estimates and superconvergence in a discrete H^1 norm also have been given in [3, 17, 21, 22, 25] under various assumptions on the above form for equations or triangulations.

More recently, the authors of [7, 8] presented a framework based on functional analysis to analyze the FVE approximations. The authors in [11] obtained some new error estimates by extending the techniques of [20]. The authors of [14, 15] considered FVE approximations for parabolic integrodifferential equations, covering the above boundary value problems as a special case, in both one and two dimensions. All the authors obtained optimal-order H^1 and $W^{1,\infty}$ error estimates and superconvergence in H^1 and $W^{1,\infty}$ norms. In addition, they found an optimal-order L^∞ error estimate in the following form:

$$||u - u_h||_\infty \leq Ch^2\bigg(||u||_{2,\infty} + ||u||_3\bigg),$$

which is in fact an error estimate without any logarithmic factor. However, all the estimates obtained by these authors require that the exact solution have H^3 regularity.

To the best of our knowledge, there have been no results indicating whether the above $W^{3,p}(\Omega)$ regularity is necessary for the FVE solution with conforming linear finite elements to have the optimal-order convergence rate. On the other hand, it is well known that in many applications the exact solution of the boundary value problem cannot have $W^{3,p}$ or H^3 regularity. In fact, the regularities of the source term f, the coefficient, and the solution domain all can abate the regularity of the exact solution. A typical case is the regularity of the solution domain that may force the exact solution not to be in $W^{3,p}$ or H^3 even for the best possible coefficient A and source term f, such as constant functions.

It has been noticed that the regularity of the source term may affect the convergence rate of an FVE solution. The counterexample in [18] showed that the FVE solution with the conforming linear elements cannot have the optimal L^2 convergence rate if the exact solution is in H^2 but the source term f is only in L^2. On the other hand, the author of [6] found an optimal error estimate for the FVE solution with the nonconforming Crouzeix–Raviart linear element under the assumption that the exact solution is in H^2 and the source term f is in H^1, but did not state whether this H^1 regularity of f is necessary for the FVE method presented there.

The central aim of this paper is to show how, by both error estimates and counterexamples, the regularity of the source term f can affect the convergence rate of the FVE solution with conforming linear elements. The results indicate that, unlike the finite element method, the H^2 regularity of the exact solution indeed cannot guarantee the optimal convergence rate of the conforming linear FVE method if the source term has a regularity worse than H^1, assuming that the coefficient is smooth enough. Namely, we will present the following error estimate:

$$||u - u_h||_0 \leq C\bigg(h^2||u||_2 + h^{1+\beta}||f||_\beta\bigg),$$

which leads to the optimal convergence rate of the FVE method only if $f \in H^\beta$ with $\beta \geq 1$. Note first that, except for special cases such as when the dimension of Ω is one or the solution domain has a boundary smooth enough, the H^1 regularity of the source term does not automatically imply the H^3 regularity of the exact solution. On the other hand, the H^3 regularity of the exact solution will lead to the H^1 regularity of the source term when the coefficient is smooth enough, and this error estimate reduces to one similar to estimates obtained in [11, 20]. Also, this error estimate is optimal from the point of view of the best possible convergence rate and the regularity of the exact solution. Moreover, counterexamples given in this paper indicate that the regularity of the source term cannot be reduced. Hence, we believe this is a more general error estimate than those in the literature.

In fact, the FVE method is a Petrov–Galerkin finite element method in which the test functions are piecewise constant. As we will see later, the nonsmoothness in the test function demands a stronger regularity of the source term than the Galerkin finite element method. Also, our view of the FVE method as a Petrov–Galerkin finite element method suggests that we treat the FVE method as a perturbation of the Galerkin finite element method [6, 20] so that we can derive optimal-order L^2, H^1, and L^∞ error estimates with a minimal regularity requirement just like finite element methods except for the additional smoothness assumption on the source term f. This error estimation framework also enables us to investigate superconvergence of the FVE method in both H^1 and $W^{1,\infty}$ norms using the regularized Green's functions [23, 29] and to obtain the uniform convergence of the FVE method similar to that in [24] for the finite element method. To summarize, we observe that the FVE method not only preserves the local conservation of certain quantities of the solution (problem dependent), but also has optimal-order convergence rates in all usual norms. The additional smoothness requirement on the source term f is necessary due to the formulation of the method.

The results of this paper can easily be extended to cover more complicated models. For example, most of the results and analysis framework are still valid if the differential equation contains a convection term $\nabla \cdot (\mathbf{b}\, u)$ (see [21] and [22]) and the symmetry of the tensor coefficient $A(x)$ is not critical. Also, one may consider Neumann and Robin boundary conditions on the whole or a part of the boundary $\partial\Omega$. In fact, the FVE method was introduced in [2] as a consistent and systematic way to handle the flux boundary conditions for finite difference methods. We also refer readers to [1, 19] for FVE approximations of nonlinear problems, to [12] for an immersed FVE method to treat boundary value problems with discontinuous coefficients, and to [13] for the mortar FVE methods with domain decomposition.

This paper is organized as follows. In section 2, we introduce some notation, formulate our FVE approximations in piecewise linear finite element spaces defined on a triangulation, and recall some basic estimates from the literature. All error estimates are presented in the pertinent subsections of section 3. Section 4 is devoted to counterexamples demonstrating that smoothness of the source term is necessary in order for the FVE method to have the optimal-order convergence rate.

2. Preliminaries.

2.1. Basic notation. We will use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with $1 \le p \le \infty$ consisting of functions that have generalized derivatives of order s in the spaces $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$||u||_{s,p,\Omega} = ||u||_{s,p} = \left(\int_\Omega \sum_{|\alpha| \le s} |D^\alpha u|^p dx \right)^{1/p} \quad \text{for } 1 \le p < \infty$$

with the standard modification for $p = \infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and skip the index $p = 2$ and Ω whenever possible; i.e., we will use $||u||_{s,2,\Omega} = ||u||_{s,\Omega} = ||u||_s$. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$ in the sense of traces. Finally, $H^{-1}(\Omega)$ denotes the space of all bounded linear functionals on $H_0^1(\Omega)$. For a functional $f \in H^{-1}(\Omega)$, its action on a function $u \in H_0^1(\Omega)$ is denoted by (f, u), which represents the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. To avoid confusion, we use $(\cdot, \cdot)$ to denote both the $L^2(\Omega)$-inner product and the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

For the polygonal domain Ω, we now consider a quasi-uniform triangulation T_h consisting of closed triangle elements K such that $\overline{\Omega} = \cup_{K \in T_h} K$. We will use N_h to denote the set of all nodes or vertices of T_h,

$$N_h = \{p : \ \ p \ \text{ is a vertex of element } \ K \in T_h \ \text{ and } \ p \in \bar{\Omega}\},$$

and we let $N_h^0 = N_h \cap \Omega$. For a vertex $x_i \in N_h$, we denote by $\Pi(i)$ the index set of those vertices that, along with x_i, are in some element of T_h.

We then introduce a dual mesh T_h^* based on T_h; the elements of T_h^* are called control volumes. There are various ways to introduce the dual mesh. Almost all approaches can be described by the following general scheme: In each element $K \in T_h$ consisting of vertices x_i, x_j, and x_k, select a point q in K, and select a point x_{ij} on each of the three edges $\overline{x_i x_j}$ of K. Then connect q to the points x_{ij} by straight lines $\gamma_{ij,K}$. Then for a vertex x_i we let V_i be the polygon whose edges are $\gamma_{ij,K}$ in which x_i is a vertex of the element K. We call V_i a control volume centered at x_i. Obviously, we have

$$\cup_{x_i \in N_h} V_i = \overline{\Omega},$$

and the dual mesh T_h^* is then defined as the collection of these control volumes. Figure 1 gives a sketch of a control volume centered at a vertex x_i.

We call the control volume mesh T_h^* *regular* or *quasi-uniform* if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \le \text{meas}(V_i) \le Ch^2 \quad \text{ for all } \quad V_i \in T_h^*;$$

here h is the maximum diameter of all elements $K \in T_h$.

There are various ways to introduce a regular dual mesh T_h^* depending on the choices of the point q in an element $K \in T_h$ and the points x_{ij} on its edges. In

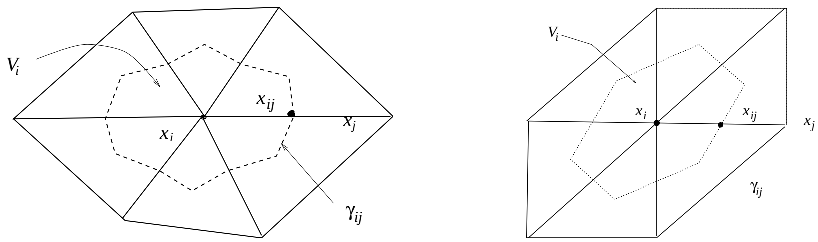

FIG. 1. *Control volumes with barycenter as internal point and interface γ_{ij} of V_i and V_j.*

this paper, we use a popular configuration in which q is chosen to be the barycenter of an element $K \in T_h$, and the points x_{ij} are chosen to be the midpoints of the edges of K. This type of control volume can be introduced for any triangulation T_h and leads to relatively simple calculations for both two- and three-dimensional problems. In addition, if T_h is locally regular, i.e., there is a constant C such that $Ch_K^2 \le \text{meas}(K) \le h_K^2, \text{diam}(K) = h_K$ for all elements $K \in T_h$, then this dual mesh T_h^* is also locally regular. Other dual meshes also may be used. For example, the analysis and results of this paper for all the error estimates in the H^1 norm are still valid if the dual mesh is of the so-called Voronoi type [21].

2.2. The FVE method. We now let S_h be the standard linear finite element space defined on the triangulation T_h,

$$S_h = \{v \in C(\Omega) : \ v|_K \quad \text{is linear for all} \ \ K \in T_h \ \ \text{and } v|_{\partial\Omega} = 0\},$$

and its dual volume element space S_h^*,

$$S_h^* = \{v \in L^2(\Omega) : \ v|_V \quad \text{is constant for all} \ \ V \in T_h^* \ \ \text{and } v|_{\partial\Omega} = 0\}.$$

Obviously, $S_h = \text{span}\{\phi_i(x) : \ x_i \in N_h^0\}$ and $S_h^* = \text{span}\{\chi_i(x) : \ x_i \in N_h^0\}$, where ϕ_i are the standard nodal basis functions associated with the node x_i, and χ_i are the characteristic functions of the volume V_i. Let $I_h : C(\Omega) \to S_h$ and $I_h^* : C(\Omega) \to S_h^*$ be the usual interpolation operators, i.e.,

$$I_h u = \sum_{x_i \in N_h} u(x_i)\phi_i(x) \quad \text{and} \quad I_h^* u = \sum_{x_i \in N_h} u_i \chi_i(x),$$

where $u_i = u(x_i)$.

Then, the FVE approximation u_h of (1.1) is defined as a solution to the following problem: Find $u_h \in S_h$ such that

$$a(u_h, I_h^* v_h) = (f, I_h^* v_h), \qquad v_h \in S_h, \tag{2.1}$$

or

$$a(u_h, v_h) = (f, v_h), \qquad v_h \in S_h^*. \tag{2.2}$$

Here the bilinear form $a(u, v)$ is defined as follows:

$$a(u,v) = \begin{cases} -\displaystyle\sum_{x_i \in N_h} v_i \int_{\partial V_i} A\nabla u \cdot \mathbf{n} dS_x, & (u,v) \in ((H_0^1 \cap H^2) \cup S_h) \times S_h^*, \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v dx, & (u,v) \in H_0^1 \times H_0^1, \end{cases} \tag{2.3}$$

where $\mathbf{n}$ is the outer-normal vector of the involved integration domain. Note that the bilinear form $a(u,v)$ has different definition formulas according to the function spaces involved. We hope that this will not lead to serious confusion but rather will simplify tremendously the notation and the overall exposition of the material.

To describe features of the bilinear forms defined in (2.3), we first define some discrete norms on S_h and S_h^*,

$$|u_h|_{0,h}^2 = (u_h, u_h)_{0,h} \text{ with } (u_h, v_h)_{0,h} = \sum_{x_i \in N_h} \text{meas}(V_i) u_{hi} v_{hi} = (I_h^* u_h, I_h^* v_h),$$
$$|u_h|_{1,h}^2 = \sum_{x_i \in N_h} \sum_{x_j \in \Pi(i)} \text{meas}(V_i) \left((u_{hi} - u_{hj})/d_{ij}\right)^2,$$
$$||u_h||_{1,h}^2 = |u_h|_{0,h}^2 + |u_h|_{1,h}^2, \qquad |||u_h|||_0^2 = (u_h, I_h^* u_h),$$

where $d_{ij} = d(x_i, x_j)$ is the distance between x_i and x_j.

In the lemmas below, we assume that the lines of discontinuity (if any) of the matrix $A(x)$ are aligned with edges of the elements in the triangulation T_h and that the entries of the matrix $A(x)$ are C^1-functions over each element of T_h.

LEMMA 2.1 (see, e.g., [7, 21]). *There exist two positive constants $C_0, C_1 > 0$, independent of h, such that*

$$C_0 |v_h|_{0,h} \le ||v_h||_0 \le C_1 |v_h|_{0,h}, \qquad v_h \in S_h,$$
$$C_0 |||v_h|||_0 \le ||v_h||_0 \le C_1 |||v_h|||_0, \qquad v_h \in S_h,$$
$$C_0 ||v_h||_{1,h} \le ||v_h||_1 \le C_1 ||v_h||_{1,h}, \qquad v_h \in S_h.$$

LEMMA 2.2 (see, e.g., [7, 21]). *There exist two positive constants $C_0, C_1 > 0$, independent of h and $h_0 > 0$, such that for all $0 < h \le h_0$,*

$$|a(u_h, I_h^* v_h)| \le C_1 ||u_h||_{1,h} ||v_h||_{1,h}, \qquad u_h, v_h \in S_h, \tag{2.4}$$
$$a(u_h, I_h^* u_h) \ge C_0 ||u_h||_{1,h}^2, \qquad u_h, v_h \in S_h. \tag{2.5}$$

3. Error estimates for the FVE method.

3.1. Optimal-order H^1 error estimates. We first consider the error of the FVE solution u_h in the H^1 norm. We start with the following two lemmas.

LEMMA 3.1. *For any $u_h, v_h \in S_h$, we have*

$$a(u_h, I_h^* v_h) = a(u_h, v_h) + E_h(u_h, v_h) \tag{3.1}$$

with

$$E_h(u_h, v_h) = -\sum_{K \in T_h} \int_K (A - A_K) \nabla u_h \cdot \nabla v_h \, dx$$
$$- \sum_{j \in N_h} \sum_{i \in \Pi(j)} \frac{1}{2} \int_{\gamma_{ij}} (A - A_K) \nabla u_h \cdot \mathbf{n} dS (v_i - v_j),$$

and

$$A_K = \frac{1}{\text{meas}(K)} \int_K A(x) dx, \qquad K \in T_h.$$

Moreover, if A is in $W^{1,\infty}(\Omega)$, then there is a positive constant $C > 0$, independent of h, such that

$$|E_h(u_h, v_h)| \le Ch||u_h||_{1,h}||v_h||_{1,h}.$$

Proof. For the proof, see [12, 13]. ∎

LEMMA 3.2. *Assume that u_h is the FVE solution defined by* (2.1). *Then we have*

$$a(u_h, v_h) = (f, I_h^* v_h) - E_h(u_h, v_h), \qquad v_h \in S_h. \tag{3.2}$$

Proof. The proof follows directly from Lemma 3.1. ∎

THEOREM 3.3. *Assume that u and u_h are the solutions of* (1.1) *and* (2.1), *respectively, $u \in H^{1+\alpha}(\Omega)$, $f \in H^{-1+\beta}(\Omega)$ with $0 < \alpha \le \beta \le 1$, and $A \in W^{1,\infty}(\Omega)$. Then we have*

$$||u - u_h||_1 \le C\left(h^\beta ||f||_{-1+\beta} + h^\alpha ||u||_{1+\alpha}\right). \tag{3.3}$$

Proof. By (3.1) and (1.1), we see that for $\phi_h = I_h u - u_h$,

$$\begin{aligned} C_0||u - u_h||_1^2 &\le a(u - u_h, u - I_h u) + a(u - u_h, \phi_h) \\ &= a(u - u_h, u - I_h u) + (f, \phi_h - I_h^* \phi_h) + E_h(u_h, \phi_h) \\ &\le Ch^\alpha ||u - u_h||_1 ||u||_{1+\alpha} + Ch^\beta ||f||_{-1+\beta} ||\phi_h||_{1,h} + Ch||u_h||_{1,h}||\phi_h||_{1,h}. \end{aligned}$$

Notice that from Lemma 2.2 and the approximation theory we have

$$\begin{aligned} &||u_h||_{1,h} \le C||f||_{-1} \le C||f||_{-1+\beta}, \\ &||\phi_h||_{1,h} \le ||u - u_h||_1 + Ch^\alpha ||u||_{1+\alpha}; \end{aligned}$$

the proof is then completed by combining these inequalities. ∎

Remark. The main idea in the proof above is motivated by [6], which is somewhat different from those ideas in [3, 17, 20, 21, 25]. The approach is also more direct and simpler because the key identity (3.2) allows us to employ the standard error estimation procedures developed for finite element methods. In particular, the estimate for $||I_h u - u_h||$ is not needed in this proof. Moreover, the estimate here describes how the regularities of the exact solution and the source term can independently affect the accuracy of the FVE solution.

3.2. Optimal-order L^2 error estimates. In this section, we derive an optimal-order L^2 error estimate for the FVE method with the minimal regularity assumption for the exact solution u. This error estimate also will show how the error in the L^2 norm depends on the regularity of the source term.

The following lemma gives another key feature of the bilinear form in the FVE method.

LEMMA 3.4. *Assume that $u_h, v_h \in S_h$. Then we have*

$$\begin{aligned} a(u_h, v_h) = a(u_h, I_h^* v_h) &+ \sum_{K \in T_h} \int_{\partial K} (A\nabla u_h \cdot \mathbf{n})\,(v_h - I_h^* v_h) dS \\ &- \sum_{K \in T_h} \int_K (\nabla \cdot A\nabla u_h)\;\,(v_h - I_h^* v_h) dx. \end{aligned} \tag{3.4}$$

Proof. It follows from Green's formula that

$$\sum_{K\in T_h}\Big(\nabla\cdot A\nabla u_h, v_h\Big)_K = \sum_{K\in T_h}\int_K \nabla\cdot A\nabla u_h v_h dx$$
$$= \sum_{K\in T_h}\int_{\partial K}(A\nabla u_h\cdot\mathbf{n})\, v_h dS - a(u_h,v_h)$$

and

$$\sum_{K\in T_h}\Big(\nabla\cdot A\nabla u_h, I_h^* v_h\Big)_K$$
$$= \sum_{K\in T_h}\sum_{j\in N_h}\Big(\nabla\cdot A\nabla u_h, I_h^* v_h\Big)_{K\cap V_j}$$
$$= \sum_{K\in T_h}\int_{\partial K}(A\nabla u_h\cdot\mathbf{n}) I_h^* v_h dS + \sum_{j\in N_j}\int_{\partial V_j}(A\nabla u_h\cdot\mathbf{n}) I_h^* v_h dS$$
$$= \sum_{K\in T_h}\int_{\partial K}(A\nabla u_h\cdot\mathbf{n}) I_h^* v_h dS - a(u_h, I_h^* v_h).$$

Then the proof is completed by taking the difference of these two identities. ∎

THEOREM 3.5. *Assume that u and u_h are the solutions of* (1.1) *and* (2.1)*, respectively, and $u\in H^2(\Omega)$, $f\in H^\beta$ $(0\le\beta\le 1)$, and $A\in W^{2,\infty}(\Omega)$. Then there exists a positive constant $C>0$ such that*

$$||u-u_h||_0 \le C\Big(h^2||u||_2 + h^{1+\beta}||f||_\beta\Big). \tag{3.5}$$

Proof. Let $w\in H_0^1(\Omega)$ be the solution of

$$-\nabla\cdot A\nabla w = u-u_h, \qquad x\in\Omega, \qquad \text{and } w=0 \text{ on } \partial\Omega.$$

Then we have $||w||_2\le ||u-u_h||_0$. By Theorem 3.3 we have

$$\begin{aligned}||u-u_h||_0^2 &= a(u-u_h, w-w_h) + a(u-u_h, w_h)\\ &\le C(h^\alpha||u||_{1+\alpha} + h^{1+\beta}\,\|f\|_\beta)||w-w_h||_1 + a(u-u_h,w_h),\ (0\le\alpha\le 1).\end{aligned}$$

Then by Lemma 3.4,

$$a(u-u_h,w_h) = J_1(u_h,w_h) + J_2(u_h,w_h) + J_3(u_h,w_h),$$

where the J_i's are defined for $u_h, w_h\in S_h$ by

$$\begin{aligned}J_1(u_h,w_h) &= \sum_{K\in T_h}(f, w_h - I_h^* w_h)_K,\\ J_2(u_h,w_h) &= \sum_{K\in T_h}\Big(\nabla\cdot A\nabla u_h,\ w_h - I_h^* w_h\Big)_K,\\ J_3(u_h,w_h) &= -\sum_{K\in T_h}\int_{\partial K}\Big(A\nabla(u-u_h)\cdot\mathbf{n}\Big)\Big(w_h - I_h^* w_h\Big)dS,\end{aligned} \tag{3.6}$$

and the continuity of $\nabla u \cdot \mathbf{n}$ on each ∂K is used.

Since the dual mesh is formed by the barycenters, we have

$$\int_K (w_h - I_h^* w_h) dx = 0 \qquad \text{for all } K \in T_h$$

so that

$$J_1 = \sum_{K \in T_h} (f - f_K, w_h - I_h^* w_h)_K \le C h^{1+\beta} ||f||_\beta ||w_h||_{1,h},$$

where f_K is the average value of f on K. Similarly, using the fact that $A \in W^{2,\infty}$, we have

$$\begin{aligned} J_2 &= -\sum_{K \in T_h} \Big(\nabla \cdot A \nabla u_h - (\nabla \cdot A \nabla u_h)_K,\ (w_h - I_h^* w_h) \Big)_K \\ &\le C h^{1+\alpha} ||u_h||_{1,h} ||w_h||_{1,h}. \end{aligned}$$

For J_3, according to the continuity of $\nabla u \cdot \mathbf{n}$ and the shape of the control volume, we have

$$J_3 = \sum_{K \in T_h} \int_{\partial K} \Big((A - \bar{A}_K) \nabla (u - u_h) \cdot \mathbf{n} \Big) (w_h - I_h^* w_h) dS,$$

where $\bar{A}_K$ is a function designed in a piecewise manner such that for any edge E of a triangle $K \in T_h$,

$$\bar{A}_K(x) = A(x_c), \qquad x \in E,$$

and x_c is the middle point of E. Since $|A(x) - \bar{A}_K| \le h ||A||_{1,\infty}$, we have from Theorem 3.3 that

$$\begin{aligned} J_3 &\le Ch \sum_{K \in T_h} \int_{\partial K} |\nabla (u - u_h) \cdot \mathbf{n}|\ |w_h - I_h^* w_h| dS \\ &\le Ch \sum_{K \in T_h} \Big\{ h_k^{1/2} ||u||_{2,K} + h_k^{-1/2} ||u - u_h||_{1,K} \Big\} \\ &\qquad \times \Big\{ h_k^{1/2} ||w_h||_{1,K} + h_k^{-1/2} ||w_h - I_h^* w_h||_{0,K} \Big\} \\ &\le C h^2 ||u||_2 ||w_h||_{1,h}. \end{aligned}$$

Thus, it follows by taking $w_h = I_h w$ that

$$\begin{aligned} J_1 + J_2 + J_3 &\le C \Big(h^2 ||u||_2 + h^{1+\beta} ||f||_\beta \Big) ||w_h||_{1,h} \\ &\le C \Big(h^2 ||u||_2 + h^{1+\beta} ||f||_\beta \Big) ||u - u_h||_0; \end{aligned}$$

therefore, we have

$$||u - u_h||_0 \le C \Big(h^2 ||u||_2 + h^{1+\beta} ||f||_\beta \Big)$$

and the proof is completed. ∎

COROLLARY 3.6. *Assume that* $u \in H^{1+\alpha}(\Omega)$, $f \in H^{\alpha}(\Omega)$ *with* $0 < \alpha \leq 1$, *and* $A \in W^{2,\infty}(\Omega)$. *Then we have*

$$||u - u_h||_0 \leq Ch^{2\alpha}\bigg(||u||_{1+\alpha} + ||f||_{\alpha}\bigg).$$

Proof. Let f_h be the L^2 projection of f into S_h and consider $S(u, f) = (u - u_h, f - f_h)$ as a linear operator from $H^s \times H^{-1+s}$ to $H^0 \times H^{-1}$ for any $s > 0$. For any $(u, f) \in H^s \times H^{-1+s}$, we let

$$||(u, f)||_s^2 = ||u||_s^2 + ||f||_{-1+s}^2.$$

Then, by Theorem 3.5, we have

$$||S(u, f)||_0 \leq Ch^2||(u, f)||_2 \quad \text{and} \quad ||S(u, f)||_0 \leq C||(u, f)||_1.$$

Hence, according to the theory of interpolation spaces [4, 5], we have

$$||S(u, f)||_0 \leq Ch^{2\alpha}||(u, f)||_{1+\alpha},$$

which in fact is (3.7). ∎

Remark. When the source term f is in H^1, the order of convergence in Theorem 3.5 is optimal with respect to the approximation capability of finite element space. Note that, in many applications, the H^1 regularity of f does not imply the $W^{3,\infty}$ or H^3 regularity of the exact solution required by the L^2 norm error estimates in the literature. Moreover, counterexamples presented in the next subsection indicate that the regularity assumption on f cannot be reduced. The result in Theorem 3.5 reveals how the regularities of the exact solution and the source term can affect the error of the FVE solution in the L^2 norm, and this is a more general result than those in the literature.

3.3. Superconvergence in the H^1 norm. In a way similar to the finite element solution with linear elements, we can show that the FVE solution has a certain superconvergence in the H^1 norm when the exact solution has a stronger regularity and the partition used has a better quality. Specifically, throughout this subsection we assume that the involved partition for the FVE solution is uniform or piecewise uniform without any interior meeting points. This requirement might be relaxed (see, for example, [29]), but we would rather use this simpler assumption to present our basic idea.

We first recall the following superconvergence estimates for the Lagrange interpolation [9, 28, 29, 30] from finite element theory.

LEMMA 3.7. *Assume that* $u \in W^{3,p}(\Omega) \cap H_0^1(\Omega)$. *We have*

$$|a(u - I_h u, v_h)| \leq Ch^2||u||_{W^{3,p}}||v_h||_{W^{1,q}}, \qquad v_h \in S_h,$$

where $1 \leq p,\ q \leq \infty$, *and* $p^{-1} + q^{-1} = 1$.

THEOREM 3.8. *Assume that* $f \in H^1(\Omega)$, $u \in H^3(\Omega) \cap H_0^1(\Omega)$, *and* $A \in W^{2,\infty}(\Omega)$. *Then we have*

$$||I_h u - u_h||_1 \leq Ch^2\bigg(||f||_1 + ||u||_3\bigg).$$

Proof. It follows from Lemma 3.7 that

$$\begin{aligned} C_0||I_hu-u_h||_1^2 &\le a(I_hu-u_h, I_hu-u_h) \\ &= a(I_hu-u, I_hu-u_h) + a(u-u_h, I_hu-u_h) \\ &\le Ch^2||u||_3||I_hu-u_h||_1 + a(u-u_h, I_hu-u_h). \end{aligned}$$

Following a similar argument used in the proof of Theorem 3.5, we see that

$$a(u-u_h, I_hu-u_h) \le Ch^2\left(||f||_1 + ||u||_2\right)||I_hu-u_h||_1$$

because I_hu-u_h is in S_h. The result of this theorem follows by combining these two inequalities. ∎

We can use one of the applications of the above superconvergence property of the FVE solution to obtain a maximum norm error estimate.

COROLLARY 3.9. *Under the assumptions of Theorem* 3.8 *and* $u \in W^{2,\infty}(\Omega) \cap H^3(\Omega)$*, we have*

$$||u-u_h||_\infty \le Ch^2\ \left(\log\frac{1}{h}\right)^{1/2}(||u||_{2,\infty} + ||u||_3 + ||f||_1)\,.$$

Proof. The proof follows from Theorem 3.8 and from the approximation theory stating that

$$\begin{aligned} ||u-u_h||_\infty &\le ||u-I_hu||_\infty + ||u_h-I_hu||_\infty \\ &\le Ch^2||u||_{2,\infty} + C\left(\log\frac{1}{h}\right)^{1/2}||u_h-I_hu||_{1,h} \\ &\le Ch^2||u||_{2,\infty} + Ch^2\ \left(\log\frac{1}{h}\right)^{1/2}\left(||f||_1 + ||u||_3\right). \end{aligned}$$ ∎

We remark that this result is not optimal with respect to the regularity required on the exact solution u. This excessive regularity can be removed according to the result in the following subsection.

3.4. Error estimates in maximum norm. Now we turn to the L^∞ norm and $W^{1,\infty}$ norm error estimates for the FVE solution. First, we recall from [10, 16, 23, 29] the definition and estimates on the regularized Green's functions.

For a point $z = (z_1, z_2) \in \Omega$, we define $G^z \equiv G(x,z) \in H_0^1(\Omega) \cap H^2(\Omega)$ to be the solution of the equation

$$-\nabla \cdot A\nabla G^z = \delta_h^z(x) \qquad \text{in } \Omega, \tag{3.7}$$

where $\delta_h^z(x) \in S_h$ is a smoothed δ-function associated with the point z, which has the following properties:

$$(\delta_h^z v_h) = v_h(z) \quad \text{ for all } v_h \in S_h, |\delta_h^z(x)| \le Ch^{-2}, \quad \text{supp}(\delta_h^z) \subset \{x;\ |x-z| \le Ch\}.$$

Let G_h^z be the finite element approximation of the regularized Green's function, i.e.,

$$a(G^z - G_h^z, \chi) = 0, \qquad \chi \in S_h.$$

Following [29], for a given point $z \in \Omega$ we define $\partial_z G^z$ by

$$\partial_z G^z = \lim_{\Delta z \to 0, \Delta z // L} \frac{G^{z+\Delta z} - G^z}{|\Delta z|}$$

for any fixed direction L in R^2, where $\Delta z // L$ means that Δz is parallel to L. Clearly, $\partial_z G^z$ satisfies

$$a(\partial_z G^z, \chi) = -\Big(\partial_z \delta_h, \chi\Big) = \partial_z \chi(z), \qquad \chi \in S_h.$$

The finite element approximation $\partial_z G_h^z$ of $\partial_z G^z$ is then defined by

$$a(\partial_z G^z - \partial_z G_h^z, \chi) = 0, \qquad \chi \in S_h.$$

It is well known that the functions G^z and $\partial_z G^z$ have the following properties [29]: For any $w \in H_0^1(\Omega)$,

$$P_h w(z) = a(G^z(t), w), \qquad \partial_z P_h w(z) = a(\partial_z G^z(t), w), \tag{3.8}$$

where P_h is an L^2-projection operator on S_h, i.e., $(u - P_h u, v_h) = 0$ for all $v_h \in S_h$.

Moreover, the following estimates have been established in the literature [10, 16, 23, 29]:

$$\|G^z - G_h^z\|_{1,1} \le Ch\log\frac{1}{h}, \tag{3.9}$$

$$\|\partial_z G^z - \partial_z G_h^z\|_{1,1} \le C, \tag{3.10}$$

$$||G_h^z||_{1,1} \le C\log\frac{1}{h}, \tag{3.11}$$

$$||\partial_z G^z||_{1,1} \le C\log\frac{1}{h}, \tag{3.12}$$

$$\|P_h u - u\|_{0,\infty} + h\,\|P_h u - u\|_{1,\infty} \le Ch^2\,\|u\|_{2,\infty} \tag{3.13}$$

with constant $C > 0$ independent of h and z.

First, let us consider the $W^{1,\infty}$ norm error estimate.

THEOREM 3.10. *Assume that $u \in W^{2,\infty}(\Omega)$, $f \in L^\infty(\Omega)$, and $A \in W^{1,\infty}(\Omega)$. Then there exist positive constants $C > 0$ and $h_0 > 0$ independent of u such that for all $0 < h \le h_0$,*

$$||u - u_h||_{1,\infty} \le Ch\ \log\left(\frac{1}{h}\right)\Big(||u||_{2,\infty} + ||f||_\infty\Big).$$

Proof. It follows from (3.8) that

$$\begin{aligned} \partial_z(P_h u - u_h)(z) &= a(u - u_h, \partial_z G^z) \\ &= a\Big(u - u_h, \partial_z G^z - \partial_z G_h^z + \partial_z G_h^z\Big) \\ &= a(u - u_h, \partial_z G^z - \partial_z G_h^z) + a(u - u_h, \partial_z G_h^z) \\ &= a(u - I_h u, \partial_z G^z - \partial_z G_h^z) + a(u - u_h, \partial_z G_h^z) \\ &\le Ch||u||_{2,\infty}\|\partial_z G^z - \partial_z G_h^z\|_{1,1} \\ &\quad + (f, \partial_z G_h^z - I_h^* \partial_z G_h^z) + E_h(u_h, \partial_z G_h^z). \end{aligned}$$

For the second term on the right-hand side, we have

$$(f, \partial_z G_h^z - I_h^* \partial_z G_h^z) \le ||f||_\infty ||\partial_z G_h^z - I_h^* \partial_z G_h^z||_{L^1} \le Ch \left(\log \frac{1}{h}\right) ||f||_\infty.$$

For the third term, by the definition of E_h given in Lemma 3.1 and the fact that $\partial_z G_h^z$ is a piecewise linear polynomial, we have

$$\begin{aligned} E_h(u_h, \partial_z G_h^z) &= E_h(u_h - I_h u + I_h u, \partial_z G_h^z) \\ &\le Ch\Big(||u_h - I_h u||_{1,\infty} + ||I_h u||_{1,\infty}\Big)||\partial_z G_h^z||_{1,1} \\ &\le Ch \log\frac{1}{h}||u_h - I_h u||_{1,\infty} + Ch \log\frac{1}{h}||u||_{1,\infty}. \end{aligned}$$

Thus, we obtain

$$||P_h u - u_h||_{1,\infty} \le Ch \log\frac{1}{h}\Big(||u||_{2,\infty} + ||f||_\infty\Big) + Ch \log\frac{1}{h}||P_h u - u_h||_{1,\infty}$$

so that we have for some $h_0 > 0$, such that $0 < h \le h_0$,

$$||P_h u - u_h||_{1,\infty} \le Ch \log\frac{1}{h}\Big(||u||_{2,\infty} + ||f||_\infty\Big).$$

Applying this inequality and (3.13) in

$$||u - u_h||_{1,\infty} \le ||P_h u - u_h||_{1,\infty} + ||P_h u - u||_{1,\infty}$$

leads to the result of this theorem. □

The following theorem gives a maximum norm error estimate for the FVE solution.

THEOREM 3.11. *Assume that $u \in W^{2,\infty}(\Omega)$, $f \in W^{1,\infty}(\Omega)$, and $A \in W^{2,\infty}(\Omega)$. Then there exist constants $C > 0$ and $h_0 > 0$, independent of u, such that for all $0 < h \le h_0$,*

$$||u - u_h||_\infty \le Ch^2 \log\left(\frac{1}{h}\right)\Big(||u||_{2,\infty} + ||f||_{1,\infty}\Big).$$

Proof. We follow an idea similar to the proof of the previous theorem, but we now use the regularized Green's function G^z and its finite element approximation G_h^z as follows:

$$\begin{aligned} (P_h u - u_h)(z) &= a(u - u_h, G^z + G_h^z - G_h^z) \\ &= a(u - u_h, G^z - G_h^z) + a(u - u_h, -G_h^z) \\ &= a(u - I_h u, G^z - G_h^z) + a(u - u_h, G_h^z) \\ &= Ch||u||_{2,\infty}||G^z - G_h^z||_{1,1} \\ &\qquad + J_1(u_h, G_h^z) + J_2(u_h, G_h^z) + J_2(u_h, G_h^z). \end{aligned}$$

The functionals J_1, J_2, and J_3 above are defined in the same way as given in the proof of Theorem 3.5. For $J_1(u_h, G_h^z)$, from (3.6) we have

$$\begin{aligned} J_1(u_h, G_h^z) &= \sum_{K\in T_h} (f - f_K, G_h^z - I_h^* G_h^z)_K \\ &\le Ch||f||_{1,\infty} \sum_{K\in T_h} ||G_h^z - I_h^* G_h^z||_{L^1(K)} \\ &\le Ch^2 ||f||_{1,\infty} ||G_h^z||_{1,1} \le Ch^2 \log\frac{1}{h} ||f||_{1,\infty}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} J_2(u_h, G_h^z) &= \sum_{K\in T_h} \Big(\nabla A\cdot\nabla u_h, G_h^z - I_h^* G_h^z\Big) \\ &= \sum_{K\in T_h} \Big((\nabla A - (\nabla A)_K)\cdot\nabla u_h, G_h^z - I_h^* G_h^z\Big) \\ &\le Ch||A||_{2,\infty}||u_h||_{1,\infty} \sum_{K\in T_h} ||G_h^z - I_h^* G_h^z||_{L^1(K)}. \end{aligned}$$

We know by Theorem 3.10 that

$$\begin{aligned} ||u_h||_{1,\infty} &\le ||u - u_h||_{1,\infty} + ||u||_{1,\infty} \\ &\le Ch\ \log\frac{1}{h}\,(||u||_{2,\infty} + ||f||_{\infty}) + ||u||_{1,\infty}. \end{aligned}$$

Therefore, there exists a small $h_0 > 0$ such that for $0 < h \le h_0$,

$$J_2(u_h, G_h^z) \le Ch^2\ \log\frac{1}{h}\Big(||u||_{2,\infty} + ||f||_{\infty}\Big).$$

As for $J_3(u_h, G_h^z)$, we note that G_h^z is a piecewise linear polynomial and

$$J_3(u_h, G_h^z) = \sum_{K\in T_h} \int_{\partial K} (A - A_K)\nabla(u - u_h)\cdot\mathbf{n}\Big(G_h^z - I_h^* G_h^z\Big) dS.$$

Thus, it is easy to see from Theorem 3.10 and (3.11) that

$$\begin{aligned} J_3(u_h, G_h^z) &\le Ch||A||_{1,\infty}||u - u_h||_{1,\infty} \sum_{K\in T_h} ||G_h^z - I_h^* G_h^z||_{L^1(\partial K)} \\ &\le Ch||u - u_h||_{1,\infty}||G_h^z||_{1,1} \\ &\le Ch^2 \log\frac{1}{h}\Big(||u||_{2,\infty} + ||f||_{\infty}\Big). \end{aligned}$$

Combining the estimates obtained above for the J_i's, we have

$$||P_h u - u_h||_{\infty} \le h^2\ \log\frac{1}{h}\Big(||u||_{2,\infty} + ||f||_{1,\infty}\Big).$$

This together with (3.13) completes the proof. ∎

The following theorem gives a superconvergence property in the maximum norm for the FVE solution.

THEOREM 3.12. *Under the same conditions as in Theorem* 3.11*, we have*

$$||I_h u - u_h||_{1,\infty} \le Ch^2 \ \log\left(\frac{1}{h}\right)\left(||u||_{3,\infty} + ||f||_{1,\infty}\right).$$

Proof. It follows from the properties of $\partial_z G_h^z$ and $\partial_z G^z$ and from Lemma 3.7 that

$$\begin{aligned}
\partial_z(I_h u - u_h)(z) &= a(I_h u - u_h, \partial_z G^z - \partial_z G_h^z + \partial_z G_h^z)\\
&= a(I_h u - u_h, \partial_z G_h^z)\\
&= a(I_h u - u, \partial_z G_h^z) + a(u - u_h, \partial_z G_h^z)\\
&= Ch^2 ||u||_{3,\infty} ||\partial_z G_h^z||_{1,1}\\
&\quad + J_1(u_h, \partial_z G_h^z) + J_2(u_h, \partial_z G_h^z) + J_2(u_h, \partial_z G_h^z).
\end{aligned}$$

We see from (3.6) and (3.11) that

$$J_1(u_h, \partial_z G_h^z) \le Ch^2 \log\frac{1}{h} ||f||_{1,\infty}.$$

When $h > 0$ is small, we also have

$$\begin{aligned}
J_2(u_h, \partial_z G_h^z) &\le Ch^2 \log\frac{1}{h} ||u_h||_{1,\infty}\\
&\le Ch^2 \ \log\frac{1}{h}\left(||u||_{2,\infty} + ||f||_{1,\infty}\right).
\end{aligned}$$

For $J_3(u_h, \partial_z G_h^z)$, we have

$$\begin{aligned}
J_3(u_h, \partial_z G_h^z) &\le Ch^2 ||u - u_h||_{1,\infty} \sum_{k \in T_h} \int_{\partial K} |\partial_z G_h^z - I_h^* \partial_z G_h^z| dS\\
&\le Ch^2 ||u - u_h||_{1,\infty} ||\partial_z G_h^z||_{1,1} \le Ch^2 \log\frac{1}{h} ||u - u_h||_{1,\infty},
\end{aligned}$$

because $\partial_z G_h^z$ is piecewise linear in each element $K \in T_h$. Finally, the proof is completed by combining the above estimates. □

3.5. Uniform convergence for u in $H_0^1(\Omega)$. In many applications, the exact solution u of (1.1) may be in the space $H^1(\Omega)$, but not in $H^{1+\alpha}(\Omega)$ for any $\alpha > 0$. In this situation, the authors of [24] showed that for any $\epsilon > 0$, there exists $h_0 = h_0(\epsilon) > 0$ such that for all $0 < h \le h_0$, we have

$$||u - u_h||_1 \le \epsilon ||f||$$

for the Galerkin finite element solution $u_h \in S_h$ (or the Ritz projection of u into S_h of the exact solution of (1.1)). This implies that u_h converges to u uniformly even though there is no order of convergence for u_h.

The following theorem shows that the FVE solution also has this uniform convergence feature.

THEOREM 3.13. *Assume that A is uniformly continuous and $f \in L^2(\Omega)$. Let $u \in H_0^1(\Omega)$ and $u_h \in S_h$ be the solutions of* (1.1) *and* (2.1)*, respectively. Then for any $\epsilon > 0$, there exists $h^* = h^*(\epsilon) > 0$ such that for all $0 < h \le h^*$, the following holds:*

$$||u - u_h||_1 \le \epsilon ||f||_0.$$

Proof. As in the proof in Theorem 3.3, we have

$$\begin{aligned} C||u-u_h||_1^2 &\le a(u-u_h,u-u_h) = a(u-u_h,u-v)+a(u-u_h,\phi) \\ &\le C||u-u_h||_1||u-v||_1 + a(u-u_h,\phi), \end{aligned}$$

where $\phi = v - u_h \in S_h$ for any $v \in S_h$. Since $A(x)$ is uniformly continuous in Ω, for any $\epsilon_0 > 0$, there exists $h_0 = h_0(\epsilon_0) > 0$ such that $|A(x) - A(y)| \le \epsilon_0$ for all $|x-y| \le h_0$. Thus, by Lemma 3.7 we can take $h \in (0, h_0)$ to obtain

$$|E_h(u_h,\phi)| \le C\epsilon_0||u_h||_{1,h}||\phi||_{1,h},$$

where E_h is defined in Lemma 3.1. By Lemma 3.2, we have

$$\begin{aligned} a(u-u_h,\phi) &= (f,\phi - I_h^*\phi) + E_h(u_h,\phi) \\ &\le C||f||_0 h||\phi||_{1,h} + C\epsilon_0||u_h||_{1,h}||\phi||_{1,h} \\ &\le C(h+\epsilon_0)||f||_0||\phi||_{1,h} \\ &\le C(h+\epsilon_0)||f||_0\Big(||u-v||_1 + ||u-u_h||_1\Big). \end{aligned}$$

Thus it follows from the triangle inequality that

$$||u-u_h||_1 \le C\left((h+\epsilon_0)||f||_0 + \inf_{v\in S_h}||u-v||_1\right).$$

Lemma 2 of [24] indicates that for any $\epsilon_1 > 0$, there exists $h_1 = h_1(\epsilon_1) > 0$ such that

$$\inf_{v\in S_h}||u-v||_1 \le \epsilon_1||f||_0.$$

Notice that the constant $C > 0$ above is independent of u, f, and A; therefore, the theorem follows from the last two inequalities. ◻

4. Counterexamples. In this section, we will present two examples to show that, when the source term $f(x,y)$ is only in $L^2(\Omega)$, the FVE solution generally cannot have the optimal second-order convergence rate even if the exact solution $u(x,y)$ has the usual H^2 regularity. The first example is based on theoretical error estimates, while the second is presented through numerical computations. We also provide an example to corroborate the optimal error estimate obtained in this paper under the condition that the exact solution u is in H^2 and the source term f is in H^1.

4.1. A one-dimensional example. First, we consider an example in one dimension,

$$-u'' = f = x^{-\alpha}, \quad 0 < x < 1, \quad u(0) = u(1) = 0, \tag{4.1}$$

where $f \in L^2(0,1)$ but is not in $H^1(0,1)$ if $0 \le \alpha < 1/2$. Clearly this problem has an exact solution,

$$u = \frac{x^{2-\alpha} - x}{(1-\alpha)(2-\alpha)},$$

which is in the space $H^2(0,1)$.

Let T_h be the uniform partition of the interval $[0,1]$ such that $x_j = hj, j = 0,1,\dots,N$ and $x_{j+1/2} = h(j+1/2), j = 0,1,\dots,N-1$. Let S_h be the piecewise linear finite element space. Let $u_f \in S_h$ be the finite element solution of (4.1) defined by

$$a(u_f, v_h) = (f, v_h), \qquad v_h \in S_h,$$

and let u_h be the FVE solution. Then we have

$$a(e_h, v_h) = (f - f_h, v_h) + (f_h, v_h - I_h^* v_h), \qquad v_h \in S_h, \tag{4.2}$$

with $e_h = u_f - u_h$ and

$$f_h = x_h^{-\alpha} = \begin{cases} \dfrac{1}{h}\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} x^{-\alpha} dx, & x \in (x_{j-1/2}, x_{j+1/2}), \quad j = 1, \ldots, N-1, \\ 0, & x \in (0, x_{1/2}) \cup (x_{N-1/2}, 1). \end{cases}$$

Our main task is to show that there exists a constant $C > 0$ such that

$$||u_h - u_f||_0 \geq Ch^{2-\alpha}.$$

This inequality and

$$\begin{aligned} ||u - u_h||_0 &\geq ||u_h - u_f||_0 - ||u - u_f||_0 \\ &\geq ||u_h - u_f||_0 - Ch^2||u||_2 \geq ||u_h - u_f||_0 - Ch^2||f||_0 \end{aligned} \tag{4.3}$$

together imply that the FVE solution cannot have the optimal L^2 norm convergence rate for $0 < \alpha < 1/2$.

We start with the estimates of the error function $e(x)$ at the nodes. Let $G(x, y)$ be the Green's function defined by

$$G(x, y) = \begin{cases} x(1 - y), & 0 < y < x, \\ y(1 - x), & x < y < 1. \end{cases} \tag{4.4}$$

Then, we have

$$\begin{aligned} &e_h(x_k) \\ &\quad= \Big(f - f_h, G(\cdot, x_k)\Big) + (f_h, G(\cdot, x_k) - I_h^* G(\cdot, x_k)) \\ &\quad= \int_0^1 (x^\alpha - x_h^\alpha) G(x, x_k) dx + (f_h, G(\cdot, x_k) - I_h^* G(\cdot, x_k)) \\ &\quad= \int_0^{h/2} x^{-\alpha} x(1 - x_k) dx + \sum_{j=1}^{k-1} \int_{x_{j-1/2}}^{x_{j+1/2}} (x^{-\alpha} - x_h^{-\alpha}) x(1 - x_k) dx \\ &\qquad + \int_{x_{k-1/2}}^{x_k} (x^{-\alpha} - x_h^{-\alpha}) x(1 - x_k) dx + \int_{x_k}^{x_{k+1/2}} (x^{-\alpha} - x_h^{-\alpha})\, x_k\, (1 - x) dx \\ &\qquad + \sum_{j=k+1}^{N-1} \int_{x_{j-1/2}}^{x_{j+1/2}} (x^{-\alpha} - x_h^{-\alpha})\, x_k\, (1 - x) dx \\ &\qquad + \int_{1-h/2}^1 (x^{-\alpha} - x_h^{-\alpha})\, x_k\, (1 - x) dx + (f_h, G(\cdot, x_k) - I_h^* G(\cdot, x_k)) \\ &\quad= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \end{aligned} \tag{4.5}$$

Now we will estimate the J_l's one by one under the assumptions that

$$0 \le \alpha < \frac{1}{2}, \quad x_k \in \left[\frac{1}{3}, \frac{2}{3}\right].$$

For J_1 and J_6 it easily follows from a simple calculation that

$$J_1 = \frac{1-x_k}{2-\alpha}\left(\frac{h}{2}\right)^{2-\alpha},$$
$$|J_6| \le 2\left(1-\frac{h}{2}\right)^{-\alpha} x_k \int_{1-h/2}^{1} (1-x)dx \le C_6 x_k h^2 \le C_6 h^2.$$

For J_5, using the definition of x_h^α and integration by parts, we have

$$\begin{aligned} J_5 &= -x_k \sum_{j=k+1}^{N-1} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(x^{-\alpha} - x_h^{-\alpha}\right) x dx \\ &= -x_k \sum_{j=k+1}^{N-1} \left(\int_{x_{j-1/2}}^{x_{j+1/2}} x^{1-\alpha} dx - h x_j x_h^{-\alpha} \right) \\ &= -x_k \sum_{j=k+1}^{N-1} \left(\int_{x_{j-1/2}}^{x_{j+1/2}} x^{1-\alpha} dx - x_j \int_{x_{j-1/2}}^{x_{j+1/2}} x^{-\alpha} dx \right) \\ &= -x_k \sum_{j=k+1}^{N-1} \int_{x_{j-1/2}}^{x_{j+1/2}} x^{-\alpha}(x - x_j) dx \\ &= \frac{-x_k}{1-\alpha} \left(\sum_{j=k+1}^{N-1} \frac{x_{j-1/2}^{1-\alpha} + x_{j+1/2}^{1-\alpha}}{2} h - \int_{x_{k+1/2}}^{x_{N-1/2}} x^{1-\alpha} dx \right). \end{aligned}$$

Note that

$$\frac{d^2}{dx^2}(x^{1-\alpha}) = (1-\alpha)(-\alpha)x^{-1-\alpha}. \tag{4.6}$$

Thus there is a positive constant C_5 independent of h such that

$$|J_5| \le C_5 h^2$$

because of the error estimate for the trapezoidal quadrature formula. Now consider J_3 and J_4. First rewrite $J_3 + J_4$ as

$$\begin{aligned} J_3 + J_4 & \\ &= \int_{x_{k-1/2}}^{x_k} (x^{-\alpha} - x_h^{-\alpha}) x (1 - x_k) dx + \int_{x_k}^{x_{k+1/2}} \left(x^{-\alpha} - x_h^{-\alpha}\right) x_k \, (1-x) dx \\ &= -\int_{x_{k-1/2}}^{x_{k+1/2}} (x^{-\alpha} - x_h^{-\alpha}) x x_k dx \\ &\quad + \int_{x_{k-1/2}}^{x_k} (x^{-\alpha} - x_h^{-\alpha}) x dx + \int_{x_k}^{x_{k+1/2}} \left(x^{-\alpha} - x_h^{-\alpha}\right) x_k dx \\ &= \int_{x_{k-1/2}}^{x_k} (x^{-\alpha} - x_h^{-\alpha})(x - x_k) dx - x_k \int_{x_{k-1/2}}^{x_{k+1/2}} (x^{-\alpha} - x_h^{-\alpha}) x dx \\ &= N_1 + N_2. \end{aligned} \tag{4.7}$$

Clearly, we have

$$|N_1| \le C\alpha \left(x_k - \frac{h}{2}\right)^{-\alpha-1} h^2$$

and

$$|N_2| \le C\alpha x_k \left(x_k - \frac{h}{2}\right)^{-1-\alpha} h^2.$$

Hence

$$|J_3 + J_4| \le C_3 h^2.$$

For J_2, following a calculation similar to that for J_5 we have

$$J_2 = \frac{1-x_k}{1-\alpha}\left(\sum_{j=1}^{k-1} \frac{x_{j-1/2}^{1-\alpha} + x_{j+1/2}^{1-\alpha}}{2} h - \int_{x_{1/2}}^{x_{k-1/2}} x^{1-\alpha} dx\right).$$

Letting $g(x) = x^{1-\alpha}$, and applying the error formula for the trapezoidal quadrature rule, we have

$$\begin{aligned}
|J_2| &= \frac{1-x_k}{1-\alpha}\frac{h^3}{12}\sum_{j=1}^{k-1} |g''(\xi_j)| \\
&= \frac{1-x_k}{1-\alpha}\frac{h^2}{12}\sum_{j=1}^{k-1}\left(|g''(\xi)|\, h - \int_{x_{1/2}}^{x_{k-1/2}} |g''(x)|\, dx + \int_{x_{1/2}}^{x_{k-1/2}} |g''(x)|\, dx\right) \\
&\le \frac{1-x_k}{1-\alpha}\frac{h^2}{12}\left[\left(\sum_{j=1}^{k-1}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{x}^{\xi_j} |g'''(y)|\, dy\right) + \int_{x_{1/2}}^{x_{k-1/2}} |g''(x)|\, dx\right] \\
&\le \frac{1-x_k}{1-\alpha}\frac{h^2}{12}\left[\left(\sum_{j=1}^{k-1} h\int_{x_{j-1/2}}^{x_{j+1/2}} |g'''(x)|\, dx\right) + \int_{x_{1/2}}^{x_{k-1/2}} |g''(x)|\, dx\right] \\
&= \frac{(1-x_k)}{3}(2\alpha+1)\left(\frac{h}{2}\right)^{2-\alpha} - \frac{(1-x_k)h^2}{12}\left(\alpha h x_{k-1/2}^{-1-\alpha} + x_{k-1/2}^{-\alpha}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
J_1 + J_2 &\ge J_1 - |J_2| \\
&\ge \frac{1-x_k}{2-\alpha}\left(\frac{h}{2}\right)^{2-\alpha} - \frac{(1-x_k)}{3}(2\alpha+1)\left(\frac{h}{2}\right)^{2-\alpha} \\
&\qquad + \frac{(1-x_k)h^2}{12}\left(\alpha h x_{k-1/2}^{-1-\alpha} + x_{k-1/2}^{-\alpha}\right) \\
&= (1-x_k)\left(\frac{h}{2}\right)^{2-\alpha}\left(\frac{1}{2-\alpha} - \frac{2\alpha+1}{3}\right) + \frac{(1-x_k)h^2}{12}\left(\alpha h x_{k-1/2}^{-1-\alpha} + x_{k-1/2}^{-\alpha}\right) \\
&\ge C_1\left(\frac{h}{2}\right)^{2-\alpha} - C_2 h^2
\end{aligned}$$

for $0 \le \alpha < 1/2$ and $x_k \in [1/3, 2/3]$.

For J_7 we have

$$\begin{aligned}J_7 &= \sum_{j=1}^{N-1} \int_{x_{j-1/2}}^{x_{j+1/2}} f_{h,j}\Big(G(x,x_k) - G(x,x_k)\Big)dx \\ &= f_{h,k}\left(\int_{x_{k-1/2}}^{x_k} (1-x_k)(x-x_k)dx + \int_{x_k}^{x_{k+1/2}} (x_k)(x_k - x)dx\right) \\ &= f_{h,k}\frac{h^2}{8},\end{aligned}$$

where $f_{h,j} = f_h, x \in (x_{j-1/2}, x_{j+1/2})$ for $j = 1, 2, \dots, N-1$. It is obvious that

$$f_{h,k} \le \left(x_k - \frac{h}{2}\right)^{-\alpha}.$$

Hence

$$|J_7| \le C_7 h^2.$$

Finally, it follows from the above estimates for the J_i's that there is a positive constant $C_0 > 0$, independent of h, such that for all $x_k \in [1/3, 2/3]$,

$$\begin{aligned}e(x_k) &\ge J_1 + J_2 - |J_3 + J_4| - |J_5| - |J_6| - |J_7| \\ &\ge C_8 h^{2-\alpha} - C_9 h^2,\end{aligned}$$

which in turn implies that

$$||e_h||_0 \ge C_0 h^{2-\alpha}$$

for all small $h > 0$ due to the equivalence of the discrete and continuous norms on S_h given in Lemma 2.1. This clearly indicates that the convergence rate of the FVE solution for this example cannot be $O(h^2)$ if $0 \le \alpha < 1/2$.

On the other hand, our discussion in subsection 3.2 shows that the FVE solution can have the optimal convergence rate when the exact solution u is in H^2 and the source term f is in H^1. This is supported by the following example. We consider the following boundary value problem:

$$\begin{aligned}&-(a(x)u')' = f, \quad x \in (0,1), \\ &u(0) = b_0, u(1) = b_1,\end{aligned}$$

where

$$a(x) = \begin{cases} 1 + \sqrt{2}\ \mathrm{arctan}\left(\frac{x}{\sqrt{2}}\right), & x \in (0, \frac{1}{2}), \\ a_0 - 2\sqrt{\frac{2}{5}}\ \mathrm{arctanh}\left(\sqrt{\frac{2}{5}}(x-1)\right), & x \in (\frac{1}{2}, 1), \end{cases}$$

$$a_0 = 1 + \sqrt{2}\ \mathrm{arctan}\left(\frac{1}{2\sqrt{2}}\right) - 2\sqrt{\frac{2}{5}}\ \mathrm{arctanh}\left(\frac{1}{\sqrt{10}}\right),$$

$$f(x) = \begin{cases} -x, & x \in (0, \frac{1}{2}), \\ x - 1, & x \in (\frac{1}{2}, 1). \end{cases}$$

Table 1
L^2 errors of the FVE solutions for various partition sizes h.

h	$e(h)$
1/10	0.62438965467175e-003
1/20	0.15638308463200e-003
1/40	0.03911376554587e-003
1/80	0.00977956824638e-003
1/160	0.00244496252767e-003
1/320	0.00061124506134e-003

The boundary conditions are chosen so that

$$u(x) = \begin{cases} \int_0^x \dfrac{1+\frac{t^2}{2}}{a(t)}\,dt, & x \in [0, \frac{1}{2}], \\ u_0 + \int_{1/2}^x \dfrac{\frac{3}{4}+t-\frac{t^2}{2}}{a(t)}\,dt \end{cases}$$

is the exact solution to this boundary value problem. Note that u is piecewise smooth, u' is continuous, but u'' is discontinuous at $x = 1/2$. Hence, in this example, the right-hand side function f is $H^1(0,1)$, but the exact solution to the boundary value problem is only in $H^2(0,1)$. The L^2 errors of the FVE solutions with linear finite elements corresponding to various mesh sizes h are listed in Table 1. The involved calculations were carried out such that $x = 1/2$ is one of the mesh points in the partitions used. Linear regression indicates that the data in this table satisfy

$$e(h) = \sqrt{\int_0^1 (u(x) - u_h(x))^2 dx} \approx 0.06241h^{1.99942},$$

which suggests the optimal convergence rate, and the data are in agreement with the error estimate obtain in subsection 3.2.

4.2. A two-dimensional example. We consider the following boundary value problem:

$$-\Delta u(x) = -\frac{24}{25}x_1^{-\frac{2}{5}}, \quad x = (x_1, x_2)^t \in \Omega,$$
$$u(x) = x_1^{\frac{8}{5}}, \quad (x_1, x_2)^t \in \partial\Omega,$$

where Ω is the unit square $(0,1) \times (0,1)$. It is easy to see that the exact solution to this boundary value problem is

$$u(x) = x_1^{\frac{8}{5}},$$

which is in $H^2(\Omega)$ but not in $H^3(\Omega)$. On the other hand, the source term $f(x) = -\frac{24}{25}x^{-\frac{2}{5}}$ is just in $L^2(\Omega)$.

We have applied the FVE method (2.1) to generate the FVE solution $u_h(x,y)$ to this boundary value problem by the usual uniform partition T_h of the unit square with the partition size h. Due to the lack of regularity in the source term, an exact

TABLE 2
Errors of the FVE solutions for various partition sizes h.

h	$e(h)$
1/10	0.0020009047803123
1/20	0.0005653708096634
1/40	0.0001617344656601
1/80	0.0000470141958737
1/160	0.0000139164337159
1/320	0.0000041963842193

integration formula is used to carry out all the quadratures in (2.1) that involve the source term $f(x, y)$. In fact, we can show that for each triangle $\Delta A_1 A_2 A_2$ with vertices

$$A_1 = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \; A_2 = \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}, \; A_3 = \begin{pmatrix} y_3 \\ z_3 \end{pmatrix},$$

we have

$$\int_{\Delta A_1 A_2 A_2} f(x)dx$$
$$= -M \left(\frac{y_1^{\frac{8}{5}}}{(y_1 - y_2)(y_1 - y_3)} + \frac{y_2^{\frac{8}{5}}}{(y_2 - y_1)(y_2 - y_3)} + \frac{y_3^{\frac{8}{5}}}{(y_3 - y_1)(y_3 - y_2)} \right)$$

with

$$M = |y_3(z_1 - z_2) + y_1(z_2 - z_3) + y_2(z_3 - z_1)| \, .$$

Note that this formula is valid only if the vertices of the triangle $\Delta A_1 A_2 A_2$ have distinct coordinate values. This is true when $\Delta A_1 A_2 A_2$ is a triangle used in the integration over a control volume.

Table 2 contains the errors of the FVE solutions for this boundary problem with various typical partition sizes h. In this table,

$$e(h) = \sqrt{\int_\Omega |u_h(x) - u(x)|^2 \, dx}$$

is the usual L^2 error of an FVE solution $u_h(x, y)$. Obviously, the FVE solutions in these computations do not seem to have the standard second-order convergence because the error is not reduced by a factor of 4 when the partition size is reduced by a factor of 2. Also see the counterexample in [18].

5. Conclusion. In this paper, we have considered the accuracy of FVE methods for solving second-order elliptic boundary value problems. The approach presented herein combines traditional finite element and finite difference methods as a variation of the Galerkin finite element method, revealing regularities in the exact solution and establishing that the source term can affect the accuracy of FVE methods. Optimal-order H^1 and L^2 error estimates and superconvergence also have been discussed. The examples presented above show that the FVE method cannot have the standard $O(h^2)$ convergence rate in the L^2 norm when the source term has the minimum regularity in L^2, even if the exact solution is in H^2.

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