FULLY-DISCRETE FINITE ELEMENT ANALYSIS OF MULTIPHASE FLOW IN GROUNDWATER HYDROLOGY

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Abstract. This paper deals with development and analysis of a fully discrete finite element method for a nonlinear differential system for describing an air-water system in groundwater hydrology. The nonlinear system is written in a fractional flow formulation, i.e., in terms of a saturation and a global pressure. The saturation equation is approximated by a finite element method, while the pressure equation is treated by a mixed finite element method. The analysis is carried out first for the case where the capillary diffusion coefficient is assumed to be uniformly positive, and is then extended to a degenerate case where the diffusion coefficient can be zero. It is shown that error estimates of optimal order in the L^2-norm and almost optimal order in the L^∞-norm can be obtained in the nondegenerate case. In the degenerate case we consider a regularization of the saturation equation by perturbing the diffusion coefficient. The norm of error estimates depends on the severity of the degeneracy in diffusivity, with almost optimal order convergence for non-severe degeneracy. Implementation of the fractional flow formulation with various nonhomogeneous boundary conditions is also discussed. Results of numerical experiments using the present approach for modeling groundwater flow in porous media are reported.

Key words. time discretization, mixed method, finite element, compressible flow, porous media, error estimate, air-water system, numerical experiments

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1. Introduction. In this paper we develop and analyze a fully-discrete finite element procedure for solving the flow equations for an air-water system in groundwater hydrology, $\alpha = a, w$ [3], [12], [31]:

$$\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla\cdot(\rho_\alpha u_\alpha) = f_\alpha, \qquad x\in\Omega,\, t>0, \tag{1.1}$$

$$u_\alpha = -\frac{kk_{r\alpha}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g), \qquad x\in\Omega,\, t>0, \tag{1.2}$$

where $\Omega \subset \Re^d$, $d \leq 3$ is a porous medium, ϕ and k are the porosity and absolute permeability of the porous system, ρ_α, s_α, p_α, u_α, and μ_α are the density, saturation, pressure, volumetric velocity, and viscosity of the α-phase, f_α is the source/sink term, $k_{r\alpha}$ is the relative permeability of the α-phase, and g is the gravitational, downward-pointing, constant vector.

Flow simulation in groundwater reservoirs has been extensively studied in past years (see, e.g., [26], [28] and the bibliographies therein). However, in most previous works the air-phase equation is eliminated by the assumption that the air-phase remains essentially at atmospheric pressure. This assumption, as mentioned in [13], is reasonable in most cases because the mobility of air is much larger than that of water, due to the viscosity difference between the two fluids. When the air-phase pressure is assumed constant, the air-phase mass balance equation can be eliminated and thus only the water-phase equation remains. Namely, the Richards equation is used to model the movement of water in groundwater reservoirs. However, it provides

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no information on the motion of air. If contaminant transport is the main concern and the contaminant can be transported in the air-phase, the air-phase needs to be included to determine the advective component of air-phase contaminant transport [7]. Furthermore, the dynamic interaction between the air and water phases is also important in vapor extraction systems. Hence in these cases the coupled system of nonlinear equations for the air-water system must be solved. It is the purpose of this paper that is to develop and analyze a finite element procedure for approximating the solution of the coupled system of nonlinear equations for the air-water system in groundwater hydrology.

In petroleum reservoir simulation the governing equations that describe fluid flow are usually written in a fractional flow formulation, i.e., in terms of a saturation and a global pressure [1], [8]. The main reason for this fractional flow approach is that efficient numerical methods can be devised to take advantage of many physical properties inherent in the flow equations. However, this pressure-saturation formulation has not yet achieved application in groundwater hydrology. In petroleum reservoirs total flux type boundary conditions are conveniently imposed and often used, but in groundwater reservoirs boundary conditions are very complicated. The most commonly encountered boundary conditions for a groundwater reservoir are of first-type (Dirichlet), second-type (Neumann), third-type (mixed), and "well" type [8]. The problem of incorporating these nonhomogeneous boundary conditions into the fractional flow formulation has been a challenge [12]. In particular, in using the fractional flow approach a difficulty arises when the Dirichlet boundary condition is imposed for one phase (e.g. air) and the Neumann type is used for another phase (e.g. water).

This paper follows the fractional flow formulation. Based on this approach, we develop a fully-discrete finite element procedure for the saturation and pressure equations. The saturation equation is approximated by a Galerkin finite element method, while the pressure equation is treated by a mixed finite element method. It is well known that the physical transport dominates the diffusive effects in incompressible flow in petroleum reservoirs. In the air-water system studied here, the transport again dominates the entire process. Hence it is important to obtain good approximate velocities. This motivates the use of the parabolic mixed method, as in [17], in the computation of the pressure and the velocity. Also, due to its convection-dominated feature, more efficient approximate procedures should be used to solve the saturation equation. However, since this is the first time to carry out an analysis for the present problem, it is of some importance to establish that the standard finite element method for this model converges at an asymptotically optimal rate for smooth problems. Characteristic Petrov-Galerkin methods based on operator splitting [20], transport diffusion methods [32], and other characteristic based methods will be considered in forthcoming papers.

The main part of this paper deals with an asymptotical analysis for the fully discrete finite element method for the first-type and second-type boundary conditions

$$p_\alpha = p_{\alpha D}(x,t), \qquad x \in \Gamma_1,\ t > 0, \tag{1.3}$$

$$u_\alpha \cdot \nu = d_\alpha(x,t), \qquad x \in \Gamma_2,\ t > 0, \tag{1.4}$$

where $p_{\alpha D}$ and d_α are given functions, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with Γ_1 and Γ_2 being disjoint, and ν is the outer unit normal to $\partial\Omega$. We point out that petroleum reservoir simulation is different from groundwater reservoir simulation. The flow of two incompressible fluids (e.g. water and oil) is usually considered in the former case, while the latter system consists of the air and water phases. Consequently, the finite element analyses for these two cases differ. As shown here, compressibility and combination of the boundary conditions (1.3) and (1.4) complicate error analyses. Indeed, if optimality is to be preserved for the finite element method, the standard error argument just fails unless we work with higher order time-differentiated forms of error equations, which require properly scaling initial conditions. Also, we mention that a slightly compressible miscible displacement problem was treated in [14], [18], [23], [33]; however, only the single phase was handled, gravitational terms were omitted, and total flux type boundary conditions were assumed. Furthermore, the so-called "quadratic" terms in velocity were neglected. The dropping of these quadratic terms may not be valid near wells, and so the miscible displacement model was oversimplified both physically and mathematically. The analysis of this paper includes these terms. Finally, only the Raviart-Thomas mixed finite element spaces [34] have been considered in these earlier papers. We are here able to discuss all existing mixed spaces.

The error analysis is given first for the case where the capillary diffusion coefficient is assumed to be uniformly positive. In this case, we show error estimates of optimal order in the L^2-norm and almost optimal order in the L^∞-norm. Then we treat a degenerate case where the diffusion coefficient vanishes for two values of saturation. In the degenerate case we consider a regularization of the saturation equation by perturbing the diffusion coefficient to obtain a nondegenerate problem with smooth solutions. It is shown that the regularized solutions converge to the original solution as the perturbation parameter goes to zero with specific convergence rates given. The norm of error estimates depends on the severity of the degeneracy in diffusivity, with almost optimal order convergence for the degeneracy under consideration.

The rest of this paper is concerned with implementation of the fractional flow formulation with various nonhomogeneous boundary conditions. We show that all the commonly encountered boundary conditions can be incorporated in the fractional flow formulation. Normally the "global" boundary conditions are highly nonlinear functions of the physical boundary conditions for the original two flow phases. This means that we have to iterate on these global boundary conditions as part of the solution process. We here develop a general solution approach to handle these boundary conditions. Results of numerical experiments using the present approach for modeling groundwater flow are reported here.

The paper is organized as follows. In §2, we define a fractional flow formulation for equations (1.1)–(1.4). Then, in §3 we introduce weak forms of the pressure-saturation equations, and in §4 a fully-discrete finite element procedure for solving these equations. An asymptotical analysis is given in §5 and §6 for the nondegenerate case and the degenerate case, respectively. Finally, in §7 we discuss implementation of various nonhomogeneous boundary conditions and present the results of numerical experiments.

2. A pressure-saturation formulation. In addition to (1.1)–(1.4), we impose the customary property that the fluid fills the volume:

$$s_a + s_w = 1, \tag{2.1}$$

and define the capillary pressure function p_c by

$$p_c(s_w) = p_a - p_w. \tag{2.2}$$

Introduce the phase mobilities

$$\lambda_\alpha = k_{r\alpha}/\mu_\alpha, \quad \alpha = a, w,$$

and the total mobility

$$\lambda = \lambda_a + \lambda_w.$$

To devise our numerical method, it is important to choose a reasonable set of dependent variables. Since $p_w = -\infty$ if s_w is equal to the water residual saturation [3], p_w cannot generally be expected to lie in any Sobolev space. Air being a continuous phase implies that p_a is well behaved. Hence, as mentioned in the introduction, we define the global pressure [1] with $s = s_w$:

$$\begin{aligned} p =& p_a - \int_{s_c}^{s} \frac{\lambda_w}{\lambda} \frac{dp_c}{d\xi} d\xi \\ =& p_a - \int_0^{p_c(s)} \big(\frac{\lambda_w}{\lambda}\big) \left(p_c^{-1}(\xi)\right) d\xi, \end{aligned} \tag{2.3}$$

where $p_c(s_c) = 0$. The integral in the right-hand side of (2.3) is well defined [1], [8]. As usual, assume that ρ_α depends on p [8]. Then we define the total velocity

$$u = -k\lambda \left(\nabla p - G(s,p)\right), \tag{2.4}$$

where

$$G(s,p) = \frac{\lambda_a \rho_a + \lambda_w \rho_w}{\lambda} g.$$

Now it can be easily seen that

$$u_w = q_w u + k\lambda_a q_w \nabla p_c - k\lambda_a q_w \tilde{\rho}, \tag{2.5a}$$

$$u_a = q_a u - k\lambda_w q_a \nabla p_c + k\lambda_w q_a \tilde{\rho}, \tag{2.5b}$$

where $q_\alpha = \lambda_\alpha/\lambda$, $\alpha = a, w$, and $\tilde{\rho} = (\rho_a - \rho_w)g$. Consequently,

$$u = u_a + u_w. \tag{2.6}$$

Equations (1.1) and (1.2) can be manipulated using (2.1)–(2.6) to have the pressure equation

$$\nabla \cdot u = -\frac{\partial \phi}{\partial t} - \sum_{\alpha=w}^{a} \frac{1}{\rho_\alpha} \left(\phi s_\alpha \frac{\partial \rho_\alpha}{\partial t} + u_\alpha \cdot \nabla \rho_\alpha - f_\alpha \right), \tag{2.7}$$

and the saturation equation

$$\begin{aligned} \phi \frac{\partial s_w}{\partial t} &+ \nabla \cdot \left(q_w u + k\lambda_a q_w (\nabla p_c - \tilde{\rho})\right) \\ &= -s_w \frac{\partial \phi}{\partial t} - \frac{1}{\rho_w} \left(\phi s_w \frac{\partial \rho_w}{\partial t} + u_w \cdot \nabla \rho_w - f_w \right). \end{aligned} \tag{2.8}$$

Terms of the form $u_\alpha \cdot \nabla \rho_\alpha$, $\alpha = a, w$ have been neglected in compressible miscible displacement problems [14], [18], [23], [33]. The dropping of these terms may not be valid near wells. Also, if they are neglected, the model may not be qualitatively equivalent to the usual formulation of two phase flow. Hence we keep them in this paper. However, the water phase is usually assumed to be incompressible. With the incompressibility of the water phase and the following notation:

$$c(s,p) = \frac{\phi(1-s)}{\rho_a}\frac{d\rho_a}{dp}, \qquad D(s) = -k\lambda_a q_w \frac{dp_c}{ds},$$
$$a(s) = k\lambda, \qquad A(s,p) = \frac{q_a a^{-1}(s)}{\rho_a}\frac{d\rho_a}{dp},$$
$$\tilde{f}_w = \frac{f_w}{\rho_w}, \qquad b(s,p) = -k\lambda_a q_w \tilde{\rho},$$
$$B(s,p) = -\frac{1}{\rho_a}\frac{d\rho_a}{dp}\left(q_a G(s,p) + a^{-1}(s)k\lambda_w q_a(\nabla p_c - \tilde{\rho})\right),$$
$$f(s,p) = \frac{1}{\rho_a}\frac{d\rho_a}{dp}k\lambda_w q_a(\nabla p_c - \tilde{\rho})\cdot G(s,p) + \frac{f_a}{\rho_a} + \frac{f_w}{\rho_w} - \frac{\partial \phi}{\partial t},$$

equations (2.7) and (2.8) can be now written as

$$c(s,p)\frac{\partial p}{\partial t} + \nabla\cdot u = A(s,p)u^2 + B(s,p)\cdot u + f(s,p), \tag{2.9}$$
$$u = -a(s)\left(\nabla p - G(s,p)\right), \tag{2.10}$$
$$\phi\frac{\partial s}{\partial t} - \nabla\cdot\left(D(s)\nabla s - q_w u - b(s,p)\right) = \tilde{f}_w - s\frac{\partial \phi}{\partial t}. \tag{2.11}$$

The boundary conditions for the pressure-saturation equations become

$$p = p_D(x,t), \qquad x\in\Gamma_1,\ t>0, \tag{2.12}$$
$$u\cdot\nu = \tilde{d}(x,t), \qquad x\in\Gamma_2,\ t>0, \tag{2.13}$$
$$s = s_D(x,t), \qquad x\in\Gamma_1,\ t>0, \tag{2.14}$$
$$\left(D(s)\nabla s - q_w u - b(s,p)\right)\cdot\nu = -d_w(x,t), \qquad x\in\Gamma_2,\ t>0, \tag{2.15}$$

where s_D and p_D are the transforms of p_{wD} and p_{aD} by (2.2) and (2.3), and $\tilde{d} = d_a + d_w$.

The model is completed by specifying the initial conditions

$$p(x,0) = p^0(x), \qquad x\in\Omega, \tag{2.16}$$
$$s(x,0) = s^0(x), \qquad x\in\Omega. \tag{2.17}$$

The later analysis for the nondegenerate case in §5 is given under a number of assumptions. First, the solution is assumed smooth; i.e., the external source terms are smoothly distributed, the coefficients are smooth, the boundary and initial data satisfy the compatibility condition, and the domain has at least the regularity required for a standard elliptic problem to have $H^2(\Omega)$-regularity and more if error estimates of order bigger than one are required. Second, the coefficients $a(s)$, ϕ, and $c(s,p)$ are assumed bounded below positively:

$$0 < a_* \le a(s) \le a^* < \infty, \tag{2.18}$$
$$0 < \phi_* \le \phi(x) \le \phi^* < \infty, \tag{2.19}$$
$$0 < c_* \le c(s,p) \le c^* < \infty. \tag{2.20}$$

Finally, the capillary diffusion coefficient $D(s)$ is assumed to satisfy

$$0 < D_* \le D(s) \le D^* < \infty. \tag{2.21}$$

While the phase mobilities can be zero, the total mobility is always positive [31]. The assumptions (2.18) and (2.19) are physically reasonable. Also, the present analysis obviously applies to the incompressible case where $c(s,p)=0$. In this case, the analysis is simpler since we have an elliptic pressure equation instead of the parabolic equation (2.9). Thus we assume condition (2.20) for the compressible case under consideration. Next, although the reasonableness of the assumption (2.21) is discussed in [16], the diffusion coefficient $D(s)$ can be zero in reality. It is for this reason that section six is devoted to consideration of the case where the solution is not required smooth and the assumption (2.21) is removed. As a final remark, we mention that for the case where point sources and sinks occur in a porous medium, an argument was given in [22] for the incompressible miscible displacement problem and can be extended to the present case.

3. Weak forms. To handle the difficulty associated with the inhomogeneous Neumann boundary condition (2.13) in the analysis of the mixed finite element method, let $\overline{d}$ be such that $\overline{d}\cdot\nu=\tilde{d}$ and introduce the change of variable $u=\tilde{u}+\overline{d}$ in equations (2.9)–(2.11). Then the homogeneous Neumann boundary condition holds for $\tilde{u}$. Thus, without loss of generality, we assume that $\tilde{d}\equiv 0$. To be compatible, we also require that this homogeneous condition holds when $t=0$.

In the two-dimensional case, let

$$H(\mathrm{div},\Omega)=\{v\in(L^2(\Omega))^2:\nabla\cdot v\in L^2(\Omega)\},$$

while it is accordingly defined in the three-dimensional case as follows:

$$H(\mathrm{div},\Omega)=\{v\in(L^2(\Omega))^3:\nabla\cdot v\in L^2(\Omega)\}.$$

Also, set

$$\begin{aligned}V&=\{v\in H(\mathrm{div},\Omega):v\cdot\nu=0\text{ on }\Gamma_2\},\\ M&=\{w\in H^1(\Omega):w=0\text{ on }\Gamma_1\}.\end{aligned}$$

The weak form of (2.9)–(2.11) on which the finite element procedure is based is given below. Let $J=(0,T]$ $(T>0)$ is the time interval of interest. The mixed formulation for the pressure is defined by seeking a pair of maps $\{u,p\}:J\to V\times L^2(\Omega)$ such that

$$(\alpha(s)u,v)-(\nabla\cdot v,p)=(G(s,p),v)-\langle p_D,v\cdot\nu\rangle_{\Gamma_1},\quad\forall v\in V,\tag{3.1a}$$

$$\begin{aligned}&(c(s,p)\frac{\partial p}{\partial t},\psi)+(\nabla\cdot u,\psi)\\ &\qquad=(A(s,p)u^2+B(s,p)\cdot u+f(s,p),\psi),\quad\forall\psi\in L^2(\Omega),\end{aligned}\tag{3.1b}$$

where $\alpha(s)=a(s)^{-1}$, the inner products $(\cdot,\cdot)$ are to be interpreted to be in $L^2(\Omega)$ or $(L^2(\Omega))^d$, as appropriate, and $\langle\cdot,\cdot\rangle_{\Gamma_1}$ denotes the duality between $H^{1/2}(\Gamma_1)$ and $H^{-1/2}(\Gamma_1)$. The weak form for the saturation $s:J\to M+s_D$ is given by

$$\begin{aligned}&(\phi\frac{\partial s}{\partial t},v)+\big(D(s)\nabla s-q_w(s)u-b(s,p),\nabla v\big)\\ &\qquad=\big(\tilde{f}_w-s\frac{\partial\phi}{\partial t},v\big)-\langle d_w,v\rangle_{\Gamma_2},\quad\forall v\in M,\end{aligned}\tag{3.2}$$

where the boundary condition (2.15) is used. Finally, to treat the nonzero initial conditions imposed on s and p in (2.16) and (2.17), we introduce the following transformations in (3.1) and (3.2):

$$\begin{aligned} s(x,t) =& \underline{s}(x,t) + s^0(x), \\ p(x,t) =& \underline{p}(x,t) + p^0(x), \\ u(x,t) =& \underline{u}(x,t) + u^0(x), \end{aligned}$$

where $u^0 = -a(s^0)(\nabla p^0 - G(s^0, p^0))$ and $\underline{u} = -a(\underline{s} + s^0)(\nabla(\underline{p} + p^0) - G(\underline{s} + s^0, \underline{p} + p^0)) - u^0(x)$. Then we have zero initial conditions for $\underline{s}$, $\underline{p}$, and $\underline{u}$. Hence, without loss of generality again, we assume that

$$s^0 = p^0 = u^0 \equiv 0. \tag{3.3}$$

The reason for introducing these transformations to have zero initial conditions is to validate equation (5.15) later.

4. Fully-discrete finite element procedures. Let Ω be a polygonal domain. For $0 < h_p < 1$ and $0 < h < 1$, let T_{h_p} and T_h be quasi-uniform partitions into elements, say, simplexes, rectangular parallelepipeds, and/or prisms. In both partitions, we also need that adjacent elements completely share their common edge or face. Let $M_h \subset W^{1,\infty}(\Omega) \cap M$ be a standard C^0-finite element space associated with T_h such that

$$\inf_{\psi \in M_h} \|v - \psi\|_{1,q} \le C\Big(\sum_K h_K^{2k} \|v\|_{k+1,q,K}^2\Big)^{1/2}, \qquad k \ge 1,\ 1 \le q \le \infty, \tag{4.1}$$

where h_K =diam(K), $K \in T_h$ and $\|v\|_{k,q,K}$ is the norm in the Sobolev space $W^{k,q}(K)$ (we omit K when $K = \Omega$ and $\|v\|_{k,K} = \|v\|_{k,2,K}$). Also, let $V_h \times W_h = V_{h_p} \times W_{h_p} \subset V \times L^2(\Omega)$ be the Raviart-Thomas-Nedelec [34], [29], the Brezzi-Douglas-Fortin-Marini [5], the Brezzi-Douglas-Marini [6] (if $d = 2$), the Brezzi-Douglas-Durán-Fortin [4] (if $d = 3$), or the Chen-Douglas [11] mixed finite element space associated with the partition T_{h_p} of index such that the approximation properties below are satisfied:

$$\inf_{\psi \in V_h} \|v - \psi\| \le C\Big(\sum_K h_{p,K}^{2r} \|v\|_{r,K}^2\Big)^{1/2}, \qquad 0 \le r \le k^* + 1, \tag{4.2}$$

$$\inf_{\psi \in V_h} \|\nabla \cdot (v - \psi)\| \le C\Big(\sum_K h_{p,K}^{2r} \|\nabla \cdot v\|_{r,K}^2\Big)^{1/2}, \quad 0 \le r \le k^{**}, \tag{4.3}$$

$$\inf_{\psi \in W_h} \|w - \psi\| \le C\Big(\sum_K h_{p,K}^{2r} \|w\|_{r,K}^2\Big)^{1/2}, \qquad 0 \le r \le k^{**}, \tag{4.4}$$

where $h_{p,K}$ =diam(K), $K \in T_{h_p}$, $\|v\| = \|v\|_0$, $k^{**} = k^* + 1$ for the first two spaces, $k^{**} = k^*$ for the second two spaces, and both cases are included in the last space. Finally, let $\{t^n\}_{n=0}^{n_T}$ be a quasi-uniform partition of J with $t^0 = 0$ and $t^{n_T} = T$, and set $\Delta t^n = t^n - t^{n-1}$, $\Delta t = \max\{\Delta t^n, 1 \le n \le n_T\}$, and

$$\psi^n = \psi(t^n), \quad \partial \psi^n = (\psi^n - \psi^{n-1})/\Delta t^n.$$

We are now in a position to introduce our finite element procedure.

The fully-discrete finite element method is given as follows. The approximation procedure for the pressure is defined by the mixed method for a pair of maps $\{u_h^n, p_h^n\} \in$

$V_h \times W_h$, $n = 1, 2, \cdots, n_T$ such that

$$(4.5a)\quad \big(\alpha(s_h^{n-1})u_h^n, v\big) - (\nabla \cdot v, p_h^n) = \big(G(s_h^{n-1}, p_h^{n-1}), v\big) - \langle p_D^n, v\cdot\nu\rangle_{\Gamma_1}, \quad \forall v \in V_h,$$

$$(4.5b)\quad \big(c(s_h^{n-1}, p_h^{n-1})\partial p_h^n, \psi\big) + (\nabla\cdot u_h^n, \psi) = \big(A(s_h^{n-1}, p_h^{n-1})(u_h^{n-1})^2 + B(s_h^{n-1}, p_h^{n-1})\cdot u_h^{n-1} + f(s_h^{n-1}, p_h^{n-1}), \psi\big), \quad \forall \psi \in W_h,$$

and the finite element method for the saturation is given for $s_h^n \in M_h + s_D^n$, $n = 1, 2, \cdots, n_T$ satisfying

$$(4.6)\quad \begin{aligned} &\big(\phi\partial s_h^n, v\big) + \big(D(s_h^{n-1})\nabla s_h^n - q_w(s_h^{n-1})u_h^n - b(s_h^{n-1}, p_h^n), \nabla v\big) \\ &\quad = \big(\tilde f_w^n - s_h^n \frac{\partial \phi^n}{\partial t}, v\big) - \langle d_w^n, v\rangle_{\Gamma_2}, \qquad \forall v \in M_h. \end{aligned}$$

The initial conditions satisfy

$$(4.7)\quad p_h^0 = 0,\ s_h^0 = 0,\ u_h^0 = 0.$$

After startup, for $n = 1, 2, \cdots, n_T$, equations (4.5) and (4.6) are computed as follows. First, using s_h^{n-1}, p_h^{n-1}, and (4.5), evaluate $\{u_h^n, p_h^n\}$. Since it is linear, (4.5) has a unique solution for each n [10], [27]. Next, using s_h^{n-1}, $\{u_h^n, p_h^n\}$, and (4.6), calculate s_h^n. Again, (4.6) has a unique solution for Δt^n sufficiently small for each n [39].

We end this section with a remark. While the backward Euler scheme is used in (4.5b) and (4.6), the Crank-Nicolson scheme and more accurate time stepping procedures (see, e.g., [21]) can be used. The present analysis applies to these schemes.

5. An error analysis for the fully-discrete scheme. In this section we give a convergence analysis for the finite element procedure (4.5) and (4.6) under assumption (2.21). As usual, it is convenient to use an elliptic projection of the solution of (2.11) into the finite element space M_h. Let $\tilde s = \tilde s_h : J \to M_h + s_D$ be defined by

$$(5.1)\quad \big(D(s)\nabla(s - \tilde s), \nabla v\big) + (s - \tilde s, v) = 0, \quad \forall v \in M_h,\ t \in J.$$

Set

$$(5.2)\quad \zeta = s - \tilde s, \quad \xi = \tilde s - s_h.$$

Then it follows from standard results of the finite element method [15], [30], [37] that

$$(5.3a)\quad \|\zeta\| + h\|\zeta\|_1 \le C\Big(\sum_K h_K^{2(k+1)}\|s\|_{k+1,K}^2\Big)^{1/2},$$

$$(5.3b)\quad \|\zeta\|_{0,\infty} \le Ch^{k+1}\big(\log h^{-1}\big)^{\gamma}\|s\|_{k+1,\infty},$$

where $\gamma = 1$ for $k = 1$ and $\gamma = 0$ for $k > 1$. The same result applies to the time-differentiated forms of (5.1) [40]:

$$(5.4)\quad \Big\|\frac{\partial\zeta}{\partial t}\Big\| + h\Big\|\frac{\partial\zeta}{\partial t}\Big\|_1 \le C\Big(\sum_K h_K^{2(k+1)}\big(\|s\|_{k+1,K}^2 + \Big\|\frac{\partial s}{\partial t}\Big\|_{k+1,K}^2\big)\Big)^{1/2}.$$

As for the analysis of the mixed finite element method, we use the the following two projections instead of the elliptic projections introduced in [14] and [18]. So the present analysis is different from and in fact simpler than those in [14] and [18]. Each of our mixed finite element spaces [4]–[6], [11], [29], [34] has the property that there are projection operators $\Pi_h : H^1(\Omega) \to V_h$ and $P_h = L^2$-projection: $L^2(\Omega) \to W_h$ such that

$$\|v - \Pi_h v\| \le C\Big(\sum_K h_{p,K}^{2r}\|v\|_{r,K}^2\Big)^{1/2}, \qquad 0 \le r \le k^* + 1, \tag{5.5}$$

$$\|\nabla \cdot (v - \Pi_h v)\| \le C\Big(\sum_K h_{p,K}^{2r}\|\nabla \cdot v\|_{r,K}^2\Big)^{1/2}, \quad 0 \le r \le k^{**}, \tag{5.6}$$

$$\|w - P_h w\| \le C\Big(\sum_K h_{p,K}^{2r}\|w\|_{r,K}^2\Big)^{1/2}, \qquad 0 \le r \le k^{**}, \tag{5.7}$$

and (see, e.g., [9], [19])

$$(\nabla \cdot (v - \Pi_h v), w) = 0, \quad \forall w \in W_h, \tag{5.8}$$

$$(\nabla \cdot v, w - P_h w) = 0, \quad \forall v \in V_h. \tag{5.9}$$

Set $\tilde{p} = P_h p$, $\tilde{u} = \Pi_h u$, and

$$\sigma = u - \tilde{u}, \quad \beta = \tilde{u} - u_h, \tag{5.10}$$

$$\eta = p - \tilde{p}, \quad \theta = \tilde{p} - p_h. \tag{5.11}$$

Note that, by (3.3) and (4.7),

$$\theta^0 = 0,\ \xi^0 = 0,\ \beta^0 = 0. \tag{5.12}$$

Finally, we prove some bounds of the projections $\tilde{s}$ and $\tilde{p}$. Let $\overline{s} = \overline{s}_h$ be the interpolant of s in M_h. Then we see, by (4.1), (5.3b), the approximation property of $\overline{s}$, and an inverse inequality in M_h, that

$$\begin{aligned}
\|\tilde{s}\|_{1,\infty} \le & \|s - \tilde{s}\|_{1,\infty} + \|s\|_{1,\infty} \\
\le & \|\overline{s} - \tilde{s}\|_{1,\infty} + \|s - \overline{s}\|_{1,\infty} + \|s\|_{1,\infty} \\
\le & Ch^{-1}\|\overline{s} - \tilde{s}\|_{0,\infty} + \|s - \overline{s}\|_{1,\infty} + \|s\|_{1,\infty} \\
\le & Ch^{-1}\big(\|\tilde{s} - s\|_{0,\infty} + \|s - \overline{s}\|_{0,\infty}\big) + \|s - \overline{s}\|_{1,\infty} + \|s\|_{1,\infty} \\
\le & Ch^k(\log h^{-1})^\gamma \|s\|_{k+1,\infty} + \|s\|_{1,\infty},
\end{aligned}$$

where γ is given as in (5.3b). This implies that $\|\tilde{s}\|_{1,\infty}$ is bounded for sufficiently smooth solutions since $k \ge 1$. The same argument applies to $\|\partial \tilde{s}/\partial t\|_{1,\infty}$. Next, note that, by the approximation property of the projection P_h [27],

$$\|\tilde{p}_t\|_{0,\infty} \le C\|p_t\|_{0,\infty}.$$

These bounds on $\tilde{p}_t$, $\nabla \tilde{s}$, and $\nabla(\partial \tilde{s}/\partial t)$ are used below.

We are now ready to prove some results. Below ε is a generic positive constant as small as we please.

5.1. Analysis of the mixed method. We first analyze the mixed method (4.5). We set $s = p = u \equiv 0$ and $s_h = p_h = u_h \equiv 0$ when $t \le 0$. The following error equation is obtained by subtracting (4.5) from (3.1) at $t = t^n$ and applying (5.8) and (5.9):

$$\begin{aligned}
(\alpha(s_h^{n-1})\beta^n, v) - (\nabla \cdot v, \theta^n) = & \big((\alpha(s_h^{n-1}) - \alpha(s^n))u^n, v\big) - (\alpha(s_h^{n-1})\sigma^n, v) \\
& + \big(G(s^n, p^n) - G(s_h^{n-1}, p_h^{n-1}), v\big), \quad \forall v \in V_h,
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
\big(c(s_h^{n-1}, p_h^{n-1})\partial\theta^n, \psi\big) + (\nabla \cdot \beta^n, \psi) = & \big(f(s^n, p^n) - f(s_h^{n-1}, p_h^{n-1}), \psi\big) \\
& + \big(A(s^n, p^n)(u^n)^2 - A(s_h^{n-1}, p_h^{n-1})(u_h^{n-1})^2, \psi\big) \\
& + \big(B(s^n, p^n) \cdot u^n - B(s_h^{n-1}, p_h^{n-1}) \cdot u_h^{n-1}, \psi\big)
\end{aligned} \tag{5.14}$$

$$+\big((c(s_h^{n-1},p_h^{n-1})-c(s^n,p^n))\frac{\partial p^n}{\partial t},\psi\big)$$
$$-\,(c(s_h^{n-1},p_h^{n-1})\big(\frac{\partial p^n}{\partial t}-\partial\tilde p^n\big),\psi),\quad \forall\psi\in W_h.$$

Below C_i indicates a generic constant with the given dependencies.

LEMMA 5.1. *Let* (u,p) *and* (u_h,p_h) *solve* (3.1) *and* (4.5), *respectively. Then*

$$\begin{aligned}\|\partial\theta^1\|^2+&\Delta t^1\|\partial\beta^1\|^2\\ &\le C_0\{\Delta t^1(\|s^1-s^0\|^2+\|\partial\sigma^1\|^2+\|\partial G^1\|^2)\\ &\quad+\|p^1-p^0\|^2+\|s^1-s^0\|^2+\|\frac{\partial p^1}{\partial t}-\partial\tilde p^1\|^2+\|u^1-u^0\|^2\},\end{aligned}\tag{5.15}$$

where $\partial G^1=(G(s^1,p^1)-G(s^0,p^0))/\Delta t^1$ *and*

$$C_0=C_0\big(\|\frac{\partial p}{\partial t}\|_{L^\infty(J\times\Omega)},\|\frac{\partial u}{\partial t}\|_{L^\infty(J\times\Omega)},\|u\|_{L^\infty(J\times\Omega)}\big).$$

Proof. Set $v=\beta^1$ in (5.13) and $\psi=\theta^1$ in (5.14), add the resulting equations at $n=1$, and use (3.3), (4.7), and (5.12) to see that

$$(c(s^0,p^0)\partial\theta^1,\partial\theta^1)+\Delta t^1(\alpha(s^0)\partial\beta^1,\partial\beta^1)=\sum_{i=1}^{8}T_i^1,$$

where

$$T_1^1=\big((\alpha(s^0)-\alpha(s^1))(u^1-u^0),\partial\beta^1\big),\qquad T_2^1=-\big(\alpha(s^0)(\sigma^1-\sigma^0),\partial\beta^1\big),$$
$$T_3^1=\big(G(s^1,p^1)-G(s^0,p^0),\partial\beta^1\big),\qquad T_4^1=\big(A(s^1,p^1)(u^1)^2-A(s^0,p^0)(u^0)^2,\partial\theta^1\big),$$
$$T_5^1=\big((c(s^0,p^0)-c(s^1,p^1))\frac{\partial p^1}{\partial t},\partial\theta^1\big),\qquad T_6^1=-\big(c(s^0,p^0)(\frac{\partial p^1}{\partial t}-\partial\tilde p^1),\partial\theta^1\big),$$
$$T_7^1=\big(B(s^1,p^1)\cdot u^1-B(s^0,p^0)\cdot u^0,\partial\theta^1\big),\quad T_8^1=\big(f(s^1,p^1)-f(s^0,p^0),\partial\theta^1\big).$$

Then (5.15) can be easily seen. □

LEMMA 5.2. *Let* (u,p) *and* (u_h,p_h) *satisfy* (3.1) *and* (4.5), *respectively. Then*

$$\begin{aligned}\|\partial\theta^\gamma\|^2&+\sum_{n=2}^{\gamma}\|\partial\beta^n\|^2\Delta t^n\\ \le C_1\Big\{&\|\partial\theta^1\|^2+\sum_{n=1}^{\gamma}\{\|\partial(\partial\eta^n)\|^2+\|\partial\eta^n\|^2+\|\eta^{n-1}\|^2+\|\partial\zeta^{n-1}\|^2\\ &+\|\zeta^{n-1}\|^2+\|\partial\sigma^n\|^2+\|\partial\big(\frac{\partial p^n}{\partial t}-\partial p^n\big)\|^2+\|\partial\theta^n\|^2\\ &+\|\partial(p^n-p^{n-1})\|^2+\|p^n-p^{n-1}\|^2+\|\partial(s^n-s^{n-1})\|^2\\ &+\|s^n-s^{n-1}\|^2+\|\theta^{n-1}\|^2+\|\partial\xi^{n-1}\|^2+\|\xi^{n-1}\|^2\\ &+\|\partial(u^n-u^{n-1})\|^2+\|u^n-u^{n-1}\|^2\\ &+(\|\partial\xi^{n-1}\|_{0,\infty}^2+\|\partial\theta^{n-1}\|_{0,\infty}^2)\big[\|\partial\theta^{n-1}\|^2+\|\frac{\partial p^n}{\partial t}-\partial p^n\|^2\\ &\qquad+\|\eta^n\|^2+(1+\|\partial\beta^{n-1}\|_{0,\infty}^2)\|\partial\beta^{n-1}\|^2\big]\\ &+(1+\|\partial\xi^{n-1}\|_{0,\infty}^2+\|\partial\beta^{n-1}\|_{0,\infty}^2)(\|\beta^{n-1}\|^2+\|\beta^{n-2}\|^2+\|\sigma^n\|^2)\}\Delta t^n\Big\},\end{aligned}$$

for $2 \le \gamma \le n_T$, *where*

$$C_1 = C_1(\|\frac{\partial s}{\partial t}\|_{L^\infty(J\times\Omega)}, \|\frac{\partial p}{\partial t}\|_{L^\infty(J\times\Omega)}, \|\frac{\partial^2 p}{\partial t^2}\|_{L^\infty(J\times\Omega)}, \|u\|_{L^\infty(J\times\Omega)}, \|\frac{\partial u}{\partial t}\|_{L^\infty(J\times\Omega)}).$$

Proof. Difference equations (5.13) and (5.14) with respect to n, set $v = \partial\beta^n$ and $\psi = \partial\theta^n$ in the resulting equations, divide by Δt^n, and add to obtain

$$(\alpha(s_h^{n-1})\partial\beta^n, \partial\beta^n) + (c(s_h^{n-1}, p_h^{n-1})\partial(\partial\theta^n), \partial\theta^n) = \sum_{i=1}^{10} T_i^n, \tag{5.16}$$

where

$$\begin{aligned}
T_1^n =& \frac{1}{\Delta t^n}\big((f(s^n, p^n) - f(s_h^{n-1}, p_h^{n-1})) - (f(s^{n-1}, p^{n-1}) - f(s_h^{n-2}, p_h^{n-2})), \partial\theta^n\big),\\
T_2^n =& \frac{1}{\Delta t^n}\big([A(s^n, p^n)(u^n)^2 - A(s_h^{n-1}, p_h^{n-1})(u_h^{n-1})^2]\\
&- [A(s^{n-1}, p^{n-1})(u^{n-1})^2 - A(s_h^{n-2}, p_h^{n-2})(u_h^{n-2})^2], \partial\theta^n\big),\\
T_3^n =& \frac{1}{\Delta t^n}\big([B(s^n, p^n)\cdot u^n - B(s_h^{n-1}, p_h^{n-1})\cdot u_h^{n-1}]\\
&- [B(s^{n-1}, p^{n-1})\cdot u^{n-1} - B(s_h^{n-2}, p_h^{n-2})\cdot u_h^{n-2}], \partial\theta^n\big),\\
T_4^n =& \frac{1}{\Delta t^n}\big((c(s_h^{n-1}, p_h^{n-1}) - c(s^n, p^n))\frac{\partial p^n}{\partial t}\\
&- (c(s_h^{n-2}, p_h^{n-2}) - c(s^{n-1}, p^{n-1}))\frac{\partial p^{n-1}}{\partial t}, \partial\theta^n\big),\\
T_5^n =& -\frac{1}{\Delta t^n}\big(c(s_h^{n-1}, p_h^{n-1})(\frac{\partial p^n}{\partial t} - \partial\tilde{p}^n)\\
&- c(s_h^{n-2}, p_h^{n-2})(\frac{\partial p^{n-1}}{\partial t} - \partial\tilde{p}^{n-1}), \partial\theta^n\big),\\
T_6^n =& \frac{1}{\Delta t^n}\big((c(s_h^{n-2}, p_h^{n-2}) - c(s_h^{n-1}, p_h^{n-1}))\partial\theta^{n-1}, \partial\theta^n\big),\\
T_7^n =& \frac{1}{\Delta t^n}\big((\alpha(s_h^{n-2}) - \alpha(s_h^{n-1}))\beta^{n-1}, \partial\beta^n\big),\\
T_8^n =& -\frac{1}{\Delta t^n}\big(\alpha(s_h^{n-1})\sigma^n - \alpha(s_h^{n-2})\sigma^{n-1}, \partial\beta^n\big),\\
T_9^n =& \frac{1}{\Delta t^n}\big((\alpha(s_h^{n-1}) - \alpha(s^n))u^n - (\alpha(s_h^{n-2}) - \alpha(s^{n-1}))u^{n-1}, \partial\beta^n\big),\\
T_{10}^n =& \frac{1}{\Delta t^n}\big((G(s^n, p^n) - G(s_h^{n-1}, p_h^{n-1}))\\
&- (G(s^{n-1}, p^{n-1}) - G(s_h^{n-2}, p_h^{n-2})), \partial\beta^n\big).
\end{aligned}$$

Observe that the left-hand side of (5.16) is larger than the quantity

$$\begin{aligned}
&\frac{1}{2\Delta t^n}\big\{(c(s_h^{n-1}, p_h^{n-1})\partial\theta^n, \partial\theta^n) - (c(s_h^{n-2}, p_h^{n-2})\partial\theta^{n-1}, \partial\theta^{n-1})\big\}\\
&\quad + (\alpha(s_h^{n-1})\partial\beta^n, \partial\beta^n) + T_{11}^n,
\end{aligned} \tag{5.17}$$

where

$$T_{11}^n = \frac{1}{2\Delta t^n}\big((c(s_h^{n-2}, p_h^{n-2}) - c(s_h^{n-1}, p_h^{n-1}))\partial\theta^{n-1}, \partial\theta^{n-1}\big).$$

We estimate the new term T_2^n in detail. Other terms can be bounded by a simpler argument. To estimate T_2^n, we write

$$\begin{aligned}
T_2^n = & \frac{1}{\Delta t^n}\big(\{[A(s^n,p^n) - A(s^{n-1},p^{n-1})] \\
& \quad - [A(s_h^{n-1},p_h^{n-1}) - A(s_h^{n-2},p_h^{n-2})]\}(u^n)^2, \partial\theta^n\big) \\
& + \frac{1}{\Delta t^n}\big([A(s_h^{n-1},p_h^{n-1}) - A(s_h^{n-2},p_h^{n-2})]((u^n)^2 - (u_h^{n-1})^2), \partial\theta^n\big) \\
& + \frac{1}{\Delta t^n}\big([A(s^{n-1},p^{n-1}) - A(s_h^{n-2},p_h^{n-2})]((u^n)^2 - (u^{n-1})^2), \partial\theta^n\big) \\
& + \frac{1}{\Delta t^n}\big(A(s_h^{n-2},p_h^{n-2})\{[(u^n)^2 - (u^{n-1})^2] - [(u_h^{n-1})^2 - (u_h^{n-2})^2]\}, \partial\theta^n\big) \\
\equiv & \sum_{i=1}^{4} T_{2,i}^n.
\end{aligned}$$

Note that

$$\begin{aligned}
& [A(s_h^{n-1},p_h^{n-1}) - A(s_h^{n-2},p_h^{n-2})] - [A(s^n,p^n) - A(s^{n-1},p^{n-1})] \\
& = \frac{\partial A}{\partial s}(\widehat{s}_h^{n-1},p_h^{n-1})(s_h^{n-1} - s_h^{n-2}) + \frac{\partial A}{\partial p}(s_h^{n-1},\widehat{p}_h^{n-1})(p_h^{n-1} - p_h^{n-2}) \\
& \quad - \frac{\partial A}{\partial s}(\widehat{s}^n,p^n)(s^n - s^{n-1}) - \frac{\partial A}{\partial p}(s^{n-1},\widehat{p}^n)(p^n - p^{n-1}),
\end{aligned}$$

where

$$\begin{aligned}
& \min\{p_h^{n-1},p_h^{n-2}\} \le \widehat{p}_h^{n-1} \le \max\{p_h^{n-1},p_h^{n-2}\}, \\
& \min\{p^n,p^{n-1}\} \le \widehat{p}^n \le \max\{p^n,p^{n-1}\},
\end{aligned}$$

and similar inequalities hold for $\widehat{s}_h^{n-1}$, $\widehat{s}^n$. Consequently, with $\lambda^n = \Delta t^{n-1}/\Delta t^n$ we see that

$$\begin{aligned}
T_{2,1}^n = & -\big(\frac{\partial A}{\partial s}\{\lambda^n[\partial\zeta^{n-1} + \partial\xi^{n-1}] - \partial(s^n - s^{n-1})\}(u^n)^2, \partial\theta^n\big) \\
& - \big([\frac{\partial^2 A}{\partial s^2}(\widehat{s}_h^{n-1} - \widehat{s}^n) + \frac{\partial^2 A}{\partial p\partial s}(p_h^{n-1} - p^n)](u^n)^2\frac{s^n - s^{n-1}}{\Delta t^n}, \partial\theta^n\big) \\
& - \big(\frac{\partial A}{\partial p}\{\lambda^n[\partial\eta^{n-1} + \partial\theta^{n-1}] - \partial(p^n - p^{n-1})\}(u^n)^2, \partial\theta^n\big) \\
& - \big([\frac{\partial^2 A}{\partial p^2}(\widehat{p}_h^{n-1} - \widehat{p}^n) + \frac{\partial^2 A}{\partial s\partial p}(s_h^{n-2} - s^{n-1})](u^n)^2\frac{p^n - p^{n-1}}{\Delta t^n}, \partial\theta^n\big),
\end{aligned}$$

so that

$$\begin{aligned}
|T_{2,1}^n| \le C_1\big(& \|\partial\zeta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\partial(s^n - s^{n-1})\|^2 + \|\widehat{s}_h^{n-1} - \widehat{s}^n\|^2 \\
& + \|p_h^{n-1} - p^n\|^2 + \|\partial\eta^{n-1}\|^2 + \|\partial\theta^{n-1}\|^2 + \|\partial(p^n - p^{n-1})\|^2 \\
& + \|\widehat{p}_h^{n-1} - \widehat{p}^n\|^2 + \|s_h^{n-2} - s^{n-1}\|^2 + \|\partial\theta^n\|^2\big),
\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}
\|\widehat{p}_h^{n-1} - \widehat{p}^n\| \le C_1\big(& \|p^n - p_h^{n-1}\| + \|p^n - p_h^{n-2}\| \\
& + \|p^{n-1} - p_h^{n-1}\| + \|p^{n-1} - p_h^{n-2}\|\big),
\end{aligned} \tag{5.19}$$

and an analogous inequality holds for $\widehat{s}_h^{n-1} - \widehat{s}^n$. Also, we see that

$$[A(s_h^{n-1}, p_h^{n-1}) - A(s_h^{n-2}, p_h^{n-2})]((u^n)^2 - (u_h^{n-1})^2)$$
$$= \big\{ \frac{\partial A}{\partial s}(s_h^{n-1} - s_h^{n-2}) + \frac{\partial A}{\partial p}(p_h^{n-1} - p_h^{n-2}) \big\} (u^n - u_h^{n-1}) \cdot (u^n + u_h^{n-1}),$$

which implies that

$$|T_{2,2}^n| \le C_1 \big((1 + \|\partial \xi^{n-1}\|_{0,\infty}^2 + \|\partial \theta^{n-1}\|_{0,\infty}^2)(1 + \|\beta^{n-1}\|_{0,\infty}^2)\|\beta^{n-1}\|^2$$
$$+ \|s_h^{n-1} - s_h^{n-2}\|^2 + \|p_h^{n-1} - p_h^{n-2}\|^2 + \|\partial \theta^n\|^2 \big).$$

Next, it can be easily seen that

$$|T_{2,3}^n| \le C_1 \big(\|s^{n-1} - s_h^{n-2}\|^2 + \|p^{n-1} - p_h^{n-2}\|^2 + \|\partial \theta^n\|^2 \big).$$

Finally, since

$$[(u_h^{n-1})^2 - (u_h^{n-2})^2] - [(u^n)^2 - (u^{n-1})^2]$$
$$= ([u_h^{n-1} - u_h^{n-2}] - [u^{n-1} - u^{n-2}])(u_h^{n-1} + u^{n-1})$$
$$+ (u^{n-2} - u_h^{n-2})([u_h^{n-2} - u_h^{n-1}] + [u^{n-2} - u^{n-1}])$$
$$+ ([u^{n-1} - u^{n-2}] - [u^n - u^{n-1}])(u^{n-1} + u^{n-2})$$
$$+ (u^n - u^{n-1})(u^n - u^{n-2}),$$

we find that

$$|T_{2,4}^n| \le C_1 \big((1 + \|\partial \beta^{n-1}\|_{0,\infty}^2)(\|\beta^{n-1}\|^2 + \|\beta^{n-2}\|^2)$$
$$+ \|\partial \sigma^{n-1}\|^2 + \|\partial (u^n - u^{n-1})\|^2 + \|u^n - u^{n-1}\|^2$$
$$+ \|\sigma^{n-2}\|^2 + \|\partial \theta^n\|^2 \big) + \varepsilon \|\partial \beta^{n-1}\|^2.$$

Hence T_2^n can be bounded in terms of $T_{2,i}^n$, $i = 1, \cdots, 4$. Other terms are bounded as follows:

$$|T_1^n| \le C_1 \big(\|\partial \zeta^{n-1}\|^2 + \|\partial \xi^{n-1}\|^2 + \|\partial (s^n - s^{n-1})\|^2 + \|\widehat{s}_{1,h}^{n-1} - \widehat{s}_1^n\|^2$$
$$+ \|p_h^{n-1} - p^n\|^2 + \|\partial \eta^{n-1}\|^2 + \|\partial \theta^{n-1}\|^2 + \|\partial (p^n - p^{n-1})\|^2$$
$$+ \|\widehat{p}_{1,h}^{n-1} - \widehat{p}_1^n\|^2 + \|s_h^{n-2} - s^{n-1}\|^2 + \|\partial \theta^n\|^2 \big),$$

$$|T_3^n| \le C_1 \big(\|\partial \zeta^{n-1}\|^2 + \|\partial \xi^{n-1}\|^2 + \|\partial (s^n - s^{n-1})\|^2 + \|\widehat{s}_{2,h}^{n-1} - \widehat{s}_2^n\|^2$$
$$+ \|p_h^{n-1} - p^n\|^2 + \|\partial \eta^{n-1}\|^2 + \|\partial \theta^{n-1}\|^2 + \|\partial (p^n - p^{n-1})\|^2$$
$$+ \|\widehat{p}_{2,h}^{n-1} - \widehat{p}_2^n\|^2 + \|s_h^{n-2} - s^{n-1}\|^2 + \|\partial \theta^n\|^2$$
$$+ (1 + \|\partial \xi^{n-1}\|_{0,\infty}^2 + \|\partial \theta^{n-1}\|_{0,\infty}^2)\|\beta^{n-1}\|^2$$
$$+ \|s^{n-1} - s_h^{n-2}\|^2 + \|p^{n-1} - p_h^{n-2}\|^2 + \|\partial \sigma^{n-1}\|^2$$
$$+ \|\partial (u^n - u^{n-1})\|^2 \big) + \varepsilon \|\partial \beta^{n-1}\|^2,$$

$$|T_4^n| \le C_1 \big(\|s^n - s_h^{n-1}\|^2 + \|p^n - p_h^{n-1}\|^2 + \|\partial \theta^n\|^2 + \|\partial \zeta^{n-1}\|^2$$
$$+ \|\partial \eta^{n-1}\|^2 + \|\partial (p^n - p^{n-1})\|^2 + \|\partial (s^n - s^{n-1})\|^2$$
$$+ \|\partial \theta^{n-1}\|^2 + \|\partial \xi^{n-1}\|^2 + \|\widehat{p}_{3,h}^{n-1} - \widehat{p}_3^n\|^2 + \|\widehat{s}_{3,h}^{n-1} - \widehat{s}_3^n\|^2 \big),$$

$$|T_5^n| \le C_1 \big((\|\partial \eta^n\|^2 + \|\frac{\partial p^n}{\partial t} - \partial p^n\|^2)(\|\partial \xi^{n-1}\|_{0,\infty}^2 + \|\partial \theta^{n-1}\|_{0,\infty}^2)$$

$$
\begin{aligned}
&\quad + \|\partial(\frac{\partial p^n}{\partial t} - \partial p^n)\|^2 + \|\partial(\partial\eta^n)\|^2 + \|\partial\theta^n\|^2\big),\\
|T_6^n| &\le C_1\big((1+\|\partial\xi^{n-1}\|^2_{0,\infty} + \|\partial\theta^{n-1}\|^2_{0,\infty})\|\partial\theta^{n-1}\|^2 + \|\partial\theta^n\|^2\big),\\
|T_7^n| &\le C_1\big(1+\|\partial\xi^{n-1}\|^2_{0,\infty}\big)\|\beta^{n-1}\|^2 + \varepsilon\|\partial\beta^n\|^2,\\
|T_8^n| &\le C_1\big((1+\|\partial\xi^{n-1}\|^2_{0,\infty})\|\sigma^n\|^2 + \|\partial\sigma^n\|^2\big) + \varepsilon\|\partial\beta^n\|^2,\\
|T_9^n| &\le C_1\big(\|\partial\zeta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\partial(s^n - s^{n-1})\|^2\\
&\quad + \|s^n - s_h^{n-1}\|^2 + \|\widehat{s}^{n-1}_{4,h} - \widehat{s}^n_4\|^2\big) + \varepsilon\|\partial\beta^n\|^2,\\
|T_{10}^n| &\le C_1\big(\|\partial\zeta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\partial(s^n - s^{n-1})\|^2 + \|s_h^{n-2} - s^{n-1}\|^2\\
&\quad + \|\partial\eta^{n-1}\|^2 + \|\partial\theta^{n-1}\|^2 + \|\partial(p^n - p^{n-1})\|^2 + \|p_h^{n-1} - p^n\|^2\\
&\quad + \|\widehat{p}^{n-1}_{5,h} - \widehat{p}^n_5\|^2 + \|\widehat{s}^{n-1}_{5,h} - \widehat{s}^n_5\|^2 + \varepsilon\|\partial\beta^n\|^2,\\
|T_{11}^n| &\le C_1\big(1+\|\partial\xi^{n-1}\|^2_{0,\infty} + \|\partial\theta^{n-1}\|^2_{0,\infty}\big)\|\partial\theta^{n-1}\|^2,
\end{aligned}
$$

where $\widehat{s}^{n-1}_{i,h} - \widehat{s}^n_i$ and $\widehat{p}^{n-1}_{i,h} - \widehat{p}^n_i$ $(i = 1, \cdots, 5)$ can be bounded as in (5.19), e.g.,

$$
\begin{aligned}
\|\widehat{s}^{n-1}_{i,h} - \widehat{s}^n_i\| \le C_1\big(&\|s^n - s_h^{n-1}\| + \|s^n - s_h^{n-2}\|\\
&+ \|s^{n-1} - s_h^{n-1}\| + \|s^{n-1} - s_h^{n-2}\|\big).
\end{aligned} \tag{5.20}
$$

Now, apply these inequalities and (5.17)–(5.20), multiply (5.16) by Δt^n, sum n, and properly arrange terms to complete the proof of the lemma. □

The error equations (5.13) and (5.14) are usually exploited to derive error estimates in the parabolic mixed finite element method [18], [27]. To handle the difficulty arising from the combination of the Dirichlet boundary condition (1.3) and the nonlinearity of the differential system (2.9)–(2.11), we must use their time-differentiated forms, as mentioned before. Also, the three terms T_i^n, $i = 1, 2, 3$ take care of the quadratic terms in the velocities, which require more regularity on u than those without these quadratic terms, as seen from Lemma 5.2.

5.2. Analysis of the saturation equation. We now turn to analyzing the finite element method (4.6).

LEMMA 5.3. *Let s and s_h solve* (3.2) *and* (4.6), *respectively. Then*

$$
\begin{aligned}
&\|\nabla\xi^\gamma\|^2 + \sum_{n=1}^{\gamma}\|\partial\xi^n\|^2\Delta t^n\\
&\quad\le C_2\Big\{\|s^\gamma - s^{\gamma-1}\|^2 + \|\xi^{\gamma-1}\|^2 + \|\zeta^{\gamma-1}\|^2 + \|\theta^\gamma\|^2 + \|\eta^\gamma\|^2 + \|\beta^\gamma\|^2 + \|\sigma^\gamma\|^2\\
&\qquad + \sum_{n=0}^{\gamma}\big(\|\frac{\partial s^n}{\partial t} - \partial s^n\|^2 + \|\partial(s^{n+1} - s^n)\|^2 + \|s^{n+1} - s^n\|^2 + \|p^{n+1} - p^n\|^2\\
&\qquad + \|\partial\zeta^n\|^2 + \|\zeta^n\|^2 + \|\partial\sigma^n\|^2 + \|\sigma^n\|^2 + \|\eta^n\|^2 + \|\xi^n\|_1^2 + \|\theta^n\|^2\\
&\qquad + \|\beta^n\|^2 + \|\partial\theta^n\|^2 + \|\partial\eta^n\|^2\big)\Delta t^n + \sum_{n=1}^{\gamma-1}\|\nabla\xi^n\|^2\|\partial\xi^n\|^2_{0,\infty}\Delta t^n\Big\}\\
&\quad + \varepsilon\sum_{n=1}^{\gamma}\|\partial\beta^n\|^2\Delta t^n,
\end{aligned}
$$

for $1 \le \gamma \le n_T$, *where*

$$C_2 = C_2\big(\|\frac{\partial s}{\partial t}\|_{L^\infty(J\times\Omega)}, \|\nabla\frac{\partial s}{\partial t}\|_{L^\infty(J\times\Omega)}, \|\nabla s\|_{L^\infty(J\times\Omega)}, \|u\|_{L^\infty(J\times\Omega)}\big).$$

Proof. Subtract (4.6) from (3.2) at $t = t^n$, use (5.1) at $t = t^n$, and set the test function $v = \partial\xi^n$ to see that

$$(\phi\partial\xi^n, \partial\xi^n) + (D(s_h^{n-1})\nabla\xi^n, \nabla\partial\xi^n) = \sum_{i=1}^{7} B_i^n, \tag{5.21}$$

where

$$\begin{aligned}
B_1^n &= -\big(\phi(\frac{\partial s^n}{\partial t} - \partial\tilde{s}^n), \partial\xi^n\big), & B_2^n &= (\zeta^n, \partial\xi^n),\\
B_3^n &= \big((q_w(s^n) - q_w(s_h^{n-1}))u^n, \nabla\partial\xi^n\big), & B_4^n &= \big((u^n - u_h^n)q_w(s_h^{n-1}), \nabla\partial\xi^n\big),\\
B_5^n &= \big(b(s^n, p^n) - b(s_h^{n-1}, p_h^n), \nabla\partial\xi^n\big), & B_6^n &= -\big(\frac{\partial\phi^n}{\partial t}(s^n - s_h^n), \partial\xi^n\big),\\
B_7^n &= -\big((D(s^n) - D(s_h^{n-1}))\nabla\tilde{s}^n, \nabla\partial\xi^n\big).
\end{aligned}$$

The left-hand side of (5.21) is bigger than the quantity

$$\begin{aligned}
&(\phi\partial\xi^n, \partial\xi^n) + \frac{1}{2\Delta t^n}(D(s_h^{n-1})\nabla\xi^n, \nabla\xi^n)\\
&\quad - \frac{1}{2\Delta t^n}(D(s_h^{n-2})\nabla\xi^{n-1}, \nabla\xi^{n-1}) + B_8^n,
\end{aligned}\tag{5.22}$$

where B_8^n is defined by

$$B_8^n = \frac{1}{2\Delta t^n}\big((D(s_h^{n-2}) - D(s_h^{n-1}))\nabla\xi^{n-1}, \nabla\xi^{n-1}\big),$$

and is bounded by

$$|B_8^n| \le C_2\big(1 + \|\partial\xi^{n-1}\|_{0,\infty}^2\big)\|\nabla\xi^{n-1}\|^2. \tag{5.23}$$

Next, it can be easily seen that

$$|B_1^n| + |B_2^n| + |B_6^n| \le C_2\big(\|\frac{\partial s^n}{\partial t} - \partial s^n\|^2 + \|\partial\zeta^n\|^2 + \|\zeta^n\|^2 + \|\xi^n\|^2\big) + \varepsilon\|\partial\xi^n\|^2. \tag{5.24}$$

To avoid an apparent loss of a factor h in B_i^n, $i = 3, 4, 5, 7$, we use summation by parts on these items. We work on B_3^n in detail, and other quantities can be estimated similarly. Applying summation by parts in n and the fact that $\xi^0 = 0$, we see that

$$\begin{aligned}
&\sum_{n=1}^{\gamma}\big((q_w(s^n) - q_w(s_h^{n-1}))u^n, \nabla\partial\xi^n\big)\Delta t^n\\
&\quad = \sum_{n=1}^{\gamma-1}\big(\{(q_w(s^n) - q_w(s_h^{n-1})) - ((q_w(s^{n+1}) - q_w(s_h^n))\}u^n, \nabla\xi^n\big)\\
&\qquad + \sum_{n=1}^{\gamma-1}\big((q_w(s^{n+1}) - q_w(s_h^n))(u^n - u^{n+1}), \nabla\xi^n\big)\\
&\qquad + \big((q_w(s^\gamma) - q_w(s_h^{\gamma-1}))u^\gamma, \nabla\xi^\gamma\big),
\end{aligned}$$

so that, using the same argument as for (5.18),

$$\left|\sum_{n=1}^{\gamma} B_3^n \Delta t^n\right| \leq C_2 \Big\{ \sum_{n=1}^{\gamma-1} \big(\|\partial\zeta^n\|^2 + \|\partial(s^{n+1}-s^n)\|^2 + \|\widehat{s}_h^n - \widehat{s}^{n+1}\|^2 + \|s^{n+1}-s_h^n\|^2 + \|\nabla\xi^n\|^2\big)\Delta t^n + \|s^\gamma - s_h^{\gamma-1}\|^2 \Big\} + \varepsilon\big(\|\nabla\xi^\gamma\|^2 + \sum_{n=1}^{\gamma-1}\|\partial\xi^n\|^2\big), \tag{5.25}$$

where $\|\widehat{s}_h^n - \widehat{s}^{n+1}\|$ can be estimated as in (5.20). The term $\sum_{n=1}^{\gamma} B_7^n \Delta t^n$ has the same bound as in (5.25). Also, we find that

$$\left|\sum_{n=1}^{\gamma} B_4^n \Delta t^n\right| \leq C_2 \Big\{ \sum_{n=1}^{\gamma-1} \big(\|\partial\sigma^{n+1}\|^2 + \|\sigma^n\|^2 + \|\beta^n\|^2 + (1+\|\partial\xi^n\|_{0,\infty}^2)\|\nabla\xi^n\|^2\big)\Delta t^n + \|\sigma^\gamma\|^2 + \|\beta^\gamma\|^2 \Big\} + \varepsilon\big(\|\nabla\xi^\gamma\|^2 + \sum_{n=1}^{\gamma-1}\|\partial\beta^{n+1}\|^2\big), \tag{5.26}$$

and

$$\left|\sum_{n=1}^{\gamma} B_5^n \Delta t^n\right| \leq C_2 \Big\{ \sum_{n=1}^{\gamma-1} \big(\|\partial\zeta^n\|^2 + \|\partial(s^{n+1}-s^n)\|^2 + \|\widehat{s}_h^n - \widehat{s}^{n+1}\|^2 + \|\partial\eta^n\|^2 + \|\partial\theta^n\|^2 + \|\widehat{p}_h^n - \widehat{p}^n\|^2 + \|s^{n+1}-s_h^n\|^2 + \|p^n - p_h^n\|^2 + \|\nabla\xi^n\|^2\big)\Delta t^n + \|s^\gamma - s_h^{\gamma-1}\|^2 + \|p^\gamma - p_h^\gamma\|^2 \Big\} + \varepsilon\big(\|\nabla\xi^\gamma\|^2 + \sum_{n=1}^{\gamma-1}\|\partial\xi^n\|^2\big). \tag{5.27}$$

Now, multiply (5.21) by Δt^n, sum n, and use (5.22)–(5.27) to complete the proof of the lemma. ∎

5.3. L^2-error estimates. We now prove the main result in this section. Define

$$\begin{aligned}
\mathcal{E}(t) = &\sum_{K\in T_{h_p}} h_{p,K}^{k^{**}}\big(\|p\|_{L^\infty(0,t;H^{k^{**}}(K))} + \|\frac{\partial p}{\partial t}\|_{L^\infty(0,t;H^{k^{**}}(K))} + \|\frac{\partial^2 p}{\partial t^2}\|_{L^2(0,t;H^{k^{**}}(K))}\big)\\
&+ \sum_{K\in T_{h_p}} h_{p,K}^{k^*+1}\big(\|u\|_{L^\infty(0,t;H^{k^*+1}(K))} + \|\frac{\partial u}{\partial t}\|_{L^2(0,t;H^{k^*+1}(K))}\big)\\
&+ \sum_{K\in T_h} h_K^{k+1}\big(\|s\|_{L^\infty(0,t;H^{k+1}(K))} + \|\frac{\partial s}{\partial t}\|_{L^2(0,t;H^{k+1}(K))}\big)\\
&+ \Delta t \sum_{i=1}^{2}\big(\|\frac{\partial^i p}{\partial t^i}\|_{L^2(J;L^2(\Omega))} + \|\frac{\partial^i s}{\partial t^i}\|_{L^2(J;L^2(\Omega))} + \|\frac{\partial^i u}{\partial t^i}\|_{L^2(J;L^2(\Omega))}\big)\\
&+ \Delta t\|\frac{\partial^3 p}{\partial t^3}\|_{L^2(J;L^2(\Omega))}, \quad t\in J.
\end{aligned}$$

THEOREM 5.4. *Let (u,p,s) and (u_h,p_h,s_h) satisfy* (3.1), (3.2) *and* (4.5), (4.6), *respectively. Then, if the parameters Δt, h_p, and h satisfy*

$$(h^{-d/2}+h_p^{-d/2})\big(\Delta t+h_p^{k^*+1}+h_p^{k^{**}}+h^{k+1}\big)\to 0 \text{ as } \Delta t,\ h\to 0, \tag{5.28}$$

we have

$$\max_{0\le n\le n_T}\{\|u^n-u_h^n\|+\|p^n-p_h^n\|+\|s^n-s_h^n\|+h\|\nabla(s^n-s_h^n)\|+\|\frac{\partial p^n}{\partial t}-\partial p_h^n\|\}$$
$$+\{\sum_{n=1}^{n_T}\|\frac{\partial s^n}{\partial t}-\partial s_h^n\|^2\Delta t^n\}^{1/2}\le C\mathcal{E}(T),$$

where $C=C(C_1,C_2,T)$.

Proof. Take a (C_1+1)-multiple of the inequality in Lemma 5.3, add the resulting inequality and the inequality in Lemma 5.2, and use (5.3)–(5.7), (5.15), and the extension of the solution for $t\le 0$ to obtain

$$\begin{aligned}&\|\nabla\xi^\gamma\|^2+\|\partial\theta^\gamma\|^2+\sum_{n=1}^{\gamma}(\|\partial\xi^n\|^2+\|\partial\beta^n\|^2)\Delta t^n\\&\le C_3\Big\{\mathcal{E}^2(t^\gamma)+\|\xi^{\gamma-1}\|^2+\|\theta^\gamma\|^2+\|\beta^\gamma\|^2\\&\qquad+\frac12\sum_{n=1}^{\gamma}\big(\|\xi^n\|_1^2+\|\beta^n\|^2+\|\theta^n\|^2+\|\partial\theta^n\|^2\\&\qquad\quad+(\|\partial\theta^{n-1}\|^2+\|\beta^{n-2}\|^2+\|\nabla\xi^{n-1}\|^2\\&\qquad\qquad+(1+\|\beta^{n-1}\|_{0,\infty}^2)\|\beta^{n-1}\|^2+\mathcal{E}^2(t^\gamma))\\&\qquad\quad\times(\|\partial\xi^{n-1}\|_{0,\infty}^2+\|\partial\theta^{n-1}\|_{0,\infty}^2+\|\partial\beta^{n-1}\|_{0,\infty}^2)\big)\Delta t^n\Big\}.\end{aligned}\tag{5.29}$$

where $C_3=C_3(C_1,C_2)$. In deriving (5.29), we required that the ε appearing in Lemma 5.3 be sufficiently small that $(C_1+1)\varepsilon\le 1/2$; this increases C_2, but not C_1. Observe that, by (5.12),

$$\|\theta^\gamma\|^2\le C\sum_{n=1}^{\gamma}\|\theta^n\|^2\Delta t^n+\varepsilon\sum_{n=1}^{\gamma}\|\partial\theta^n\|^2\Delta t^n.\tag{5.30}$$

The same result holds for ξ^γ and β^γ. Combine (5.29), (5.30), and an inverse inequality to see that

$$\begin{aligned}&\|\xi^\gamma\|_1^2+\|\theta^\gamma\|^2+\|\partial\theta^\gamma\|^2+\|\beta^\gamma\|^2+\sum_{n=1}^{\gamma}(\|\partial\xi^n\|^2+\|\partial\beta^n\|^2)\Delta t^n\\&\le C_3\Big\{\mathcal{E}^2(t^\gamma)+\frac12\sum_{n=1}^{\gamma}\big(\|\xi^n\|_1^2+\|\beta^n\|^2+\|\theta^n\|^2+\|\partial\theta^n\|^2\\&\qquad+(h^{-d}+h_p^{-d})(\|\partial\theta^{n-1}\|^2+\|\beta^{n-2}\|^2+\|\nabla\xi^{n-1}\|^2\\&\qquad\quad+(1+h_p^{-d}\|\beta^{n-1}\|^2)\|\beta^{n-1}\|^2+\mathcal{E}^2(t^\gamma))\\&\qquad\times(\|\partial\xi^{n-1}\|^2)+\|\partial\theta^{n-1}\|^2+\|\partial\beta^{n-1}\|^2)\big)\Delta t^n\Big\}.\end{aligned}\tag{5.31}$$

We now make the induction hypothesis that

$$\max_{n\le\gamma-1}(\|\xi^n\|_1^2+\|\theta^n\|^2+\|\partial\theta^n\|^2+\|\beta^n\|^2) + \sum_{n=1}^{\gamma-1}(\|\partial\xi^n\|^2+\|\partial\beta^n\|^2)\Delta t^n \le C_4\mathcal{E}^2(T), \tag{5.32}$$

where $C_4 = 2C_3e^{TC_3}$. Note that, by (5.12), (5.32) holds trivially for $\gamma = 1$. Then, by (5.32), (5.31) becomes

$$\begin{aligned}\|\xi^\gamma\|_1^2+\|\theta^\gamma\|^2+\|\partial\theta^\gamma\|^2+\|\beta^\gamma\|^2+\sum_{n=1}^{\gamma}(\|\partial\xi^n\|^2+\|\partial\beta^n\|^2)\Delta t^n \\ \le C_3\Big\{\mathcal{E}^2(t^\gamma)+\frac{1}{2}\sum_{n=1}^{\gamma}\big(\|\xi^n\|_1^2+\|\beta^n\|^2+\|\theta^n\|^2+\|\partial\theta^n\|^2 \\ +2(h^{-d}+h_p^{-d})C_4\mathcal{E}^2(T)(1+C_4h_p^{-d}\mathcal{E}^2(T)) \\ \times(\|\partial\xi^{n-1}\|^2+\|\partial\theta^{n-1}\|^2+\|\partial\beta^{n-1}\|^2)\big)\Delta t^n\Big\}.\end{aligned} \tag{5.33}$$

Using (5.28), we choose the discretization parameters so small that

$$2(h^{-d}+h_p^{-d})C_3C_4\mathcal{E}^2(T)(1+C_4h_p^{-d}\mathcal{E}^2(T)) \le 1/2.$$

Then it follows from (5.33) that

$$\begin{aligned}\|\xi^\gamma\|_1^2+\|\theta^\gamma\|^2+\|\partial\theta^\gamma\|^2+\|\beta^\gamma\|^2+\sum_{n=1}^{\gamma}(\|\partial\xi^n\|^2+\|\partial\beta^n\|^2)\Delta t^n \\ \le C_3\Big\{\mathcal{E}^2(t^\gamma)+\sum_{n=1}^{\gamma}\big(\|\xi^n\|_1^2+\|\beta^n\|^2+\|\theta^n\|^2+\|\partial\theta^n\|^2\big)\Delta t^n\Big\},\end{aligned}$$

which, together with Gronwall's inequality, implies that

$$\|\xi^\gamma\|_1^2+\|\theta^\gamma\|^2+\|\partial\theta^\gamma\|^2+\|\beta^\gamma\|^2+\sum_{n=1}^{\gamma}(\|\partial\xi^n\|^2+\|\partial\beta^n\|^2)\Delta t^n \le C_5\mathcal{E}^2(T), \tag{5.34}$$

where

$$C_5 = C_3(1-C_3\Delta t)^{-T/\Delta t} \le 2C_3e^{TC_3} \equiv C_4,$$

for Δt not too large. Consequently, the induction argument is completed and the theorem follows. □

We remark that, if h and h_p are of the same order as they tend to zero, then

$$(h^{-d/2}+h_p^{-d/2})\big(h_p^{k^*+1}+h_p^{k^{**}}+h^{k+1}\big) \le Ch^{-d/2}\big(h^{k^{**}}+h^{k+1}\big),$$

since $k^{**} \le k^*+1$. Since $k \ge 1$,

$$h^{-d/2}h^{k+1} \to 0 \text{ as } h \to 0,\ d = 2,3.$$

Also, if $k^{**} \ge 2$, we see that

$$h^{-d/2}h^{k^{**}} \to 0 \text{ as } h \to 0,\ d = 2,3.$$

Thus, for (5.28) to be satisfied, we assume that $k^{**} \ge 2$. This excludes the mixed finite element spaces of lowest order, i.e., $k^{**} = 1$. The lowest order case has to be treated using different techniques. If the nonlinear coefficients $\alpha(s)$ and $c(s,p)$ in (4.5) are projected into the finite element space W_h, the technique developed in [10] can be used to handle the lowest order case. We shall not pursue this here.

5.4. L^∞-error estimates. The main objective of this paper is to establish the L^2-error estimates given in Theorem 5.4. For completeness, we end this section with a statement of L^∞-estimates for the errors $s - s_h$ and $p - p_h$ in the two-dimensional case.

THEOREM 5.5. *Assume that* (p, s) *and* (p_h, s_h) *satisfy* (3.1), (3.2) *and* (4.5), (4.6), *respectively, and the parameters* h_p *and* h *satisfy* (5.28). *Then*

$$(5.35)\qquad \max_{0\le n\le n_T} \|p^n - p_h^n\|_{0,\infty} \le C \log h_p^{-1}\big(\mathcal{E}(T) + h_p^{k^{**}}\|p\|_{L^\infty(J;H^{k^{**}+1}(\Omega))}\big),$$

$$(5.36)\qquad \max_{0\le n\le n_T} \|s^n - s_h^n\|_{0,\infty} \le C\big(\log h^{-1}\big)^\gamma\big(\mathcal{E}(T) + h^{k+1}\|s\|_{L^\infty(J;W^{k+1,\infty}(\Omega))}\big),$$

where $C = C(C_1, C_2, T)$, $\gamma = 1$ *for* $k = 1$, *and* $\gamma = 1/2$ *for* $k > 1$.

Proof. First, it follows from the approximation property of the projection P_h [27] that

$$(5.37)\qquad \|p^n - \tilde{p}^n\|_{0,\infty} \le Ch_p^{k^{**}}\big(\log h_p^{-1}\big)^{1/2}\|p^n\|_{k^{**}+1}.$$

Also, from [27, Lemma 1.2] and (5.13), we see that

$$\begin{aligned}\|\theta^n\|_{0,\infty} \le C\log h_p^{-1}\|\alpha(s_h^{n-1})\beta^n + (\alpha(s^n) - \alpha(s_h^{n-1}))u^n\\ + \alpha(s_h^{n-1})\sigma^n + (G(s_h^{n-1}, p_h^{n-1}) - G(s^n, p^n))\|,\end{aligned}$$

so that, by Theorem 5.4,

$$\max_{0\le n\le n_T} \|\theta^n\|_{0,\infty} \le C\log h_p^{-1}\mathcal{E}(T).$$

This, together with (5.37), implies (5.35). Finally, apply the embedding inequality [36]

$$\|\xi^n\|_{0,\infty} \le C\big(\log h^{-1}\big)^{1/2}\|\xi^n\|_1,$$

(5.3b), and (5.34) to obtain (5.36). ∎

6. Finite elements for a degenerate problem. In this section we consider a degenerate case where the diffusion coefficient $D(s)$ can be zero. Since the pressure equation is the same as before, we here focus on the saturation equation. For simplicity we neglect gravity. Then the saturation equation (2.11) can be written as

$$(6.1)\qquad \phi\frac{\partial s}{\partial t} - \nabla\cdot(D(s)\nabla s - q_w(s)u) = \tilde{f}_w - s\frac{\partial \phi}{\partial t}, \quad (x,t) \in \Omega\times J.$$

For technical reasons we only consider the Neumann boundary condition (2.15):

$$(6.2a)\qquad (D(s)\nabla s - q_w(s)u)\cdot\nu = -d_w(x,t), \quad (x,t)\in\partial\Omega\times J,$$

and the initial condition is given by

$$(6.2b)\qquad s(x,0) = s^0(x), \quad x\in\Omega,$$

where $0 \le s^0(x) \le 1$, $x \in \Omega$. We impose the following conditions on the degeneracy of $D(s)$:

$$D(s) \ge \begin{cases} \beta_1 |s|^{\mu_1}, & 0 \le s \le \alpha_1, \\ \beta_2, & \alpha_1 \le s \le \alpha_2, \\ \beta_3 |1-s|^{\mu_2}, & \alpha_2 \le s \le 1, \end{cases} \tag{6.3}$$

where the β_i are positive constants and α_j and μ_j $(j = 1, 2)$ satisfy the conditions:

$$0 < \alpha_1 < 1/2 < \alpha_2 < 1,\ 0 < \mu_j \le 2.$$

Difficulties arise when trying to derive error estimates for the approximate solution of (6.1) and (6.2) with $D(s)$ satisfying the condition (6.3). To get around this problem, we consider the perturbed diffusion coefficient $D_\kappa(s)$ defined by [13], [24], [35], [38]

$$D_\kappa(s) = \max\{D(s), \kappa^\mu\},$$

where $\kappa > 0$ and $\mu = \max\{\mu_1, \mu_2\}$. Since the coefficient $D_\kappa(s)$ is bounded away from zero, the previous error analysis applies to the perturbed problem:

$$\phi \frac{\partial s_\kappa}{\partial t} - \nabla \cdot (D_\kappa(s_\kappa)\nabla s_\kappa - q_w(s_\kappa)u) = \tilde{f}_w - s_\kappa \frac{\partial \phi}{\partial t}, \quad (x,t) \in \Omega \times J, \tag{6.4a}$$

$$(D_\kappa(s_\kappa)\nabla s_\kappa - q_w(s_\kappa)u) \cdot \nu = -d_w(x,t), \quad (x,t) \in \partial\Omega \times J, \tag{6.4b}$$

$$s_\kappa(x,0) = s^0(x), \quad x \in \Omega. \tag{6.4c}$$

We now state a result on the convergence of s_κ to s as κ tends to zero. Its proof is given in [24] for the case where $d_w \equiv 0$ and the right-hand side of (6.1) is zero, and can be easily extended to the present case.

THEOREM 6.1. *Assume that $D(s)$ satisfies* (6.3) *and there is a constant $C^* > 0$ such that*

$$C^*|q_w(s_1) - q_w(s_2)|^2 \le (\mathcal{D}(s_1) - \mathcal{D}(s_2))(s_1 - s_2), \quad 0 \le s_1, s_2 \le 1, \tag{6.5}$$

where

$$\mathcal{D}(s) = \int_0^s D(\xi) d\xi.$$

Then there is C independent of κ, s, and μ such that

$$||s - s_\kappa||_{L^{2+\mu}(J;L^{2+\mu}(\Omega))} \le C\kappa. \tag{6.6}$$

As shown in [24], the requirement (6.5) is reasonable. We now consider a fully-discrete finite element method for (6.4). Let M_h be the standard C^0 piecewise linear polynomial space associated with T_h; due to the roughness of the solution to (6.1) and (6.2), no improvements in the asymptotic convergence rates result from taking higher order finite element spaces. Also, we extend the domain of D_κ and q_w as follows:

$$D_\kappa(\xi) = \begin{cases} D_\kappa(1) & \text{if } \xi \ge 1, \\ D_\kappa(-\xi) & \text{if } \xi \le 0, \end{cases}$$

and

$$q_w(\xi) = 0, \quad \forall \xi \in (-\infty, 0) \cup (1, \infty).$$

Now the finite element solution $s_h^n : J \to M_h$, $n = 1, 2, \cdots, n_T$ to (6.4) is given by

$$\text{(6.7a)} \qquad \big(\phi \partial s_h^n, v\big) + \big(D_\kappa(s_h^n)\nabla s_h - q_w(s_h^n)u^n, \nabla v\big) = \big(\tilde{f}_w^n - s_h^n \frac{\partial \phi^n}{\partial t}, v\big) - \langle d_w^n, v\rangle_{\partial\Omega}, \quad \forall v \in M_h,$$

$$\text{(6.7b)} \qquad s_h^0 = \mathcal{P}_h s^0,$$

where $\mathcal{P}_h$ is the L^2-projection onto M_h. The following theorem states the convergence of s_h to s. For (6.8) below to be satisfied, we see from (6.6) that the perturbation parameter κ need to satisfy the relation $\kappa = O(h^{\lambda_1})$, where λ_1 is given by

$$\lambda_1 = (4 + 2\mu)/(2 + 4\mu + \mu^2).$$

THEOREM 6.2. *Let s and s_h solve* (6.1), (6.2) *and* (6.7), *respectively, and let the hypotheses of Theorem* 6.1 *be satisfied. Then there is C independent of κ, s, and μ such that*

$$\text{(6.8)} \qquad \max_{0\le n\le n_T} ||s(t^n) - s_h^n||^2_{H^{-1}(\Omega)} + \sum_{n=0}^{n_T} ||s(t^n) - s_h^n||^{2+\mu}_{L^{2+\mu}(\Omega)} \Delta t^n \le C\big(h^{(2+\mu)\lambda_1} \left(\log h^{-1}\right)^{\frac{\mu}{1+\mu}} + \Delta t^{\frac{\lambda_2+2}{2}}\big),$$

where $\lambda_2 = (2 + \mu)/(1 + \mu)$.

The proof can be carried out as in [25], [35], and [38]; we omit the details.

7. Simulation with various boundary conditions. Let $\partial\Omega$ be a set of four disjoint regions Γ_i, $i = 1, \cdots, 4$, and let $\Gamma_3 = \cup_j \Gamma_{3,j}$ where each $\Gamma_{3,j}$ is connected. As mentioned in the introduction, the most commonly encountered boundary conditions for the two-pressure equations are of first-type, second-type, third-type, and well type. Then we consider for $\alpha = w, a$

$$\text{(7.1)} \qquad p_\alpha = p_{\alpha D}(x, t), \qquad x \in \Gamma_1,\ t > 0,$$

$$\text{(7.2)} \qquad u_\alpha \cdot \nu + \chi_\alpha(x, t, s)p_\alpha = \upsilon_\alpha(x, t, s), \qquad x \in \Gamma_2,\ t > 0,$$

$$\text{(7.3a)} \qquad \int_{\Gamma_{3,j}} (u_w + u_\alpha)\cdot \nu = \upsilon_j(t), \qquad x \in \Gamma_{3,j},\ t > 0,$$

$$\text{(7.3b)} \qquad p_\alpha = p_{\alpha D}(x, t) + d_j(t), \qquad x \in \Gamma_{3,j},\ t > 0,$$

$$\text{(7.4a)} \qquad p_a = p_{aD}(x, t), \qquad x \in \Gamma_4,\ t > 0,$$

$$\text{(7.4b)} \qquad u_w \cdot \nu + \chi_w(x, t, s)p_w = \upsilon_w(x, t, s), \qquad x \in \Gamma_4,\ t > 0,$$

where $p_{\alpha D}$, χ_α, υ_α, and υ_j are given functions, d_j is an arbitrary scaling constant, and ν is the outer unit normal to $\partial\Omega$. Note that Γ_1 is of the first type, Γ_2 is of the third type (it reduces to the second type as $\chi_\alpha \equiv 0$), Γ_3 is of the well type, and on Γ_4 we have the Dirichlet condition for the air phase and the Neumann condition for the water phase. Let $\Gamma_{p,i} = \Gamma_i$, $i = 1, \cdots, 4$, $\Gamma_{s,1} = \Gamma_1 \cup \Gamma_3$, and $\Gamma_{s,2} = \Gamma_2 \cup \Gamma_4$. Then the global boundary conditions for the pressure-saturation equations (2.9)–(2.11) become

$$\text{(7.5)} \qquad p = p_D(x, t), \qquad x \in \Gamma_{p,1},\ t > 0,$$

$$\text{(7.6)} \qquad u \cdot \nu + \chi(x, t, s)p = \Upsilon(x, t, s), \qquad x \in \Gamma_{p,2},\ t > 0,$$

$$\text{(7.7a)} \qquad \int_{\Gamma_{p,3,j}} u \cdot \nu = \upsilon_j(t), \qquad x \in \Gamma_{p,3,j},\ t > 0,$$

$$\text{(7.7b)} \qquad p = p_D(x, t) + d_j(t), \qquad x \in \Gamma_{p,3,j},\ t > 0,$$

$$
\begin{aligned}
&(7.8) \qquad p = p_{aD}(x,t) + \varphi(s), && x \in \Gamma_{p,4},\ t > 0,\\
&(7.9) \qquad s = s_D(x,t), && x \in \Gamma_{s,1},\ t > 0,\\
&(7.10) \qquad (q_w u + k\lambda_a q_w(\nabla p_c - \tilde{\rho})) \cdot \nu \\
&\qquad\qquad + \chi_w(x,t,s)p = \Upsilon_w(x,t,s), && x \in \Gamma_{s,2},\ t > 0,
\end{aligned}
$$

where p_D and s_D are the transforms of p_{wD} and p_{aD} by (2.2) and (2.3), and

$$
\begin{aligned}
\chi &= \chi_w + \chi_a,\\
\Upsilon &= \upsilon_w + \upsilon_a - \chi_a p_c + \chi \int_0^{p_c(s)} q_a\left(p_c^{-1}(\xi)\right) d\xi,\\
\Upsilon_w &= \upsilon_w + \chi_w \int_0^{p_c(s)} q_a\left(p_c^{-1}(\xi)\right) d\xi,\\
\varphi(s) &= -\int_0^{p_c(s)} q_w\left(p_c^{-1}(\xi)\right) d\xi.
\end{aligned}
$$

We now incorporate the boundary conditions (7.5)–(7.10) in the finite element scheme given in (4.5) and (4.6). The constraint $V_h \subset V$ says that the normal components of the members of V_h are continuous across the interior boundaries in T_{h_p}. Following [2], [9], we relax this constraint on V_h by introducing Lagrange multipliers over interior boundaries. Since the mixed space V_h is finite dimensional and defined locally on each element K in T_{h_p}, let $V_h(K) = V_h|_K$. Then we define

$$
\begin{aligned}
\tilde{V}_h &= \{v \in (L^2(\Omega))^d : v|_K \in V_h(K) \text{ for each } K \in T_{h_p}\},\\
L_{h,\pi_1,\{\pi_j\},\pi_3} &= \Big\{ r \in L^2\Big(\bigcup_{e\in\partial T_{h_p}} e\Big) : r|_e \in V_h\cdot\nu|_e \text{ for each } e \in \partial T_{h_p};\\
&\qquad (r-\pi_1, r_1)_e = 0,\ r_1 \in V_h\cdot\nu|_e,\ \forall e \in \Gamma_{p,1},\\
&\qquad (r-\pi_j, r_2)_e = 0,\ r_2 \in V_h\cdot\nu|_e,\ \forall e \in \Gamma_{p,3,j},\ \text{ for each } j,\\
&\qquad (r-\pi_3, r_3)_e = 0,\ r_3 \in V_h\cdot\nu|_e,\ \forall e \in \Gamma_{p,4} \Big\},
\end{aligned}
$$

and W_h and M_h are given as before. The mixed finite element solution of the pressure equation is $\{u_h^n, p_h^n, \ell_h^n\} \in \tilde{V}_h \times W_h \times L_{h,p_D^n,\{p_D^n+d_j^n\},p_{aD}^n+\varphi^{n-1}}$, $n = 1,2,\cdots,n$ satisfying

$$
\begin{aligned}
&(c(s_h^{n-1},p_h^{n-1})\partial p_h^n,\psi) + \sum_K (\nabla\cdot u_h^n,\psi)_K = (f(p_h^{n-1}),\psi), \quad \forall \psi \in W_h,\\
&(\alpha(s_h^{n-1})u_h^n, v) - \sum_K \left\{(\nabla\cdot v, p_h^n)_K - (\ell_h^n, v\cdot\nu_K)_{\partial K}\right\} = (G(s_h^{n-1},p_h^{n-1}),v), \quad \forall v \in \tilde{V}_h,\\
&\sum_K (u_h^n\cdot\nu_K, r)_{\partial K\setminus(\Gamma_{p,1}\cup\Gamma_{p,4})} = (\Upsilon(s_h^{n-1}) - \chi(s_h^{n-1})\ell_h^n, r)_{\Gamma_{p,2}} + \sum_j \frac{(\upsilon_j^n, r)_{\Gamma_{p,3,j}}}{|\Gamma_{p,3,j}|},\\
&\qquad\qquad \forall r \in L_{h,0,\{0\},0},
\end{aligned}
$$

and the finite element method for the saturation is given for $s_h^n \in M_h + s_D^n$ satisfying

$$
\begin{aligned}
&\left(\phi\partial s_h^n, v\right) + \left(D(s_h^{n-1})\nabla s_h^n - q_w(s_h^{n-1})u_h^n - b(s_h^{n-1},p_h^n), \nabla v\right)\\
&\quad = \left(\tilde{f}_w^n - s_h^n\frac{\partial\phi^n}{\partial t}, v\right) - (\Upsilon_w(s_h^{n-1}) - \chi_w(s_h^{n-1})\ell_h^n, v)_{\Gamma_2}, \qquad \forall v \in M_h,
\end{aligned}
$$

for $n = 1, 2, \cdots, n_T$. The computation of these equations can be carried out as in (4.5) and (4.6). Note that the last equation in the unconstrained mixed formulation above enforces the continuity requirement on u_h, so in fact $u_h \in V_h$. It is well known [2], [9] that the linear system arising from this unconstrained mixed formulation leads to a symmetric, positive definite system for the Lagrange multipliers, which can be easily solved. Also, the introduction of the Lagrange multipliers makes it easier to incorporate the boundary conditions (7.5)–(7.10).

We now present a numerical example. The relative permeability functions are taken as follows:

$$k_{rw} = s - s_{rw}, \qquad k_{ra} = 1 - s - s_{ra},$$

where s_{rw} and s_{ra} are the irreducible saturations of the water and air phases, respectively. The capillary pressure function is of the form

$$p_c(s) = (1-s)\{\gamma(s^{-1} - 1) + \Theta\},$$

where γ and Θ are functions of the irreducible saturations. The water and air viscosities and densities are set to be $1cP$ and $0.8cP$, and $100kg/m^3$ and $1.3kg/m^3$, respectively. The permeability rate is $1 \times 10^{-12} m^2$. A two-dimensional domain of $4m$ width by $1m$ depth is simulated. Finally, the boundary of the domain is divided into the following segments:

$$\begin{aligned}
\Gamma_1 &= \{(x,y) : x = 0,\ 0 < y < 1\},\\
\Gamma_2 &= \{(x,y) : x = 4,\ 0 \le y \le 1\} \cup \{(x,y) : y = 0,\ 0 \le x < 4\},\\
\Gamma_3 &= \emptyset,\\
\Gamma_4 &= \{(x,y) : y = 1,\ 0 \le x < 4\}.
\end{aligned}$$

A uniform partition of Ω into rectangles with $h = \Delta x = \Delta y$ is taken, and the time step Δt is required to satisfy (5.28). The Raviart-Thomas space of lowest-order over rectangles is chosen. Tables 1 and 2 describe the errors and convergence orders for the pressure and saturation at time $t = 1min$, respectively. Experiments at other times and on finer meshes are also carried out; similar results are observed and not reported here.

$1/h$	L^∞-error	$L^\infty-$order	L^2-error	L^2-order
10	0.0570	-	0.0501	-
20	0.0343	0.73	0.0245	1.02
40	0.0186	0.88	0.0122	1.00
80	0.0090	1.05	0.0059	1.02

Table 1. Convergence of p_h at $T = 1min$.

$1/h$	L^∞-error	$L^\infty-$order	L^2-error	L^2-order
10	0.0766	-	0.0695	-
20	0.0526	0.55	0.0482	0.53
40	0.0295	0.83	0.0271	0.83
80	0.0167	0.82	0.0152	0.84

Table 2. Convergence of s_h at $T = 1min$.

From Table 1, we see that the scheme is first-order accurate both in L^2 and L^∞ norms for the pressure, i.e., optimal order. Table 2 shows that the scheme is almost optimal order for the saturation. Thus the numerical experiments in the two tables are in agreement with our earlier analytic results.

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