

## FULLY DISCRETE FINITE ELEMENT ANALYSIS OF MULTIPHASE FLOW IN GROUNDWATER HYDROLOGY\*

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**Abstract.** This paper deals with the development and analysis of a fully discrete finite element method for a nonlinear differential system for describing an air-water system in groundwater hydrology. The nonlinear system is written in a fractional flow formulation, i.e., in terms of a saturation and a global pressure. The saturation equation is approximated by a finite element method, while the pressure equation is treated by a mixed finite element method. The analysis is carried out first for the case where the capillary diffusion coefficient is assumed to be uniformly positive, and is then extended to a degenerate case where the diffusion coefficient can be zero. It is shown that error estimates of optimal order in the  $L^2$ -norm and almost optimal order in the  $L^\infty$ -norm can be obtained in the nondegenerate case. In the degenerate case we consider a regularization of the saturation equation by perturbing the diffusion coefficient. The norm of error estimates depends on the severity of the degeneracy in diffusivity, with almost optimal order convergence for nonsevere degeneracy. Implementation of the fractional flow formulation with various nonhomogeneous boundary conditions is also discussed. Results of numerical experiments using the present approach for modeling groundwater flow in porous media are reported.

**Key words.** time discretization, mixed method, finite element, compressible flow, porous media, error estimate, air-water system, numerical experiments

**AMS subject classifications.** 65N30, 76S05

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**1. Introduction.** In this paper we develop and analyze a fully discrete finite element procedure for solving the flow equations for an air-water system in groundwater hydrology,  $\alpha = a, w$  [3], [14], [32]:

$$(1.1) \quad \frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha u_\alpha) = f_\alpha, \quad x \in \Omega, t > 0,$$

$$(1.2) \quad u_\alpha = -\frac{kk_{r\alpha}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g), \quad x \in \Omega, t > 0,$$

where  $\Omega \subset \mathbb{R}^d$ ,  $d \leq 3$  is a porous medium,  $\phi$  and  $k$  are the porosity and absolute permeability of the porous system,  $\rho_\alpha$ ,  $s_\alpha$ ,  $p_\alpha$ ,  $u_\alpha$ , and  $\mu_\alpha$  are the density, saturation, pressure, volumetric velocity, and viscosity of the  $\alpha$ -phase,  $f_\alpha$  is the source/sink term,  $k_{r\alpha}$  is the relative permeability of the  $\alpha$ -phase, and  $g$  is the gravitational, downward-pointing, constant vector.

Flow simulation in groundwater reservoirs has been extensively studied in past years (see, e.g., [27], [29], and the bibliographies therein). However, in most previous works the air-phase equation is eliminated by the assumption that the air phase remains essentially at atmospheric pressure. This assumption, as mentioned in [12], is reasonable in most cases because the mobility of air is much larger than that of water, due to the viscosity difference between the two fluids. When the air-phase

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pressure is assumed constant, the air-phase mass balance equation can be eliminated and thus only the water-phase equation remains. Namely, the Richards equation is used to model the movement of water in groundwater reservoirs. However, it provides no information on the motion of air. If contaminant transport is the main concern and the contaminant can be transported in the air phase, the air phase needs to be included to determine the advective component of air-phase contaminant transport [7]. Furthermore, the dynamic interaction between the air and water phases is also important in vapor extraction systems. Hence, in these cases the coupled system of nonlinear equations for the air-water system must be solved. It is the purpose of this paper to develop and analyze a finite element procedure for approximating the solution of the coupled system of nonlinear equations for the air-water system in groundwater hydrology.

In petroleum reservoir simulation the governing equations that describe fluid flow are usually written in a fractional flow formulation, i.e., in terms of a saturation and a global pressure [1], [8]. The main reason for this fractional flow approach is that efficient numerical methods can be devised to take advantage of many physical properties inherent in the flow equations. However, this pressure-saturation formulation has not yet achieved application in groundwater hydrology. In petroleum reservoirs total flux-type boundary conditions are conveniently imposed and often used, but in groundwater reservoirs boundary conditions are very complicated. The most commonly encountered boundary conditions for a groundwater reservoir are of first type (Dirichlet), second type (Neumann), third type (mixed), and “well” type [8]. The problem of incorporating these nonhomogeneous boundary conditions into the fractional flow formulation has been a challenge [14]. In particular, in using the fractional flow approach a difficulty arises when the Dirichlet boundary condition is imposed for one phase (e.g., air) and the Neumann type is used for another phase (e.g., water).

This paper follows the fractional flow formulation. Based on this approach, we develop a fully discrete finite element procedure for the saturation and pressure equations. The saturation equation is approximated by a Galerkin finite element method, while the pressure equation is treated by a mixed finite element method. It is well known that the physical transport dominates the diffusive effects in incompressible flow in petroleum reservoirs. In the air-water system studied here, the transport again dominates the entire process. Hence, it is important to obtain good approximate velocities. This motivates the use of the parabolic mixed method, as in [18], in the computation of the pressure and the velocity. Also, due to its convection-dominated feature, more efficient approximate procedures should be used to solve the saturation equation. However, since this is the first time to carry out an analysis for the present problem, it is of some importance to establish that the standard finite element method for this model converges at an asymptotically optimal rate for smooth problems. Characteristic Petrov–Galerkin methods based on operator splitting [21], transport diffusion methods [33], and other characteristic-based methods will be considered in forthcoming papers.

The main part of this paper deals with an asymptotical analysis for the fully discrete finite element method for the first-type and second-type boundary conditions

$$(1.3) \quad p_\alpha = p_{\alpha D}(x, t), \quad x \in \Gamma_1, t > 0,$$

$$(1.4) \quad u_\alpha \cdot \nu = d_\alpha(x, t), \quad x \in \Gamma_2, t > 0,$$

where  $p_{\alpha D}$  and  $d_\alpha$  are given functions,  $\partial\Omega = \Gamma_1 \cup \Gamma_2$  with  $\Gamma_1$  and  $\Gamma_2$  being disjoint, and  $\nu$  is the outer unit normal to  $\partial\Omega$ . We point out that petroleum reservoir simulation

is different from groundwater reservoir simulation. In the latter case the pressures are much lower, the variety of species is much larger, the topography plays important role, and the needed accuracy for numerical approximations is often high (in particular, for the concentration of pollutants). Also, the flow of two incompressible fluids (e.g., water and oil) is usually considered in the former case, while the latter system consists of the air and water phases. Consequently, the finite element analyses for these two cases differ. As shown here, compressibility and combination of the boundary conditions (1.3) and (1.4) complicate error analyses. Indeed, if optimality is to be preserved for the finite element method, the standard error argument just fails unless we work with higher order time-differentiated forms of error equations, which require properly scaling initial conditions. Next, we mention that a slightly compressible miscible displacement problem was treated in [15], [19], [24], [34]; however, only the single phase was handled, gravitational terms were omitted, and total flux-type boundary conditions were assumed. Furthermore, the so-called “quadratic” terms in velocity were neglected. The dropping of these quadratic terms may not be valid near wells, and so the miscible displacement model was oversimplified both physically and mathematically. The analysis of this paper includes these terms. Finally, only the Raviart–Thomas mixed finite element spaces [35] have been considered in these earlier papers. We are here able to discuss all existing mixed spaces.

The error analysis is given first for the case where the capillary diffusion coefficient is assumed to be uniformly positive. In this case, we show error estimates of optimal order in the  $L^2$ -norm and almost optimal order in the  $L^\infty$ -norm. Then we treat a degenerate case where the diffusion coefficient vanishes for two values of saturation. In the degenerate case we consider a regularization of the saturation equation by perturbing the diffusion coefficient to obtain a nondegenerate problem with smooth solutions. It is shown that the regularized solutions converge to the original solution as the perturbation parameter goes to zero with specific convergence rates given. The norm of error estimates depends on the severity of the degeneracy in diffusivity, with almost optimal order convergence for the degeneracy under consideration.

The rest of this paper is concerned with implementation of the fractional flow formulation with various nonhomogeneous boundary conditions. We show that all the commonly encountered boundary conditions can be incorporated in the fractional flow formulation. Normally the “global” boundary conditions are highly nonlinear functions of the physical boundary conditions for the original two flow phases. This means that we have to iterate on these global boundary conditions as part of the solution process. We here develop a general solution approach to handle these boundary conditions. Results of numerical experiments using the present approach for modeling groundwater flow are reported here.

The paper is organized as follows. In section 2, we define a fractional flow formulation for equations (1.1)–(1.4). Then, in section 3 we introduce weak forms of the pressure-saturation equations, and in section 4 a fully discrete finite element procedure for solving these equations. An asymptotical analysis is given in sections 5 and 6 for the nondegenerate case and the degenerate case, respectively. Finally, in section 7 we discuss implementation of various nonhomogeneous boundary conditions and present the results of numerical experiments.

**2. A pressure-saturation formulation.** In addition to (1.1)–(1.4), we impose the customary property that the fluid fills the volume,

$$(2.1) \quad s_a + s_w = 1,$$

and define the capillary pressure function  $p_c$  by

$$(2.2) \quad p_c(s_w) = p_a - p_w.$$

Introduce the phase mobilities

$$\lambda_\alpha = k_{r\alpha}/\mu_\alpha, \quad \alpha = a, w,$$

and the total mobility

$$\lambda = \lambda_a + \lambda_w.$$

To devise our numerical method, it is important to choose a reasonable set of dependent variables. Since  $p_w = -\infty$  if  $s_w$  is equal to the water residual saturation [3],  $p_w$  cannot generally be expected to lie in any Sobolev space. Air being a continuous phase implies that  $p_a$  is well behaved. Hence, as mentioned in the introduction, we define the global pressure [1] with  $s = s_w$ :

$$(2.3) \quad \begin{aligned} p &= p_a - \int_{s_c}^s \frac{\lambda_w}{\lambda} \frac{dp_c}{d\xi} d\xi \\ &= p_a - \int_0^{p_c(s)} \left( \frac{\lambda_w}{\lambda} \right) (p_c^{-1}(\xi)) d\xi, \end{aligned}$$

where  $p_c(s_c) = 0$ . The integral in the right-hand side of (2.3) is well defined [1], [8]. As usual, assume that  $\rho_\alpha$  depends on  $p$  [8]. Then we define the total velocity

$$(2.4) \quad u = -k\lambda(\nabla p - G(s, p)),$$

where

$$G(s, p) = \frac{\lambda_a \rho_a + \lambda_w \rho_w}{\lambda} g.$$

Now it can be easily seen that

$$(2.5a) \quad u_w = q_w u + k\lambda_a q_w \nabla p_c - k\lambda_a q_w \tilde{\rho},$$

$$(2.5b) \quad u_a = q_a u - k\lambda_w q_a \nabla p_c + k\lambda_w q_a \tilde{\rho},$$

where  $q_\alpha = \lambda_\alpha/\lambda$ ,  $\alpha = a, w$ , and  $\tilde{\rho} = (\rho_a - \rho_w)g$ . Consequently,

$$(2.6) \quad u = u_a + u_w.$$

Equations (1.1) and (1.2) can be manipulated using (2.1)–(2.6) to have the pressure equation

$$(2.7) \quad \nabla \cdot u = -\frac{\partial \phi}{\partial t} - \sum_{\alpha=w}^a \frac{1}{\rho_\alpha} \left( \phi s_\alpha \frac{\partial \rho_\alpha}{\partial t} + u_\alpha \cdot \nabla \rho_\alpha - f_\alpha \right),$$

and the saturation equation

$$(2.8) \quad \begin{aligned} \phi \frac{\partial s_w}{\partial t} + \nabla \cdot (q_w u + k\lambda_a q_w (\nabla p_c - \tilde{\rho})) \\ = -s_w \frac{\partial \phi}{\partial t} - \frac{1}{\rho_w} \left( \phi s_w \frac{\partial \rho_w}{\partial t} + u_w \cdot \nabla \rho_w - f_w \right). \end{aligned}$$

Terms of the form  $u_\alpha \cdot \nabla \rho_\alpha$ ,  $\alpha = a, w$  have been neglected in compressible miscible displacement problems [15], [19], [24], [34]. The dropping of these terms may not be valid near wells. Also, if they are neglected, the model may not be qualitatively equivalent to the usual formulation of two-phase flow. Hence, we keep them in this paper. However, the water phase is usually assumed to be incompressible. With the incompressibility of the water phase and the notation

$$\begin{aligned} c(s, p) &= \frac{\phi(1-s)}{\rho_a} \frac{d\rho_a}{dp}, & D(s) &= -k\lambda_a q_w \frac{dp_c}{ds}, \\ a(s) &= k\lambda, & A(s, p) &= \frac{q_a a^{-1}(s)}{\rho_a} \frac{d\rho_a}{dp}, \\ \tilde{f}_w &= \frac{f_w}{\rho_w}, & b(s, p) &= -k\lambda_a q_w \tilde{p}, \\ B(s, p) &= -\frac{1}{\rho_a} \frac{d\rho_a}{dp} (q_a G(s, p) + a^{-1}(s) k\lambda_w q_a (\nabla p_c - \tilde{p})), \\ f(s, p) &= \frac{1}{\rho_a} \frac{d\rho_a}{dp} k\lambda_w q_a (\nabla p_c - \tilde{p}) \cdot G(s, p) + \frac{f_a}{\rho_a} + \frac{f_w}{\rho_w} - \frac{\partial \phi}{\partial t}, \end{aligned}$$

equations (2.7) and (2.8) can now be written as

$$(2.9) \quad c(s, p) \frac{\partial p}{\partial t} + \nabla \cdot u = A(s, p) u^2 + B(s, p) \cdot u + f(s, p),$$

$$(2.10) \quad u = -a(s) (\nabla p - G(s, p)),$$

$$(2.11) \quad \phi \frac{\partial s}{\partial t} - \nabla \cdot (D(s) \nabla s - q_w u - b(s, p)) = \tilde{f}_w - s \frac{\partial \phi}{\partial t}.$$

The boundary conditions for the pressure-saturation equations become

$$(2.12) \quad p = p_D(x, t), \quad x \in \Gamma_1, t > 0,$$

$$(2.13) \quad u \cdot \nu = \tilde{d}(x, t), \quad x \in \Gamma_2, t > 0,$$

$$(2.14) \quad s = s_D(x, t), \quad x \in \Gamma_1, t > 0,$$

$$(2.15) \quad (D(s) \nabla s - q_w u - b(s, p)) \cdot \nu = -d_w(x, t), \quad x \in \Gamma_2, t > 0,$$

where  $s_D$  and  $p_D$  are the transforms of  $p_{wD}$  and  $p_{aD}$  by (2.2) and (2.3), and  $\tilde{d} = d_a + d_w$ .

The model given in equations (2.9)–(2.15) for the pressure  $p$ , velocity  $u$ , and saturation  $s$  is completed by specifying the initial conditions

$$(2.16) \quad p(x, 0) = p^0(x), \quad x \in \Omega,$$

$$(2.17) \quad s(x, 0) = s^0(x), \quad x \in \Omega.$$

The later analysis for the nondegenerate case in section 5 is given under a number of assumptions. First, the solution is assumed smooth; i.e., the external source terms are smoothly distributed, the coefficients are smooth, the boundary and initial data satisfy the compatibility condition, and the domain has at least the regularity required for a standard elliptic problem to have  $H^2(\Omega)$ -regularity and more if error estimates of order bigger than one are required. Second, the coefficients  $a(s)$ ,  $\phi$ , and  $c(s, p)$  are assumed bounded below positively:

$$(2.18) \quad 0 < a_* \leq a(s) \leq a^* < \infty,$$

$$(2.19) \quad 0 < \phi_* \leq \phi(x) \leq \phi^* < \infty,$$

$$(2.20) \quad 0 < c_* \leq c(s, p) \leq c^* < \infty.$$

The capillary diffusion coefficient  $D(s)$  is assumed to satisfy

$$(2.21) \quad 0 < D_* \leq D(s) \leq D^* < \infty.$$

Finally, an example of the air density function  $\rho_a$  is given by the relation [3]

$$(2.22) \quad \rho_a(p_a) = \rho_{0a} \left( 1 + \frac{p_a}{p_{0a}} \right),$$

where  $\rho_{0a}$  is the density of the air phase at the reference pressure  $p_{0a}$ .

While the phase mobilities can be zero, the total mobility is always positive [32]. The assumptions (2.18) and (2.19) are physically reasonable. Also, the present analysis obviously applies to the incompressible case where  $c(s, p) = 0$ . In this case, the analysis is simpler since we have an elliptic pressure equation instead of the parabolic equation (2.9). Thus we assume condition (2.20) for the compressible case under consideration. Next, although the reasonableness of the assumption (2.21) is discussed in [17], the diffusion coefficient  $D(s)$  can be zero in reality [13]. It is for this reason that section 6 is devoted to consideration of the case where the solution is not required smooth and the assumption (2.21) is removed. As a final remark, we mention that for the case where point sources and sinks occur in a porous medium, an argument was given in [23] for the incompressible miscible displacement problem and can be extended to the present case.

**3. Weak forms.** To handle the difficulty associated with the inhomogeneous Neumann boundary condition (2.13) in the analysis of the mixed finite element method, let  $\bar{d}$  be such that  $\bar{d} \cdot \nu = \tilde{d}$  and introduce the change of variable  $u = \tilde{u} + \bar{d}$  in equations (2.9)–(2.11). Then the homogeneous Neumann boundary condition holds for  $\tilde{u}$ . Thus, without loss of generality, we assume that  $\tilde{d} \equiv 0$ . To be compatible, we also require that this homogeneous condition holds when  $t = 0$ .

In the two-dimensional case, let

$$H(\text{div}, \Omega) = \{v \in (L^2(\Omega))^2 : \nabla \cdot v \in L^2(\Omega)\},$$

while it is accordingly defined in the three-dimensional case as follows:

$$H(\text{div}, \Omega) = \{v \in (L^2(\Omega))^3 : \nabla \cdot v \in L^2(\Omega)\}.$$

Also, set

$$V = \{v \in H(\text{div}, \Omega) : v \cdot \nu = 0 \text{ on } \Gamma_2\},$$

$$M = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_1\}.$$

The weak form of (2.9)–(2.11) on which the finite element procedure is based is given below. Let  $J = (0, T]$  ( $T > 0$ ) is the time interval of interest. The mixed formulation for the pressure is defined by seeking a pair of maps  $\{u, p\} : J \rightarrow V \times L^2(\Omega)$  such that

$$(3.1a) \quad (\alpha(s)u, v) - (\nabla \cdot v, p) = (G(s, p), v) - \langle p_D, v \cdot \nu \rangle_{\Gamma_1} \quad \forall v \in V,$$

$$(3.1b) \quad \left( c(s, p) \frac{\partial p}{\partial t}, \psi \right) + (\nabla \cdot u, \psi) \\ = (A(s, p)u^2 + B(s, p) \cdot u + f(s, p), \psi) \quad \forall \psi \in L^2(\Omega),$$

where  $\alpha(s) = a(s)^{-1}$ , the inner products  $(\cdot, \cdot)$  are to be interpreted to be in  $L^2(\Omega)$  or  $(L^2(\Omega))^d$ , as appropriate, and  $\langle \cdot, \cdot \rangle_{\Gamma_1}$  denotes the duality between  $H^{1/2}(\Gamma_1)$  and  $H^{-1/2}(\Gamma_1)$ . The weak form for the saturation  $s : J \rightarrow M + s_D$  is given by

$$(3.2) \quad \begin{aligned} & \left( \phi \frac{\partial s}{\partial t}, v \right) + (D(s)\nabla s - q_w(s)u - b(s, p), \nabla v) \\ & = \left( \tilde{f}_w - s \frac{\partial \phi}{\partial t}, v \right) - \langle d_w, v \rangle_{\Gamma_2} \quad \forall v \in M, \end{aligned}$$

where the boundary condition (2.15) is used. Finally, to treat the nonzero initial conditions imposed on  $s$  and  $p$  in (2.16) and (2.17), we introduce the following transformations in (3.1) and (3.2):

$$\begin{aligned} s(x, t) &= \underline{s}(x, t) + s^0(x), \\ p(x, t) &= \underline{p}(x, t) + p^0(x), \\ u(x, t) &= \underline{u}(x, t) + u^0(x), \end{aligned}$$

where  $u^0 = -a(s^0)(\nabla p^0 - G(s^0, p^0))$  and  $\underline{u} = -a(\underline{s} + s^0)(\nabla(\underline{p} + p^0) - G(\underline{s} + s^0, \underline{p} + p^0)) - u^0(x)$ . Then we have zero initial conditions for  $\underline{s}$ ,  $\underline{p}$ , and  $\underline{u}$ . Hence, without loss of generality again, we assume that

$$(3.3) \quad s^0 = p^0 = u^0 \equiv 0.$$

The reason for introducing these transformations to have zero initial conditions is to validate equation (5.15) later.

**4. Fully discrete finite element procedures.** Let  $\Omega$  be a polygonal domain. For  $0 < h_p < 1$  and  $0 < h < 1$ , let  $T_{h_p}$  and  $T_h$  be quasi-uniform partitions into elements, say, simplexes, rectangular parallelepipeds, and/or prisms. In both partitions, we also need that adjacent elements completely share their common edge or face. Let  $M_h \subset W^{1,\infty}(\Omega) \cap M$  be a standard  $C^0$ -finite element space associated with  $T_h$  such that

$$(4.1) \quad \inf_{\psi \in M_h} \|v - \psi\|_{1,q} \leq C \left( \sum_K h_K^{2k} \|v\|_{k+1,q,K}^2 \right)^{1/2}, \quad k \geq 1, 1 \leq q \leq \infty,$$

where  $h_K = \text{diam}(K)$ ,  $K \in T_h$  and  $\|v\|_{k,q,K}$  is the norm in the Sobolev space  $W^{k,q}(K)$  (we omit  $K$  when  $K = \Omega$  and  $\|v\|_{k,K} = \|v\|_{k,2,K}$ ). Also, let  $V_h \times W_h = V_{h_p} \times W_{h_p} \subset V \times L^2(\Omega)$  be the Raviart–Thomas–Nedelec [35], [30], the Brezzi–Douglas–Fortin–Marini [5], the Brezzi–Douglas–Marini [6] (if  $d = 2$ ), the Brezzi–Douglas–Durán–Fortin [4] (if  $d = 3$ ), or the Chen–Douglas [11] mixed finite element space associated with the partition  $T_{h_p}$  of index such that the approximation properties below are satisfied:

$$(4.2) \quad \inf_{\psi \in V_h} \|v - \psi\| \leq C \left( \sum_K h_{p,K}^{2r} \|v\|_{r,K}^2 \right)^{1/2}, \quad 0 \leq r \leq k^* + 1,$$

$$(4.3) \quad \inf_{\psi \in V_h} \|\nabla \cdot (v - \psi)\| \leq C \left( \sum_K h_{p,K}^{2r} \|\nabla \cdot v\|_{r,K}^2 \right)^{1/2}, \quad 0 \leq r \leq k^{**},$$

$$(4.4) \quad \inf_{\psi \in W_h} \|w - \psi\| \leq C \left( \sum_K h_{p,K}^{2r} \|w\|_{r,K}^2 \right)^{1/2}, \quad 0 \leq r \leq k^{**},$$

where  $h_{p,K} = \text{diam}(K)$ ,  $K \in T_{h_p}$ ,  $\|v\| = \|v\|_0$ ,  $k^{**} = k^* + 1$  for the first two spaces,  $k^{**} = k^*$  for the second two spaces, and both cases are included in the last space. Finally, let  $\{t^n\}_{n=0}^{n_T}$  be a quasi-uniform partition of  $J$  with  $t^0 = 0$  and  $t^{n_T} = T$ , and set  $\Delta t^n = t^n - t^{n-1}$ ,  $\Delta t = \max\{\Delta t^n, 1 \leq n \leq n_T\}$ , and

$$\psi^n = \psi(t^n), \quad \partial \psi^n = (\psi^n - \psi^{n-1})/\Delta t^n.$$

We are now in a position to introduce our finite element procedure.

The fully discrete finite element method is given as follows. The approximation procedure for the pressure is defined by the mixed method for a pair of maps  $\{u_h^n, p_h^n\} \in V_h \times W_h$ ,  $n = 1, 2, \dots, n_T$ , such that

$$(4.5a) \quad (\alpha(s_h^{n-1})u_h^n, v) - (\nabla \cdot v, p_h^n) = (G(s_h^{n-1}, p_h^{n-1}), v) - \langle p_D^n, v \cdot \nu \rangle_{\Gamma_1} \quad \forall v \in V_h,$$

$$(4.5b) \quad (c(s_h^{n-1}, p_h^{n-1})\partial p_h^n, \psi) + (\nabla \cdot u_h^n, \psi) = (A(s_h^{n-1}, p_h^{n-1})(u_h^{n-1})^2 + B(s_h^{n-1}, p_h^{n-1}) \cdot u_h^{n-1} + f(s_h^{n-1}, p_h^{n-1}), \psi) \quad \forall \psi \in W_h,$$

and the finite element method for the saturation is given for  $s_h^n \in M_h + s_D^n$ ,  $n = 1, 2, \dots, n_T$ , satisfying

$$(4.6) \quad (\phi \partial s_h^n, v) + (D(s_h^{n-1})\nabla s_h^n - q_w(s_h^{n-1})u_h^n - b(s_h^{n-1}, p_h^n), \nabla v) = \left( \tilde{f}_w^n - s_h^n \frac{\partial \phi^n}{\partial t}, v \right) - \langle d_w^n, v \rangle_{\Gamma_2} \quad \forall v \in M_h.$$

The initial conditions satisfy

$$(4.7) \quad p_h^0 = 0, \quad s_h^0 = 0, \quad u_h^0 = 0.$$

After startup, for  $n = 1, 2, \dots, n_T$ , equations (4.5) and (4.6) are computed as follows. First, using  $s_h^{n-1}$ ,  $p_h^{n-1}$ , and (4.5), evaluate  $\{u_h^n, p_h^n\}$ . Since it is linear, (4.5) has a unique solution for each  $n$  [10], [28]. Next, using  $s_h^{n-1}$ ,  $\{u_h^n, p_h^n\}$ , and (4.6), calculate  $s_h^n$ . Again, (4.6) has a unique solution for  $\Delta t^n$  sufficiently small for each  $n$  [40].

We end this section with two remarks. First, while the backward Euler scheme is used for the time discretization terms in (4.5b) and (4.6), the Crank–Nicolson scheme and more accurate time stepping procedures (see, e.g., [22]) can be used, and the present analysis applies to these schemes. Second, the nonlinearities in the pressure and saturation equations are handled by lagging in time. Consequently, a linear system of algebraic equations is solved at each time step instead of a nonlinear system. We point out that the analysis below extends to the nonlinear version where we use  $s_h^n$ ,  $p_h^n$ , and  $u_h^n$  in the coefficients of equations (4.5) and (4.6) instead of  $s_h^{n-1}$ ,  $p_h^{n-1}$ , and  $u_h^{n-1}$  (see the scheme (6.7) in section 6). In this case the time step  $\Delta t$  in the condition (5.28) below would disappear.

**5. An error analysis for the fully discrete scheme.** In this section we give a convergence analysis for the finite element procedure (4.5) and (4.6) under assumption (2.21). As usual, it is convenient to use an elliptic projection of the solution of (2.11) into the finite element space  $M_h$ . Let  $\tilde{s} = \tilde{s}_h : J \rightarrow M_h + s_D$  be defined by

$$(5.1) \quad (D(s)\nabla(s - \tilde{s}), \nabla v) + (s - \tilde{s}, v) = 0 \quad \forall v \in M_h, \quad t \in J.$$

Set

$$(5.2) \quad \zeta = s - \tilde{s}, \quad \xi = \tilde{s} - s_h.$$



Then it follows from standard results of the finite element method [16], [31], [38] that

$$(5.3a) \quad \|\zeta\| + h\|\zeta\|_1 \leq C \left( \sum_K h_K^{2(k+1)} \|s\|_{k+1,K}^2 \right)^{1/2},$$

$$(5.3b) \quad \|\zeta\|_{0,\infty} \leq Ch^{k+1} (\log h^{-1})^\gamma \|s\|_{k+1,\infty},$$

where  $\gamma = 1$  for  $k = 1$  and  $\gamma = 0$  for  $k > 1$ . The same result applies to the time-differentiated forms of (5.1) [41]:

$$(5.4) \quad \left\| \frac{\partial \zeta}{\partial t} \right\| + h \left\| \frac{\partial \zeta}{\partial t} \right\|_1 \leq C \left( \sum_K h_K^{2(k+1)} \left( \|s\|_{k+1,K}^2 + \left\| \frac{\partial s}{\partial t} \right\|_{k+1,K}^2 \right) \right)^{1/2}.$$

As for the analysis of the mixed finite element method, we use the following two projections instead of the elliptic projections introduced in [15] and [19]. So the present analysis is different from and in fact simpler than those in [15] and [19]. Each of our mixed finite element spaces [4]–[6], [11], [30], [35] has the property that there are projection operators  $\Pi_h : H^1(\Omega) \rightarrow V_h$  and  $P_h = L^2$ -projection:  $L^2(\Omega) \rightarrow W_h$  such that

$$(5.5) \quad \|v - \Pi_h v\| \leq C \left( \sum_K h_{p,K}^{2r} \|v\|_{r,K}^2 \right)^{1/2}, \quad 0 \leq r \leq k^* + 1,$$

$$(5.6) \quad \|\nabla \cdot (v - \Pi_h v)\| \leq C \left( \sum_K h_{p,K}^{2r} \|\nabla \cdot v\|_{r,K}^2 \right)^{1/2}, \quad 0 \leq r \leq k^{**},$$

$$(5.7) \quad \|w - P_h w\| \leq C \left( \sum_K h_{p,K}^{2r} \|w\|_{r,K}^2 \right)^{1/2}, \quad 0 \leq r \leq k^{**},$$

and (see, e.g., [9], [20])

$$(5.8) \quad (\nabla \cdot (v - \Pi_h v), w) = 0 \quad \forall w \in W_h,$$

$$(5.9) \quad (\nabla \cdot v, w - P_h w) = 0 \quad \forall v \in V_h.$$

Set  $\tilde{p} = P_h p$ ,  $\tilde{u} = \Pi_h u$ , and

$$(5.10) \quad \sigma = u - \tilde{u}, \quad \beta = \tilde{u} - u_h,$$

$$(5.11) \quad \eta = p - \tilde{p}, \quad \theta = \tilde{p} - p_h.$$

Note that, by (3.3) and (4.7),

$$(5.12) \quad \theta^0 = 0, \quad \xi^0 = 0, \quad \beta^0 = 0.$$

Finally, we prove some bounds of the projections  $\tilde{s}$  and  $\tilde{p}$ . Let  $\bar{s} = \bar{s}_h$  be the interpolant of  $s$  in  $M_h$ . Then we see, by (4.1), (5.3b), the approximation property of  $\bar{s}$ , and an inverse inequality in  $M_h$ , that

$$\begin{aligned} \|\tilde{s}\|_{1,\infty} &\leq \|s - \tilde{s}\|_{1,\infty} + \|s\|_{1,\infty} \\ &\leq \|\bar{s} - \tilde{s}\|_{1,\infty} + \|s - \bar{s}\|_{1,\infty} + \|s\|_{1,\infty} \\ &\leq Ch^{-1} \|\bar{s} - \tilde{s}\|_{0,\infty} + \|s - \bar{s}\|_{1,\infty} + \|s\|_{1,\infty} \\ &\leq Ch^{-1} (\|\tilde{s} - s\|_{0,\infty} + \|s - \bar{s}\|_{0,\infty}) + \|s - \bar{s}\|_{1,\infty} + \|s\|_{1,\infty} \\ &\leq Ch^k (\log h^{-1})^\gamma \|s\|_{k+1,\infty} + \|s\|_{1,\infty}, \end{aligned}$$

where  $\gamma$  is given as in (5.3b). This implies that  $\|\tilde{s}\|_{1,\infty}$  is bounded for sufficiently smooth solutions since  $k \geq 1$ . The same argument applies to  $\|\partial\tilde{s}/\partial t\|_{1,\infty}$ . Next, note that, by the approximation property of the projection  $P_h$  [28],

$$\|\tilde{p}_t\|_{0,\infty} \leq C\|p_t\|_{0,\infty}.$$

These bounds on  $\tilde{p}_t$ ,  $\nabla\tilde{s}$ , and  $\nabla(\partial\tilde{s}/\partial t)$  are used below.

We are now ready to prove some results. Below  $\varepsilon$  is a generic positive constant as small as we please.

**5.1. Analysis of the mixed method.** We first analyze the mixed method (4.5). We set  $s = p = u \equiv 0$  and  $s_h = p_h = u_h \equiv 0$  when  $t \leq 0$ . The following error equation is obtained by subtracting (4.5) from (3.1) at  $t = t^n$  and applying (5.8) and (5.9):

$$(5.13) \quad (\alpha(s_h^{n-1})\beta^n, v) - (\nabla \cdot v, \theta^n) = ((\alpha(s_h^{n-1}) - \alpha(s^n))u^n, v) - (\alpha(s_h^{n-1})\sigma^n, v) \\ + (G(s^n, p^n) - G(s_h^{n-1}, p_h^{n-1}), v) \quad \forall v \in V_h,$$

$$(5.14) \quad (c(s_h^{n-1}, p_h^{n-1})\partial\theta^n, \psi) + (\nabla \cdot \beta^n, \psi) = (f(s^n, p^n) - f(s_h^{n-1}, p_h^{n-1}), \psi) \\ + (A(s^n, p^n)(u^n)^2 - A(s_h^{n-1}, p_h^{n-1})(u_h^{n-1})^2, \psi) \\ + (B(s^n, p^n) \cdot u^n - B(s_h^{n-1}, p_h^{n-1}) \cdot u_h^{n-1}, \psi) \\ + \left( (c(s_h^{n-1}, p_h^{n-1}) - c(s^n, p^n)) \frac{\partial p^n}{\partial t}, \psi \right) \\ - (c(s_h^{n-1}, p_h^{n-1}) \left( \frac{\partial p^n}{\partial t} - \partial\tilde{p}^n \right), \psi) \quad \forall \psi \in W_h.$$

Below  $C_i$  indicates a generic constant with the given dependencies.

LEMMA 5.1. *Let  $(u, p)$  and  $(u_h, p_h)$  solve (3.1) and (4.5), respectively. Then*

$$(5.15) \quad \|\partial\theta^1\|^2 + \Delta t^1 \|\partial\beta^1\|^2 \\ \leq C_0 \{ \Delta t^1 (\|s^1 - s^0\|^2 + \|\partial\sigma^1\|^2 + \|\partial G^1\|^2) \\ + \|p^1 - p^0\|^2 + \|s^1 - s^0\|^2 + \left\| \frac{\partial p^1}{\partial t} - \partial\tilde{p}^1 \right\|^2 + \|u^1 - u^0\|^2 \},$$

where  $\partial G^1 = (G(s^1, p^1) - G(s^0, p^0))/\Delta t^1$  and

$$C_0 = C_0 \left( \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(J \times \Omega)}, \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(J \times \Omega)}, \|u\|_{L^\infty(J \times \Omega)} \right).$$

*Proof.* Set  $v = \beta^1$  in (5.13) and  $\psi = \theta^1$  in (5.14), add the resulting equations at  $n = 1$ , and use (3.3), (4.7), and (5.12) to see that

$$(c(s^0, p^0)\partial\theta^1, \partial\theta^1) + \Delta t^1 (\alpha(s^0)\partial\beta^1, \partial\beta^1) = \sum_{i=1}^8 T_i^1,$$

where

$$T_1^1 = ((\alpha(s^0) - \alpha(s^1))(u^1 - u^0), \partial\beta^1), \quad T_2^1 = -(\alpha(s^0)(\sigma^1 - \sigma^0), \partial\beta^1), \\ T_3^1 = (G(s^1, p^1) - G(s^0, p^0), \partial\beta^1), \quad T_4^1 = (A(s^1, p^1)(u^1)^2 - A(s^0, p^0)(u^0)^2, \partial\theta^1), \\ T_5^1 = \left( (c(s^0, p^0) - c(s^1, p^1)) \frac{\partial p^1}{\partial t}, \partial\theta^1 \right), \quad T_6^1 = -(c(s^0, p^0) \left( \frac{\partial p^1}{\partial t} - \partial\tilde{p}^1 \right), \partial\theta^1), \\ T_7^1 = (B(s^1, p^1) \cdot u^1 - B(s^0, p^0) \cdot u^0, \partial\theta^1), \quad T_8^1 = (f(s^1, p^1) - f(s^0, p^0), \partial\theta^1).$$

Then (5.15) can be easily seen.  $\square$

LEMMA 5.2. *Let  $(u, p)$  and  $(u_h, p_h)$  satisfy (3.1) and (4.5), respectively. Then*

$$\begin{aligned} & \|\partial\theta^\gamma\|^2 + \sum_{n=2}^\gamma \|\partial\beta^n\|^2 \Delta t^n \\ & \leq C_1 \left\{ \|\partial\theta^1\|^2 + \sum_{n=1}^\gamma \left\{ \|\partial(\partial\eta^n)\|^2 + \|\partial\eta^n\|^2 + \|\eta^{n-1}\|^2 + \|\partial\zeta^{n-1}\|^2 \right. \right. \\ & \quad + \|\zeta^{n-1}\|^2 + \|\partial\sigma^n\|^2 + \left\| \partial \left( \frac{\partial p^n}{\partial t} - \partial p^n \right) \right\|^2 + \|\partial\theta^n\|^2 \\ & \quad + \|\partial(p^n - p^{n-1})\|^2 + \|p^n - p^{n-1}\|^2 + \|\partial(s^n - s^{n-1})\|^2 \\ & \quad + \|s^n - s^{n-1}\|^2 + \|\theta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\xi^{n-1}\|^2 \\ & \quad + \|\partial(u^n - u^{n-1})\|^2 + \|u^n - u^{n-1}\|^2 \\ & \quad + (\|\partial\xi^{n-1}\|_{0,\infty}^2 + \|\partial\theta^{n-1}\|_{0,\infty}^2) \left[ \|\partial\theta^{n-1}\|^2 + \left\| \frac{\partial p^n}{\partial t} - \partial p^n \right\|^2 \right. \\ & \quad \left. \left. + \|\eta^n\|^2 + (1 + \|\partial\beta^{n-1}\|_{0,\infty}^2) \|\partial\beta^{n-1}\|^2 \right] \right. \\ & \quad \left. + (1 + \|\partial\xi^{n-1}\|_{0,\infty}^2 + \|\partial\beta^{n-1}\|_{0,\infty}^2) (\|\beta^{n-1}\|^2 + \|\beta^{n-2}\|^2 + \|\sigma^n\|^2) \right\} \Delta t^n \end{aligned}$$

for  $2 \leq \gamma \leq n_T$ , where

$$C_1 = C_1 \left( \left\| \frac{\partial s}{\partial t} \right\|_{L^\infty(J \times \Omega)}, \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(J \times \Omega)}, \left\| \frac{\partial^2 p}{\partial t^2} \right\|_{L^\infty(J \times \Omega)}, \|u\|_{L^\infty(J \times \Omega)}, \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(J \times \Omega)} \right).$$

*Proof.* Using difference equations (5.13) and (5.14) with respect to  $n$ , set  $v = \partial\beta^n$  and  $\psi = \partial\theta^n$  in the resulting equations, divide by  $\Delta t^n$ , and add to obtain

$$(5.16) \quad (\alpha(s_h^{n-1})\partial\beta^n, \partial\beta^n) + (c(s_h^{n-1}, p_h^{n-1})\partial(\partial\theta^n), \partial\theta^n) = \sum_{i=1}^{10} T_i^n,$$

where

$$T_1^n = \frac{1}{\Delta t^n} ((f(s^n, p^n) - f(s_h^{n-1}, p_h^{n-1})) - (f(s^{n-1}, p^{n-1}) - f(s_h^{n-2}, p_h^{n-2})), \partial\theta^n),$$

$$\begin{aligned} T_2^n &= \frac{1}{\Delta t^n} ([A(s^n, p^n)(u^n)^2 - A(s_h^{n-1}, p_h^{n-1})(u_h^{n-1})^2] \\ & \quad - [A(s^{n-1}, p^{n-1})(u^{n-1})^2 - A(s_h^{n-2}, p_h^{n-2})(u_h^{n-2})^2], \partial\theta^n), \end{aligned}$$

$$\begin{aligned} T_3^n &= \frac{1}{\Delta t^n} ([B(s^n, p^n) \cdot u^n - B(s_h^{n-1}, p_h^{n-1}) \cdot u_h^{n-1}] \\ & \quad - [B(s^{n-1}, p^{n-1}) \cdot u^{n-1} - B(s_h^{n-2}, p_h^{n-2}) \cdot u_h^{n-2}], \partial\theta^n), \end{aligned}$$

$$\begin{aligned} T_4^n &= \frac{1}{\Delta t^n} \left( (c(s_h^{n-1}, p_h^{n-1}) - c(s^n, p^n)) \frac{\partial p^n}{\partial t} \right. \\ & \quad \left. - (c(s_h^{n-2}, p_h^{n-2}) - c(s^{n-1}, p^{n-1})) \frac{\partial p^{n-1}}{\partial t}, \partial\theta^n \right), \end{aligned}$$

$$\begin{aligned}
T_5^n &= -\frac{1}{\Delta t^n} \left( c(s_h^{n-1}, p_h^{n-1}) \left( \frac{\partial p^n}{\partial t} - \partial \bar{p}^n \right) \right. \\
&\quad \left. - c(s_h^{n-2}, p_h^{n-2}) \left( \frac{\partial p^{n-1}}{\partial t} - \partial \bar{p}^{n-1} \right), \partial \theta^n \right), \\
T_6^n &= \frac{1}{\Delta t^n} ((c(s_h^{n-2}, p_h^{n-2}) - c(s_h^{n-1}, p_h^{n-1})) \partial \theta^{n-1}, \partial \theta^n), \\
T_7^n &= \frac{1}{\Delta t^n} ((\alpha(s_h^{n-2}) - \alpha(s_h^{n-1})) \beta^{n-1}, \partial \beta^n), \\
T_8^n &= -\frac{1}{\Delta t^n} (\alpha(s_h^{n-1}) \sigma^n - \alpha(s_h^{n-2}) \sigma^{n-1}, \partial \beta^n), \\
T_9^n &= \frac{1}{\Delta t^n} ((\alpha(s_h^{n-1}) - \alpha(s_h^n)) u^n - (\alpha(s_h^{n-2}) - \alpha(s_h^{n-1})) u^{n-1}, \partial \beta^n), \\
T_{10}^n &= \frac{1}{\Delta t^n} ((G(s_h^n, p_h^n) - G(s_h^{n-1}, p_h^{n-1})) \\
&\quad - (G(s_h^{n-1}, p_h^{n-1}) - G(s_h^{n-2}, p_h^{n-2})), \partial \beta^n).
\end{aligned}$$

Observe that the left-hand side of (5.16) is larger than the quantity

$$(5.17) \quad \frac{1}{2\Delta t^n} \{ (c(s_h^{n-1}, p_h^{n-1}) \partial \theta^n, \partial \theta^n) - (c(s_h^{n-2}, p_h^{n-2}) \partial \theta^{n-1}, \partial \theta^{n-1}) \} \\
+ (\alpha(s_h^{n-1}) \partial \beta^n, \partial \beta^n) + T_{11}^n,$$

where

$$T_{11}^n = \frac{1}{2\Delta t^n} ((c(s_h^{n-2}, p_h^{n-2}) - c(s_h^{n-1}, p_h^{n-1})) \partial \theta^{n-1}, \partial \theta^{n-1}).$$

We estimate the new term  $T_2^n$  in detail. Other terms can be bounded by a simpler argument. To estimate  $T_2^n$ , we write

$$\begin{aligned}
T_2^n &= \frac{1}{\Delta t^n} (\{ [A(s_h^n, p_h^n) - A(s_h^{n-1}, p_h^{n-1})] \\
&\quad - [A(s_h^{n-1}, p_h^{n-1}) - A(s_h^{n-2}, p_h^{n-2})] \} (u^n)^2, \partial \theta^n) \\
&\quad + \frac{1}{\Delta t^n} ([A(s_h^{n-1}, p_h^{n-1}) - A(s_h^{n-2}, p_h^{n-2})] ((u^n)^2 - (u_h^{n-1})^2), \partial \theta^n) \\
&\quad + \frac{1}{\Delta t^n} ([A(s_h^{n-1}, p_h^{n-1}) - A(s_h^{n-2}, p_h^{n-2})] ((u^n)^2 - (u^{n-1})^2), \partial \theta^n) \\
&\quad + \frac{1}{\Delta t^n} (A(s_h^{n-2}, p_h^{n-2}) \{ [(u^n)^2 - (u^{n-1})^2] - [(u_h^{n-1})^2 - (u_h^{n-2})^2] \}, \partial \theta^n) \\
&\equiv \sum_{i=1}^4 T_{2,i}^n.
\end{aligned}$$

Note that

$$\begin{aligned}
&[A(s_h^{n-1}, p_h^{n-1}) - A(s_h^{n-2}, p_h^{n-2})] - [A(s_h^n, p_h^n) - A(s_h^{n-1}, p_h^{n-1})] \\
&= \frac{\partial A}{\partial s}(\widehat{s}_h^{n-1}, p_h^{n-1})(s_h^{n-1} - s_h^{n-2}) + \frac{\partial A}{\partial p}(s_h^{n-1}, \widehat{p}_h^{n-1})(p_h^{n-1} - p_h^{n-2}) \\
&\quad - \frac{\partial A}{\partial s}(\widehat{s}_h^n, p_h^n)(s_h^n - s_h^{n-1}) - \frac{\partial A}{\partial p}(s_h^{n-1}, \widehat{p}_h^n)(p_h^n - p_h^{n-1}),
\end{aligned}$$

where

$$\begin{aligned} \min\{p_h^{n-1}, p_h^{n-2}\} &\leq \widehat{p}_h^{n-1} \leq \max\{p_h^{n-1}, p_h^{n-2}\}, \\ \min\{p^n, p^{n-1}\} &\leq \widehat{p}^n \leq \max\{p^n, p^{n-1}\}, \end{aligned}$$

and similar inequalities hold for  $\widehat{s}_h^{n-1}, \widehat{s}^n$ . Consequently, with  $\lambda^n = \Delta t^{n-1}/\Delta t^n$  we see that

$$\begin{aligned} T_{2,1}^n &= - \left( \frac{\partial A}{\partial s} \{ \lambda^n [\partial \zeta^{n-1} + \partial \xi^{n-1}] - \partial (s^n - s^{n-1}) \} (u^n)^2, \partial \theta^n \right) \\ &\quad - \left( \left[ \frac{\partial^2 A}{\partial s^2} (\widehat{s}_h^{n-1} - \widehat{s}^n) + \frac{\partial^2 A}{\partial p \partial s} (p_h^{n-1} - p^n) \right] (u^n)^2 \frac{s^n - s^{n-1}}{\Delta t^n}, \partial \theta^n \right) \\ &\quad - \left( \frac{\partial A}{\partial p} \{ \lambda^n [\partial \eta^{n-1} + \partial \theta^{n-1}] - \partial (p^n - p^{n-1}) \} (u^n)^2, \partial \theta^n \right) \\ &\quad - \left( \left[ \frac{\partial^2 A}{\partial p^2} (\widehat{p}_h^{n-1} - \widehat{p}^n) + \frac{\partial^2 A}{\partial s \partial p} (s_h^{n-2} - s^{n-1}) \right] (u^n)^2 \frac{p^n - p^{n-1}}{\Delta t^n}, \partial \theta^n \right), \end{aligned}$$

so that

$$\begin{aligned} |T_{2,1}^n| &\leq C_1 (\|\partial \zeta^{n-1}\|^2 + \|\partial \xi^{n-1}\|^2 + \|\partial (s^n - s^{n-1})\|^2 + \|\widehat{s}_h^{n-1} - \widehat{s}^n\|^2 \\ (5.18) \quad &\quad + \|p_h^{n-1} - p^n\|^2 + \|\partial \eta^{n-1}\|^2 + \|\partial \theta^{n-1}\|^2 + \|\partial (p^n - p^{n-1})\|^2 \\ &\quad + \|\widehat{p}_h^{n-1} - \widehat{p}^n\|^2 + \|s_h^{n-2} - s^{n-1}\|^2 + \|\partial \theta^n\|^2), \end{aligned}$$

where

$$\begin{aligned} \|\widehat{p}_h^{n-1} - \widehat{p}^n\| &\leq C_1 (\|p^n - p_h^{n-1}\| + \|p^n - p_h^{n-2}\| \\ (5.19) \quad &\quad + \|p^{n-1} - p_h^{n-1}\| + \|p^{n-1} - p_h^{n-2}\|), \end{aligned}$$

and an analogous inequality holds for  $\widehat{s}_h^{n-1} - \widehat{s}^n$ . Also, we see that

$$\begin{aligned} &[A(s_h^{n-1}, p_h^{n-1}) - A(s_h^{n-2}, p_h^{n-2})] ((u^n)^2 - (u_h^{n-1})^2) \\ &= \left\{ \frac{\partial A}{\partial s} (s_h^{n-1} - s_h^{n-2}) + \frac{\partial A}{\partial p} (p_h^{n-1} - p_h^{n-2}) \right\} (u^n - u_h^{n-1}) \cdot (u^n + u_h^{n-1}), \end{aligned}$$

which implies that

$$\begin{aligned} |T_{2,2}^n| &\leq C_1 ((1 + \|\partial \xi^{n-1}\|_{0,\infty}^2 + \|\partial \theta^{n-1}\|_{0,\infty}^2) (1 + \|\beta^{n-1}\|_{0,\infty}^2) \|\beta^{n-1}\|^2 \\ &\quad + \|s_h^{n-1} - s_h^{n-2}\|^2 + \|p_h^{n-1} - p_h^{n-2}\|^2 + \|\partial \theta^n\|^2). \end{aligned}$$

Next, it can be easily seen that

$$|T_{2,3}^n| \leq C_1 (\|s^{n-1} - s_h^{n-2}\|^2 + \|p^{n-1} - p_h^{n-2}\|^2 + \|\partial \theta^n\|^2).$$

Finally, since

$$\begin{aligned} &[(u_h^{n-1})^2 - (u_h^{n-2})^2] - [(u^n)^2 - (u^{n-1})^2] \\ &= ([u_h^{n-1} - u_h^{n-2}] - [u^{n-1} - u^{n-2}]) (u_h^{n-1} + u^{n-1}) \\ &\quad + (u^{n-2} - u_h^{n-2}) ([u_h^{n-2} - u_h^{n-1}] + [u^{n-2} - u^{n-1}]) \\ &\quad + ([u^{n-1} - u^{n-2}] - [u^n - u^{n-1}]) (u^{n-1} + u^{n-2}) \\ &\quad + (u^n - u^{n-1}) (u^n - u^{n-2}), \end{aligned}$$

we find that

$$\begin{aligned} |T_{2,4}^n| &\leq C_1 \left( (1 + \|\partial\beta^{n-1}\|_{0,\infty}^2) (\|\beta^{n-1}\|^2 + \|\beta^{n-2}\|^2) \right. \\ &\quad + \|\partial\sigma^{n-1}\|^2 + \|\partial(u^n - u^{n-1})\|^2 + \|u^n - u^{n-1}\|^2 \\ &\quad \left. + \|\sigma^{n-2}\|^2 + \|\partial\theta^n\|^2 \right) + \varepsilon \|\partial\beta^{n-1}\|^2. \end{aligned}$$

Hence  $T_2^n$  can be bounded in terms of  $T_{2,i}^n$ ,  $i = 1, \dots, 4$ . Other terms are bounded as follows:

$$\begin{aligned} |T_1^n| &\leq C_1 \left( \|\partial\zeta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\partial(s^n - s^{n-1})\|^2 + \|\widehat{s}_{1,h}^{n-1} - \widehat{s}_1^n\|^2 \right. \\ &\quad + \|p_h^{n-1} - p^n\|^2 + \|\partial\eta^{n-1}\|^2 + \|\partial\theta^{n-1}\|^2 + \|\partial(p^n - p^{n-1})\|^2 \\ &\quad \left. + \|\widehat{p}_{1,h}^{n-1} - \widehat{p}_1^n\|^2 + \|s_h^{n-2} - s^{n-1}\|^2 + \|\partial\theta^n\|^2 \right), \\ |T_3^n| &\leq C_1 \left( \|\partial\zeta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\partial(s^n - s^{n-1})\|^2 + \|\widehat{s}_{2,h}^{n-1} - \widehat{s}_2^n\|^2 \right. \\ &\quad + \|p_h^{n-1} - p^n\|^2 + \|\partial\eta^{n-1}\|^2 + \|\partial\theta^{n-1}\|^2 + \|\partial(p^n - p^{n-1})\|^2 \\ &\quad + \|\widehat{p}_{2,h}^{n-1} - \widehat{p}_2^n\|^2 + \|s_h^{n-2} - s^{n-1}\|^2 + \|\partial\theta^n\|^2 \\ &\quad + (1 + \|\partial\xi^{n-1}\|_{0,\infty}^2 + \|\partial\theta^{n-1}\|_{0,\infty}^2) \|\beta^{n-1}\|^2 \\ &\quad + \|s^{n-1} - s_h^{n-2}\|^2 + \|p^{n-1} - p_h^{n-2}\|^2 + \|\partial\sigma^{n-1}\|^2 \\ &\quad \left. + \|\partial(u^n - u^{n-1})\|^2 \right) + \varepsilon \|\partial\beta^{n-1}\|^2, \\ |T_4^n| &\leq C_1 \left( \|s^n - s_h^{n-1}\|^2 + \|p^n - p_h^{n-1}\|^2 + \|\partial\theta^n\|^2 + \|\partial\zeta^{n-1}\|^2 \right. \\ &\quad + \|\partial\eta^{n-1}\|^2 + \|\partial(p^n - p^{n-1})\|^2 + \|\partial(s^n - s^{n-1})\|^2 \\ &\quad \left. + \|\partial\theta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\widehat{p}_{3,h}^{n-1} - \widehat{p}_3^n\|^2 + \|\widehat{s}_{3,h}^{n-1} - \widehat{s}_3^n\|^2 \right), \\ |T_5^n| &\leq C_1 \left( \left( \|\partial\eta^n\|^2 + \left\| \frac{\partial p^n}{\partial t} - \partial p^n \right\|^2 \right) (\|\partial\xi^{n-1}\|_{0,\infty}^2 + \|\partial\theta^{n-1}\|_{0,\infty}^2) \right. \\ &\quad \left. + \left\| \partial \left( \frac{\partial p^n}{\partial t} - \partial p^n \right) \right\|^2 + \|\partial(\partial\eta^n)\|^2 + \|\partial\theta^n\|^2 \right), \\ |T_6^n| &\leq C_1 \left( (1 + \|\partial\xi^{n-1}\|_{0,\infty}^2 + \|\partial\theta^{n-1}\|_{0,\infty}^2) \|\partial\theta^{n-1}\|^2 + \|\partial\theta^n\|^2 \right), \\ |T_7^n| &\leq C_1 \left( 1 + \|\partial\xi^{n-1}\|_{0,\infty}^2 \right) \|\beta^{n-1}\|^2 + \varepsilon \|\partial\beta^n\|^2, \\ |T_8^n| &\leq C_1 \left( (1 + \|\partial\xi^{n-1}\|_{0,\infty}^2) \|\sigma^n\|^2 + \|\partial\sigma^n\|^2 \right) + \varepsilon \|\partial\beta^n\|^2, \\ |T_9^n| &\leq C_1 \left( \|\partial\zeta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\partial(s^n - s^{n-1})\|^2 \right. \\ &\quad \left. + \|s^n - s_h^{n-1}\|^2 + \|\widehat{s}_{4,h}^{n-1} - \widehat{s}_4^n\|^2 \right) + \varepsilon \|\partial\beta^n\|^2, \\ |T_{10}^n| &\leq C_1 \left( \|\partial\zeta^{n-1}\|^2 + \|\partial\xi^{n-1}\|^2 + \|\partial(s^n - s^{n-1})\|^2 + \|s_h^{n-2} - s^{n-1}\|^2 \right. \\ &\quad + \|\partial\eta^{n-1}\|^2 + \|\partial\theta^{n-1}\|^2 + \|\partial(p^n - p^{n-1})\|^2 + \|p_h^{n-1} - p^n\|^2 \\ &\quad \left. + \|\widehat{p}_{5,h}^{n-1} - \widehat{p}_5^n\|^2 + \|\widehat{s}_{5,h}^{n-1} - \widehat{s}_5^n\|^2 + \varepsilon \|\partial\beta^n\|^2 \right), \\ |T_{11}^n| &\leq C_1 \left( 1 + \|\partial\xi^{n-1}\|_{0,\infty}^2 + \|\partial\theta^{n-1}\|_{0,\infty}^2 \right) \|\partial\theta^{n-1}\|^2, \end{aligned}$$

where  $\widehat{s}_{i,h}^{n-1} - \widehat{s}_i^n$  and  $\widehat{p}_{i,h}^{n-1} - \widehat{p}_i^n$  ( $i = 1, \dots, 5$ ) can be bounded as in (5.19), e.g.,

$$(5.20) \quad \begin{aligned} \|\widehat{s}_{i,h}^{n-1} - \widehat{s}_i^n\| &\leq C_1 \left( \|s^n - s_h^{n-1}\| + \|s^n - s_h^{n-2}\| \right. \\ &\quad \left. + \|s^{n-1} - s_h^{n-1}\| + \|s^{n-1} - s_h^{n-2}\| \right). \end{aligned}$$

Now, apply these inequalities and (5.17)–(5.20), multiply (5.16) by  $\Delta t^n$ , sum  $n$ , and properly arrange terms to complete the proof of the lemma.  $\square$

The three terms  $T_i^n$ ,  $i = 1, 2, 3$ , take care of the quadratic terms in the velocities, which require more regularity on  $u$  than those without these quadratic terms, as seen from Lemma 5.2.

**5.2. Analysis of the saturation equation.** We now turn to analyzing the finite element method (4.6).

LEMMA 5.3. *Let  $s$  and  $s_h$  solve (3.2) and (4.6), respectively. Then*

$$\begin{aligned} & \|\nabla \xi^\gamma\|^2 + \sum_{n=1}^{\gamma} \|\partial \xi^n\|^2 \Delta t^n \\ & \leq C_2 \left\{ \|s^\gamma - s^{\gamma-1}\|^2 + \|\xi^{\gamma-1}\|^2 + \|\zeta^{\gamma-1}\|^2 + \|\theta^\gamma\|^2 + \|\eta^\gamma\|^2 + \|\beta^\gamma\|^2 + \|\sigma^\gamma\|^2 \right. \\ & \quad + \sum_{n=0}^{\gamma} \left( \left\| \frac{\partial s^n}{\partial t} - \partial s^n \right\|^2 + \|\partial(s^{n+1} - s^n)\|^2 + \|s^{n+1} - s^n\|^2 + \|p^{n+1} - p^n\|^2 \right. \\ & \quad + \|\partial \zeta^n\|^2 + \|\zeta^n\|^2 + \|\partial \sigma^n\|^2 + \|\sigma^n\|^2 + \|\eta^n\|^2 + \|\xi^n\|_1^2 + \|\theta^n\|^2 \\ & \quad \left. \left. + \|\beta^n\|^2 + \|\partial \theta^n\|^2 + \|\partial \eta^n\|^2 \right) \Delta t^n + \sum_{n=1}^{\gamma-1} \|\nabla \xi^n\|^2 \|\partial \xi^n\|_{0,\infty}^2 \Delta t^n \right\} \\ & \quad + \varepsilon \sum_{n=1}^{\gamma} \|\partial \beta^n\|^2 \Delta t^n \end{aligned}$$

for  $1 \leq \gamma \leq n_T$ , where

$$C_2 = C_2 \left( \left\| \frac{\partial s}{\partial t} \right\|_{L^\infty(J \times \Omega)}, \left\| \nabla \frac{\partial s}{\partial t} \right\|_{L^\infty(J \times \Omega)}, \|\nabla s\|_{L^\infty(J \times \Omega)}, \|u\|_{L^\infty(J \times \Omega)} \right).$$

*Proof.* Subtract (4.6) from (3.2) at  $t = t^n$ , use (5.1) at  $t = t^n$ , and set the test function  $v = \partial \xi^n$  to see that

$$(5.21) \quad (\phi \partial \xi^n, \partial \xi^n) + (D(s_h^{n-1}) \nabla \xi^n, \nabla \partial \xi^n) = \sum_{i=1}^7 B_i^n,$$

where

$$\begin{aligned} B_1^n &= - \left( \phi \left( \frac{\partial s^n}{\partial t} - \partial \bar{s}^n \right), \partial \xi^n \right), & B_2^n &= (\zeta^n, \partial \xi^n), \\ B_3^n &= ((q_w(s^n) - q_w(s_h^{n-1}))u^n, \nabla \partial \xi^n), & B_4^n &= ((u^n - u_h^n)q_w(s_h^{n-1}), \nabla \partial \xi^n), \\ B_5^n &= (b(s^n, p^n) - b(s_h^{n-1}, p_h^n), \nabla \partial \xi^n), & B_6^n &= - \left( \frac{\partial \phi^n}{\partial t} (s^n - s_h^n), \partial \xi^n \right), \\ B_7^n &= - ((D(s^n) - D(s_h^{n-1})) \nabla \bar{s}^n, \nabla \partial \xi^n). \end{aligned}$$

The left-hand side of (5.21) is bigger than the quantity

$$(5.22) \quad \begin{aligned} & (\phi \partial \xi^n, \partial \xi^n) + \frac{1}{2\Delta t^n} (D(s_h^{n-1}) \nabla \xi^n, \nabla \xi^n) \\ & - \frac{1}{2\Delta t^n} (D(s_h^{n-2}) \nabla \xi^{n-1}, \nabla \xi^{n-1}) + B_8^n, \end{aligned}$$

where  $B_8^n$  is defined by

$$B_8^n = \frac{1}{2\Delta t^n} ((D(s_h^{n-2}) - D(s_h^{n-1}))\nabla\xi^{n-1}, \nabla\xi^{n-1})$$

and is bounded by

$$(5.23) \quad |B_8^n| \leq C_2(1 + \|\partial\xi^{n-1}\|_{0,\infty}^2)\|\nabla\xi^{n-1}\|^2.$$

Next, it can be easily seen that

$$(5.24) \quad |B_1^n| + |B_2^n| + |B_6^n| \leq C_2 \left( \left\| \frac{\partial s^n}{\partial t} - \partial s^n \right\|^2 + \|\partial\zeta^n\|^2 + \|\zeta^n\|^2 + \|\xi^n\|^2 \right) + \varepsilon\|\partial\xi^n\|^2.$$

To avoid an apparent loss of a factor  $h$  in  $B_i^n$ ,  $i = 3, 4, 5, 7$ , we use summation by parts on these items. We work on  $B_3^n$  in detail, and other quantities can be estimated similarly. Applying summation by parts in  $n$  and the fact that  $\xi^0 = 0$ , we see that

$$\begin{aligned} & \sum_{n=1}^{\gamma} ((q_w(s^n) - q_w(s_h^{n-1}))u^n, \nabla\partial\xi^n)\Delta t^n \\ &= \sum_{n=1}^{\gamma-1} (\{(q_w(s^n) - q_w(s_h^{n-1})) - ((q_w(s^{n+1}) - q_w(s_h^n))\}u^n, \nabla\xi^n) \\ & \quad + \sum_{n=1}^{\gamma-1} ((q_w(s^{n+1}) - q_w(s_h^n))(u^n - u^{n+1}), \nabla\xi^n) \\ & \quad + ((q_w(s^\gamma) - q_w(s_h^{\gamma-1}))u^\gamma, \nabla\xi^\gamma), \end{aligned}$$

so that, using the same argument as for (5.18),

$$(5.25) \quad \left| \sum_{n=1}^{\gamma} B_3^n \Delta t^n \right| \leq C_2 \left\{ \sum_{n=1}^{\gamma-1} (\|\partial\zeta^n\|^2 + \|\partial(s^{n+1} - s^n)\|^2 + \|\widehat{s}_h^n - \widehat{s}^{n+1}\|^2 + \|s^{n+1} - s_h^n\|^2 + \|\nabla\xi^n\|^2) \Delta t^n + \|s^\gamma - s_h^{\gamma-1}\|^2 \right\} + \varepsilon \left( \|\nabla\xi^\gamma\|^2 + \sum_{n=1}^{\gamma-1} \|\partial\xi^n\|^2 \right),$$

where  $\|\widehat{s}_h^n - \widehat{s}^{n+1}\|$  can be estimated as in (5.20). The term  $\sum_{n=1}^{\gamma} B_7^n \Delta t^n$  has the same bound as in (5.25). Also, we find that

$$(5.26) \quad \left| \sum_{n=1}^{\gamma} B_4^n \Delta t^n \right| \leq C_2 \left\{ \sum_{n=1}^{\gamma-1} (\|\partial\sigma^{n+1}\|^2 + \|\sigma^n\|^2 + \|\beta^n\|^2 + (1 + \|\partial\xi^n\|_{0,\infty}^2)\|\nabla\xi^n\|^2) \Delta t^n + \|\sigma^\gamma\|^2 + \|\beta^\gamma\|^2 \right\} + \varepsilon \left( \|\nabla\xi^\gamma\|^2 + \sum_{n=1}^{\gamma-1} \|\partial\beta^{n+1}\|^2 \right),$$



and

$$\begin{aligned}
 \left| \sum_{n=1}^{\gamma} B_5^n \Delta t^n \right| \leq C_2 \left\{ \sum_{n=1}^{\gamma-1} (\|\partial \zeta^n\|^2 + \|\partial(s^{n+1} - s^n)\|^2 + \|\widehat{s}_h^n - \widehat{s}^{n+1}\|^2 \right. \\
 + \|\partial \eta^n\|^2 + \|\partial \theta^n\|^2 + \|\widehat{p}_h^n - \widehat{p}^n\|^2 \\
 + \|s^{n+1} - s_h^n\|^2 + \|p^n - p_h^n\|^2 + \|\nabla \xi^n\|^2) \Delta t^n \\
 \left. + \|s^\gamma - s_h^{\gamma-1}\|^2 + \|p^\gamma - p_h^\gamma\|^2 \right\} \\
 + \varepsilon \left( \|\nabla \xi^\gamma\|^2 + \sum_{n=1}^{\gamma-1} \|\partial \xi^n\|^2 \right).
 \end{aligned}
 \tag{5.27}$$

Now, multiply (5.21) by  $\Delta t^n$ , sum  $n$ , and use (5.22)–(5.27) to complete the proof of the lemma.  $\square$

Note that in order to avoid an apparent loss of a fact  $h$ , summation by parts (i.e., integration by parts) has been exploited to estimate the  $B_i^n$  ( $i = 3, 4, 5, 7$ ) terms. If the usual error equations (5.13) and (5.14) are used, a difficulty arises from the combination of the Dirichlet boundary condition (1.3) and the nonlinearity of the differential system (2.9)–(2.11) since the global pressure and the total velocity appear in these terms and boundary integrals result from the integration by parts (see [19] for treating a simpler problem with no flow boundary condition using the usual argument). To handle this difficulty we use the time-differentiated forms of these error equations and the homogeneous initial conditions, as mentioned before.

**5.3.  $L^2$ -error estimates.** We now prove the main result in this section. Define

$$\begin{aligned}
 \mathcal{E}(t) = & \sum_{K \in T_{h_p}} h_{p,K}^{k^{**}} \left( \|p\|_{L^\infty(0,t;H^{k^{**}}(K))} + \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(0,t;H^{k^{**}}(K))} + \left\| \frac{\partial^2 p}{\partial t^2} \right\|_{L^2(0,t;H^{k^{**}}(K))} \right) \\
 & + \sum_{K \in T_{h_p}} h_{p,K}^{k^*+1} \left( \|u\|_{L^\infty(0,t;H^{k^*+1}(K))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,t;H^{k^*+1}(K))} \right) \\
 & + \sum_{K \in T_h} h_K^{k+1} \left( \|s\|_{L^\infty(0,t;H^{k+1}(K))} + \left\| \frac{\partial s}{\partial t} \right\|_{L^2(0,t;H^{k+1}(K))} \right) \\
 & + \Delta t \sum_{i=1}^2 \left( \left\| \frac{\partial^i p}{\partial t^i} \right\|_{L^2(J;L^2(\Omega))} + \left\| \frac{\partial^i s}{\partial t^i} \right\|_{L^2(J;L^2(\Omega))} + \left\| \frac{\partial^i u}{\partial t^i} \right\|_{L^2(J;L^2(\Omega))} \right) \\
 & + \Delta t \left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(J;L^2(\Omega))}, \quad t \in J.
 \end{aligned}$$

**THEOREM 5.4.** *Let  $(u, p, s)$  and  $(u_h, p_h, s_h)$  satisfy (3.1), (3.2) and (4.5), (4.6), respectively. Then, if the parameters  $\Delta t, h_p,$  and  $h$  satisfy*

$$(5.28) \quad (h^{-d/2} + h_p^{-d/2})(\Delta t + h_p^{k^*+1} + h_p^{k^{**}} + h^{k+1}) \rightarrow 0 \text{ as } \Delta t, h \rightarrow 0,$$

we have

$$\begin{aligned}
 \max_{0 \leq n \leq n_T} \left\{ \|u^n - u_h^n\| + \|p^n - p_h^n\| + \|s^n - s_h^n\| + h \|\nabla(s^n - s_h^n)\| + \left\| \frac{\partial p^n}{\partial t} - \partial p_h^n \right\| \right\} \\
 + \left\{ \sum_{n=1}^{n_T} \left\| \frac{\partial s^n}{\partial t} - \partial s_h^n \right\|^2 \Delta t^n \right\}^{1/2} \leq C \mathcal{E}(T),
 \end{aligned}$$

where  $C = C(C_1, C_2, T)$ .

*Proof.* Take a  $(C_1 + 1)$ -multiple of the inequality in Lemma 5.3, add the resulting inequality and the inequality in Lemma 5.2, and use (5.3)–(5.7), (5.15), and the extension of the solution for  $t \leq 0$  to obtain

$$\begin{aligned}
 & \|\nabla \xi^\gamma\|^2 + \|\partial \theta^\gamma\|^2 + \sum_{n=1}^{\gamma} (\|\partial \xi^n\|^2 + \|\partial \beta^n\|^2) \Delta t^n \\
 & \leq C_3 \left\{ \mathcal{E}^2(t^\gamma) + \|\xi^{\gamma-1}\|^2 + \|\theta^\gamma\|^2 + \|\beta^\gamma\|^2 \right. \\
 (5.29) \quad & \quad + \frac{1}{2} \sum_{n=1}^{\gamma} (\|\xi^n\|_1^2 + \|\beta^n\|^2 + \|\theta^n\|^2 + \|\partial \theta^n\|^2 \\
 & \quad + (\|\partial \theta^{n-1}\|^2 + \|\beta^{n-2}\|^2 + \|\nabla \xi^{n-1}\|^2 \\
 & \quad + (1 + \|\beta^{n-1}\|_{0,\infty}^2) \|\beta^{n-1}\|^2 + \mathcal{E}^2(t^\gamma)) \\
 & \quad \left. \times (\|\partial \xi^{n-1}\|_{0,\infty}^2 + \|\partial \theta^{n-1}\|_{0,\infty}^2 + \|\partial \beta^{n-1}\|_{0,\infty}^2) \Delta t^n \right\},
 \end{aligned}$$

where  $C_3 = C_3(C_1, C_2)$ . In deriving (5.29), we required that the  $\varepsilon$  appearing in Lemma 5.3 be sufficiently small that  $(C_1 + 1)\varepsilon \leq 1/2$ ; this increases  $C_2$  but not  $C_1$ . Observe that, by (5.12),

$$(5.30) \quad \|\theta^\gamma\|^2 \leq C \sum_{n=1}^{\gamma} \|\theta^n\|^2 \Delta t^n + \varepsilon \sum_{n=1}^{\gamma} \|\partial \theta^n\|^2 \Delta t^n.$$

The same result holds for  $\xi^\gamma$  and  $\beta^\gamma$ . Combine (5.29), (5.30), and an inverse inequality to see that

$$\begin{aligned}
 & \|\xi^\gamma\|_1^2 + \|\theta^\gamma\|^2 + \|\partial \theta^\gamma\|^2 + \|\beta^\gamma\|^2 + \sum_{n=1}^{\gamma} (\|\partial \xi^n\|^2 + \|\partial \beta^n\|^2) \Delta t^n \\
 (5.31) \quad & \leq C_3 \left\{ \mathcal{E}^2(t^\gamma) + \frac{1}{2} \sum_{n=1}^{\gamma} (\|\xi^n\|_1^2 + \|\beta^n\|^2 + \|\theta^n\|^2 + \|\partial \theta^n\|^2 \right. \\
 & \quad + (h^{-d} + h_p^{-d})(\|\partial \theta^{n-1}\|^2 + \|\beta^{n-2}\|^2 + \|\nabla \xi^{n-1}\|^2 \\
 & \quad + (1 + h_p^{-d} \|\beta^{n-1}\|^2) \|\beta^{n-1}\|^2 + \mathcal{E}^2(t^\gamma)) \\
 & \quad \left. \times (\|\partial \xi^{n-1}\|^2 + \|\partial \theta^{n-1}\|^2 + \|\partial \beta^{n-1}\|^2) \Delta t^n \right\}.
 \end{aligned}$$

We now make the induction hypothesis that

$$\begin{aligned}
 (5.32) \quad & \max_{n \leq \gamma-1} (\|\xi^n\|_1^2 + \|\theta^n\|^2 + \|\partial \theta^n\|^2 + \|\beta^n\|^2) \\
 & + \sum_{n=1}^{\gamma-1} (\|\partial \xi^n\|^2 + \|\partial \beta^n\|^2) \Delta t^n \leq C_4 \mathcal{E}^2(T),
 \end{aligned}$$

where  $C_4 = 2C_3e^{TC_3}$ . Note that, by (5.12), (5.32) holds trivially for  $\gamma = 1$ . Then, by (5.32), (5.31) becomes

$$\begin{aligned}
 & \|\xi^\gamma\|_1^2 + \|\theta^\gamma\|^2 + \|\partial\theta^\gamma\|^2 + \|\beta^\gamma\|^2 + \sum_{n=1}^{\gamma} (\|\partial\xi^n\|^2 + \|\partial\beta^n\|^2)\Delta t^n \\
 (5.33) \quad & \leq C_3 \left\{ \mathcal{E}^2(t^\gamma) + \frac{1}{2} \sum_{n=1}^{\gamma} (\|\xi^n\|_1^2 + \|\beta^n\|^2 + \|\theta^n\|^2 + \|\partial\theta^n\|^2 \right. \\
 & \quad \left. + 2(h^{-d} + h_p^{-d})C_4\mathcal{E}^2(T)(1 + C_4h_p^{-d}\mathcal{E}^2(T)) \right. \\
 & \quad \left. \times (\|\partial\xi^{n-1}\|^2 + \|\partial\theta^{n-1}\|^2 + \|\partial\beta^{n-1}\|^2)\Delta t^n \right\}.
 \end{aligned}$$

Using (5.28), we choose the discretization parameters so small that

$$2(h^{-d} + h_p^{-d})C_3C_4\mathcal{E}^2(T)(1 + C_4h_p^{-d}\mathcal{E}^2(T)) \leq 1/2.$$

Then it follows from (5.33) that

$$\begin{aligned}
 & \|\xi^\gamma\|_1^2 + \|\theta^\gamma\|^2 + \|\partial\theta^\gamma\|^2 + \|\beta^\gamma\|^2 + \sum_{n=1}^{\gamma} (\|\partial\xi^n\|^2 + \|\partial\beta^n\|^2)\Delta t^n \\
 & \leq C_3 \left\{ \mathcal{E}^2(t^\gamma) + \sum_{n=1}^{\gamma} (\|\xi^n\|_1^2 + \|\beta^n\|^2 + \|\theta^n\|^2 + \|\partial\theta^n\|^2)\Delta t^n \right\},
 \end{aligned}$$

which, together with Gronwall's inequality, implies that

$$(5.34) \quad \|\xi^\gamma\|_1^2 + \|\theta^\gamma\|^2 + \|\partial\theta^\gamma\|^2 + \|\beta^\gamma\|^2 + \sum_{n=1}^{\gamma} (\|\partial\xi^n\|^2 + \|\partial\beta^n\|^2)\Delta t^n \leq C_5\mathcal{E}^2(T),$$

where

$$C_5 = C_3(1 - C_3\Delta t)^{-T/\Delta t} \leq 2C_3e^{TC_3} \equiv C_4,$$

for  $\Delta t$  not too large. Consequently, the induction argument is completed and the theorem follows.  $\square$

We remark that, if  $h$  and  $h_p$  are of the same order as they tend to zero, then

$$(h^{-d/2} + h_p^{-d/2})(h^{k^*+1} + h_p^{k^{**}} + h^{k+1}) \leq Ch^{-d/2}(h^{k^{**}} + h^{k+1}),$$

since  $k^{**} \leq k^* + 1$ . Since  $k \geq 1$ ,

$$h^{-d/2}h^{k+1} \rightarrow 0 \text{ as } h \rightarrow 0, \quad d = 2, 3.$$

Also, if  $k^{**} \geq 2$ , we see that

$$h^{-d/2}h^{k^{**}} \rightarrow 0 \text{ as } h \rightarrow 0, \quad d = 2, 3.$$

Thus, for (5.28) to be satisfied, we assume that  $k^{**} \geq 2$ . This excludes the mixed finite element spaces of lowest order, i.e.,  $k^{**} = 1$ . The lowest order case has to be treated using different techniques. If the nonlinear coefficients  $\alpha(s)$  and  $c(s, p)$  in (4.5) are projected into the finite element space  $W_h$ , the technique developed in [10] can be used to handle the lowest order case. We shall not pursue this here. Also, the time step  $\Delta t$  is required to satisfy the condition (5.28), which comes from the nonlinearity of the pressure and saturation equations and the linearization scheme. It is clear from (5.28) that this condition is not very restrictive for  $\Delta t$ .

**5.4.  $L^\infty$ -error estimates.** The main objective of this paper is to establish the  $L^2$ -error estimates given in Theorem 5.4. For completeness, we end this section with a statement of  $L^\infty$ -estimates for the errors  $s - s_h$  and  $p - p_h$  in the two-dimensional case.

**THEOREM 5.5.** *Assume that  $(p, s)$  and  $(p_h, s_h)$  satisfy (3.1), (3.2) and (4.5), (4.6), respectively, and the parameters  $h_p$  and  $h$  satisfy (5.28). Then*

$$(5.35) \quad \max_{0 \leq n \leq n_T} \|p^n - p_h^n\|_{0,\infty} \leq C \log h_p^{-1} (\mathcal{E}(T) + h_p^{k^{**}} \|p\|_{L^\infty(J; H^{k^{**}+1}(\Omega))}),$$

$$(5.36) \quad \max_{0 \leq n \leq n_T} \|s^n - s_h^n\|_{0,\infty} \leq C (\log h^{-1})^\gamma (\mathcal{E}(T) + h^{k+1} \|s\|_{L^\infty(J; W^{k+1,\infty}(\Omega))}),$$

where  $C = C(C_1, C_2, T)$ ,  $\gamma = 1$  for  $k = 1$ , and  $\gamma = 1/2$  for  $k > 1$ .

*Proof.* First, it follows from the approximation property of the projection  $P_h$  [28] that

$$(5.37) \quad \|p^n - \tilde{p}^n\|_{0,\infty} \leq C h_p^{k^{**}} (\log h_p^{-1})^{1/2} \|p^n\|_{k^{**}+1}.$$

Also, from [28, Lemma 1.2] and (5.13), we see that

$$\begin{aligned} \|\theta^n\|_{0,\infty} &\leq C \log h_p^{-1} \|\alpha(s_h^{n-1})\beta^n + (\alpha(s^n) - \alpha(s_h^{n-1}))u^n \\ &\quad + \alpha(s_h^{n-1})\sigma^n + (G(s_h^{n-1}, p_h^{n-1}) - G(s^n, p^n))\|, \end{aligned}$$

so that, by Theorem 5.4,

$$\max_{0 \leq n \leq n_T} \|\theta^n\|_{0,\infty} \leq C \log h_p^{-1} \mathcal{E}(T).$$

This, together with (5.37), implies (5.35). Finally, apply the embedding inequality [37]

$$\|\xi^n\|_{0,\infty} \leq C (\log h^{-1})^{1/2} \|\xi^n\|_1,$$

(5.3b), and (5.34) to obtain (5.36).  $\square$

**6. Finite elements for a degenerate problem.** In this section we consider a degenerate case where the diffusion coefficient  $D(s)$  can be zero. Since the pressure equation is the same as before, we here focus on the saturation equation. For simplicity we neglect gravity. Then the saturation equation (2.11) can be written as

$$(6.1) \quad \phi \frac{\partial s}{\partial t} - \nabla \cdot (D(s)\nabla s - q_w(s)u) = \tilde{f}_w - s \frac{\partial \phi}{\partial t}, \quad (x, t) \in \Omega \times J.$$

For technical reasons we consider only the Neumann boundary condition (2.15)

$$(6.2a) \quad (D(s)\nabla s - q_w(s)u) \cdot \nu = -d_w(x, t), \quad (x, t) \in \partial\Omega \times J,$$

and the initial condition is given by

$$(6.2b) \quad s(x, 0) = s^0(x), \quad x \in \Omega,$$

where  $0 \leq s^0(x) \leq 1$ ,  $x \in \Omega$ . We impose the following conditions on the degeneracy of  $D(s)$ :

$$(6.3) \quad D(s) \geq \begin{cases} \beta_1 |s|^{\mu_1}, & 0 \leq s \leq \alpha_1, \\ \beta_2, & \alpha_1 \leq s \leq \alpha_2, \\ \beta_3 |1-s|^{\mu_2}, & \alpha_2 \leq s \leq 1, \end{cases}$$

where the  $\beta_i$  are positive constants and  $\alpha_j$  and  $\mu_j$  ( $j = 1, 2$ ) satisfy the conditions

$$0 < \alpha_1 < 1/2 < \alpha_2 < 1, \quad 0 < \mu_j \leq 2.$$

Difficulties arise when trying to derive error estimates for the approximate solution of (6.1) and (6.2) with  $D(s)$  satisfying the condition (6.3). To get around this problem, we consider the perturbed diffusion coefficient  $D_\kappa(s)$  defined by [12], [25], [36], [39],

$$D_\kappa(s) = \max\{D(s), \kappa^\mu\},$$

where  $\kappa > 0$  and  $\mu = \max\{\mu_1, \mu_2\}$ . Since the coefficient  $D_\kappa(s)$  is bounded away from zero, the previous error analysis applies to the perturbed problem:

$$(6.4a) \quad \phi \frac{\partial s_\kappa}{\partial t} - \nabla \cdot (D_\kappa(s_\kappa) \nabla s_\kappa - q_w(s_\kappa)u) = \tilde{f}_w - s_\kappa \frac{\partial \phi}{\partial t}, \quad (x, t) \in \Omega \times J,$$

$$(6.4b) \quad (D_\kappa(s_\kappa) \nabla s_\kappa - q_w(s_\kappa)u) \cdot \nu = -d_w(x, t), \quad (x, t) \in \partial\Omega \times J,$$

$$(6.4c) \quad s_\kappa(x, 0) = s^0(x), \quad x \in \Omega.$$

We now state a result on the convergence of  $s_\kappa$  to  $s$  as  $\kappa$  tends to zero. Its proof is given in [25] for the case where  $d_w \equiv 0$  and the right-hand side of (6.1) is zero, and can be easily extended to the present case.

**THEOREM 6.1.** *Assume that  $D(s)$  satisfies (6.3) and there is a constant  $C^* > 0$  such that*

$$(6.5) \quad C^* |q_w(s_1) - q_w(s_2)|^2 \leq (\mathcal{D}(s_1) - \mathcal{D}(s_2))(s_1 - s_2), \quad 0 \leq s_1, s_2 \leq 1,$$

where

$$\mathcal{D}(s) = \int_0^s D(\xi) d\xi.$$

Then there is  $C$  independent of  $\kappa$ ,  $s$ , and  $\mu$  such that

$$(6.6) \quad \|s - s_\kappa\|_{L^{2+\mu}(J; L^{2+\mu}(\Omega))} \leq C\kappa.$$

As shown in [25], the requirement (6.5) is reasonable. We now consider a fully discrete finite element method for (6.4). Let  $M_h$  be the standard  $C^0$  piecewise linear polynomial space associated with  $T_h$ ; due to the roughness of the solution to (6.1) and (6.2), no improvements in the asymptotic convergence rates result from taking higher order finite element spaces. Also, we extend the domain of  $D_\kappa$  and  $q_w$  as follows:

$$D_\kappa(\xi) = \begin{cases} D_\kappa(1) & \text{if } \xi \geq 1, \\ D_\kappa(-\xi) & \text{if } \xi \leq 0, \end{cases}$$

and

$$q_w(\xi) = 0 \quad \forall \xi \in (-\infty, 0) \cup (1, \infty).$$

Now the finite element solution  $s_h^n : J \rightarrow M_h$ ,  $n = 1, 2, \dots, n_T$ , to (6.4) is given by

$$(6.7a) \quad \begin{aligned} & (\phi \partial s_h^n, v) + (D_\kappa(s_h^n) \nabla s_h^n - q_w(s_h^n)u^n, \nabla v) \\ & = \left( \tilde{f}_w^n - s_h^n \frac{\partial \phi^n}{\partial t}, v \right) - \langle d_w^n, v \rangle_{\partial\Omega} \quad \forall v \in M_h, \end{aligned}$$

$$(6.7b) \quad s_h^0 = \mathcal{P}_h s^0,$$

where  $\mathcal{P}_h$  is the  $L^2$ -projection onto  $M_h$ . The following theorem states the convergence of  $s_h$  to  $s$ . For (6.8) below to be satisfied, we see from (6.6) that the perturbation parameter  $\kappa$  needs to satisfy the relation  $\kappa = O(h^{\lambda_1})$ , where  $\lambda_1$  is given by

$$\lambda_1 = (4 + 2\mu)/(2 + 4\mu + \mu^2).$$

**THEOREM 6.2.** *Let  $s$  and  $s_h$  solve (6.1), (6.2), and (6.7), respectively, and let the hypotheses of Theorem 6.1 be satisfied. Then there is  $C$  independent of  $\kappa$ ,  $s$ , and  $\mu$  such that*

$$(6.8) \quad \begin{aligned} & \max_{0 \leq n \leq n_T} \|s(t^n) - s_h^n\|_{H^{-1}(\Omega)}^2 + \sum_{n=0}^{n_T} \|s(t^n) - s_h^n\|_{L^{2+\mu}(\Omega)}^{2+\mu} \Delta t^n \\ & \leq C(h^{(2+\mu)\lambda_1} (\log h^{-1})^{\frac{\mu}{1+\mu}} + \Delta t^{\frac{\lambda_2+2}{2}}), \end{aligned}$$

where  $\lambda_2 = (2 + \mu)/(1 + \mu)$ .

The proof can be carried out as in [26], [36], and [39]; we omit the details.

**7. Simulation with various boundary conditions.** Let  $\partial\Omega$  be a set of four disjoint regions  $\Gamma_i$ ,  $i = 1, \dots, 4$ , and let  $\Gamma_3 = \cup_j \Gamma_{3,j}$  where each  $\Gamma_{3,j}$  is connected. As mentioned in the introduction, the most commonly encountered boundary conditions for the two-pressure equations are of first type, second type, third type, and well type. Then we consider for  $\alpha = w, a$

$$(7.1) \quad p_\alpha = p_{\alpha D}(x, t), \quad x \in \Gamma_1, t > 0,$$

$$(7.2) \quad u_\alpha \cdot \nu + \chi_\alpha(x, t, s)p_\alpha = v_\alpha(x, t, s), \quad x \in \Gamma_2, t > 0,$$

$$(7.3a) \quad \int_{\Gamma_{3,j}} (u_w + u_\alpha) \cdot \nu = v_j(t), \quad x \in \Gamma_{3,j}, t > 0,$$

$$(7.3b) \quad p_\alpha = p_{\alpha D}(x, t) + d_j(t), \quad x \in \Gamma_{3,j}, t > 0,$$

$$(7.4a) \quad p_a = p_{aD}(x, t), \quad x \in \Gamma_4, t > 0,$$

$$(7.4b) \quad u_w \cdot \nu + \chi_w(x, t, s)p_w = v_w(x, t, s), \quad x \in \Gamma_4, t > 0,$$

where  $p_{\alpha D}$ ,  $\chi_\alpha$ ,  $v_\alpha$ , and  $v_j$  are given functions,  $d_j$  is an arbitrary scaling constant, and  $\nu$  is the outer unit normal to  $\partial\Omega$ . Note that  $\Gamma_1$  is of the first type,  $\Gamma_2$  is of the third type (it reduces to the second type as  $\chi_\alpha \equiv 0$ ),  $\Gamma_3$  is of the well type, and on  $\Gamma_4$  we have the Dirichlet condition for the air phase and the Neumann condition for the water phase. Let  $\Gamma_{p,i} = \Gamma_i$ ,  $i = 1, \dots, 4$ ,  $\Gamma_{s,1} = \Gamma_1 \cup \Gamma_3$ , and  $\Gamma_{s,2} = \Gamma_2 \cup \Gamma_4$ . Then the global boundary conditions for the pressure-saturation equations (2.9)–(2.11) become

$$(7.5) \quad p = p_D(x, t), \quad x \in \Gamma_{p,1}, t > 0,$$

$$(7.6) \quad u \cdot \nu + \chi(x, t, s)p = \Upsilon(x, t, s), \quad x \in \Gamma_{p,2}, t > 0,$$

$$(7.7a) \quad \int_{\Gamma_{p,3,j}} u \cdot \nu = v_j(t), \quad x \in \Gamma_{p,3,j}, t > 0,$$

$$(7.7b) \quad p = p_D(x, t) + d_j(t), \quad x \in \Gamma_{p,3,j}, t > 0,$$

$$(7.8) \quad p = p_{aD}(x, t) + \varphi(s), \quad x \in \Gamma_{p,4}, t > 0,$$

$$(7.9) \quad s = s_D(x, t), \quad x \in \Gamma_{s,1}, t > 0,$$

$$(7.10) \quad \begin{aligned} & (q_w u + k\lambda_a q_w (\nabla p_c - \tilde{p})) \cdot \nu \\ & + \chi_w(x, t, s)p = \Upsilon_w(x, t, s), \quad x \in \Gamma_{s,2}, t > 0, \end{aligned}$$

where  $p_D$  and  $s_D$  are the transforms of  $p_{wD}$  and  $p_{aD}$  by (2.2) and (2.3), and

$$\begin{aligned}\chi &= \chi_w + \chi_a, \\ \Upsilon &= v_w + v_a - \chi_a p_c + \chi \int_0^{p_c(s)} q_a(p_c^{-1}(\xi)) d\xi, \\ \Upsilon_w &= v_w + \chi_w \int_0^{p_c(s)} q_a(p_c^{-1}(\xi)) d\xi, \\ \varphi(s) &= - \int_0^{p_c(s)} q_w(p_c^{-1}(\xi)) d\xi.\end{aligned}$$

We now incorporate the boundary conditions (7.5)–(7.10) in the finite element scheme given in (4.5) and (4.6). The constraint  $V_h \subset V$  says that the normal components of the members of  $V_h$  are continuous across the interior boundaries in  $T_{h_p}$ . Following [2], [9], we relax this constraint on  $V_h$  by introducing Lagrange multipliers over interior boundaries. Since the mixed space  $V_h$  is finite dimensional and defined locally on each element  $K$  in  $T_{h_p}$ , let  $V_h(K) = V_h|_K$ . Then we define

$$\begin{aligned}\tilde{V}_h &= \{v \in (L^2(\Omega))^d : v|_K \in V_h(K) \text{ for each } K \in T_{h_p}\}, \\ L_{h,\pi_1,\{\pi_j\},\pi_3} &= \left\{ r \in L^2\left(\bigcup_{e \in \partial T_{h_p}} e\right) : r|_e \in V_h \cdot \nu|_e \text{ for each } e \in \partial T_{h_p}; \right. \\ &\quad (r - \pi_1, r_1)_e = 0, r_1 \in V_h \cdot \nu|_e, \forall e \in \Gamma_{p,1}, \\ &\quad (r - \pi_j, r_2)_e = 0, r_2 \in V_h \cdot \nu|_e, \forall e \in \Gamma_{p,3,j}, \text{ for each } j, \\ &\quad \left. (r - \pi_3, r_3)_e = 0, r_3 \in V_h \cdot \nu|_e, \forall e \in \Gamma_{p,4} \right\},\end{aligned}$$

and  $W_h$  and  $M_h$  are given as before. The mixed finite element solution of the pressure equation is  $\{u_h^n, p_h^n, \ell_h^n\} \in \tilde{V}_h \times W_h \times L_{h,p_D,\{p_D^n+d_j^n\},p_{aD}^n+\varphi^{n-1}}$ ,  $n = 1, 2, \dots, n$ , satisfying

$$\begin{aligned}(c(s_h^{n-1}, p_h^{n-1}) \partial p_h^n, \psi) + \sum_K (\nabla \cdot u_h^n, \psi)_K &= (f(p_h^{n-1}), \psi) \quad \forall \psi \in W_h, \\ (\alpha(s_h^{n-1}) u_h^n, v) - \sum_K \{(\nabla \cdot v, p_h^n)_K - (\ell_h^n, v \cdot \nu)_K\} &= (G(s_h^{n-1}, p_h^{n-1}), v) \quad \forall v \in \tilde{V}_h, \\ \sum_K (u_h^n \cdot \nu_K, r)_{\partial K \setminus (\Gamma_{p,1} \cup \Gamma_{p,4})} &= (\Upsilon(s_h^{n-1}) - \chi(s_h^{n-1}) \ell_h^n, r)_{\Gamma_{p,2}} + \sum_j \frac{(v_j^n, r)_{\Gamma_{p,3,j}}}{|\Gamma_{p,3,j}|} \\ &\quad \forall r \in L_{h,0,\{0\},0},\end{aligned}$$

and the finite element method for the saturation is given for  $s_h^n \in M_h + s_D^n$  satisfying

$$\begin{aligned}(\phi \partial s_h^n, v) + (D(s_h^{n-1}) \nabla s_h^n - q_w(s_h^{n-1}) u_h^n - b(s_h^{n-1}, p_h^n), \nabla v) \\ = (\tilde{f}_w^n - s_h^n \frac{\partial \phi^n}{\partial t}, v) - (\Upsilon_w(s_h^{n-1}) - \chi_w(s_h^{n-1}) \ell_h^n, v)_{\Gamma_2} \quad \forall v \in M_h,\end{aligned}$$

for  $n = 1, 2, \dots, n_T$ . The computation of these equations can be carried out as in (4.5) and (4.6). Note that the last equation in the unconstrained mixed formulation above enforces the continuity requirement on  $u_h$ , so in fact  $u_h \in V_h$ . It is well known [2], [9] that the linear system arising from this unconstrained mixed formulation leads to a symmetric, positive definite system for the Lagrange multipliers, which can be

TABLE 1  
Convergence of  $p_h$  at  $T = 1min.$

| $1/h$ | $L^\infty$ -error | $L^\infty$ -order | $L^2$ -error | $L^2$ -order |
|-------|-------------------|-------------------|--------------|--------------|
| 10    | 0.0570            | –                 | 0.0501       | –            |
| 20    | 0.0343            | 0.73              | 0.0245       | 1.02         |
| 40    | 0.0186            | 0.88              | 0.0122       | 1.00         |
| 80    | 0.0090            | 1.05              | 0.0059       | 1.02         |

TABLE 2  
Convergence of  $s_h$  at  $T = 1min.$

| $1/h$ | $L^\infty$ -error | $L^\infty$ -order | $L^2$ -error | $L^2$ -order |
|-------|-------------------|-------------------|--------------|--------------|
| 10    | 0.0766            | –                 | 0.0695       | –            |
| 20    | 0.0526            | 0.55              | 0.0482       | 0.53         |
| 40    | 0.0295            | 0.83              | 0.0271       | 0.83         |
| 80    | 0.0167            | 0.82              | 0.0152       | 0.84         |

easily solved. Also, the introduction of the Lagrange multipliers makes it easier to incorporate the boundary conditions (7.5)–(7.10).

We now present a numerical example. The relative permeability functions are taken as follows:

$$k_{rw} = s - s_{rw}, \quad k_{ra} = 1 - s - s_{ra},$$

where  $s_{rw}$  and  $s_{ra}$  are the irreducible saturations of the water and air phases, respectively. The capillary pressure function is of the form

$$p_c(s) = (1 - s)\{\gamma(s^{-1} - 1) + \Theta\},$$

where  $\gamma$  and  $\Theta$  are functions of the irreducible saturations. The water and air viscosities and densities are set to be  $1cP$  and  $0.8cP$ , and  $100kg/m^3$  and  $1.3kg/m^3$ , respectively. The permeability rate is  $1 \times 10^{-12}m^2$ . A two-dimensional domain of  $4m$  width by  $1m$  depth is simulated. Finally, the boundary of the domain is divided into the following segments:

$$\begin{aligned} \Gamma_1 &= \{(x, y) : x = 0, 0 < y < 1\}, \\ \Gamma_2 &= \{(x, y) : x = 4, 0 \leq y \leq 1\} \cup \{(x, y) : y = 0, 0 \leq x < 4\}, \\ \Gamma_3 &= \emptyset, \\ \Gamma_4 &= \{(x, y) : y = 1, 0 \leq x < 4\}. \end{aligned}$$

A uniform partition of  $\Omega$  into rectangles with  $h = \Delta x = \Delta y$  is taken, and the time step  $\Delta t$  is required to satisfy (5.28). The Raviart–Thomas space of lowest-order over rectangles is chosen. Tables 1 and 2 describe the errors and convergence orders for the pressure and saturation at time  $t = 1min$ , respectively. Experiments at other times and on finer meshes are also carried out; similar results are observed and not reported here.

From Table 1, we see that the scheme is first-order accurate both in  $L^2$  and  $L^\infty$  norms for the pressure, i.e., optimal order. Table 2 shows that the scheme is almost optimal order for the saturation. Thus the numerical experiments in the two tables are in agreement with our earlier analytic results.



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