

Kerr-Schild Structure and Harmonic 2-forms on (A)dS-Kerr-NUT Metrics

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ABSTRACT

We demonstrate that the general (A)dS-Kerr-NUT solutions in D dimensions with $([D/2], [(D+1)/2])$ signature admit $[D/2]$ linearly-independent, mutually-orthogonal and affinely-parameterised null geodesic congruences. This enables us to write the metrics in a multi-Kerr-Schild form, where the mass and all of the NUT parameters enter the metrics linearly. In the case of $D = 2n$, we also obtain n harmonic 2-forms, which can be viewed as charged (A)dS-Kerr-NUT solution at the linear level of small-charge expansion, for the higher-dimensional Einstein-Maxwell theories. In the BPS limit, these 2-forms reduce to $n - 1$ linearly-independent ones, whilst the resulting Calabi-Yau metric acquires a Kähler 2-form, leaving the total number the same.

1 Introduction

One intriguing feature of General Relativity is that, despite its high degree of non-linearity, many exact solutions can be cast into a Kerr-Schild form [1] where non-trivial parameters such as mass, charge, or cosmological constant enter the metrics as a linear perturbation of flat spacetime. A simple example is the (A)dS metric, which can be written as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_n^2 + \Lambda r^2 (dt - dr)^2, \quad (1)$$

where the first three terms describe the $(n + 2)$ -dimensional Minkowski spacetime and the cosmological constant enters the last term linearly. More complicated examples include the Plebanski metric [2]; in (2,2) signature, the Plebanski metric can have a double Kerr-Schild form where both the mass and the NUT charge enter the metric linearly [3].

The most general higher-dimensional (A)dS-Kerr-NUT solutions, which can be viewed as higher-dimensional generalisations of the Plebanski metric, were recently obtained in [4]. The solutions are parameterised by the mass, multiple NUT charges and arbitrary orthogonal rotations. The metrics have $U(1)^n$ isometries, where $n = [(D + 1)/2]$. They are demonstrated [5] to be of type D in the higher-dimensional generalisation [6] of the Petrov classification.

Many further interesting properties of the metrics were obtained, such as the separability of the Hamiltonian-Jacobi and Klein-Gordon equations [7], and the existence of Killing-Yano tensors [8]. The metrics also admit BPS limits where the Killing spinors can emerge [4]. In the odd $2n + 1$ dimensions, this leads to a large class of Einstein-Sasaki metrics with $U(1)^n$ isometry, generalising the previously known $Y^{p,q}$ [9] and L^{pqr} [10] spaces. In the even $2n$ dimensions, this leads to the non-compact Calabi-Yau metrics that can provide a resolution of the cone over the Einstein-Sasaki metrics constructed in the odd dimensions [11, 12].

In this letter, we demonstrate in section 2 that the D -dimensional (A)dS-Kerr-NUT solution admits $[D/2]$ linearly-independent, mutually-orthogonal and affinely parameterised null geodesic congruences upon Wick-rotation of the metric to $([D/2], [(D+1)/2])$ signature. This enables us to cast the metric into the multi-Kerr-Schild form, where the mass and all of the NUT parameters enter the metric linearly. In section 3, we obtain n harmonic 2-forms on the (A)dS-Kerr-NUT metrics in $D = 2n$ dimensions. In the BPS limit, these n harmonic 2-forms becomes linearly dependent, and the number of linearly-independent ones becomes $n - 1$. However, a Kähler 2-form emerges under the BPS limit, and hence the total number of harmonic 2-forms remains n . We conclude the letter in section 4.

2 Multi-Kerr-Schild structure

Let us first consider the case of $D = 2n + 1$ dimensions, for which the metric was given in [4]. In order to put the metric in a Kerr-Schild form, it is necessary to Wick rotate to $(n, n + 1)$ signature. This can be easily achieved by Wick rotating all the spatial $U(1)$ coordinates. The corresponding metric is then given by

$$ds^2 = \sum_{\mu=1}^n \left\{ \frac{dx_{\mu}^2}{Q_{\mu}} - Q_{\mu} \left(\sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_k \right)^2 \right\} + \frac{c}{\left(\prod_{\nu=1}^n x_{\nu}^2 \right)} \left(\sum_{k=0}^n A^{(k)} d\psi_k \right)^2, \quad (2)$$

where

$$\begin{aligned} Q_{\mu} &= \frac{X_{\mu}}{U_{\mu}}, & U_{\mu} &= \prod_{\nu=1}^m (x_{\nu}^2 - x_{\mu}^2), & X_{\mu} &= \sum_{k=1}^n c_k x_{\mu}^{2k} + \frac{c}{x_{\mu}^2} - 2b_{\mu}, \\ A_{\mu}^{(k)} &= \sum_{\nu_1 < \nu_2 < \dots < \nu_k}^{\prime} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, & A^{(k)} &= \sum_{\nu_1 < \nu_2 < \dots < \nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, \end{aligned} \quad (3)$$

The prime on the summation and product symbols in the definition of $A_{\mu}^{(k)}$ and U_{μ} indicates that the index value μ is omitted in the summations of the ν indices over the range $[1, n]$. Note that ψ_0 was denoted as t in [4], playing the rôle of the time like coordinate in the $(1, 2n)$ spacetime signature. In this way of writing the metric, all of the integration constants of the solution enter only in the functions X_{μ} . The constant $c_n = (-1)^n \Lambda$ is fixed by the value of the cosmological constant, with $R_{\mu\nu} = 2n\Lambda g_{\mu\nu}$. The other $2n$ constants c_k , c and b_{μ} are arbitrary. These are related to the n rotation parameters, the mass and the $(n - 1)$ NUT parameters, with one parameter being trivial and removable through a scaling symmetry [4]. Note that in $(n, n + 1)$ signature, the NUT charges are really masses with respect to different time-like Killing vectors. However, we shall continue to refer them as NUT charges.

We now re-arrange the metric (2) into the form

$$\begin{aligned} ds^2 &= - \sum_{\mu=1}^n \frac{X_{\mu}}{U_{\mu}} \left[\sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_k + \frac{U_{\mu}}{X_{\mu}} dx_{\mu} \right] \left[\sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_k - \frac{U_{\mu}}{X_{\mu}} dx_{\mu} \right] \\ &\quad + \frac{c}{\left(\prod_{\nu=1}^n x_{\nu}^2 \right)} \left(\sum_{k=0}^n A^{(k)} d\psi_k \right)^2. \end{aligned} \quad (4)$$

If we perform the following coordinate transformation,

$$d\hat{\psi}_k = d\psi_k + \sum_{\mu=1}^n \frac{(-x_{\mu}^2)^{n-k-1}}{X_{\mu}} dx_{\mu}, \quad k = 0, \dots, n, \quad (5)$$

the metric can then be cast into the n-Kerr-Schild form, namely

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^n \frac{2b_{\mu}}{U_{\mu}} \left[\sum_{k=0}^{n-1} A_{\mu}^{(k)} d\hat{\psi}_k \right]^2, \quad (6)$$

where

$$\begin{aligned}
d\bar{s}^2 &= -\sum_{\mu=1}^n \left\{ \frac{\bar{X}_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\hat{\psi}_k \right]^2 - 2 \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\hat{\psi}_k \right] dx_\mu \right\} \\
&\quad + \frac{c}{\left(\prod_{\nu=1}^n x_\nu^2 \right)} \left(\sum_{k=0}^n A^{(k)} d\hat{\psi}_k \right)^2, \\
\bar{X}_\mu &= \sum_{k=1}^n c_k x_\mu^{2k} + \frac{c}{x_\mu^2}. \tag{7}
\end{aligned}$$

It is straightforward to verify that the metric $d\bar{s}^2$ is that of pure (A)dS spacetime. The mass and NUT parameters b_μ appear linearly in the metric ds^2 . It should be emphasised that although the constants c and c_k with $k < n$ are trivial in the metric $d\bar{s}^2$, they provide non-trivial angular momentum parameters in the metric ds^2 . It is interesting to note that all of the constants c_k , including c_n that is related to the cosmological constant, appear linearly in the metric, and can all be extracted from $d\bar{s}^2$ and grouped in the second term of (6). This implies that all the parameters, the mass, NUTs and angular momenta and cosmological constant can enter the metric linearly as a perturbation of flat spacetime. In this letter, we shall consider in detail only the Kerr-Schild form where the mass and NUT parameters enter the metric linearly as a perturbation of pure (A)dS spacetime.

The (A)dS metric (7) can be diagonalised, in a way that the second term of (6) remains simple. To do so, let us first rewrite the \bar{X}_μ as follows

$$\bar{X}_\mu = \frac{(1 + \Lambda x_\mu^2)}{x_\mu^2} \prod_{k=1}^n (a_k^2 - x_\mu^2). \tag{8}$$

Then we complete the square in $d\bar{s}^2$:

$$d\bar{s}^2 = \sum_{\mu=1}^n \left\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\hat{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \right]^2 \right\} + \frac{c}{\left(\prod_{\nu=1}^n x_\nu^2 \right)} \left(\sum_{k=0}^n A^{(k)} d\hat{\psi}_k \right)^2, \tag{9}$$

and make the coordinate transformation,

$$d\tilde{\psi}_k = -d\hat{\psi}_k + \sum_{\mu=1}^n \frac{(-x_\mu^2)^{n-k-1}}{\bar{X}_\mu} dx_\mu, \quad k = 0, \dots, n. \tag{10}$$

The metric can be put into a new form,

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^n \frac{2b_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\tilde{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \right]^2, \tag{11}$$

where

$$d\bar{s}^2 = \sum_{\mu=1}^n \left\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\tilde{\psi}_k \right]^2 \right\} + \frac{c}{\left(\prod_{\nu=1}^n x_\nu^2 \right)} \left(\sum_{k=0}^n A^{(k)} d\tilde{\psi}_k \right)^2. \tag{12}$$

Performing a recombination of the $U(1)$ coordinates, namely

$$\tau = \sum_{k=0}^n B^{(k)} d\tilde{\psi}_k, \quad \frac{\varphi_i}{a_i} = \sum_{k=1}^n B_i^{(k-1)} d\tilde{\psi}_k - \Lambda \sum_{k=0}^{n-1} B_i^{(k)} d\tilde{\psi}_k, \quad i = 1, \dots, n, \quad (13)$$

where

$$B_i^{(k)} = \sum_{j_1 < j_2 < \dots < j_k} a_{j_1}^2 a_{j_2}^2 \dots a_{j_k}^2, \quad B^{(k)} = \sum_{j_1 < j_2 < \dots < j_k} a_{j_1}^2 a_{j_2}^2 \dots a_{j_k}^2, \quad (14)$$

the odd dimensional (A)dS-Kerr-NUT metrics can be expressed as

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^n \frac{2b_\mu}{U_\mu} (k_{(\mu)\alpha} dx^\alpha)^2, \quad (15)$$

$$d\bar{s}^2 = \frac{W}{\prod_{i=1}^n \Xi_i} d\tau^2 + \sum_{\mu=1}^n \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \sum_{i=1}^n \frac{\gamma_i}{\Xi_i \prod_{k=1}^n (a_i^2 - a_k^2)} d\varphi_i^2, \quad (16)$$

$$k_{(\mu)\alpha} dx^\alpha = \frac{W}{1 + \Lambda x_\mu^2} \frac{d\tau}{\prod_{i=1}^n \Xi_i} - \frac{U_\mu dx_\mu}{\bar{X}_\mu} - \sum_{i=1}^n \frac{a_i \gamma_i d\varphi_i}{(a_i^2 - x_\mu^2) \Xi_i \prod_{k=1}^n (a_i^2 - a_k^2)}, \quad (17)$$

where

$$\Xi_i = 1 + \Lambda a_i^2, \quad \gamma_i = \prod_{\nu=1}^n (a_i^2 - x_\nu^2), \quad W = \prod_{\nu=1}^n (1 + \Lambda x_\nu^2). \quad (18)$$

If we set all but one of the b_μ to zero, the result reduces to the Kerr-Schild form for rotating (A)dS black holes obtained previously in [13].

We now turn our attention to the the case of $D = 2n$ dimensions. The corresponding (A)dS-Kerr-NUT metrics were obtained in [4]. After performing Wick rotations, the metric with (n, n) signature is given by

$$ds^2 = \sum_{\mu=1}^n \left\{ \frac{dx_\mu^2}{Q_\mu} - Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 \right\}, \quad (19)$$

where we Q_μ , U_μ and $A_\mu^{(k)}$ have the same form as those in the even dimensions, given in (3). The functions X_μ are given by

$$X_\mu = \sum_{k=0}^n c_k x_\mu^{2k} + 2b_\mu x_\mu. \quad (20)$$

The constants c_k and b_μ are arbitrary, except for $c_n = (-1)^n \Lambda$, which is fixed by the value of the cosmological constant, $R_{\mu\nu} = (2n - 1)\Lambda g_{\mu\nu}$. The metric can be re-arranged into the form

$$ds^2 = - \sum_{\mu=1}^n \frac{X_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k + \frac{U_\mu}{X_\mu} dx_\mu \right] \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k - \frac{U_\mu}{X_\mu} dx_\mu \right]. \quad (21)$$

After performing the coordinate transformation

$$d\hat{\psi}_k = d\psi_k + \sum_{\mu=1}^n \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} dx_\mu, \quad k = 0, \dots, n-1, \quad (22)$$

the metric can be cast into the n -Kerr-Schild form,

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^n \frac{2b_\mu x_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\hat{\psi}_k \right]^2 \quad (23)$$

where

$$\begin{aligned} d\bar{s}^2 &= - \sum_{\mu=1}^n \left\{ \frac{\bar{X}_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\hat{\psi}_k \right]^2 - 2 \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\hat{\psi}_k \right] dx_\mu \right\}, \\ \bar{X}_\mu &= \sum_{k=0}^n c_k x_\mu^{2k}. \end{aligned} \quad (24)$$

It is straightforward to verify that $d\bar{s}^2$ is the metric for pure (A)dS spacetime. As in the odd dimensions, this metric can be put into a diagonal form, while keeping the second term of (23) simple. To do that, we first reparameterise X_μ as

$$\bar{X}_\mu = -(1 - g^2 x_\mu^2) \prod_{k=1}^{n-1} (a_k^2 - x_\mu^2). \quad (25)$$

We then complete the square in $d\bar{s}^2$, *i.e.*

$$d\bar{s}^2 = \sum_{\mu=1}^n \left\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\hat{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \right]^2 \right\} \quad (26)$$

and make the coordinate transformation

$$d\tilde{\psi}_k = -d\hat{\psi}_k + \sum_{\mu=1}^n \frac{(-x_\mu^2)^{n-k-1}}{\bar{X}_\mu} dx_\mu, \quad k = 0, \dots, n-1. \quad (27)$$

The metric (23) can then be put into a new form:

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^n \frac{2b_\mu x_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\tilde{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \right]^2, \quad (28)$$

where

$$d\bar{s}^2 = \sum_{\mu=1}^n \left\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\tilde{\psi}_k \right]^2 \right\}. \quad (29)$$

The $d\bar{s}^2$ metric can now straightforwardly be diagonalised by means of the coordinate transformation

$$\tau = \sum_{k=0}^{n-1} B^{(k)} d\tilde{\psi}_k, \quad \frac{\varphi_i}{a_i} = \sum_{k=1}^{n-1} B_i^{(k-1)} d\tilde{\psi}_k + g^2 \sum_{k=0}^{n-2} B_i^{(k)} d\tilde{\psi}_k \quad i = 1, \dots, n-1, \quad (30)$$

where

$$B_i^{(k)} = \sum_{j_1 < j_2 < \dots < j_k} a_{j_1}^2 a_{j_2}^2 \dots a_{j_k}^2, \quad B^{(k)} = \sum_{j_1 < j_2 < \dots < j_k} a_{j_1}^2 a_{j_2}^2 \dots a_{j_k}^2. \quad (31)$$

The even dimensional (A)-dS Kerr-NUT metrics can now be expressed as

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^n \frac{2b_\mu x_\mu}{U_\mu} (k_{(\mu)\alpha} dx^\alpha)^2, \quad (32)$$

where

$$d\bar{s}^2 = \frac{W}{\prod_{i=1}^{n-1} \Xi_i} d\tau^2 + \sum_{\mu=1}^n \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \sum_{i=1}^{n-1} \frac{\gamma_i}{a_i^2 \Xi_i \prod_{k=1}^{n-1} (a_i^2 - a_k^2)} d\varphi_i^2, \quad (33)$$

$$k_{(\mu)\alpha} dx^\alpha = \frac{W}{1 - g^2 x_\mu^2} \frac{d\tau}{\prod_{i=1}^{n-1} \Xi_i} - \frac{U_\mu dx_\mu}{\bar{X}_\mu} - \sum_{i=1}^{n-1} \frac{\gamma_i d\varphi_i}{(a_i^2 - x_\mu^2) a_i \Xi_i \prod_{k=1}^{n-1} (a_i^2 - a_k^2)}, \quad (34)$$

where Ξ_i , γ_i and W have the same structure as that in the even dimensions, given by (18). When all but one of the b_μ vanishes, the metric reduces to the Kerr-Schild form of the rotating (A)dS black hole obtained in [13].

To summarise, we find that in both even and odd dimensions, the (A)dS-Kerr-NUT solution can be cast into the following multi-Kerr-Schild form:

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^n \frac{2b_\mu f(x_\mu)}{U_\mu} (k_{(\mu)\alpha} dx^\alpha)^2, \quad (35)$$

where $f(x_\mu) = 1$ for odd dimensions and $f(x_\mu) = x_\mu$ for even dimensions. The vectors $k_{(\mu)\alpha}$ are n linearly-independent, mutually-orthogonal and affinely-parameterised null geodesic congruences, satisfying

$$k_{(\mu)\alpha} k_{(\nu)}^\alpha = 0, \quad k_{(\mu)}^\alpha \bar{\nabla}_\alpha k_{(\mu)\beta} = 0. \quad (36)$$

Note that the index α in $k_{\alpha(\mu)}$ can be raised with either $g^{\alpha\beta}$ or $\bar{g}^{\alpha\beta}$ for the above conditions to be satisfied.

3 Harmonic 2-forms in $D = 2n$ dimensions

In this section, we find n harmonic 2-forms $G_{(2)}^{(\mu)} = dB_{(1)}^{(\mu)}$ on the $2n$ -dimensional (A)dS-Kerr-NUT metric (19), where we use the index $\mu = 1, 2, \dots, n$ to label the harmonic 2-forms. The potentials have a rather simple form, given by

$$B_{(1)}^{(\mu)} = \frac{x_\mu}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right). \quad (37)$$

The metric (19) admits a natural vielbein basis, namely

$$e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad \tilde{e}^\mu = \sqrt{Q_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right). \quad (38)$$

In this vielbein basis, the harmonic 2-forms $G_{(2)}^{(\mu)}$ are given by

$$G_{(2)}^{(\mu)} = \sum f_\nu^{(\mu)} e^\nu \wedge \tilde{e}^\nu, \quad (39)$$

where the coefficients are

$$\begin{aligned} f_\mu^{(\mu)} &= \frac{1}{U_\mu^2} \left[A^{(n-1)} + \sum_{k=1}^{n-2} (-1)^k (2k+1) x_\mu^{2(k+1)} A_\mu^{(n-k-2)} \right], \\ f_\nu^{(\mu)} &= -\frac{2x_\mu x_\nu}{U_\mu^2} \prod_{\rho \neq \mu, \nu} (x_\rho^2 - x_\mu^2), \quad \text{with } \mu \neq \nu. \end{aligned} \quad (40)$$

We verify with low-lying examples that all of the $G_{(2)}^{(\mu)}$ are harmonic, *i.e.* $dG_{(2)}^{(\mu)} = 0 = d * G_{(2)}^{(\mu)}$. It is worth observing that these 2-forms are harmonic regardless of the detailed structure of the functions X_μ .

It was shown in [4] that the BPS limit of the metric (19) gives rise to the non-compact Calabi-Yau metric that can provide a resolutions of the cone over the Einstein-Sasaki spaces. Under suitable coordinate transformation, the metric is given by

$$ds^2 = \sum_{\mu=1}^n \left\{ \frac{dx_\mu^2}{Q_\mu} + Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 \right\}, \quad (41)$$

where we define

$$\begin{aligned} Q_\mu &= \frac{4X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1}^m (x_\nu - x_\mu), \quad X_\mu = x_\mu \prod_{k=1}^{n-1} (x_\mu + \alpha_k) + 2b_\mu, \\ A_\mu^{(k)} &= \sum_{\nu_1 < \nu_2 < \dots < \nu_k} x_{\nu_1} x_{\nu_2} \dots x_{\nu_k}. \end{aligned} \quad (42)$$

Note that we have Wick rotated the metric to have Euclidean signature. We can choose the same form of the vielbein basis as in (38). The Kähler 2-form is then given by

$$J_{(2)} = \sum_{\mu=1}^n e^\mu \wedge \tilde{e}^\mu. \quad (43)$$

The 1-form potentials for the harmonic 2-forms are given by

$$B_{(1)}^{(\mu)} = \frac{1}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right). \quad (44)$$

The corresponding harmonic 2-forms $G_{(2)}^{(\mu)}$ have the same form as in (39), with the functions $f_{\nu}^{(\mu)}$ are given by

$$f_{\nu}^{(\mu)} = \frac{2}{U^2} \prod_{\rho \neq \mu, \nu} (x_{\rho} - x_{\mu}), \text{ with } \mu \neq \nu, \quad f_{\mu}^{(\mu)} = - \sum_{\nu \neq \mu} f_{\nu}^{(\mu)}. \quad (45)$$

Note that $G_{(2)}^{(\mu)}$ satisfy the linear relation $\sum_{\mu=1}^n G_{(2)}^{(\mu)} = 0$. Thus, in the BPS limit, there are $(n - 1)$ linearly independent such harmonic 2-forms. Together with the Kähler 2-form, the total number of harmonic 2-forms is n again.

4 Conclusion

In this letter, we explicitly express the general (A)dS-Kerr-NUT metrics in Kerr-Schild form for both even and odd dimensions. We demonstrate that, in a suitable coordinate system the mass, NUT and angular momentum parameters enter linearly in the metric, and hence they can be viewed as a linear perturbation of pure (A)dS spacetime.

We also obtain n harmonic 2-forms on the $2n$ -dimensional (A)dS-Kerr-NUT metrics. An interesting property of these harmonic 2-forms is that the closure and co-closure do not depend on the detailed structure of the functions X_{μ} . This provides a potential ansatz for charged (A)dS-Kerr-NUT solutions for pure Einstein-Maxwell theories in higher dimensions, whose explicit analytical solutions remain elusive. In the case of four dimensions, the back-reaction of the gauge field to the Einstein equations gives precisely the charged Plebanski metric [2], where only the functions X_{μ} in the metric have extra contributions from the electric and magnetic charges. However, the same phenomenon does not occur in higher dimensions; nevertheless, the harmonic 2-forms we constructed can be viewed as charged (A)dS-Kerr-NUT solutions at the linear level for small-charge expansion. Together with the charged slowly-rotating black holes obtained in [14, 15], our results may lead to the general charged (A)dS-Kerr-NUT solutions.

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