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Exact Absorption Probability in the Extremal Six-Dimensional Dyonic String Background

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ABSTRACT

We show that the minimally coupled massless scalar wave equation in the background of an six-dimensional extremal dyonic string (or D1-D5 brane intersection) is exactly solvable, in terms of Mathieu functions. Using this fact, we calculate absorption probabilities for these scalar waves, and present the explicit results for the first few low energy corrections to the leading-order expressions. For a specific tuning of the dyonic charges one can reach a domain where the low energy absorption probability goes to zero with inverse powers of the logarithm of the energy. This is a dividing domain between the regime where the low energy absorption probability approaches zero with positive powers of energy and the regime where the probability is an oscillatory function of the logarithm of the energy. By the conjectured AdS/CFT correspondence, these results shed novel light on the strongly coupled two-dimensional field theory away from its infrared conformally invariant fixed point (the strongly coupled “non-critical” string).

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Contents

1	Introduction	2
2	Scalar wave equation for the dyonic string	4
3	Solving the Mathieu equation	6
3.1	The Floquet expansion	6
3.2	The Floquet exponents	8
3.3	Asymptotic behaviour and absorption probabilities	10
4	Fixed-energy absorption, and “phase transitions”	14
4.1	Generic $\alpha^2 > 0$	15
4.2	$\alpha = \frac{1}{2}L + 1$	16
4.3	$\alpha^2 = \frac{1}{4}$	17
4.4	$\alpha^2 = 0$	17
4.5	$\alpha^2 < 0$	18
5	Conclusions	19

1 Introduction

One of the more intriguing recent developments in string theory and M-theory has been the conjecture that when the theory is placed in a background that is of the form of a product of an m -dimensional anti-de Sitter AdS_m spacetime and an n -sphere S^n , it is dual to a superconformal theory on the boundary of the AdS_m spacetime [1]. The $\text{AdS}_m \times S^n$ sphere structure arises in the near-horizon limit of non-dilatonic extremal p -branes [2, 3], and thus one arrives at a conjectured relation between aspects of the p -brane supergravity solution and the superconformal theory on the AdS_m boundary.

This conjectured relationship has been explored in a number of contexts. One particular aspect has been probed by studying the absorption probabilities and scattering cross-sections for massless fields propagating in the supergravity background describing p -brane solitons [4, 5, 6]. In particular, one is principally interested in studying the absorption probability for incident waves with low frequency ω . For example, the leading low-energy absorption probability for $\ell = 0$ partial wave of the the minimally coupled massless scalar field (dilaton) determines its two-point correlation function at the conformally invariant infrared fixed point of the strongly coupled $(m - 1)$ -dimensional field theory [7, 8, 9, 10]. Sub-leading corrections are expected to shed light on the effects of (irrelevant) perturbations away from the conformally invariant fixed point.

In many of the examples that have been studied, the wave equations turn out not to be explicitly solvable in closed form, and as a result various techniques for matching solutions in an overlap between approximately-solvable inner and outer regions have been used [11, 12, 13, 14, 15, 16, 17, 18, 5, 6, 19, 20, 21, 22, 23, 24]. The possibility of making such overlapping approximations arises because of the low-frequency nature of the incident waves. One can then obtain expressions for absorption probabilities as power-series in ω . In most cases where such approximations must be made, it is difficult to obtain more than a few sub-leading corrections to the leading-order absorption probabilities.

On the other hand, in certain special cases (of extremal p -brane systems) the wave equations turn out to be exactly solvable, and in these situations one has more complete control over the calculation of corrections to the leading-order absorption probability. One particularly interesting example arises for the extremal D3-brane, where the wave equation for massless scalars turns out to be precisely the (modified) Mathieu equation [25]. This equation has been much studied by mathematicians, and techniques have been developed that are directly adaptable to the problem of calculating the absorption probability. Indeed, the work of [26] formed the basis of the approach used in [25] for calculating the D3-brane

absorption probability.

In this paper we consider another example where the wave equation is exactly solvable, namely the extremal dyonic string [28, 29] in six dimensions. This configuration can alternatively be viewed, by diagonal dimensional reduction, as being equivalent to an extremal 2-charge black hole in $D = 5$, a D1-brane/D5-brane intersection in $D = 10$, or an M2-brane/M5-brane intersection in $D = 11$ [27]. From the point of view of the AdS/CFT conjecture this is an especially interesting case, since the near-horizon geometry of the dyonic string approaches $\text{AdS}_3 \times S^3$, and thus the superconformal field theory [30] is defined on the two-dimensional boundary of AdS_3 . A controlled method to calculate the absorption probability to an arbitrary order in energy corrections is thus of interest; such a result would provide information on the strongly coupled two-dimensional field theory perturbed away from the infrared conformally-invariant fixed point by (irrelevant) operators, *i.e.* a strongly-coupled “non-critical string” would be probed.

Low energy absorption probabilities for the D1-brane/D5-brane system, or equivalently the six-dimensional dyonic string, were considered in [22], by using the approximate methods described above, which involve a division of the spacetime into overlapping inner and outer regions. In particular, the next-to-leading order energy correction to the s-wave absorption probability was given there.

In this paper we show that the wave equation for the minimally coupled massless scalar in the extremal six-dimensional dyonic string background has in fact an exact solution; a straightforward change of variables transforms it into the modified Mathieu equation. Thus we can again, as was done for the D3-brane [25], make use of the results in [26], and calculate the absorption probabilities to an arbitrary order in the energy expansion. However, there are some significant differences between the nature of the low energy expansion of the results presented in this paper and those of the extremal D3-brane, which we shall explain later in the paper.

The paper is organised as follows. In section 2, we write down the scalar wave equation in the background of an extremal dyonic string, and show that it can be transformed into the Mathieu equation. In section 3, we discuss the details of how to solve the Mathieu equation, paying particular attention to some subtleties associated with the particular parameter values that arise in the dyonic background, and their energy dependence. We obtain the absorption probability as an energy expansion for a fixed dyonic string configuration. In section 4, we explore some different regions in the parameter space, for which the small expansion parameter is governed by the ratio of the two dyonic string charges. By tuning

the ratio of the two charges one probes regimes whose absorption probabilities exhibit novel functional dependence on the energy parameter.

2 Scalar wave equation for the dyonic string

We shall consider a minimally coupled scalar field propagating in the background of an extremal dyonic string in six dimensions, for which the metric is

$$ds_6^2 = (H_e H_m)^{-1/2} (-dt^2 + dx^2) + (H_e H_m)^{1/2} (dr^2 + r^2 d\Omega_3^2), \quad (2.1)$$

where

$$H_e = 1 + \frac{Q_e}{r^2}, \quad H_m = 1 + \frac{Q_m}{r^2}. \quad (2.2)$$

The wave equation $\square\Phi = 0$ can be separated, by assuming coordinate dependence

$$\Phi(t, r, \theta_i) = \phi(r) Y(\theta_i) e^{-i\omega t}, \quad (2.3)$$

where $Y(\theta_i)$ denotes an harmonic on S^3 , satisfying $\nabla^2 Y = -\ell(\ell+2)Y$ on the unit 3-sphere.

The radial function $\phi(r)$ therefore satisfies the equation

$$\frac{d^2\phi}{dr^2} + \frac{3}{r} \frac{d\phi}{dr} + \left(\omega^2 H_e H_m - \frac{\ell(\ell+2)}{r^2} \right) \phi = 0. \quad (2.4)$$

Defining $\rho = \omega r$, the equation becomes

$$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho} \frac{d\phi}{d\rho} + \left(1 + \frac{\lambda_e^2 + \lambda_m^2 - \ell(\ell+2)}{\rho^2} + \frac{\lambda_e^2 \lambda_m^2}{\rho^4} \right) \phi = 0, \quad (2.5)$$

where $\lambda_e^2 = \omega^2 Q_e$ and $\lambda_m^2 = \omega^2 Q_m$. Note that we would obtain exactly the same equation (2.5) if we were to consider the intersection of a D1-brane and a D5-brane in $D = 10$, or for such an intersection with NS-NS branes, or an intersection of an M2-brane and an M5-brane in $D = 11$. We would also encounter the same equation if we considered a 2-charge black hole in $D = 5$. These equivalences hold because the associated brane configurations are related to one another by diagonal dimensional reduction, and the wave equation is invariant under such reductions on the world volume.

It is easy to see that the equation (2.5) can be transformed into the Mathieu equation, by defining

$$\phi(\rho) = \frac{1}{\rho} \Psi(\rho), \quad \rho = \sqrt{\lambda_e \lambda_m} e^{-z}, \quad (2.6)$$

whence we obtain

$$\Psi'' + (8\lambda^2 \cosh(2z) - 4\alpha^2) \Psi = 0. \quad (2.7)$$

Here, we have defined

$$\begin{aligned}
\alpha^2 &\equiv \frac{1}{4}(\ell + 1)^2 - \lambda^2 \Delta , \\
\lambda^2 &\equiv \frac{1}{4}\lambda_e \lambda_m = \frac{1}{4}\omega^2 \sqrt{Q_e Q_m} , \\
\Delta &\equiv \frac{\lambda_e}{\lambda_m} + \frac{\lambda_m}{\lambda_e} = \sqrt{\frac{Q_e}{Q_m}} + \sqrt{\frac{Q_m}{Q_e}} = \frac{Q_e + Q_m}{\sqrt{Q_e Q_m}} .
\end{aligned}
\tag{2.8}$$

Note that $\Delta \geq 2$, and that λ will be treated as a small parameter (corresponding to low-energy waves).

The result that we have found here is in many respects similar to the one for the D3-brane obtained in [25], where it was shown that the scalar wave equation in the D3-brane background could be reduced to the Mathieu equation. However, a significant difference arises in our case, due to the fact that now the parameter α , appearing in the Mathieu equation, itself depends on the small (energy dependent) parameter λ . Although we shall be able to follow the same general strategy for solving the equation as the one described in [26, 25], some adjustments will be required in order to take into account the additional λ (energy) dependence of parameter α .

The solutions to the Mathieu equation are constructed as an iterative sequence of terms which correspond to a power series in the small parameter λ . There are various ways in which one can view the relation between α and λ in equation (2.8). The simplest of these is to think of it as defining α as a power series in λ , so that one has $\alpha = \frac{1}{2}(\ell + 1) + \mathcal{O}(\lambda^2)$. In this viewpoint, one ensures that λ is small by choosing the frequency ω to be sufficiently small, with no constraints on the charges. We calculate the absorption probabilities within this framework in section 3. One can alternatively think of α as being a freely-specifiable parameter (independent of λ), where one ensures that λ , now viewed as being *defined* by the relation between α and λ in equation (2.8), is small, by choosing the charges so that Δ is sufficiently large. We calculate the absorption probabilities within this framework in section 4. (In order to distinguish the somewhat different interpretations of the λ parameter in the two viewpoints, we replace λ by Λ in section 4 when we are viewing it as a derived quantity.) It is of particular interest is to explore the limit where $\alpha \rightarrow 0$, since this approaches the boundary of the region $\alpha^2 \leq 0$, which has a novel low-energy behaviour for the absorption probability.

3 Solving the Mathieu equation

A procedure for solving the Mathieu equation was developed in [26], and formed the basis of the approach used in [25]. In order for us to present our notation and results, and to explain certain subtleties that arise, it will be necessary for us to review the procedures developed in [26, 25]. At the same time we shall elaborate on some delicacies in the calculations that were not explicitly discussed in those papers.

3.1 The Floquet expansion

The Mathieu equation can be solved by mapping the problem into one of solving a difference equation for coefficients in a series expansion. Specifically, the solution can be expressed in the Floquet form

$$\Psi(z) = \sum_{n=-\infty}^{\infty} (-1)^n C(n + \mu) e^{2(n+\mu)z} , \quad (3.1)$$

where μ is a certain constant, related somewhat non-trivially to the parameters in the Mathieu equation, which will be determined later. The quantity 2μ is known as the Floquet exponent, or Mathieu characteristic exponent. Substituting (3.1) into (2.7), one finds that the coefficients $C(n + \mu)$ satisfy the recursion relation

$$C(x + 1) + C(x - 1) = \frac{x^2 - \alpha^2}{\lambda^2} C(x) . \quad (3.2)$$

Convergence of the series (3.1) requires that $C(x + 1) \ll C(x - 1)$, and hence an approximation $C^{(0)}(x)$ to the solution of this three-term recursion relation can be obtained by solving the two-term relation

$$C^{(0)}(x - 1) = \frac{x^2 - \alpha^2}{\lambda^2} C^{(0)}(x) . \quad (3.3)$$

This is easily seen to lead to

$$C^{(0)}(x) = \frac{\lambda^{2x}}{\Gamma(x + \alpha + 1) \Gamma(x - \alpha + 1)} . \quad (3.4)$$

A solution to the full recursion relation (3.2) can now be obtained by writing

$$C(x) = C^{(0)}(x) B(x) , \quad (3.5)$$

where, substituting into (3.2), one finds that the coefficients $B(x)$ must satisfy

$$B(x) - B(x + 1) = -\frac{\lambda^4}{((x + 1)^2 - \alpha^2)((x + 2)^2 - \alpha^2)} B(x + 2) . \quad (3.6)$$

Since λ is treated as a small parameter, we can therefore solve this to any desired accuracy by an iterative procedure, in which the right-hand side is treated as a “source term” constructed using the solution for the $B(x)$ obtained at the previous iteration. Thus we solve

$$B^{(i)}(x) - B^{(i)}(x+1) = a(x) B^{(i-1)}(x+2) , \quad (3.7)$$

where

$$a(x) \equiv -\frac{\lambda^4}{((x+1)^2 - \alpha^2)((x+2)^2 - \alpha^2)} \quad (3.8)$$

and $B^{(0)}(x) = 1$. The solution to (3.6) is then given by

$$B(x) = \sum_{i \geq 0} B^{(i)}(x) = 1 + B^{(1)}(x) + B^{(2)}(x) + \dots . \quad (3.9)$$

Each successive term in this sum is smaller than its predecessor by a multiplicative factor of λ^4 , the small expansion parameter of the iterative solution.¹ It should be emphasised that in the case we are considering here for the dyonic string, unlike the D3-brane discussed in [25], the constant α in (2.7) is itself λ dependent. However, this does not alter the validity of the iterative procedure.

It is straightforward to see from the above discussion that the solutions for the quantities $B_n^{(i)}$ are given by

$$\begin{aligned} B^{(1)}(x) &= \sum_{p \geq 0} a(x+p) , \\ B^{(2)}(x) &= \sum_{p_1 \geq 0} \sum_{p_2 \geq 2} a(x+p_1) a(x+p_1+p_2) , \\ B^{(3)}(x) &= \sum_{p_1 \geq 0} \sum_{p_2 \geq 2} \sum_{p_3 \geq 2} a(x+p_1) a(x+p_1+p_2) a(x+p_1+p_2+p_3) , \\ &\vdots \\ B^{(i)}(x) &= \sum_{p_1 \geq 0} \sum_{p_2 \geq 2} \dots \sum_{p_i \geq 2} a(x+p_1) a(x+p_1+p_2) \dots a(x+p_1+\dots+p_i) . \end{aligned} \quad (3.10)$$

Defining

$$S(m_0, m_1, \dots, m_q) \equiv \sum_{i=0}^{\infty} \prod_{j=0}^q (a(x+i+j))^{m_j} , \quad (3.11)$$

it can then be straightforwardly shown [26] that the coefficients $B^{(i)}(x)$ are given by

$$B^{(1)}(x) = S(1) , \quad B^{(2)}(x) = \frac{1}{2}S(1)^2 - \frac{1}{2}S(2) - S(1, 1) , \quad \text{etc.} . \quad (3.12)$$

¹There is a subtlety concerning this point, which we shall address in detail later. For now, we just remark that the statement is true provided that $B^{(i)}(x)$ is evaluated for a sufficiently large (positive) value of x , and that we can arrange for this criterion to be met.

The sums $S(m_0, m_1, \dots, m_q)$ can all be performed explicitly, with the results expressed in terms of the Digamma and Polygamma functions. The expressions for $S(1)$, $S(2)$ and $S(1, 1)$ were given in [25]. We shall just present the expression for $S(1)$ here, since it is the only sum needed for our present purposes:

$$S(1) = \frac{(2x+3)\lambda^4}{(4\alpha^2-1)(\alpha^2-(x+1)^2)} + \frac{\psi(1+\alpha+x) - \psi(1-\alpha+x)}{\alpha(4\alpha^2-1)} \lambda^4, \quad (3.13)$$

where $\psi(x) \equiv \Gamma'(x)/\Gamma(x)$ is the Digamma function. (Our definition of $S(m_0, m_1, \dots, m_q)$ includes a factor of $(-\lambda^4)^n$ relative to the one in [25], where $n = \sum_{i=0}^q m_i$ is the rank of $S(m_0, m_1, \dots, m_q)$. Thus our $S(1)$ is $(-\lambda^4)$ times the one in [25]; $S(2)$ and $S(1, 1)$ have additional factors of λ^8 , and so on.) As we shall discuss below, the series in (3.11) defining $S(1)$ is convergent, and free of singularities, provided that the denominators in $a(x)$, given by (3.8), do not happen to vanish for some particular terms in the summation. In particular, this means that for generic x the expression (3.13) should be finite for all values of the parameter α . There are apparent divergences at $\alpha = \pm\frac{1}{2}$, but these are actually spurious, and by taking the limit $\alpha \rightarrow \pm\frac{1}{2}$ with due care, the expression (3.13) can be seen to be perfectly finite.²

3.2 The Floquet exponents

The constant μ , related to the Floquet exponent 2μ , can be determined from the observation that $\Psi(z)$ with exponent -2μ and $\Psi(-z)$ with exponent 2μ are two Floquet solutions that must be proportional to one another. From (3.1) one then reads off that the coefficients $C(n + \mu)$ must satisfy the relation

$$C(n + \mu) = \beta C(-n - \mu), \quad \text{for all } n, \quad (3.14)$$

where β is an n -independent constant. In fact it suffices to impose (3.14) for just two consecutive values of n , say $n = 0$ and $n = 1$, since all the rest can then be deduced using (3.2). Thus the content of (3.14) is just two independent equations, which determine the two unknowns β and μ . In particular, we may conclude that μ is determined by the equation

$$\frac{C(\mu)C(-\mu+1)}{C(\mu-1)C(-\mu)} = 1. \quad (3.15)$$

²A similar phenomenon arises also in the expressions that one obtains for the higher functions $S(m_0, m_1, \dots, m_q)$. For example the expression for the sum $S(1, 1)$, given in [25], has apparent divergences not only at $\alpha = \pm\frac{1}{2}$, but also at $\alpha = \pm 1$. Again, these are spurious, and a careful limiting procedure reveals that $S(1, 1)$ is indeed perfectly finite at these values.

The recursion relation (3.2) can be re-expressed as $G(x) = (V(x) - G(x+1))^{-1}$, where $G(x) \equiv C(x)/C(x-1)$ and $V(x) \equiv (x^2 - \alpha^2)/\lambda^2$. This may be solved to give the continued fraction

$$G(x) = \frac{1}{V(x) - \frac{1}{V(x+1) - \frac{1}{V(x+2) - \dots}}} . \quad (3.16)$$

It follows that (3.15), which can be rewritten as $G(\mu)G(1-\mu) = 1$, can be solved to give μ as a power series in λ :

$$\mu = \sum_{n \geq 0} \mu_n \lambda^{2n} = \mu_0 + \mu_1 \lambda^2 + \mu_2 \lambda^4 + \dots . \quad (3.17)$$

Note that in our case we obtain a power series in λ^2 , by contrast to the power series in λ^4 that arose for the D3-brane in [25]. This is because our coefficient α^2 in the Mathieu equation for the dyonic string is itself λ dependent, as given in (2.8). For the first four partial waves, $\ell = 0, 1, 2, 3$, we find, up to order λ^8 :

$$\begin{aligned} \ell = 0: \quad \mu &= \frac{1}{2} - \sqrt{\Delta^2 - 1} \lambda^2 - \frac{\Delta(2\Delta^2 - 3)}{2\sqrt{\Delta^2 - 1}} \lambda^4 - \frac{(2\Delta^2 - 3)(8\Delta^4 - 12\Delta^2 + 5)}{8(\Delta^2 - 1)^{3/2}} \lambda^6 \\ &\quad - \frac{\Delta(240\Delta^8 - 1080\Delta^6 + 1798\Delta^4 - 1316\Delta^2 + 355)}{48(\Delta^2 - 1)^{5/2}} \lambda^8 + \mathcal{O}(\lambda^{10}) , \\ \ell = 1: \quad \mu &= 1 - \frac{1}{2}\Delta \lambda^2 - \frac{3\Delta^2 + 8}{24} \lambda^4 - \frac{9\Delta^4 + 88\Delta^2 - 36}{144\Delta} \lambda^6 \\ &\quad - \frac{135\Delta^6 + 3248\Delta^4 - 2032\Delta^2 + 576}{3456\Delta^2} \lambda^8 + \mathcal{O}(\lambda^{10}) , \\ \ell = 2: \quad \mu &= \frac{3}{2} - \frac{1}{3}\Delta \lambda^2 - \frac{4\Delta^2 + 9}{108} \lambda^4 - \frac{\Delta(16\Delta^2 + 117)}{1944} \lambda^6 - \frac{1600\Delta^4 + 25380\Delta^2 + 18387}{699840} \lambda^8 + \mathcal{O}(\lambda^{10}) , \\ \ell = 3: \quad \mu &= 2 - \frac{1}{4}\Delta \lambda^2 - \frac{15\Delta^2 + 32}{960} \lambda^4 - \frac{\Delta(225\Delta^2 + 1504)}{115200} \lambda^6 \\ &\quad - \frac{16875\Delta^4 + 235712\Delta^2 + 140288}{55296000} \lambda^8 + \mathcal{O}(\lambda^{10}) . \end{aligned} \quad (3.18)$$

The general formula, up to order λ^8 , turns out to be

$$\begin{aligned} \mu &= \frac{1}{2}(\ell + 1) - \frac{\Delta}{\ell + 1} \lambda^2 - \frac{2 + \ell(\ell + 2)(\Delta^2 + 2)}{\ell(\ell + 1)^3(\ell + 2)} \lambda^4 - \frac{4(\ell + 1)^2(3\ell^2 + 6\ell + 2)\Delta + 2\ell^2(\ell + 2)^2\Delta^3}{\ell^2(\ell + 1)^5(\ell + 2)^2} \lambda^6 \\ &\quad - \frac{\lambda^8}{(\ell - 1)\ell^3(\ell + 1)^7(\ell + 2)^3(\ell + 3)} \left\{ 2(\ell + 1)^4(15\ell^4 + 60\ell^3 + 55\ell^2 - 10\ell - 12) \right. \\ &\quad \quad \left. + 4(\ell - 1)(\ell + 1)^2(\ell + 3)(15\ell^4 + 60\ell^3 + 80\ell^2 + 40\ell + 8)\Delta^2 \right. \\ &\quad \quad \left. + 5(\ell - 1)\ell^3(\ell + 2)^3(\ell + 3)\Delta^4 \right\} + \mathcal{O}(\lambda^{10}) , \end{aligned} \quad (3.19)$$

when $\ell = 2, 3, 4, \dots$. In fact if ℓ takes a generic non-integer value, (3.19) is universally valid. The integer values for ℓ have to be treated carefully because the general solution for

μ develops poles at the integer values, and one has to resort to a case-by-case analysis. It seems that in fact only $\ell = 0$ and $\ell = 1$ are not described by the general formula (3.19).³

3.3 Asymptotic behaviour and absorption probabilities

A remarkable property of the Floquet solution of the Mathieu equation is that one can re-express the series expansion (3.1) in terms of Bessel functions rather than exponentials, with the *same* coefficients $C(n + \mu)$ (up to $(-1)^n$ factors). (The explanation can be traced back to the fact that exponential functions and Bessel functions are closely related *via* the Laplace transform [32].) The required solution of the Mathieu equation, which is in-going on the horizon at $z = \infty$, can be written in the form

$$\Psi(z) = \mathcal{M}(\mu, z) \equiv \sum_{n=-\infty}^{\infty} \frac{C(n + \mu)}{C(\mu)} J_n(2\lambda e^{-z}) H_{n+2\mu}^{(1)}(2\lambda e^z), \quad (3.20)$$

(The second solution is obtained by choosing an independent Bessel function, such as $H_{n+2\mu}^{(2)}$, in place of the first Hankel function $H_{n+2\mu}^{(1)}$.)

The series (3.9) for $\Psi(z)$ is dominated by the $n = 0$ term near the horizon, where $z \rightarrow \infty$, and this allows the asymptotic behaviour in the $z \rightarrow \infty$ region to be read off easily. To determine the behaviour at spatial infinity, corresponding to $z \rightarrow -\infty$, one can make use of the fact, mentioned earlier, that $\Psi(-z)$ with Floquet exponent 2μ is proportional to $\Psi(z)$ with Floquet exponent -2μ . In fact from (3.14) one obtains the result that

$$\mathcal{M}(-\mu, z) = \frac{C(-\mu)}{C(\mu)} \mathcal{M}(\mu, -z). \quad (3.21)$$

After some further steps described in detail in [25], in which one extracts the ratio between the amplitudes of the ingoing and outgoing waves at spatial infinity, one concludes that the

³In this section, we shall present explicit results for corrections up to and including $\mathcal{O}(\lambda^6)$ with respect to the leading-order results for absorption probabilities. One might think, therefore, that it would be sufficient to determine μ up to $\mathcal{O}(\lambda^6)$, since it has a λ -independent constant term for its leading order. However, as will be seen in the calculation of the absorption probability P in (3.22) and (3.23) below, there is a factor of $|e^{2i\mu\pi} - e^{-2i\mu\pi}|^2$ in the expression for P , and so the fact that the constant terms in μ are integers or half-integers means that $|e^{2i\mu\pi} - e^{-2i\mu\pi}|$ is a quantity of $\mathcal{O}(\lambda^2)$, rather than the naively-expected $\mathcal{O}(1)$. Consequently, in the calculation of $|e^{2i\mu\pi} - e^{-2i\mu\pi}|$ it is necessary to work with μ up to $\mathcal{O}(\lambda^8)$, for the purposes of eventually obtaining corrections up to $\mathcal{O}(\lambda^6)$ in the absorption probabilities. Similarly, the occurrence of a pole in $\Gamma(-\mu - \alpha + 1)$ in (3.4) means that one must also work with μ to order λ^8 in that expression too, when calculating $C(-\mu)$ using (3.5). By the same token, in the D3-brane calculations described in [25], one should calculate μ up to $\mathcal{O}(\lambda^{16})$ in order to obtain corrections up to $\mathcal{O}(\lambda^{12})$ in the absorption probabilities.

absorption probability is given by

$$P = \frac{|\eta - \eta^{-1}|^2}{|\eta - \eta^{-1}|^2 + |\chi - \chi^{-1}|^2}, \quad (3.22)$$

where

$$\eta = e^{2i\mu\pi}, \quad \chi = \frac{C(-\mu)}{C(\mu)}. \quad (3.23)$$

Note that when determining the orders in λ to which one must work so as to obtain corrections to P to a specified order in λ , one must be careful of a number of points. One such point was discussed in footnote 3, relating to the fact that the leading-order terms in μ are always integers or half-integers. Another point is that if the leading-order term in χ is a λ -independent constant, then if this constant happens to equal ± 1 then the quantity $|\chi - \chi^{-1}|$, which would naively have been of $\mathcal{O}(1)$, would actually instead be of higher order in λ . This in turn would necessitate evaluating χ to a higher order in λ than one might naively have expected, for the purposes of finding the corrections to the absorption probability up to some particular order in λ . This particular subtlety does not arise in our calculations in this section, but it will play a rôle in the calculations in section 4.

The computation of the absorption probability is now reduced to the problem of calculating the ratio $C(-\mu)/C(\mu)$, where $C(x)$ is determined from (3.5), (3.9), (3.11) and (3.12). It is at this point that we meet a subtlety, which was not explicitly encountered in the general discussion in [26]. It arises because the parameters that occur in our Mathieu equation are not generic, but are instead rather specific ones that are fortuitously inconvenient.

It is evident from (3.8) that the sums in (3.11) are rapidly convergent, and that the possibility of divergent behaviour would arise only if the denominators in $a(x)$ were to go to zero for some particular low term in the summations. Since x , and α , are functions of the small parameter λ , the issue that concerns us here is therefore not that the sums $S(m_0, m_1, \dots, m_q)$ might be infinite, but rather that they might acquire inverse powers of λ that could counteract the manifest λ^4 numerator in (3.8). If this were to occur, then our implicit assumption that each term $B^{(i)}(x)$ in (3.9) were smaller than its predecessor $B^{(i-1)}(x)$ by a factor of λ^4 would be invalid. At the very least this would complicate the discussion because it would mean that we would have to work to higher order in the iterative solutions $B^{(i)}(x)$ than naively appeared necessary. At worst, it could be difficult to determine whether one had included sufficiently many terms in the iterative solution to give an accurate result to the desired order in λ .

First, let us note that no difficulty of this kind arises in the calculation of $C(\mu)$. This is because $\mu = \frac{1}{2}(\ell + 1) + \mathcal{O}(\lambda^2)$, and $\alpha = \frac{1}{2}(\ell + 1) + \mathcal{O}(\lambda^2)$, and hence from (3.8) it follows

that all the quantities $a(x+i+j)/\lambda^4$ appearing in the sums (3.11) are strictly finite as λ tends to zero. If we define the rank $n = \sum_{i=0}^q m_i$ for the sum $S(m_0, m_1, \dots, m_q)$, then it follows that the quantities $\lambda^{-4n} S(m_0, m_1, \dots, m_q)$, evaluated at $x = \mu$, are all of the form of a non-zero constant $+\mathcal{O}(\lambda^2)$ as λ tends to zero, and so the iterative solutions $B^{(i)}(\mu)$ do indeed have the “naively expected” form $B^{(i)}(\mu) = \mathcal{O}(\lambda^{4i})$.

The potential difficulties arise when one considers $C(-\mu)$, since now we see from (3.8) that for sufficiently small values of the summation variables in (3.11) one will encounter denominators in $a(x)$ that tend to zero as λ tends to zero. This means that these sums $S(m_0, m_1, \dots, m_q)$, evaluated at $x = -\mu$, and the associated iterative solutions $B^{(i)}(-\mu)$, will have λ dependence whose leading order is at a more dominant power than the naively expected one.

The easiest way to handle this problem is to make use once again of the recursion relation (3.6) for $B(x)$. By iterating it n times, we can express $B(-\mu)$ in terms of $B(n-\mu)$ and $B(n+1-\mu)$. Thus we can choose $n = \ell + 1$, so that the leading-order constant term in the λ expansion of $n-\mu$ is precisely the same positive constant as occurs in the λ expansion of μ itself. By this means, we reduce the problem of calculating $B(-\mu)$, and hence $C(-\mu)$, to the problem of calculating $B(\frac{1}{2}(\ell+1) + \mathcal{O}(\lambda^2))$ and $B(\frac{1}{2}(\ell+1) + 1 + \mathcal{O}(\lambda^2))$. By our previous discussion, these cases are easily evaluated, in terms of quantities $S(m_0, m_1, \dots, m_q)$ that have precisely the “expected” λ^{4k} leading-order dependences, where $k = \sum_i m_i$. Note that, as mentioned in footnote 3, one must be careful to include one additional order in λ^2 in the expansion for μ , over and above the naively expected one, on account of the pole at $\lambda = 0$ in the quantity $\Gamma(-\mu - \alpha + 1)$ that arises in the computation of $C(-\mu)$ using (3.4) and (3.5).

The upshot from the above discussion is that provided we handle the $C(-\mu)$ calculation carefully, it is necessary to include only the first iterative correction $B^{(1)}(x) = S(1)$ in order to obtain our results for absorption probabilities with corrections up to $\mathcal{O}(\lambda^6)$ to the leading-order result. (By the same token, the results up to $\mathcal{O}(\lambda^{12})$ corrections for the D3-brane, discussed in [25], require the inclusion only of the first three iterations $B^{(1)}(x)$, $B^{(2)}(x)$ and $B^{(3)}(x)$.) Of course the fact that each term $B^{(i)}(x)$ comes with an overall leading-order λ^{4i} factor means that one need calculate the associated sums $S(m_0, m_1, \dots, m_q)$ only up to λ^{n-4i} corrections in order to obtain a results for $B(x)$ to order n . (Assuming, as usual, that the necessary recursion using (3.6) is performed first, so that the $B^{(i)}(x)$ are evaluated at arguments $x \geq \alpha$.)

We find that the absorption probability P_ℓ for the ℓ 'th partial wave has the general

structure⁴

$$P_\ell = \frac{4\pi^2 \lambda^{4+4\ell}}{(\ell+1)^2 \Gamma(\ell+1)^4} \sum_{n \geq 0} \sum_{k=0}^n b_{n,k} \lambda^{2n} (\log \bar{\lambda})^k, \quad (3.24)$$

where $\bar{\lambda} = e^\gamma \lambda$, and γ is Euler's constant. The prefactor is chosen so that $b_{0,0} = 1$. Our results for the coefficients $b_{n,k}$ with $k \leq n \leq 3$, for the first four partial waves $\ell = 0, 1, 2, 3$, are as follows:

$$\begin{aligned} \ell = 0: \quad & b_{1,0} = 4\Delta, \quad b_{1,1} = -8\Delta, \\ & b_{2,0} = \frac{1}{2}(32\Delta^2 - 7) - \frac{4}{3}(\Delta^2 + 2)\pi^2, \quad b_{2,1} = -4(10\Delta^2 + 1), \quad b_{2,2} = 16(2\Delta^2 + 1), \\ & b_{3,0} = \Delta(64\Delta^2 - 31) - \frac{4}{3}\Delta(6\Delta^2 + 17)\pi^2 - \frac{32}{3}\Delta(\Delta^2 - 1)\zeta(3), \\ & b_{3,1} = -8\Delta(22\Delta^2 + 1) + \frac{32}{3}\Delta(\Delta^2 + 5)\pi^2, \\ & b_{3,2} = 32\Delta(6\Delta^2 + 5), \quad b_{3,3} = -\frac{256}{3}\Delta(\Delta^2 + 2), \end{aligned}$$

$$\begin{aligned} \ell = 1: \quad & b_{1,0} = 5\Delta, \quad b_{1,1} = -4\Delta, \\ & b_{2,0} = \frac{1}{9}(144\Delta^2 + 14) - \frac{1}{3}\Delta^2\pi^2, \quad b_{2,1} = -\frac{1}{3}(63\Delta^2 + 8), \quad b_{2,2} = 8\Delta^2, \\ & b_{3,0} = \frac{1}{54}\Delta(2241\Delta^2 + 704) - \frac{1}{18}\Delta(33\Delta^2 + 8)\pi^2 - \frac{4}{3}\Delta^3\zeta(3), \\ & b_{3,1} = -\frac{1}{18}\Delta(1251\Delta^2 + 440) + \frac{4}{3}\Delta^3\pi^2, \\ & b_{3,2} = \frac{4}{3}\Delta(33\Delta^2 + 8), \quad b_{3,3} = -\frac{32}{3}\Delta^3, \end{aligned} \quad (3.25)$$

$$\begin{aligned} \ell = 2: \quad & b_{1,0} = \frac{40}{9}\Delta, \quad b_{1,1} = -\frac{8}{3}\Delta, \\ & b_{2,0} = \frac{1}{648}(7472\Delta^2 + 477) - \frac{4}{27}\Delta^2\pi^2, \quad b_{2,1} = -\frac{2}{27}(164\Delta^2 + 9), \quad b_{2,2} = \frac{32}{9}\Delta^2, \\ & b_{3,0} = \frac{1}{5832}\Delta(132992\Delta^2 + 25443) - \frac{2}{81}\Delta(28\Delta^2 + 3)\pi^2 - \frac{32}{81}\Delta^3\zeta(3), \\ & b_{3,1} = -\frac{2}{243}\Delta(3904\Delta^2 + 657) + \frac{32}{81}\Delta^3\pi^2, \\ & b_{3,2} = \frac{16}{27}\Delta(28\Delta^2 + 3), \quad b_{3,3} = -\frac{256}{81}\Delta^3, \end{aligned}$$

$$\begin{aligned} \ell = 3: \quad & b_{1,0} = \frac{47}{12}\Delta, \quad b_{1,1} = -2\Delta, \\ & b_{2,0} = \frac{1}{450}(3875\Delta^2 + 171) - \frac{1}{12}\Delta^2\pi^2, \quad b_{2,1} = -\frac{1}{120}(955\Delta^2 + 32), \quad b_{2,2} = 2\Delta^2, \\ & b_{3,0} = \frac{1}{432000}\Delta(6053125\Delta^2 + 786432) - \frac{1}{1440}\Delta(485\Delta^2 + 32)\pi^2 - \frac{1}{6}\Delta^3\zeta(3), \\ & b_{3,1} = -\frac{2}{14400}\Delta(255275\Delta^2 + 27488) + \frac{1}{6}\Delta^3\pi^2, \\ & b_{3,2} = \frac{1}{60}\Delta(485\Delta^2 + 32), \quad b_{3,3} = -\frac{4}{3}\Delta^3, \end{aligned}$$

⁴The absorption cross-section σ_ℓ is related to the absorption probability P_ℓ by [31]: $\sigma_\ell = \frac{4\pi}{\omega^3}(l+1)^2 P_\ell$.

Note that our result for $b_{1,1}$ at $\ell = 0$ agrees in magnitude, but not in sign, with the result obtained in [22].

For general values of ℓ it is quite straightforward to obtain expressions for the coefficients $b_{1,0}$ and $b_{1,1}$ in (3.24), using procedures analogous to those that we described previously. The only new feature is that in order to calculate $B(-\mu)$ in terms of $B(x)$ with appropriately positive arguments $(\ell + 1 - \mu)$ and $(\ell + 2 - \mu)$, one must now apply the recursion relation (3.6) a total of $(\ell + 1)$ times. Up to order λ^2 , it is easy to see that one need only solve (3.6) to linear order in $a(x)$, and evaluate $B(\ell + 1 - \mu)$ and $B(\ell + 2 - \mu)$ to zero'th order, $B(\ell + 1 - \mu) = 1 + \mathcal{O}(\lambda^4)$ and $B(\ell + 2 - \mu) = 1 + \mathcal{O}(\lambda^4)$, giving

$$B(-\mu) = 1 + \sum_{i=0}^{\ell} a(-\mu + i) + \dots . \quad (3.26)$$

Furthermore, it is clear that only those terms for which the denominator in $a(x)$ in (3.8) has zeros at $\lambda = 0$, namely $i = \ell$ and $i = \ell - 1$, will give $\mathcal{O}(\lambda^2)$ contributions. This leads to the result

$$B(-\mu) = 1 + \frac{\lambda^2}{\Delta \ell (\ell + 2)} + \mathcal{O}(\lambda^4) . \quad (3.27)$$

After further straightforward calculations, we find that the absorption probability is given by

$$P_{\ell} = \frac{4\pi^2 \lambda^{4+4\ell}}{(\ell + 1)^2 \Gamma(\ell + 1)^4} \left[1 - \frac{8\Delta}{\ell + 1} \lambda^2 \log \lambda + \frac{4\Delta \lambda^2}{(\ell + 1)^2} \left(1 + 2(\ell + 1) \psi(\ell + 1) \right) + \dots \right] , \quad (3.28)$$

where $\psi(x) \equiv \Gamma'(x)/\Gamma(x)$ is the Digamma function. Note that in this case we have not absorbed the Euler gamma constant γ in a rescaling of λ in the logarithm. However, for all integer ℓ the Digamma function has the form $\psi(\ell + 1) = R - \gamma$, where R is a rational number, and so the usual $\bar{\lambda} = e^{\gamma} \lambda$ rescaling will enable the Euler constant to be absorbed in a natural fashion.

4 Fixed-energy absorption, and “phase transitions”

So far, we have calculated absorption probabilities in the regime where Δ is fixed, and λ is a small parameter (see eqn. (2.8) for definitions of these quantities). These results describe the probabilities as continuous functions of low-energy incident waves, in a given dyonic string background. In this section, we shall study a different regime, where α , defined in (2.8), is fixed. Namely, since the background depends on two charge parameters Q_e and Q_m , one can adjust the dyonic background by tuning Δ in such a way that α is fixed (for

a fixed value of λ). This implies that

$$\frac{1}{4}(\ell + 1)^2 - \alpha^2 = \lambda^2 \Delta = \frac{1}{4}\omega^2 M \quad (4.1)$$

is fixed, where $M = Q_e + Q_m$ is the mass per unit length of the string. The low energy expansion with $\lambda \ll 1$ implies that $\Delta \gg 1$ ($1/\Delta$ can be viewed as the expansion parameter). Note that in this case $\omega^2 M$ can be large, provided that Δ is sufficiently large.

In order not to cause confusion with the previous section, we write the Mathieu equation as follows:

$$\Psi'' + (8\Lambda^2 \cosh(2z) - 4\alpha^2) \Psi = 0, \quad (4.2)$$

where

$$\Lambda^2 \equiv \frac{\frac{1}{4}(\ell + 1)^2 - \alpha^2}{\Delta}. \quad (4.3)$$

In other words Λ is now viewed as a function of Δ , having first specified α . It is arranged so that Λ is small by choosing the charges so that Δ is sufficiently large. This should be contrasted with the viewpoint in section 3, where the parameter λ is chosen to be small by taking ω to be small, with the charges unconstrained and with α a derived quantity following from the expression in equation (2.8).

This equation is analogous to the Mathieu equation discussed in [25], in that α^2 is now a given fixed parameter, independent of the expansion parameter Λ . However, in our case α^2 can be arbitrary, whereas in [25] the analogous parameter could take only discrete integer or half-integer values. We see from (2.8) that depending on the value of Δ , we can have α^2 either positive, negative or zero.

4.1 Generic $\alpha^2 > 0$

The Floquet exponents in this case turn out to be given by

$$\begin{aligned} \mu = \alpha - \frac{\Lambda^4}{\alpha(4\alpha^2 - 1)} - \frac{(60\alpha^4 - 35\alpha^2 + 2) \Lambda^8}{4\alpha^3(\alpha^2 - 1)(4\alpha^2 - 1)^3} \\ - \frac{(6720\alpha^{10} - 18480\alpha^8 + 15260\alpha^6 - 4705\alpha^4 + 413\alpha^2 - 18) \Lambda^{12}}{4\alpha^5(4\alpha^2 - 1)^5(\alpha^2 - 1)^2(4\alpha^2 - 9)} + \mathcal{O}(\Lambda^{16}). \end{aligned} \quad (4.4)$$

This result will be valid for all generic values of α , but breaks down at the poles occurring when α is an integer or half-integer. We shall return to a discussion of these special cases later, and concentrate for now on the generic situation when (4.4) is valid. It is useful to define

$$\alpha = \frac{1}{2}L + 1, \quad (4.5)$$

so that for now, we shall consider cases where L is not an integer. For such generic values of the parameter α there are no subtleties involving the occurrence of poles or zeros in the calculation of $C(-\mu)$, nor in the expression $|\eta - \eta^{-1}|$ appearing in (3.22). We shall just present results for the first two corrections to the leading-order absorption. One has to distinguish between two cases, according to whether $\alpha > \frac{1}{2}$ or $\alpha < \frac{1}{2}$ (we shall consider the case $\alpha = \frac{1}{2}$ later). The reason for this distinction is that when $\alpha > \frac{1}{2}$ the next-to-leading order corrections are always of the form Λ^4 and $\Lambda^4 \log \Lambda$, whilst if $\alpha < \frac{1}{2}$ the next-to-leading order correction is of the form $\Lambda^{8\alpha}$. For $\alpha > \frac{1}{2}$, we find that the absorption probability is given by

$$P_\ell(\alpha) = \frac{\pi^2 \Lambda^{8\alpha}}{\alpha^2 [\Gamma(-2\alpha)^2 - \Gamma(2\alpha)^2]} \left(1 + b_{1,1} \Lambda^4 \log \Lambda + b_{1,0} \Lambda^4 + \dots \right), \quad (4.6)$$

where

$$\begin{aligned} b_{1,1} &= \frac{8 [\Gamma(-2\alpha)^2 + \Gamma(2\alpha)^2]}{\alpha (4\alpha^2 - 1) [\Gamma(-2\alpha)^2 - \Gamma(2\alpha)^2]}, \\ b_{1,0} &= -\frac{4\pi \cot 2\pi\alpha}{\alpha (4\alpha^2 - 1)} + \frac{1}{2} b_{1,1} \left(\frac{4\alpha^2}{4\alpha^2 - 1} - \psi(-2\alpha) - \psi(2\alpha) \right), \end{aligned} \quad (4.7)$$

where ψ is again the Digamma function.

When $\alpha < \frac{1}{2}$, we find that up to the first correction to the leading order, the absorption probability is

$$P_\ell(\alpha) = \frac{4\Lambda^{8\alpha} \Gamma(-2\alpha)^2 \sin^2 2\pi\alpha}{\Gamma(2\alpha)^2} \left(1 + 2\Lambda^{8\alpha} \frac{\Gamma(-2\alpha)^2}{\Gamma(2\alpha)} \cos 4\pi\alpha + \dots \right). \quad (4.8)$$

The next correction will be either $\Lambda^{16\alpha}$, or $\Lambda^4 \log \Lambda$, depending on whether $\alpha < \frac{1}{4}$ or $\alpha \geq \frac{1}{4}$.

4.2 $\alpha = \frac{1}{2}L + 1$

In the previous subsection, we saw that for generic values of $\alpha \equiv \frac{1}{2}L + 1$, corresponding to L non-integer, the leading-order absorption probability is of the form

$$P_L \sim \Lambda^{4L}. \quad (4.9)$$

This result is not applicable if L is an integer. This can be seen both from the resulting poles in the expression (4.4) for μ , and the singular behaviour of $\Gamma(-2\alpha)$ and the trigonometric functions in (4.6) and (4.8). In fact the absorption probabilities for the special cases $L =$ integer are precisely those calculated in [25]. These have the leading-order form

$$P_L \sim \Lambda^{8+4L}. \quad (4.10)$$

To be precise, the absorption probabilities in these cases are given by [25]

$$P_L = \frac{4\pi^2 \Lambda^{8+4L}}{(L+1)^2 \Gamma(L+1)^4} \sum_{n \geq 0} \sum_{k=0}^n b_{n,k} \Lambda^{4n} (\log \bar{\Lambda})^k, \quad (4.11)$$

where $\bar{\Lambda} = e^\gamma \Lambda$. The explicit results for the coefficients $b_{n,k}$ with $k \leq n \leq 3$ were given for $L = 0, 1, 2$ in [25]. In order to calculate the corrections up to this Λ^{12} order, it is necessary to calculate the Floquet exponent 2μ up to order Λ^{16} , for the reasons that we discussed in footnote 3. They are given by

$$\begin{aligned} L = 0: \quad \mu &= 1 - \frac{i\sqrt{5}}{6} \Lambda^4 + \frac{7i}{216\sqrt{5}} \Lambda^8 + \frac{11851i}{62208\sqrt{5}} \Lambda^{12} - \frac{1243157i}{55987200\sqrt{5}} \Lambda^{16} + \mathcal{O}(\Lambda^{20}), \\ L = 1: \quad \mu &= \frac{3}{2} - \frac{1}{12} \Lambda^4 + \frac{133}{8640} \Lambda^8 + \frac{311}{3110400} \Lambda^{12} + \frac{908339}{1254113280} \Lambda^{16} + \mathcal{O}(\Lambda^{20}), \\ L = 2: \quad \mu &= 2 - \frac{1}{30} \Lambda^4 - \frac{137}{54000} \Lambda^8 + \frac{305843}{1360800000} \Lambda^{12} - \frac{64347197}{1714608000000} \Lambda^{16} + \mathcal{O}(\Lambda^{20}). \end{aligned} \quad (4.12)$$

It is also necessary to calculate the sums $S(m_0, m_1, \dots, m_q)$ up to rank $n \equiv \sum_i m_i = 3$. The expressions for $n = 1$ and $n = 2$ are given in [25].

There are in fact special values of the integer L which themselves must be treated as special cases. One of these is $L = -2$, corresponding to $\alpha^2 = 0$. The other is $L = -1$, corresponding to $\alpha^2 = \frac{1}{4}$. These two cases are discussed in the next two subsections.

4.3 $\alpha^2 = \frac{1}{4}$

In this case, the Floquet exponent 2μ is given by

$$\mu = \frac{1}{2} + i \Lambda^2 - \frac{15i}{8} \Lambda^6 + \frac{9707i}{1152} \Lambda^{10} + \mathcal{O}(\Lambda^{14}). \quad (4.13)$$

We find that the absorption probability is then given by

$$P = 4\pi \Lambda^4 \left(1 + 16\Lambda^4 (\log \Lambda)^2 - 4\Lambda^4 \log \Lambda - \left(\frac{7}{2} + \frac{8}{3} \pi^2 \right) \Lambda^4 + \dots \right). \quad (4.14)$$

In fact this result can also be read off directly from the $\ell = 0$ expressions in (3.25), by setting $\Delta = 0$.

4.4 $\alpha^2 = 0$

When $\alpha = 0$, we find that the Floquet exponent is given by

$$\mu = \sqrt{2} \Lambda^2 \left(1 + \frac{25}{8} \Lambda^4 + \frac{31783}{1152} \Lambda^8 + \frac{26498155}{82944} \Lambda^{12} + \mathcal{O}(\Lambda^{16}) \right). \quad (4.15)$$

We then obtain the absorption probability

$$P = \frac{\pi^2}{\pi^2 + (2 \log \bar{\Lambda})^2} \left(1 - \frac{32}{3} \Lambda^4 (\log \bar{\Lambda})^2 - \frac{16}{3} (4\zeta(3) - 3) \frac{\Lambda^4 \log \bar{\Lambda}}{\pi^2 + (2 \log \bar{\Lambda})^2} + \mathcal{O}(\Lambda^8) \right), \quad (4.16)$$

where $\bar{\Lambda} = e^\gamma \Lambda$. Note that as $\Lambda \rightarrow 0$ the absorption probability (4.16) approaches zero not as a positive (integer) power of Λ , but instead as $(\log \bar{\Lambda})^{-2}$. This result indicates that on the field theory side, the infrared limit with the specific choice of the dyonic background that ensures $\alpha = 0$ exhibits new features. For example, the two-point correlation function for the minimally-coupled scalar [7] has a leading non-zero contribution that is inherently quantum in nature (as signalled by the $\log \bar{\Lambda}$ dependence). Note that the $\alpha^2 = 0$ case that we have considered here is an example of the kind we mentioned in section 3.2, where the Λ -independent term in the expansion for χ is equal to 1, implying that one loses the naively-expected leading-order constant when one evaluates $|\chi - \chi^{-1}|$.

It would be interesting to explore the effects of infinitesimal deviations from the limit $\alpha = 0$. It turns out that if one takes a special slice $\alpha^2 = b^2 \Lambda^4$, where $b = \text{constant}$, one obtains a systematic expansion of μ around small Λ :

$$\mu = \sqrt{2} \Lambda^2 \left(\sqrt{1 + \frac{b^2}{2}} + \frac{\frac{25}{8} + 2b^2}{\sqrt{1 + \frac{b^2}{2}}} \Lambda^4 + \mathcal{O}(\Lambda^8) \right). \quad (4.17)$$

In this case the absorption probability has the following form:

$$P = \frac{\pi^2}{\pi^2 + (2 \log \bar{\Lambda})^2} \left(1 - \frac{8}{3} \left(1 + \frac{b^2}{2}\right) \Lambda^4 (2 \log \bar{\Lambda})^2 - \frac{8}{3} [\zeta(3)(8 + b^2) - 6] \frac{\Lambda^4 \log \bar{\Lambda}}{\pi^2 + (2 \log \bar{\Lambda})^2} + \mathcal{O}(\Lambda^8) \right), \quad (4.18)$$

Note that as one approaches $\alpha^2 \sim b^2 \Lambda^4$, only the sub-leading term in the Λ expansion depends on b^2 .

4.5 $\alpha^2 < 0$

One can see from (4.3) that it is possible to choose the parameters so that α^2 becomes negative, while keeping Λ small. In such a case, we may write

$$\alpha = i\beta, \quad (4.19)$$

where β is real, and related to Δ by

$$\Lambda^2 = \frac{\frac{1}{4}(\ell + 1)^2 + \beta^2}{\Delta}. \quad (4.20)$$

In this case the Floquet exponents are given by (4.4) with $\alpha = i\beta$. The absorption probability is now qualitatively different in form. In particular, the leading-order contribution becomes oscillatory as a function of Λ ; we find that it is given by

$$P = \frac{\sinh^2 2\pi\beta}{\sinh^2 2\pi\beta + \sin^2(\theta - 4\beta \log \Lambda)} + \dots, \quad (4.21)$$

where

$$\theta = \arg \frac{\Gamma(2i\beta)}{\Gamma(-2i\beta)}. \quad (4.22)$$

The oscillations become insignificant if $\beta \gg 1$. Again this region exhibits novel functional dependence of the absorpton probability on Λ , thus signalling novel field theoretical phenomena on the dual side.

5 Conclusions

In this paper, we have studied the solutions of the minimally coupled scalar wave equation in the background of a six-dimensional extremal dyonic string. Owing to the fact that the equation is equivalent to the modified Mathieu equation, the wave-functions can be constructed exactly, in the sense that one can obtain exact solutions as power series in the parameter λ^2 appearing in the Mathieu equation (2.7). This provides a systematic power-series expansion that is quite different in nature from a power-series solution of a differential equation in some restricted range of the independent variable. In particular, it allows one to have complete control over the relation between the small-distance and large-distance asymptotics of the wave functions; it is this information that is needed in order to calculate the absorption probabilities.

Since the dyonic string background depends on two charge parameters Q_e and Q_m , there are two independent dimensionless quantities that can play the rôle of ordering parameters in the iterative solution of the Mathieu equation. One of these is the quantity $\lambda = \frac{1}{4}\omega^2 \sqrt{Q_e Q_m}$, which can be arranged to be small by taking the frequency ω of the wave sufficiently small. The other parameter is $1/\Delta = \sqrt{Q_e Q_m}/(Q_e + Q_m) \leq 1/2$, which provides a measure of the ratio between the electric and magnetic charges Q_e and Q_m . In section 3 we calculated the absorption probability at low frequency, with Δ fixed, *i.e.* with a fixed dyonic string background. In this case the leading-order constant term in the expansion for α can take only discrete values, namely the integers and half-integers $\frac{1}{2}(\ell + 1)$, where ℓ characterises the angular dependence of the ℓ 'th partial wave. The low-frequency expansion arises in even powers of λ along with the characteristic dependence on powers of $\log \bar{\lambda}$ ($\bar{\lambda} = e^\gamma \lambda$, where γ is Euler's constant). These results are analogous in structure to those for other extremal black-hole or p -brane backgrounds. For example, the occurrence of $\log \bar{\lambda}$ in sub-leading terms in the energy expansion for the absorption probability is analogous to what is seen in other examples, such as the M-branes and D3-brane [20, 25]. By the AdS/CFT correspondence, the results of section 3 should yield information on correlation

functions in strongly coupled two-dimensional quantum field theory perturbed away from the infrared conformally invariant fixed point.

In section 4 we calculated absorption probabilities in a different parameter regime, namely where α is held fixed, with $1/\Delta$ being used as the small order-parameter of the iterative solution of the Mathieu equation. This can be viewed as an expansion in which an incident wave of fixed frequency (satisfying $\lambda \ll 1$) is absorbed in the dyonic string background with a range of charge ratios, in a limit where Δ is sufficiently large, implying that the charge ratio is very large. In this case the value of α is fixed, but can be chosen to have an arbitrary value. We first obtained results for generic positive values of α^2 . The expressions for Floquet exponents and absorption probabilities become singular when α takes integer or half-integer values, and these cases have to be treated separately; this was discussed in section 4.2.

A case of particular interest which itself lies outside the general discussion of special cases in section 4.2 is when $\alpha = 0$, discussed in section 4.4. In this situation, we find that the absorption probabilities depend inversely on powers of $\log \Lambda$, which seems to indicate a correspondence to an intrinsically quantum regime of the two-dimensional field theory. The $\alpha = 0$ point can be viewed as a “critical point”, since the absorption probabilities for $\alpha^2 > 0$ and for $\alpha^2 < 0$ (discussed in section 4.5) are qualitatively different. When $\alpha^2 > 0$, the absorption probability vanishes in the limit $\lambda \rightarrow 0$ and approaches this limit with a positive (multiple of 4) power of λ . On the other hand when $\alpha^2 < 0$, the leading-order behaviour becomes an oscillatory function of $\log \lambda$. We also studied the region close to the critical point $\alpha = 0$, by taking $\alpha^2 = b^2 \lambda^4$, with b constant.

Owing to the fact that the boundary conformal field theory for the extremal dyonic string is two-dimensional, this example provides an especially manageable arena for making concrete tests of the AdS/CFT conjecture. The fact that on the supergravity side one has full control over the study of the scattering problem provides a starting point for addressing the corresponding strongly coupled non-critical two-dimensional field theory.

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