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# Super Yang-Mills in $(11, 3)$ Dimensions

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ABSTRACT

A supersymmetric Yang-Mills system in $(11, 3)$ dimensions is constructed with the aid of two mutually orthogonal null vectors which naturally arise in a generalized spacetime superalgebra. An obstacle encountered in an attempt to extend this result to beyond 14 dimensions is described. A null reduction of the $(11, 3)$ model is shown to yield the known super Yang-Mills model in $(10, 2)$ dimensions. An $(8, 8)$ supersymmetric super Yang-Mills system in $(3, 3)$ dimensions is obtained by an ordinary dimensional reduction of the $(11, 3)$ model, and it is suggested there may exist a superbrane with $(3, 3)$ dimensional worldvolume propagating in $(11, 3)$ dimensions.

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## 1 $(3,3)$ Superbrane in $(11,3)$ Dimensions?

There are several reasons for exploring supersymmetry in higher than eleven dimensions. Many of our motivations for considering supersymmetry in $(10,2)$ dimensions in particular have already been discussed in [1], where additional references can be found. Here, we will especially emphasize the fact that a) the F-theory considerations [2] have shown the power of $(10,2)$ dimensional framework for unifying a large class of string vacua in a nontrivial way, b) the (2,1) string approach to $M$-theory [3] has also pointed at a $(10,2)$ dimensional target space, and c) in an algebraic approach to unifying the perturbative and nonperturbative superstring states [4, 5, 6] evidence has been put forward for a $(10,2)$ dimensional structure [7]. Later, possible existence of hidden symmetries descending from (11,2) dimensions was proposed [8].

Very recently [9], it has been suggested that the fundamental supersymmetric theory may admit as many as 11 spacelike and 3 timelike dimensions [10]. This observation has motivated us to look for an extension of the work presented in [1], in search of a supersymmetric field theory in $(11,3)$ dimensions, the simplest one being super Yang-Mills theory. Interestingly enough, we find that the construction of [1] generalizes naturally to $(11,3)$ dimensions, while an extension beyond $(11,3)$ dimensions runs into an obstacle.

The existence of the model constructed here seems to require two mutually orthogonal null vectors. These are essential for a null reduction to $(10,2)$ dimensions, yielding the results of [1], or to the $(9,1)$ dimensional super Yang- Mills theory by a further null reduction. It is also possible to obtain an $(8,8)$ supersymmetric Yang-Mills system in $(3,3)$ dimensions by means of an ordinary dimensional reduction. The result contains the two mutually orthogonal null vectors inherited from $(11,3)$ dimensions.

The $(11,3) \to (3,3)$ reduction is similar to the $(10,2) \to$ (2,2) reduction, where the resulting theory in $(2,2)$ dimensions is relevant to the target space of the $(2,1)$ string of [3]. This fact and the close relationship observed between the $(8+n,n)$ theories and their $(n,n)$ reductions for $n=1,2,3$ prompts us to suggest that there may exist a superbrane with a $(3,3)$ dimensional worldvolume propagating in $(11,3)$ dimensions. In view of the intricate and fascinating results that continue to emerge in six dimensional physics, it is conceivable that this proposal finds a realization. This, in turn, may play a significant role in unifying a large class of duality symmetries in an interesting way. In particular, the fact that the isometry group $SO(3,3) \sim SL(4,R)$ is the conformal group for a $(2,2)$ dimensional world may be of relevance in this picture.

While we are not aware of any literature on supersymmetric field theories in $(3,3)$ dimensions, we note ref. [11] where a generalized self-duality condition on the Yang-Mills curvature in $(3,3)$ dimensions was shown to produce the KP equation.

We now turn to the description of the super Yang-Mills system in $(11,3)$ dimensions. As a prelude to doing so, we first describe a general class of superalgebras in $(8+n,n)$ dimensions, then reviewing briefly the $(10,2)$ results. After describing the $(11,3)$ model and its dimensional reductions, we will explain the obstacle to the construction in higher than 14 dimensions in the final section.

## 2 $(1,0)$ Superalgebras in $(8+n,n)$ Dimensions

Let us consider a spacetime superalgebra which contains a single Majorana-Weyl spinor generator $Q_\alpha$ which has $2^{n+3}$ real components [2] and all possible bosonic generators that can occur in their anticommutation relation. Hence there are $2^{n+2}\left(2^{n+3}+1\right)$ bosonic generators, in addition to the Lorentz generators. The chirality and symmetry properties of the $\gamma$-matrices are identical for $n$ mod 4. Hence, we have the following algebras

$$n = 0 \bmod 4 :$$
$$\{Q_\alpha, Q_\beta\} = \eta_{\alpha\beta}\ Z + (\gamma^{\mu_1\cdots\mu_4})_{\alpha\beta}\ Z_{\mu_1\cdots\mu_4} + \cdots + (\gamma^{\mu_1\cdots\mu_{n+4}})_{\alpha\beta}\ Z_{\mu_1\cdots\mu_{n+4}}\ , \tag{1}$$

$$n = 1 \bmod 4 :$$
$$\{Q_\alpha, Q_\beta\} = (\gamma^{\mu})_{\alpha\beta}\ P_\mu + (\gamma^{\mu_1\cdots\mu_5})_{\alpha\beta}\ Z_{\mu_1\cdots\mu_5} + \cdots + (\gamma^{\mu_1\cdots\mu_{n+4}})_{\alpha\beta}\ Z_{\mu_1\cdots\mu_{n+4}}\ , \tag{2}$$

$$n = 2 \bmod 4 :$$
$$\{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu})_{\alpha\beta}\ Z_{\mu\nu} + (\gamma^{\mu_1\cdots\mu_6})_{\alpha\beta}\ Z_{\mu_1\cdots\mu_6} + \cdots + (\gamma^{\mu_1\cdots\mu_{n+4}})_{\alpha\beta}\ Z_{\mu_1\cdots\mu_{n+4}}\ , \tag{3}$$

$$n = 3 \bmod 4 :$$
$$\{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu\rho})_{\alpha\beta}\ Z_{\mu\nu\rho} + (\gamma^{\mu_1\cdots\mu_7})_{\alpha\beta}\ Z_{\mu_1\cdots\mu_7} + \cdots + (\gamma^{\mu_1\cdots\mu_{n+4}})_{\alpha\beta}\ Z_{\mu_1\cdots\mu_{n+4}}\ , \tag{4}$$

where $Z_{m_1\cdots m_p}$ are the bosonic $p$-form generators, all of which commute with each other and with $P_\mu$. The commutators involving the Lorentz generators are the usual ones. These algebras are mapped into each other by a dimensional reduction on a (1,1) dimensional internal space, followed by a chiral truncation.

[2] One can equally well work with pseudo Majorana-Weyl spinors. It can be easily verified that this choice does not alter the form of the algebra. Note also that the symmetry property of the matrices $\gamma^{m_1\cdots\mu_r}C^{-1}$ always repeats itself for $r$ mod 4. See [12] for further details.

The $\gamma$-matrices $\gamma^{\mu_1\cdots\mu_p}$ are actually the chirally projected $(\gamma^{\mu_1\cdots\mu_p}C^{-1})$, where $C$ is the charge conjugation matrix in $(8+n,n)$ dimensions. The $\eta$-matrix occurring in (1) is the chiral projection of $C$. In this chiral notation, the spinor index takes the values $\alpha=1,...,2^{n+3}$. In a given dimension, all possible symmetric $\gamma$- matrices that survive the chiral projection occur on the right hand side, and in all cases the maximal rank $\gamma$-matrix has a definite duality property. Hence, we take the generator $Z_{\mu_1\cdots\mu_{n+4}}$ to be self-dual. Taking this into account, one can easily confirm that the r.h.s. of the above algebras span the full symmetric space of relevant dimension.

Notice that a vector momentum operator $P_\mu$ occurs only in the case of $n=1$ *mod* 4. Since all the $p$-form generators $Z_{\mu_1\cdots\mu_p}$ correspond to charges that can be carried by $p$-branes, one should also add $Z_\mu$ to the algebra to allow string charges. While this may be redefined away by shifting $P_\mu$, there are some global subtleties in doing so, and they have an interesting role to play in the description of string winding states [5].

It is worth mentioning that the case of (1,0) algebra in $(9,1)$ dimensions admits a non Abelian extension which involves a super 1-form and a super 5-form generator [13]. Whether the $(8+n,n)$ algebras with $n>1$ admit a similar non Abelian extension is an interesting open question.

Given the *mod* 4 repetitive character of the above algebras, there is a sense in which the $(11,3)$ dimensions is a natural maximum dimension, namely it is the last member of the first quartet. We will consider the case of (12,4) later, but to keep the discussion and calculations tractable, let us consider the cases of $n=0,1,2,3$, and keep the lowest rank bosonic generators:

$$(8,0): \qquad \{Q_\alpha,Q_\beta\}=\eta_{\alpha\beta}\ Z\ , \tag{5}$$

$$(9,1): \qquad \{Q_\alpha,Q_\beta\}=(\gamma^\mu)_{\alpha\beta}\ P_\mu\ , \tag{6}$$

$$(10,2): \qquad \{Q_\alpha,Q_\beta\}=(\gamma^{\mu\nu})_{\alpha\beta}\ Z_{\mu\nu}\ , \tag{7}$$

$$(11,3): \qquad \{Q_\alpha,Q_\beta\}=(\gamma^{\mu\nu\rho})_{\alpha\beta}\ Z_{\mu\nu\rho}\ . \tag{8}$$

Note that in each one of these cases there is only one more $Z$-generator, which is of rank $n+4$ and self-dual. This fact will be significant later, when we discuss the obstacle to constructing super Yang-Mills in higher than 14 dimensions (see section 6).

The (8,0) algebra, though interesting in its own right and may as well have certain applications, it can not provide a basis for an acceptable spacetime since it is timeless. The $(9,1)$ algebra is the well known Poincaré superalgebra. In the case of $(10, 2)$ dimensions, while there is no vector momentum generator, there is a way to introduce it by introducing a constant (null) vector $n_\mu$ into the algebra as follows [1, 8]

$$(10, 2): \qquad \{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu})_{\alpha\beta}\ P_\mu\ n_\nu\ . \tag{9}$$

The constancy of the vector $n_\mu$ should be considered as a special case of a more general situation where $n_\mu$ is another momentum generator. Indeed, in [9, 14] just such a scenario has been advocated, and an interesting two-particle interpretation has been put forward. It is beyond the scope of this paper to review these ideas in any detail here.

In the next section, we will recall the results of [1], after which we will present the generalization to $(11, 3)$ dimensions.

## 3 Recalling Super Yang-Mills in $(10, 2)$ Dimensions

The Yang-Mills equations of motion are given by [1]

$$\gamma^\mu D_\mu\lambda = 0\ , \tag{10}$$

$$D^\mu F_{\mu[\rho}n_{\sigma]} - \tfrac{1}{2}\bar{\lambda}\gamma_{\rho\sigma}\lambda = 0\ , \tag{11}$$

where the fields are Lie algebra valued and in the adjoint representation of the Yang-Mills gauge group, and $D_\mu\lambda = \partial_\mu\lambda + [A_\mu, \lambda]$. Due to the symmetry of $\gamma^{\mu\nu}C^{-1}$, the last term in (11) involves a commutator of the Lie algebra generators. In addition to the manifest Yang-Mills gauge symmetry, these equations are invariant under the supersymmetry transformations [1]

$$\delta_Q A_\mu = \bar{\epsilon}\gamma_\mu\lambda\ , \tag{12}$$

$$\delta_Q\lambda = -\tfrac{1}{4}\gamma^{\mu\nu\rho}\epsilon F_{\mu\nu}n_\rho\ , \tag{13}$$

and the extra bosonic local gauge transformation [1]

$$\delta_\Omega A_\mu = \Omega\ n_\mu\ , \qquad \delta_\Omega\lambda = 0\ , \tag{14}$$

provided that the following conditions hold [1]

$$n^\mu D_\mu\lambda = 0\ , \tag{15}$$

$$n^\mu\gamma_\mu\lambda = 0\ , \tag{16}$$

$$n^\mu F_{\mu\nu} = 0\ , \tag{17}$$

$$n^\mu n_\mu = 0\ , \tag{18}$$

$$n^\mu D_\mu\Omega = 0\ , \tag{19}$$

One can check that the field equations as well as the constraints are invariant under supersymmetry as well as extra gauge transformations.

Finally, we recall that the commutator of two supersymmetry transformations closes on shell, and yields a generalized translation, the usual Yang-Mills gauge transformation and an extra gauge transformation with parameters $\xi^\mu$, $\Lambda$, $\Omega$, respectively, as follows:

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_\xi + \delta_\Lambda + \delta_\Omega \ , \tag{20}$$

where the composite parameters are given by [1]

$$\xi^\mu = \bar{\epsilon}_2 \gamma^{\mu\nu} \epsilon_1 \ n_\nu \ , \tag{21}$$

$$\Lambda = -\xi^\mu \ A_\mu \ , \tag{22}$$

$$\Omega = \tfrac{1}{2} \bar{\epsilon}_2 \gamma^{\mu\nu} \epsilon_1 \ F_{\mu\nu} \ . \tag{23}$$

Note that the global part of the algebra (20) is given by (9).

The closure of the supersymmetry algebra on the fermion requires the constraints (15) and (16), while the supersymmetry and $\Omega$- symmetry of the field equations and constraints require the remaining constraints as well [1]. A superspace formulation of this model, as well as its null reductions to $(9,1)$ and (2,2) can be found in [1]. As we will see in the next section, most of these results have a natural generalization to $(11,3)$ dimensions.

## 4 Super Yang-Mills in $(11,3)$ Dimensions

We begin by introducing the momentum generator to the algebra (8), by making use of a constant tensor $v_{\mu\nu}$ as follows:

$$\{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu\rho})_{\alpha\beta} \ P_\mu \ v_{\nu\rho} \ . \tag{24}$$

The next step is to postulate the supersymmetry transformation rules which make use of $v_{\mu\nu}$. The strategy is then to obtain the field equations, and any additional constraints by demanding the closure of these transformation rules. At the end the (extra) gauge and supersymmetry of all the resulting equations must be established. In what follows, we will first present the results that emerge out of this procedure. Later, we will explain the step by step derivation of these results.

The super Yang-Mills equations take the form

$$\gamma^\mu D_\mu \lambda = 0 \ , \tag{25}$$

$$D^\sigma F_{\sigma[\mu} v_{\nu\rho]} + \tfrac{1}{12} \bar{\lambda} \gamma_{\mu\nu\rho} \lambda = 0 \ . \tag{26}$$

In addition to the manifest Yang-Mills gauge symmetry, these equations are invariant under the supersymmetry transformations

$$\delta_Q A_\mu = \bar{\epsilon}\gamma_\mu\lambda \ , \tag{27}$$

$$\delta_Q\lambda = -\tfrac{1}{4}\gamma^{\mu\nu\rho\sigma}\epsilon F_{\mu\nu}v_{\rho\sigma} \ , \tag{28}$$

and the extra bosonic local gauge transformation

$$\delta_\Omega A_\mu = -v_{\mu\nu}\ \Omega^\nu \ , \qquad \delta_\Omega\lambda = 0 \ , \tag{29}$$

provided that the following conditions hold:

$$v_\mu{}^\nu D_\nu\lambda = 0 \ , \tag{30}$$

$$v_{\mu\nu}\gamma^\nu\lambda = 0 \ , \tag{31}$$

$$v_\mu{}^\nu F_{\nu\rho} = 0 \ , \tag{32}$$

$$v_\mu{}^\rho v_{\rho\nu} = 0 \ , \tag{33}$$

$$v_{[\mu\nu}v_{\rho\sigma]} = 0 \ , \tag{34}$$

$$v_\mu{}^\rho v_\nu{}^\sigma D_\rho\Omega_\sigma = 0 \ , \tag{35}$$

$$v^{\mu\nu}D_\mu\Omega_\nu = 0 \ . \tag{36}$$

The commutator of two supersymmetry transformations closes on shell, and yields a generalized translation, the usual Yang-Mills gauge transformation and an extra gauge transformation with parameters $\xi^\mu$, $\Lambda$, $\Omega^\mu$, respectively, as follows:

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_\xi + \delta_\Lambda + \delta_\Omega \ , \tag{37}$$

where the composite parameters are given by

$$\xi^\mu = \bar{\epsilon}_2\gamma^{\mu\nu\rho}\epsilon_1\ v_{\nu\rho} \ , \tag{38}$$

$$\Lambda = -\xi^\mu\ A_\mu \ , \tag{39}$$

$$\Omega^\mu = \tfrac{1}{2}\bar{\epsilon}_2\gamma^{\mu\nu\rho}\epsilon_1\ F_{\nu\rho} \ . \tag{40}$$

The global part of the algebra (37) indeed agrees with (24). Note the symmetry between the parameters $\xi^\mu$ and $\Omega^\mu$. The former involves a contraction with $v_{\mu\nu}$, and the latter one with $F_{\mu\nu}$.

The derivation of these results proceeds as follows. First, it is easy to check that the closure on the gauge field requires an additional local gauge transformation (29) with the composite parameter (40). Next, one checks the closure on the gauge fermion. In doing so, the following

Fierz-rearrangement formula is useful:

$$\epsilon_{[1}\bar{\epsilon}_{2]} = \frac{1}{64}\left(\frac{1}{3!}\ \bar{\epsilon}_2\gamma^{\mu\nu\rho}\epsilon_1\ \gamma_{\mu\nu\rho} + \frac{1}{7!2}\ \bar{\epsilon}_2\gamma^{\mu_1\cdots\mu_7}\epsilon_1\ \gamma_{\mu_1\cdots\mu_7}\right)\ , \tag{41}$$

where $\epsilon_1$ and $\epsilon_2$ are Majorana-Weyl spinors of the same chirality. Using this formula, and after a little bit of algebra, one finds that:

(a) The closure on the gauge fermion holds provided that the fermionic field equation (25), along with the constraints (30) and (31) are satisfied.

(b) The supersymmetry of the constraint (30) requires the constraint (32), and a further variation of this constraint does not yield new information.

(c) The supersymmetry of the constraint (31) requires the further constraints (34) and (35).

(d) The equations of motion (25) and (26) transform into each other under supersymmetry. This can be shown with the use the constraints (31) and (32).

(e) Finally the invariance of the full system, i.e. equations of motion and constraints, under the extra gauge transformation (29) has to be verified. The invariance of the fermionic field equation (25), as well as the constraints (30) and (31) do not impose new conditions. However, the invariance of the constraint (32) imposes the condition (35), and the invariance of the bosonic field equation (26) imposes the condition (36) on the parameter $\Omega^\mu$. Both of these conditions are gauge invariant.

In summary, equations (24)-(40) form a consistent and closed system of supersymmetric, Yang-Mills gauge and $\Omega$-gauge invariant equations. The similarity of these equations to the corresponding ones in $(10,2)$ dimensions is evident. One expects, therefore, a natural reduction of these equations to those in $(10,2)$ dimensions. This will indeed turn out to be the case, as we will see in the next section.

The important next step is to establish that the constant tensor $v_{\mu\nu}$ satisfying the conditions (34) and (33) actually exists. Fortunately this is the case, and we have the solution

$$v_{\mu\nu} = m_{[\mu}n_{\nu]}\ , \tag{42}$$

where $m_\mu$ and $n_\nu$ are mutually orthogonal null vectors, i.e. they satisfy

$$m_\mu m^\mu = 0\ , \qquad n_\mu n^\mu = 0\ , \qquad m^\mu n_\mu = 0\ . \tag{43}$$

Given the signature of the 14-dimensional spacetime, finding two mutually orthogonal null vectors, of course, does not present a problem. Indeed, this solution suggests a that an *ordinary* dimensional reduction to $(9,1)$ dimensions should yield the usual super Yang-Mills system. In the next section we will show that this is indeed the case.

# 5 Dimensional Reductions to $(10,2)$ and $(3,3)$

The simplest way to show that the ordinary dimensional reduction of the $(11,3)$ system to $(9,1)$ dimensions yields the usual super Yang-Mills equations is to establish that the full system of equations in $(10,3)$ dimensions reduce to those in $(10,2)$ dimensions. Since the latter have already been shown to reduce to the usual super Yang-Mills equations in $(9,1)$ dimensions [1], we need not repeat the second step of dimensional reduction.

In this section we shall use hats for the fields and indices in $(11,3)$, to distinguish them from the unhatted ones in $(10,2)$. The coordinates are $(x^\mu, x^{13}, x^{14})$ and the metric is $(\widehat{\eta}_{\hat\mu\hat\nu}) = (\eta_{\mu\nu}, +, -)$, where $x^\mu$ are the coordinates of the $(10,2)$ dimensional space with metric $\eta_{\mu\nu} = \mathrm{diag.}\,(-,+,\cdots,+,-)$. It is convenient to define the coordinates $x^\pm \equiv (x^{11} \pm x^{12})/\sqrt{2}$.

The $(11,3)$ $\gamma$-matrices satisfy $\{\widehat{\gamma}_{\hat\mu}, \widehat{\gamma}_{\hat\nu}\} = 2\widehat{\eta}_{\hat\mu\hat\nu}$. A convenient choice for the $\gamma$-matrix is

$$\widehat{\gamma}^{\hat\mu} = \begin{cases} \widehat{\gamma}^\mu = \gamma^\mu \otimes \sigma_3 \quad , \\ \widehat{\gamma}^{13} = I \otimes \sigma_1 \quad , \\ \widehat{\gamma}^{14} = I \otimes i\sigma_2 \quad . \end{cases} \tag{44}$$

Here $I$ is the $64 \times 64$ unit matrix and the $\sigma$'s are the Pauli matrices and $\gamma^\mu$ are the $64 \times 64$ Dirac $\gamma$-matrices in $(10,2)$ dimensions. The charge conjugation matrix and the chirality operators can defined as

$$\widehat{C} = C_{12} \otimes i\sigma_2 \ , \qquad \widehat{\gamma}_{15} = \gamma_{13} \otimes \sigma_3 \ , \tag{45}$$

where $C_{12}$ is the antisymmetric charge conjugation matrix in $(10,2)$ such that $\gamma^\mu (C_{12})^{-1}$ is antisymmetric, and $\gamma^{13}$ is the chirality operator in $(10,2)$ which squares to one. The mutually orthogonal null vectors $m_\mu$ and $n_\mu$ are taken to be

$$(\widehat{m}_{\hat\mu}) = (\vec{0}, 1, 1) \ , \qquad (\widehat{n}_{\hat\mu}) = (n_\mu, 0, 0) \ , \tag{46}$$

where $n_\mu$ is a null vector in $(10,2)$ dimensions.

With these choices, the $v$-tensor has the components $v_{\mu\nu} = 0$, $v_{-\mu} = 0$ and $v_{+\mu} = n_\mu$. Without making any assumption on the $x^\pm$ dependence of the fields and parameters, the bosonic constraint (32) reduces to

$$F_{-\mu} = 0 \ , \qquad F_{-+} = 0 \ , \qquad n^\mu F_{+\mu} = 0 \ , \tag{47}$$

$$n^\mu F_{\mu\nu} = 0 \ . \tag{48}$$

Similarly the gauge transformations (29), including the Yang-Mills gauge transformations become

$$\delta A_- = D_-\Lambda \ , \qquad \delta A_+ = D_+\Lambda - n_\mu \Omega^\mu \ , \tag{49}$$

$$\delta A_\mu = D_\mu \Lambda + n_\mu \Omega^+ \ , \tag{50}$$

with the constraint (35) on the parameter $\Omega^\mu$ reducing to

$$D_-\Omega^+ = 0 \ , \qquad D_-\left(n_\rho\Omega^\rho\right) = 0 \ , \qquad n^\mu D_\mu\left(n_\rho\Omega^\rho\right) = 0 \ , \tag{51}$$

$$n^\mu D_\mu \Omega^+ = 0 \ . \tag{52}$$

The reduction of the fermionic constraints (30) and (31) is also straightforward. The latter one gives $\gamma^+\lambda = 0$, where $\gamma^+ = I \otimes (\sigma_1 + i\sigma_2)/\sqrt{2}$. Together with the $(11,3)$ chirality condition, this implies that $\widehat{\lambda}$ can be written as

$$\widehat{\lambda} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \ , \qquad \gamma_{13}\lambda = \lambda \ . \tag{53}$$

The fermionic constraint (31) now reduces to

$$n_\mu \gamma^\mu \lambda = 0 \ , \tag{54}$$

and the constraint (30) gives

$$D_-\lambda = 0 \ , \tag{55}$$

$$n^\mu D_\mu \lambda = 0 \ . \tag{56}$$

There remains the reduction of the field equations and the supersymmetry transformation rules. It is easily seen that equations of motion reduce to

$$\gamma^\mu D_\mu \lambda = 0 \ , \tag{57}$$

$$D^\mu F_{\mu[\rho} n_{\sigma]} - \tfrac{1}{8}\bar{\lambda}\gamma_{\rho\sigma}\lambda = 0 \ . \tag{58}$$

and the supersymmetry transformation take the form

$$\delta_Q A_- = 0 \ , \qquad \delta_Q A_+ = \bar{\eta}\lambda \ , \tag{59}$$

$$\delta_Q A_\mu = \bar{\epsilon}\gamma_\mu\lambda \ , \tag{60}$$

$$\delta_Q \lambda = \gamma^{\mu\nu\rho}\epsilon F_{\mu\nu} n_\rho \ , \tag{61}$$

where we have used the notation

$$\widehat{\epsilon} = \begin{pmatrix} \eta \\ \epsilon \end{pmatrix} \ . \tag{62}$$

$\eta$ is a Majorana-Weyl spinor in $(10,2)$ with chirality opposite to that of $\epsilon$. The $\eta$-transformation of $A_+$ was omitted in [1] but this is inconsequential since $A_+$ can be gauged away by an $\Omega$-transformation.

Up to global issues which may involve large gauge transformations and nontrivial topologies, equations (47)-(62) describe the super Yang-Mills system in $(10,2)$ dimensions, in which an arbitrary dependence on $x^+$ is introduced but the derivative $\partial_+$ does not occur. Integrating over $x^+$ then yields precisely the super Yang-Mills equations of [1]. Though the details may differ slightly, this phenomenon is similar in essence to the null reduction from $(10,2)$ to $(9,1)$ discussed in [1]. Thus, a two-step reduction to $(9,1)$ dimensions is expected to yield a doubly affinized version of the usual super Yang-Mill theory in $(9,1)$ dimensions. Applied to the present case, the single step reduction argument goes as follows.

Firstly, the field $A_+$ can be gauged away by using the last two constraints in (47) and the second gauge transformation in (49). In doing so, the last two equations in (51) needs to be taken into account. Second, the field $A_-$ can be gauged away by using the first two constraints in (47) and the first gauge transformation in (49). The surviving fields are $A_\mu(x^+,\vec{x})$ and $\lambda(x^+,\vec{x})$ obeying the constraints (48), (54) and (56). Here $\vec{x}$ represents the $(10,2)$ coordinates. The surviving symmetries are the rigid supersymmetry transformations (60) and (61), and the gauge transformations (50) with parameters $\Lambda(x^+,\vec{x})$ and $\Omega^+(x^+,\vec{x})$ subject to the condition (52). With the identification $\Omega^+\equiv\Omega$, and suitable rescalings of the fields and parameters, this system is exactly the super Yang-Mills system of [1] as summarized in eqs. (10)-(23), with the additional and arbitrary $x^+$ dependence inherited from $(11,3)$ dimensions.

To conclude this section, we describe the ordinary dimensional reduction to $(3,3)$ dimensions, in which case the extra coordinates become part of the resulting spacetime. To begin with, let us label the coordinates as

$$x^{\hat\mu}=(x^\mu,x^i)\ ,\quad \mu=0,1,11,...,14\ ,\quad i=2,...,9\ . \tag{63}$$

Thus the signature of the $(3,3)$ spacetime is $(-++-+-)$, and the internal space is Euclidean. The ordinary dimensional reduction from $(11,3)$ to $(3,3)$ is achieved simply by setting

$$\partial_i=0\ ,\quad v_{ij}=0\ ,\quad v_{i\mu}=0\ , \tag{64}$$

and using the Dirac matrices $\widehat\gamma_\mu=\gamma_\mu\otimes I$ and $\widehat\gamma_i=\Gamma_7\otimes\gamma_i$, where $\gamma_\mu$ and $\gamma_i$ are the $8\times 8$ $SO(3,3)$ and $16\times16$ $SO(8)$ Dirac matrices, respectively, and $\Gamma_7$ is the chirality operator in $(3,3)$ dimensions and $I$ is the unit matrix.

The resulting fields are $(A_\mu,\phi^i,\lambda^A,\lambda^{\dot A})$, where the eight scalars are defined as $A_i\equiv\phi_i$, and the spinors are Majorana-Weyl in $(3,3)$ dimensions, with the indices $A,\dot A=1,...,8$ labelling the left and right handed spinors of the internal $SO(8)$ group. It is a straightforward matter to apply (64) to all the equations from (25) to (40). The resulting system is clearly of the same form as the one in $(11,3)$ dimensions in that it contains the tensor $v_{\mu\nu}$ in a similar fashion. There are, of course, the contributions of the eight scalars $A_i$ to the equations which are easily obtained from the $(11,3)$ equations.

The superalgebra in $(3,3)$ dimensions that underlies this model is an (8,8) type superalgebra with 8 left-handed and 8 right-handed Majorana-Weyl spinor generators. Using the chiral notation in which the lower and upper SO(3,3) spinor indices refer to left and right handed projections, the superalgebra takes the form

$$\{Q^A_\alpha, Q^B_\beta\} = \delta^{AB}\,(\gamma^{\mu\nu\rho})_{\alpha\beta}\; P_\mu\; v_{\nu\rho}\ , \tag{65}$$

$$\{Q^A_\alpha, Q^{\beta\dot{B}}\} = 0\ , \tag{66}$$

$$\{Q^{\alpha\dot{A}}, Q^{\beta\dot{B}}\} = \delta^{\dot{A}\dot{B}}\,(\gamma_{\mu\nu\rho})^{\alpha\beta}\; P^\mu\; v^{\nu\rho}\ . \tag{67}$$

Although $(p,q)$ type chiral version of this algebra perfectly exists, just as in $(11,3)$ dimensions, the field theoretic realization obtained by ordinary dimensional reduction is vectorlike, unless one imposes the self-duality condition

$$p^{\mu_1} v^{\mu_2\mu_3} = \tfrac{1}{3!}\epsilon^{\mu_1\cdots\mu_6}\, p_{\mu_4} v_{\mu_5\mu_6}\ , \tag{68}$$

in which case the algebra becomes $(8,0)$ with the anticommutation relation (65).

## 6 Beyond 14 Dimensions and Comments

The $(11,3)$ model suggests a generalization to $(8+n,n)$ dimensions for all values of $n$. One way to proceed is to keep the lowest rank bosonic generators, namely $P_\mu$ for $n=1$ mod 4; $Z_{\mu\nu}$ for $n=2$ mod 4 and $Z_{\mu\nu\rho}$ for $n=3$ mod 4. However, supersymmetry transformations fail to close on the gauge fermion for $n>3$, due to the appearance of new and unwanted contributions to the relevant Fierz identity. Another approach would be to introduce higher rank generators into the algebra that take the form

$$v_{\mu_1\cdots\mu_p} = n_{[\mu_1}\cdots n_{\mu_p]}\ , \tag{69}$$

for one or more suitable values of $p$. If such generators are introduced in addition to the lowest rank generators mentioned above, the problem with the closure of the algebra on the gauge fermion persists. A third approach would be to introduce the $v$-tensor for a particular $p$-form generator. The simplest case to consider is the superalgebra in $(12,4)$ dimensions:

$$\{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu\rho\sigma})_{\alpha\beta}\; P_\mu v_{\nu\rho\sigma}\ . \tag{70}$$

Postulating the obvious analogs of the supersymmetry transformation rules (27) and (28) for this case, one finds that the algebra indeed closes modulo constraints analogous to (30)-(36), and the fermionic field equation (25). However, a supersymmetric variation of the fermionic field equation $\gamma^\mu D_\mu \lambda = 0$ does not yield a bosonic field equation, and therefore the system fails to be supersymmetric. The problem arises in the variation of the gauge field in the covariant derivative. It requires the Fierz rearrangement formula

$$\epsilon_{[1}\bar{\epsilon}_{2]} = \frac{1}{128}\left(\bar{\epsilon}_2\epsilon_1 + \frac{1}{4!}\ \bar{\epsilon}_2\gamma^{\mu_1\cdots\mu_4}\epsilon_1\ \gamma_{\mu_1\cdots\mu_4} + \frac{1}{8!2}\ \bar{\epsilon}_2\gamma^{\mu_1\cdots\mu_8}\epsilon_1\ \gamma_{\mu_1\cdots\mu_8}\right)\ , \tag{71}$$

where $\epsilon_1$ and $\epsilon_2$ are Majorana-Weyl spinors of the same chirality. The last term is acceptable because $\gamma^\rho\gamma_{\mu_1\cdots\mu_8}\gamma_\rho = 0$. The second term is also acceptable because it produces the $\bar{\lambda}\gamma_{\mu_1\cdots\mu_4}\lambda$ term in the Yang-Mills equation. However the first term in (71) gives an unwanted contribution which can not be interpreted as part of the Yang-Mills field equation[3]. The reason why the construction works for $n = 1, 2, 3$ is that the Fierz rearrangement formula in those cases gives only two terms, one of which is harmless due to the formula $\gamma^\rho\gamma_{\mu_1\cdots\mu_{n+4}}\gamma_\rho = 0$ in $(2n+8)$-dimensions, and the other gives rise to a fermionic bilinear term in the Yang-Mills equation.

The obstacle mentioned above arises for all $n > 3$, since the unwanted lower rank $\gamma$-matrix contributions to the relevant Fierz rearrangement formula would set in. While this is a tentative analysis and in principle new structures may be introduced to the algebra to avoid the obstacle, it is nonetheless interesting to see that there is something special about the lowest triplet of theories in the $(8+n, n)$ dimensional class, namely the ones corresponding to $n = 1, 2, 3$, at least within the current framework.

To conclude, we turn to the case of $(11, 3)$ dimensions and note that the algebra (24) can be generalized to

$$\{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu\rho})_{\alpha\beta}\ P_{1\mu}\ P_{2\nu}\ P_{3\rho}\ , \tag{72}$$

where $P_{1\mu}$, $P_{2\nu}$ and $P_{3\rho}$ are to be considered on equal footing as momentum generators, just as it has been suggested in [7, 8, 9, 14] for the case of $(10, 2)$ dimensions where two such momenta arise. Recently an interesting two-particle interpretation has been given for that case [9]. This approach is indeed promising because among its premises is a manifestly $SO(10, 2)$ invariant action, albeit with the introduction of bi-local fields. The arguments of [9] suggest, however, that bi-local fields need not necessarily suffer from the old problems. A suitable application of these ideas to the present case would presumably involve tri-local fields, and possibly a notion of a new kind of triality symmetry. It would be very interesting to see if such a picture might emerge within the framework of a $(3, 3)$ superbrane propagating in $(11, 3)$ dimensions mentioned in the introduction.

[3] As this problem does not arise in the Abelian case, the construction works for super Maxwell for any $n$. However, we do not consider this to be interesting since it is a free theory.

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