INFINITE DIMENSIONAL ALGEBRAS IN CHERN-SIMONS QUANTUM MECHANICS

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Abstract

We study a charged particle in an electromagnetic field without kinetic or potential term. Although dynamically trivial, this system is interesting because it has an infinite dimensional symmetry group. We discuss the way in which this group behaves under quantization.

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It is a challenging task to quantize nonlinear sigma models with an action given only by a Wess-Zumino-Witten term. We have already studied these models from a classical point of view in [1]. Here we discuss the quantization of the $0+1$-dimensional case. This is the simplest version of such theories, describing a particle moving on an n-dimensional manifold N, with action

$$S=\int dt\ \dot{q}^{\alpha}\mathcal{A}_{\alpha}(q)\ , \tag{1}$$

where $\mathcal{A}$ is a fixed background electromagnetic potential. This model is also related to Chern-Simons theories and therefore has come to be known as "Chern-Simons quantum mechanics" [2,3]. We will now recall the main features of the model, and refer the reader to [1] for more details.

The Euler-Lagrange equations which follow from (1) are

$$\dot{q}^{\alpha}\mathcal{F}_{\alpha\beta}=0 \tag{2}$$

where $\mathcal{F}_{\alpha\beta}=\partial_{\alpha}\mathcal{A}_{\beta}-\partial_{\beta}\mathcal{A}_{\alpha}$ is the field strength. These equations demand that $\dot{q}^{\alpha}$ be a null eigenvector of $\mathcal{F}$. Without loss of generality, in this paper we will consider only the case in which $\mathcal{F}$ is nondegenerate (*i.e.* it is a symplectic form). In fact, if $\mathcal{F}$ was degenerate, there would be a nontrivial gauge group $\mathcal{G}$ and the model would be equivalent to a particle moving on $N/\mathcal{G}$ in a nondegenerate electromagnetic field. Note that when $\mathcal{F}$ is nondegenerate, eq.(2) states simply that $\dot{q}^{\alpha}=0$. In this case the Hamiltonian is identically zero.

The action (1) is invariant under the group $S(\mathcal{F})$ of symplectic diffeomorphisms of N, also called the canonical transformations of N. The Lie algebra of $S(\mathcal{F})$, denoted $X(\mathcal{F})$, is the algebra of vectorfields v on N such that the Lie derivative of $\mathcal{F}$ along v vanishes; equivalently, if N is simply connected,

$$v^{\alpha}\mathcal{F}_{\alpha\beta}=\partial_{\beta}\Omega_{v}\ , \tag{3}$$

for some (globally defined) real function Ω_v on N ($X(\mathcal{F})$ is called the algebra of hamiltonian vectorfields on N). The Noether charge correponding to an infinitesimal symmetry transformation v is $-\Omega_v$. The symplectic diffeomorphisms are genuine symmetries and not gauge invariances. Thus we are in a very unusual situation in which a finite dimensional system possesses an infinite dimensional symmetry group.

The phase space of this theory is N itself, with symplectic form $\mathcal{F}$ [1,4]. The Poisson bracket of two functions f, g on N is

$$\{f,g\}=(\mathcal{F}^{-1})^{\alpha\beta}\partial_{\alpha}f\,\partial_{\beta}g\ . \tag{4}$$

The Lie algebra of all smooth real functions on N with this bracket will be denoted by Γ. Given any $\Omega\in\Gamma$ we can construct a unique hamiltonian vectorfield v using eq.(3). Since functions differing by a constant give the same v, Γ is a central extension of $X(\mathcal{F})$. Conversely given a hamiltonian vectorfield v, eq.(3) defines Ω_v only up to a constant. If it is possible to fix this ambiguity in such a way that

$$\{\Omega_v,\Omega_w\}=\Omega_{[v,w]}\ , \tag{5}$$

then, as an algebra, $\Gamma = \mathbf{R} \oplus X(\mathcal{F})$. In this case the center $\mathbf{R}$ can be dropped and the Noether charges provide a realization of the abstract algebra $X(\mathcal{F})$. However, as we shall see later, this is sometimes impossible and a nonremovable central term $c(v, w)$ may appear on the r.h.s. of eq.(5). Note that all this is still at the classical level.

Since the dynamics of the system is trivial, the only interesting thing to discuss are its symmetries. Note further that the classical observables of a dynamical system are the functions on phase space; since in our case all functions are generators of symmetries, there follows that the discussion of the algebra of observables coincides with the discussion of the symmetry algebra.

Now, there is a well-known theorem, originally due to van Hove, which implies that there is no way of quantizing this system preserving the whole classical symmetry algebra. More precisely, there is no irreducible representation of the observables $f \in \Gamma$ as self-adjoint operators $\hat{f}$ on a Hilbert space such that

$$[\hat{f}, \hat{g}] = i\widehat{\{f, g\}} \ . \tag{6}$$

For a clear exposition, see [5]. This impossibility of preserving a classical symmetry algebra at the quantum level is reminiscent of anomalies in gauge theories. However, the situation here is quite different. In gauge theories the anomalies manifest themselves as non conservation of certain charges. On the other hand in our model all Noether charges are conserved because of the vanishing Hamiltonian. Instead, the algebra obeyed by these charges is modified.

In what follows we will consider various special cases for N and discuss the fate of the symmetry group when the model is quantized.

$N = \mathbf{R}^2$

We take $\mathcal{A}_\alpha = -\frac{1}{2}\varepsilon_{\alpha\beta}q^\beta$, with $\alpha, \beta = 1, 2$ and $\varepsilon_{12} = 1$. Then $\mathcal{F}_{\alpha\beta} = \varepsilon_{\alpha\beta}$ is the euclidean invariant volume form on $\mathbf{R}^2$ and $S(\mathcal{F})$ is the group $\mathcal{SD}iff\mathbf{R}^2$ of volume-preserving diffeomorphisms of $\mathbf{R}^2$. A constant factor g in front of $\mathcal{F}$ can be absorbed by a rescaling of the coordinates. Thus the theory is independent of the strength of the magnetic field.

The maximal finite dimensional subgroup of $\mathcal{SD}iff\mathbf{R}^2$ is the semidirect product of the group $\mathbf{R} \times \mathbf{R}$ of translations (with generators $v^{(1)} = \partial_1$, $v^{(2)} = \partial_2$) and the group $SL(2, \mathbf{R}) \sim Sp(1, \mathbf{R})$ (with generators $v^{(0)} = q^2\partial_2 - q^1\partial_1$, $v^{(+)} = q^2\partial_1$ and $v^{(-)} = q^1\partial_2$). The nonvanishing Lie brackets of these vectorfields are $[v^{(+)}, v^{(-)}] = v^{(0)}$, $[v^{(0)}, v^{(\pm)}] = \pm 2v^{(\pm)}$, $[v^{(1)}, v^{(0)}] = -v^{(1)}$, $[v^{(2)}, v^{(0)}] = v^{(2)}$, $[v^{(1)}, v^{(-)}] = v^{(2)}$, $[v^{(2)}, v^{(+)}] = v^{(1)}$. Note that $v^{(1)}$, $v^{(2)}$ and $v^{(-)} - v^{(+)}$ generate the Euclidean group.

Already at the classical level, the Noether charges do not give a representation of the symmetry algebra $X(\mathcal{F})$, but rather of its central extension Γ. A basis for Γ is given by the monomials $f^{(m,n)} = (q^1)^m(q^2)^n$, with $m, n = 0, 1, 2 \ldots$. The functions which generate the finite dimensional subgroup are $\Omega_{(1)} = f^{(0,1)}$, $\Omega_{(2)} = -f^{(1,0)}$, $\Omega_{(0)} = -f^{(1,1)}$, $\Omega_{(+)} = \frac{1}{2}f^{(0,2)}$, $\Omega_{(-)} = -\frac{1}{2}f^{(2,0)}$. The Poisson algebra of these generators is the same as the Lie bracket algebra of the corresponding vectorfields, except for $\{\Omega_{(1)}, \Omega_{(2)}\} = -1$. Clearly the center cannot be eliminated by redefining the generators. Therefore in this model the generators of translations fail to commute already at the classical level and the algebra of translations is enlarged to the Heisenberg algebra $h(1)$.

The system can be quantized in the Schrödinger picture replacing q^2 and q^1 by the operators $\hat{q}^2 = q$ and $\hat{q}^1 = -i\frac{\partial}{\partial q}$ acting on the Hilbert space of square integrable functions of q. The quantization of all functions in Γ can be achieved by replacing each monomial $f^{(m,n)}$ by the symmetrically ordered operator $\hat{f}^{(m,n)} = S^{(m,n)}_{\alpha_1,\ldots,\alpha_{m+n}}\hat{q}^{\alpha_1}\ldots\hat{q}^{\alpha_{m+n}}$, where $S^{(m,n)}$ is the totally symmetric tensor with components

$$S^{(m,n)}_{\alpha_1,\ldots,\alpha_{m+n}} = \begin{cases} m!n!/(m+n)! & \text{if } \alpha_i = 1 \text{ for } m \text{ values of } i; \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The algebra of these operators closes, but is different from the classical symmetry algebra. We begin by observing that the maximal finite dimensional subalgebra is not modified by quantization; in particular, the operators $\hat{\Omega}^{(0)}$ and $\hat{\Omega}^{(\pm)}$ give a representation of the algebra $sl(2,\mathbf{R})$ with Casimir operator $\frac{1}{2}(\hat{\Omega}_{(+)}\hat{\Omega}_{(-)} + \hat{\Omega}_{(-)}\hat{\Omega}_{(+)}) + \frac{1}{4}(\hat{\Omega}_{(0)})^2 = \frac{3}{16}$. This is known as the metaplectic representation [6]. The complete quantum symmetry algebra is the universal enveloping algebra of $h(1) \oplus\!\!\!\!\!\supset sl(2,\mathbf{R})$. The realization of this algebra on the Hilbert space has a nontrivial kernel generated by the elements of the form

$$\hat{\Omega}_{(0)} - \frac{1}{2}(\hat{\Omega}_{(1)}\hat{\Omega}_{(2)} + \hat{\Omega}_{(2)}\hat{\Omega}_{(1)}) \quad , \quad \hat{\Omega}_{(+)} - \frac{1}{2}(\hat{\Omega}_{(1)})^2 \quad , \quad \hat{\Omega}_{(-)} + \frac{1}{2}(\hat{\Omega}_{(2)})^2 \ . \tag{8}$$

The algebra which is realized faithfully is obtained by factoring out this kernel. It is isomorphic to the universal enveloping algebra $\mathcal{U}(h(1))$.

We note that any ordering prescription would give rise to a closed algebra of quantum operators. All these algebras are isomorphic. In fact, a choice of ordering prescription is equivalent to a choice of basis in the universal enveloping algebra.

Another possibility is to introduce a $\mathbf{Z}_2$ grading in the space of the quantum operators $\hat{f}^{(m,n)}$: we call them fermionic or bosonic depending on whether $m+n$ is odd or even. Then, introducing a corresponding graded bracket, the operators $\hat{f}^{(m,n)}$ with $m+n = 1,2$, generate the noncompact version of the superalgebra $osp(1,2)$. Note that in this approach it is not necessary to introduce the constant multiples of unity to close the algebra. This representation of $osp(1,2)$ is also known as the metaplectic representation [7]. With this graded bracket the entire algebra generated by the operators $\hat{f}^{(m,n)}$, with $m+n > 0$, is the universal enveloping algebra of $osp(1,2)$. Note that this is also known as the higher spin algebra $shs(2)$ [8]. Again, there is a nontrivial kernel given by (8).

Finally, we observe that all that has been said so far for the case $N = \mathbf{R}^2$ can be generalized in a straightforward manner to the case $N = \mathbf{R}^{2n}$. We choose $\mathcal{F}_{2i-1,2i} = -\mathcal{F}_{2i,2i-1} = 1$, for $i = 1,\ldots,n$. Then, the quantum symmetry algebra is the universal enveloping algebra of $h(n) \oplus\!\!\!\!\!\supset sp(n,\mathbf{R})$ if only commutators are used, and of $osp(1,2n)$ if the graded brackets are used.

$N = S^2 = SO(3)/O(2)$

In this case we take $\mathcal{F}$ to be the field strength of a monopole with magnetic charge g. Unlike the previous case, here it is not possible to eliminate the constant g from the discussion by redefining coordinates. In polar coordinates θ and φ, $\mathcal{A}_\theta = 0$, $\mathcal{A}_\varphi = -g(1+\cos\theta)$, and $\mathcal{F}_{\theta\varphi} = g\sin\theta$. The symmetry group $S(\mathcal{F})$ is now the group $\mathcal{SD}iffS^2$ of volume-preserving diffeomorphisms of the sphere. Its maximal finite dimensional subgroup is $SO(3)$.

The Noether charges can be uniquely defined by requiring that their integral over S^2 be zero. In this way no center arises [1]. This gives a realization of the Lie algebra of $\mathcal{SD}iffS^2$ as the Poisson algebra of functions on S^2 with vanishing integral. A suitable basis for these functions are the spherical harmonics Y_l^m with $l = 1, 2\ldots$. The structure constants of the Lie algebra of $\mathcal{SD}iffS^2$ have been computed in this basis in [9]. We note for future reference that the spherical harmonics can be thought of as homogeneous polynomials in the variables x_1, x_2, x_3, if S^2 is embedded in $\mathbf{R}^3$ by the equation $x_1^2 + x_2^2 + x_3^2 = 1$. The three harmonics with $l = 1$ (corresponding to the polynomials x_1, x_2, x_3) generate the group $SO(3)$. In fact, defining

$$J_1 = -\, gx_1 = -g\sin\theta\cos\varphi = \ g\sqrt{\frac{2\pi}{3}}(Y_1^1 + Y_1^{-1}) \ , \tag{9a}$$

$$J_2 = -\, gx_2 = -g\sin\theta\sin\varphi = -ig\sqrt{\frac{2\pi}{3}}(Y_1^1 - Y_1^{-1}) \ , \tag{9b}$$

$$J_3 = -\, gx_3 = -g\cos\theta \qquad = -g\sqrt{\frac{4\pi}{3}}Y_1^0 \ , \tag{9c}$$

and using the Poisson bracket (4), we get $\{J_\alpha, J_\beta\} = \varepsilon_{\alpha\beta\gamma}J_\gamma$.

For the quantization of a classical system with compact phase space the best approach is that of geometric quantization [10]. This method yields a Hilbert space and a set of quantum observables whose algebra is isomorphic to the Poisson algebra of the corresponding classical observables. Because of van Hove's theorem, this procedure can not work for all observables. In the case of S^2 it works successfully only for the functions J_α.

Geometric quantization requires that $2g$ be an integer, *i.e.* that the Dirac condition be satisfied. Then, the Hilbert space $\mathcal{H}_g$ is the space of polynomials of order $\leq 2g$ in the complex variable $z = e^{i\varphi}\cot(\theta/2)$, with inner product [11]

$$\langle\Psi, \Phi\rangle = \frac{2g+1}{2\pi i}\int_{S^2}\frac{dz\,d\bar{z}}{(1+z\bar{z})^{2g+2}}\Psi^*(z)\Phi(z) \ . \tag{10}$$

The dimension of $\mathcal{H}_g$ is $2g + 1$ and a basis is given by the functions $1, z, z^2, \ldots, z^{2g}$.

The operators corresponding to J_1, J_2 and J_3 are

$$\hat{J}_1 = \frac{1}{2}(1 - z^2)\partial_z + gz \ , \tag{11a}$$

$$\hat{J}_2 = \frac{i}{2}(1 + z^2)\partial_z - igz \ , \tag{11b}$$

$$\hat{J}_3 = z\partial_z - g \ , \tag{11c}$$

obeying $[\hat{J}_\alpha, \hat{J}_\beta] = i\varepsilon_{\alpha\beta\gamma}\hat{J}_\gamma$. Notice that the value of the Casimir operator $\hat{J}^2$ in the representation (11) is $g(g+1)$. Therefore, the Hilbert space $\mathcal{H}_g$ is the familiar representation space of angular momentum spanned by kets $|j, m\rangle$, with $j = g$. The function z^n in $\mathcal{H}_g$ corresponds to the state $|g, n - g\rangle$.

As we have mentioned, the remaining functions on the sphere cannot be quantized in such a way that their classical Poisson algebra is preserved. Nevertheless, as in the case of $\mathbf{R}^2$, one could envisage a more general scheme.

It follows from previous remarks that a basis for all smooth functions on the sphere is given by polynomials in the three harmonics with $l = 1$. In order to achieve a quantization of the whole set of classical observables it is therefore sufficient to quantize powers of J's. We associate to each monomial $f^{(n_1,n_2,n_3)} = (J_1)^{n_1}(J_2)^{n_2}(J_3)^{n_3}$ an operator $\hat{f}^{(n_1,n_2,n_3)}$, in which the factors $\hat{J}_\alpha$ have been ordered according to some given prescription. In this way the whole algebra Γ of functions on S^2 is turned under quantization into the universal enveloping algebra $\mathcal{U}(so(3))$. The quantum symmetry algebra consists of the operators in $\mathcal{H}_g$ corresponding to functions with vanishing integral. These form the algebra $\mathcal{U}(so(3))/\mathbf{R}$, where $\mathbf{R}$ are the constant multiples of unity.

Since the Hilbert space is finite dimensional, this representation will have an infinite dimensional kernel. A classical theorem of Burnside [12] says that factoring out this kernel we remain with the finite dimensional algebra of linear transformations in $\mathcal{H}_g$. Thus one can regard the quantum symmetry algebra to be $sl(2g+1,\mathbf{C})$ or $sl(2g+1,\mathbf{R})$, depending on whether the representation of $so(3)$ is complex or real.

$N = S^{(1,1)} = SO(1,2)/O(2)$

The hyperboloid $S^{(1,1)}$ is the surface in $\mathbf{R}^3$ defined by the equation $x_1^2 + x_2^2 - x_0^2 = -1$. On this surface we define coordinates χ, φ by $x_1 = \sinh\chi\cos\varphi$, $x_2 = \sinh\chi\sin\varphi$ and $x_0 = \cosh\chi$. As electromagnetic field we take $\mathcal{A}_\chi = 0$, $\mathcal{A}_\varphi = g\cosh\chi$ and $\mathcal{F}_{\chi\varphi} = g\sinh\chi$. The symmetry group $S(\mathcal{F}) = \mathcal{SD}iff S^{(1,1)}$ consists of volume-preserving diffeomorphisms of $S^{(1,1)}$. Its maximal finite dimensional subgroup is $SO(1,2)$. As in the case of the sphere, a basis for functions on the hyperboloid is given by the homogeneous polynomials in the variables x_1, x_2, x_0. The polynomials of degree ≥ 1 clearly give rise (through eq.(3)) to all hamiltonian vectorfields, and close under Poisson bracket. Therefore also in this case the Poisson algebra of the Noether charges does not get a central extension.

The maximal compact subalgebra $so(1,2)$ is generated by the functions

$$K_1 = gx_1 = g\sinh\chi\cos\varphi \ , \tag{13a}$$

$$K_2 = gx_2 = g\sinh\chi\sin\varphi \ , \tag{13b}$$

$$K_0 = gx_0 = g\cosh\chi \ , \tag{13c}$$

obeying $\{K_1, K_2\} = -K_0$, $\{K_0, K_1\} = K_2$, $\{K_0, K_2\} = -K_1$. This subalgebra can be quantized without modification. It is convenient to define the complex coordinate $z = \tanh(\chi/2)e^{i\varphi}$ ($|z| < 1$). For $g \geq 1/2$ we define the Hilbert space $\mathcal{H}_g$ as the space of holomorphic functions on the unit disk with inner product

$$\langle\Psi,\Phi\rangle = \frac{(2g-1)}{2\pi i}\int_{|z|<1} dz\, d\bar{z}(1-z\bar{z})^{2g-2}\Psi^*(z)\Phi(z) \ . \tag{14}$$

The condition on g is dictated by the requirement that the inner product be well defined (similar conditions have been discussed in [13]). For $g < 1/2$ one can use other representations of $so(1,2)$[14], but we will not discuss this here. The operators corresponding to K_1, K_2 and K_3 are [14,11]

$$\hat{K}_1 = \frac{1}{2}(1+z^2)\partial_z + gz \ , \tag{15a}$$

$$\hat{K}_2 = \frac{i}{2}(1 - z^2)\partial_z - igz \ , \tag{15b}$$

$$\hat{K}_3 = z\partial_z + g \ . \tag{15c}$$

The value of the Casimir operator $\hat{K}_1^2 + \hat{K}_2^2 - \hat{K}_0^2$ in this representation is $g(1-g)$. For non-integer values of $2g$, the above representation of the algebra $so(1,2)$ leads to multi-valued representations of the group $SO(1,2)$ [14].

Proceeding as in the case of the sphere, we now associate to each monomial $f^{(n_1,n_2,n_3)} = (K_1)^{n_1}(K_2)^{n_2}(K_3)^{n_3}$ the operator $\hat{f}^{(n_1,n_2,n_3)}$ with a given ordering of the factors $\hat{K}_\alpha$. Then the quantum algebra of the Noether charges will be $\mathcal{U}(so(1,2))/\mathbf{R}$. It is represented faithfully on $\mathcal{H}_g$.

The examples we have considered suggest that the quantization of Chern-Simons quantum mechanics will follow the same general pattern, at least for a wide class of manifolds N. The classical symmetry group of canonical transformations has a maximal finite dimensional subgroup G. One can build a Hilbert space $\mathcal{H}$ carrying a realization of this group; thus the algebra of the corresponding Noether charges can be quantized in accordance with eq.(6). We assume that all the symmetry charges (all functions on N) can be expanded in polynomials in the generators of G (this is called the "strong generating principle" in [15]). Then, choosing any ordering, all these charges can be realized as quantum operators on $\mathcal{H}$. However, their algebra will now be different from the classical one: it is the universal enveloping algebra of the Lie algebra of G. This realization of the universal enveloping algebra might have a nontrivial kernel, so to obtain a faithfully realized symmetry algebra one has to factor it out. If N is compact, the resulting quantum symmetry algebra is finite dimensional.

We conclude by observing that this way of quantizing a classical dynamical system is not restricted to Chern-Simons quantum mechanics, but may be used also in the presence of a kinetic term. Our approach allows to quantize all functions on phase space, *i.e.* to construct a quantum operator acting on Hilbert space for each classical observable. The obstruction to quantization given in van Hove's theorem is circumvented by relaxing the condition that (6) holds for all f, g. As a consequence the algebra of quantum observables is a deformation of the classical Poisson algebra and only reduces to the latter in the classical limit. This is different from the usual point of view on quantization, where one insists on (6) but does not seek to quantize all functions on phase space. Typically, one tries only to quantize those functions which generate the symmetries of the theory. When a kinetic or potential term is present, the symmetry group is finite dimensional and, as we have observed, the algebra of its generators is quantized without deformation. It is only for "topological" theories with vanishing Hamiltonian that every function on phase space is a symmetry generator, and the necessity of quantizing all observables arises.

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