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Infinite-Dimensional Symmetries of Two-Dimensional Coset Models

H. Lü,[1] Malcolm J. Perry[2] and C.N. Pope[1,2]

[1] George P. & Cynthia W. Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843-4242, USA.

[2] DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, England.

Abstract

It has long been appreciated that the toroidal reduction of any gravity or supergravity to two dimensions gives rise to a scalar coset theory exhibiting an infinite-dimensional global symmetry. This symmetry is an extension of the finite-dimensional symmetry $\mathcal{G}$ in three dimensions, after performing a further circle reduction. There has not been universal agreement as to exactly what the extended symmetry algebra is, with different arguments seemingly concluding either that it is $\hat{\mathcal{G}}$, the affine Kac-Moody extension of $\mathcal{G}$, or else a subalgebra thereof. We take the very explicit approach of Schwarz as our starting point for studying the simpler situation of two-dimensional flat-space sigma models, which nonetheless capture all the essential details. We arrive at the conclusion that the full symmetry is described by the Kac-Moody algebra $\hat{\mathcal{G}}$, whilst the subalgebra obtained by Schwarz arises as a gauge-fixed truncation. We then consider the explicit example of the $SL(2,\mathbb{R})/O(2)$ coset, and relate Schwarz's approach to an earlier discussion that goes back to the work of Geroch.

Contents

1 Introduction

The study of supergravity theories, and their symmetries, have played a very important rôle in uncovering the underlying structures of string theory. Especially significant are the U-duality symmetries of the string, which have their origin in the classical global symmetries exhibited by eleven-dimensional supergravity and type IIA and IIB supergravities after toroidal dimensional reduction. For example, if one reduces eleven-dimensional supergravity on an n-torus, for $n \le 8$, the resulting $D = (11 - n)$-dimensional theory exhibits a global

E_n symmetry [1–3]. In the cases $n \geq 3$ this symmetry arises in quite a subtle way, involving an interplay between the original eleven-dimensional metric and the 3-form potential.

In view of the large E_8 symmetry that one finds after reduction to three dimensions, it is natural to push further and investigate the symmetries after further reduction to two dimensions, and even beyond. It turns out that the analysis of the global symmetry for a reduction to two dimensions is considerably more complicated than the higher-dimensional ones. There are two striking new features that lead to this complexity. The first is that, unlike a reduction to $D \geq 3$ dimensions, one can no longer use a reduction scheme in which the metric is reduced from an Einstein-frame metric in the higher dimension to an Einstein-frame metric in the lower dimension. (In the Einstein conformal frame, the Lagrangian for gravity itself takes the form $\mathcal{L} \sim \sqrt{-g}R$, with no scalar conformal factor.) The inability to reach the Einstein conformal frame in two dimensions is intimately connected to the fact that $\sqrt{-g}R$ is a conformal invariant in two dimensions. It has the consequence that the metric in two dimensions is not invariant under the global symmetries.

The second striking new feature is that an axionic scalar field (*i.e.* a scalar appearing everywhere covered by a derivative) can be dualised to give another axionic scalar field in the special case of two dimensions. This has the remarkable consequence that the global symmetry group actually becomes infinite in dimension. This was seen long ago by Geroch, in the context of four-dimensional gravity reduced to two. There are degrees of freedom in two dimensions that are described by the sigma model $SL(2,\mathbb{R})/O(2)$, and under dualisation this yields another $SL(2,\mathbb{R})/O(2)$ sigma model. Geroch showed that the two associated global $SL(2,\mathbb{R})$ symmetries do not commute, and that if one takes repeated commutators of the two sets of transformations, an infinite-dimensional algebra results [4]. The precise nature of this symmetry, now known as the *Geroch Group*, was not uncovered in [4].

The feature of having an infinite-dimensional symmetry in two dimensions is not restricted to situations where gravity is involved, and in fact the same essential mechanism operates in a similar fashion if one considers a sigma model in a flat two-dimensional spacetime. Thus, a natural preliminary to investigating the symmetries of two-dimensional reductions in supergravity is to study the symmetry of a flat two-dimensional sigma model $\mathbf{G}/\mathbf{H}$. Considerable simplifications arise if one restricts attention to symmetric-space sigma models, and since these in any case always arise in supergravity dimensional reductions, the specialisation to this class of models is a very natural one. We shall use the acronym SSM to denote a symmetric-space sigma model.

There is quite a considerable literature on the subject of the infinite dimensional symmetries of two-dimensional symmetric-space sigma models, both in the flat and the curved spacetime cases (see, for example, [5–15], some of which considers also principal chiral models). A very clear and explicit presentation of the global symmetry algebras of two-dimensional SSMs has been provided by Schwarz, whose papers formulate the problem in a very transparent way. He first considers the problem of two-dimensional theories in flat spacetime in [16], and then generalises to the case of a curved two-dimensional spacetime in [17]. He also gives an extended history of the earlier literature, and rather than attempting to repeat that here, we refer the reader to his papers for further details.

Our work in the present paper is concerned entirely with the case of symmetric-space sigma models in flat two-dimensional spacetime, and we follow very closely the approach taken by Schwarz in [16]. The results in [16] differ somewhat from those in much of the literature, where the infinite-dimensional global symmetry algebra of the SSM $\mathbf{G}/\mathbf{H}$ is found to be $\hat{\mathcal{G}}$, the affine Kac-Moody extension of the underlying algebra $\mathcal{G}$ of the "manifest" global symmetry group $\mathbf{G}$. The generators of $\hat{\mathcal{G}}$ may be represented by J^i_n, satisfying

$$[J^i_m, J^j_n] = f^{ij}{}_k\, J^k_{m+n}\,, \tag{1.1}$$

where $f^{ij}{}_k$ are the structure constants for the Lie algebra $\mathcal{G}$, whose generators T^i satisfy $[T^i, T^j] = f^{ij}{}_k\, T^k$.

By contrast, Schwarz obtained a certain subalgebra $\hat{\mathcal{G}}_H$ of $\hat{\mathcal{G}}$ as the global symmetry algebra, essentially generated by $J'^i_n = J^i_n \pm J^i_{-n}$, where the $+$ sign is chosen if i lies in the denominator algebra $\mathcal{H}$, and the $-$ sign if i lies in the coset $\mathcal{K} = \mathcal{G}/\mathcal{H}$.

We find that by extending the techniques developed by Schwarz, we can construct explicit global symmetries for the entire $\hat{\mathcal{G}}$ Kac-Moody algebra, expressed purely in terms of local field transformations. As far as we are aware, it is only through the use of the construction that Schwarz developed that it has become possible to obtain explicit local transformations for the entire Kac-Moody algebra. We find also that the subalgebra obtained by Schwarz can be viewed as a gauge-fixed version of this full Kac-Moody symmetry.

In order to understand this, we recall that the possibility of dualising axions to give new axions in two dimensions means that the original theory can be reformulated in terms of new fields that are non-locally related to the original ones (since the process of dualisation requires differentiation and Hodge dualisation, followed by integration, to obtain the new variables). A convenient way to handle this is to enlarge the system by introducing auxiliary fields, so that the manifest global symmetries of the original and the dualised sigma models can be exhibited simultaneously, in purely local terms. In fact to do this, one has to introduce an infinite number of auxiliary fields. The full set of Kac-Moody symmetries, generated by J^i_n with $-\infty \le n \le \infty$, acts on the complete set of original plus auxiliary fields. However, the "negative half" of the Kac-Moody algebra $\hat{\mathcal{G}}$, generated by J^i_n with $n < 0$, acts exclusively on the auxiliary fields, whilst leaving the original sigma-model fields inert. In fact these symmetries are essentially constant shift transformations of the auxiliary fields, reflecting the arbitrariness of the choice of constants of integration that arose when the non-local dualisation was recast into a local form in terms of the auxiliary fields.

The subalgebra $\hat{\mathcal{G}}_H$ of symmetries found by Schwarz can be viewed as a gauge fixing in which the values of the original and the auxiliary fields are all set to prescribed values at some chosen point in the two-dimensional spacetime. Effectively, the level-0 transformations J_0^i that lie in $\mathcal{K}$ are used up in gauge fixing the original fields to their prescribed values, and the entirety of the J_n^i transformations with $n < 0$ are used up in doing the same for the auxiliary fields.

The formulation in which the auxiliary fields are added has been developed considerably by Nicolai, and Julia [18, 19]. However, the work of Schwarz provided a procedure for obtaining explicit expressions for the transformations associated with the "upper half" of the Kac-Moody algebra. We are able to draw the two approaches together and provide a fully explicit and local description of the entire Kac-Moody algebra of symmetries.

In order to illustrate these ideas in detail, it is useful to examine an example. For this purpose we choose the simplest non-trivial symmetric-space sigma model, $SL(2,\mathbb{R})/O(2)$. We show how one needs to introduce an infinity of auxiliary fields in order to describe simultaneously the original $SL(2,\mathbb{R})$ symmetry and the $SL(2,\mathbb{R})$ symmetry of the dualised version (we denote this by $\overline{SL(2,\mathbb{R})}$). We also show how each generator of each copy of $SL(2,\mathbb{R})$ can be precisely matched with a corresponding generator in the Kac-Moody algebra, and this allows us to show explicitly that the Geroch algebra generated by taking multiple commutators of $SL(2,\mathbb{R})$ and $\overline{SL(2,\mathbb{R})}$ transformations is exactly the same as the full Kac-Moody algebra $\widehat{SL(2,\mathbb{R})}$.

We also examine a further symmetry of two-dimensional symmetric-space sigma models $\mathbf{G}/\mathbf{H}$, again basing our analysis on the work of Schwarz [16]. This is again an infinite-dimensional symmetry, but this time a singlet under the original $\mathcal{G}$ symmetry. It turns out to be related to the centreless Virasoro algebra.

2 Lax Equation and Infinite-Dimensional Symmetries

2.1 Basic formalism

We shall begin by considering an arbitrary symmetric-space sigma model (SSM) in a flat two-dimensional spacetime background, with coset given by $\mathbf{K} = \mathbf{G}/\mathbf{H}$, where $\mathbf{G}$ is a Lie group with subgroup $\mathbf{H}$. The commutation relations for the corresponding generators of the algebra take the form

$$[\mathcal{H},\mathcal{H}]=\mathcal{H}\,,\qquad [\mathcal{H},\mathcal{K}]=\mathcal{K}\,,\qquad [\mathcal{K},\mathcal{K}]=\mathcal{H}\,. \tag{2.1}$$

The condition that $\mathbf{K}$ is a symmetric space is reflected in the absence of $\mathcal{K}$ generators on the right-hand side of the last commutation relation. The symmetric-space algebra implies that there is an involution $\sharp$ under which

$$\mathcal{K}^\sharp=\mathcal{K}\,,\qquad \mathcal{H}^\sharp=-\mathcal{H}\,. \tag{2.2}$$

In many cases, such as when $\mathbf{G}=SL(n,\mathbb{R})/O(n)$, the involution map is given by Hermitean conjugation,

$$\mathcal{K}^\dagger=\mathcal{K}\,,\qquad \mathcal{H}^\dagger=-\mathcal{H}\,, \tag{2.3}$$

and later, we shall typically write formulae under this assumption. In some cases, such as $\mathbf{G}=E_{(8,8)}$, $\mathcal{H}=O(16)$, the involution $\sharp$ is more involved.

Let $\mathcal{V}$ be a coset representative in $\mathbf{K}$. We may then define

$$M=\mathcal{V}^\sharp\,\mathcal{V}\,,\qquad A=M^{-1}dM\,. \tag{2.4}$$

Under transformations

$$\mathcal{V}\longrightarrow h\mathcal{V}g\,, \tag{2.5}$$

where g is a global element in the group $\mathbf{G}$ and h is a local element in the denominator subgroup $\mathbf{H}$, we have shall have

$$M\longrightarrow g^\sharp M g\,,\qquad A\longrightarrow g^{-1}Ag\,, \tag{2.6}$$

since it follows from $\mathcal{H}^\sharp=-\mathcal{H}$ that $h^\sharp=h^{-1}$.

The Cartan-Maurer equation $d(M^{-1}dM)=-(M^{-1}dM)\wedge(M^{-1}dM)$ implies that the field strength for A vanishes:

$$F\equiv dA+A\wedge A=0\,. \tag{2.7}$$

The Lagrangian for the coset model may be written as $L=-\frac{1}{4}\mathrm{tr}({*A}\wedge A)$ (or, using indices, $L=-\frac{1}{4}\eta^{\mu\nu}\mathrm{tr}(A_\mu A_\nu)$) and hence the equation of motion is

$$d{*A}=0\,. \tag{2.8}$$

The Lagrangian is clearly invariant under the global $\mathbf{G}$ transformations, and the equations (2.7) and (2.8) transform covariantly under $\mathbf{G}$.

As discussed in [16], the equations (2.7) and (2.8) can both be derived from the integrability condition for the *Lax Pair* of linear equations

$$\Big(\partial_+ +\frac{t}{t-1}A_+\Big)X=0\,,\qquad \Big(\partial_- +\frac{t}{t+1}A_-\Big)X=0\,, \tag{2.9}$$

to admit a solution $X(x;t)$, where t is an arbitrary constant *spectral parameter*. These equations are written in light-cone coordinates on the two-dimensional flat spacetime, in which the metric is $ds^2 = 2dx^+\, dx^-$. We prefer to use the language of differential forms, for which $A = A_+ dx^+ + A_- dx^-$. On 1-forms we have $*^2 = +1$, where $*$ is the Hodge dual operator, and

$$*dx^\pm = \pm dx^\pm\,, \qquad \text{and so} \qquad *A = A_+ dx^+ - A_- dx^-\,. \tag{2.10}$$

It is useful also to record the following properties for 1-forms u and v:

$$*u \wedge v = *v \wedge u\,, \qquad *u \wedge *v = -u \wedge v\,, \tag{2.11}$$

and for Lie-algebra valued 1-forms A and B:

$$*A \wedge B = -A \wedge *B\,, \qquad *A \wedge *A = -A \wedge A\,. \tag{2.12}$$

In terms of differential forms, the Lax pair (2.9) becomes simply the single equation

$$t(d + A)X = *dX\,. \tag{2.13}$$

We shall call this the *Lax Equation*. By taking the appropriate linear combination of this and its dual, we obtain

$$dXX^{-1} = \frac{t}{1-t^2}\, *A + \frac{t^2}{1-t^2}\, A\,. \tag{2.14}$$

Thus the integrability condition for the existence of a solution $X(x;t)$ to the Lax equation, which follows from the Cartan-Maurer equation $d(dXX^{-1}) = (dXX^{-1}) \wedge (dXX^{-1})$, gives

$$d{*A} + t(dA + A \wedge A) = 0\,. \tag{2.15}$$

Since this must hold for all t we indeed derive (2.7) and (2.8). Note that (2.14) is an equivalent formulation of the Lax equation; an appropriate linear combination of (2.14) and its dual gives back (2.13). Thus we may use the term "Lax equation" interchangeably for (2.13) and (2.14).

2.2 Infinite-dimensional extension of the global G symmetry

We have already noted that the global **G** transformations (2.6) are a symmetry of the zero-curvature condition (2.7) and the equations of motion (2.8) of the two-dimensional coset model. In fact, these symmetries are merely the tip of an infinite-dimensional "iceberg" of global symmetries. These extended symmetries are a special feature that arises because the coset model lives in a two-dimensional world volume, and they may be understood in a variety of ways. An intuitive understanding, which we shall turn into a concrete discussion in section 4 for the example of the coset $SL(2,\mathbb{R})/O(2)$, is that the axionic scalars can be dualised into new, non-locally related sets of axions in two dimensions, and that the manifest global symmetries in the different duality pictures do not commute, but instead their commutators close only on an infinite-dimensional extension of the finite-dimensional symmetries that are manifest in each individual duality choice.[1]

In the present section, we shall begin by following a construction given in [16], which shows how the formalism of the Lax equation may be used to derive the infinite-dimensional algebra. Our description will be formulated in the language of differential forms rather than light-cone coordinates. The details of our calculation differ somewhat from those in [16], and our conclusions differ also. Specifically, we find that the full symmetry of the symmetric-space sigma model is precisely the affine Kac-Moody extension $\hat{\mathcal{G}}$ of the manifest $\mathcal{G}$ global symmetry, and not merely the subalgebra of $\hat{\mathcal{G}}$ that was found in [16]. (We shall comment further about this later in this subsection, and in appendix A.)

At the infinitesimal level, the transformation (2.5) becomes

$$\delta\mathcal{V} = \mathcal{V}\epsilon + \delta h\,\mathcal{V}\,, \tag{2.16}$$

where ϵ is an infinitesimal global element of the Lie algebra $\mathcal{G}$ and δh is a local element of $\mathcal{H}$. In order to exhibit the infinite-dimensional extension of this symmetry algebra, one may consider more general transformations of the form [16]

$$\delta\mathcal{V} = \mathcal{V}\eta + \delta h\,\mathcal{V}\,, \qquad \text{where} \qquad \eta = X(t)\epsilon X(t)^{-1}\,. \tag{2.17}$$

The meaning of this equation is as follows. As before, $\mathcal{V}$ is a coset representative for $\mathbf{G}/\mathbf{H}$, and thus it depends on the scalar fields parameterising the coset, which themselves depend on the two spacetime coordinates x, but it does not depend on the spectral parameter t. The function $X(t)$ is the solution of the Lax equation (2.13) and thus it depends on the spacetime coordinates x (we are now suppressing the explicit indication of this dependence) and on the spectral parameter t. The quantity δh, in the denominator algebra $\mathcal{H}$, is a function of the spacetime fields and it may now depend upon t. On the left-hand side of (2.17) there is t-dependence only in the variational symbol δ itself, and it is to be interpreted as

$$\delta = \delta(\epsilon, t) = \sum_{n\geq 0} t^n\,\delta_{(n)}(\epsilon)\,. \tag{2.18}$$

Thus by equating powers of t on the two sides of (2.17) we obtain a hierarchy of transformations $\delta_{(n)}$ that act upon the scalar fields in the coset representative $\mathcal{V}$. The lowest set of

[1] This idea dates back to a paper on four-dimensional gravity reduced to two dimensions, by Geroch [4], although at that time the precise nature of the infinite-dimensional algebra was not addressed.

transformations, *i.e.* for $n = 0$, just correspond to the original infinitesimal $\mathbf{G}$ transformations that were manifest in the coset model from the outset. By contrast the transformations $\delta_{(n)}$ with $n > 0$, which all involve t-dependent terms in $X(t)$, are non-local expressions in terms of the original fields of the scalar coset.[2]

To check that (2.17) does indeed give symmetries of the theory, one must check that the corresponding variation of the equation of motion (2.8) vanishes. First, one sees from[3] $M = \mathcal{V}^\dagger \mathcal{V}$ and $A = M^{-1} dM$ that (2.17) implies

$$\delta M = M\eta + \eta^\dagger M\,, \qquad \delta A = D\eta + D(M^{-1}\eta^\dagger M)\,, \tag{2.19}$$

where the $\mathbf{G}$-covariant exterior derivative is defined on any $\mathcal{G}$-valued function f by

$$Df = df + [A, f]\,. \tag{2.20}$$

It can also be seen from the definition of η in (2.17), after making use of the Lax equation (2.13), that

$$D\eta = \frac{1}{t} *d\eta\,, \qquad D(M^{-1}\eta^\dagger M) = t *d(M^{-1}\eta^\dagger M)\,. \tag{2.21}$$

Thus we conclude that under (2.17),

$$\delta A = *d\Big(\frac{1}{t}\,\eta + t\,M^{-1}\eta^\dagger M\Big)\,, \tag{2.22}$$

which indeed verifies that $d{*}\delta A = 0$.

In order to read off the symmetry algebra one needs to calculate commutators of the form $[\delta_{(m)}, \delta_{(n)}]$. Since, as noted above, the variations $\delta_{(n)}$ involve $X(t)$, which itself depends non-locally on the fields of the scalar coset, one first needs to calculate the variations of $X(t)$ with respect to the hierarchy of transformations $\delta_{(n)}$. This was obtained in [16], and with a small but important modification that we shall discuss later, it is given by

$$\delta_1 X_2 = \frac{t_2}{t_1 - t_2}\,(\eta_1 X_2 - X_2 \epsilon_1) + \frac{t_1 t_2}{1 - t_1 t_2}\, M^{-1} \eta_1^\dagger M X_2\,. \tag{2.23}$$

Here δ_1 (with no parentheses around the 1) denotes $\delta(\epsilon_1, t_1) = \sum_{n\ge 0} t_1^n \delta_{(n)}(\epsilon_1)$, whilst X_2 denotes $X(t)$ for a different and independent choice of spectral parameter t_2. By equating

[2] Note, however, that all the transformations become local if one introduces an infinite set of auxiliary fields, as we shall do later.

[3] From this point onwards, we shall assume for simplicity, and to make the expressions look more palatable, that the involution of the symmetric space algebra is implemented by Hermitean conjugation, as in (2.3). In a case where the more general $\sharp$ involution operator is required, all $\dagger$ symbols in what follows should be replaced by $\sharp$.

the coefficients of $t_1^m t_2^n$ on both sides of (2.23), one can read off the variation under $\delta_{(m)}$ of the t_2^n term in the series expansion of $X(t_2)$.

In order to derive (2.23), we follow the method used in [16], which amounts to varying the Lax equation (2.13) under (2.17), with δA given in (2.19) and δX given by (2.23), and verifying that the varied equation is also satisfied. Thus, one must substitute (2.23) into

$$[t_2(d+A) - *d](\delta_1 X_2) + t_2(\delta_1 A)X_2 = 0\,, \tag{2.24}$$

or in other words, after using (2.19), into

$$[t_2(d+A) - *d](\delta_1 X_2) + t_2[D\eta_1 + D(M^{-1}\eta_1^\dagger M)]X_2 = 0\,. \tag{2.25}$$

After some algebra, again involving the use of the Lax equation, the desired result follows.

Using (2.23) one can calculate the commutator of transformations on $M = \mathcal{V}^\dagger\,\mathcal{V}$, finding (in a similar manner to [16]) that

$$[\delta_1, \delta_2]M = \frac{t_1\,\delta(\epsilon_{12}, t_1) - t_2\,\delta(\epsilon_{12}, t_2)}{t_1 - t_2}\,M\,, \tag{2.26}$$

where $\epsilon_{12} = [\epsilon_1, \epsilon_2]$. It is also straightforward to show, after some lengthy algebra, that

$$[\delta_1, \delta_2]X_3 = \frac{t_1\,\delta(\epsilon_{12}, t_1) - t_2\,\delta(\epsilon_{12}, t_2)}{t_1 - t_2}\,X_3\,, \tag{2.27}$$

If the transformations δ given in (2.17) and (2.23) were the only ones extending G then we would have essentially "half" of the affine Kac-Moody extension $\hat{G}$. However, there are additional transformations, which we shall denote by $\tilde\delta$, that also extend G. These leave M invariant but they do act non-trivially on X. They are given by

$$\tilde\delta_1 M = 0\,, \qquad \tilde\delta_1 X_2 = \frac{t_1 t_2}{1 - t_1 t_2} X_2 \epsilon_1\,. \tag{2.28}$$

Again, the notation here is that $\tilde\delta_1 = \tilde\delta(\epsilon_1, t_1) = \sum_{n\ge 1} t_1^n \tilde\delta_{(n)}$, and $X_2 = X(t_2)$. (Note that there is no $n = 0$ term here in the expansion of $\tilde\delta_1$, as can be seen from the absence of a t_1^0 term on the right-hand side of (2.28).) It is easy to verify that (2.28) describes symmetries of the Lax equation. The easiest way to do this is to note that (2.28) implies $\tilde\delta(dXX^{-1}) = 0$, and so since $\tilde\delta A = 0$, it is evident that the Lax equation (2.14) is indeed invariant under $\tilde\delta$.

The commutators of the $\tilde\delta$ transformations give

$$[\tilde\delta_1, \tilde\delta_2]X_3 = \frac{t_2}{t_1 - t_2}\,\tilde\delta(\epsilon_{12}, t_1)X_3 - \frac{t_1}{t_1 - t_2}\,\tilde\delta(\epsilon_{12}, t_2)X_3\,, \tag{2.29}$$

where again, $\epsilon_{12} = [\epsilon_1, \epsilon_2]$. (This commutation relation is vacuous, of course, when acting on M.) Finally, we may calculate the commutators of δ and $\tilde\delta$ transformations, finding

$$[\delta_1, \tilde\delta_2] X_3 = \frac{t_1 t_2}{1 - t_1 t_2}\, \delta(\epsilon_{12}, t_1) X_3 + \frac{1}{1 - t_1 t_2}\, \tilde\delta(\epsilon_{12}, t_2) X_3 \,. \tag{2.30}$$

(The commutator on M is the same, except that there is no $\tilde\delta$ term on the right-hand side since $\tilde\delta M = 0$.)

In summary, we therefore have in total the commutation relations

$$\begin{aligned}
[\delta_1, \delta_2] &= \frac{t_1}{t_1 - t_2}\, \delta(\epsilon_{12}, t_1) - \frac{t_2}{t_1 - t_2}\, \delta(\epsilon_{12}, t_2)\,, && (2.31)\\
[\delta_1, \tilde\delta_2] &= \frac{t_1 t_2}{1 - t_1 t_2}\, \delta(\epsilon_{12}, t_1) + \frac{1}{1 - t_1 t_2}\, \tilde\delta(\epsilon_{12}, t_2)\,, && (2.32)\\
[\tilde\delta_1, \tilde\delta_2] &= \frac{t_2}{t_1 - t_2}\, \tilde\delta(\epsilon_{12}, t_1) - \frac{t_1}{t_1 - t_2}\, \tilde\delta(\epsilon_{12}, t_2)\,. && (2.33)
\end{aligned}$$

From these, one can read off the towers of modes in the t-expansions, using $\delta(\epsilon, t) = \sum_n t^n \delta_n(\epsilon)$, etc. For example, (2.31) gives

$$\begin{aligned}
\sum_{m\ge 0}\sum_{n\ge 0} t_1^m t_2^n\, [\delta_{(m)}(\epsilon_1), \delta_{(n)}(\epsilon_2)] &= \frac{1}{t_1 - t_2} \sum_{p\ge 0} (t_1^{p+1} - t_2^{p+1})\, \delta_{(p)}(\epsilon_{12})\,,\\
&= \sum_{p\ge 0}\sum_{q=0}^{p} t_1^q t_2^{p-q}\, \delta_{(p)}(\epsilon_{12})\,,\\
&= \sum_{m\ge 0}\sum_{n\ge 0} t_1^m t_2^n\, \delta_{(m+n)}(\epsilon_{12})\,, && (2.34)
\end{aligned}$$

whence we obtain

$$[\delta_{(m)}(\epsilon_1), \delta_{(n)}(\epsilon_2)] = \delta_{(m+n)}(\epsilon_{12})\,, \qquad m \ge 0\,,\ n \ge 0\,. \tag{2.35}$$

The analogous calculations for (2.32) and (2.33) give

$$\begin{aligned}
[\delta_{(m)}(\epsilon_1), \tilde\delta_{(n)}(\epsilon_2)] &= \delta_{(m-n)}(\epsilon_{12}) + \tilde\delta_{(n-m)}(\epsilon_{12})\,, \qquad m \ge 0\,,\ n \ge 1\,, && (2.36)\\
[\tilde\delta_{(m)}(\epsilon_1), \tilde\delta_{(n)}(\epsilon_2)] &= \tilde\delta_{(m+n)}(\epsilon_{12})\,, \qquad m \ge 1\,,\ n \ge 1\,, && (2.37)
\end{aligned}$$

where in (2.36) it is to be understood that $\delta_{(n)} = 0$ for $n \le -1$ and $\tilde\delta_{(n)} = 0$ for $n \le 0$.

The three sets of commutation relations can be combined into one by introducing a new set $\Delta_{(n)}$ of variations, defined for all n with $-\infty \le n \le \infty$, according to

$$\begin{aligned}
\Delta_{(n)} &= \delta_{(n)}\,, \qquad n \ge 0\,,\\
\Delta_{(-n)} &= \tilde\delta_{(n)}\,, \qquad n \ge 1\,. && (2.38)
\end{aligned}$$

It is then easily seen that (2.35), (2.36) and (2.37) become

$$[\Delta_{(m)}(\epsilon_1), \Delta_{(n)}(\epsilon_2)] = \Delta_{(m+n)}(\epsilon_{12})\,, \qquad m, n \in \mathbb{Z}\,, \tag{2.39}$$

with $\epsilon_{12} = [\epsilon_1, \epsilon_2]$. This defines the affine Kac-Moody algebra $\hat{\mathcal{G}}$. In terms of currents $J^i(\sigma)$ defined on a circle, with

$$J^i(\sigma) = \sum_{n=-\infty}^{\infty} e^{in\sigma}\, J^i_n\,, \tag{2.40}$$

the commutation relations (2.39) are equivalent to

$$[J^i_m, J^j_n] = f^{ij}{}_k\, J^k_{m+n}\,, \tag{2.41}$$

where $f^{ij}{}_k$ are the structure constants of the Lie algebra $\mathcal{G}$. Specifically, we have the association

$$\Delta_{(n)}(\epsilon^i) \leftrightarrow J^i_n\,, \tag{2.42}$$

where $\epsilon = \epsilon^i\, T_i$, and T_i are the generators of the Lie algebra $\mathcal{G}$.

Since we have arrived at a somewhat different conclusion from Schwarz, who finds only a subalgebra of the Kac-Moody algebra $\hat{\mathcal{G}}$ as a symmetry of the SSM [16], we shall discuss in appendix A exactly why the difference has arisen. In essence, the key distinction is that we include the transformations $\tilde{\delta}$ defined in (2.28) as independent symmetries. They are non-trivial symmetries of the Lax equation, even though they act trivially on the scalar fields in the coset representative $\mathcal{V}$ itself. In section 4, we shall study the explicit example of the $SL(2,\mathbb{R})/O(2)$ coset model, in order to illustrate this point in greater detail. We shall show that a natural formulation of the model involves introducing an infinite number of additional scalar fields, in terms of which X appearing in the Lax equation (2.13) can be expressed as a local quantity. The $\tilde{\delta}$ transformations act on this infinite tower of additional fields. We shall also show how this infinity of extra scalars can be interpreted as fields that one introduces in order to exhibit in a local fashion the symmetries arising from the closure of the two non-commuting $SL(2,\mathbb{R})$ symmetries of the original theory and a dualised version.

A further remark about the Kac-Moody transformations δ and $\tilde{\delta}$ is also in order. The $\tilde{\delta}$ transformation defined in (2.28) are of the general form $\tilde{\delta}X \sim X\epsilon$. It can be seen that the second of the three terms on the right-hand side of the δX transformation given in (2.23) is also of this general form. This means that as far as obtaining symmetries of the Lax equation is concerned, one could have omitted the second term in (2.23) altogether, since it is itself a distinct symmetry in its own right. However, it actually serves an important purpose in (2.23), namely to subtract out what would otherwise be a pole at $t_1 = t_2$ if one had only $t_2\eta_1 X_2/(t_1 - t_2)$ rather than $t_2(\eta_1 X_2 - X_2\epsilon_1)/(t_1 - t_2)$. (The third term is in

(2.23) is necessary in addition, in order to get a symmetry, but there is no pole associated with this term, since we expand t_1 and t_2 around zero.) Now, the derivations of the δX and $\tilde{\delta} X$ transformations as symmetries involved considering the variation of (dXX^{-1}) in the Lax equation (2.14). In the case of the $\tilde{\delta}$ transformation we have $\tilde{\delta} A = 0$, and one may view $\tilde{\delta} X$ as the solution of the homogeneous equation $\tilde{\delta}(dXX^{-1}) = 0$, whilst δX is the solution of the inhomogeneous equation $\delta(dXX^{-1}) =$ (non-zero source). Thus the inclusion of a $\tilde{\delta} X$ contribution as the second term in (2.23) can be viewed as the necessary addition of a solution of the homogeneous solution that is needed in order to ensure that the inhomogeneous solution satisfies the necessary boundary condition (*i.e.* that $\delta_1 X_2$ be regular at $t_1 = t_2$).

This discussion also emphasises the point that it is really the δ transformations found by Schwarz, appearing in our slightly modified form in (2.23), that lie at the heart of the Kac-Moody symmetries of the symmetric-space sigma models. The $\tilde{\delta}$ transformations, although they are of course equally necessary in order to obtain the complete Kac-Moody symmetry, are somewhat secondary in nature since they are already present within the construction of the δ transformations.

It is also worth remarking that we have obtained the full Kac-Moody algebra as a symmetry of the SSM by means of a purely perturbative analysis, which involved a small-t expansion of $X(t)$ around $t = 0$. One may also consider instead a large-t expansion of $X(t)$, around $t = \infty$. The result is in fact equivalent. This can be seen by letting $t = \tilde{t}^{-1}$, whereupon the Lax equation (2.14) becomes

$$dXX^{-1} = -\frac{\tilde{t}}{1-\tilde{t}^2} *A - \frac{1}{1-\tilde{t}^2} A \,. \tag{2.43}$$

If we let $X = M^{-1}(\widetilde{X}^{-1})^\dagger$, we arrive at a Lax equation that is identical in form to the original expression (2.14), namely

$$d\widetilde{X}\widetilde{X}^{-1} = \frac{\tilde{t}}{1-\tilde{t}^2} *A + \frac{\tilde{t}^2}{1-\tilde{t}^2} A \,, \tag{2.44}$$

showing that the large-t expansion is equivalent to the small-$\tilde{t}$ expansion. One would therefore reach identical conclusions had one performed a large-t expansion instead of a small-t expansion. It would be interesting to study the regime where the small-t and large-t expansions overlap. Although the Lax equation is regular in both regions, it becomes singular at $t = \pm 1$. Even if such a non-perturbative analysis could be performed, we would not necessarily expect to find a larger symmetry algebra than the full Kac-Moody algebra, which is already found in our perturbative approach.

2.3 Virasoro-like symmetry

The symmetry discussed in section 2.2 is an infinite-dimensional extension of the manifest $\mathcal{G}$ symmetry of the $\mathbf{G/H}$ symmetric-space sigma model. As such, the transformation parameters ϵ in (2.17) are themselves $\mathcal{G}$ valued. There is an additional infinite-dimensional symmetry of the SSM, with transformation parameters that are singlets under $\mathcal{G}$, which turns out to be a subalgebra of the Virasoro algebra. Our discussion here again begins by using an approach that is very close to that of Schwarz [16], although with certain modifications and elaborations.

The transformations in question act on the coset representative $\mathcal{V}$ as follows[4] [16]:

$$\delta^V(t)\mathcal{V} = \mathcal{V}\xi\,, \quad \text{where} \quad \xi = -t\dot{X}(t)X(t)^{-1}\,. \tag{2.45}$$

By equating the coefficients of each power of t in (2.45), one obtains an infinite set of transformations $\delta^V_{(n)}$ of the scalar fields in the SSM, with[5]

$$\delta^V(t) = \sum_{n\geq 1} t^n \delta^V_{(n)}\,. \tag{2.46}$$

Note that it is because of the explicit t factor in the definition of ξ in (2.45) that the sum in (2.46) does not include $n = 0$.

To see that (2.45) indeed describes symmetries of the theory, one must show that the equation of motion $d{*A} = 0$ is preserved. It follows from (2.45) that

$$\delta^V A = D\xi + M^{-1}d\xi^\dagger M = D\xi + D(M^{-1}\xi^\dagger M)\,, \tag{2.47}$$

where as usual $D\xi = d\xi + [A, \xi]$. Differentiating the Lax equation (2.13) with respect to t, and subtracting the Lax equation premultiplied by $(\dot{X}X^{-1})$ and postmultiplied by X^{-1}, one finds that

$$D(\dot{X}X^{-1}) = \frac{1}{t}\left[{*d}(\dot{X}X^{-1}) - \frac{1}{1-t^2}\,A - \frac{t}{1-t^2}\,{*A}\right], \tag{2.48}$$

[4]We should really include an infinitesimal parameter as a prefactor in the definition of ξ in equation (2.45). However, since it is a singlet it plays no significant rôle, and so it may be omitted without any risk of ambiguity.

[5]Our transformations (2.45) differ slightly from those given in [16], in which the lowest-order term is subtracted out and the overall t-dependent factor is different. Our choice for the explicit t-dependent factor is made so that the algebra takes the simplest possible form. The subtraction was shown to be necessary in the context of principal chiral models in [16], and was carried over into the discussion of the SSM case in that paper. In fact, the subtraction becomes optional in the SSM case, which amounts to saying that the SSM has an additional mode in the symmetry transformation. We shall discuss this in further in appendix B.

From this, one can also show that

$$D(M^{-1}(\dot{X}X^{-1})^\dagger M) = t *d(M^{-1}(\dot{X}X^{-1})^\dagger M) + \frac{1}{1-t^2} *A + \frac{t}{1-t^2}\,A\,. \tag{2.49}$$

Substituting into (2.47), we find

$$\delta^V A = *d\left(\frac{1}{t}\xi + tM^{-1}\xi^\dagger M\right) + A\,, \tag{2.50}$$

and from this is follows that $d{*}\delta^V A = 0$, thus proving that δ^V is a symmetry of the equations of motion.[6]

The next step is to calculate the commutator of the δ^V transformations, in order to determine their algebra. As a preliminary, we need an expression for $\delta_1^V X_2$. Guided by the discussion in [16], we find that it is given by

$$\delta_1^V X_2 = Y X_2\,, \qquad Y = \frac{1}{t_1 - t_2}\left[t_2\,\xi_1 + \frac{t_1(t_2^2-1)}{1-t_1t_2}\,\xi_2\right] + \frac{t_1t_2}{1-t_1t_2}\,M^{-1}\xi_1^\dagger M\,. \tag{2.51}$$

The verification that (2.51) is correct is achieved by substituting (2.47) and (2.51) into the Lax equation (2.14).

After lengthy calculations of the commutators $[\delta_1^V, \delta_2^V]M$ and $[\delta_1^V, \delta_2^V]X_3$, we find that

$$[\delta_1^V, \delta_2^V] = -2t_1t_2\left[\frac{1}{(t_1-t_2)^2} + \frac{1}{(1-t_1t_2)^2}\right]\delta_1^V + \frac{t_1t_2(1-t_1^2)}{(t_1-t_2)(1-t_1t_2)}\,\dot{\delta}_1^V - [1 \leftrightarrow 2]\,, \tag{2.52}$$

where $\dot{\delta}_1^V$ denotes the derivative of δ_1^V with respect to its argument t_1, and the symbol $[1 \leftrightarrow 2]$ indicates the subtraction of two terms obtained from those that are displayed by exchanging the 1 and 2 subscripts everywhere.

To derive the mode algebra, we substitute the mode expansion (2.46) into (2.52), and collect terms associated with each power of t_1 and t_2. We then find that the abstract algebra of the δ^V transformations is given by

$$[\delta_{(m)}^V, \delta_{(n)}^V] = (m-n)\delta_{(m+n)}^V - (m+n)\delta_{(m-n)}^V\,, \tag{2.53}$$

where it is understood that $\delta_{(n)}^V$ with negative mode numbers n is *defined* to be given by

$$\delta_{(-n)}^V \equiv -\delta_{(n)}^V\,, \qquad n \geq 1\,. \tag{2.54}$$

[6] It is because of the cancellation in (2.50) of the contributions proportional to $*A$ coming from the two terms in (2.47) that there is no need to make the lowest-order subtraction that was found in [16] to be necessary in the PCM case.

One might have thought that the ostensible occurrence of pole terms at $t_1 = t_2$ in (2.52) would have presented difficulties in interpreting the algebra, but in fact one finds that cancellations imply there are no such poles. One way to make this manifest is to note that (2.52) can be rewritten as

$$[\delta_1^V, \delta_2^V] = -\frac{2t_1t_2(\delta_1^V - \delta_2^V)}{(1-t_1t_2)^2} - \frac{t_1t_2(t_1-t_2)}{1-t_1t_2}\,\frac{\partial^2}{\partial t_1 \partial t_2}\Big[\frac{t_2(t_1-t_1^{-1})\delta_1^V - t_1(t_2-t_2^{-1})\delta_2^V}{t_1-t_2}\Big]\,. \tag{2.55}$$

We may define a current $K(\sigma)$ in which we associate the mode K_n with the symmetry transformation $\delta_{(n)}^V$:

$$K(\sigma) = \sum_{n=-\infty}^{\infty} e^{in\sigma}\,K_n\,. \tag{2.56}$$

The reflection condition (2.54) implies that the modes K_n satisfy $K_n = -K_{-n}$, and from (2.53), they satisfy the algebra

$$[K_m, K_n] = (m-n)K_{m+n} - (m+n)K_{m-n}\,, \tag{2.57}$$

This is clearly not the Virasoro algebra, but it is closely related to it. Specifically, if we introduce generators L_m for a centreless Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n}\,, \tag{2.58}$$

then we find that the modes K_m may be represented as

$$K_m = L_m - L_{-m}\,, \qquad m \neq 0 \tag{2.59}$$

(Recall that that (2.45) contains no $\delta_{(0)}^V$ transformation, and so K_0 is not present in the algebra.) If we define the usual Virasoro current

$$T(\sigma) = \sum_{m=-\infty}^{\infty} L_m\,e^{im\sigma}\,, \tag{2.60}$$

then it follows from (2.56) and (2.59) that

$$K(\sigma) = 2i\,\Im\left(T(\sigma)\right)\,. \tag{2.61}$$

It is interesting to contrast this result with the analogous one that was obtained in [16] for the case of a principal chiral model, where it was shown that $K_m = L_{m+1} - L_{m-1}$ and hence $K(\sigma) = -2i\sin\sigma\,T(\sigma)$. In that case, one could view this relation as a definition of the energy-momentum tensor $T(\sigma)$ in terms of $K(\sigma)$, save for the degenerate points $\sigma = 0$

and $\sigma = \pi$ at the ends of the line segment. By contrast, the relation (2.61) for the SSM cannot be used to define the whole of $T(\sigma)$, but only its imaginary part. Thus the Virasoro algebra itself is not described by the symmetry transformations (2.45).

We may also calculate the commutators of the Virasoro-like transformations δ^V with the Kac-Moody transformations δ and $\tilde{\delta}$ of section 2.2. These commutators must be evaluated on X, and not merely on M, in order to capture the resulting terms that correspond to $\tilde{\delta}$ transformations, since M is inert under these.

By calculating the commutator $[\delta_1^V, \tilde{\delta}_2]$ acting on M and on X_3, we find that

$$\begin{aligned}[\delta_1^V, \tilde{\delta}_2] &= \frac{t_1 t_2}{(1-t_1t_2)^2}\,\delta(t_1,\epsilon_2) + t_1t_2\left[\frac{1}{(1-t_1t_2)^2} + \frac{1}{(t_1-t_2)^2}\right]\tilde{\delta}(t_2,\epsilon_2) \\ &\quad -\frac{t_1t_2}{(t_1-t_2)^2}\,\tilde{\delta}(t_1,\epsilon_2) - \frac{t_1t_2(t_2^2-1)}{(t_1-t_2)(1-t_1t_2)}\,\dot{\tilde{\delta}}(t_2,\epsilon_2)\,, \end{aligned} \tag{2.62}$$

where $\dot{\tilde{\delta}}(t_2,\epsilon_2)$ denotes the derivative of $\tilde{\delta}(t_2,\epsilon_2)$ with respect to t_2.

Similarly, calculating the commutator $[\delta_1^V, \delta_2]$ acting on M and on X_3, we find

$$\begin{aligned}[\delta_1^V, \delta_2] &= \frac{t_1 t_2}{(1-t_1t_2)^2}\,\tilde{\delta}(t_1,\epsilon_2) + t_1t_2\left[\frac{1}{(1-t_1t_2)^2} + \frac{1}{(t_1-t_2)^2}\right]\delta(t_2,\epsilon_2) \\ &\quad -\frac{t_1t_2}{(t_1-t_2)^2}\,\delta(t_1,\epsilon_2) - \frac{t_1t_2(t_2^2-1)}{(t_1-t_2)(1-t_1t_2)}\,\dot{\delta}(t_2,\epsilon_2)\,, \end{aligned} \tag{2.63}$$

As in the case of (2.52), although there are ostensibly poles in (2.62) and (2.63) at $t_1 = t_2$, these in fact cancel. Expanding in powers of t_1 and t_2, and making use of the definition (2.38) for the full set of Kac-Moody transformations Δ_m, we find that

$$[\delta^V_{(m)}, \Delta_{(n)}] = -n(\Delta_{(n+m)} - \Delta_{(n-m)})\,. \tag{2.64}$$

In terms of the Kac-Moody current-algebra modes J^i_n and Virasoro-like modes K_n that we introduced earlier, we therefore find

$$[K_m, J^i_n] = -n(J^i_{n+m} - J^i_{n-m})\,. \tag{2.65}$$

One may verify that this is consistent with the Jacobi identity $[K_m,[K_n,J^i_p]] + \cdots = 0$, after using our result (2.57) for the commutator $[K_m,K_n]$.

3 An Alternative Description

A slightly different approach to describing the symmetries of two-dimensional symmetric-space coset models was taken in [18], and it is useful to summarise some salient aspects here, since we shall make use of some of the formalism in section 4. It is again an approach where the SSM is viewed as an integrable system, and it is essentially equivalent to the description in [16] in terms of the Lax equation.

Starting from the coset representative $\mathcal{V}$ that we introduced previously, one may define

$$d\mathcal{V}\mathcal{V}^{-1} = Q + P\,, \tag{3.1}$$

where Q is the projection into the denominator algebra $\mathcal{H}$ and P is the projection into the coset algebra $\mathcal{K}$. From the Cartan-Maurer equation $d(d\mathcal{V}\mathcal{V}^{-1}) = (d\mathcal{V}\mathcal{V}^{-1}) \wedge (d\mathcal{V}\mathcal{V}^{-1})$, one can then read off the equations

$$\begin{aligned} dQ - Q\wedge Q - P\wedge P &= 0\,, &(3.2)\\ DP \equiv dP - Q\wedge P - P\wedge Q &= 0\,. &(3.3)\end{aligned}$$

Under the transformations (2.5) one has

$$Q \longrightarrow hQh^{-1} + dhh^{-1}\,, \qquad P \longrightarrow hPh^{-1}\,, \tag{3.4}$$

which shows that $D = d - Q\wedge - \wedge Q$ can be viewed as an $\mathcal{H}$-covariant connection. P transforms covariantly under $\mathcal{H}$ and is invariant under the global right-acting $\mathcal{G}$ transformations.

From (2.4), and making the convenient assumption again that the involution $\sharp$ is implemented by Hermitean conjugation, we see that with $M = \mathcal{V}^\dagger\mathcal{V}$

$$A = M^{-1}dM = \mathcal{V}^{-1}\left(d\mathcal{V}\mathcal{V}^{-1} + (d\mathcal{V}\mathcal{V}^{-1})^\dagger\right)\mathcal{V} = \mathcal{V}^{-1}(Q+P+Q^\dagger+P^\dagger)\mathcal{V} = 2\mathcal{V}^{-1}P\mathcal{V}\,, \tag{3.5}$$

since under the involution we shall have $Q^\dagger = -Q$, $P^\dagger = P$. It follows from (2.14) that

$$\begin{aligned} \mathcal{V}dXX^{-1}\mathcal{V}^{-1} &= \frac{2t}{1-t^2}*P + \frac{2t^2}{1-t^2}P\,,\\ &= \frac{2t}{1-t^2}*P + \frac{1+t^2}{1-t^2}P - P\,,\\ &= \frac{2t}{1-t^2}*P + \frac{1+t^2}{1-t^2}P + Q - d\mathcal{V}\mathcal{V}^{-1}\,, \end{aligned} \tag{3.6}$$

and hence

$$d\hat{\mathcal{V}}(t)\hat{\mathcal{V}}(t)^{-1} = Q + \frac{2t}{1-t^2}*P + \frac{1+t^2}{1-t^2}P\,, \tag{3.7}$$

where we define

$$\hat{\mathcal{V}}(t) \equiv \mathcal{V}X(t)\,. \tag{3.8}$$

The Kac-Moody transformations δ and $\tilde{\delta}$, which we defined in (2.17), (2.23) and (2.28), can now be applied to $\hat{\mathcal{V}}$. We find

$$\begin{aligned}\delta_1\hat{\mathcal{V}}_2 &= \frac{t_1}{t_1-t_2}\hat{\mathcal{V}}_2X_2^{-1}\eta_1X_2 - \frac{t_2}{t_1-t_2}\hat{\mathcal{V}}_2\epsilon_1 + \frac{t_1t_2}{1-t_1t_2}\hat{\mathcal{V}}_2(MX_2)^{-1}\eta_1^\dagger MX_2 + \delta h\hat{\mathcal{V}}_2\,, &(3.9)\\ \tilde{\delta}_1\hat{\mathcal{V}}_2 &= \frac{t_1t_2}{1-t_1t_2}\hat{\mathcal{V}}_2\epsilon_1\,, &(3.10)\end{aligned}$$

where as usual $\eta_1 = X_1\epsilon_1X_1^{-1}$, δh is an $\mathcal{H}$ compensating transformation and $\hat{\mathcal{V}}_2 = \mathcal{V}X_2$.

The quantity $A = M^{-1}dM$ can be thought of as a $\mathcal{G}$-valued conserved current, since as we noted in section (2.1), it transforms under global $\mathbf{G}$ transformations $\mathcal{V}\to h\mathcal{V}g$ as $A\to g^{-1}Ag$, and it satisfies $d{*}A = 0$. We see from (3.5) that $A = 2\mathcal{V}^{-1}P\mathcal{V}$. One can construct a hierarchy of conserved currents $\hat{\mathcal{J}}(t)$, for which $\hat{\mathcal{J}}(0) = A$, by defining

$$\hat{\mathcal{J}}(t) = \frac{2}{1-t^2}\,\hat{\mathcal{V}}^{-1}\left(\frac{1+t^2}{1-t^2}P + \frac{2t}{1-t^2}{*P}\right)\hat{\mathcal{V}}\,. \tag{3.11}$$

That $\hat{\mathcal{J}}$ is conserved can be seen from the following calculation, which also provides [18] a simpler expression for the currents:

$$\begin{aligned}\hat{\mathcal{J}} &= \frac{2}{1-t^2}\,\hat{\mathcal{V}}^{-1}*\left(\frac{2t}{1-t^2}P + \frac{1+t^2}{1-t^2}{*P}\right)\hat{\mathcal{V}}\,,\\ &= \hat{\mathcal{V}}^{-1}*\frac{\partial}{\partial t}\left(\frac{1+t^2}{1-t^2}P + \frac{2t}{1-t^2}{*P}\right)\hat{\mathcal{V}}\,,\\ &= \hat{\mathcal{V}}^{-1}*\frac{\partial}{\partial t}\left(Q + \frac{1+t^2}{1-t^2}P + \frac{2t}{1-t^2}{*P}\right)\hat{\mathcal{V}}\,,\\ &= \hat{\mathcal{V}}^{-1}*\frac{\partial}{\partial t}\left(d\hat{\mathcal{V}}\hat{\mathcal{V}}^{-1}\right)\hat{\mathcal{V}}\,,\\ &= *d\left(\hat{\mathcal{V}}^{-1}\frac{\partial\hat{\mathcal{V}}}{\partial t}\right)\,. \end{aligned} \tag{3.12}$$

Note that using (3.8), we can also write $\hat{\mathcal{J}}$ as

$$\hat{\mathcal{J}} = *d(X^{-1}\dot{X})\,. \tag{3.13}$$

It is also useful to define the quantity

$$v(t) = X^{-1}(t)\dot{X}(t) = \sum_{n\geq 0} t^n v_{(n)}\,, \tag{3.14}$$

such that $\mathcal{J} = *dv$.

The quantity $v(t)$ has a simple transformation under the $\tilde{\delta}$ Kac-Moody symmetries, with

$$\tilde{\delta}_1 v(t_2) = \frac{t_1}{(1-t_1t_2)^2}\,\epsilon_1 + \frac{t_1t_2}{1-t_1t_2}\,[v(t_2),\epsilon_1]\,. \tag{3.15}$$

In terms of the mode expansion in (3.14), this implies

$$\tilde{\delta}_{(m)}(\epsilon)v_{(n)} = m\delta_{m,n+1}\epsilon + [v_{(n-m)},\epsilon]\,. \tag{3.16}$$

The generalised currents $\hat{\mathcal{J}} = *dv$ also transform nicely under the Kac-Moody transformations $\tilde{\delta}$. From (3.15) we find

$$\tilde{\delta}_1 \hat{\mathcal{J}}_2 = \frac{t_1t_2}{1-t_1t_2}\,[\hat{\mathcal{J}}_2,\epsilon_1]\,, \tag{3.17}$$

where $\hat{\mathcal{J}}_2 \equiv *d(X_2^{-1}\dot{X}_2)$. If we expand $\hat{\mathcal{J}}$ as a power series

$$\hat{\mathcal{J}}(t) = \sum_{n\ge 0} t^n \hat{\mathcal{J}}_{(n)}\,, \tag{3.18}$$

then (3.17) implies that

$$\tilde{\delta}_{(m)}(\epsilon)\hat{\mathcal{J}}_{(n)} = [\hat{\mathcal{J}}_{(n-m)},\epsilon]\,, \qquad n\ge m\,. \tag{3.19}$$

One might be tempted therefore to regard $\hat{\mathcal{J}}$ as defining a hierarchy of Kac-Moody currents. However, although they transform covariantly under the "lower half" of the Kac-Moody symmetries corresponding to $\tilde{\delta}$, their transformations in general under the "upper half" of the Kac-Moody symmetries, corresponding to δ, are very complicated, and one cannot express $\delta_{(m)}(\epsilon)\hat{\mathcal{J}}_{(n)}$ as any linear combination of $\hat{\mathcal{J}}_{(p)}$ currents with field-independent coefficients.

4 An Explicit Example: $SL(2,\mathbb{R})/O(2)$ Coset Model

4.1 Infinitely many fields

The simplest non-trivial example that illustrates the constructions we have described in this paper is provided by the symmetric-space sigma model $SL(2,\mathbb{R})/O(2)$. We begin by defining the $SL(2,\mathbb{R})$ generators

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \qquad E^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,. \tag{4.1}$$

The $O(2)$ denominator group is generated by the anti-Hermitean combination $E^+ - E^-$, whilst the generators in the coset are the Hermitean matrices H and $E^+ + E^-$. A convenient way to parametrise the coset representative $\mathcal{V}$ is in the Borel gauge, for which

$$\mathcal{V} = e^{\frac{1}{2}\phi_0 H} e^{\chi_0 E^+} \,. \tag{4.2}$$

The fields ϕ_0 and χ_0 are the standard dilaton and axion of the $SL(2,\mathbb{R})/O(2)$ sigma model, with the Lagrangian

$$L = -\tfrac{1}{4}\mathrm{tr}(A^\mu A_\mu) = -\tfrac{1}{2}(\partial\phi_0)^2 - \tfrac{1}{2}e^{2\phi_0}(\partial\chi_0)^2 \,. \tag{4.3}$$

From (3.1) we find

$$Q = \tfrac{1}{2}(E^+ - E^-)\widetilde{Q} \,, \qquad P = \tfrac{1}{2}HP_\phi + \tfrac{1}{2}(E^+ + E^-)P_\chi \,, \tag{4.4}$$

with

$$\widetilde{Q} = e^{\phi_0} d\chi_0 \,, \qquad P_\phi = d\phi_0 \,, \qquad P_\chi = e^{\phi_0} d\chi_0 \,. \tag{4.5}$$

The standard $SL(2,\mathbb{R})$ symmetry of the sigma model is given by

$$\delta(\epsilon)\mathcal{V} = \delta h \mathcal{V} + \mathcal{V}\epsilon \,, \tag{4.6}$$

with $\epsilon = \epsilon^0 H + \epsilon^- E^+ + \epsilon^+ E^-$, where δh is the appropriate $O(2)$ compensating transformation to restore the Borel gauge choice. Thus we have

$$\delta\phi_0 = 2\epsilon^0 + 2\epsilon^+ \chi_0 \,, \qquad \delta\chi_0 = \epsilon^- - 2\epsilon^0 \chi_0 + \epsilon^+ (e^{-2\phi_0} - \chi_0^2) \,. \tag{4.7}$$

The next step is to define $\hat{\mathcal{V}}$, whose relation to X is given in (3.8). Following the general idea described in [18], we do this by introducing scalar fields $\hat{\phi}$, $\hat{\chi}$ and $\hat{\psi}$, which depend on the spectral parameter t as well as the spacetime coordinates, and writing

$$\hat{\mathcal{V}}(t) = e^{\frac{1}{2}\phi(t) H} e^{\chi(t) E^+} e^{\psi(t) E^-} \,. \tag{4.8}$$

We require that $\hat{\mathcal{V}}$ smoothly approach $\mathcal{V}$, defined in (4.2), as t goes to zero, and so

$$\phi(0) = \phi_0 \,, \qquad \chi(0) = \chi_0 \,, \qquad \psi(0) = 0 \,. \tag{4.9}$$

In terms of power-series expansions for ϕ, χ and ψ, we may therefore write

$$\begin{aligned} \phi(t) &= \phi_0 + t\phi_1 + t^2\phi_2 + \cdots \,, \\ \chi(t) &= \chi_0 + t\chi_1 + t^2\chi_2 + \cdots \,, \\ \psi(t) &= t\psi_1 + t^2\psi_2 + \cdots \,. \end{aligned} \tag{4.10}$$

Since Q, P and $*P$ in (3.7) are independent of the spectral parameter t, it follows that by substituting (4.8) into (3.7) we can read off a hierarchy of equations for the fields ϕ_i, χ_i and ψ_i. At order t^0, we simply obtain the expressions for $\widetilde{Q}$, P_ϕ and P_χ already given in (4.5). At order t^1, we find

$$\begin{aligned} *P_\phi &= \tfrac{1}{2}d\phi_1 + \chi_0 d\psi_1\,, && (4.11)\\ *P_\chi &= \tfrac{1}{2}e^{\phi_0}(d\chi_1 + \phi_1 d\chi_0 - \chi_0^2 d\psi_1) + \tfrac{1}{2}e^{-\phi_0}d\psi_1\,, && (4.12)\\ 0 &= e^{\phi_0}(d\chi_1 + \phi_1 d\chi_0 - \chi_0^2 d\psi_1) - e^{-\phi_0}d\psi_1\,, && (4.13) \end{aligned}$$

where the last equation comes from the absence of t-dependence in the denominator group term $\widetilde{Q}$. It can be used to simplify the $*P_\chi$ expression, to give

$$*P_\chi = e^{-\phi_0}d\psi_1\,. \tag{4.14}$$

By equating the t^1 expressions (4.11) and (4.14) for $*P_\phi$ and $*P_\chi$ to the duals of the t^0 expressions for P_ϕ and P_χ in (4.5), we obtain, together with (4.13), the t^1 equations of motion

$$\begin{aligned} *d\phi_0 &= \tfrac{1}{2}d\phi_1 + \chi_0\, d\psi_1\,, && (4.15)\\ e^{2\phi_0} *d\chi_0 &= d\psi_1\,, && (4.16)\\ 0 &= d\chi_1 + \phi_1 d\chi_0 - (\chi_0^2 + e^{-2\phi_0})d\psi_1\,. && (4.17) \end{aligned}$$

At order t^2 we find

$$\begin{aligned} P_\phi &= \tfrac{1}{2}d\phi_2 + \chi_1 d\psi_1 + \chi_0 d\psi_2\,, && (4.18)\\ P_\chi &= e^{-\phi_0}\,(d\psi_2 - \phi_1 d\psi_1)\,, && (4.19)\\ 0 &= d\chi_2 + \phi_1 d\chi_1 + (\phi_2 + \tfrac{1}{2}\phi_1^2)d\chi_0 - (\chi_0^2 + e^{-2\phi_0})d\psi_2 \\ &\quad +[\phi_1 e^{-2\phi_0} - \chi_0(2\chi_1 + \phi_1\chi_0)]d\psi_1\,, && (4.20) \end{aligned}$$

and we therefore obtain in total 3 equations at this order, after equating these expressions for P_ϕ and P_χ to those in (4.5). One can continue this process to any desired order in t.

The $SL(2,\mathbb{R})$ symmetry $\delta(\epsilon)$ in (4.7) extends to the higher-level fields via the construction (2.23), with $t_1 = 0$. Thus we have $\delta(\epsilon)X = [X, \epsilon]$, and so using (3.8) to write $X = \mathcal{V}^{-1}\hat{\mathcal{V}}$, together with (4.8), we find we can write the $SL(2,R)$ transformations as

$$\begin{aligned} \delta\phi &= -2\epsilon^-\psi + 2\epsilon^0\phi + 2\epsilon^+\chi e^{\phi-\phi_0}\,,\\ \delta\chi &= \epsilon^-(1 + 2\chi\psi) - 2\epsilon^0\chi + \epsilon^+ e^{-\phi-\phi_0}(1 - \chi^2 e^{2\phi})\,,\\ \delta\psi &= -\epsilon^-\psi^2 + 2\epsilon^0\psi + \epsilon^+(1 - e^{\phi-\phi_0})\,. \end{aligned} \tag{4.21}$$

Note that these transformations are linear when acting on v defined in (3.14): $\delta v = [v, \epsilon]$.

Expanding out (4.21) in powers of t, using (4.10), we recover (4.7) at order t^0, and at the next couple of orders we find

$$\begin{aligned}
\delta\psi_1 &= -\epsilon^+\phi_1 + 2\epsilon^0\psi_1\,,\\
\delta\phi_1 &= 2\epsilon^+(\chi_1 + \chi_0\,\phi_1) - 2\epsilon^-\psi_1\,,\\
\delta\chi_1 &= -\epsilon^+(2\chi_0\chi_1 + \chi_0^2\phi_1 + e^{-2\phi_0}\phi_1) - 2\epsilon^0\chi_1 + 2\epsilon^-\chi_0\psi_1\,,\\
\delta\psi_2 &= -\epsilon^+(\phi_2 + \tfrac12\phi_1^2) + 2\epsilon^0\psi_2 - \epsilon^-\psi_1^2\,,\\
\delta\phi_2 &= \epsilon^+\Big(2\chi_2 + 2\chi_1\phi_1 + \chi_0(\phi_1^2 + 2\phi_2)\Big) - 2\epsilon^-\psi_2\,,\\
\delta\chi_2 &= \epsilon^+\Big(-2\chi_0(\chi_2 + \chi_1\phi_1) - \tfrac12\chi_0^2(\phi_1^2 + 2\phi_2) - \chi_1^2 + e^{-2\phi_0}(\tfrac12\phi_1^2 - \phi_2)\Big)\\
&\qquad -2\epsilon^0\chi_2 + 2\epsilon^-(\chi_1\psi_1 + \chi_0\psi_2)\,.
\end{aligned}\tag{4.22}$$

The hierarchy of equations of motion for the higher-level fields, for which we presented the first two orders in (4.15)–(4.17), and (4.18)–(4.20), are invariant under the $SL(2,\mathbb{R})$ transformations (4.21).

4.2 The Geroch group

An interpretation of the higher-level fields can be given as follows. The equations of motion for the original level-0 fields, following from the Lagrangian (4.3), are

$$d{*}d\phi_0 + e^{2\phi_0}\,{*}d\chi_0\wedge d\chi = 0\,,\qquad d(e^{2\phi_0}\,{*}d\chi_0) = 0\,,\tag{4.23}$$

Since we are in two dimensions, the axion χ_0 can be dualised to another axion $\bar\chi_0$, such that

$$d\bar\chi_0 = e^{2\phi_0}\,{*}d\chi_0\,.\tag{4.24}$$

Substituting this into the ϕ_0 equation of motion, we can remove a derivative from this equation too, obtaining

$${*}d\phi_0 = d\sigma + \chi_0 d\bar\chi_0\,,\tag{4.25}$$

for some new field σ. Defining $\bar\phi_0 = -\phi_0$, the original Lagrangian (4.3) can be written in a dualised form, terms of the barred fields, as

$$\mathcal{L} = -\tfrac12(\partial\bar\phi_0)^2 - \tfrac12 e^{2\bar\phi_0}(\partial\bar\chi_0)^2\,.\tag{4.26}$$

We see, comparing (4.24) and (4.25) with (4.15) and (4.16), that

$$\bar\chi_0 = \psi_1\,,\qquad \sigma = \tfrac12\phi_1\,.\tag{4.27}$$

The dualised Lagrangian (4.26) clearly also has an $SL(2,\mathbb{R})$ symmetry, which we shall denote by $\overline{SL(2,\mathbb{R})}$. Denoting its infinitesimal parameters by $\bar\epsilon^{\,\pm}$ and $\bar\epsilon^{\,0}$, this symmetry acts on $\bar\phi_0$ and $\bar\chi_0$ exactly analogously to the action of the original $SL(2,\mathbb{R})$ on ϕ_0 and χ_0:

$$\bar\delta\bar\phi_0 = 2\bar\epsilon^{\,0} + 2\bar\epsilon^{\,-}\,\bar\chi_0\,, \qquad \bar\delta\bar\chi_0 = \bar\epsilon^{\,+} - 2\bar\epsilon^{\,0}\,\bar\chi_0 + \bar\epsilon^{\,-}\,(e^{-2\bar\phi_0} - \bar\chi_0^2)\,. \tag{4.28}$$

(For notational reasons that will become clear shortly, we switch the + and − indices on $\bar\epsilon^{\,\pm}$, relative to $\epsilon^\pm$, when passing to this barred version of (4.7).)

One may also define an infinite tower of higher-level barred fields for the dualised sigma model, precisely analogous to the unbarred ones defined above. For example, in order to obtain the barred version of (4.15)–(4.17), we should make the identifications

$$\bar\phi_0 = -\phi_0\,, \qquad \bar\chi_0 = \psi_1\,, \qquad \bar\phi_1 = -\phi_1 - 2\chi_0\psi_1\,. \tag{4.29}$$

The barring operation is an involution, with the bar of a bar being the identity operator, and so there is an analogous version of (4.29) in which all barred and unbarred fields are exchanged. The relations (4.29) can be extended to all levels, as we shall now discuss.

What we are seeing here is that although the original (ϕ_0,χ_0) fields are non-locally related to the dual fields $(\bar\phi_0,\bar\chi_0)$ (because of the differential relation (4.24) expressing $\bar\chi_0$ in terms of χ_0), there exists a purely local relation between the full hierarchy of fields (ϕ_i,χ_i,ψ_i) and their barred analogues. This relation can be established to any desired higher order in level number, by systematically examining the systems of equations that follow from (3.7), which we presented at level-1 in (4.15)–(4.17) and level-2 in (4.18)–(4.20). There is, however, a simpler way of presenting the entire hierarchy of relations in a compact form.

To do this, we first introduce a barred version of $\hat{\mathcal{V}}$, which was defined in equation (4.30):

$$\hat{\bar{\mathcal{V}}}(t) = e^{\frac12\bar\phi(t)\bar H}\, e^{\bar\chi(t)\bar E^+}\, e^{\bar\psi(t)\bar E^-}\,. \tag{4.30}$$

Here $\bar H$ and $\bar E^\pm$ are $SL(2,\mathbb{R})$ generators that satisfy identical commutation relations to H and $E^\pm$, namely

$$[H,E^\pm] = \pm 2E^\pm\,, \quad [E^+,E^-] = H\,; \qquad [\bar H,\bar E^\pm] = \pm 2\bar E^\pm\,, \quad [\bar E^+,\bar E^-] = \bar H\,. \tag{4.31}$$

This is already enough to ensure that the barred hierarchy of fields will satisfy identical equations of motion to the unbarred hierarchy; they are derived from the barred version of (3.7). Next, we note that we may make the following choice for the barred generators in terms of the unbarred ones:

$$\bar E^+ = t\,E^-\,, \qquad \bar E^- = \frac{1}{t}\,E^+\,, \qquad \bar H = -H\,, \tag{4.32}$$

since this is consistent with (4.31). Thus we have

$$\hat{\bar{\mathcal{V}}}(t) = e^{-\frac{1}{2}\bar{\phi}(t)H} e^{t\,\bar{\chi}(t)E^-} e^{t^{-1}\,\bar{\psi}(t)E^+}\,. \tag{4.33}$$

We now impose the relation

$$\hat{\bar{\mathcal{V}}}(t) = \hat{\mathcal{V}}(t) \tag{4.34}$$

which therefore establishes a relation between these barred and unbarred fields, which have already been established to satisfy the same system of equations. This is easy to solve explicitly, since one has only to exponentiate 2×2 matrices in this example. We find (suppressing the explicit indication of the t-dependence of all the fields)

$$\psi = \frac{t\bar{\chi}}{1+\bar{\chi}\bar{\psi}}\,,\quad \chi = \frac{1}{t}\,\bar{\psi}\left(1+\bar{\chi}\bar{\psi}\right)\,,\quad \phi = -\bar{\phi} - 2\log\left(1+\bar{\chi}\bar{\psi}\right)\,. \tag{4.35}$$

Expanding in powers of t allows us to read off the relation between the entire hierarchies of barred and unbarred fields. At the leading order, we find precisely the relations (4.29) that we obtained previously when we started the level-by-level process of mapping the unbarred equations of motion into barred ones. If one carries out such a sequential calculation, one finds that the entire hierarchy of relations between barred and unbarred fields uniquely follows, once the leading-order relations (4.29) are fed in. Thus, we may conclude that since the all-level relations (4.35) match (4.29) at the leading order, they represent the unique completion of this relation to all orders.

The barred hierarchy of fields transforms under $\overline{SL(2,\mathbb{R})}$ in precisely the same way as the unbarred hierarchy transforms under $SL(2,\mathbb{R})$. For example, for the first couple of levels, the barred fields will transform under the dual $\overline{SL(2,\mathbb{R})}$ symmetry according to the barred version of (4.22) (with the exchange of $\bar{\epsilon}^{\,+}$ and $\bar{\epsilon}^{\,-}$, as we discussed previously for $\bar{\phi}_0$ and $\bar{\chi}_0$ in (4.28)). The $\overline{SL(2,\mathbb{R})}$ transformations of the entire hierarchy of dual fields can be succinctly expressed as the barred analogue of (4.21), which is therefore given by

$$\begin{aligned}
\bar{\delta}\bar{\phi} &= -2\bar{\epsilon}^{\,+}\bar{\psi} + 2\bar{\epsilon}^{\,0}\bar{\phi} + 2\bar{\epsilon}^{\,-}\bar{\chi}e^{\bar{\phi}-\bar{\phi}_0}\,,\\
\bar{\delta}\bar{\chi} &= \bar{\epsilon}^{\,+}(1+2\bar{\chi}\bar{\psi}) - 2\bar{\epsilon}^{\,0}\bar{\chi} + \bar{\epsilon}^{\,-}e^{-\bar{\phi}-\bar{\phi}_0}(1-\bar{\chi}^2e^{2\bar{\phi}})\,,\\
\bar{\delta}\bar{\psi} &= -\bar{\epsilon}^{\,+}\bar{\psi}^2 + 2\bar{\epsilon}^{\,0}\bar{\psi} + \bar{\epsilon}^{\,-}(1-e^{\bar{\phi}-\bar{\phi}_0})\,.
\end{aligned} \tag{4.36}$$

Since we also have the relation (4.35) between the barred and the unbarred fields, it is now a straightforward matter to work out the transformations of the original unbarred fields under the dual $\overline{SL(2,R)}$ symmetry. From (4.35) and (4.36) we find

$$\begin{aligned}\bar\delta\phi &= -2\bar\epsilon^{\,0} - 2\bar\epsilon^{\,-}\left[t\chi e^{\phi+\phi_0} + \frac{1}{t}\psi\right]\,,\\ \bar\delta\chi &= 2\bar\epsilon^{\,0}\chi + \bar\epsilon^{\,-}\left[t\chi^2 e^{\phi+\phi_0} + \frac{1}{t}(1+2\chi\psi - e^{-\phi+\phi_0})\right]\,,\\ \bar\delta\psi &= t\bar\epsilon^{\,+} - 2\bar\epsilon^{\,0}\psi + \bar\epsilon^{\,-}\left[t e^{\phi+\phi_0} - \frac{1}{t}\psi^2\right]\,.\end{aligned} \tag{4.37}$$

Expanded, as usual, in powers of t, these equations give the transformations of the entire hierarchy of original fields (ϕ_i, χ_i, ψ_i) under the dual $\overline{SL(2,R)}$ symmetry. Note that there are no negative powers of t in the expansions.

It is evident from (4.37) that the $\bar\epsilon^{\,0}$ transformation in $\overline{SL(2,\mathbb{R})}$ is the same (modulo a sign) as the ϵ^0 transformation with respect to the original $SL(2,\mathbb{R})$ (see equation (4.21)). The $\bar\epsilon^{\,+}$ transformation in (4.37) is also very simple, with

$$\bar\delta(\bar\epsilon^{\,+})\phi = 0\,,\qquad \bar\delta(\bar\epsilon^{\,+})\chi = 0\,,\qquad \bar\delta(\bar\epsilon^{\,+})\psi = t\bar\epsilon^{\,+}\,. \tag{4.38}$$

In terms of the expansions (4.10), this means that all fields (ϕ_i, χ_i, ψ_i) in the hierarchy are inert except for ψ_1, which suffers the shift transformation

$$\bar\delta(\bar\epsilon^{\,+})\psi_1 = \bar\epsilon^{\,+}\,. \tag{4.39}$$

It is easy to see that this is precisely the same as the transformation given by $\tilde\delta_1 X_2$ in equation (2.28), at order t_1^1 and with ϵ_1 taken to be just ϵ^+, *i.e.*

$$\tilde\delta_{(1)}(\bar\epsilon^{\,+})X_2 = t_2\, X_2\bar\epsilon^{\,+}\,. \tag{4.40}$$

This shows that the $\bar\epsilon^{\,+}$ transformation in $\overline{SL(2,\mathbb{R})}$ is implemented by the Kac-Moody generator J^+_{-1} (see (2.42)).

This leaves the $\bar\epsilon^{\,-}$ transformation in $\overline{SL(2,\mathbb{R})}$ still to be identified. In fact, this is precisely a $\delta_1 X_2$ transformation as given in (2.23), at order t_1^1 and with ϵ_1 taken to be just $\bar\epsilon^{\,-}$. Using (2.23), this is given by

$$\delta_{(1)}(\epsilon_1)X_2 = \frac{1}{t_2}[X_2, \epsilon_1] - \dot\eta_1(\epsilon_1, 0)X_2 + t_2 M^{-1}\epsilon_1^\dagger M X_2\,, \tag{4.41}$$

with $\epsilon_1 = \bar\epsilon^{\,-}$, where $\eta_1(\epsilon_1, t) \equiv X(t)\epsilon_1 X^{-1}(t)$. Substituting $X = \mathcal{V}^{-1}\hat{\mathcal{V}}$ into this, and using (4.8), one straightforwardly reproduces the $\bar\epsilon^{\,-}$ transformation in (4.37). This shows that the $\bar\epsilon^{\,-}$ transformation in $\overline{SL(2,\mathbb{R})}$ is implemented by the Kac-Moody generator J^-_1 (see (2.42)).

At this stage, we have arrived at a complete understanding of all six transformations in the original and dual symmetry groups $SL(2,\mathbb{R})$ and $\overline{SL(2,\mathbb{R})}$. The original $SL(2,\mathbb{R})$ transformations $\epsilon^\pm$ and ϵ^0 of course correspond to the level-0 Kac-Moody generators $J_0^\pm$ and J_0^0. We have also shown that the dual $\overline{SL(2,\mathbb{R})}$ transformations $\bar\epsilon^{\,+}$, $\bar\epsilon^{\,-}$ and $\bar\epsilon^{\,0}$ correspond to the Kac-Moody generators J_{-1}^+, J_1^- and J_0^0:

$$
\begin{aligned}
SL(2,\mathbb{R}):&\qquad (J_0^+, J_0^-, J_0^0)\,,\\[1em]
\overline{SL(2,\mathbb{R})}:&\qquad (J_{-1}^+, J_1^-, J_0^0)\,.
\end{aligned}
\tag{4.42}
$$

It is indeed clear from the Kac-Moody algebra (2.41) that both these triplets selected from the generators J_n^i form $SL(2,\mathbb{R})$ subalgebras. It is also clear that the two triplets do not commute. In fact, from the two triplets one can fill out the entire Kac-Moody algebra, by taking appropriate sequences of multiple commutators.

Thus we have shown in a very explicit and precise way that the affine $\widehat{SL(2,\mathbb{R})}$ Kac-Moody symmetry of the two-dimensional $SL(2,\mathbb{R})/O(2)$ symmetric-space sigma model is generated by taking multiple commutators of the two $SL(2,\mathbb{R})$ symmetries of the original and the dualised formulations of the theory.

It is interesting to note that the entire "negative half" of the Kac-Moody symmetry (*i.e.* J_n^i with $n<0$), which can be generated by multiple commutation of J_{-1}^+ with J_n^i with $n\ge 0$, emerges from the humble shift symmetry $\bar\delta\psi_1 = \bar\epsilon^{\,+}$ that we obtained in (4.39). This emphasises the point, which we remarked on earlier, that the negative half of the Kac-Moody algebra arises through symmetries that are realised only on the infinite tower of fields (ϕ_i,χ_i,ψ_i) with $i>0$ that were introduced in order to allow the symmetries of the sigma model to be expressed in a local, as opposed to non-local, manner. (See appendix A for further discussion of this point.)

4.3 Explicit formulae for $\tilde\delta$ and some example δ transformations

It is not hard to work out the explicit form of all the $\tilde\delta$ transformations on the fields (ϕ_i,χ_i,ψ_i). From (2.28), (3.8) and (4.8) we find

$$
\begin{aligned}
\tilde\delta_1\psi(t_2) &= \frac{t_1t_2}{1-t_1t_2}\,\left(\epsilon^+ + 2\epsilon^0\psi(t_2) - \epsilon^-\,\psi(t_2)^2\right)\,,\\
\tilde\delta_1\chi(t_2) &= \frac{t_1t_2}{1-t_1t_2}\,\left(-2\epsilon^0\chi(t_2) + \epsilon^-\,(1+2\chi(t_2)\psi(t_2))\right)\,,\\
\tilde\delta_1\phi(t_2) &= \frac{2t_1t_2}{1-t_1t_2}\,(\epsilon^0 - \epsilon^-\,\psi(t_2))\,.
\end{aligned}
\tag{4.43}
$$

Collecting the powers of t_1 and t_2, we find for $n \geq m \geq 1$ that

$$\begin{aligned}
\tilde{\delta}_{(m)}(\epsilon)\phi_n &= 2\delta_{mn}\,\epsilon^0 - 2\epsilon^-\,\psi_{n-m}\,, \\
\tilde{\delta}_{(m)}(\epsilon)\chi_n &= -2\delta_{mn}\,\epsilon^0\,\chi_n + \delta_{mn}\,\epsilon^- + 2\epsilon^- \sum_{p=0}^{n-m-1} \chi_p\,\psi_{n-m-p}\,, \\
\tilde{\delta}_m(\epsilon)\psi_n &= \delta_{mn}\,\epsilon^+ + 2\epsilon^0\psi_{n-m} - \epsilon^- \sum_{p=1}^{n-m-1} \psi_p\,\psi_{n-m-p}\,,
\end{aligned} \tag{4.44}$$

where it is understood that on the right-hand side $\chi_n = 0$ for $n < 0$ and $\psi_n = 0$ for $n < 1$. Note that $\tilde{\delta}_{(m)}\phi_n = 0$, $\tilde{\delta}_{(m)}\chi_n = 0$ and $\tilde{\delta}_{(m)}\psi_n = 0$ whenever $m < n$. Of course we also have $\tilde{\delta}_{(m)}\phi_0 = 0$, $\tilde{\delta}_{(m)}\chi_0 = 0$.

The symmetries $\tilde{\delta}$ in (4.44) are essentially just shift transformations of ϕ_n, χ_n and ψ_n by constant parameters ϵ^0, ϵ^- and ϵ^+ (with independent sets of these $SL(2,\mathbb{R})$ parameters at each of the negative Kac-Moody levels), with the extra terms being the necessary "dressings" that ensure that the transformations leave the equations of motion invariant. In accordance with an observation we made previously, the $\tilde{\delta}$ transformations could therefore be used in order to "gauge fix" the auxiliary fields (*i.e.* (ϕ_i, χ_i, ψ_i) for $i \geq 1$ in this $SL(2,\mathbb{R})/O(2)$ example) to any desired set of values at one chosen point in spacetime. Since the auxiliary fields also transform under the δ symmetries, one could view the $\tilde{\delta}$ transformations, in such a gauge-fixed situation, as compensating transformations that restored the fields to these chosen values after having performed δ transformations. This is effectively what happens in the construction of Schwarz's subalgebra of the full Kac-Moody algebra.

As we observed in section 3, the $\tilde{\delta}$ transformations become more elegant if they are applied to the quantities $v_{(n)}$ defined in (3.14), for which we have (3.16). In fact $v(t)$ is easily calculated in terms of $\phi(t)$, $\chi(t)$ and $\psi(t)$, giving

$$v^- = \dot{\chi} + \chi\dot{\phi}\,, \quad v^0 = \tfrac{1}{2}\dot{\phi} + \psi\dot{\chi} + \chi\psi\dot{\phi}\,, \quad v^+ = \dot{\psi} - (1 + \chi\psi)\psi\dot{\phi} - \psi^2\dot{\chi}\,. \tag{4.45}$$

Thus, as can be seen by expanding in powers of t, the $v^{\pm}_{(n)}$ and $v^0_{(n)}$ are are just certain combinations of the ϕ_m, χ_m and ψ_m fields,

$$v^-_{(0)} = \chi_1 + \chi_0\phi_1\,, \qquad v^0_{(0)} = \tfrac{1}{2}\phi_1\,, \qquad v^+_{(0)} = \psi_1\,, \qquad \text{etc.} \tag{4.46}$$

The δ symmetries in (2.17) and (2.23) are more non-trivial, but again they are completely local transformations of the fields (ϕ_i, χ_i, ψ_i), which can be read off explicitly to any desired order of non-negative Kac-Moody level, and to any desired order in the t-expansion of the fields. For example, we find for the $SL(2,\mathbb{R})/O(2)$ example that at Kac-Moody level 1, the transformations on $(\phi_0, \chi_0, \psi_1, \chi_1, \psi_1)$ are given by

$$
\begin{aligned}
\delta_{(1)}(\epsilon)\phi_0 &= 2\epsilon^+\,\chi_1 + 4\epsilon^0\,\chi_0\psi_1 - 2\epsilon^-\,\psi_1\,,\\
\delta_{(1)}(\epsilon)\chi_0 &= -\epsilon^+\,(\phi_1 e^{-2\phi_0} + 2\chi_0\chi_1 + \chi_0^2\phi_1) + \epsilon^-\,(\phi_1 + 2\chi_0\psi_1)\\
&\quad +2\epsilon^0\,(\psi_1 e^{-2\phi_0} - \chi_1 - \chi_0\phi_1 - \chi_0^2\psi_1)\,,\\
\delta_{(1)}(\epsilon)\phi_1 &= \epsilon^+\,\big(2\chi_2 + \chi_0(2 + 2\phi_2 - \phi_1^2) + 2\chi_0^3 e^{2\phi_0}\big) + 2\epsilon^-\,(\psi_2 + \chi_0 e^{2\phi_0})\\
&\quad +2\epsilon^0\,(1 + 2\chi_0^2 e^{2\phi_0} + 2\chi_1\psi_1 + 2\chi_0\phi_1\psi_1)\,,\\
\delta_{(1)}(\epsilon)\chi_1 &= \epsilon^+\,\big((1+\phi_1^2)e^{-2\phi_0} - \chi_1^2 - 2\chi_0\chi_2 + \chi_0^2(\phi_1^2 - 2\phi_2) - \chi_0^4 e^{2\phi_0}\big)\\
&\quad +\epsilon^-\,(\phi_2 + 2\chi_0\psi_2 + 2\chi_1\psi_1 - \tfrac{1}{2}\phi_1^2 + \chi_0^2 e^{2\phi_0})\\
&\quad +\epsilon^0\,\big(-2\chi_2 + \chi_0(\phi_1^2 - 2\phi_2 - 4\chi_1\psi_1) - 2\chi_0^2\phi_1\psi_1 - 2\phi_1\psi_1 e^{-2\phi_0} - 2\chi_0^3 e^{2\phi_0}\big)\,,\\
\delta_{(1)}(\epsilon)\psi_1 &= -\epsilon^+\,(\phi_2 - \tfrac{1}{2}\phi_1^2 + \chi_0^2 e^{2\phi_0}) + \epsilon^-\,(e^{2\phi_0} - \psi_1^2)\\
&\quad +2\epsilon^0\,(\psi_2 - \phi_1\psi_1 - \chi_0 e^{2\phi_0})\,.
\end{aligned}
\tag{4.47}
$$

5 Conclusions

In this paper, we have studied the global symmetries of flat two-dimensional symmetric-space sigma models. This can be viewed as a preliminary to studying the somewhat more intricate problem of curved-space two-dimensional sigma models, which arise in the toroidal compactification of supergravity theories. Both the curved and the flat cases share the common feature that the global symmetries include an infinite-dimensional extension of the manifest $\mathcal{G}$ symmetry of the $\mathbf{G/H}$ sigma model.

There has been some controversy over the precise nature of the infinite-dimensional extension. Whilst most authors have asserted that the symmetry is the affine Kac-Moody extension $\hat{\mathcal{G}}$ of $\mathcal{G}$, Schwarz [16] found instead a certain subalgebra $\hat{\mathcal{G}}_H$ of the Kac-Moody algebra. One of our goals in this paper has been to resolve the discrepancies.

In our work we made extensive use of Schwarz's results which have, it seems for the first time, provided explicit expressions for the key transformations that underlie the positive half of the Kac-Moody symmetry algebra. By synthesising this with earlier work where the idea of introducing an infinity of auxiliary fields in order to provide a local formulation was developed, we have been able to construct a fully local description of the entire Kac-Moody algebra of global symmetry transformations.

We have also shown how the subalgebra found by Schwarz can be viewed as a consequence of making a gauge choice, in which the values of the complete set of fields are fixed to prescribed values at a chosen distinguished point in the two-dimensional spacetime.

In order to make some of the ideas more concrete, we also studied a simple explicit example, where the coset of the sigma model is taken to be $SL(2,\mathbb{R})/O(2)$. We showed how our present analysis could be related to much earlier work by Geroch [4], in which the infinite-dimensional symmetry was obtained by commuting $SL(2,\mathbb{R})$ symmetry transformations of the original sigma model and its dual version. In particular, we were able to exhibit the precise correspondence between the two sets of $SL(2,\mathbb{R})$ transformations and certain generators of the Kac-Moody algebra. This provides an explicit demonstration that the Geroch algebra formed by taking commutators of the two $SL(2,\mathbb{R})$ transformations is the same as the Kac-Moody algebra $\widehat{SL(2,\mathbb{R})}$.

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A Schwarz Algebra $\hat{\mathcal{G}}_H$ Versus Kac-Moody Algebra $\hat{\mathcal{G}}$

In [16], the Lax equation (2.14) is solved for X as a non-local function of the original sigma-model fields, by writing

$$X(x;t) = \mathcal{P}\exp\Big[\int_{x_0}^{x}\Big(\frac{t}{1-t^2}\,{*A} + \frac{t^2}{1-t^2}\,A\Big)\Big]\,, \tag{A.1}$$

where $\mathcal{P}$ denotes path ordering along the integration path, and x_0 is an arbitrarily-chosen point. This is a significantly different approach from the one we have followed, where X is expressed locally in terms of an infinity of auxiliary fields.

Our transformation (2.23) for $\delta_1 X_2$ is not quite the same as the one given in Schwarz's discussion [16]. Let us denote his expression by $\delta_1' X_2$; it is given by

$$\delta_1' X_2 = \frac{t_2}{t_1 - t_2}\,(\eta_1 X_2 - X_2\epsilon_1) + \frac{t_1 t_2}{1 - t_1 t_2}\,(M^{-1}\eta_1^\dagger M X_2 - X_2 M_0^{-1}\epsilon_1^\dagger M_0)\,, \tag{A.2}$$

where $M_0 = M(x_0)$, and x_0 is chosen as the lower limit of the integral expression (A.1) for $X(t)$. Thus the relation between δ'_1 and our expression δ_1 is

$$\delta'_1 = \delta_1 - \frac{t_1 t_2}{1 - t_1 t_2}\, X_2 M_0^{-1} \epsilon_1^\dagger M_0\,. \tag{A.3}$$

In [16], Schwarz calculates the commutator $[\delta'_1, \delta'_2]M$, finding

$$[\delta'_1, \delta'_2]M = \frac{t_1\, \delta'(\epsilon_{12}, t_1) - t_2\, \delta'(\epsilon_{12}, t_2)}{t_1 - t_2}\, M - \frac{t_1\, t_2}{1 - t_1\, t_2}\, \big(\delta'(\epsilon'_{12}, t_1) - \delta'(\epsilon'_{12}, t_2)\big)\, M\,, \tag{A.4}$$

where

$$\epsilon_{12} = [\epsilon_1, \epsilon_2]\,, \qquad \epsilon'_{12} = [M_0^{-1} \epsilon_1^\dagger M_0, \epsilon_2]\,. \tag{A.5}$$

(In obtaining this result, one must hold M_0 fixed.) The right-hand side of (A.4) involves δ' transformations again, and so the algebra appears to be closing. However, Schwarz does not calculate $[\delta'_1, \delta'_2]X_3$. Let us denote his result in (A.4) as $[\delta'_1, \delta'_2]M = \delta^S M$. After some algebra, we find that

$$[\delta'_1, \delta'_2]X_3 = \delta^S X_3 + \frac{t_1 t_3}{1 - t_1 t_3}\, X_3 (M_0^{-1} \epsilon_{12}^\dagger M_0 - \epsilon'_{12}) + \frac{t_2 t_3}{1 - t_2 t_3}\, X_3 (M_0^{-1} \epsilon_{12}^\dagger M_0 + \epsilon'_{21})\,. \tag{A.6}$$

This shows that on X_3, the commutator of δ' transformations does not close merely on δ', but instead it gives transformations of the form $X_3 \tilde\epsilon$ as well, for certain $\tilde\epsilon$. In fact, such transformations are of the type $\tilde\delta$ that we introduced in (2.28), and (A.6) may be written abstractly as

$$[\delta'_1, \delta'_2] = \delta^S + \tilde\delta(M_0^{-1} \epsilon_{12}^\dagger M_0 - \epsilon'_{12}, t_1) + \tilde\delta(M_0^{-1} \epsilon_{12}^\dagger M_0 + \epsilon'_{21}, t_2)\,. \tag{A.7}$$

Of course, the extra $\tilde\delta$ terms on the right-hand side was not seen in Schwarz's calculations, because he calculated the commutator only on M, for which we know $\tilde\delta M = 0$, but not on X.

The conclusion from (A.7) is that if all the δ' transformations (A.3) are included in the symmetry algebra, then it is necessary to extend the algebra further by including the $\tilde\delta$ transformations too, in order to achieve closure. As may be seen from (A.3), Schwarz's δ' transformations are themselves a combination of our δ and $\tilde\delta$ transformations; in fact, one has

$$\delta'(\epsilon_1) = \delta(\epsilon_1) - \tilde\delta(M_0^{-1} \epsilon_1^\dagger M_0)\,. \tag{A.8}$$

The upshot is that once one has extended Schwarz's transformations to comprise not only δ' but also $\tilde\delta$, one has, equivalently, extended to the full set of δ and $\tilde\delta$ transformations that we considered in section 2.2. These, as we showed, generate the complete affine Kac-Moody extension $\hat{\mathcal{G}}$ of the original $\mathcal{G}$ algebra.

One can, alternatively, take a more restrictive viewpoint, which is effectively the one that was adopted by Schwarz in [16]. Namely, the commutation relations (A.7) imply that it is only if either δ'_1 or δ'_2 is a level-0 transformation that the $\tilde{\delta}$ transformations are generated. (This follows from the fact that the second term on the right-hand side of (A.7) is independent of t_2, and the third term is independent of t_1.) Thus, we have

$$[\delta'_{(m)}(\epsilon_1), \delta'_{(n)}(\epsilon_2)] = \delta^S_{(m+n)}(\epsilon_{12})\,, \qquad \text{for} \qquad m > 0\,, \quad n > 0\,, \tag{A.9}$$

$$[\delta'_{(0)}(\epsilon_1), \delta'_{(n)}(\epsilon_2)] = \delta^S_{(n)}(\epsilon_{12}) + \tilde{\delta}_{(n)}(M_0^{-1}\epsilon_{12}^\dagger M_0 + \epsilon'_{21})\,, \qquad n > 0\,. \tag{A.10}$$

(We have taken δ'_1 to be a level-0 transformation in the second equation, for definiteness.) One can therefore avoid generating any $\tilde{\delta}$ transformations if one restricts the level-0 transformations in δ' to be such that

$$M_0^{-1}\epsilon_1^\dagger M_0 + \epsilon_1 = 0\,. \tag{A.11}$$

This equation is essentially the condition that ϵ should belong to the denominator algebra $\mathcal{H}$ of the coset model. This is most immediately clear if one chooses, as one may, the "gauge" in which $M_0 = 1$. Equation (A.11) then implies that ϵ is anti-Hermitean, which is precisely the standard condition for it to lie in the denominator algebra $\mathcal{H}$. If some other gauge choice is made for M_0, then ϵ is again required to be in the denominator algebra, in a basis conjugated by M_0. The upshot of this discussion is that the necessity of including all the $\tilde{\delta}$ symmetries as well in order to achieve closure of the algebra (A.7) can be avoided if one truncates to that subset of the δ' transformations in which the $\mathcal{K}$ transformations at 0-level are omitted.

This, therefore, accounts for the symmetry algebra that was found by Schwarz in [16]. The full Kac-Moody symmetry algebra $\hat{\mathcal{G}}$ is generated by our δ and $\tilde{\delta}$ transformations, whilst Schwarz's subalgebra, which he denoted by $\hat{\mathcal{G}}_H$, corresponds to the transformations δ' given in (A.8), with the further restriction that at level-0 the $\mathcal{K}$ transformations are omitted. Omitting these particular transformations is precisely what is needed in order to maintain a fixed boundary condition for M_0 (such as $M_0 = 1$). In the gauge choice $M_0 = 1$, we see from (A.8) that $\delta'(\epsilon) = \delta(\epsilon) \pm \tilde{\delta}(\epsilon)$, with the plus sign occurring when ϵ lies in $\mathcal{H}$ and the minus sign when ϵ lies in $\mathcal{K}$. The generators ${J'}_n^i$ of the Schwarz subalgebra are therefore given in terms of the Kac-Moody generators J_n^i by

$$\begin{aligned} {J'}_n^i &= J_n^i + J_{-n}^i\,, \qquad \text{for} \qquad i \in \mathcal{H}\,, \\ {J'}_n^i &= J_n^i - J_{-n}^i\,, \qquad \text{for} \qquad i \in \mathcal{K}\,. \end{aligned} \tag{A.12}$$

One sees immediately that the level-0 generators ${J'}_0^i$ vanish if t lies in $\mathcal{K}$. It can easily be verified directly that the generators ${J'}_n^i$ form a closed subalgebra of the full Kac-Moody algebra (2.41).

The Schwarz subalgebra of the Kac-Moody algebra can be interpreted as follows. By writing $X(t)$ as in (A.1), a choice has been made to set $X(t) = 1$ at the point x_0 in the two-dimensional spacetime. This can be viewed as a gauge-fixing that is achieved by using the $\tilde{\delta}$ transformations. Furthermore, as we remarked below (A.5), M_0 must be held fixed, which is a further gauge fixing (of the original sigma-model fields), achieved by using the $\mathcal{K}$ part of the original $\mathcal{G}$ Lie algebra transformations. In other words, only the $\mathcal{H}$ part of the original $\mathcal{G}$ symmetry survives. If we wish instead to retain the full algebra $\mathcal{G}$ of original symmetries, then Schwarz's subalgebra will necessarily have to be extended to the full Kac-Moody algebra $\hat{\mathcal{G}}$.

It is instructive to look at this truncated subalgebra in the concrete example of the $SL(2,\mathbb{R})/O(2)$ sigma model that we studied in section 4. Especially, it is interesting to look at the transformations of the original $SL(2,\mathbb{R})$ symmetry and the dual $\overline{SL(2,\mathbb{R})}$ symmetry, to see which are retained and which are truncated out in the subalgebra.

The combinations of Kac-Moody generators J_n^i that lie in $\mathcal{K}$ and in $\mathcal{H}$ are given, respectively, by

$$\begin{aligned}
\mathcal{K}: &\qquad J_n^0\,, \qquad (J_n^+ + J_n^-)\,, \\
\mathcal{H}: &\qquad (J_n^+ - J_n^-)\,.
\end{aligned} \tag{A.13}$$

It then follows from (A.12) that the generators ${J'}_n^i$ that are retained in the truncated algebra of [16] are

$$\begin{aligned}
\mathcal{K}: &\qquad {J'}_n^{(1)} = J_n^0 - J_{-n}^0\,, \qquad {J'}_n^{(2)} = J_n^+ + J_n^- - J_{-n}^+ - J_{-n}^-\,, \\
\mathcal{H}: &\qquad {J'}_n^{(3)} = J_n^+ - J_n^- + J_{-n}^+ - J_{-n}^-\,.
\end{aligned} \tag{A.14}$$

Since the $SL(2,\mathbb{R})$ transformations correspond to the Kac-Moody generators J_0^i, and the $\overline{SL(2,\mathbb{R})}$ transformations correspond to the generators J_{-1}^+, J_0^0 and J_1^-, it suffices to consider just the levels $m = 0$ and $m = 1$ in (A.14). These give the four following non-vanishing generators:

$$\begin{aligned}
n = 0: &\qquad {J'}_0^{(3)} = 2(J_0^+ - J_0^-)\,, \\
n = 1: &\qquad {J'}_1^{(1)} = J_1^0 - J_{-1}^0\,, \\
&\qquad {J'}_1^{(2)} = J_1^+ + J_1^- - J_{-1}^+ - J_{-1}^-\,, \\
&\qquad {J'}_1^{(3)} = J_1^+ - J_1^- + J_{-1}^+ - J_{-1}^-\,.
\end{aligned} \tag{A.15}$$

We see that just two of the five inequivalent transformations in $SL(2,\mathbb{R})$ and $\overline{SL(2,\mathbb{R})}$ are retained within the truncated algebra:

$$J_0'^{(3)} \leftrightarrow (\epsilon^+ - \epsilon^-)\,, \qquad (J_1'^{(2)} - J_1'^{(3)}) \leftrightarrow (\bar\epsilon^+ - \bar\epsilon^-)\,. \tag{A.16}$$

Thus, the infinite-dimensional subalgebra of the full Kac-Moody algebra that is retained in the truncation (A.12) omits not only the $\mathcal{K}$ generators in the original $SL(2,\mathbb{R})$, but also the $\mathcal{K}$ generators in the dual symmetry algebra $\overline{SL(2,\mathbb{R})}$. If one wants to have a symmetry algebra that at least contains all the generators of the original and the dual $SL(2,\mathbb{R})$ algebras then, as we showed in section 4.2, this will necessarily be the full Kac-Moody algebra.

B The Virasoro-type Symmetry and the Schwarz Approach

In section 2.3 we obtained a Virasoro-like symmetry of the symmetric-space sigma models, with generators K_n satisfying the algebra (2.57). Our construction was closely related to one given in [16] but there were significant differences, which we shall elaborate on here.

The first respect in which our discussion diverges from that in [16] is that in that paper, the quantity $\xi(t)$ appearing in the our transformation $\delta^V(t)\mathcal{V} = \mathcal{V}\xi(t)$ (see (2.45)) is replaced by

$$\tilde\xi(t) = (t^2-1)\dot X(t) X(t)^{-1} + \mathcal{I}\,, \tag{B.1}$$

where

$$\mathcal{I} = \int *A\,. \tag{B.2}$$

One can see from the path-ordered integral expression (A.1) for $X(t)$ that

$$X(t) = 1 + t\int *A + \mathcal{O}(t^2)\,, \tag{B.3}$$

and so in fact $\mathcal{I} = \dot X(0) = \dot X(0) X(0)^{-1}$. Thus from (B.1) we see that Schwarz's $\tilde\xi$ and our ξ are related by

$$\tilde\xi(t) = \frac{1-t^2}{t}\,\xi(t) - \Big[\frac{1-t^2}{t}\,\xi(t)\Big]_{t=0}\,. \tag{B.4}$$

Thus the lowest mode in our transformation is excluded in the PCM analysis in [16].

The lowest mode had to be excluded in [16] for the principal chiral model, as opposed to the symmetric-space sigma model, in order to ensure that the transformation was a symmetry of the equations of motion. In brief, the transformation of A under ξ (defined as in (2.45)) in the PCM case is simply $\delta^V A = D\xi$, rather than (2.47) of the SSM case, and so using (2.48) one finds

$$\delta^V A = *d(t^{-1}\xi) + \frac{1}{1-t^2}\,A + \frac{t}{1-t^2}\,{*A}\,. \tag{B.5}$$

This means that $d{*}\delta^V A = t/(1-t^2)dA$, and so the equation of motion $d{*}A = 0$ is not preserved. However, if the lowest-order term in $\delta^V(t)$ is subtracted out, as is done in (B.4), then the resulting transformation $\tilde{\delta}^V$ *does* give a symmetry.

Although Schwarz carried over the assumption that the lowest mode should also be subtracted out when he then considered the SSM case, it is actually no longer necessary to do so, as we explained in section 2.3. As we showed there, with the transformation $\delta^V A$ now given by (2.47), one finds using (2.48) and (2.49) that the contributions to $\delta^V A$ of the form $*A$ coming from the two terms in (2.47) cancel out, and so $d{*}\delta^V A = 0$ automatically, without the need to subtract the lowest mode term. The upshot is that the set of Virasoro-like symmetries that we find for the symmetric-space sigma models is actually larger then the set obtained by Schwarz in [16], by virtue of the inclusion of the lowest mode in $\delta^V(t)$.

A second difference between our results and those in [16] is concerned with the precise form of the Virasoro-like algebra in the two cases. We were able to make a convenient choice of $-t$ as the prefactor of $\dot{X}X^{-1}$ in (2.45) which gave the algebra in the form (2.57), which is very close in structure to the Virasoro algebra. On the other hand, in [16] the choice of t-dependent prefactor was apparently constrained by certain requirements of matching between left and right acting transformations on the group manifold of the PCM (a consideration that does not apply in the SSM case). This led to the choice of (t^2-1) prefactor that was made in [16], and this in turn led to the rather different algebra

$$[K_m, K_n] = (m-n)(K_{m+n+1} - K_{m+n-1}) \tag{B.6}$$

for the PCM case.

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