UPR/0847-T CTP TAMU-19/99 RU99-6-B May 1999
hep-th/9905096

Domain-wall Supergravities from Sphere Reduction

M. Cvetič [†1], James T. Liu [⋆2], H. Lü [†1] and C.N. Pope [‡3]

[†] *Dept. of Phys. and Astro., University of Pennsylvania, Philadelphia, PA 19104*

[‡] *Center for Theoretical Physics, Texas A&M University, College Station, TX 77843*

[⋆] *Department of Physics, The Rockefeller University, New York, NY 10021*

ABSTRACT

Kaluza-Klein sphere reductions of supergravities that admit AdS×Sphere vacuum solutions are believed to be consistent. The examples include the S^4 and S^7 reductions of eleven-dimensional supergravity, and the S^5 reduction of ten-dimensional type IIB supergravity. In this paper we provide evidence that sphere reductions of supergravities that admit instead Domain-wall×Sphere vacuum solutions are also consistent, where the background can be viewed as the near-horizon structure of a dilatonic p-brane of the theory. The resulting lower-dimensional theory is a gauged supergravity that admits a domain wall, rather than AdS, as a vacuum solution. We illustrate this consistency by taking the singular limits of certain modulus parameters, for which the original S^n compactifying spheres ($n = 4, 5$ or 7) become $S^p \times R^q$, with $p = n - q < n$. The consistency of the S^4, S^7 and S^5 reductions then implies the consistency of the S^p reductions of the lower-dimensional supergravities. In particular, we obtain explicit non-linear ansätze for the S^3 reduction of type IIA and heterotic supergravities, restricting to the $U(1)^2$ subgroup of the $SO(4)$ gauge group of S^3. We also study the black hole solutions in the lower-dimensional gauged supergravities with domain-wall backgrounds. We find new domain-wall black holes which are not the singular-modulus limits of the AdS black holes of the original theories, and we obtain their Killing spinors.

[1] Research supported in part by DOE grant DOE-FG02-95ER40893

[2] Research supported in part by DOE grant DOE-91ER40651-TASKB

[3] Research supported in part by DOE grant DOE-FG03-95ER40917

Contents

1 Introduction **2**

2 S^4 reduction of M-theory, and its limits **5**

2.1 Limit to flat space 7

2.2 Limit to $S^3 \times \mathbb{R}$ 10

2.3 New domain-wall black holes in $D = 7$ 13

3 S^7 reduction of M-theory, and its limits **15**

3.1 Limit to $S^3 \times \mathbb{R}^4$ 16

3.2 Domain-wall black holes in $D = 4$ 17

3.3 Limit to $S^5 \times \mathbb{R}^2$ 20

3.4 Limits of solutions 21

4 S^5 reduction of type IIB theory, and its limit **22**

4.1 Limit to $S^3 \times \mathbb{R}^2$ 22

4.2 New domain-wall black holes in $D = 5$ 23

5 Conclusions **24**

A A class of domain-wall black holes **26**

1 Introduction

Kaluza-Klein sphere reductions of supergravities that admit AdS$\times$Sphere vacuum solutions are believed to be consistent. The examples include the S^7 [1, 2] and S^4 [3] reductions of eleven-dimensional supergravity, and the S^5 [4, 5, 6] reduction of ten-dimensional type IIB supergravity. The consistency is important since one can then be assured that the solutions of the resulting lower dimensional gauged supergravities are also solutions of the M-theory or type IIB theory. These solutions of the gauged supergravities then provide one side of the picture of the duality [7, 8, 9, 10] between Anti-de Sitter space and conformal field theories on its boundary. The complete proof of consistency is still lacking, owing to the complexity of the reduction ansätze. The S^7 reduction of eleven-dimensional supergravity is better understood than the other two cases, as a consequence of its having received more attention owing to its connection with four dimensions. The full non-linear reduction ansatz, although highly implicit, was given in [11, 12]. Recently, explicit non-linear reduction ansätze that focus on the Cartan subgroups of the full gauge groups were obtained for these three examples [13]. The construction shows that the reduction must be performed at the level of the equations of motion, rather than in the Lagrangian (even when it exists). Furthermore, the consistency of the reduction depends on a delicate balance between the contributions of the metric and the relevant antisymmetric tensor field strength in the higher-dimensional supergravity theory. This conspiracy between the higher-dimensional fields is equivalent to the one that governs the supersymmetry of the theory.

Note that by far the most involved part of establishing the consistency of any sphere-reduction ansatz is concerned with the contributions of scalar fields that parameterise inhomogeneous deformations of the sphere metric. Specifically, it is for these inhomogeneous deformations that the conspiracies between terms in the higher-dimensional Lagrangian are needed in order to achieve consistency of the reduction. This is in contrast to the situation in the reduction on a group manifold G, as described in [14], where the scalars parameterise only homogeneous deformations, and the consistency of the reduction is guaranteed for each individual term in the higher-dimensional Lagrangian, by virtue of the invariance of the reduction ansatz under the right action of the group G.

The consistency of the above sphere reductions raises the question as to whether it is also possible to perform consistent Kaluza-Klein sphere reductions on other supergravity theories that do not admit AdS$\times$Sphere solutions. AdS$\times$Sphere spacetimes can arise as the near-horizon structures of the M2-brane, M5-brane and D3-brane. In these examples there are no couplings to scalar fields. However for a generic p-brane that couples to a dilaton, the near-horizon structure of the metric is instead a product of a domain wall[1] and a sphere, with a certain warp factor in the metric. (In this paper we shall refer to this spacetime as a Domain-wall×Sphere, even though the warp-factor implies that the spacetime metric is not a direct product.) Note that super Yang-Mills field theories do not only arise at the boundary of AdS space, but also emerge as the world-volume theories of D-branes. A generic D-brane has a near-horizon structure which is a Domain-wall×Sphere. This raises the question as to whether supergravity in the D-brane near-horizon background has anything to do with the super Yang-Mills theory on the D-brane world-volume. The correspondence of supergravity on a domain-wall background with quantum field theory on the wall was proposed in [15].

In this paper, we shall address the following question: Is it consistent to dimensionally reduce the theory with a domain wall×Sphere solution on the associated sphere? If so, what will be the lower-dimensional theory? In the AdS×Sphere compactification, the lower-dimensional theory is a gauged supergravity that allows an AdS spacetime as a vacuum solution. In other words, the scalar potential of the spherically-reduced theory has at least one stationary point. We refer to these theories as "AdS supergravities." In the new cases that we wish to consider here, we would expect that the lower-dimensional theory would instead admit a domain wall, rather than an AdS spacetime, as its vacuum solution. We refer to such theories as "Domain-wall supergravities." Before we address the question of the consistency of such a reduction, we shall first examine whether there exist lower dimensional theories that admit such domain-wall solutions.

Let us consider gauged $D = 7$, $N = 2$ supergravity as an example. The bosonic sector of the supergravity multiplet contains the metric, a dilaton a 3-form vector potential and $SU(2)$ Yang-Mills gauge fields. The Lagrangian has the following form [16, 17]

$$\begin{aligned} e^{-1}\mathcal{L}_7 &= R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\, e^{\frac{8}{\sqrt{10}}\phi} + 4g\,m\,e^{\frac{3}{\sqrt{10}}\phi} + 4g^2\,e^{-\frac{2}{\sqrt{10}}\phi}\ , \\ &\quad -\tfrac{1}{48}e^{-\frac{4}{\sqrt{10}}\phi}\,(F_{(4)})^2 - \tfrac{1}{4}e^{\frac{2}{\sqrt{10}}\phi}\,(F^a_{(2)})^2 \\ &\quad +\tfrac{1}{2}F_{(4)}\wedge F^a_{(2)}\wedge A^a_{(1)} + \tfrac{1}{2}m\,F_{(4)}\wedge A_{(3)}\ . \end{aligned} \tag{1.1}$$

Note that the 3-form gauge potential has a topological mass term, with mass parameter m. The scalar potential has a supersymmetric maximum, and a non-supersymmetric minimum. The scalar sector admits a supersymmetric domain-wall solution, given by [18]

$$ds_7^2 = e^{2A}\,dx^\mu dx^\nu \eta_{\mu\nu} + e^{8A}\,dy^2\ , \qquad e^{-\frac{3}{\sqrt{10}}\phi} = H\ ,$$

[1] We use the term "domain wall" to mean a $(D-2)$-brane in D dimensions.

$$e^{-4A} = \frac{8m}{5(H^{4/3})'} + \frac{2g}{5\,(H^{-1/3})'}\ , \qquad H = e^{-\frac{3}{\sqrt{10}}\phi_0} + q\,|y|\ , \tag{1.2}$$

where a prime denotes a derivative with respect to y. For generic non-vanishing m and g, we can obtain an AdS_7 spacetime by sending q to zero. In either of the special cases where m=0 or $g = 0$, the domain wall has no AdS_7 limit.

Let us first examine the case when $g = 0$. It was observed in [19] that the corresponding domain wall can be obtained from the vertical dimensional reduction of an M5-brane, whose charges are uniformly distributed over four of the transverse dimensions, which are taken as the compactification coordinates. The eleven-dimensional 5-brane has the form

$$\begin{aligned} ds_{11}^2 &= H^{-1/3} dx^\mu dx^\nu \eta_{\mu\nu} + H^{2/3}(dy^2 + ds_4^2)\ , \\ F_4 &= q\,\epsilon_4 \end{aligned} \tag{1.3}$$

where ds_4^2 is the metric of a four-dimensional flat space, with ϵ_4 being its volume form. Thus the $g = 0$ limit can be viewed as M-theory compactified on a 4-dimensional flat space. The topological mass term of the 3-form gauge potential has its origin in the FFA term of eleven-dimensional supergravity.

The $m = 0$ limit is quite different. We find that its higher-dimensional origin is from the near-horizon limit of the NS-5brane in $D = 10$, which has the metric

$$ds_{10}^2 = H^{-1/4} dx^\mu dx^\nu \eta_{\mu\nu} + H^{3/4}(dr^2 + r^2\, d\Omega_3^2)\ , \tag{1.4}$$

with $H = 1 + Q/r^2$. In the near-horizon limit, we shall have $H \to Q/r^2$.

The seven-dimensional theory (1.1) can be obtained from the S^4 reduction of M-theory. With the parameters g and m both non-zero, it allows AdS_7 as a vacuum solution. In one singular limit, where $g = 0$, it can be obtained from M-theory on a 4-dimensional flat background, with the 4-form field strength having a term proportional to the volume form of the internal space. Such a reduction is indeed consistent, giving rise to a 3-form gauge potential with a topological mass term [19]. The vacuum domain-wall solution in $D = 7$ has its origin as a 5-brane in $D = 11$, with its charges uniformly distributed over the internal space. In the other singular limit, with $m = 0$, the vacuum domain-wall solution has its origin instead as the S^3 reduction of the NS-5brane in $D = 10$.

The above example illustrates that there should exist two different singular limits of the S^4 reduction of M-theory, in one of which the internal 4-space becomes flat, while in the other the internal space limits to 4-space that contains a 3-sphere factor. The consistency of the S^4 reduction of M-theory thus implies the consistency of the S^3 reduction of the ten-dimensional theory.

The main purpose of this paper is to examine the above conjecture in detail. In section 2, we show that there exist singular limits in certain modulus parameters such that S^4 becomes either a flat four-space $\mathbb{R}^4$, or else $S^3 \times \mathbb{R}$. In the flat-space limit, the M-theory becomes a massive supergravity in $D = 7$ with a topologically massive 3-form potential. In the $S^3 \times \mathbb{R}$ limit, the M-theory becomes a gauged supergravity, but with no massive 3-form potential. We also obtain new domain-wall black holes in these limiting cases. These are analogous to the AdS black holes of gauged anti-de Sitter supergravities, but with asymptotic structures that approach domain walls rather than AdS spacetimes.

In section 3, we study the S^7 reduction of M-theory, and show that there exist singular limits in certain modulus parameters such that S^7 becomes $S^3 \times \mathbb{R}^4$ or $S^5 \times \mathbb{R}^2$. We also obtain supersymmetric domain-wall black holes in the original theory before taking the limit. We study how the solutions behave under these limiting procedures. In section 4, we turn out attention to the S^5 reduction of the type IIB theory, and we find a singular limit where S^5 becomes $S^3 \times \mathbb{R}^2$. We also obtain new supersymmetric domain-wall solutions in the limiting theory. In the appendix, we obtain a general class of domain-wall black holes.

2 S^4 reduction of M-theory, and its limits

In this section, we show that there exist singular limits in certain modulus parameters such that S^4 becomes either a flat four-space $\mathbb{R}^4$, or else $S^3 \times \mathbb{R}$. In the flat space limit, the M-theory becomes a massive supergravity in $D = 7$ with a topological massive 3-form potential. In the $S^3 \times \mathbb{R}$ limit, the M-theory becomes a gauged supergravity, but with no massive 3-form potential. This observation give a geometrical interpretation of the massive, but ungauged supergravity in $D = 7$ and the gauged massless $D = 7$ gauged supergravity, in terms of taking limits of certain modulus parameters. We also obtain new domain-wall black holes in these limiting cases. For appropriate choice of parameters these solutions are supersymmetric.

We begin this section with a review of the Kaluza-Klein reduction from $D = 11$ to $D = 7$ on S^4. As we said in the introduction, it is believed, although it is strictly speaking still only a conjecture, that the maximal $N = 4$, $SO(5)$ gauged supergravity in $D = 7$ can be obtained by performing a Kaluza-Klein reduction on S^4 accompanied by a truncation to the massless supermultiplet of fields. We shall consider the further restriction discussed in [13], where the $SO(5)$ gauge fields are truncated down to the abelian $U(1) \times U(1)$ subgroup, and correspondingly just two of the scalar fields of the full $N = 4$ theory are retained. The full non-linear ansatz for this reduction, found in [13], is given by

$$ds_{11}^2 = \widetilde{\Delta}^{1/3}\, ds_7^2 + g^{-2}\, \widetilde{\Delta}^{-2/3} \Big(X_0^{-1}\, d\mu_0^2 + \sum_{i=1}^{2} X_i^{-1}\, (d\mu_i^2 + \mu_i^2\, (d\psi_i + g\, A_{(1)}^i)^2) \Big) \,, \tag{2.1}$$

$$\begin{aligned} {*F_{(4)}} &= 2g \sum_{\alpha=0}^{2} \Big(X_\alpha^2\, \mu_\alpha^2 - \widetilde{\Delta}\, X_\alpha \Big)\, \epsilon_{(7)} + g\, \widetilde{\Delta}\, X_0\, \epsilon_{(7)} + \frac{1}{2g} \sum_{\alpha=0}^{2} X_\alpha^{-1}\, \bar{*} dX_\alpha \wedge d(\mu_\alpha^2) \\ &\quad + \frac{1}{2g^2} \sum_{i=1}^{2} X_i^{-2}\, d(\mu_i^2) \wedge (d\psi_i + g\, A_{(1)}^i) \wedge \bar{*} F_{(2)}^i \,, \end{aligned} \tag{2.2}$$

where $\bar{*}$ denotes the Hodge dual with respect to the seven-dimensional metric ds_7^2, $\epsilon_{(7)}$ denotes its volume form, and $*$ denotes the Hodge dualisation in the eleven-dimensional metric. The quantity $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta} = \sum_{\alpha=0}^{2} X_\alpha\, \mu_\alpha^2 \,, \tag{2.3}$$

where μ_0, μ_1 and μ_2 satisfy

$$\mu_0^2 + \mu_1^2 + \mu_2^2 = 1 \,. \tag{2.4}$$

fields are parameterised by X_1 and X_2, with X_0 introduced for convenience as an auxiliary variable, defined by $X_0 \equiv (X_1 X_2)^{-2}$. The two scalar fields X_i can be parameterised in terms of two canonically-normalised dilatons $\vec{\phi} = (\phi_1, \phi_2)$ by writing

$$X_i = e^{-\frac{1}{2}\vec{a}_i \cdot \vec{\phi}} \,, \tag{2.5}$$

where the dilaton vectors satisfy the relations $\vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - \frac{8}{5}$. A convenient parameterisation is given by

$$\vec{a}_1 = (\sqrt{2}, \sqrt{\tfrac{2}{5}}) \,, \qquad \vec{a}_2 = (-\sqrt{2}, \sqrt{\tfrac{2}{5}}) \,. \tag{2.6}$$

The μ_α and ψ_i coordinates parameterise the compactifying 4-sphere. When the seven-dimensional gauge fields and scalars are set to zero, the compactifying metric becomes simply

$$d\Omega_4^2 = d\mu_0^2 + \sum_{i=1}^{2} \Big(d\mu_i^2 + \mu_i^2\, d\psi_i^2 \Big) \,, \tag{2.7}$$

which is nothing but the metric on a unit-radius round 4-sphere.

For future reference, note that more generally, the metric on any even-dimensional unit sphere S^{2n} can be written as [20]

$$ds^2 = d\mu_0^2 + \sum_{i=1}^{n} \Big(d\mu_i^2 + \mu_i^2\, d\psi_i^2 \Big) \,, \tag{2.8}$$

where $\sum_{\alpha=0}^n \mu_\alpha^2 = 1$. For odd-dimensional spheres, the unit S^{2n-1} metric can be written as [20]

$$ds^2 = \sum_{i=1}^n \Big(d\mu_i^2 + \mu_i^2\, d\psi_i^2\Big)\ , \tag{2.9}$$

where $\sum_{i=1}^n \mu_i^2 = 1$.

It was shown in [13] that after substituting (2.1) and (2.2) into the eleven-dimensional equations of motion, one obtains seven-dimensional equations that can be derived from the Lagrangian

$$e^{-1}\mathcal{L}_7 = R - \tfrac12 (\partial\vec\phi)^2 - V - \tfrac14 \sum_{i=1}^2 e^{\vec a_i\cdot\vec\phi}\, (F^i_{(2)})^2\ , \tag{2.10}$$

where the potential V is given by

$$V = g^2(\,-4X_1X_2 - 2X_1^{-1}\,X_2^{-2} - 2X_2^{-1}\,X_1^{-2} + \tfrac12 (X_1X_2)^{-4})\ . \tag{2.11}$$

The system can be further consistently truncated to $X_1 = X_2 \equiv X = e^{-\phi_2/\sqrt{10}}$. If we make a constant shift of the dilaton ϕ_2 such that $X \to X\, e^{\lambda/5}$, and define

$$m' = g\, e^{-\frac45\lambda}\ , \qquad g' = g\, e^{\frac15\lambda}\ , \tag{2.12}$$

the potential V becomes exactly the same as the one given in (1.1), after dropping the primes on the rescaled coupling constants.

2.1 Limit to flat space

In this subsection, we consider a limiting procedure in which the S^4 reduction reviewed above becomes a reduction on a flat internal space, giving rise to a seven-dimensional theory that is equivalent to one obtained by a certain toroidal compactification from $D = 11$. To do this, it is useful first to apply the appropriate limiting procedure in the seven-dimensional theory itself.

From (2.5) and (2.6), we see that the scalar potential (2.11) is given by

$$V = g^2\Big(-4e^{-\frac12(\vec a_1+\vec a_2)\cdot\vec\phi} - 2e^{\frac12(\vec a_1+2\vec a_2)\cdot\vec\phi} - 2e^{\frac12(\vec 2a_1+\vec a_2)\cdot\vec\phi} + \tfrac12 e^{2(\vec a_1+\vec a_2)\cdot\vec\phi}\Big)\ . \tag{2.13}$$

We now make the following redefinitions of fields and the gauge coupling constant g:

$$\vec\phi = \vec\phi' - \tfrac12(\vec a_1+\vec a_2)\,\lambda\ , \qquad A^i_{(1)} = e^{\frac15\lambda}\, A^i_{(1)}{}'\ , \qquad g = e^{\frac45\lambda}\, m'\ , \tag{2.14}$$

where λ is a constant. It follows from (2.6) and (2.5) that X_1 and X_2, and the auxiliary quantity X_0, will suffer the rescalings

$$X_i = e^{\frac15\lambda}\, X_i'\ , \qquad X_0 = e^{-\frac45\lambda}\, X_0'\ , \tag{2.15}$$

where i runs over the values 1 and 2. Substituting into the seven-dimensional Lagrangian (2.10), and then taking the limit $\lambda \longrightarrow -\infty$, we arrive at the Lagrangian

$$e^{-1}\mathcal{L}_7 = R - \tfrac{1}{2}(\partial\vec{\phi})^2 - \tfrac{1}{2}m^2\, e^{\frac{8}{\sqrt{10}}\phi_2} - \tfrac{1}{4}\sum_{i=1}^{2} e^{\vec{a}_i\cdot\vec{\phi}}\,(F^i_{(2)})^2\ , \tag{2.16}$$

where, having taken the limit, we have then suppressed the primes on the redefined fields. This corresponds to the limit where the parameter g in (1.1) is set to zero.

The theory described by the Lagrangian (2.16) is one that can be obtained by means of a toroidal reduction from $D=11$, in which the usual Kaluza-Klein ansatz is generalised somewhat by allowing the inclusion of a constant 4-volume term $q\,\epsilon_{(4)}$ in the ansatz for the eleven-dimensional $F_{(4)}$. In terms of an ansatz on the 3-form potential, this corresponds to allowing a linear dependence on one or more of the toroidal compactification coordinates [19, 21]. Here, our goal will be to show how this theory can be obtained by taking an appropriate limit in the S^4 reduction described above.

The required limiting process that we shall apply to the S^4 reduction ansatz given in (2.1) and (2.2) is governed by what we already determined in $D=7$, where the appropriate scalings of fields and the gauge-coupling g were established. It is evident that to obtain a regular limit, it is necessary also to apply an appropriate rescaling to the coordinates μ_α involved in the parameterisation of the 4-sphere. We find that the necessary rescalings are as follows:

$$\mu_0 = \mu_0'\ , \qquad \mu_i = e^{\frac{1}{2}\lambda}\,\mu_i'\ . \tag{2.17}$$

Note that as we take the limit $\lambda \longrightarrow -\infty$, the quadratic constraint (2.4), which becomes ${\mu_0'}^2 + e^\lambda\,\mu_i'\,\mu_i' = 1$, will imply that ${\mu_0'}^2 = 1$, while the two quantities μ_i' will become unconstrained. Also, the quantity $\widetilde{\Delta}$ defined in (2.3) will, in the limit, be given by $\widetilde{\Delta} \longrightarrow e^{-\frac{4}{5}\lambda}\,X_0'$.

Applying this limiting procedure to the S^4 reduction ansätze (2.1) and (2.2), we therefore find that in the limit where $\lambda \longrightarrow -\infty$, the dominant terms in the eleven-dimensional metric and 4-form become

$$ds_{11}^2 = e^{-\frac{4}{15}\lambda}\Big(X_0^{1/3}\,ds_7^2 + m^{-2}\,X_0^{-2/3}\sum_{i=1}^{2} X_i^{-1}\,(d\mu_i^2 + \mu_i^2\,d\psi_i^2)\Big)\ , \tag{2.18}$$

$$*F_{(4)} = e^{-\frac{4}{5}\lambda}\Big(m\,X_0^2\,\epsilon_{(7)} + \frac{1}{2m^2}\sum_{i=1}^{2} X_i^{-2}\,d(\mu_i^2)\wedge d\psi_i\wedge \bar{*}F^i_{(2)}\Big)\ . \tag{2.19}$$

Here, we have dropped the primes on all the fields and the coupling constant, after having taken the limit.

Note that the overall λ-dependent prefactors in the above expressions, although divergent as $\lambda \longrightarrow -\infty$, do not give rise to any singularity in the seven-dimensional equations of motion. The reason for this is that there is a scaling symmetry of the eleven-dimensional supergravity equations of motion, referred to as a "trombone" symmetry in [22], under which the metric and the 3-form potential scale by the constant factors

$$g_{MN} \longrightarrow k^2\, g_{MN}\ , \qquad A_{(3)} \longrightarrow k^3\, A_{(3)}\ . \tag{2.20}$$

(More generally, any theory with an Einstein-Hilbert term and quadratic field-strength kinetic terms will have such a scaling symmetry if the metric and the gauge potentials each scale with a power of k equal to the number of indices carried by the metric or potential.) This is a symmetry of the equations of motion. It is not a symmetry of the action, which scales by a uniform constant factor under this transformation. Bearing in mind that the ansatz for $F_{(4)}$ given in (2.19) is for the dual of $F_{(4)}$, which is a 7-form in $D=11$ and thus would have a 6-form potential, the powers of e^λ in the prefactors in (2.18) and (2.19) are seen to be precisely of the correct form to factor out in the equations of motion by virtue of the scaling symmetry.

It is now quite straightforward to see that in this limiting situation, the metric and 4-form ansätze are in fact equivalent to standard ones on a 4-torus. To see this, let us define new coordinates (z_1, z_2, z_3, z_4) in place of $(\mu_1, \mu_2, \psi_1, \psi_2)$, where

$$\begin{aligned} z_1 &= \mu_1\, \cos\psi_1\ , \qquad & z_2 &= \mu_1\, \sin\psi_1\ ,\\ z_3 &= \mu_2\, \cos\psi_2\ , \qquad & z_4 &= \mu_2\, \sin\psi_2\ , \end{aligned} \tag{2.21}$$

in terms of which the metric ansatz (2.18) becomes

$$ds_{11}^2 = e^{-\frac{4}{15}\lambda}\, \Big(X_0^{1/3}\, ds_7^2 + m^2\, X_0^{-2/3}\, (X_1^{-1}\, (dz_1^2 + dz_2^2) + X_2^{-1}\, (dz_3^2 + dz_4^2))\Big)\ . \tag{2.22}$$

Similarly, we may express the 4-form ansatz in terms of the z coordinates. It is convenient at the same time to perform a dualisation, so that we express the ansatz on the 4-form itself, rather than its dual. Upon doing so, we find that (2.19) becomes

$$F_{(4)} = e^{-\frac{2}{5}\lambda}\, \Big(m^5\, d^4 z + \frac{1}{m^2}\, dz_3\wedge dz_4 \wedge F_{(2)}^1 - \frac{1}{m^2}\, dz_1 \wedge dz_2 \wedge F_{(2)}^2\Big)\ . \tag{2.23}$$

The reduction ansätze (2.22) and (2.23) that we have arrived at in this limiting case are equivalent to those for a Kaluza-Klein reduction from $D=11$ on a 4-torus, with certain of the fields set to zero, and with the inclusion of a Scherk-Schwarz type generalisation in the reduction ansatz for the 4-form field strength. In such a reduction, before setting any of the fields to zero, we would have

$$\begin{aligned} d\hat s_{11}^2 &= e^{\frac{1}{3}\vec g\cdot\vec\phi}\, ds_7^2 + \sum_{i=1}^4 e^{2\vec\gamma_i\cdot\vec\phi}\, (h^i)^2\ ,\\ \hat F_{(4)} &= F_{(4)} + F_{(3)i}\wedge h^i + \tfrac{1}{2} F_{(2)ij}\wedge h^i\wedge h^j + \tfrac{1}{6} F_{(1)ijk}\wedge h^i\wedge h^j\wedge h^k + q\, d^4 z\ , \end{aligned} \tag{2.24}$$

where $h^i = dz^i + \mathcal{A}^i_{(1)} + \mathcal{A}^i_{(0)j}\, dz^j$, and the various dilaton-coupling vectors $\vec g$ and $\vec\gamma_i$ are defined in [23, 24]. If the Kaluza-Klein vectors $\mathcal{A}^i_{(1)}$, and their subsequently-descendant axions $\mathcal{A}^i_{(0)j}$ are truncated out, along with certain of the other fields, it is evident that this toroidal reduction is equivalent to the reduction defined by the ansätze (2.22) and (2.23). Specifically, the two 2-form field strengths that survive in (2.23) correspond to the two field strengths $F_{(2)12}$ and $F_{(2)34}$ of the toroidal reduction. The two dilatons $\vec\phi = (\phi_1, \phi_2)$, parameterised as X_1 and X_2 in the ansätze (2.22) and (2.23), correspond to two specific surviving combinations of the four dilatons $\vec\phi$ of the 4-torus reduction.

2.2 Limit to $S^3 \times \mathbb{R}$

The other limiting case that we wish to consider corresponds to setting the parameter m to zero in (1.1). In terms of the description of the seven-dimensional theory at the beginning of this section, this limit is achieved by making rescalings analogous to (2.14) and (2.15), but now given by

$$\vec\phi = \vec\phi' - \tfrac12(\vec a_1 + \vec a_2)\,\lambda\,, \qquad A^i_{(1)} = e^{\frac15\lambda}\, {A^i_{(1)}}'\,, \qquad g = e^{-\frac15\lambda}\, g'\,, \tag{2.25}$$

where λ is a constant. Thus the crucial change is that g is now rescaled differently from before. It follows from (2.6) and (2.5) that X_1 and X_2, and the auxiliary quantity X_0, will rescale as

$$X_i = e^{\frac15\lambda}\, X_i'\,, \qquad X_0 = e^{-\frac45\lambda}\, X_0'\,, \tag{2.26}$$

Now, if we take the limit where $\lambda \longrightarrow +\infty$, we see that the Lagrangian (2.10) reduces to

$$e^{-1}\mathcal{L}_7 = R - \tfrac12(\partial\vec\phi)^2 + 4g^2\, e^{-\frac{2}{\sqrt{10}}\phi_2} - \tfrac14\sum_{i=1}^2 e^{\vec a_i\cdot\vec\phi}\,(F^i_{(2)})^2\,, \tag{2.27}$$

where, having taken the limit, we have again suppressed the primes on the redefined fields. This corresponds to the limit where the parameter m in (1.1) is set to zero.

We may again now consider the limiting procedure from the viewpoint of the ansätze for the eleven-dimensional fields. It is easily seen in this case that we should rescale the coordinates μ_α in the 4-sphere as follows:

$$\mu_0 = e^{-\frac12\lambda}\, \mu_0'\,, \qquad \mu_i = \mu_i'\,. \tag{2.28}$$

This has the effect that the constraint (2.4) becomes

$$e^{-\lambda}\, {\mu_0'}^2 + \mu_i'\, \mu_i' = 1\,, \tag{2.29}$$

where as usual, i runs over 1 and 2. In the limit when we send $\lambda \longrightarrow +\infty$, we will therefore have

$$\mu_i'\, \mu_i' = 1\ , \tag{2.30}$$

with μ_0' unconstrained. The quantity $\widetilde{\Delta}$ defined in (2.3) will, in the limit where $\lambda \longrightarrow +\infty$, tend to the form

$$\widetilde{\Delta} \longrightarrow e^{\frac{1}{5}\lambda}\, \widetilde{\Delta}'\ , \tag{2.31}$$

where

$$\widetilde{\Delta}' = \sum_{i=1}^{2} X_i'\, \mu_i'^{\,2}\ . \tag{2.32}$$

Applying this limiting procedure to the ansätze (2.1) and (2.2) for the eleven-dimensional fields, we find that they become

$$\begin{aligned} ds_{11}^2 &= e^{\frac{1}{15}\lambda}\Big\{\widetilde{\Delta}^{1/3}\, ds_7^2 \\ &\quad + g^{-2}\, \widetilde{\Delta}^{-2/3}\, \Big(X_0^{-1}\, d\mu_0^2 + \sum_{i=1}^{2} X_i^{-1}\, (d\mu_i^2 + \mu_i^2\, (d\psi_i + g\, A^i_{(1)})^2)\Big)\Big\}\ , \qquad (2.33) \\ {*F_{(4)}} &= e^{\frac{1}{5}\lambda}\Big\{2g \sum_{i=1}^{2} \Big(X_i^2\, \mu_i^2 - \widetilde{\Delta}\, X_i\Big)\, \epsilon_{(7)} + \frac{1}{2g} \sum_{i=1}^{2} X_i^{-1}\, \bar{*} dX_\alpha \wedge d(\mu_i^2) \\ &\quad + \frac{1}{2g^2} \sum_{i=1}^{2} X_i^{-2}\, d(\mu_i^2) \wedge (d\psi_i + g\, A^i_{(1)}) \wedge \bar{*} F^i_{(2)}\Big\}\ , \qquad (2.34) \end{aligned}$$

where again we have suppressed the primes on the various fields and coordinates, and on g, after having taken the limit.

The four coordinates μ_i and ψ_i, together with the quadratic constraint (2.30), can be recognised as parameterising a 3-sphere in the internal space (see equation (2.9)). The coordinate μ_0, which is now unconstrained, parameterises the fourth of the directions in the internal space. Initially, the μ_0 coordinate was one of three that were subject to the quadratic constraint (2.4) in the original 4-sphere. A parameterisation of the original μ_α in terms of two unconstrained angles could be taken to be $\mu_0 = \cos\theta_1$, $\mu_1 = \sin\theta_1\, \cos\theta_2$, $\mu_2 = \sin\theta_1\, \sin\theta_2$. Thus μ_0 originally ranged over a finite line segment. As the limit $\lambda \longrightarrow +\infty$ is taken, this line segment expands to cover the entire real line.

We can use the $S^3 \times \mathbb{R}$ limit of the S^4 reduction ansatz that we have just obtained in order to describe an S^3 reduction of ten-dimensional type IIA supergravity. To do this, we break the $S^3 \times \mathbb{R}$ reduction up into two steps, the first of which consists of reducing from $D = 11$ to $D = 10$ on the $\mathbb{R}$ direction. Of course since this corresponds to a Killing symmetry of the ansatz, generated by $\partial/\partial\mu_0$, we can choose to re-interpret μ_0 as an angular ignorable coordinate on S^1, rather than a coordinate on the real line. At the same time, we can exploit the trombone scaling symmetry of the $D = 11$ equations of motion to eliminate the e^{λ} factors in the limiting form of the ansätze.

To implement the resulting S^1 reduction, we use the standard Kaluza-Klein metric ansatz

$$d\hat{s}_{11}^2 = e^{-\frac{1}{6}\phi}\, ds_{10}^2 + e^{\frac{4}{3}\phi}\, (dz + \mathcal{A}_{(1)})^2 \ . \tag{2.35}$$

For the 4-form, we have $\hat{F}_4 = F_{(4)} + F_{(3)} \wedge (dz + \mathcal{A}_{(1)})$, with $F_4 = dA_{(3)} - dA_{(2)} \wedge \mathcal{A}_{(1)}$ and $F_{(3)} = dA_{(2)}$. From these ansätze, it is easy to see that we shall have

$$\hat{*}\hat{F}_{(4)} = e^{\frac{1}{2}\phi}\, {*F_{(4)}} \wedge (dz + \mathcal{A}_{(1)}) + e^{-\phi}\, {*F_{(3)}} \ , \tag{2.36}$$

where we are using $\hat{*}$ here to denote an eleven-dimensional Hodge dual, and $*$ to denote a ten-dimensional one.

From the above, it follows that the $S^3 \times \mathbb{R}$ metric and 4-form reduction ansätze (2.33) and (2.34) can be re-interpreted as type IIA S^3 reduction ansätze with

$$\begin{aligned}
ds_{10}^2 &= \widetilde{\Delta}^{1/4}\, (X_1 X_2)^{1/4}\, ds_7^2 + g^{-2} \widetilde{\Delta}^{-3/4}\, (X_1 X_2)^{1/4} \sum_{i=1}^{2} X_i^{-1}\, (d\mu_i^2 + \mu_i^2\, (d\psi_i + g\, A_{(1)}^i)^2) \ , \\
e^{-\phi}\, {*F_{(3)}} &= 2g \sum_{i=1}^{2} \Big(X_i^2\, \mu_i^2 - \widetilde{\Delta}\, X_i \Big)\, \epsilon_{(7)} + \frac{1}{2g} \sum_{i=1}^{2} X_i^{-1}\, \bar{*} dX_i \wedge d(\mu_i^2) \\
&\quad + \frac{1}{2g^2} \sum_{i=1}^{2} X_i^{-2}\, d(\mu_i^2) \wedge (d\psi_i + g\, A_{(1)}^i) \wedge \bar{*} F_{(2)}^i \ , \\
e^{2\phi} &= \frac{(X_1 X_2)^3}{\widetilde{\Delta}} \ , \qquad F_{(4)} = 0 \ , \qquad \mathcal{F}_{(2)} = 0 \ ,
\end{aligned} \tag{2.37}$$

where ϕ is the dilaton of the type IIA theory.

We arrived at the above ansätze for the S^3 reduction of type IIA supergravity by taking a singular limit of the S^4 reduction of $D = 11$ supergravity. However, having obtained it by this method, we can now view it as a valid reduction procedure in its own right. In particular, by virtue of the previously-established consistency of the S^4 reduction, we now know that this procedure has provided us with a consistent S^3 reduction scheme for the type IIA theory, leading to the seven-dimensional Lagrangian given in (2.27). In particular, this is an example of a non-trivial sphere reduction (*i.e.* one that includes scalars describing inhomogeneous sphere deformations) that gives rise to a gauged supergravity with no AdS vacuum.

Note that the Ramond-Ramond fields $F_{(4)}$ and $\mathcal{F}_{(2)}$ are set to zero in the ansätze (2.37) for the S^3 reduction of type IIA supergravity. This means that we can also interpret it as an S^3 reduction of the $N = 1$ or heterotic theories in $D = 10$. Although μ_0 naturally became a coordinate on the entire real line in the taking of the singular limit of the S^4 compactification, the consistency of the Kaluza-Klein reduction does not of itself require that μ_0 cover the infinite interval. We could instead impose a cut-off on the interval, and require instead that μ_0 range only over a finite line segment, S^1/Z_2. This could then be combined with a projection in which the Ramond-Ramond fields were set to zero, precisely in the fashion of the S^1/Z_2 reduction of M-theory introduced in [25]. Thus the reduction ansätze (2.33) and (2.34), where the Ramond-Ramond fields in $D = 10$ have already been projected out, can also be viewed as a consistent reduction ansatz for M-theory on $S^3 \times (S^1/Z_2)$. The limit that was taken in arriving at (2.33) and (2.34) from the S^4 reduction ansatz however implies that the line segment S^1/Z_2 expands to become $\mathbb{R}$; this corresponds to strongly coupled heterotic string theory (and the topological information encoded in S^1/Z_2 is lost).

2.3 New domain-wall black holes in $D = 7$

Having obtained the two singular limits of the 4-sphere reduction of eleven-dimensional supergravity, it is of interest to study how the solutions behave in these limits. One example was already illustrated in the introduction, where a domain-wall solution supported purely by the scalar potential was presented. As we discussed there, the limits associated with $m = 0$ and $g = 0$ give rise to domain walls that can be respectively interpreted as the near horizon of an isotropic NS-NS 5-brane or an M5-brane, with charges uniformly distributed over a flat four-space. In this subsection, we obtain a new domain-wall black-hole solution, which is not merely the singular-modulus limit of the previously known AdS black hole.

The Lagrangian (1.1) admits AdS$_7$ black hole solutions, given by

$$
\begin{aligned}
ds_7^2 &= -H^{-8/5}\, f\, dt^2 + H^{2/5}\,(f^{-1} dr^2 + r^2\, d\Omega_{5,k}^2)\ ,\\
f &= k - \frac{\mu}{r^4} + \tfrac{1}{4} m^2 r^2\, H^2\ , \qquad e^{\frac{5}{\sqrt{10}}\phi} = H\ ,\\
A_{(1)} &= \frac{\sqrt{2}(1 + k\sinh^2\alpha)^{1/2}}{\sinh\alpha}\, H^{-1}\, dt\ , \qquad H = \frac{g}{m}\left(1 + \frac{\mu\sinh^2\alpha}{r^4}\right)\ ,
\end{aligned}
\qquad (2.38)
$$

where $d\Omega_{5,k}^2$ is the metric on a unit S^5, T^5 or H^5 according to whether $k = 1, 0$ or -1. (Note that the solution was previously obtained in [13, 26] for $g = m$.) In particular the $k = 0$ solution can be oxidised back to $D = 11$ and becomes [27, 13] the near horizon structure of the rotating M5-brane [28]. We see that the solution (2.38) does not admit an $m = 0$ limit. It does, however, allow us to take the limit $g \to 0$, provided that $g \sinh^2\alpha \equiv Q$ is held fixed. The solution then reduces to a domain-wall black hole.

Although the solution (2.38) does not admit an $m=0$ limit, there does exist a new domain-wall black hole solution for the Lagrangian (1.1) when $m=0$. In fact when $m=0$, the Lagrangian fits precisely the general pattern in (A.5) in the appendix. It follows that the domain-wall black hole is given by

$$
\begin{aligned}
ds_7^2 &= -f\,dt^2 + f^{-1}\,dr^2 + r\,(dx_1^2 + \cdots + dx_5^2)\ ,\\
f &= 2r(\tfrac{8}{25}g^2 + \frac{\mu}{r^{5/2}} + \frac{\lambda^2}{4r^5})\ , \qquad e^{\frac{2}{\sqrt{10}}\phi} = r\ ,\\
A_{(1)} &= \lambda\, r^{-5/2}\, dt\ .
\end{aligned} \tag{2.39}
$$

To examine the supersymmetry of this domain-wall black hole, we note that the $N=2$ supersymmetry transformations on the fermions for the Lagrangian (1.1) are given by [26]

$$
\begin{aligned}
\delta\psi_\mu &= [\nabla_\mu + \tfrac{\mathrm{i}}{\sqrt{2}} g A_\mu^a\sigma^a + \tfrac{1}{20}(m e^{\frac{4}{\sqrt{10}}\phi} + 4g e^{-\frac{1}{\sqrt{10}}\phi})\gamma_\mu - \tfrac{\mathrm{i}}{20\sqrt{2}}(\gamma_\mu{}^{\nu\lambda} - 8\delta_\mu^\nu\gamma^\lambda)e^{\frac{1}{\sqrt{10}}\phi}F^a_{\nu\lambda}\sigma^a\\
&\qquad + \tfrac{1}{160}(\gamma_\mu{}^{\nu\rho\sigma\lambda} - \tfrac{8}{3}\delta_\mu^\nu\gamma^{\rho\sigma\lambda})e^{-\frac{2}{\sqrt{10}}\phi}F_{\nu\rho\sigma\lambda}]\epsilon\ ,\\
\delta\lambda &= [-\tfrac{1}{2\sqrt{2}}\gamma^\mu\partial_\mu\phi + \tfrac{1}{\sqrt{5}}(m\, e^{\frac{4}{\sqrt{10}}\phi} - g\, e^{-\frac{1}{\sqrt{10}}\phi}) + \tfrac{\mathrm{i}}{4\sqrt{10}}e^{\frac{1}{\sqrt{10}}\phi}F^a_{\mu\nu}\gamma^{\mu\nu}\sigma^a\\
&\qquad + \tfrac{1}{48\sqrt{5}}e^{-\frac{2}{\sqrt{10}}\phi}F_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}]\epsilon\ ,
\end{aligned} \tag{2.40}
$$

where the σ^a are the $SU(2)$ Pauli matrices. In the absence of $F_{(4)}$, and with only the $U(1)$ subgroup of $SU(2)$ excited (as is appropriate for the above black hole solutions), the supersymmetry transformations reduce to

$$
\begin{aligned}
\delta\psi_\mu &= [\nabla_\mu + \tfrac{\mathrm{i}}{\sqrt{2}} g A_\mu + \tfrac{1}{20}(m e^{\frac{4}{\sqrt{10}}\phi} + 4g e^{-\frac{1}{\sqrt{10}}\phi})\gamma_\mu - \tfrac{\mathrm{i}}{20\sqrt{2}}(\gamma_\mu{}^{\nu\lambda} - 8\delta_\mu^\nu\gamma^\lambda)e^{\frac{1}{\sqrt{10}}\phi}F_{\nu\lambda}]\epsilon\ ,\\
\delta\lambda &= [-\tfrac{1}{2\sqrt{2}}\gamma^\mu\partial_\mu\phi + \tfrac{1}{\sqrt{5}}(m\, e^{\frac{4}{\sqrt{10}}\phi} - g\, e^{-\frac{1}{\sqrt{10}}\phi}) + \tfrac{\mathrm{i}}{4\sqrt{10}}e^{\frac{1}{\sqrt{10}}\phi}F_{\mu\nu}\gamma^{\mu\nu}]\epsilon\ .
\end{aligned} \tag{2.41}
$$

(Here, we have adopted the natural complex notation for the supersymmetry parameter, in this $U(1)$ truncation.)

Examination of the $\delta\lambda$ variation indicates that the domain-wall black hole (2.39) is in general non-supersymmetric unless $\mu=0$, so that $f=\frac{16}{25}g^2 r + \frac{1}{2}\lambda^2 r^{-4}$. In this case the Killing spinors must satisfy the half-supersymmetry projection $P\epsilon=0$ where

$$
P = \tfrac{1}{2}[1 + f^{-1/2}(\tfrac{4}{5}g r^{1/2}\gamma_1 - \tfrac{\mathrm{i}}{\sqrt{2}}\lambda r^{-2}\gamma_0)]\ . \tag{2.42}
$$

Note that γ_0 and γ_1 denote the Dirac matrices with vielbein indices in the t and r directions respectively. For such spinors ϵ, the gravitino variation reduces to

$$
\begin{aligned}
\delta\psi_0 &= \partial_0\epsilon\ ,\\
\delta\psi_r &= [\partial_r + \frac{1}{r} + g f^{-1/2} r^{-1/2}\gamma_1]\epsilon\ ,\\
\delta\psi_{\theta_i} &= \partial_{\theta_i}\epsilon\ .
\end{aligned} \tag{2.43}
$$

The above equations are easily solved to obtain the $N=2$ Killing spinors

$$\epsilon = [\sqrt{f^{1/2} - \tfrac{1}{\sqrt{2}}\lambda r^{-2}} - \sqrt{f^{1/2} + \tfrac{1}{\sqrt{2}}\lambda r^{-2}}\,\gamma_1](1+\mathrm{i}\gamma_0)\epsilon_0\ , \tag{2.44}$$

where ϵ_0 is an arbitrary constant spinor. Note that the projection $P_0 \equiv \frac{1}{2}(1+\mathrm{i}\gamma_0)$ indicates that the domain-wall black hole preserves exactly half of the supersymmetries when $\mu = 0$.

To summarise, we have seen that there are two distinct domain-wall black hole solutions in $D=7$, corresponding to the limiting cases where $g=0$ or $m=0$. The former, describing a domain-wall black hole with $k=1$, corresponds to a solution of the theory obtained by compactifying $D=11$ supergravity on $\mathbb{R}^4$; this itself, as we saw, is a singular limit of the S^4 compactification. The other domain-wall black hole, with $m=0$, has $k=0$. This corresponds to a solution of the $S^3\times\mathbb{R}$ compactification from $D=11$, which is another singular limit of the S^4 compactification.

3 S^7 reduction of M-theory, and its limits

Let us now turn to the four-dimensional theory obtained by the dimensional reduction of M-theory on S^7. We shall consider two limits that correspond to consistent reductions on $S^3\times\mathbb{R}^4$ and $S^5\times\mathbb{R}^2$. We also find new domain-wall black hole solutions.

As shown in [13], there is an $N=2$, $U(1)^4$ truncation of the full $N=8$ gauged supergravity, for which the explicit Kaluza-Klein reduction ansatz can be given.[2] The reduction ansatz is

$$ds_{11}^2 = \widetilde{\Delta}^{2/3}\, ds_4^2 + g^{-2}\,\widetilde{\Delta}^{-1/3}\sum_{i=1}^4 X_i^{-1}\Big(d\mu_i^2 + \mu_i^2\,(d\psi_i + g\,A_{(1)}^i)^2\Big)\ , \tag{3.1}$$

$$\begin{aligned} F_{(4)} &= 2g\sum_{i=1}^4 \Big(X_i^2\,\mu_i^2 - \widetilde{\Delta}\,X_i\Big)\,\epsilon_{(4)} + \frac{1}{2g}\sum_{i=1}^4 X_i^{-1}\,{\bar*}dX_i\wedge d(\mu_i^2) \\ &\quad -\frac{1}{2g^2}\sum_{i=1}^4 X_i^{-2}\,d(\mu_i^2)\wedge(d\psi_i + g\,A_{(1)}^i)\wedge{\bar*}F_{(2)}^i\ , \end{aligned} \tag{3.2}$$

where $\widetilde{\Delta} = \sum_{i=1}^4 X_i\,\mu_i^2$. The four quantities μ_i satisfy $\sum_i \mu_i^2 = 1$. Here, $\bar*$ denotes the Hodge dual with respect to the four-dimensional metric ds_4^2, and $\epsilon_{(4)}$ denotes its volume form.

The four X_i, which satisfy $X_1X_2X_3X_4 = 1$, can be parameterised in terms of three

[2] A fully consistent truncation would require the inclusion of 3 axionic scalars as well as the 3 dilatonic scalars that are present in the $U(1)^4$ ansatz given in [13]. If one is considering solutions such as the 4-charge AdS black holes of $D=4$ gauge supergravity [29], for which the axions can be set to zero, this truncation of the axions is justified.

dilatonic scalars $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ as

$$X_i = e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\phi}} \ , \tag{3.3}$$

where the $\vec{a}_i$ satisfy the dot products

$$M_{ij} \equiv \vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - 1 \ . \tag{3.4}$$

A convenient choice is

$$\vec{a}_1 = (1,1,1) \ , \quad \vec{a}_2 = (1,-1,-1) \ , \quad \vec{a}_3 = (-1,1,-1) \ , \quad \vec{a}_4 = (-1,-1,1) \ . \tag{3.5}$$

In terms of this basis, the scalar potential in $D = 4$ is given by

$$V = -4g^2 \sum_{i<j} X_i\, X_j = -8g^2\,(\cosh\phi_1 + \cosh\phi_2 + \cosh\phi_3) \ , \tag{3.6}$$

and in fact the reduction ansatz given above leads [13] to the four-dimensional theory described by the Lagrangian

$$e^{-1}\mathcal{L}_4 = R - \tfrac{1}{2}(\partial\vec{\phi})^2 + 8g^2(\cosh\phi_1 + \cosh\phi_2 + \cosh\phi_3) - \tfrac{1}{4}\sum_{i=1}^{4} e^{\vec{a}_i\cdot\vec{\phi}}\,(F^i_{(2)})^2 \ . \tag{3.7}$$

3.1 Limit to $S^3 \times \mathbb{R}^4$

Let us first consider a limit where the dilaton ϕ_1 is subjected to the following shift:

$$\phi_1 = \phi_1' - \lambda \ , \tag{3.8}$$

with ϕ_2 and ϕ_3 unchanged. This will give a finite Lagrangian as $\lambda \longrightarrow +\infty$, provided that the coupling constant and the gauge potentials are scaled as follows:

$$g = e^{-\frac{1}{2}\lambda}\, g' \ , \qquad A^a_{(1)} = e^{\frac{1}{2}\lambda}\, A^a_{(1)}{}' \ , \qquad A^{\bar a}_{(1)} = e^{-\frac{1}{2}\lambda}\, A^{\bar a}_{(1)}{}' \ , \tag{3.9}$$

where we have introduced the notation that the index $i = (1,2,3,4)$ is split into $a = (1,2)$ and $\bar a = (3,4)$. Note that the quantities X_i scale as

$$X_a = e^{\frac{1}{2}\lambda}\, X_a' \ , \qquad X_{\bar a} = e^{-\frac{1}{2}\lambda}\, X_{\bar a}' \ . \tag{3.10}$$

After these redefinitions, the scalar potential in (3.7) becomes simply

$$V = -4{g'}^2\, e^{-\phi_1'} \ . \tag{3.11}$$

We find that these scalings can be implemented in the metric and 4-form ansätze (3.1) and (3.2) if we also make the scalings

$$\mu_a = \mu_a' \ , \qquad \mu_{\bar a} = e^{-\frac{1}{2}\lambda}\, \mu_{\bar a}' \ . \tag{3.12}$$

In particular, this means that the condition $\sum_i \mu_i^2 = 1$ becomes, in the limit $\lambda \longrightarrow +\infty$, simply $\sum_a {\mu'_a}^2 = 1$, with $\mu'_{\bar a}$ unconstrained. Defining $\widetilde\Delta' = \sum_a X'_a\, {\mu'_a}^2$, we see that as $\lambda \longrightarrow +\infty$, the ansätze become

$$\begin{aligned} ds_{11}^2 &= e^{\frac13\lambda}\Big\{\widetilde\Delta^{2/3}\, ds_4^2 + g^{-2}\,\widetilde\Delta^{-1/3}\Big(\sum_{a=1}^2 X_a^{-1}\,(d\mu_a^2 + \mu_a^2\,(d\psi_a + g\,A_{\scriptscriptstyle (1)}^a)^2)\\ &\qquad + \sum_{\bar a=3}^4 X_{\bar a}^{-1}\,(d\mu_{\bar a}^2 + \mu_{\bar a}^2\,d\psi_{\bar a}^2)\Big)\Big\}\,, \end{aligned} \tag{3.13}$$

$$\begin{aligned} F_{\scriptscriptstyle (4)} &= e^{\frac12\lambda}\Big\{2g\sum_{a=1}^2\Big(X_a^2\,\mu_a^2 - \widetilde\Delta\,X_a\Big)\,\epsilon_{\scriptscriptstyle (4)} + \frac{1}{2g}\sum_{a=1}^2 X_a^{-1}\,{\bar *}dX_a\wedge d(\mu_a^2)\\ &\qquad -\frac{1}{2g^2}\sum_{a=1}^2 X_a^{-2}\,d(\mu_a^2)\wedge(d\psi_a + g\,A_{\scriptscriptstyle (1)}^a)\wedge{\bar *}F_{\scriptscriptstyle (2)}^a\\ &\qquad -\frac{1}{2g^2}\sum_{\bar a=3}^4 X_{\bar a}^{-2}\,d(\mu_{\bar a}^2)\wedge d\psi_{\bar a}\wedge{\bar *}F_{\scriptscriptstyle (2)}^{\bar a}\Big\}\,, \end{aligned} \tag{3.14}$$

where, as usual, we have dropped the primes after taking the limit. This can be recognised as a reduction on $S^3\times{\rm I\!R}^4$.

Substituting these ansätze into the eleven-dimensional equations of motion, we will obtain equations of motion that can be derived from the following four-dimensional Lagrangian:

$$\begin{aligned} e^{-1}\mathcal{L}_4 &= R - \tfrac12(\partial\vec\phi)^2 + 4g^2\,e^{-\phi_1} - \tfrac14 e^{\phi_1}\,\Big(e^{\phi_2+\phi_3}\,(F_{\scriptscriptstyle (2)}^1)^2 + e^{-\phi_2-\phi_3}\,(F_{\scriptscriptstyle (2)}^2)^2\Big)\\ &\qquad -\tfrac14 e^{-\phi_1}\,\Big(e^{\phi_2-\phi_3}\,(F_{\scriptscriptstyle (2)}^3)^2 + e^{-\phi_2+\phi_3}\,(F_{\scriptscriptstyle (2)}^4)^2\Big)\,. \end{aligned} \tag{3.15}$$

For simplicity, we may now consistently set $\phi_2=\phi_3=0$, provided that we also set $F_{\scriptscriptstyle (2)}^1 = F_{\scriptscriptstyle (2)}^2 \equiv F_{\scriptscriptstyle (2)}/\sqrt2$ and $F_{\scriptscriptstyle (2)}^3 = F_{\scriptscriptstyle (2)}^4 = \mathcal{F}_{\scriptscriptstyle (2)}/\sqrt2$. Thus we arrive at the truncated Lagrangian

$$e^{-1}\mathcal{L}_4 = R - \tfrac12(\partial\phi_1)^2 - \tfrac14 e^{\phi_1}\,(F_{\scriptscriptstyle (2)})^2 - \tfrac14 e^{-\phi_1}\,(\mathcal{F}_{\scriptscriptstyle (2)})^2 + 4g^2\,e^{-\phi_1}\,. \tag{3.16}$$

The field strengths $F_{\scriptscriptstyle (2)}$ and $\mathcal{F}_{\scriptscriptstyle (2)}$ play quite different rôles in this truncated theory. As can be seen from the ansätze (3.13) and (3.14), the field $F_{\scriptscriptstyle (2)}$, which is associated with $F_{\scriptscriptstyle (2)}^a$, is in the Cartan subgroup of the original gauged supergravity Yang-Mills group. On the other hand $\mathcal{F}_{\scriptscriptstyle (2)}$, associated with $F_{\scriptscriptstyle (2)}^{\bar a}$, is in the ungauged sector corresponding to the ${\rm I\!R}^4$ part of the $S^3\times{\rm I\!R}^4$ limiting compactification.

3.2 Domain-wall black holes in $D=4$

In the previous section, we singled out the scalar ϕ_1, making a rescaling under which the 7-sphere becomes $S^3\times R^4$. We then consistently set ϕ_2 and ϕ_3 to zero, provided that

$F^1_{(2)} = F^2_{(2)} \equiv F_{(2)}/\sqrt{2}$ and $F^3_{(2)} = F^4_{(2)} = \mathcal{F}_{(2)}/\sqrt{2}$. For further simplicity, we shall set $\mathcal{F}_{(2)} = 0$. In this section, we shall accordingly look for solutions to the Lagrangian

$$e^{-1}\mathcal{L}_4 = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{\phi}\,(F_{(2)})^2 + 4m^2\,e^{\phi} + 4g^2\,e^{-\phi} + 16m\,g\;. \tag{3.17}$$

Note that we have introduced an additional constant parameter m, by making a constant shift of the dilaton ϕ, accompanied by a redefinition of g. For non-vanishing m and g, the scalar potential is equivalent to the one obtained by setting $\phi_2 = \phi_3 = 0$ in (3.7). The $N = 2$ supersymmetry transformation rules for the fermionic superpartners are given, in a purely bosonic background, by

$$\begin{aligned}
\delta\psi^i_\mu &= D_\mu\,\epsilon^i - \tfrac{1}{\sqrt{2}}\,g\,A_\mu\,\epsilon^{ij}\,\epsilon_j + \tfrac{1}{2}(m\,e^{\frac{1}{2}\phi} + g\,e^{-\frac{1}{2}\phi})\,\gamma_\mu\,\epsilon^i + \frac{1}{8\sqrt{2}}\,e^{\frac{1}{2}\phi}\,F_{\nu\rho}\,\gamma^{\nu\rho}\,\gamma_\mu\,\epsilon^{ij}\,\epsilon_j\;,\\
\delta\chi^i &= -\gamma^\mu\,\partial_\mu\phi\,\epsilon^{ij}\,\epsilon_j + 2(m\,e^{\frac{1}{2}\phi} - g\,e^{-\frac{1}{2}\phi})\,\epsilon^{ij}\,\epsilon_j + \tfrac{1}{2\sqrt{2}}\,e^{\frac{1}{2}\phi}\,F_{\mu\nu}\,\gamma^{\mu\nu}\,\epsilon^i\;,
\end{aligned} \tag{3.18}$$

where ϵ^i denotes the two supersymmetry parameters. These can be rewritten in terms of complex spinors as

$$\begin{aligned}
\delta\psi_\mu &= D_\mu\,\epsilon - \tfrac{\mathrm{i}}{\sqrt{2}}\,g\,A_\mu\,\epsilon + \tfrac{1}{2}(m\,e^{\frac{1}{2}\phi} + g\,e^{-\frac{1}{2}\phi})\,\gamma_\mu\,\epsilon + \frac{\mathrm{i}}{8\sqrt{2}}\,e^{\frac{1}{2}\phi}\,F_{\nu\rho}\,\gamma^{\nu\rho}\,\gamma_\mu\,\epsilon\;,\\
\delta\chi &= -\mathrm{i}\,\gamma^\mu\,\partial_\mu\phi\,\epsilon + 2\mathrm{i}\,(m\,e^{\frac{1}{2}\phi} - g\,e^{-\frac{1}{2}\phi})\,\epsilon + \tfrac{1}{2\sqrt{2}}\,e^{\frac{1}{2}\phi}\,F_{\mu\nu}\,\gamma^{\mu\nu}\,\epsilon\;,
\end{aligned} \tag{3.19}$$

The full Lagrangian (3.17) admits an AdS_4 black hole solution, given by

$$\begin{aligned}
ds_4^2 &= -H^{-1}\,f\,dt^2 + H\,(f^{-1}\,dr^2 + r^2\,d\Omega^2_{2,k})\;,\\
f &= k - \frac{\mu}{r} + 4m^2 r^2\,H^2\;, \qquad e^{\phi} = H\;,\\
A_{(1)} &= \frac{\sqrt{2}(1 + k\sinh^2\alpha)^{1/2}}{\sinh\alpha}\,H^{-1}\,dt\;, \qquad H = \frac{g}{m}\left(1 + \frac{\mu\sinh^2\alpha}{r}\right)\;.
\end{aligned} \tag{3.20}$$

Here $d\Omega^2_{2,k}$ denotes the metric on a two-dimensional space of positive, zero or negative constant curvature, according to whether k is positive, zero or negative. By convention, we take the spaces with $k = +1$ and $k = -1$ to be the unit S^2 and hyperbolic space H^2 respectively. Note that the solution with $k = 1$ corresponds to a special case of the 4-charge AdS black holes found in [29], where just two equal charges are turned on. The solution with $k = 0$ can be obtained from the S^7 reduction of the near horizon structure of the rotating M2-brane [13]. It is straightforward to check that in the extremal limit, where μ is sent to zero while holding $\mu\,\sinh^2\alpha$ fixed, these AdS black-hole solutions preserve one half of the $N = 2$ supersymmetry.

We see that these solutions do not allow an $m = 0$ limit to be taken, but they do permit us instead to take the $g \to 0$ limit provided $\alpha \to \infty$ with $g\sinh^2\alpha \equiv Q$ held fixed. In this limit we find

$$A_{(1)} \to \sqrt{2k} H^{-1} dt \ , \qquad H \to \frac{Q\mu}{m\, r} \ , \tag{3.21}$$

which describes an electrically charged domain-wall black hole for non-vanishing k. On the other hand, when $k=0$ the above solution is a pure domain-wall black hole with vanishing electric charge.

The scalar potential in (3.17) has a symmetry $\phi \leftrightarrow -\phi + 2\log g - 2\log m$. Accordingly we find a solution where $e^{-\phi} = H$ rather than $e^{\phi} = H$ given by

$$\begin{aligned} ds_4^2 &= -H^{-1}\, f\, dt^2 + H\, (f^{-1}\, dr^2 + r^2\, d\Omega_{2,k}^2) \ , \\ f &= k - \frac{\mu}{r} + 4g^2 r^2\, H^2 + \frac{\lambda^2}{2r^2} \ , \qquad e^{-\phi} = H \ , \\ A_{(1)} &= \frac{\lambda}{r}\, dt \ , \qquad H = \frac{m}{g} + \frac{Q}{r} \ , \end{aligned} \tag{3.22}$$

where the constant parameters satisfy

$$m^2\, \lambda^2 + 2g\, Q\, (m\, \mu + g\, k\, Q) = 0 \ . \tag{3.23}$$

Thus we have two possibilities. If $m \neq 0$ this solution is in fact equivalent to (3.20). This may be seen by performing a coordinate transformation

$$t = \frac{m}{g}\tilde{t} \ , \qquad r = \frac{g}{m}(\tilde{r} - Q) \ , \tag{3.24}$$

and making the identification

$$\mu = \frac{g\,\tilde{\mu}}{m}(1 + 2k\sinh^2\alpha) \ , \qquad Q = -\tilde{\mu}\sinh^2\alpha \ , \qquad \lambda = \frac{\sqrt{2}g\tilde{\mu}}{m}\sinh\alpha(1 + k\sinh^2\alpha)^{1/2} \ . \tag{3.25}$$

After dropping the tilde this may be seen to be identical to (3.20). On the other hand, if $m=0$ then λ is no longer constrained by (3.23). In this case the constraint requires $k=0$ for non-vanishing Q. For $m=0$, $k=0$ the solution cannot be obtained from the original AdS_4 black hole of (3.20). Nevertheless, the Lagrangian (3.17) fits the pattern of the generic class of Lagrangians (A.5) given in the appendix, and the solution is a special case of (A.6). In this case, the parameter Q can be absorbed into a rescaling of coordinates. The solution with $m=0$ was obtained in [30].

From the transformation rules (3.19), we find that these $\phi = -\log H$ domain-wall black holes preserve half the $N=2$ supersymmetry if the following condition hold:

$$Q = -\frac{m\,\mu}{2k\, g} \ , \qquad \lambda = \frac{\mu}{\sqrt{2k}} \ . \tag{3.26}$$

The conditions (3.26) imply that the function f in the metric is given by

$$f = \frac{g^2}{m^2}\,(k + 4m^2\,r^2)\,H^2\ . \tag{3.27}$$

When $k=1$, so that we have S^2 surfaces with $d\Omega_{2,1}^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$, the Killing spinors take the form

$$\epsilon = H^{1/4} e^{-\mathrm{i}gt}[\sqrt{h^{1/2}+1} + \sqrt{h^{1/2}-1}\,\gamma_1] e^{-\frac{\mathrm{i}}{2}\theta\gamma_{012}} e^{\frac{1}{2}\varphi\gamma_{23}}(1+\mathrm{i}\gamma_0)\epsilon_0\ , \tag{3.28}$$

where $h = 1 + 4m^2r^2$.

In the special case where $m=0$ and $k=0$, for which the 2-metric $d\Omega_{2,0}^2$ can be taken to be $dy^i\,dy^i$, the solution is supersymmetric if

$$\mu = 0\ , \qquad H = \frac{Q}{r}\ , \qquad f = \frac{\lambda^2}{2r^2} + 4g^2\,Q^2\ . \tag{3.29}$$

The Killing spinors are now given by

$$\epsilon = [\sqrt{\tilde h^{1/2} + \lambda r^{-1/2}} - \sqrt{\tilde h^{1/2} - \lambda r^{-1/2}}\,\gamma_1](1+\mathrm{i}\gamma_0)\epsilon_0\ , \tag{3.30}$$

where $\tilde h = 8g^2Q^2r + \lambda^2 r^{-1}$. Note that these Killing spinors have the same structure as (2.44) corresponding to the domain-wall black hole solution in the $m=0$ limit of $D=7$, $N=2$ gauged supergravity.

To summarize, we have seen that in addition to the known AdS_4 black hole solutions to (3.17) for non-vanishing g and m, there are also domain-wall black hole solutions in the two distinct limits: $g=0$ with $k=1$, and $m=0$ with $k=0$. In both cases, these correspond to new solutions in the $S^3\times\mathbb{R}^4$ compactification of M-theory. When $m=0$ the domain-wall black hole is electrically charged in the Cartan subgroup of the gauge fields arising from S^3. When $g=0$ the charge is instead carried by a field strength coming from the $\mathbb{R}^4$ reduction.

3.3 Limit to $S^5\times\mathbb{R}^2$

There is a different limit which we may consider, in which three exponentials, rather than only one, survive in the original potential in (3.7). To do this, we redefine $\vec\phi$ according to

$$\vec\phi = \vec\phi' + \vec a_1\,\lambda\ , \tag{3.31}$$

and then send $\lambda \longrightarrow +\infty$. The necessary scalings now imply

$$\begin{aligned}
&g = e^{-\frac{1}{2}\lambda}\,g'\ , \qquad X_1 = e^{-\frac{3}{2}\lambda}\,X_1'\ , \qquad X_m = e^{\frac{1}{2}\lambda}\,X_m'\ ,\\
&A^1_{(1)} = e^{-\frac{3}{2}\lambda}\,A^1_{(1)}{}'\ , \qquad A^m_{(1)} = e^{\frac{1}{2}\lambda}\,A^m_{(1)}{}'\ ,\\
&\mu_1 = e^{-\lambda}\,\mu_1'\ , \qquad \mu_m = \mu_m'\ ,
\end{aligned} \tag{3.32}$$

where we now split the $i = (1, 2, 3, 4)$ index as $i = 1$ and $i = m = (2, 3, 4)$. The scalar potential in (3.7) now becomes

$$V = -4g'^2 \left(e^{\phi_1} + e^{\phi_2} + e^{\phi_3}\right) . \tag{3.33}$$

Implementing these scalings in the ansätze (3.1) and (3.2), we find

$$\begin{aligned} ds_{11}^2 &= e^{\frac{1}{3}\lambda} \Big\{ \tilde{\Delta}^{2/3}\, ds_4^2 + g^{-2}\, \tilde{\Delta}^{-1/3} \Big(X_1^{-1}\, (d\mu_1^2 + \mu_1^2\, d\psi_1^2) \\ &\quad + \sum_{m=2}^{4} X_i^{-m} \Big(d\mu_m^2 + \mu_m^2\, (d\psi_m + g\, A_{(1)}^m)^2 \Big) \Big\} , \qquad (3.34) \\ F_{(4)} &= e^{\frac{1}{2}\lambda} \Big\{ 2g \sum_{m=2}^{4} \Big(X_m^2\, \mu_m^2 - \tilde{\Delta}\, X_m \Big)\, \epsilon_{(4)} + \frac{1}{2g} \sum_{m=2}^{4} X_m^{-1}\, {\bar{*}} dX_m \wedge d(\mu_m^2) \qquad (3.35) \\ &\quad - \frac{1}{2g^2}\, X_1^{-2}\, d(\mu_1^2) \wedge d\psi_1 \wedge {\bar{*}} F_{(2)}^1 - \frac{1}{2g^2} \sum_{m=2}^{4} X_m^{-2}\, d(\mu_m^2) \wedge (d\psi_m + g\, A_{(1)}^m) \wedge {\bar{*}} F_{(2)}^m , \end{aligned}$$

where $\sum_{m=2}^{4} \mu_m^2 = 1$. We have again dropped the primes after taking the $\lambda \longrightarrow +\infty$ limit. In this case, we can recognise the reduction as being on $S^5 \times \mathbb{R}^2$.

3.4 Limits of solutions

As we have seen in the previous section, the $S^5 \times R^2$ limit is obtained by uniformly shifting the three dilatonic scalars ϕ_i. In fact it is possible to have a consistent truncation of the Lagrangian (3.7) such that all the three scalars are equal, *i.e.* $\phi_i = \phi/\sqrt{3}$, provided that all but one of the vector potentials are set to zero. It follows that the Lagrangian becomes

$$e^{-1}\mathcal{L}_4 = R - \tfrac{1}{2}(\partial\phi)^2 + 12m^2\, e^{\phi/\sqrt{3}} + 12g^2\, e^{-\phi/\sqrt{3}} - \tfrac{1}{4} e^{\sqrt{3}\,\phi}\, F_{(2)}^2 . \tag{3.36}$$

Note that we have again introduced the additional parameter m, by shifting ϕ and rescaling g appropriately. For non-vanishing m and g, the values of m and g are unimportant, since they can be changed by a constant shift of the dilaton ϕ. The Lagrangian admits a single charge AdS$_4$ black hole, (*i.e.* $a = \sqrt{3}$,) given by

$$\begin{aligned} ds_4^2 &= -H^{-1/2}\, f\, dt^2 + H^{1/2}\, f^{-1} (dr^2 + r^2 (dy_1^2 + dy_2^2)) , \\ f &= -\frac{\mu}{r} + 4m^2 r^2\, H , \qquad e^{\frac{2}{\sqrt{3}}\phi} = H , \\ A_{(1)} &= \frac{(1 + k \sinh^2\alpha)^{1/2}}{\sinh\alpha} H^{-1}\, dt , \qquad H = \frac{g}{m} \left(1 + \frac{\mu \sinh^2\alpha}{r} \right) . \end{aligned} \tag{3.37}$$

As in the previous cases, the solution does not admit an $m = 0$ limit, but it does allow a limit where instead $g \to 0$, $\alpha \to \infty$ with $g \sinh^2\alpha$ held fixed, in which case the solution becomes a domain-wall black hole.

4 S^5 reduction of type IIB theory, and its limit

The Kaluza-Klein reduction of type IIB supergravity on S^5 is expected to admit a consistent truncation to gauged $N=8$ supergravity in $D=5$, with $SO(6)$ Yang-Mills gauge fields in the supergravity multiplet. As usual, the complexity of the reduction procedure has prevented any complete results from being obtained. However, in [13], it was shown that a further truncation to $N=2$ supergravity in $D=5$, with just the $U(1)^3$ Cartan subgroup of the $SO(6)$ Yang-Mills gauge fields surviving, *is* explicitly embeddable in $D=10$ type IIB supergravity. The ansatz found in [13], which gives a consistent truncation to the $N=2$ supermultiplet comprising the graviton, three $U(1)$ gauge fields, and two scalar fields is

$$\begin{aligned} ds_{10}^2 &= \sqrt{\widetilde{\Delta}}\, ds_5^2 + \frac{1}{g^2\sqrt{\widetilde{\Delta}}} \sum_{i=1}^3 X_i^{-1} \Big(d\mu_i^2 + \mu_i^2\, (d\psi_i + g\, A^i)^2 \Big)\ , \qquad (4.1)\\ G_{(5)} &= 2g \sum_i \Big(X_i^2\, \mu_i^2 - \widetilde{\Delta}\, X_i \Big)\, \epsilon_{(5)} - \frac{1}{2g} \sum_i X_i^{-1}\, \bar{*} dX_i \wedge d(\mu_i^2) \\ &\quad + \frac{1}{2g^2} \sum_i X_i^{-2}\, d(\mu_i^2) \wedge (d\psi_i + g\, A_{(1)}^i) \wedge \bar{*} F_{(2)}^i\ . \qquad (4.2) \end{aligned}$$

where the self-dual 5-form of $D=10$ is given by $F_{(5)} = G_{(5)} + *G_{(5)}$. The two scalars are parameterised in terms of the three quantities X_i, which are subject to the constraint $X_1\, X_2\, X_3 = 1$. They can be parameterised in terms of two dilatons $\vec\phi = (\phi_1, \phi_2)$ as

$$X_i = e^{-\frac{1}{2}\vec a_i\cdot\vec\phi}\ , \qquad (4.3)$$

A convenient choice for the dilaton vectors $\vec a_i$ is

$$\vec a_1 = (\tfrac{2}{\sqrt6}, \sqrt2)\ , \qquad \vec a_2 = (\tfrac{2}{\sqrt6}, -\sqrt2)\ , \qquad \vec a_3 = (-\tfrac{4}{\sqrt6}, 0)\ . \qquad (4.4)$$

The three quantities μ_i are subject to the constraint $\sum_i \mu_i^2 = 1$.

Substituting these ansätze into the equations of motion of the type IIB theory, it was shown in [13] that one consistently gets five-dimensional equations of motion that can be derived from the Lagrangian

$$e^{-1}\,\mathcal{L}_5 = R - \tfrac12 (\partial\phi_1)^2 - \tfrac12 (\partial\phi_2)^2 + 4g^2 \sum_i X_i^{-1} - \tfrac14 \sum_i X_i^{-2}\, (F_{(2)}^i)^2 + \tfrac14 \epsilon^{\mu\nu\rho\sigma\lambda}\, F_{\mu\nu}^1\, F_{\rho\sigma}^2\, A_\lambda^3\ . \qquad (4.5)$$

4.1 Limit to $S^3 \times \mathbb{R}^2$

Let us consider a limit under which the dilatonic scalar ϕ_1 in $\vec\phi = (\phi_1, \phi_2)$ is shifted by a constant λ, according to

$$\phi_1 = \phi_1' - \sqrt6\, \lambda\ , \qquad (4.6)$$

with ϕ_2 left unchanged. This implies that the quantities X_i will scale as

$$X_a = e^{\lambda}\, X_a' \,, \qquad X_3 = e^{-2\lambda}\, X_3' \,, \tag{4.7}$$

where we have split the $i = (1,2,3)$ index as $i = a = (1,2)$ and $i = 3$. If we also make the scalings

$$g = e^{-\lambda}\, g' \,, \qquad A^a_{(1)} = e^{\lambda}\, A^a_{(1)}{}' \,, \qquad A^3_{(1)} = e^{-2\lambda}\, A^3_{(1)} \,, \tag{4.8}$$

then we can take the limit $\lambda \longrightarrow +\infty$, to obtain the Lagrangian

$$e^{-1}\,\mathcal{L}_5 = R - \tfrac12 (\partial\phi_1)^2 - \tfrac12 (\partial\phi_2)^2 + 4g^2\, e^{-\frac{2}{\sqrt6}\phi_1} - \tfrac14 \sum_i X_i^{-2}\, (F^i_{(2)})^2 + \tfrac14 \epsilon^{\mu\nu\rho\sigma\lambda}\, F^1_{\mu\nu}\, F^2_{\rho\sigma}\, A^3_\lambda \,, \tag{4.9}$$

where we have dropped the primes after taking the limit.

If we additionally make the rescalings

$$\mu_a = \mu_a' \,, \qquad \mu_3 = e^{-\frac32\lambda}\, \mu_3' \tag{4.10}$$

in the reduction ansätze (4.1) and (4.2), then we find that they become

$$\begin{aligned} ds_{10}^2 &= e^{\frac12\lambda} \Big\{ \sqrt{\widetilde\Delta}\, ds_5^2 + \frac{1}{g^2\,\sqrt{\widetilde\Delta}} \Big(\sum_{a=1}^{2} X_a^{-1}\, (d\mu_a^2 + \mu_a^2\, (d\psi_a + g\, A^a)^2 \\ &\qquad + X_3^{-1}\, (d\mu_3^2 + \mu_3^2\, d\psi_3^2) \Big) \Big\} \,, \qquad\qquad (4.11) \\ G_{(5)} &= e^{\lambda} \Big\{ 2g \sum_{a=1}^{2} \Big(X_a^2\, \mu_a^2 - \widetilde\Delta\, X_a \Big)\, \epsilon_{(5)} - \frac{1}{2g} \sum_{a=1}^{2} X_a^{-1}\, {*d}X_a \wedge d(\mu_a^2) \qquad\qquad (4.12) \\ &\qquad + \frac{1}{2g^2} \sum_{a=1}^{2} X_a^{-2}\, d(\mu_a^2) \wedge (d\psi_a + g\, A^a_{(1)}) \wedge {*F}^a_{(2)} + \frac{1}{2g^2}\, X_3^{-2}\, d(\mu_3^2) \wedge d\psi_3 \wedge {*F}^3_{(2)} \Big\} \,, \end{aligned}$$

where, having dropped the primes as usual, we now have $\sum_{a=1}^2 \mu_a^2 = 1$, with μ_3 unconstrained, and $\widetilde\Delta = \sum_a X_a\, \mu_a^2$. We can recognise this as an $S^3 \times \mathbb{R}^2$ reduction of the type IIB theory. The resulting five-dimensional Lagrangian follows from (4.5) by applying the limiting procedure that we have used here, giving

$$e^{-1}\,\mathcal{L}_5 = R - \tfrac12 (\partial\phi_1)^2 - \tfrac12 (\partial\phi_2)^2 + 4g^2\, e^{-\frac{2}{\sqrt6}\phi_1} - \tfrac14 \sum_i X_i^{-2}\, (F^i_{(2)})^2 + \tfrac14 \epsilon^{\mu\nu\rho\sigma\lambda}\, F^1_{\mu\nu}\, F^2_{\rho\sigma}\, A^3_\lambda \,. \tag{4.13}$$

4.2 New domain-wall black holes in $D = 5$

In the S^5 limit to $S^3 \times R^2$ discussed in the previous section, we have rescaled the dilaton ϕ_1, while ϕ_2 is left alone. For simplicity, the dilaton ϕ_2 can be consistently truncated out, provided $F^1_{(2)} = F^2_{(2)} = F_{(2)}/\sqrt2$. Thus we shall consider the relevant Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_5 &= R - \tfrac12 (\partial\phi)^2 + 8m^2\, e^{\frac{1}{\sqrt6}\phi} + 4g^2 e^{-\frac{2}{\sqrt6}\phi} - \tfrac14 e^{\frac{2}{\sqrt6}\phi} (F_{(2)})^2 - \tfrac14 e^{-\frac{4}{\sqrt6}\phi} (F^3_{(2)})^2 \\ &\quad + \tfrac18 \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} A^3_\lambda \,, \end{aligned} \tag{4.14}$$

which, if we take the limit $m = 0$, corresponds to the $S^3 \times \mathbb{R}^2$ compactification of the previous subsection. The $N = 2$ fermion supersymmetry transformations are given by

$$\begin{aligned}
\delta\psi_\mu &= [\nabla_\mu - \tfrac{\mathrm{i}}{\sqrt{2}} g A_\mu - \tfrac{\mathrm{i}m^2}{2g} A^3_\mu + (\tfrac{g}{3} e^{-\frac{1}{\sqrt{6}}\phi} + \tfrac{m^2}{6g} e^{\frac{2}{\sqrt{6}}\phi})\gamma_\mu \\
&\qquad + \tfrac{\mathrm{i}}{24}(\gamma_\mu{}^{\nu\lambda} - 4\delta^\nu_\mu \gamma^\lambda)(e^{\frac{1}{\sqrt{6}}\phi} F_{\nu\lambda} + e^{-\frac{2}{\sqrt{6}}\phi} F^3_{\nu\lambda})]\epsilon\ , \qquad (4.15)\\
\delta\lambda &= [-\tfrac{\mathrm{i}}{4}\gamma^\mu \partial_\mu \phi - \tfrac{\mathrm{i}}{\sqrt{6}}(g e^{-\frac{1}{\sqrt{6}}\phi} - \tfrac{m^2}{g} e^{\frac{2}{\sqrt{6}}\phi}) + \tfrac{1}{8\sqrt{6}}(e^{\frac{1}{\sqrt{6}}\phi} F_{\mu\nu} - 2e^{-\frac{2}{\sqrt{6}}\phi} F^3_{\mu\nu})\gamma^{\mu\nu}]\epsilon\ .
\end{aligned}$$

This Lagrangian admits an AdS_5 black hole solution, given by

$$\begin{aligned}
ds_5^2 &= -H^{4/3} f\, dt^2 + H^{2/3}(f^{-1}\, dr^2 + r^2\, (dy^i dy_i))\ ,\\
f &= -\frac{\mu}{r^2} + m^2 r^2 H^2\ , \qquad e^{\sqrt{6}\phi} = H\ ,\\
A_{(1)} &= \frac{\sqrt{2}(1 + k \sinh^2\alpha)^{1/2}}{\sinh\alpha} H^{-1} dt\ , \qquad A^3_{(1)} = 0\ , \qquad H = \frac{g}{m}\left(1 + \frac{\mu \sinh^2\alpha}{r^2}\right) \qquad (4.16)
\end{aligned}$$

Note that this solution was obtained in [31] for $m = g$. When $k = 0$, it can also be obtained [27, 13] from the S^5 reduction of the rotating D3-brane [32, 33, 13]. It follows from (4.16) that the solution does not have an $m = 0$ limit, but it does have a $g \to 0$ limit with $g \sinh^2\alpha$ held fixed, which gives rise to a domain-wall black hole solution with $k \neq 0$.

When $m = 0$, the Lagrangian fits the general pattern of the Lagrangian (A.5) in the appendix. Thus in this limit, we can find a new domain-wall black hole solution, given by

$$\begin{aligned}
ds_5^2 &= -f\, dt^2 + f^{-1} dr^2 + r\, (dy_1^2 + dy_2^2 + dy_3^2)\ , \qquad e^{\frac{2}{\sqrt{6}}\phi} = r\ ,\\
f &= 2r\, (\tfrac{8}{9} g^2 + \frac{\lambda^2}{4r^3} + \frac{\mu}{r^{3/2}})\ , \qquad A_{(1)} = \lambda\, r^{-3/2}\, dt \qquad A^3_{(1)} = 0\ . \qquad (4.17)
\end{aligned}$$

This solution preserves half of the $N = 2$ supersymmetry provided $\mu = 0$. As in the $D = 7$ and the $D = 4$ cases, we find that the Killing spinors are given by

$$\epsilon = [\sqrt{h^{1/2} - \tfrac{1}{\sqrt{2}}\lambda r^{-1}} - \sqrt{h^{1/2} + \tfrac{1}{\sqrt{2}}\lambda r^{-1}}\, \gamma_1](1 + \mathrm{i}\gamma_0)\epsilon_0\ , \qquad (4.18)$$

where $h = \frac{16}{9} g^2 r + \frac{1}{2}\lambda^2 r^{-2}$. This $m = 0$ solution, with $k = 0$, corresponds to a domain-wall black hole of the $S^3 \times \mathbb{R}^2$ compactification obtained in the previous subsection.

5 Conclusions

We studied in detail a class of consistent Kaluza-Klein sphere compactifications of string/M-theory whose effective theories reduce to gauged supergravity theories that do not admit anti-de Sitter (AdS) space-time as a vacuum solution. We refer to them as domain-wall supergravities. This class of supergravity solutions can be viewed as particular limits of the (standard) AdS gauged supergravity theories in $D = 7, 4, 5$, which are obtained as Kaluza-Klein compactifications on S^4 and S^7 of eleven-dimensional supergravity and on S^5 of ten-dimensional Type IIB supergravity.

We obtained specific limits of AdS gauged supergravity theories by taking certain moduli, associated with the non-homogeneous deformations of S^n, to their boundary values, which result in compactifications of string/M-theory on $S^p \times R^q$ $(p + q = n)$ spaces and correspond to domain-wall supergravities. We classified such possible limits for the abelian truncations of the gauged AdS supergravities in $D = 7, 4, 5$. A particular instructive example is a limit of M-theory compactifications on S^4 which reduces to a compactification on $S^3 \times R$, and a domain-wall supergravity in $D = 7$. This compactification can be reinterpreted as a consistent compactification of Type IIA theory on S^3. The limiting procedure we employed highlights a geometrical interpretation of the massive, but ungauged supergravity in $D = 7$ and the massless $D = 7$ gauged supergravity.

Since the sphere compactifications of the AdS supergravities are believed to be consistent, it follows that the resulting domain-wall supergravities are consistent sphere compactifications of higher dimensional supergravities as well. While these domain-wall supergravities were obtained as certain limits of AdS supergravities, they now stand as valid sphere compactifications in their own right.

Typical solutions of domain-wall supergravities are of the (black-hole) domain-wall-type, *i.e.* asymptotically the configurations have non-constant scalar fields. We have shown how some of these solutions can be obtained as limits of the existing (black-hole) solutions of the corresponding AdS gauged supergravities. However, we have also found new classes of domain-wall black holes that cannot be obtained as special limits of the AdS supergravity solutions and have shown that they are supersymmetric. These sets of solutions may play an important role in the study of the dual quantum field theories (QFT), according to the proposed domain-wall/QFT correspondence [15].

Note added:

After this work was completed, a paper appeared in which the complete non-linear ansatz for the S^4 reduction of $D = 11$ supergravity was presented [34].

Appendix

A A class of domain-wall black holes

Let us consider a bosonic system with the metric, a vector potential and a dilaton, with the Lagrangian given by

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{a\phi}\, F_{(2)}^2 - V(\phi) \ , \tag{A.1}$$

where $V(\phi)$ is the scalar potential. We consider the following ansätze

$$\begin{aligned} ds^2 &= -e^{2u}\, dt^2 + e^{2A}\, dy^i dy^i + e^{2v}\, dr^2 \ , \\ A_{(1)} &= w\, dt \ . \end{aligned} \tag{A.2}$$

where u, v, w, C and the dilaton ϕ are functions of r only. The dimension of the space $dy^i dy^i$ is $p = D - 2$. The equations of motion is then given by

$$\begin{aligned} u'' + u'(u' - v' + pA') &= \frac{p-1}{2p} w'^2\, e^{a\phi - 2u} - \frac{1}{p} V\, e^{2v} \ , \\ A'' + A'(u' - v' + pA') &= -\frac{1}{2p} w'^2\, e^{a\phi - 2u} - \frac{1}{p} V\, e^{2v} \ , \\ A'' - A'(u' + v' - A') &= -\frac{1}{2p}\phi'^2 \ , \\ \phi'' + \phi'(u' - v' + pA') &= -\tfrac{1}{2} a\, w'^2\, e^{a\phi - 2u} + V_{,\phi}\, e^{2v} \ , \\ (w'\, e^{a\phi - u - v + pA})' &= 0 \ , \end{aligned} \tag{A.3}$$

where a prime denotes a derivative with respect to r. This set of equations are self-consistent, implying the existence of solutions.

In this paper, we showed that there exists a modulus limit for generic gauged supergravities, where the relevant part of the Lagrangian has the form (A.1) with

$$V(\phi) = -\tfrac{1}{2} g^2 e^{-a\phi} \qquad \text{and} \qquad a^2 = \frac{2}{p} \ . \tag{A.4}$$

In other words, the Lagrangian has the form

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{\sqrt{2/p}\,\phi}\, F_{(2)}^2 + \tfrac{1}{2} g^2\, e^{-\sqrt{2/p}\,\phi} \ . \tag{A.5}$$

In this case, the equations of motion (A.3) admit a closed-form solution, given by

$$\begin{aligned} ds^2 &= -f\, dt^2 + f^{-1}\, dr^2 + r\, dy^i dy^i \ , \\ f &= 2r\Big(\frac{g^2}{p^2} + \frac{\lambda^2}{4r^p} + \frac{\mu}{r^{p/2}}\Big) \ , \qquad A_{(1)} = \lambda\, r^{-p/2}\, dt \ , \qquad e^{\sqrt{2/p}\,\phi} = r \ . \end{aligned} \tag{A.6}$$

The solution describes a domain-wall black hole, whose geometry approaches a pure domain-wall spacetime as $r \to \infty$.

References

[1] M.J. Duff and C.N. Pope, *Kaluza-Klein supergravity and the seven sphere*, in: Supersymmetry and supergravity 82, eds. S. Ferrara, J.G. Taylor and P. van Nieuwenhuizen (World Scientific, Singapore, 1983).

[2] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Kaluza-Klein supergravity*, Phys. Rep. **130** (1986) 1.

[3] K. Pilch, P. K. Townsend and P. van Nieuwenhuizen, *Compactification of $d = 11$ supergravity on S^4 (or $11 = 7 + 4$, too)*, Nucl. Phys. **B242** (1984) 377.

[4] J.H. Schwarz, *Covariant field equations of chiral $N = 2$ $D = 10$ supergravity*, Nucl. Phys. **B226** (1983) 269.

[5] M. Gunaydin and N. Marcus, *The spectrum of the S^5 compactification of the $N = 2$, $D = 10$ supergravity and the unitary supermultiplet*, Class. Quant. Grav. **2** (1985) L11.

[6] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *Mass spectrum of chiral ten-dimensional $N = 2$ supergravity on S^5*, Phys. Rev. **D32** (1985) 389.

[7] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.

[8] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** (1998) 105, hep-th/9802109.

[9] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[10] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. **2** (1998) 505, hep-th/9803131.

[11] B. de Wit, H. Nicolai and N.P. Warner, *The embedding of gauged $N = 8$ supergravity into $D = 11$ supergravity*, Nucl. Phys. B255 (1985) 29.

[12] B. de Wit and H. Nicolai, *The consistency of the S^7 truncation in $D = 11$ supergravity*, Nucl. Phys. B281 (1987) 211.

[13] M. Cvetič, M.J. Duff, P. Hoxha, J.T. Liu, H. Lü, J.X. Lu, R. Martinez-Acosta, C.N. Pope, H. Sati and T.A. Tran, *Embedding AdS black holes in ten and eleven dimensions*, hep-th/9903214.

[14] J. Scherk and J.H. Schwarz, *How to get massses from extra dimensions*, Nucl. Phys. **B153** (1979) 61.

[15] H.J. Boonstra, K. Skenderis and P.K. Townsend, *The domain wall/QFT correspondence*, JHEP, **9901** (1999) 003, hep-th/9807137.

[16] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged maximally extended supergravity in seven dimensions*, Phys. Lett. **B143** (1984) 103.

[17] P.K. Townsend and P. van Nieuwenhuizen, *Gauged seven-dimensional supergravity*, Phys. Lett. **B125** (1983) 41-6.

[18] H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, *Dilatonic p-brane solitons*, Phys. Lett. **B371** (1996) 46, hep-th/9511203.

[19] H. Lü and C.N. Pope, *Domain walls from M-branes*, Mod. Phys. Lett. **A12** (1997) 1087, hep-th/9611079.

[20] R.C. Myers and M.J. Perry, *Black holes in higher dimensional spacetimes*, Ann. Phys. **172** (1986) 304.

[21] I.A. Lavrinenko, H. Lü and C.N. Pope, *From topology to generalised dimensional reduction*, Nucl. Phys. **B492** (1997) 278, hep-th/9611134.

[22] E. Cremmer, H. Lü, C.N. Pope and K.S. Stelle, *Spectrum generating symmetries for BPS solitons*, Nucl. Phys. **B520** (1998) 132, hep-th/9707207.

[23] H. Lü and C.N. Pope, *p-brane solitons in maximal supergravities*, Nucl. Phys. **B465** (1996) 127, hep-th/9512012.

[24] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities*, Nucl. Phys. **B523** (1998) 73, hep-th/9710119.

[25] P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. **B486** (1996) 506, hep-th/9510209.

[26] J.T. Liu and R. Minasian, *Black holes and membranes in* AdS_7, hep-th/9903269.

[27] M. Cvetič and S.S. Gubser, *Phases of R charged black holes, spinning branes and strongly coupled gauge theories*, hep-th/9902195.

[28] M. Cvetič and D. Youm, *Rotating intersecting M-branes*, Nucl. Phys. **B499** (1997) 253, hep-th/9612229.

[29] M.J. Duff and J.T. Liu, *Anti-de Sitter black holes in gauged N=8 supergravity*, hep-th/9901149.

[30] D. Klemm, *BPS black holes in gauged* $N = 4$, $D = 4$ *supergravity*, hep-th/9810090.

[31] K. Behrndt, M. Cvetič and W.A. Sabra, *Non-extreme black holes five-dimensional* $N = 2$ *AdS supergravity*, hep-th/9810227.

[32] P. Kraus, F. Larsen and S.P. Trivedi, *The Coulomb branch of gauge theory from rotating branes*, hep-th/9811120.

[33] J.G. Russo and K. Sfetsos, *Rotating D3-branes and QCD in three dimensions*, hep-th/9901056.

[34] H. Nastase, D. Vaman and P. van Nieuwenhuizen, *Consistent nonlinear KK reduction of 11d supergravity on* $AdS_7 \times S^4$ *and self-duality in odd dimensions*, hep-th/9905075.