Gauge Dyonic Strings and Their Global Limit

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ABSTRACT

We show that six-dimensional supergravity coupled to tensor and Yang-Mills multiplets admits not one but two different theories as global limits, one of which was previously thought not to arise as a global limit and the other of which is new. The new theory has the virtue that it admits a global anti-self-dual string solution obtained as the limit of the curved-space gauge dyonic string, and can, in particular, describe tensionless strings. We speculate that this global model can also represent the worldvolume theory of coincident branes. We also discuss the Bogomol'nyi bounds of the gauge dyonic string and show that, contrary to expectations, zero eigenvalues of the Bogomol'nyi matrix do not lead to enhanced supersymmetry and that negative tension does not necessarily imply a naked singularity.

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1 Introduction

This paper is devoted to certain properties of the six-dimensional gauge dyonic string [1] and in particular to its global limit in which it becomes anti-self dual. An important special case corresponds to the tensionless string, which has been the subject of much interest lately [2,3,4,5,6,7,8,9,10,11,12,13,1], especially in the context of phase transitions [14,11,1,15].[1] This global limit is particularly interesting because one might then expect to be able to find an anti-self-dual string solution by directly solving the global supersymmetric theory in six-dimensions [11] describing an anti-self-dual tensor multiplet coupled to Yang-Mills. However, an apparently paradoxical claim was made in [18] that no such global limit exists. Here we resolve the paradox, and show that not only does the limit exist but that there are in fact two different limits, each giving different globally supersymmetric theories. One of these is the theory constructed in [18], which we shall refer to as the "BSS theory". The other flat-space theory, which for reasons described below we shall refer to as the "interacting theory", appears to be new, and admits an anti-self-dual string solution which can indeed be obtained as the flat-space limit of the dyonic string of the supergravity theory.

A surprising feature of the BSS theory constructed in [18] is that there is an asymmetry in the interactions between the Yang-Mills multiplet and the anti-self-dual tensor multiplet. In particular, the Yang-Mills multiplet satisfies free equations of motion, whereas the equations of motion for the tensor multiplet do involve couplings to the Yang-Mills fields. By contrast, the interactions in the "interacting" theory obtained in the present paper here are more symmetrical, in that they occur in all the equations of motion. Interestingly, however, the additional interaction terms of the new theory cancel in the special case of its anti-self-dual string solution, and so the same configuration is also a solution of the BSS theory. Curiously, however, it is not tensionless in that theory, and indeed the BSS theory is inappropriate for describing any tensionless string solution.

Another intriguing aspect of the gauge dyonic string concerns the counter-intuitive relations between its Bogomol'nyi bound, unbroken supersymmetry and its singularity structure [1]. We confirm:

(1) The dyonic string continues to preserve just half of the supersymmetry even in the tensionless limit, notwithstanding the standard Bogomol'nyi argument that a BPS state

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[1]But note that, contrary to some claims in the literature, the tensionless string corresponds to the (quasi)-anti-self-dual limit of the dyonic string of [16], where the string couples dominantly to the 3-form field strength of the tensor matter multiplet, and not the self-dual string of [17] where the string couples only to the 3-form field strength of the gravity multiplet.

with vanishing central charge leads to completely unbroken supersymmetry.

(2) A solution with *negative* tension can be completely non-singular, contrary to the folk-wisdom that negative mass necessarily implies naked singularities.

Finally, six dimensional global models are also important as fivebrane worldvolume theories [19, 20, 21, 18] and as the worlvolume theories of coincident higher-dimensional branes with six dimensions in common [25, 24]. We speculate that the interacting anti-self-dual-tensor Yang-Mills system is indeed such a worldvolume theory. Hence the global gauge anti-self-dual string, and in particular the tensionless string, may also be regarded as a string on the worldvolume. In the case of the tensionless string, in the limit as the size ρ of the Yang-Mills instanton shrinks to zero, one recovers the global limit of the neutral tensionless string [16, 1] which is also a solution of the $(2,0)$ theory that resides on the worldvolume of the M-theory fivebrane. It is curious, therefore, that we find in this limit that the tension really is zero, as opposed to the infinite tension of the string solution of the free $(2,0)$ theory [22, 23].

2 $N = 1$ supergravity and the gauge dyonic string

The low-energy $D = 6$ $N = (1,0)$ supergravity is generated by a pair of symplectic Majorana-Weyl spinors ϵ transforming in the 2 of $Sp(2)$. This theory has the unusual feature in that the antisymmetric tensor breaks up into self-dual and anti-self-dual components. The basic supergravity theory consists of the graviton multiplet $(g_{\mu\nu}, \psi_\mu, B^+_{\mu\nu})$ coupled to n_T tensor multiplets $(B^-_{\mu\nu}, \chi, \phi)$. When $n_T = 1$, corresponding to the heterotic string compactified on $K3$, these multiplets may be combined, yielding a single ordinary antisymmetric tensor $B_{\mu\nu}$.

We are interested, however, in the general case with n_T tensor multiplets coupled to an arbitrary number of vector multiplets (A_μ, λ). Due to the presence of chiral antisymmetric tensor fields, there is no manifestly covariant Lagrangian formulation of this theory. Nevertheless, the equations of motion may be constructed, and were studied in [26, 27]. With n_T tensor multiplets, there are n_T scalars parametrizing the coset $SO(1, n_T)/SO(n_T)$. This may be described in terms of a $(n_T + 1) \times (n_T + 1)$ vielbein transforming as vectors of both $SO(1, n_T)$ and $SO(n_T)$. Following the conventions of [27], the vielbein may be decomposed as

$$V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} = \begin{bmatrix} v_0 & v_M \\ x^m{}_0 & x^m{}_M \end{bmatrix}, \tag{2.1}$$

satisfying the condition $V^{-1} = \eta V^T \eta$ where η is the $SO(1, n_T)$ metric, $\eta = \text{diag}(1, -I_{n_T})$.

Below, we use indices $r, s, \ldots = \{0, M\}$ to denote $SO(1, n_T)$ vector indices. The composite $SO(n_T)$ connection is then given by

$$
\begin{aligned}
S_\mu^{[mn]} &= (\partial_\mu V_- \eta V_-^T)^{[mn]} \\
&= -x^m{}_0 \partial_\mu x^n{}_0 + x^m{}_M \partial_\mu x^n{}_M \ ,
\end{aligned} \tag{2.2}
$$

so that the fully covariant derivative acting on $SO(n_T)$ vectors is given by $\mathcal{D}_\mu = \nabla_\mu + S_\mu$.

To describe the combined supergravity plus tensor system, we introduce $(n_T + 1)$ anti-symmetric tensors $B_{\mu\nu}$ transforming as a vector of $SO(1, n_T)$. In the presence of Yang-Mills fields, the three-form field strengths pick up a Chern-Simons coupling

$$
\mathcal{H} = dB + c\,\omega_3 \ , \tag{2.3}
$$

where $\omega_3 = AdA + \frac{2}{3}A^3$, so that $d\mathcal{H} = c\,\mathrm{tr}\,F^2$. The constants c form a $(n_T + 1) \times n_V$ matrix where n_V is the number of vector multiplets[2]. Note that this coupling of the vector and tensor multiplets is dictated by supersymmetry and encompasses both tree-level and one-loop Yang-Mills corrections. Furthermore, the supersymmetry guarantees that there are no higher-loop corrections. The vielbein is then used to transform the field strengths $\mathcal{H}$ into their chiral components $H = v_r \mathcal{H}^r$ and $K^m = x^m{}_r \mathcal{H}^r$ so that the (anti-)self-duality conditions for the tensors become $H = *H$ and $K^m = - *K^m$.

With the above conventions, the bosonic equations of motion are

$$
\begin{aligned}
G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R &= T_{\mu\nu} \\
\mathcal{D}_\mu P^{m\,\mu} &= -\tfrac{\sqrt{2}}{3} H_{\mu\nu\rho} K^{m\,\mu\nu\rho} - \tfrac{1}{\sqrt{2}} x^m{}_r c^r \,\mathrm{tr}\,(F_{\mu\nu} F^{\mu\nu}) \\
dH &= -\sqrt{2} P^m K^m + v_r c^r \,\mathrm{tr}\,F^2 \\
(d\delta^{mn} + S^{mn}) K^n &= -\sqrt{2} P^m H + x^m{}_r c^r \,\mathrm{tr}\,F^2 \\
v_r c^r D^\mu F_{\mu\nu} &= \sqrt{2} P^{m\,\mu} x^m{}_r c^r F_{\mu\nu} + H_{\nu\rho\sigma} v_r c^r F^{\rho\sigma} + K^m_{\nu\rho\sigma} x^m{}_r c^r F^{\rho\sigma} \ , \quad (2.4)
\end{aligned}
$$

where

$$
\begin{aligned}
P_\mu^m &= \tfrac{1}{\sqrt{2}} (\partial_\mu V_+ \eta V_-^T)^m \\
&= \tfrac{1}{\sqrt{2}} (x^m{}_0 \partial_\mu v_0 - x^m{}_M \partial_\mu v_M) \ ,
\end{aligned} \tag{2.5}
$$

and S and P are the 1-forms, $S = S_\mu dx^\mu$, $P = P_\mu dx^\mu$. The symmetric stress tensor is given by

$$
T_{\mu\nu} = H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + K^m_{\mu\rho\sigma} K_\nu^{m\,\rho\sigma} + 2[P_\mu^m P_\nu^m - \tfrac{1}{2} g_{\mu\nu} P_\rho^m P^{m\,\rho}] + 4 v_r c^r \,\mathrm{tr}\,[F_{\mu\lambda} F_\nu{}^\lambda - \tfrac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}] \ . \tag{2.6}
$$

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[2]For non-abelian gauge fields, instead of having n_V independent quantities, there is a single set of c's for each factor of the gauge group.

For the antisymmetric tensors, Eqn. (2.4) along with the (anti-)self-duality constraint may be viewed as the equivalent of the combined Bianchi identities and equations of motion. Finally, the fermionic equations of motion are

$$\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho = -H^{\mu\nu\rho}\gamma_\nu\psi_\rho + \tfrac{i}{2}K^{m\,\mu\nu\rho}\gamma_{\nu\rho}\chi^m - \tfrac{i}{\sqrt{2}}P_\nu^m\gamma^\nu\gamma^\mu\chi^m - \tfrac{1}{\sqrt{2}}\gamma^{\sigma\tau}\gamma^\mu v_r c^r \,\mathrm{tr}\,F_{\sigma\tau}\lambda$$

$$\gamma^\mu\nabla_\mu\chi^m = \tfrac{i}{2}K^{m\,\mu\nu\rho}\gamma_{\mu\nu}\psi_\rho + \tfrac{1}{12}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\chi^m + \tfrac{i}{\sqrt{2}}P_\nu^m\gamma^\mu\gamma^\nu\psi_\mu - \tfrac{i}{\sqrt{2}}\gamma^{\mu\nu}x^m_{\ r}c^r\,\mathrm{tr}\,F_{\mu\nu}\lambda$$

$$v_r c^r\gamma^\mu D_\mu\lambda = \tfrac{1}{\sqrt{2}}P_\mu^m\gamma^\mu x^m_{\ r}c^r\lambda - \tfrac{1}{2\sqrt{2}}v_r c^r F_{\lambda\tau}\gamma^\mu\gamma^{\lambda\tau}\psi_\mu - \tfrac{i}{2\sqrt{2}}x^m_{\ r}c^r F_{\mu\nu}\gamma^{\mu\nu}\chi^m$$
$$\qquad - \tfrac{1}{12}K^m_{\mu\nu\rho}x^m_{\ r}c^r\gamma^{\mu\nu\rho}\lambda \ . \tag{2.7}$$

In order to examine the Bogomol'nyi bound, we need the supersymmetry variations for the fermionic fields:

$$\delta\psi_\mu = [\nabla_\mu + \tfrac{1}{4}H_{\mu\nu\rho}\gamma^{\nu\rho}]\epsilon$$

$$\delta\chi^m = i[\tfrac{1}{\sqrt{2}}\gamma^\mu P_\mu^m + \tfrac{1}{12}K^m_{\mu\nu\rho}\gamma^{\mu\nu\rho}]\epsilon$$

$$\delta\lambda = -\tfrac{1}{2\sqrt{2}}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \tag{2.8}$$

(given to lowest order). For completeness, the bosonic fields transform according to

$$\delta e_\mu^{\ a} = -i\bar\epsilon\gamma^a\psi_\mu$$

$$\delta B^r_{\mu\nu} = \eta^{rs}\bar\epsilon[iv_s\gamma_{[\mu}\psi_{\nu]} - \tfrac{1}{2}x^m_{\ s}\gamma_{\mu\nu}\chi^m] + 2c^r\,\mathrm{tr}\,A_{[\mu}\delta A_{\nu]}$$

$$\delta v_r = x^m_{\ r}\bar\epsilon\chi^m$$

$$\delta A_\mu = -\tfrac{i}{\sqrt{2}}\bar\epsilon\gamma_\mu\lambda \ . \tag{2.9}$$

Careful examination of Eqns. (2.8) and (2.9) reveals the intricate interplay between terms of various chiralities necessary to maintain $D = 6$ $N = (1,0)$ supersymmetry. In particular, ϵ is a chiral spinor satisfying $P_+\epsilon = 0$ where $P_\pm = \tfrac{1}{2}(1 \pm \gamma^7)$ is the chirality projection in six dimensions. As a consequence, H and K satisfy the identities

$$(H_{\mu\nu\lambda}\gamma^{\mu\nu\lambda})\epsilon = 0$$

$$(K^m_{\mu\nu\lambda}\gamma^{\mu\nu\lambda}\gamma_\alpha)\epsilon = 0 \ , \tag{2.10}$$

which prove to be useful in manipulating Nester's form below.

2.1 The gauge dyonic string solution

It was shown in [1] that the equations of motion (2.4) admit a gauge dyonic string solution carrying both self-dual and anti-self-dual tensor charges. Under an appropriate $SO(n_T)$

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rotation, the latter charge can be put in a single tensor component, so that we may focus on the theory with $n_T = 1$. In this case, corresponding to a compactified heterotic string, the self-dual and anti-self-dual three-forms in the graviton and tensor multiplets respectively may be combined together according to

$$H = \tfrac{1}{2}e^{-\phi}(*\mathcal{H} + \mathcal{H}) , \qquad K = \tfrac{1}{2}e^{-\phi}(*\mathcal{H} - \mathcal{H}) , \tag{2.11}$$

where we have chosen a vielbein

$$V = \begin{bmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{bmatrix} . \tag{2.12}$$

For a simple gauge group, we pick the coupling vector c to be

$$c = \frac{\alpha'}{16} \begin{bmatrix} v + \tilde{v} \\ -v + \tilde{v} \end{bmatrix} , \tag{2.13}$$

so that the $\mathcal{H}$ Bianchi identity and equation of motion, given in Eqn. (2.4), may be rewritten as

$$d\mathcal{H} = \tfrac{1}{8}\alpha'\, v \operatorname{tr} F \wedge F \qquad d(e^{-2\phi} * \mathcal{H}) = \tfrac{1}{8}\alpha'\, \tilde{v} \operatorname{tr} F \wedge F . \tag{2.14}$$

The gauge dyonic string is built around a single self-dual $SU(2)$ Yang-Mills instanton in transverse space, and is given in terms of three parameters, which are the electric and magnetic charges Q and P carried by the string, and ρ which is the scale parameter of the instanton. Splitting the six-dimensional space into longitudinal $\mu, \nu = 0, 1$ and transverse $m, n, \ldots = 2, 3, 4, 5$ components, the gauge dyonic string solution is given by [1]

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A}dy^m dy^m$$

$$\mathcal{H}_{mnp} = \tfrac{1}{2}\epsilon_{mnpq}\partial_q H_1 \qquad \mathcal{H}_{\mu\nu m} = \tfrac{1}{2}\epsilon_{\mu\nu}\partial_m H_2^{-1}$$

$$e^{-\phi} = \sqrt{H_2/H_1} \qquad e^{-2A} = \sqrt{H_1 H_2} , \tag{2.15}$$

where $\epsilon_{01} = 1$, $\epsilon_{2345} = 1$. The functions H_1 and H_2 are

$$H_1 = e^{\phi_0} + \frac{P(2\rho^2 + r^2)}{(\rho^2 + r^2)^2} , \qquad H_2 = e^{-\phi_0} + \frac{Q(2\rho^2 + r^2)}{(\rho^2 + r^2)^2} , \tag{2.16}$$

and are determined by the effect of the instanton source

$$F^a = \frac{2\rho^2}{(\rho^2 + r^2)^2}\eta^a_{mn}dy^m \wedge dy^n , \tag{2.17}$$

on the three-form tensor according to (2.14). (Note that $\operatorname{tr}(F^2) = 2F^a_{mn}F^{amn}$.) In particular, the charges are thus given by $Q = 2\alpha'\tilde{v}$ and $P = 2\alpha' v$. The mass per unit length of the dyonic string is given by

$$2\pi\alpha'^2\, m = Pe^{-\phi_0} + Qe^{\phi_0} . \tag{2.18}$$

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This expression for the mass, and its relation to the Bogomol'nyi bound, will be examined in detail in the following section.

In the $\rho \to 0$ limit, we recover the neutral dyonic string obtained in [16].

3   The Bogomol'nyi bound in six dimensions

It is well known that the six-dimensional $N = (1, 0)$ supersymmetry algebra admits a single real string-like central charge, putting a lower bound on the tension of the six-dimensional string. Thus the tensionless string only arises in the limit of vanishing central charge. Before focusing on the tensionless string, we examine the Bogomol'nyi mass bound in general and determine the conditions for which it is satisfied.

For a string-like field configuration in six dimensions, we may construct the supercharge per unit length of the string from the behavior of the gravitino at infinity [28]

$$Q_\epsilon = \int_{\partial \mathcal{M}} \overline{\epsilon} \gamma^{\mu\nu\lambda} \psi_\lambda d\Sigma^{\mu\nu} \ , \tag{3.1}$$

where $\mathcal{M}$ is the four-dimensional space transverse to the string. We note that in writing the supercharge in terms of the gravitino, this expression holds only up to the equations of motion. It is for this reason that, unlike in the global case, saturation of the Bogomol'nyi bound alone is insufficient to guarantee that the bosonic background solves the supergravity equations of motion.

Using Nester's procedure [29, 28, 30], we may take the anticommutator of two supercharges to get

$$\{Q_\epsilon, Q_{\epsilon'}\} = \delta_\epsilon Q_{\epsilon'} = \int_{\partial \mathcal{M}} N^{\mu\nu} d\Sigma_{\mu\nu} \ , \tag{3.2}$$

where

$$N^{\mu\nu} = \overline{\epsilon'} \gamma^{\mu\nu\lambda} \delta_\epsilon \psi_\lambda = \overline{\epsilon'} \gamma^{\mu\nu\lambda} [\nabla_\lambda + \tfrac{1}{4} H_{\lambda\rho\sigma} \gamma^{\rho\sigma}] \epsilon \tag{3.3}$$

is a generalized Nester's form. Appealing to the supersymmetry algebra, we then see that the mass and central charge per unit length of the six-dimensional string is encoded in the surface integral of $N^{\mu\nu}$. For a string in the 0-1 direction, the ADM mass per unit length M of the string is given by the asymptotic behavior of the metric

$$ds^2 = (1 - \frac{GM}{2r^2} + \cdots)[-dt^2 + dz^2] + (1 + \frac{GM}{2r^2} + \cdots) dy^i dy^i \ , \tag{3.4}$$

where $r^2 = y^i y^i$ is the transverse radial distance from the string. Using this definition of the ADM mass, the surface integral of Nester's form becomes

$$\int_{\partial \mathcal{M}} N^{\mu\nu} d\Sigma_{\mu\nu} = 2\pi^2 \, \epsilon'^\dagger [\frac{GM}{2} - Z\gamma^0 \gamma^1] \epsilon \ , \tag{3.5}$$

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where the real string-like central charge Z is given by the self dual H charge

$$\int_{\partial\mathcal{M}} H = 2\pi^2 Z . \tag{3.6}$$

This reinforces the close relation between the central charges of a supergravity theory and the bosonic charges of the fields in the graviton multiplet.

From the point of view of the supersymmetry algebra, the left hand side of Eqn. (3.2) is non-negative for identical (commuting) spinors $\epsilon' = \epsilon$. Since $\gamma^0\gamma^1$ has eigenvalues ± 1, this gives rise to the Bogomol'nyi bound

$$GM \geq 2|Z| , \tag{3.7}$$

with saturation of the bound corresponding to (partially) unbroken supersymmetry. However an issue has arisen over the necessary conditions for this bound to apply. In particular, it has been noted that the gauge dyonic string [1] may have a tensionless limit without naked singularities when the instanton size in the gauge solution is sufficiently large. Corresponding to Eqn. (3.7), this tensionless string has vanishing central charge and is hence quasi-anti-self-dual. Nevertheless, examination of the Killing spinor equations indicates that it still breaks exactly half of the supersymmetries, in contrast to the expectation that $M = 0$ yields completely unbroken supersymmetry. In terms of singular four-dimensional solutions, this breakdown of the Bogomol'nyi argument has also been discussed in [31, 32].

In order to address the issue of where the Bogomol'nyi expression may break down, we take a closer look at the Witten-Nester proof of the positive energy theorem [33, 29]. Following [28], the charges at infinity may be related to the divergence of Nester's form:

$$\int_{\partial\mathcal{M}} N^{\mu\nu} d\Sigma_{\mu\nu} = \int_{\mathcal{M}} \nabla_\mu N^{\mu\nu} d\Sigma_\nu . \tag{3.8}$$

Proof of the Bogomol'nyi bound is then a matter of reexpressing this divergence in a manifestly non-negative form. Straightforward but tedious manipulations allow the divergence of Nester's form to be rewritten in terms of the supersymmetry variations of the fermionic fields given in Eqn. (2.8). Starting with

$$\begin{aligned}\nabla_\mu N^{\mu\nu} &= \overline{\delta_{\epsilon'}\psi_\mu}\gamma^{\mu\nu\rho}\delta_\epsilon\psi_\rho - \tfrac{1}{2}\overline{\epsilon'}G^\nu{}_\sigma\gamma^\sigma\epsilon \\ &\quad + \tfrac{1}{4}\overline{\epsilon'}\gamma^{\mu\nu\rho}\gamma^{\beta\gamma}(\nabla_\mu H_{\rho\beta\gamma})\epsilon + \tfrac{1}{16}\overline{\epsilon'}\gamma^{\beta\gamma}\gamma^{\mu\nu\rho}\gamma^{\lambda\sigma}H_{\mu\beta\gamma}H_{\rho\lambda\sigma}\epsilon , \end{aligned} \tag{3.9}$$

it is apparent that the H equations of motion must enter the calculation. Working through these equations then gives the final result

$$\nabla_\mu N^{\mu\nu} = \overline{\delta_{\epsilon'}\psi_\mu}\gamma^{\mu\nu\rho}\delta_\epsilon\psi_\rho + \overline{\delta_{\epsilon'}\chi}\gamma^\nu\delta_\epsilon\chi + v_r c^r \,\mathrm{tr}\,\overline{\delta_{\epsilon'}\lambda}\gamma^\nu\delta_\epsilon\lambda - \tfrac{1}{2}\overline{\epsilon'}[G^{\nu\sigma} - \mathcal{T}^{\nu\sigma}]\gamma_\sigma\epsilon$$

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$$-\tfrac{1}{12}\overline{\epsilon}'\gamma^{\mu\rho}\gamma^\nu\gamma^{\beta\gamma}[\partial_{[\mu}H_{\rho\beta\gamma]} + \sqrt{2}P^m_{[\mu}K^m_{\rho\beta\gamma]} - \tfrac{3}{2}v_r c^r \,\mathrm{tr}\, F_{[\mu\rho}F_{\beta\gamma]}]\epsilon$$
$$+\tfrac{1}{2}\overline{\epsilon}'[\nabla_\alpha H^{\alpha\nu\sigma} - \sqrt{2}P^m_\alpha K^{m\,\alpha\nu\sigma} - \tfrac{1}{4}\epsilon^{\nu\sigma\alpha\beta\gamma\delta}v_r c^r F_{\alpha\beta}F_{\gamma\delta}]\gamma_\sigma\epsilon \ . \qquad (3.10)$$

We wish to point out that this is an exact expression, where only kinematics has been used in rewriting the divergence. The last two lines are related to the self-dual H equation of motion (in Bianchi identity and divergence form respectively), and hence vanish on-shell. In addition to the expected terms, this divergence has the unusual feature in that the full stress tensor $\mathcal{T}_{\mu\nu}$ arising from the supersymmetry manipulations is modified by the inclusion of an antisymmetric contribution

$$
\begin{aligned}
\mathcal{T}_{\mu\nu} &= T_{\mu\nu} + T'_{\mu\nu} \\
&= T_{\mu\nu} - 2v_r c^r \,\mathrm{tr}\,[F_{\mu\alpha}F_\nu{}^\alpha - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \tfrac{1}{8}\epsilon_{\mu\nu\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta}] \ , \qquad (3.11)
\end{aligned}
$$

where $T_{\mu\nu}$, given in (2.6), is the symmetric stress tensor appearing in Einstein's equation. In particular, this antisymmetric component, which arises as a consequence of the $N = (1,0)$ supersymmetry algebra in six dimensions [34], is related to the fact that the classical equations of motion, Eqns. (2.4), are actually inconsistent in such a manner as to cancel the effects of the gauge anomalies when loop corrections are taken into account [35, 36]. As a result, the equations violate Bose symmetry in a way that would be impossible if they were derivable from a Lagrangian. There exists a Lagrangian, at least in the case $n_T = 1$, which automatically leads to Bose symmetric equations but which lacks gauge invariance [14]. As discussed in [35, 36], these two formulations are related to the difference between consistent and covariant anomalies. It is interesting to note, however, that the gauge dyonic string solves both sets of equations, since the Bose non-symmetric terms vanish in this background.

We are now in a position to examine the conditions under which the Bogomol'nyi bound, Eqn. (3.7), may hold. Based on the rewriting of the Bogomol'nyi equation in terms of a volume integral, it is apparent that the mass bound will hold provided the divergence $\nabla_\mu N^{\mu\nu}$ is positive semi-definite over the entire transverse space $\mathcal{M}$. This gives rise to the following three conditions: i) the supergravity equations of motion must be satisfied[3], ii) Witten's condition must hold globally so the gravitino variation is non-negative, and iii) the Yang-Mills contributions from both the gaugino variation and the correction $T'_{\mu\nu}$ to the stress tensor must be non-negative. While the first condition is straightforward, the other two require further explanation. Witten's condition [33] is essentially a spatial Dirac

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[3]Only Einstein's equation and the H equation of motion are relevant for the Bogomol'nyi calculation. Note that when we refer to Einstein's equation, we do not include the correction $T'_{\mu\nu}$ which is accounted for separately.

equation, $\gamma^i \delta_\epsilon \psi_i = 0$, where $i = 1, \dots, 5$. While this condition may be satisfied for a well behaved background, it is also important to ensure that such spinors are normalizable on all of $\mathcal{M}$ so that the divergence integral is well defined. In particular, this normalizability condition apparently breaks down in the presence of naked singularities, as we subsequently verify for the gauge dyonic string solution. This leads us to believe that Witten's condition is essentially equivalent to demanding that the background contains no naked singularities.

We now turn to the conditions that need to be imposed on the Yang-Mills fields. Looking at the gaugino variation in Eqn. (3.10), it is natural to impose the condition that all components of the n_V dimensional vector $v_r c^r$ are to be non-negative. Since the n_T scalars encoded in the vielbein v_r act as gauge coupling constants, this condition simply states that the Yang-Mills fields must have the correct sign kinetic terms. Starting from a weakly coupled point in moduli space, it is apparent that the only way to generate a wrong sign term is to pass through infinite coupling. Since this corresponds to a phase transition [14], driven by tensionless strings [13, 11, 1], it indicates that the Bogomol'nyi results need to be applied with care when discussing the strong coupling dynamics of six dimensional strings.

Since the Yang-Mills fields lead to a modification of the stress tensor, it is also necessary to require that $T'_{\mu\nu}$ enters non-negatively into the divergence of Nester's form. For a string-like geometry in the 0-1 direction, this condition is equivalent to demanding that $-T'_{00} \geq |T'_{01}| \geq 0$, which is automatically satisfied for gauge fields living only in transverse space (again provided $v_r c^r$ is non-negative). To see this, note that for $\mu, \nu = 0, 1$ we may write

$$T'_{\mu\nu} = \tfrac{1}{2} v_r c^r \, \mathrm{tr} \left[g_{\mu\nu} F_{mn} F^{mn} + \epsilon_{\mu\nu} F_{mn} *_4 F^{mn} \right] , \tag{3.12}$$

and use the instanton argument, $\mathrm{tr}\,(F \pm *_4 F)^2 \geq 0$, to show that the T' conditions are satisfied. Therefore as long as the Yang-Mills fields vanish in the longitudinal directions of the string-like solution, no further condition is necessary. It is perhaps not coincidental that this vanishing of the gauge fields on the string also renders unimportant the inconsistency of the classical equations of motion.

3.1 Supersymmetry of the gauge dyonic string

It is instructive to see how the Bogomol'nyi equation breaks down in the various limits of the gauge dyonic string. For this string background, given by (2.15), the supersymmetry variations of the fermions, (2.8), become

$$
\begin{aligned}
\delta\psi_\mu &= -\gamma^n \partial_n A \gamma_\mu \mathcal{P}_2^+ \epsilon \\
\delta\psi_m &= \gamma^n \partial_n A \gamma_m \mathcal{P}_2^+ \epsilon + e^{A/2} \partial_m (e^{-A/2} \epsilon)
\end{aligned}
$$

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$$\delta\chi = -i\gamma^n\partial_n\phi\mathcal{P}_2^+\epsilon$$
$$\delta\lambda = -\frac{1}{2\sqrt{2}}F_{mn}\gamma^{mn}\mathcal{P}_2^+\epsilon , \qquad (3.13)$$

where $\mathcal{P}_2^+ = \frac{1}{2}(1 + \gamma^{\overline{01}})$ is a projection onto the chiral two-dimensional world-sheet of the string-like solution (overlined symbols indicate tangent-space indices). This indicates, as noted in [1], that the Killing spinor equations are solved for spinors ϵ satisfying

$$\mathcal{P}_2^+\epsilon = 0 , \qquad \epsilon = e^{A/2}\epsilon_0 . \qquad (3.14)$$

On the other hand, the fermion zero modes are given by spinors ϵ surviving the projection, namely $\mathcal{P}_2^+\epsilon = \epsilon$. Note that for the zero modes there is no further condition on ϵ.

Based on the above supersymmetry variations, we may explicitly calculate the divergence of Nester's expression. Since this expression obviously vanishes for Killing spinors, we only concern ourselves with the fermion zero modes. For simplicity in working with the derivative term entering $\delta\psi_m$, we assume a simple scaling so that ϵ is given by

$$\epsilon = e^{\alpha A}\epsilon_0 , \qquad \mathcal{P}_2^+\epsilon = \epsilon , \qquad (3.15)$$

where ϵ_0 is a constant spinor. Working out the divergence then gives

$$\int_{\mathcal{M}} \nabla_\mu N^{\mu\nu}d\Sigma_\nu = 2\pi^2\,\epsilon_0^\dagger\epsilon_0\int e^{2(\alpha-\frac{1}{2})A}[4(\alpha-\tfrac{1}{2})\partial_m A\partial_m A + 2\partial_m\partial_m A]r^3 dr$$
$$= 2\pi^2\,\epsilon_0^\dagger\epsilon_0\frac{1}{\alpha-\frac{1}{2}}\int r^3 dr\,\partial_m\partial_m e^{2(\alpha-\frac{1}{2})A} , \qquad (3.16)$$

where the last line holds for $\alpha \neq \frac{1}{2}$ and is in fact a total derivative, which is not surprising considering the origin of this expression. Substituting in the explicit function $A(r)$, we then find

$$2\pi^2\,\epsilon_0^\dagger\epsilon_0[\frac{GM}{2} - Z] = \int_{\mathcal{M}}\nabla_\mu N^{\mu\nu}d\Sigma_\nu = 2\pi^2\,\epsilon_0^\dagger\epsilon_0[Pe^{-\phi_0} + Qe^{\phi_0}] , \qquad (3.17)$$

which is independent of α as expected. Combining this with $[\frac{GM}{2} + Z] = 0$ appropriate to Killing spinors then gives an explicit derivation of the Bogomol'nyi bound,

$$GM = -2Z = Pe^{-\phi_0} + Qe^{\phi_0} , \qquad (3.18)$$

for the gauge dyonic string.

So far we have not addressed the issue of what conditions are necessary to ensure the validity of the Bogomol'nyi bound. While the equations of motion are satisfied by construction, both Witten's condition and the positivity of the gauge function are not guaranteed. Examining first Witten's condition, we find

$$\gamma^i\delta_\epsilon\psi_i = \gamma^n\partial_n A[\alpha - \tfrac{1}{2} - \mathcal{P}_2^+]\epsilon . \qquad (3.19)$$

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Therefore, for Killing spinors, we choose $\alpha = \frac{1}{2}$ as noted previously in order to satisfy Witten's condition. On the other hand, we must choose $\alpha = \frac{3}{2}$ for the case of the fermion zero modes. Provided there are no naked singularities, this value of α gives rise to a well-behaved integral, so that there is no problem satisfying Witten's condition. However this is no longer the case whenever there are naked singularities. To see this, we note that such naked singularities develop whenever $2Pe^{-\phi_0} \le -\rho^2$ or $2Qe^{\phi_0} \le -\rho^2$ so that e^{-2A} vanishes for some $r^2 \ge 0$ [1]. Convergence of the volume integral near the singularity then requires $\alpha < -\frac{3}{2}$ (or $\alpha < -\frac{1}{2}$ for the case $Pe^{-\phi_0} = Qe^{\phi_0}$) which clearly indicates the incompatibility of Witten's condition with normalizable fermion zero modes whenever naked singularities are present.

Note that for any value of the mass given by Eqn. (3.18), it is always possible to avoid naked singularities in the gauge dyonic string by choosing a sufficiently large instanton size ρ. Therefore evasion of the Bogomol'nyi bound, Eqn. (3.7), is possible even without singularities. Whenever $M < 0$ we may see that the breakdown in Bogomol'nyi occurs because the Yang-Mills couplings have the wrong sign (this is already obvious because M itself is related to the gauge coupling at infinity). A quick check shows that this breakdown is also present for the tensionless ($M = 0$) quasi-anti-self-dual string where there is an exact cancellation between the contributions from the graviton and tensor multiplet fields and the wrong sign Yang-Mills fields. As shown below, this cancellation continues to hold when examining the energy integral for the tensionless string in the flat-space limit.

4   The flat-space limit

If the charges P and Q are such that $P = Qe^{2\phi_0}$, the anti-self-dual 3-form field strength and the dilaton decouple, *i.e.* $K^m_{\mu\nu\rho} = 0$, $\phi = \phi_0$. In other words, the matter multiplet decouples in this case, and we recover the self-dual string of [17]. On the other hand if $P = -Qe^{2\phi_0}$, the dyonic string becomes massless, as can be seen from (2.18). At first sight, one might think that in this case the self-dual 3-form $H_{\mu\nu\rho}$ and the metric of the gravity multiplet would be decoupled. However, this is not in fact what happens. This can easily be seen from the fact that the metric (2.15) does not become flat: indeed the $1/r^2$ terms cancel asymptotically, as they must since the solution is now massless, but the metric still has non-vanishing asymptotic deviations from Minkowski spacetime of order $1/r^4$. Similarly, the self-dual 3-form $H_{\mu\nu\rho}$ falls off as $1/r^4$. On the other hand, the fields $K^m_{\mu\nu\rho}$ and $\phi - \phi_0$ fall off as $1/r^2$ at large r. For this reason, the dyonic string in this limit should more

appropriately be called quasi-anti-self-dual [1], rather than anti-self-dual. However, the solution becomes anti-self-dual asymptotically, since the self-dual part of the 3-form falls off faster by a factor of $1/r^2$.

The above discussion suggests that it should be possible to take the flat-space limit of the $N = (1,0)$ supergravity theory, and the quasi-anti-self-dual solution, where Newton's constant is set to zero. In fact, as we shall show below, there are actually two distinct limits that can be taken, yielding two inequivalent flat-space theories. To show this, we shall first construct the flat-space limit of the more general $N = 1$ supergravity coupled to an arbitrary number of anti-self-dual fields $K^m_{\mu\nu\rho}$. To do this, it is convenient to re-introduce Newton's constant κ in the supergravity theory, by rescaling the fields of the tensor multiplet in the following manner:

$$V = \begin{bmatrix} v_0 & v_M \\ x^m{}_0 & x^m{}_M \end{bmatrix} \longrightarrow \begin{bmatrix} v_0 & \kappa v_M \\ \kappa x^m{}_0 & x^m{}_M \end{bmatrix} , \tag{4.1}$$
$$B^M_{\mu\nu} \longrightarrow \kappa\, B^M_{\mu\nu} ,$$
$$\chi^m \longrightarrow \kappa\, \chi^m ,$$

while the fields of the Yang-Mills multiplet have not been rescaled, the coupling constants c^r are naturally dimensionless in the global limit, and hence must be rescaled according to

$$c^r \longrightarrow \kappa\, c^r . \tag{4.2}$$

Note that $P^m_\mu \to \kappa P^m_\mu$ under the rescalings. As a result of this rescaling, the equations of motion for the supergravity fields become

$$
\begin{aligned}
G_{\mu\nu} - H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} &= \kappa^2[K^m_{\mu\rho\sigma}K^{m\,\rho\sigma}_\nu + 2(P^m_\mu P^m_\nu - \tfrac{1}{2}g_{\mu\nu}P^m_\rho P^{m\,\rho}) \\
&\quad + \kappa[4(v_0 c^0 + \kappa v_M c^M)\,\mathrm{tr}\,(F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma})] \\
dH &= -\kappa^2\sqrt{2}P^m K^m + \kappa(v_0 c^0 + \kappa v_M c^M)\,\mathrm{tr}\,F^2 \\
\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho + H^{\mu\nu\rho}\gamma_\nu\psi_\rho &= \kappa^2[\tfrac{i}{2}K^{m\,\mu\nu\rho}\gamma_{\nu\rho}\chi^m - \tfrac{i}{\sqrt{2}}P^m_\nu\gamma^\mu\gamma^\nu\chi^m] \\
&\quad - \kappa[\tfrac{1}{\sqrt{2}}\gamma^{\sigma\tau}\gamma^\mu(v_0 c^0 + \kappa v_M c^M)\,\mathrm{tr}\,F_{\sigma\tau}\lambda] ,
\end{aligned}
\tag{4.3}
$$

where now $H = v_0\mathcal{H}^0 + \kappa^2 v_M\mathcal{H}^M$, indicating that in the limit $\kappa \to 0$ we may consistently set the gravity fields to their flat-space backgrounds,

$$g_{\mu\nu} \to \eta_{\mu\nu}, \qquad B^0_{\mu\nu} \to 0, \qquad \psi_\mu \to 0 . \tag{4.4}$$

Note that the terms proportional to c^0 (the coupling of Yang-Mills to the self-dual H) enter at $O(\kappa)$. This suggests the possibility that two different limits can arise; one where c^0/κ is

held fixed, and the other where c^0 is non-vanishing and held fixed, as κ goes to zero. This may be made more transparent by examining the Yang-Mills equation of motion

$$D^\mu[(v_0 c^0 + \kappa v_M c^M) F_{\mu\nu}] \;=\; H_{\nu\rho\sigma}(v_0 c^0 + \kappa v_M c^M) F^{\rho\sigma} + \kappa K^m_{\nu\rho\sigma}(\kappa x^m{}_0 c^0 + x^m{}_M c^M) F^{\rho\sigma} \;,$$

$$(4.5)$$

from which we see that the $O(\kappa^0)$ terms survive only in the second limit, whilst the equation is of order κ in the first limit. Before proceeding with the flat-space limits, we note that the constrained vielbein matrix V simplifies greatly in the $\kappa \to 0$ limit, and the n_T degrees of freedom can be parametrized by scalar fields ϕ^m defined by $\delta^m_M v_M = x^m{}_0 = \phi^m$. The other components of V simply become $v_0 = 1$ and $x^m{}_M = \delta^m_M$. The two flat-space limits arise as follows:

Flat-space limit with c^0/κ fixed:

In this limit, it is natural to define $\tilde c^0 = c^0/\kappa$ before taking the flat-space limit. We see that there are now no κ-independent terms in (4.5), and we obtain a Yang-Mills equation that includes interactions with the anti-self-dual matter multiplets. We find that the complete set of flat-space equations is

$$\begin{aligned}
\Box \phi^m &= c^m \,\mathrm{tr}\,(F_{\mu\nu} F^{\mu\nu}) \;, \\
\partial^\mu K^m_{\mu\nu\rho} &= -\tfrac{1}{4} c^m \,\epsilon_{\mu\nu\rho\alpha\beta\gamma}\,\mathrm{tr}\,(F^{\mu\alpha} F^{\beta\gamma}) \;, \\
\gamma^\mu \partial_\mu \chi^m &= -\tfrac{i}{\sqrt 2} c^m \,\mathrm{tr}\,(F_{\mu\nu}\gamma^{\mu\nu}\lambda) \;, \\
D^\mu[(\tilde c^0 + c^m \phi^m)\, F_{\mu\nu}] &= c^m F^{\rho\sigma} K^m_{\nu\rho\sigma} \;, \\
(\tilde c^0 + c^m \phi^m)\gamma^\mu D_\mu \lambda &= -\tfrac{1}{2} c^m (\partial_\mu \phi^m)\gamma^\mu \lambda - \tfrac{i}{2\sqrt 2} c^m F_{\mu\nu}\gamma^{\mu\nu}\chi^m - \tfrac{1}{12} c^m K^m_{\mu\nu\rho}\gamma^{\mu\nu\rho}\lambda \;.
\end{aligned}$$

$$(4.6)$$

Note that the anti-self-dual field strengths are given by

$$K^m = dB^m + c^m \omega_3 \;,$$

$$(4.7)$$

where $\omega_3 = AdA + \tfrac{2}{3}A^3$. The supersymmetry transformation rules in this flat-space limit become

$$\begin{aligned}
\delta\phi^m = \bar\epsilon\chi^m \;, \qquad & \delta A_\mu = -\tfrac{i}{\sqrt 2}\bar\epsilon\gamma_\mu\lambda \;, \qquad \delta B^m_{\mu\nu} = \tfrac{1}{2}\bar\epsilon\gamma_{\mu\nu}\chi^m + 2c^m \,\mathrm{tr}\,(A_{[\mu}\delta A_{\nu]}) \;, \\
\delta\lambda = -\tfrac{1}{2\sqrt 2} F_{\mu\nu}\gamma^{\mu\nu}\epsilon \;, \qquad & \delta\chi^m = -\tfrac{i}{2}\partial_\mu\phi^m\,\gamma^\mu\epsilon + \tfrac{i}{12}K^m_{\mu\nu\rho}\,\gamma^{\mu\nu\rho}\epsilon \;.
\end{aligned}$$

$$(4.8)$$

The energy-momentum tensor for this flat-space theory may be obtained simply by applying the same limiting procedure to the right-hand side of the Einstein equation of

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the original supergravity theory, given in (4.3). By this means we obtain the flat-space expression

$$T_{\mu\nu} = K_{\mu\rho\sigma}^m \, K_\nu^{m\rho\sigma} + \partial_\mu \phi^m \, \partial_\nu \phi^m - \tfrac{1}{2}\eta_{\mu\nu} \, (\partial\phi^m)^2 + 4(\tilde{c}^0 + c^m \, \phi^m) \, \mathrm{tr} \, (F_{\mu\lambda} \, F_\nu^{\;\lambda} - \tfrac{1}{4}\eta_{\mu\nu} \, F_{\lambda\sigma} \, F^{\lambda\sigma}) \,.$$
$$(4.9)$$

This is the theory that we refer to as the "interacting theory". It is interesting to note that the bosonic equations of motion of (4.6) can be derived from the Lagrangian

$$\mathcal{L} = -(\partial\phi^m)^2 - \tfrac{1}{3}K^2 - 2(\tilde{c}^0 + c^m \, \phi^m) \, \mathrm{tr} \, (F^2) - 2c^m \, *(B^m \wedge \, \mathrm{tr} \, (F \wedge F)) \,, \qquad (4.10)$$

where K is taken to be unconstrained, with its anti-self-duality being imposed only after having obtained the equations of motion.

Flat-space limit with c^0 held fixed:

The situation is different when c^0 is non-vanishing and is held fixed when κ goes to zero. As can be seen from (4.5), the leading-order terms in the Yang-Mills equation are now independent of κ, and in fact there are now no interactions with the anti-self-dual multiplets in the $\kappa \to 0$ limit. All equations of motion, and supersymmetry transformation rules, remain the same as in the previous $c^0 \sim \kappa$ limit with the exception of the Yang-Mills equations and the gaugino equation, which are now source-free and given by

$$D^\mu \, F_{\mu\nu} = 0 \,,$$
$$\gamma^\mu \, D_\mu \, \lambda = 0 \,. \qquad (4.11)$$

Note however that the energy-momentum tensor, again obtained from the right-hand side of the Einstein equation in (4.3) by applying the limiting procedure, is now simply given by

$$T_{\mu\nu} = 4c^0 \, \mathrm{tr} \, (F_{\mu\lambda} \, F_\nu^{\;\lambda} - \tfrac{1}{4}\eta_{\mu\nu} \, F_{\lambda\sigma} \, F^{\lambda\sigma}) \,. \qquad (4.12)$$

In the case of a single self-dual tensor multiplet, this is the theory that we refer to as the "BSS theory".

A word of explanation is in order here. Firstly, it should be noted that this energy-momentum tensor arose as a term of order κ in the Einstein equation, rather than the usual order κ^2 for matter fields. Consequently the Yang-Mills contribution dominates the $O(\kappa^2)$ contributions from the tensor multiplets, and so they are absent in this flat-space limit. Indeed, it is evident that if one were to add "standard" contributions for the fields of the tensor multiplets, one would find that the resulting energy-momentum tensor was not conserved upon using the equations of motion. Effectively the tensor multiplets describe

"test fields" in a Yang-Mills background, whose energy-momentum tensor is negligible in comparison to that of the Yang-Mills field. For the same reason, they do not affect the Yang-Mills equation. The energy-momentum tensor (4.12) would cease to be appropriate in a configuration where the Yang-Mills field was zero, since now the previously-neglected matter contributions would become important. This rather pathological feature of the BSS theory is reflected also in the fact that it cannot be described by an analogue of the Lagrangian (4.10), owing to the inherent asymmetry between the occurrence of interaction terms in the matter and Yang-Mills equations.

A number of further comments are also in order. Firstly, it should be emphasised that the higher-order fermi terms are not included in the equations of motion and supersymmetry transformation rules (4.6) and (4.8); they were not included in [26,27], and indeed they have only recently been computed [37]. (See also [36,38].) Nevertheless, one can see on general grounds that the inclusion of the higher-order terms in the supergravity theory will not present any obstacle in the taking of the two inequivalent flat-space limits. Alternatively, the higher-order completion of the supersymmetry transformations may be determined in either of the flat-space theories by demanding the closure of the supersymmetry algebra on the fermi fields. For the interacting theory the supersymmetry transformation rules for the bosons remain unchanged, while the complete transformation rules for the fermions are

$$
\begin{aligned}
\delta\chi^m &= -\tfrac{i}{2}[\partial_\mu\phi^m\gamma^\mu - \tfrac{1}{6}K^m_{\mu\nu\rho}\gamma^{\mu\nu\rho}]\epsilon - \tfrac{1}{2}c^m\operatorname{tr}\left[\gamma_\mu\lambda(\bar\epsilon\gamma^\mu\lambda)\right], \qquad (4.13)\\
\delta\lambda &= -\tfrac{1}{2\sqrt{2}}F_{\mu\nu}\gamma^{\mu\nu}\epsilon + \frac{c^m}{(\tilde{c}^0 + c^n\phi^n)}\left[-\tfrac{1}{2}(\bar\chi^m\lambda)\epsilon - \tfrac{1}{4}(\bar\chi^m\epsilon)\lambda + \tfrac{1}{8}(\bar\chi^m\gamma_{\mu\nu}\epsilon)\gamma^{\mu\nu}\lambda\right],
\end{aligned}
$$

and agree with the flat-space limit of the transformations in the supergravity theory [37]. On the other hand, in the BSS theory the gaugino variation remains unmodified, and only $\delta\chi^m$ picks up a higher-order correction (identical to that of the interacting theory). Note that it is straightforward to see that this must be the case, since the lowest-order transformation for the Yang-Mills multiplet, (4.8), already closes on the source-free gaugino equation of motion, (4.11). So in fact we see that the only difference in the supersymmetry transformation rules in the two flat-space limits is in the higher-order terms in the gaugino variation, consistent with the difference in the equations of motion for the Yang-Mills multiplet between the two limits.

The complete gaugino transformation rule in the interacting theory is somewhat unusual, in that it contains a possibly singular denominator, $(\tilde{c}^0 + c^n\phi^n)$ (which was also noted in [36,37]). As in the supergravity situation, this singular denominator is just a manifestation of the strong coupling singularity already present in the lowest-order Yang-Mills equations.

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This form of the denominator also shows up in the complete equations of motion, given for the fermi fields in the interacting theory by[4]

$$
\gamma^\mu \partial_\mu \chi^m = -\frac{i}{\sqrt{2}} c^m \, \mathrm{tr}\,(F_{\mu\nu}\gamma^{\mu\nu}\lambda) + \frac{i c^m c^n}{(\tilde{c}^0 + c^p \phi^p)} \, \mathrm{tr}\,[\tfrac{3}{2}(\overline{\chi}^n \lambda)\lambda - \tfrac{1}{4}(\overline{\chi}^n \gamma_{\mu\nu}\lambda)\gamma^{\mu\nu}\lambda] \ ,
$$

$$
\begin{aligned}
(\tilde{c}^0 + c^m \phi^m)\gamma^\mu D_\mu \lambda = \ & -\tfrac{1}{2} c^m (\partial_\mu \phi^m)\gamma^\mu \lambda - \frac{i}{2\sqrt{2}} c^m \, F_{\mu\nu}\gamma^{\mu\nu}\chi^m - \tfrac{1}{12} c^m K^m_{\mu\nu\rho}\gamma^{\mu\nu\rho}\lambda \\
& - i\frac{c^m c^n}{(\tilde{c}^0 + c^p \phi^p)}[\tfrac{3}{4}(\overline{\lambda}\chi^m)\chi^n - \tfrac{1}{8}(\overline{\lambda}\gamma_{\mu\nu}\chi^m)\gamma^{\mu\nu}\chi^n] \\
& + i\alpha c^m c^{m\prime} \, \mathrm{tr}\,'[(\overline{\lambda}\gamma_\mu \lambda')\gamma^\mu \lambda'] \ ,
\end{aligned}
\tag{4.14}
$$

where the primes in the last line indicate the quantities involved in the trace. (Recall that there can be different c^m constants for each factor in a semi-simple group.) Note that α is an arbitrary parameter that is not fixed by the supersymmetry algebra [37], and appears to be related to the gauge anomaly (see [37] for a more complete discussion).

We also note that the flat-space limit when c^0 is non-vanishing and held fixed, if we specialise to the case where there is only a single anti-self-dual multiplet, coincides with the BSS theory, constructed in [18]. It was argued in [18] that this theory could not be obtained as a $\kappa \to 0$ limit of the supergravity theory, on the grounds that the Chern-Simons form ω enters the 3-form field strengths in (2.3) with a factor of κ (after restoring Newton's constant, as in (4.2)), and thus it would disappear in the flat-space limit. However, while this is indeed the case for the self-dual field of the gravity multiplet, the potentials $B^m_{\mu\nu}$ for the anti-self-dual matter fields also acquire factors of κ, with the net result that the Chern-Simons terms are of the same order, and hence they survive in the $\kappa \to 0$ limit, as we saw in (4.7) above. The non-standard dimensions of the energy-momentum tensor (4.12) is a reflection of the need for a dimensionful free parameter, which was also seen in [18]. Finally, we remark that the more general flat-space theory we obtained in the limit where $c^0 \sim \kappa$, does not conflict with the results in [18] which found only the free Yang-Mills equations (4.11), since in [18] it was assumed that the kinetic term for the Yang-Mills multiplet was described by the standard superspace free action. Note also that the BSS theory can be obtained from the interacting flat-space theory by taking k to zero after making the following rescalings of the fields of the interacting theory:

$$
\phi^m \to k\,\phi^m \ , \qquad B^m_{\mu\nu} \to k\,B^m_{\mu\nu} \ , \qquad \chi^m \to k\,\chi^m \ ,
\tag{4.15}
$$

together with the rescaling $c^m \to k\,c^m$. Thus the interacting flat-space theory encompasses the BSS theory as a singular limiting case.

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[4]While these equations of motion were obtained by taking the flat-space limit of [37], they equally well follow from closure of the supersymmetry algebra, (4.13).

Let us now consider the flat-space limit of the quasi-anti-self-dual dyonic string solution (2.15) of the supergravity theory. This solution is massless, and hence from (2.18) it follows that the magnetic charge is related to the electric charge by $P = -Q\, e^{2\phi_0}$. Consequently, the parameters $c^0 = (Q + P)/32$ and $c^1 = (Q - P)/32$ are given by

$$c^0 = -\tfrac{1}{16} Q\, e^{\phi_0} \sinh \phi_0 \ , \qquad c^1 = \tfrac{1}{16} Q\, e^{\phi_0} \cosh \phi_0 \ . \tag{4.16}$$

In the flat-space limit, where in particular ϕ was rescaled by κ, we see that $c^0 = -\tfrac{1}{16} Q\, \kappa\, \phi_0$ prior to sending κ to zero, and hence we are in the regime of the "interacting theory", corresponding to the first of the two limits discussed above. We find that the flat-space solution is given by

$$\begin{aligned}
\phi &= \phi_0 - \frac{Q(2\rho^2 + r^2)}{(\rho^2 + r^2)^2} \\
K_{mnp} &= -\tfrac{1}{2}\epsilon_{mnpq}\partial_q\phi \ , \qquad K_{\mu\nu m} = -\tfrac{1}{2}\epsilon_{\mu\nu}\partial_m\phi \ , \\
F^a &= \frac{2\rho^2}{(\rho^2 + r^2)^2}\, \eta^a_{mn}\, dy^m \wedge dy^n \ .
\end{aligned} \tag{4.17}$$

Note that ϕ_0 no longer has physical significance as a coupling constant, and it can be eliminated by making a constant shift of ϕ.

Since this solution has been obtained as the flat-space limit of a tensionless string, we expect that it should have vanishing energy. This might at first sight seem surprising, since it is described by a non-trivial field configuration. However, a straightforward calculation of T_{00} given by (4.9) yields

$$\begin{aligned}
T_{00} &= K_{00}^2 + \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{16}Q\,(\phi - \phi_0)\, \mathrm{tr}\,(F^2) \ , \\
&= \frac{4Q^2\, r^2\,(3\rho^2 + r^2)^2}{(\rho^2 + r^2)^6} - \frac{24Q^2\, \rho^4\,(2\rho^2 + r^2)}{(\rho^2 + r^2)^6} \ ,
\end{aligned} \tag{4.18}$$

where the first term in the second line comes from the (equal) contributions from K and ϕ, and the second term comes from F. It is easily verified that while T_{00} itself is non-vanishing, the integral $\int_0^\infty T_{00}\, r^3\, dr$ is equal to zero. Clearly the Yang-Mills field is giving a negative contribution to the energy, in precisely such a way that the total energy is zero. This is the flat-space analogue of the cancellation that occurs in the supergravity theory, with its associated subtleties in the Bogomol'nyi analysis, which we discussed at the end of section 3.

It should be emphasised that the vanishing energy of the flat-space tensionless string occurs for arbitrary scale size ρ of the Yang-Mills instanton. However, if we consider instead the neutral tensionless string, which can be achieved by setting $\rho = 0$ so that the instanton is not present, then the expression (4.18) becomes $T_{00} = 4Q^2/r^6$, whose integral over the

transverse space diverges at the core of the string. Thus the Yang-Mills instanton in the gauge-dyonic string can be viewed as a regulator for the total energy.

There are also massive string solutions to the interacting flat-space theory, which can also be obtained as flat-space limits of the curved-space gauge dyonic string. They arise by taking the ADM mass, as given by (2.18), to be non-zero and of the form $m_0 \kappa$. Upon taking the $\kappa \to 0$ flat-space limit, this gives a solution of the same form as (4.17), but with ϕ shifted by the constant m_0. From (4.18), this gives an extra term in T_{00} which gives rise to an energy m_0 per unit length for the flat-space string.

It is interesting to note that while the flat-space limit of the tensionless string always results in the $c^0 \sim \kappa$ limit of the interacting theory, the final solution itself, as given in (4.17), also satisfies the equations of motion of the BSS theory, where c^0 is held fixed in the flat-space limit[5]. To see this, we note that for a bosonic background, only the Yang-Mills equation differs between the two flat-space theories. In particular, both Yang-Mills equations may be expressed as $D^\mu F_{\mu\nu} = J_\nu$, where the current is

$$J_\mu = [c^m F_{\mu\nu} \partial^\nu \phi^m + c^m F^{\rho\sigma} K^m_{\mu\rho\sigma}]/[\tilde{c}^0 + c^n \phi^n] \,, \tag{4.19}$$

for the first theory, and vanishes for the latter. Because of the form of the solution, (4.17), we see that J_μ identically vanishes, and hence the background is indeed a solution to both flat-space limits. Furthermore, examination of the BPS conditions arising from (4.8) indicates that $J_\mu = 0$ for any string-like background preserving half of the supersymmetries. It should be remarked, however, that when interpreted as a solution to the c^0 fixed flat-space limit, the string no longer has vanishing energy per unit length, since in this case the stress tensor (4.12) has only a positive contribution from the Yang-Mills instanton. Thus only the interacting theory from the first flat-space limit, (4.6), provides a suitable description of the tensionless string in flat space.

Finally, we note that by taking the divergence of (4.19), we obtain

$$D^\mu J_\mu = \tfrac{1}{8} c^m c^{m\prime} \epsilon_{\mu\nu\rho\sigma\eta\lambda} F^{\mu\nu} \operatorname{tr}' F^{\rho\sigma\prime} F^{\eta\lambda\prime}/[\tilde{c}^0 + c^n \phi^n] \,, \tag{4.20}$$

indicating that the current is not conserved classically. Thus the inconsistency of the supergravity theory, which we discussed in section 3, survives in the "interacting" flat-space limit. Nevertheless since, as for the gauge dyonic string in curved space, J_μ vanishes identically for the global gauge string, this classical inconsistency does not spoil the solution.

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[5]The solution (4.17) has also been obtained in the BSS theory by directly solving its first-order BPS equations [39].

On the other hand, since the BSS theory is free of this inconsistency it is possible that such a classical inconsistency, necessary for anomaly cancellation in the quantum theory, is an integral part of a fully interacting theory.

We have not paid much attention in this paper to the question of gravitational anomalies which is always an important issue when dealing with *chiral* theories. In particular, we have for simplicity ignored the presence of hypermultiplets. A coupled supergravity-matter theory which is initailly free of gravitational anomalies theory will not remain so when the gravity multiplet is switched off because the contributions from the gravitino and self-dual 2-form, necessary for the anomaly cancellation, are no longer present. Naively, of course, one could argue that gravitational anomalies are no longer of any concern in the flat space limit. However, it may be that subtleties arise when one tries to take the global limit of a worldvolume theory which relies for its anomaly freedom on anomaly inflow from the bulk. This is deserving of further study.

In conclusion, we note that six-dimensional global models are also important as five-brane worldvolume theories [19, 20, 21, 18]. In the absence of Yang-Mills fields, the $(1, 0)$ multiplet is the only one available to describe the worldvolume theory of the $D = 7$, $N = 1$ fivebrane solution found in [40]. Six-dimensional global models also arise from configurations of higher-dimensional branes with six worldvolume dimensions in common. Indeed, the brane configurations yielding $(1, 0)$ theories with tensor multiplets, vector multiplets and hypermultiplets have been identified in [24, 25], although no field equations were written down. Here we speculate further that the interacting anti-self-dual-tensor Yang-Mills system given in this paper (together with hypermultiplets where necessary) is the appropriate one to describe these global models. The global gauge anti-self-dual string, and in particular the tensionless string, could then be regarded as strings on the worldvolume.

Note added

Global $D = 6$, $(1, 0)$ models of the type discussed in this paper have recently been shown to arise from configurations of NS fivebranes, Dirichlet sixbranes and eightbranes [41]

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