NEW LOWER BOUNDS FOR THE RANK OF MATRIX MULTIPLICATION

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Abstract. The rank of the matrix multiplication operator for $\mathbf{n}\times\mathbf{n}$ matrices is one of the most studied quantities in algebraic complexity theory. I prove that the rank is at least $3\mathbf{n}^2 - o(\mathbf{n}^2)$. More precisely, for any integer $p \leq \mathbf{n}-1$ the rank is at least $(3-\frac{1}{p+1})\mathbf{n}^2 - (1+2p\binom{2p}{p-1})\mathbf{n}$. The previous lower bound, due to Bläser, was $\frac{5}{2}\mathbf{n}^2 - 3\mathbf{n}$ (the case $p=1$). The new bounds improve Bläser's bound for all $\mathbf{n} > 84$. I also prove lower bounds for rectangular matrices significantly better than the the previous bound.

1. Introduction

Let $X = (x^i_j)$, $Y = (y^i_j)$ be $\mathbf{n}\times\mathbf{n}$-matrices with indeterminant entries. The *rank* of matrix multiplication, denoted $\mathbf{R}(M_{\langle \mathbf{n},\mathbf{n},\mathbf{n}\rangle})$, is the smallest number r of products $p_\rho = u_\rho(X)v_\rho(Y)$, $1\leq\rho\leq r$, where u_ρ, v_ρ are linear forms, such that the entries of the matrix product XY are contained in the linear span of the p_ρ. This quantity is also called the bilinear complexity of $\mathbf{n}\times\mathbf{n}$ matrix multiplication. More generally, one may define the rank $\mathbf{R}(b)$ of any bilinear map b, see §2.

From the point of view of geometry, rank is badly behaved as it is not semi-continuous. Geometers usually prefer to work with the *border rank* of matrix multiplication, which fixes the semi-continuity problem by fiat: the border rank of a bilinear map b, denoted $\underline{\mathbf{R}}(b)$, is the smallest r such that b can be approximated to arbitrary precision by bilinear maps of rank r. By definition, one has $\mathbf{R}(b) \geq \underline{\mathbf{R}}(b)$. A more formal definition is given in §2.

Let $M_{\langle \mathbf{n},\mathbf{m},\mathbf{l}\rangle}$ denote the multiplication of an $\mathbf{n}\times\mathbf{m}$ matrix by an $\mathbf{m}\times\mathbf{l}$ matrix. In [5] G. Ottaviani and I gave new lower bounds for the border rank of matrix multiplication, namely, for all $p\leq\mathbf{n}-1$, $\underline{\mathbf{R}}(M_{\langle \mathbf{n},\mathbf{n},\mathbf{m}\rangle}) \geq \frac{2p+1}{p+1}\mathbf{nm}$. Taking $p=\mathbf{n}-1$ gives the bound $\underline{\mathbf{R}}(M_{\langle \mathbf{n},\mathbf{n},\mathbf{m}\rangle}) \geq 2\mathbf{nm}-\mathbf{m}$. In this article it will be advantageous to work with a smaller value of p. The results of [5] are used here to prove:

Theorem 1.1. *Let $p\leq\mathbf{n}-1$ be a natural number. Then*

$$\mathbf{R}(M_{\langle \mathbf{n},\mathbf{n},\mathbf{m}\rangle}) \geq \frac{2p+1}{p+1}\mathbf{nm} + \mathbf{n}^2 - (1+2p\binom{2p}{p-1})\mathbf{n}.$$

The previous bound, due to Bläser [2], was $\mathbf{R}(M_{\langle \mathbf{n},\mathbf{n},\mathbf{m}\rangle}) \geq 2\mathbf{nm}-\mathbf{m}+2\mathbf{n}-2$. For square matrices Theorem 1.1 specializes to:

Theorem 1.2. *Let $p\leq\mathbf{n}-1$ be a natural number. Then*

$$\mathbf{R}(M_{\langle \mathbf{n},\mathbf{n},\mathbf{n}\rangle}) \geq (3-\frac{1}{p+1})\mathbf{n}^2 - (1+2p\binom{2p}{p-1})\mathbf{n}.$$

In particular, $\mathbf{R}(M_{\langle \mathbf{n},\mathbf{n},\mathbf{n}\rangle}) \geq 3\mathbf{n}^2 - o(\mathbf{n}^2)$.

The "in particular" follows by setting e.g., $p = \lfloor\sqrt{\log(\mathbf{n})}\rfloor$.

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This improves Bläser's bound [1] of $\frac{5}{2}\mathbf{n}^2 - 3\mathbf{n}$ (the case $p = 1$) for all $\mathbf{n} > 84$. Working under my direction, Alex Massarenti and Emanuele Raviolo [8, 7] improved the error term in Theorem 1.2. In a preprint of this article I made a mistake in computing the error term. Unfortunately this mistake was not noticed before Massarenti and Raviolo's paper [8] was published, repeating the error, although their contribution is completely correct and their correct bound will appear in [7].

Remark 1.3. If T is a tensor of border rank r, where the approximating curve of rank r tensors limits in such a way that q derivatives of the curve are used, then the rank of T is at most $(2q-1)r$, see [3, Prop. 15.26]. In [6] they give explicit, but very large upper bounds on the order of approximation h needed to write a tensor of border rank r as lying in the h-jet of a curve of tensors of rank r.

The language of tensors will be used throughout. In §2 the language of tensors is introduced and previous work of Bläser and others is rephrased in a language suitable for generalizations. In §3 I describe the equations of [5] and give a very easy proof of a slightly weaker result than Theorem 1.1. In §4 I express the equations in coordinates and prove Theorem 1.1. I work over the complex numbers throughout.

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2. Ranks and border ranks of tensors

Let A, B, C be vector spaces, of dimensions $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and with dual spaces A^*, B^*, C^*. That is, A^* is the space of linear maps $A \to \mathbb{C}$. Write $A^* \otimes B$ for the space of linear maps $A \to B$ and $A^* \otimes B^* \otimes C$ for the space of bilinear maps $A \times B \to C$. To avoid extra $*$-s, I work with bilinear maps $A^* \times B^* \to C$, i.e., elements of $A \otimes B \otimes C$. Let $T : A^* \times B^* \to C$ be a bilinear map. One may also consider T as a linear map $T : A^* \to B \otimes C$ (and similarly with the roles of A, B, C exchanged), or as a trilinear map $A^* \times B^* \times C^* \to \mathbb{C}$.

The *rank* of a bilinear map $T : A^* \times B^* \to C$, denoted $\mathbf{R}(T)$, is the smallest r such that there exist $a_1, \cdots, a_r \in A$, $b_1, \cdots, b_r \in B$, $c_1, \cdots, c_r \in C$ such that $T(\alpha, \beta) = \sum_{i=1}^r a_i(\alpha) b_i(\beta) c_i$ for all $\alpha \in A^*$ and $\beta \in B^*$. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that T may be written as a limit of a sequence of rank r tensors. Since the set of tensors of border rank at most r is closed, one can use polynomials to obtain lower bounds on border rank. That is, let P be a polynomial on $A \otimes B \otimes C$ such that P vanishes on all tensors of border rank at most r: if $T \in A \otimes B \otimes C$ is such that $P(T) \neq 0$, then $\underline{\mathbf{R}}(T) > r$.

The following proposition is a rephrasing of part of the proof in [1]:

Proposition 2.1. *Let P be a polynomial of degree d on $A \otimes B \otimes C$ such that $P(T) \neq 0$ implies $\underline{\mathbf{R}}(T) > r$. Let $T \in A \otimes B \otimes C$ be a tensor such that $P(T) \neq 0$ and $T : A^* \to B \otimes C$ is injective. Then $\mathbf{R}(T) \geq r + \mathbf{a} - d$.*

As stated, the proposition is useless, as the degrees of polynomials vanishing on on all tensors of border rank at most r are greater than r. (A general tensor of border rank r also has rank r.) However the conclusion still holds if one can find, for a given tensor T, a polynomial, or collection of polynomials on smaller spaces, such that the nonvanishing of P on T is equivalent to the non-vanishing of the new polynomials. Then one substitutes the smaller degree into the statement to obtain the nontrivial lower bound.

In our situation, first I will show $P(M_{\langle \mathbf{n},\mathbf{n},\mathbf{m} \rangle}) \neq 0$ if and only if $\tilde{P}(\tilde{M}) \neq 0$ where $\tilde{M}$ is a tensor in a smaller space of tensors and $\tilde{P}$ is a polynomial of lower degree than P, see (2). More precisely, note that in the course of the proof, $B \otimes C$ does not play a role, and we will see that the relevant polynomial, when applied to matrix multiplication $M \in A{\otimes}B{\otimes}C = A{\otimes}\mathbb{C}^{\mathbf{n}^2}{\otimes}\mathbb{C}^{\mathbf{n}^2}$, will not vanish if and only if a polynomial $\tilde{P}$ applied to $\tilde{M} \in A{\otimes}\mathbb{C}^{\mathbf{n}}{\otimes}\mathbb{C}^{\mathbf{n}}$ with $\deg(\tilde{P}) = \deg(P)/\mathbf{n}$, does not vanish, so the proof below works in this case. Then, in §4, I show that $\tilde{P}(\tilde{M}) \neq 0$ is implied by the non-vanishing of two polynomials of even smaller degrees.

This is why both Bläser's result and the result of this paper improve the bound of border rank by $\mathbf{a} = \mathbf{n}^2$ minus an error term, where Bläser improves Strassen's bound and I improve the bound of [5]. (Bläser shows Strassen's equations for border rank, when applied to the matrix multiplication tensor, are equivalent to the non-vanishing of three polynomials of degree $\mathbf{n}$, hence the error term of $3\mathbf{n}$. See [4, §11.5] for an exposition.)

To prove the Proposition, we need a standard Lemma, also used in [2], which appears in this form in [4, Lemma 11.5.0.2]:

Lemma 2.2. *Let $\mathbb{C}^{\mathbf{a}}$ be given a basis. Given a polynomial P of degree d on $\mathbb{C}^{\mathbf{a}}$, there exists a set of at most d basis vectors such that P restricted to their span is not identically zero.*

The lemma follows by simply choosing a monomial that appears in P, as it can involve at most d basis vectors.

Proof of Proposition 2.1. Let $\mathbf{R}(T) = r$ and assume we have written T as a sum of r rank one tensors. Since $T : A^* \to B{\otimes}C$ is injective we may write $T = T' + T''$ with $\mathbf{R}(T') = \mathbf{a}$, $\mathbf{R}(T'') = r - \mathbf{a}$ and $T' : A^* \to B{\otimes}C$ injective. Now consider the $\mathbf{a}$ elements of $A{\otimes}B{\otimes}C$ appearing in T'. Since they are linearly independent, by Lemma 2.2 we may choose a subset of d of them such that P, evaluated on the sum of terms in T whose A terms are in the span of these d elements, is not identically zero. Let T_1 denote the sum of the terms in T' not involving the (at most) d basis vectors needed for nonvanishing, so $\mathbf{R}(T_1) \geq \mathbf{a} - d$. Let $T_2 = T - T_1 + T''$. Now $\underline{\mathbf{R}}(T_2) \geq r$ because $P(T_2) \neq 0$. Finally $\mathbf{R}(T) = \mathbf{R}(T_1) + \mathbf{R}(T_2)$. □

Let $G(k,V) \subset \mathbb{P}\Lambda^k V$ denote the Grassmannian of k-planes through the origin in V in its Plücker embedding. That is, if a k plane is spanned by $v_1, \cdots, v_k$, we write it as $[v_1 \wedge \cdots \wedge v_k]$. One says a function on $G(k,V)$ is a polynomial of degree d if, as a function in the Plücker coordinates, it is a degree d polynomial. The Plücker coordinates (x^μ_α), $k+1 \leq \mu \leq \dim V = \mathbf{v}$, $1 \leq \alpha \leq k$ are obtained by choosing a basis $e_1, \cdots, e_{\mathbf{v}}$ of V, centering the coordinates at $[e_1 \wedge \cdots \wedge e_k]$, and writing a nearby k-plane as $[(e_1 + \sum x^\mu_1 e_\mu) \wedge \cdots \wedge (e_k + \sum x^\mu_k e_\mu)]$. If the polynomial is also homogeneous in the x^μ_α, this is equivalent to it being the restriction of a homogeneous degree d polynomial on $\Lambda^k V$. (The ambiguity of the scale does not matter as we are only concerned with its vanishing.)

Lemma 2.3. *Let A be given a basis. Given a homogeneous polynomial of degree d on the Grassmannian $G(k,A)$, there exists at least dk basis vectors such that, denoting their (at most) dk-dimensional span by A', P restricted to $G(k,A')$ is not identically zero.*

Proof. Consider the map $f : A^{\times k} \to G(k,A)$ given by $(a_1, \cdots, a_k) \mapsto [a_1 \wedge \cdots \wedge a_k]$. Then f is surjective. Take the polynomial P and pull it back by f. (The pullback $f^*(P)$ is defined by $f^*(P)(a_1, \cdots, a_k) := P(f(a_1, \cdots, a_k))$.) The pullback is of degree d in each copy of A. (I.e., fixing $k-1$ parameters, it becomes a degree d polynomial in the k-th.) Now simply apply Lemma 2.2 k times to see that the pulled back polynomial is not identically zero restricted to A', and thus P restricted to $G(k,A')$ is not identically zero. □

Remark 2.4. The bound in Lemma 2.3 is sharp, as give A a basis $a_1, \cdots, a_{\mathbf{a}}$ and consider the polynomial on $\Lambda^k A$ with coordinates $x^I = x^{i_1}, \cdots, x^{i_k}$ corresponding to the vector $\sum_I x^I a_{i_1} \wedge \cdots \wedge a_{i_k}$: $P = x^{1,\cdots,k} x^{k+1,\cdots,2k} \cdots x^{(d-1)k+1,\cdots,dk}$. Then P restricted to $G(k, \langle a_1, \cdots, a_{dk} \rangle)$ is non-vanishing but there is no smaller subspace spanned by basis vectors on which it is non-vanishing.

3. Matrix multiplication and its rank

Let $M_{\langle \mathbf{m},\mathbf{n},\mathbf{l}\rangle} : Mat_{\mathbf{m}\times\mathbf{n}} \times Mat_{\mathbf{n}\times\mathbf{l}} \to Mat_{\mathbf{m}\times\mathbf{l}}$ denote the matrix multiplication operator. Write $M = \mathbb{C}^{\mathbf{m}}$, $N = \mathbb{C}^{\mathbf{n}}$ and $L = \mathbb{C}^{\mathbf{l}}$. Then

$$M_{\langle \mathbf{m},\mathbf{n},\mathbf{l}\rangle} : (N\otimes L^*) \times (L\otimes M^*) \to N\otimes M^*$$

has the interpretation as $M_{\langle \mathbf{m},\mathbf{n},\mathbf{l}\rangle} = Id_N\otimes Id_M\otimes Id_L \in (N^*\otimes L)\otimes(L^*\otimes M)\otimes(N\otimes M^*)$, where $Id_N \in N^*\otimes N$ is the identity map. If one thinks of $M_{\langle \mathbf{m},\mathbf{n},\mathbf{l}\rangle}$ as a trilinear map $(N\otimes L^*)\times(L\otimes M^*)\times(N\otimes M^*)\to\mathbb{C}$, in bases it is $(X,Y,Z)\mapsto \mathrm{trace}(XYZ)$. If one thinks of $M_{\langle \mathbf{m},\mathbf{n},\mathbf{l}\rangle}$ as a linear map $N\otimes L^* \to (L^*\otimes M)\otimes(N\otimes M^*)$ it is just the identity map tensored with Id_M. In particular, if $\alpha \in N\otimes L^*$ is of rank q, its image, considered as a linear map $L\otimes M^* \to N\otimes M^*$, is of rank $q\mathbf{m}$.

Returning to general tensors $T \in A\otimes B\otimes C$, from now on assume $\mathbf{b} = \mathbf{c}$. When $T = M_{\langle \mathbf{m},\mathbf{n},\mathbf{l}\rangle}$, one has $A = N^*\otimes L$, $B = L^*\otimes M$, $C = N\otimes M^*$, so $\mathbf{b} = \mathbf{c}$ is equivalent to $\mathbf{l} = \mathbf{n}$.

The equations of [5] are as follows: given $T \in A\otimes B\otimes C$, with $\mathbf{b} = \mathbf{c}$, take $A' \subset A$ of dimension $2p+1 \le \mathbf{a}$. Define a linear map

$$T^{\wedge p}_{A'} : \Lambda^p A'\otimes B^* \to \Lambda^{p+1}A'\otimes C \tag{1}$$

by first considering $T|_{A'\otimes B\otimes C} : B^* \to A'\otimes C$ tensored with the identity map on $\Lambda^p A'$, which is a map $\Lambda^p A\otimes B^* \to \Lambda^p A\otimes A\otimes C$, and then projecting the image to $\Lambda^{p+1}A'\otimes C$. Then if the determinant of this linear map is nonzero, the border rank of T is at least $\frac{2p+1}{p+1}\mathbf{b}$. If there exists an A' such that the determinant is nonzero, we may think of the determinant as a nontrivial homogeneous polynomial of degree $\binom{2p+1}{p}\mathbf{b}$ on $G(2p+1, A)$.

Now consider the case $T = M_{\langle \mathbf{n},\mathbf{n},\mathbf{m}\rangle}$, and recall that $B = L^*\otimes M$, $C = N\otimes M^*$. The map $(M_{\langle \mathbf{n},\mathbf{n},\mathbf{m}\rangle})^{\wedge p}_{A'} : \Lambda^p A'\otimes L\otimes M^* \to \Lambda^{p+1}A'\otimes N\otimes M^*$ is actually a reduced map

$$\tilde{M}^{\wedge p}_{A'} : \Lambda^p A'\otimes L \to \Lambda^{p+1}A'\otimes N \tag{2}$$

tensored with the identity map $M^* \to M^*$, and thus its determinant is non-vanishing if and only if the determinant of $\tilde{M}^{\wedge p}_{A'}$ is nonvanishing. *But this is a polynomial of degree* $\binom{2p+1}{p}\mathbf{n} << \binom{2p+1}{p}\mathbf{n}^2$ *on* $G(2p+1,\mathbf{n}^2)$. Proposition 2.1 with $d = \binom{2p+1}{p}\mathbf{n}$, $\mathbf{a} = \mathbf{n}^2$ and $r = \frac{2p+1}{p+1}\mathbf{mn}$, combined with Lemma 2.3 gives the bound

$$\mathbf{R}(M_{\langle \mathbf{n},\mathbf{n},\mathbf{m}\rangle}) \ge \frac{2p+1}{p+1}\mathbf{nm} + \mathbf{n}^2 - (2p+1)\binom{2p+1}{p}\mathbf{n}.$$

Note that this already gives the $3\mathbf{n}^2 - o(\mathbf{n}^2)$ asymptotic lower bound. The remainder of the paper is dedicated to improving the error term.

4. The equations of [5] in coordinates

Let $\mathbf{a} = 3$ (so $p = 1$) and $\mathbf{b} = \mathbf{c}$, the map (1) expressed in bases is a $3\mathbf{b}\times 3\mathbf{b}$ matrix. If a_0, a_1, a_2 is a basis of A and one chooses bases of B, C, then elements of $B\otimes C$ may be written as matrices, and $T = a_0\otimes X_0 + a_1\otimes X_1 + a_2\otimes X_2$, where the X_j are size $\mathbf{b}$ square matrices. Order the basis of A by a_0, a_1, a_2 and of $\Lambda^2 A$ by $a_1\wedge a_2, a_0\wedge a_1, a_0\wedge a_2$. We compute

$$T^{\wedge 1}_A(a_0\otimes\beta) = \beta(X_0)\otimes a_0\wedge a_0 + \beta(X_1)\otimes a_1\wedge a_0 + \beta(X_2)\otimes a_2\wedge a_0 = -\beta(X_1)\otimes a_0\wedge a_1 - \beta(X_2)\otimes a_0\wedge a_2,$$

$$T^{\wedge 1}_A(a_1\otimes\beta) = \beta(X_0)\otimes a_0\wedge a_1 - \beta(X_2)\otimes a_1\wedge a_2,$$

$$T^{\wedge 1}_A(a_2\otimes\beta) = \beta(X_0)\otimes a_0\wedge a_2 + \beta(X_1)\otimes a_1\wedge a_2,$$

so the corresponding matrix for $T_A^{\wedge 1}$ is the block matrix

$$Mat(T_A^{\wedge 1}) = \begin{pmatrix} 0 & -X_2 & X_1 \\ -X_1 & X_0 & 0 \\ -X_2 & 0 & X_0 \end{pmatrix}.$$

Now assume X_0 is invertible and change bases such that it is the identity matrix. Recall the formula for block matrices

$$\det \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(W)\det(X - YW^{-1}Z), \tag{3}$$

assuming W is invertible. Then, using the $(\mathbf{b}, 2\mathbf{b}) \times (\mathbf{b}, 2\mathbf{b})$ blocking (so $X = 0$ in (3))

$$\det Mat(T_A^{\wedge 1}) = \det(X_1X_2 - X_2X_1) = \det([X_1, X_2]).$$

When $\dim A > 3$, if there exists a three dimensional subspace A' of A, such that $\det Mat(T_{A'}^{\wedge 1}) \neq 0$, then $\underline{\mathbf{R}}(T) \geq \frac{3}{2}\mathbf{b}$ as this is (1) in the case $p = 1$. These are Strassen's equations [9].

I now phrase the equations of [5] in coordinates. Let $\dim A = 2p+1$. Write $T = a_0 \otimes X_0 + \cdots + a_{2p} \otimes X_{2p}$. The expression of (1) in bases is as follows: write $a_I := a_{i_1} \wedge \cdots \wedge a_{i_p}$ for $\Lambda^p A$, require that the first $\binom{2p}{p-1}$ basis vectors have $i_1 = 0$, that the second $\binom{2p}{p}$ do not, and call these multi-indices $0J$ and K. Order the bases of $\Lambda^{p+1}A$ such that the first $\binom{2p}{p+1}$ multi-indices do not have 0, and the second $\binom{2p}{p}$ do, and furthermore that the second set of indices is ordered the same way as K, only we write $0K$ since a zero index is included. Then the resulting matrix is of the form

$$\begin{pmatrix} 0 & Q \\ \tilde{Q} & R \end{pmatrix} \tag{4}$$

where this matrix is blocked $(\binom{2p}{p+1}\mathbf{b}, \binom{2p}{p}\mathbf{b}) \times (\binom{2p}{p+1}\mathbf{b}, \binom{2p}{p}\mathbf{b})$,

$$R = \begin{pmatrix} X_0 & & \\ & \ddots & \\ & & X_0 \end{pmatrix},$$

and $Q, \tilde{Q}$ have entries in blocks consisting of $X_1, \cdots, X_{2p}$ and zero. Thus if X_0 is the identity matrix, so is R and the determinant equals the determinant of $Q\tilde{Q}$. If X_0 is the identity matrix, when $p = 1$ we have $Q\tilde{Q} = [X_1, X_2]$ and when $p = 2$

$$Q\tilde{Q} = \begin{pmatrix} 0 & [X_1, X_2] & [X_1, X_3] & [X_1, X_4] \\ [X_2, X_1] & 0 & [X_2, X_3] & [X_2, X_4] \\ [X_3, X_1] & [X_3, X_2] & 0 & [X_3, X_4] \\ [X_4, X_1] & [X_4, X_2] & [X_4, X_3] & 0 \end{pmatrix}. \tag{5}$$

In general, when X_0 is the identity matrix, $Q\tilde{Q}$ is a block $\binom{2p}{p-1}\mathbf{b} \times \binom{2p}{p-1}\mathbf{b}$ matrix whose block entries are either zero or commutators $[X_i, X_j]$.

To prove Theorem 1.1 we work with $\tilde{M}_{A'}^{\wedge p}$ of (2), so $\mathbf{b} = \mathbf{n}$. First apply Lemma 2.2 to choose $\mathbf{n}$ basis vectors such that restricted to them $\det(X_0)$ is non-vanishing, and then we consider our polynomial $\det(Q\tilde{Q})$ as defined on $G(2p, (2p+1)\mathbf{n}^2 - 1)$, and apply Lemma 2.3, using $2p\binom{2p}{p-1}\mathbf{n}$ basis vectors to insure it is non-vanishing. Our error term is thus $\mathbf{n} + 2p\binom{2p}{p-1}\mathbf{n}$, and the theorem follows.

Remark 4.1. In [8, 7], they show the matrix $Q\tilde{Q}$ can be made to have a nonzero determinant by a subtle combination of factoring and splitting it into a sum of two matrices that carries a lower cost than just taking its determinant.

References

1. Markus Bläser, *A $\frac{5}{2}n^2$-lower bound for the rank of $n\times n$-matrix multiplication over arbitrary fields*, 40th Annual Symposium on Foundations of Computer Science (New York, 1999), IEEE Computer Soc., Los Alamitos, CA, 1999, pp. 45–50. MR MR1916183
2. ———, *On the complexity of the multiplication of matrices of small formats*, J. Complexity **19** (2003), no. 1, 43–60. MR MR1951322 (2003k:68040)
3. Peter Bürgisser, Michael Clausen, and M. Amin Shokrollahi, *Algebraic complexity theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 315, Springer-Verlag, Berlin, 1997, With the collaboration of Thomas Lickteig. MR 99c:68002
4. J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012. MR 2865915
5. J.M. Landsberg and Giorgio Ottaviani, *New lower bounds for the border rank of matrix multiplication*, preprint, arXiv:1112.6007.
6. Thomas Lehmkuhl and Thomas Lickteig, *On the order of approximation in approximative triadic decompositions of tensors*, Theoret. Comput. Sci. **66** (1989), no. 1, 1–14. MR 1018840 (91f:68099)
7. Alex Massarenti and Emanuele Raviolo, *Erratum to: The rank of $n \times n$ matrix multiplication is at least $3n^2 - 2\sqrt{2}n^3/2 - 3n$*, to appear.
8. ———, *The rank of $n\times n$ matrix multiplication is at least* $3n^2-2\sqrt{2}n^{\frac{3}{2}}-3n$, Linear Algebra Appl. **438** (2013), no. 11, 4500–4509. MR 3034546
9. V. Strassen, *Rank and optimal computation of generic tensors*, Linear Algebra Appl. **52/53** (1983), 645–685. MR 85b:15039

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