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Anomaly-free W_3 gravity and critical W_3 strings

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ABSTRACT

We construct an anomaly-free theory of chiral W_3 gravity, with an arbitrary number $n \geq 2$ of scalar matter fields.

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The problem of formulating a consistent $d = 2$ quantum theory with gauged W-algebra symmetry revolves around the various types of anomalies that can occur. Indeed, the problem of anomalies has played a prominent rôle in discussions of W-gravity theories from the first papers on the subject [1–7]. This problem becomes significantly simpler in the quantisation of w_∞ gravity, where it has recently been shown [7] that the removal of matter-dependent anomalies has the inevitable consequence that the w_∞ symmetry [8] is renormalised to a W_∞ symmetry [9]. This relative simplicity is due to the linearity of these algebras, in contrast to the nonlinear structure of the W_N algebras [10,11] for finite N. After the removal of the matter-dependent anomalies in the W_∞ case, there remain the "universal" anomalies, which depend only on the gauge fields and are subject to a simple anomalous Ward identity [7]. These latter can, formally at least, be removed by treating the gauge fields as quantum fields in their own right and consequently applying the usual BRST procedure with the inclusion of the corresponding ghost fields.

The anomaly cancellation may also be viewed from the standpoint of the BRST operator Q, satisfying the fundamental nilpotency condition $Q^2 = 0$ [12]. For a linear algebra such as the Virasoro algebra or W_∞, it is known that this requirement entails both that the matter currents close properly to form an algebra and that the central charge take the appropriate critical value. The analogue of the $c = 26$ result for the matter currents in the Virasoro algebra is, after regularisation, $c = -2$ in the case of W_∞ [13,14]. This is precisely the central charge in the case of the single-scalar realisation of W_∞ gravity considered in [7].

In this paper, we shall show that W_3 gravity may be handled in an exactly parallel fashion. The first requirement is to obtain a BRST operator Q that is nilpotent. In the W_3 case, a necessary condition is that $c = 100$ for the matter currents [15]. Thus, to construct a quantum theory of W_3 gravity, one must first find matter currents that satisfy the full quantum operator-product algebra with $c_{\rm mat} = 100$. Such realisations have been constructed for arbitrary numbers $n \geq 2$ of scalar fields [16], generalising the $n = 2$ realisation of [11]. In all cases, in order to achieve $c_{\rm mat} = 100$ (or indeed any $c \neq 2$) it is necessary to include background charges*. Previous attempts [4–6] at constructing an anomaly-free W_3-gravity theory have not succeded either because the matter currents were not renormalised to close on the W_3 algebra at the quantum level, or because the central charge was not set equal to 100.

The BRST operator for a chiral W_3 theory in $d = 2$ dimensions has the form [15]

$$Q = \oint dz \Big(c(T_{\rm mat} + \tfrac{1}{2} T_{\rm gh}) + \gamma (W_{\rm mat} + \tfrac{1}{2} W_{\rm gh}) \Big), \tag{1}$$

* Only for 2 scalars can background charges be omitted, yielding a $c = 2$ realisation of W_3. In particular, background charges are necessary even for the $c = 100$ realisation in terms of $n = 100$ scalars. Moreover, because the algebra is nonlinear, one cannot make new realisations simply by tensoring together more elementary ones.

where $T_{\rm mat}$ and $W_{\rm mat}$ must generate the W_3 algebra with central charge $c_{\rm mat} = 100$. The ghost currents $T_{\rm gh}$ and $W_{\rm gh}$ are given by [15]

$$T_{\rm gh} = -2b\,\partial c - \partial b\, c - 3\beta\,\partial\gamma - 2\partial\beta\,\gamma \tag{2a}$$

$$\begin{aligned} W_{\rm gh} = &-\partial\beta\, c - 3\beta\,\partial c - \tfrac{8}{261}\big[\partial(b\,\gamma\, T_{\rm mat}) + b\,\partial\gamma\, T_{\rm mat}\big] \\ &+ \tfrac{25}{6\cdot 261}\hbar\Big(2\gamma\,\partial^3 b + 9\partial\gamma\,\partial^2 b + 15\partial^2\gamma\,\partial b + 10\partial^3\gamma\, b\Big), \end{aligned} \tag{2b}$$

where the ghost-antighost pairs (c,b) and (γ,β) correspond respectively to the T and W generators. Note that the BRST operator (1) contains ghosts but no gauge fields, since the latter have no kinetic terms in the $d = 2$ theories we are considering. The factor of $\frac{1}{261}$ that appears in ($2b$) and throughout the following has its origin in a factor of $(22 + 5c_{\rm mat})$ that would appear in the case of a general central charge $c_{\rm mat}$.

The matter currents $T_{\rm mat}$ and $W_{\rm mat}$ are required to generate the W_3 algebra [10]

$$\hbar^{-1}T_{\rm mat}(z)T_{\rm mat}(w) \sim \frac{\partial T_{\rm mat}(w)}{z-w} + \frac{2T_{\rm mat}(w)}{(z-w)^2} + \hbar\frac{c_{\rm mat}/2}{(z-w)^4} \tag{3a}$$

$$\hbar^{-1}T_{\rm mat}(z)W_{\rm mat}(w) \sim \frac{\partial W_{\rm mat}}{z-w} + \frac{3W_{\rm mat}(w)}{(z-w)^2} \tag{3b}$$

$$\begin{aligned} \hbar^{-1}W_{\rm mat}(z)W_{\rm mat}(w) \sim &\frac{1}{z-w}\left(\tfrac{1}{15}\hbar\partial^3 T_{\rm mat}(w) + \frac{16}{22+5c_{\rm mat}}\partial\Lambda(w)\right) \\ &+ \frac{1}{(z-w)^2}\left(\tfrac{3}{10}\hbar\partial^2 T_{\rm mat}(w) + \frac{32}{22+5c_{\rm mat}}\Lambda(w)\right) \\ &+ \hbar\frac{\partial T_{\rm mat}(w)}{(z-w)^3} + \hbar\frac{2T_{\rm mat}(w)}{(z-w)^4} + \hbar^2\frac{c_{\rm mat}/3}{(z-w)^6} \end{aligned} \tag{3c}$$

with central charge $c_{\rm mat} = 100$. In ($3c$), Λ is a composite current;

$$\Lambda = (T_{\rm mat}T_{\rm mat}) - \tfrac{3}{10}\hbar\partial^2 T_{\rm mat}, \tag{4}$$

where the normal ordering is taken with respect to the modes of the currents $T_{\rm mat}$, according to the prescription [17]

$$(AB)(w) \equiv \oint \frac{dz}{z-w}A(z)B(w). \tag{5}$$

We consider general realisations of the spin-2 and spin-3 currents in terms of n scalar fields φ^i:

$$T_{\rm mat} = \tfrac{1}{2}\partial\varphi^i\partial\varphi^i + \sqrt{\hbar}\alpha_i\partial^2\varphi^i, \tag{6a}$$

$$W_{\rm mat} = \tfrac{1}{3}d_{ijk}\partial\varphi^i\partial\varphi^j\partial\varphi^k + \sqrt{\hbar}e_{ij}\partial\varphi^i\partial^2\varphi^j + \hbar f_i\partial^3\varphi^i, \tag{6b}$$

where φ^i satisfies the OPE

$$\partial\varphi^i(z)\partial\varphi^j(w) \sim \frac{\hbar\delta^{ij}}{(z-w)^2}. \tag{7}$$

It should be noted that in $(6a, b)$, and throughout the paper, normal ordering with respect to the modes of fundamental fields is understood. It was shown in ref. [7] that in chiral Lorentz-invariant theories of the type considered in this paper, all infinities are removed by this normal-ordering procedure.

As shown in [16], the currents $(6a, b)$ generate the W_3 algebra at the quantum level if and only if the constants α_i, d_{ijk}, e_{ij} and f_i satisfy

$$d_{ijj} - 6e_{ij}\alpha_j + 6f_i = 0 \tag{8a}$$

$$e_{(ij)} - d_{ijk}\alpha_k = 0 \tag{8b}$$

$$3f_i - \alpha_j e_{ji} = 0 \tag{8c}$$

$$d_{ik\ell}d_{jk\ell} + 6d_{ijk}f_k - 3e_{ik}e_{jk} = \tfrac{1}{2}\delta_{ij} \tag{8d}$$

$$d_{(ij}{}^m d_{k\ell)m} = \frac{8}{(22+5c_{\rm mat})}\delta_{(ij}\delta_{k\ell)} \tag{8e}$$

$$d_{ij\ell}(e_{\ell k} - e_{k\ell}) + 2e_{(i}{}^{\ell}d_{j)k\ell} = \frac{16}{(22+5c_{\rm mat})}\delta_{ij}\alpha_k. \tag{8f}$$

Two useful consequences of this complete set of equations are

$$e_{ik}e_{kj} = -\frac{(c_{\rm mat}-2)}{2(22+5c_{\rm mat})}\delta_{ij}, \tag{9a}$$

$$6d_{ijk}f_k + e_{ki}e_{kj} = \frac{32}{(22+5c_{\rm mat})}\alpha_i\alpha_j + \frac{2(c_{\rm mat}-2)}{(22+5c_{\rm mat})}\delta_{ij}. \tag{9b}$$

The central charge $c_{\rm mat}$ is given by

$$c_{\rm mat} = n - 12\alpha_i\alpha_i, \tag{10}$$

where n is the number of scalars. We shall take

$$c_{\rm mat} = 100 \tag{11}$$

in equations $(8a$–$f)$ and $(9a, b)$, and in the rest of this paper, in order to achieve nilpotence of the BRST operator.

It was shown in [15] (and discussed further in [18]) that for matter currents $T_{\rm mat}$ and $W_{\rm mat}$ constructed with coefficients satisfying eqns $(8a$–$f)$ with $c_{\rm mat} = 100$, the BRST operator (1) is nilpotent, *i.e.* the full quantum operator will satisfy $Q^2 = 0$. Correspondingly, we may now write down the fully-renormalised action and transformation rules for W_3 gravity, for which the quantum effective action Γ will be BRST invariant, *i.e.* the Slavnov-Taylor-Ward identity for the BRST symmetry will be satisfied. The complete action for chiral anomaly-free W_3 gravity is therefore given by $I = 1/\pi\int d^2z L$, where the renormalised Lagrangian is

$$\begin{aligned} L =& \tfrac{1}{2}\bar\partial\varphi^i\partial\varphi^i - b\bar\partial c - \beta\bar\partial\gamma + \pi_b(h - h_{\rm back}) + \pi_\beta(B - B_{\rm back}) \\ &- h(T_{\rm mat} + T_{gh}) - B(W_{\rm mat} + W_{gh}). \end{aligned} \tag{12}$$

The quantum spin-2 and spin-3 gauge fields h and B are gauge fixed by the conditions $h = h_{\rm back}$ and $B = B_{\rm back}$, which are imposed by the Lagrange multipliers π_b and π_β. The *classical* action is given by the $\hbar$-independent terms in (12). The $\hbar$-dependent terms in (12) are the counterterms, which, we shall see below, are exactly those needed to cancel all the anomalies. Thus, we have

$$\begin{aligned} L_0 = &\tfrac12\bar\partial\varphi^i\partial\varphi^i - b\bar\partial c - \beta\bar\partial\gamma + \pi_b(h-h_{\rm back}) + \pi_\beta(B-B_{\rm back}) \\ &- h\Big(\tfrac12\partial\varphi^i\partial\varphi^i - 2b\,\partial c - \partial b\, c - 3\beta\,\partial\gamma - 2\partial\beta\,\gamma\Big) \\ &- B\Big(\tfrac13 d_{ijk}\partial\varphi^i\partial\varphi^j\partial\varphi^k - \partial\beta c - 3\beta\partial c - \tfrac{4}{261}\big[\partial(b\gamma\partial\varphi^i\partial\varphi^i) + b\partial\gamma\,\partial\varphi^i\partial\varphi^i\big]\Big) \end{aligned} \tag{13}$$

and

$$\begin{aligned} L_{\rm counter} = \sqrt{\hbar}\Big\{ &- h\alpha_i\partial^2\varphi^i - B\big[e_{ij}\partial\varphi^i\partial^2\varphi^j - \tfrac{8}{261}\alpha^i\Big(\partial(b\gamma\partial^2\varphi^i) + b\partial\gamma\,\partial^2\varphi^i\Big)\big]\Big\} \\ + \hbar\Big\{ &- B\Big[f_i\partial^3\varphi^i + \tfrac{25}{6\cdot261}\Big(2\gamma\,\partial^3 b + 9\partial\gamma\,\partial^2 b + 15\partial^2\gamma\,\partial b + 10\partial^3\gamma\, b\Big)\Big]\Big\}. \end{aligned} \tag{14}$$

In these equations all field products are understood to be normal ordered to implement the infinite renormalisation, as we have already discussed.

The transformation rules for the matter fields φ^i and the ghosts c and γ are obtained in the standard way using $\delta\varphi^i = \{Q,\varphi^i\}$, etc. The transformation rules for the gauge fields h and B follow by requiring BRST invariance. Thus we have

$$\begin{aligned} \delta\varphi^i &= c\,\partial\varphi^i + \gamma d_{ijk}\,\partial\varphi^j\partial\varphi^k + \tfrac{8}{261}b\,\gamma\,\partial\gamma\partial\varphi^i \\ &\quad + \sqrt{\hbar}\Big(-\alpha_i\partial c + (e_{ij}-e_{ji})\gamma\partial^2\varphi^j - e_{ji}\partial\gamma\partial\varphi^j - \tfrac{8}{261}\alpha_i\partial(b\gamma\partial\gamma)\Big) + \hbar f_i\partial^2\gamma, \quad (15a) \\ \delta h &= \bar\partial c + c\,\partial h - \partial c\, h + \tfrac{4}{261}(\gamma\partial B - \partial\gamma B)\partial\varphi^i\partial\varphi^i + \tfrac{8}{261}\sqrt{\hbar}(\gamma\partial B - \partial\gamma B)\alpha_i\partial^2\varphi^i \\ &\quad + \tfrac{25}{6\cdot261}\hbar\Big(2\gamma\,\partial^3 B - 3\partial\gamma\,\partial^2 B + 3\partial^2\gamma\,\partial B - 2\partial^3\gamma\, B\Big), \quad (15b) \\ \delta B &= \bar\partial\gamma + c\,\partial B - 2\partial c\, B + 2\gamma\,\partial h - \partial\gamma\, h, \quad (15c) \\ \delta c &= c\,\partial c + \tfrac{4}{261}\gamma\,\partial\gamma\,\partial\varphi^i\partial\varphi^i + \tfrac{8}{261}\sqrt{\hbar}\alpha_i\gamma\,\partial\gamma\,\partial^2\varphi^i + \tfrac{25}{6\cdot261}\hbar(2\gamma\,\partial^3\gamma - 3\partial\gamma\,\partial^2\gamma), \quad (15d) \\ \delta\gamma &= c\,\partial\gamma - 2\partial c\,\gamma, \quad (15e) \\ \delta b &= \pi_b, \qquad \delta\pi_b = 0, \quad (15e) \\ \delta\beta &= \pi_\beta, \qquad \delta\pi_\beta = 0. \quad (15f) \end{aligned}$$

The $\hbar$-independent terms in ($15a$–f) correspond to the classical BRST symmetry of the classical action given by (13)*. The $\hbar$-dependent terms in ($15a$–f) correspond to renormalisations of the classical transformation rules which, together with the counterterms $L_{\rm counter}$ given in (14), ensure full quantum BRST invariance of the effective action.

* The coefficient $\frac{4}{261}$ in (13), and related coefficients in ($15a, b, d$), reflect the fact that our classical normalisations are those inherited from the conventionally-normalised quantum W_3 algebra ($3a$–c), upon taking the classical limit $\hbar = 0$. At the classical level, one could rescale the fields B, γ, β, π_β and the parameters d_{ijk}, in order to remove all such factors. We choose to retain the normalisations of (13), (14) and ($15a$–f), in order to end up with the conventionally-normalised W_3 algebra at the quantum level.

Note that from the field-theoretic point of view, the occurrence of terms proportional to $\sqrt{\hbar}$ in the counterterms (14) and the transformation rules (15a-f) may seem strange. This, however, has already been seen in the renormalisation of w_∞ gravity to W_∞ gravity [7], where it played a crucial rôle in the cancellation of anomalies. In both that example and the present paper, the $\sqrt{\hbar}$ corrections have their origin in the occurrence of background charges in the matter currents. These terms are natural generalisations of the standard background-charge terms in the stress tensor. Since φ^i has dimensions of $\sqrt{\hbar}$, these terms, involving fewer fields than the leading term at a given spin, necessarily carry appropriate half-integer powers of $\hbar$.

At this stage, it is guaranteed by construction that the action (12) describes anomaly-free W_3 gravity, by virtue of the nilpotence of the BRST charge. In other words, it is guaranteed that the BRST variation of the effective action Γ is BRST-trivial. In fact, as we shall now demonstrate, the particular choices of counterterms (14) and renormalisations of the transformation rules (15a–f) ensure that Γ is actually invariant, and so the anomaly-freedom is made manifest.

In order to illustrate the BRST invariance of the effective action, we shall consider some representative potentially-anomalous Feynman diagrams, to show how the anomalies in fact cancel. Before considering loop diagrams, we note that the $\hbar$-independent terms in the variation cancel by virtue of (8e) [1]. The terms proportional to $\sqrt{\hbar}$ in the variation of the effective action cancel amongst themselves by virtue of the relations (8b) and (8f). Turning now to loops, we consider first diagrams that could have given rise to matter-dependent anomalies. The first of these is given in Fig. 1. It can be calculated by evaluating the double contractions in the operator product expansion of $1/(2\hbar)\int d^2z h(z)\partial\varphi^i\partial\varphi^i(z)$ times $1/(3\hbar)\int d^2w B(w) d_{ijk}\partial\varphi^i\partial\varphi^j\partial\varphi^k(w)$. The resulting contribution to the effective action is

$$\begin{aligned}\Gamma_{hB\varphi} &= \frac{\hbar}{\pi^2} d_{ijj}\int d^2z d^2w\ h(z)B(w)\frac{1}{(z-w)^4}\partial\varphi^i(w)\\ &= -\frac{\hbar}{6\pi} d_{ijj}\int d^2z d^2w\ h(z)B(w)\frac{\partial_z^3}{\bar\partial_{\bar z}}\delta^2(z-w)\partial\varphi^i(w)\\ &= -\frac{\hbar}{6\pi} d_{ijj}\int d^2z \left(\frac{\partial^3}{\bar\partial}h(z)\right)B(z)\partial\varphi^i(z).\end{aligned}\tag{16}$$

Under the leading order inhomogeneous terms in the BRST transformations (15b,c), ($\delta h = \bar\partial c+\cdots$, $\delta B=\bar\partial\gamma+\cdots$) the anomalous variation of $\Gamma_{hB\varphi}$ is

$$\Delta_{hB\varphi} = -\frac{\hbar}{6\pi} d_{ijj}\int d^2z (B\partial^3 c - \gamma\partial^3 h)\partial\varphi^i.\tag{17}$$

Note that in the derivation of this result, one may drop terms proportional to the φ^i field equation, since these cancel in the quantum Slavnov-Taylor-Ward identity [19] against terms involving operator insertions of the φ^i transformations into the relevant one-loop diagrams.

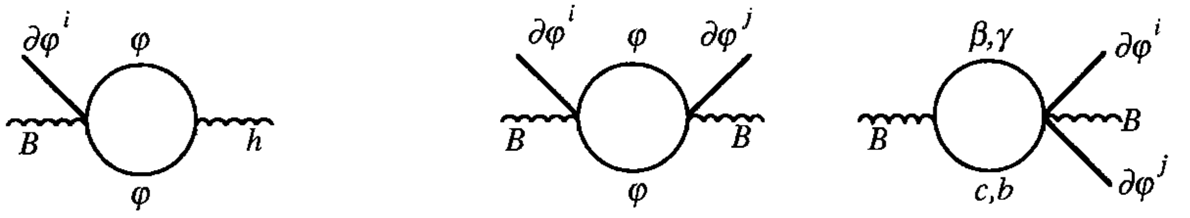

Fig. 1 Fig. 2a Fig. 2b

The anomalous variation (17) must be cancelled by the $O(\hbar)$ contributions coming from the counterterms and quantum corrections to the transformation rules. Specifically, there are contributions:

$$
\begin{aligned}
\delta_1 L_0 &= -\hbar f_i\, h\, \partial\varphi^i \partial^3\gamma, \\
\delta_{\frac{1}{2}} L_{\frac{1}{2}} &= -\hbar\alpha_i\, h\, \partial^2\Big((e_{ij} - e_{ji})\gamma\partial^2\varphi^j - e_{ji}\partial\gamma\, \partial\varphi^j\Big), \\
\delta_0 L_1 &= -\hbar f_i\, B\, \partial^3(c\partial\varphi^i) - \hbar f_i \partial^3\varphi^i\Big(c\,\partial B - 2\partial c\, B + 2\gamma\,\partial h - \partial\gamma\, h\Big),
\end{aligned}
\tag{18}
$$

where the subscripts on δ and L indicate their order in $\hbar$. Collecting these contributions together with (17), we find that the anomalies indeed cancel, by virtue of the relations (8a) and (8c).

The next two diagrams, Figs. 2a,b, comprise two types of contribution to the anomalies, with the same external-field structure. Evaluating these contributions in the same manner as above, we find that the total anomalous variation coming from Figs. 2a,b is

$$
\begin{aligned}
\Delta_{BB\varphi\varphi} = &-\frac{\hbar}{3\pi} d_{ik\ell} d_{jk\ell} \int d^2z\, B\, \partial\varphi^i \partial^3(\gamma\, \partial\varphi^j) \\
&- \frac{4\hbar}{261\pi} \int d^2z \Big(2\partial^2\gamma\, \partial B - 2\partial\gamma\, \partial^2 B - \tfrac{5}{6}\partial^3\gamma\, B + \tfrac{5}{6}\gamma\, \partial^3 B\Big)\partial\varphi^i \partial\varphi^i.
\end{aligned}
\tag{19}
$$

There is a new feature that appears for the first time in the anomalies associated with these diagrams, owing to the nonlinearity on the right-hand side of (4c). The anomalies in a quantum field theory are actually anomalies in the BRST Slavnov-Taylor-Ward identity, which has the form

$$
\int d^2z \frac{\delta\Gamma}{\delta\Phi_I(z)} \delta\Phi_I(z) \bullet \Gamma = 0, \tag{20}
$$

where Φ_I denotes all the fields. In a theory with linear field transformations, (20) reduces simply to the statement that the effective action must be invariant under the renormalised symmetry transformations. In the nonlinear case, however, the Slavnov-Taylor-Ward identity (20) includes terms where the $\delta\Phi_I \bullet \Gamma$ generating functionals include contributions from insertions of the variation operators $\delta\Phi_I$ into 1-particle-irreducible loop diagrams.

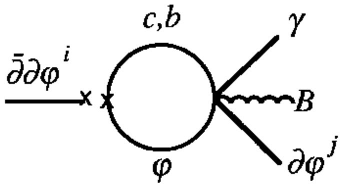

Fig. 3

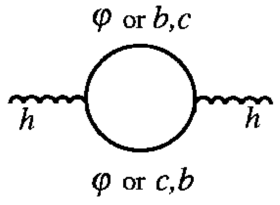

Fig. 4

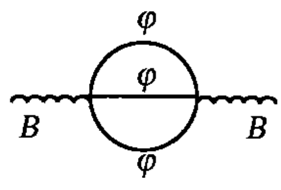

Fig. 5

In the specific case of the anomalies associated with the diagrams of Figs. 2a,b, there is a Slavnov-Taylor-Ward contribution depicted in Fig. 3 [6]. This corresponds to the contribution from the operator $\delta\varphi^i(z) = c(z)\partial\varphi^i(z)$ in $\int d^2z \frac{\delta\Gamma_0}{\delta\varphi^i(z)}\delta\varphi^i(z)\cdot\Gamma_1$, where the subscripts on Γ indicate the orders in $\hbar$. As a result, we obtain the local contribution

$$\Delta_{\varphi;\varphi B\varphi} = -\frac{4\hbar}{261\pi}\int d^2z\Big(\gamma\,\partial B - \partial\gamma\,B\Big)\partial\varphi^i\partial^3\varphi^i \tag{21}$$

which must be added to the anomalous variation (19). Including the relevant contributions from counterterms and quantum corrections to the transformation rules, we find that all the potentially-anomalous structures associated with Figs 2 and 3 cancel, by virtue of the conditions (8d), (9a) and (9b).

Turning now to the potential universal anomalies, we consider two classes of diagrams; one class corresponding to the (spin-2)-(spin-2) sector, and the other class corresponding to the (spin-3)-(spin-3) sector. For the spin-2 case, contributions arise from 1-loop shish-kebab diagrams with two external h fields and either matter fields or ghost fields in the loop (Fig. 4). There is also a $\delta_{\frac{1}{2}}L_{\frac{1}{2}}$ contribution at the same order. These contributions give a total anomalous variation

$$\Delta_{hh} = \frac{\hbar}{\pi}\Big(\alpha_i\alpha_i - \frac{n}{12} + \frac{100}{12}\Big)\int d^2z\,h\,\partial^3c. \tag{22}$$

From (10), we see that the coefficient can be written as $(100 - c_{\rm mat})/12$, and thus it vanishes at $c_{\rm mat} = 100$.

In the (spin-3)-(spin-3) sector, contributions arise from the two-loop beach-ball diagram (Fig. 5) with two external B fields and matter in the loops; 1-loop diagrams with the same external structure and either matter or ghosts in the loop (the extra factors of $\hbar$ or $\sqrt{\hbar}\sqrt{\hbar}$ arise from counterterm vertices); and a tree diagram of the form δ_1L_1. These give a total anomalous variation

$$\Delta_{BB} = -\frac{\hbar^2}{180\pi}\Big(d_{ijk}d_{ijk} - 9e_{ij}e_{ij} - 6e_{ij}e_{ji} + 180f_if_i - 50\Big)\int d^2z\,B\,\partial^5\gamma. \tag{23}$$

For $c_{\rm mat} = 100$, the total coefficient vanishes, by virtue of the relations (8a–d). In the mixed (spin-2)-(spin-3) sector, potential anomalies of the form $-\frac{1}{12}\hbar^{3/2}(e_{ii} - \alpha_i f_i)\int d^2z(B\partial^4 c - h\partial^4\gamma)$ vanish by virtue of (8a–c).

We have now demonstrated that our action (12) defines an anomaly-free chiral quantum W_3 gravity theory. Anomaly-freedom was guaranteed by the nilpotence of the BRST operator (1). Our diagramatic analysis has demonstrated that the counterterms (14) and quantum corrections to the transformation rules (15) are the correct ones for explicitly removing the anomalies in the sectors that we have examined. In fact the conditions on the coefficients α_i, d_{ijk}, e_{ij} and f_i needed for the above cancellations coincide precisely with the conditions (8a–f) and (11) needed for nilpotence of the BRST operator. On dimensional grounds there can be no further counterterms, and so it follows that the action (12) gives an explicitly anomaly-free theory. Note that this can be achieved for any number of scalars $n \geq 2$, provided that $c_{\rm mat}$ is given the value 100 by choosing the background-charge parameters appropriately.

It should be emphasised that the algebra that we have been gauging in this paper is precisely the full W_3 algebra of Zamolodchikov [10], at the quantum level, with $c = 100$. That this algebra is closed, albeit in a nonlinear manner, is a consequence of the fact that the composite spin-4 current Λ arising on the right-hand side of (3c) is defined entirely in terms of the abstract currents T and W. In other words, one may verify the consistency of the algebra without reference to the underlying field-theoretic substructure. In the field-theoretic renormalisation programme, we have used normal ordering with respect to the field modes in order to remove the infinities. In the field-theoretic realisation, the composite current Λ would naturally arise in a form written using field normal ordering, with $\partial\varphi^i$ appearing explicitly, but this may straightforwardly be rearranged into the form (4). Thus there is no conflict between writing the W_3 algebra in terms of the current normal ordering of (4), and realising W_3 in a field-theoretic context with infinities subtracted by means of field normal ordering.

Given a nilpotent BRST operator, the construction that we have used in this paper can equally be applied for gauging other algebras. Thus, for example, the problem of obtaining an anomaly-free W_N gravity is reduced to the problem of constructing a nilpotent quantum BRST operator for the W_N algebra.

The anomaly-free W_3 gravity that we have constructed in this paper is a natural generalisation of critical two-dimensional gravity, or, equivalently, $D = 26$ critical string theory. It is natural to speculate on the possibility of a similar spacetime interpretation in our case. The most general known n-scalar solutions of the conditions (8a–f) and (11) can, without loss of generality, be chosen so that only two of the scalar fields have background charges [16]. Thus we may write the matter currents $T_{\rm mat}$ and $W_{\rm mat}$ as [16]

$$T_{\rm mat} = T + \tfrac{1}{2}(\partial\varphi_1)^2 + \tfrac{1}{2}(\partial\varphi_2)^2 + \sqrt{\hbar}\big(\alpha_1\partial^2\varphi_1 + \alpha_2\partial^2\varphi_2\big) \qquad (24a)$$

$$W_{\rm mat} = \frac{2}{\sqrt{261}}\Big\{\tfrac{1}{3}(\partial\varphi_1)^3 - \partial\varphi_1(\partial\varphi_2)^2 + \sqrt{\hbar}\big(\alpha_1\partial\varphi_1\partial^2\varphi_1 - 2\alpha_2\partial\varphi_1\partial^2\varphi_2 - \alpha_1\partial\varphi_2\partial^2\varphi_2\big) + \hbar\big(\tfrac{1}{3}\alpha_1^2\partial^3\varphi_1 - \alpha_1\alpha_2\partial^3\varphi_2\big) - 2\partial\varphi_1\, T - \alpha_1\sqrt{\hbar}\,\partial T\Big\}, \tag{24b}$$

where T is a stress tensor for $D = n-2$ scalar fields without background charges,

$$T = \tfrac{1}{2}\sum_{\mu=1}^{D}\partial X^\mu\,\partial X^\mu, \tag{25}$$

and the background charges α_1 and α_2 for φ_1 and φ_2 are given by

$$\begin{aligned}\alpha_1^2 &= -\tfrac{49}{8}\\ \alpha_2^2 &= \tfrac{1}{12}(D - \tfrac{49}{2}).\end{aligned} \tag{26}$$

These conditions on the background charges ensure that

$$c_{\rm mat} = D + (1-12\alpha_1^2) + (1-12\alpha_2^2) = 100. \tag{27}$$

The fact that α_1, and, if D is chosen to satisfy $D \le 24$, α_2, are imaginary, suggests that φ_1, and, when appropriate, φ_2, should be redefined according to $\varphi \to i\varphi$, inducing a corresponding change in the signature of the scalar-field metric.

The remaining $D = n-2$ scalars X^μ enter the currents $T_{\rm mat}$ and $W_{\rm mat}$ only through their Poincaré-invariant stress tensor T, given by (25) [16]. Thus it is natural to try to associate these scalars with the coordinates of a D-dimensional flat target space [20]. One is free to replace one (or more) of the coordinates X^μ by iX^μ, in order to have an indefinite-signature target spacetime. The anomaly-free W_3 gravity theory that we have constructed in this paper may tentatively be adopted as a starting point for a critical W_3-string theory.

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Note added:

After this work was completed, we received an interesting paper by C.M. Hull and L. Palacios [21], in which background-charge terms are considered in a field-theoretic realisation without ghosts, starting from the classical W_3 algebra, in order to remove matter-dependent anomalies. They follow a different approach from ours, in which the renormalised quantum currents would close on an algebra that is different from the Zamolodchikov W_3 algebra.

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