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New Black Holes in Five Dimensions

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ABSTRACT

We construct new stationary Ricci-flat metrics of cohomogeneity 2 in five dimensions, which generalise the Myers-Perry rotating black hole metrics by adding a further non-trivial parameter. We obtain them *via* a construction that is analogous to the construction by Plebanski and Demianski in four dimensions of the most general type D metrics. Limiting cases of the new metrics contain not only the general Myers-Perry black hole with independent angular momenta, but also the single rotation black ring of Emparan and Reall. In another limit, we obtain new static metrics that describe black holes whose horizons are distorted lens spaces $L(n;m) = S^3/\Gamma(n;m)$, where $m \ge n+2 \ge 3$. They are asymptotic to Minkowski spacetime factored by $\Gamma(m;n)$. In the general stationary case, by contrast, the new metrics describe spacetimes with an horizon and with a periodicity condition on the time coordinate; these examples can be thought of as five-dimensional analogues of the four-dimensional Taub-NUT metrics.

Contents

1 Introduction

A very considerable literature exists on the subject of exact solutions in four-dimensional general relativity (see, for example, [1]). With the advent of higher-dimensional supergravity and string theory, it has become important to extend the search for exact solutions to the arena of higher dimensions. This is important because these solutions may have intrinsic significance within string theory or M-theory. An investigation of higher-dimensional solutions is also interesting because low dimensions, such as four, may have special features that no longer persist in higher dimensions.

An example of particular importance concerns the uniqueness of black-hole solutions. In four dimensions, there exist very powerful theorems which establish, among other things, that S^2 is the only allowed topology for the event horizon of a black hole. Furthermore, uniqueness theorems have been established which demonstrate that a four-dimensional black hole solution of the vacuum Einstein equations is completely characterised by its mass and angular momentum. It has been known for some time that the notion of black-hole uniqueness is very much weaker in higher dimensions. The most obvious illustration is provided by the five-dimensional black ring solution [2], which has an event horizon whose topology is $S^2 \times S^1$. This contrasts with the S^3 horizon topology of the static spherically-symmetric Schwarzschild-Tangherlini [3, 4] five-dimensional black hole, and its rotating generalisation, which was found in [5]. Many other ring-like black holes have also been found in five dimensions (see, for example, [6], for a recent summary).

There are also other simple examples of black holes with non-standard geometry in $D \geq 5$ dimensions. Starting from the usual cohomogeneity-1 Schwarzschild-Tangherlini metric, which is spherically symmetric and has an S^{D-2} horizon geometry, one can replace the round $(D-2)$-sphere by any other Einstein manifold with the same Ricci curvature. Of course the metric will be asymptotically flat (in the strict sense of approaching Minkowski spacetime at infinity) only for the case of the round $(D-2)$-sphere, but the curvature will go to zero at infinity for all choices. In five dimensions, the only options involve replacing S^3 by S^3/Γ, where Γ is some subgroup of $SO(4)$ that acts freely on S^3. Examples are the lens spaces $L(m;n) = S^3/\Gamma(m;n)$. The spacetime is asymptotic to $(\text{Minkowski})_5/\Gamma(m;n)$. One of the results in this paper is to show that aside from these trivial generalisations of the cohomogeneity-1 Schwarzschild-Tangherlini solution, there exist more complicated cohomogeneity-2 black holes that also have $(\text{Minkowski})_5/\Gamma(m;n)$ asymptotic structures.

A particularly rich class of solutions in four dimensions is provided by the Type D metrics. These include the Schwarzschild and Kerr black holes, and also the Taub-NUT metrics. A convenient formulation of many of the Type D metrics, including Kerr-Taub-NUT, was given by Plebanski in [7]. Subsequently, Plebanski and Demianski [8] gave an elegant formulation of the most general Type D metrics. Amongst vacuum solutions, the key new feature in the extension by Plebanski and Demianski was the inclusion of the acceleration parameter as well as the mass, NUT charge and angular momentum. A cosmological constant can also be included. (See [1] for a more complete discussion of these metrics.)

The generalisation of the Kerr solution to arbitrary dimension was obtained by Myers and Perry [5]. This was later generalised to include a cosmological constant; in five dimensions in [9], and in arbitrary dimensions in [10, 11]. In further generalisations, it was shown in [12, 13] that NUT charges can also be introduced in the higher-dimensional rotating black hole solutions. (The counting of parameters is different in even and in odd dimensions. In $D = 2n$ dimensions there are $(n-1)$ independent NUT charges and $(n-1)$ independent angular momenta, while in $D = 2n+1$ dimensions there are $(n-2)$ NUT charges and n angular momenta.) In the process of constructing the NUT-charged solutions [12, 13], the metrics were cast into a form that is a very natural higher-dimensional generalisation of the four-dimensional Plebanski metric of [7].

It is natural now to investigate whether yet more general solutions can be obtained in higher dimensions by procedures that generalise the construction given in [8]. In the case of five dimensions, we find that this can indeed be done, leading to a new larger class of vacuum solutions. Our new five-dimensional metrics contain one additional non-trivial parameter, over and above the parameters in the Myers-Perry black hole (making three non-trivial dimensionless parameters plus one scale parameter, in total).

After obtaining the new solutions, we give a detailed analysis of their local and global properties. First, we show by taking an appropriate limit that they contain the Myers-Perry black holes (with two independent rotation parameters) as a special case, and by taking a different limit, we show that they contain the original black ring solution of [2] as another special case. A third limit gives rise to a class of static metrics.

In a detailed analysis of the static metric limit, we show that with appropriate choices for the parameters these describe black-hole spacetimes in which the asymptotic spacetime geometry is $(\text{Minkowski})_5/\Gamma(m;n)$, where $\Gamma(m;n)$ is a certain discrete subgroup of $SO(4)$. The spatial surfaces at large radius are the lens space $L(m;n) = S^3/\Gamma(m;n)$. The horizon, on the other hand, has the topology of the lens space $L(n;m)$ (with a distorted, non-Einstein metric). Note that these new solutions are very different from the five-dimensional generalised Schwarzschild-Tangherlini "lens space black holes" mentioned earlier. In those examples, which have cohomogeneity 1, both the horizon and the spatial sections at infinity are round $L(m;n)$ spaces, whereas in our new metrics, which have cohomogeneity 2, there is a sort of "slumping" in which the horizon has $L(n;m)$ topology while the spatial sections at infinity have $L(m;n)$ topology.

The difference between these lens space black holes can be seen by looking at the dimensionless quantity $16\pi ST^4$. This is equal to $1/m$ for the factored Schwarzschild-Tangherlini black hole with $L(m;n)$ asymptotic structure, and is larger than $1/m$ for our new "slumped" black hole with the same $L(m;n)$ asymptotic structure.

We also study the global structure of the general, stationary, metrics. These turn out to have properties that are somewhat analogous to those of the four-dimensional Taub-NUT metrics, in that the time coordinate must be periodic in order to avoid conical singularities. Unlike the four-dimensional Taub-NUT metrics, however, there is no fibering of the time coordinate at infinity.

2 The General Local Solution

2.1 The new five-dimensional Ricci-flat metrics

The five-dimensional vacuum solution describing a rotating black hole with two independent angular momenta is contained within the higher-dimensional rotating black holes found in [5]. The generalisation of the five-dimensional solution to include a cosmological constant was obtained in [9]. It was then shown in [12, 13] that this solution to the Einstein equations $R_{\mu\nu} = 4\lambda g_{\mu\nu}$ could be written in the simple form

$$\begin{aligned} ds_5^2 &= \frac{x-y}{4X}\,dx^2 + \frac{y-x}{4Y}\,dy^2 + \frac{X\,(d\phi+yd\psi)^2}{x(x-y)} + \frac{Y\,(d\phi+xd\psi)^2}{y(y-x)} \\ &\quad + \frac{a_0}{xy}\Big(d\phi+(x+y)d\psi+xydt\Big)^2\,, \end{aligned} \tag{2.1}$$

where

$$X = a_0 + a_1\,x + a_2\,x^2 - \lambda x^3\,, \qquad Y = a_0 + b_1\,y + a_2\,y^2 - \lambda y^3\,. \tag{2.2}$$

The constants a_0, a_1, a_2 and b_1 are related to the two angular momenta, the mass and the NUT parameter [12, 13]. Note that the metric has a coordinate scaling symmetry $x \to \mu x$, $y \to \mu y$, which can be used to eliminate the NUT parameter [12]. To interpret this metric as a rotating black hole, appropriate Wick rotations must be performed. In particular, ϕ is Wick rotated to become the time coordinate.

The metric (2.1) is closely analogous to the form of the four-dimensional Kerr-de Sitter metric given in [7]. It is therefore natural to seek a generalisation of (2.1), analogous to the four-dimensional generalisation with acceleration parameter that was given in [8]. Accordingly, we may try an ansatz of the form

$$\begin{aligned} ds_5^2 &= \Omega_1(xy)\Big[\frac{x-y}{4X}dx^2 + \frac{y-x}{4Y}dy^2 + \frac{X\,(d\phi+yd\psi)^2}{x(x-y)} + \frac{Y\,(d\phi+xd\psi)^2}{y(y-x)}\Big] \\ &\quad + \frac{a_0}{xy}\,\Omega_2(xy)\Big(d\phi+(x+y)d\psi+xydt\Big)^2\,, \end{aligned} \tag{2.3}$$

where again we assume $X = X(x)$ and $Y = Y(y)$.

We find that provided we assume the cosmological constant vanishes, Ricci-flat solutions of the form (2.3) can arise, in the case that

$$\Omega_1(xy) = \frac{1}{(1-xy)^2}\,, \qquad \Omega_2(xy) = 1\,. \tag{2.4}$$

Specifically, we find that the metric

$$\begin{aligned} ds_5^2 &= \frac{1}{(1-xy)^2}\Big[\frac{x-y}{4X}dx^2 + \frac{y-x}{4Y}dy^2 + \frac{X\,(d\phi+yd\psi)^2}{x(x-y)} + \frac{Y\,(d\phi+xd\psi)^2}{y(y-x)}\Big] \\ &\quad + \frac{a_0}{xy}\Big(d\phi+(x+y)d\psi+xydt\Big)^2\,, \end{aligned} \tag{2.5}$$

is Ricci flat, provided that X and Y are given by

$$\begin{aligned} X &= a_0 + a_3\,x + a_2\,x^2 + a_1\,x^3 + a_0\,x^4 \\ Y &= a_0 + a_1\,y + a_2\,y^2 + a_3\,y^3 + a_0\,y^4\,. \end{aligned} \tag{2.6}$$

In what follows, it will sometimes turn out to be convenient to make a change of coordinates in which we send

$$x \to 1/x\,,\quad t \to \mathrm{i}\,t\,,\quad \phi \to \mathrm{i}\,\phi\,,\quad \psi \to \mathrm{i}\,\psi\,. \tag{2.7}$$

After doing so, the metric (2.5) can be written as

$$\begin{aligned} ds_5^2 = {}& \frac{1}{(x-y)^2}\Big[\frac{x(1-xy)dx^2}{4G(x)} - \frac{x(1-xy)dy^2}{4G(y)} - \frac{G(x)(d\phi+yd\psi)^2}{(1-xy)} + \frac{xG(y)(d\psi+xd\phi)^2}{y(1-xy)}\Big] \\ & -\frac{a_0\,y}{x}\Big(dt + \frac{x}{y}\,d\phi + (x+y^{-1})d\psi\Big)^2\,, \end{aligned} \tag{2.8}$$

where

$$G(\xi) \equiv a_0 + a_1\,\xi + a_2\,\xi^2 + a_3\,\xi^3 + a_0\,\xi^4\,. \tag{2.9}$$

Since a Ricci-flat metric remains Ricci-flat when scaled by any constant factor, one can absorb one of the four parameters in (2.9) into an overall dimensionful scale. This then implies that the local metric (2.8) has three non-trivial continuous parameters.[1] These three parameters are (by definition) dimensionless.

As we shall now show, the metric (2.5) admits several limiting forms that are of interest.

2.2 Limits of the new five-dimensional metrics

We find that there are three limiting cases that are of particular interest:

Case I:

In this limit, we start from the general metric in the form (2.5) and then send

$$\begin{aligned} & x \to \epsilon^2\,x\,,\quad y \to \epsilon^2\,y\,,\quad \phi \to \epsilon^{-1}\,\phi\,,\quad \psi \to \epsilon^{-3}\,\psi\,,\quad t \to \epsilon^{-5}\,t\,, \\ & a_0 \to \epsilon^6\,a_0\,,\quad a_1 \to \epsilon^4\,a_1\,,\quad a_2 \to \epsilon^2\,a_2\,,\quad a_3 \to \epsilon^4\,a_3\,. \end{aligned} \tag{2.10}$$

Upon sending ϵ to zero, the metric reduces to (2.1), with

$$X = a_0 + a_1\,x + a_2\,x^2\,,\qquad Y = a_0 + a_3\,y + a_2\,y^2\,, \tag{2.11}$$

[1] In the same sense, the Schwarzschild solution has no non-trivial continuous parameters, since the scale of the mass can be absorbed through an overall rescaling of the metric.

which is of the form (2.2) with vanishing cosmological constant. Thus in the $\epsilon \to 0$ limit of (2.10), the general metric reduces to the five-dimensional Myers-Perry black hole, in the form given in [12, 13]. It has two non-trivial continuous (dimensionless) parameters. (One of the four parameters in (2.11) can be absorbed by means of a coordinate transformation, and a second by making an overall constant scaling of the metric.)

Case II:

To describe this limit, it is convenient to start from the general metric in the form (2.8). We then scale the coordinates and parameters according to

$$\begin{aligned} &x \to \epsilon^2\, x\,, \quad y \to \epsilon^2\, y\,, \quad \phi \to \epsilon\, \phi\,, \quad \psi \to \epsilon\, \psi\,, \quad t \to \epsilon^{-1}\, t\,, \\ &a_0 \to \epsilon^2\, a_0\,, \quad a_1 \to a_1\,, \quad a_2 \to \epsilon^{-2}\, a_2\,, \quad a_3 \to \epsilon^{-4}\, a_3\,. \end{aligned} \tag{2.12}$$

Upon sending ϵ to zero, this leads to the metric

$$ds_5^2 = \frac{1}{(x-y)^2}\Big[\frac{xdx^2}{4G(x)} - \frac{xdy^2}{4G(y)} - G(x)d\phi^2 + \frac{xG(y)d\psi^2}{y}\Big] - \frac{a_0\, y}{x}\,\Big(dt + y^{-1}\, d\psi\Big)^2\,, \tag{2.13}$$

where

$$G(\xi) \equiv a_0 + a_1\, \xi + a_2\, \xi^2 + a_3\, \xi^3\,. \tag{2.14}$$

The local metric in this limit has two non-trivial continuous (dimensionless) parameters. (One of the four parameters in (2.14) can be absorbed by means of a coordinate transformation, and a second by making an overall constant scaling of the metric.)

As we shall discuss in appendix A, this metric contains the original black ring, found in [2]. The *Case II* limit is a five-dimensional analogue of the limit in which the Plebanski-Demianski metric gives rise to the C-metric in four dimensions (see, for example, [1]). In fact, the local black ring solution was obtained from Wick rotation of the Kaluza-Klein lifting [14] of a dilatonic generalisation of the four-dimensional C-metric [15].

Case III:

The third limiting form that we shall consider is obtained from (2.8) by making the scaling

$$a_0 \to \epsilon^2\, a_0\,, \qquad a_0^{1/2}\, t \to \epsilon^{-1}\, t\,, \qquad \phi \to \mathrm{i}\phi\,, \qquad \psi \to \mathrm{i}\psi\,, \tag{2.15}$$

with all other coordinates and parameters left unscaled. Upon sending ϵ to zero, we obtain the metric

$$\begin{aligned} ds_5^2 = \frac{1}{(x-y)^2}\Big[&\frac{(1-xy)dx^2}{4G(x)} - \frac{x(1-xy)dy^2}{4yG(y)} + \frac{xG(x)(d\phi + yd\psi)^2}{(1-xy)} - \frac{xG(y)(d\psi + xd\phi)^2}{(1-xy)}\Big] \\ &-\frac{y}{x}\, dt^2\,, \end{aligned} \tag{2.16}$$

where

$$G(\xi) \equiv a_1 + a_2\,\xi + a_2\,\xi^2\,. \tag{2.17}$$

Note that the Wick rotations of ϕ and ψ in (2.16) are performed just for later convenience; the same effect could be achieved by sending $x \to -x$, $y \to -y$ and $\psi \to -\psi$.

The metric (2.16) has two non-trivial continuous (dimensionless) parameters. (One of the three parameters in (2.17) can be absorbed by an overall constant scaling of the metric.) As we shall discuss below, these static metrics describe a rather wide class of black holes.

3 Static Black Holes with New Geometry

The local metric of the static black holes that we are going to discuss is given by (2.16). We choose to parameterise the constants a_i in such a way that the function G becomes

$$G(\xi) = -\mu^2(\xi - \xi_1)(\xi - \xi_2)\,, \tag{3.1}$$

where

$$0 < \xi_1 \le \xi_2\,, \qquad \xi_1\xi_2 \le 1\,. \tag{3.2}$$

The coordinates x and y lie in the ranges

$$\xi_1 \le x \le \xi_2\,, \qquad -\infty \le y \le \xi_1\,. \tag{3.3}$$

The asymptotic region at infinity occurs at $x = \xi_1 = y$, and the horizon is located at $y = 0$. There is a power-law singularity at $y = \infty$, which is hidden by the horizon when the ξ_i parameters are chosen as described above. (There would also be a power-law singularities at $x = 0$ and $xy = 1$, but these do not lie within the spacetime manifold, for the choice of coordinate ranges and parameters we are making.) The metric contains no closed time-like circles outside the horizon.

Two special cases arise. One case is when $\xi_1\xi_2 = 1$, for which the solution reduces to standard five-dimensional Schwarzshild-Tangherlini black hole. The other special case is when $\xi_1 = \xi_2$. This gives rise to the Kaluza-Klein monopole, and it is discussed in appendix B. Our focus in this section, therefore, will be when the parameters lie in the range

$$0 < \xi_1 < \xi_2\,, \qquad \xi_1\xi_2 < 1\,. \tag{3.4}$$

To determine the periods that the azimuthal coordinates ϕ and ψ must take in order to avoid any possible conical singularities, we need to investigate the spacelike Killing vectors that degenerate to zero length at each of the three locations $x = \xi_1$, $x = \xi_2$ and $y =$

ξ_1. We normalise them by requiring that each have unit Euclidean surface gravity at its corresponding degeneration surface. (This ensures that each is associated with a 2π period.) We find that the three degenerate Killing vectors are given by

$$\begin{aligned} x = \xi_1 : \qquad & \ell_1 = \frac{\partial}{\partial \phi_1}\,, \\ x = \xi_2 : \qquad & \ell_2 = \alpha\,\frac{\partial}{\partial \phi_1} + \beta\,\frac{\partial}{\partial \phi_2}\,, \\ y = \xi_1 : \qquad & \ell_3 = \frac{\partial}{\partial \phi_2}\,, \end{aligned} \tag{3.5}$$

where we have defined two new azimuthal coordinates ϕ_1 and ϕ_2, related to the original ψ and ϕ coordinates by

$$\phi_1 = \frac{\mu^2\sqrt{\xi_1}(\xi_2-\xi_1)(\phi+\xi_1\psi)}{1-\xi_1^2}\,, \qquad \phi_2 = \frac{\mu^2\sqrt{\xi_1}(\xi_2-\xi_1)(\psi+\xi_1\phi)}{1-\xi_1^2}\,. \tag{3.6}$$

The constants α and β in (3.5) are given by

$$\alpha = \frac{(1-\xi_1\xi_2)\sqrt{\xi_1}}{(1-\xi_1^2)\sqrt{\xi_2}}\,, \qquad \beta = -\frac{(\xi_2-\xi_1)\sqrt{\xi_1}}{(1-\xi_1^2)\sqrt{\xi_2}}\,. \tag{3.7}$$

It is clear that the Killing vectors ℓ_1, ℓ_2 and ℓ_3 are linearly dependent. In order to avoid conical singularities, it is necessary that the coefficients of the linear dependence be coprime integers, *i.e.*

$$p\ell_1 + m\ell_2 + n\,\ell_3 = 0\,. \tag{3.8}$$

(See [16] for a discussion of this technique for studying the removal of conical singularities in metrics with degeneration surfaces.) Furthermore, note that ℓ_2 and ℓ_3 can be simultaneously degenerate when $x = \xi_2$ and $y = \xi_1$, which implies that any linear combination of ℓ_2 and ℓ_3 is also a degenerate Killing vector at this surface. For the coprime integer pair (m, n), the minimum period generated by $m\ell_2 + n\ell_3$ is 2π. It follows that in order to avoid a conical singularity, we must have $p = \pm 1$. Without loss of generality, let $p = -1$, and hence

$$\ell_1 = m\ell_2 + n\ell_3\,. \tag{3.9}$$

It follows from (3.5) and (3.7) that we have

$$\frac{(1-\xi_1^2)\sqrt{\xi_2}}{(1-\xi_1\xi_2)\sqrt{\xi_1}} = m\,, \qquad \frac{\xi_2-\xi_1}{1-\xi_1\xi_2} = n\,. \tag{3.10}$$

Thus the solution space is parameterised by a pair of coprime integers (m, n). For the parameter range specified in (3.4), the integers (m, n) must obey the inequalities

$$m \ge n+2 \ge 3\,. \tag{3.11}$$

(The case $n = 1$ occurs when $\xi_2 = 1$.)

To understand the global structure of the spacetime, we first examine the region at the double degeneration ($x = \xi_2, y = \xi_1$) in more detail. It is useful to introduce two coordinates ρ and ϑ, related to x and y by

$$x = \xi_2 - \frac{\rho^2 \sin^2\vartheta}{\xi_1}\,, \qquad y = \xi_1 - \frac{\rho^2 \cos^2\vartheta}{\xi_2}\,. \tag{3.12}$$

The double degeneration will occur at $\rho = 0$. We also introduce new azimuthal coordinates χ_1 and χ_2, defined by

$$\chi_1 = m\phi_1\,, \qquad \chi_2 = \phi_2 + n\phi_1\,. \tag{3.13}$$

These are defined so that the Killing vectors ℓ_2 and ℓ_3 defined in (3.5) are simply given by

$$\ell_2 = \frac{\partial}{\partial\chi_1}\,, \qquad \ell_3 = \frac{\partial}{\partial\chi_2}\,. \tag{3.14}$$

For small ρ, we then find that the metric (2.16) approaches

$$ds^2 = -\frac{\xi_1}{\xi_2}\,dt^2 + \frac{(1-\xi_1\xi_2)}{\mu^2\xi_1(\xi_3-\xi_1)^3}\Big[d\rho^2 + \rho^2(d\vartheta^2 + \sin^2\vartheta\,d\chi_1^2 + \cos^2\vartheta\,d\chi_2^2)\Big]\,. \tag{3.15}$$

In order not to have conical singularities, we see that χ_1 and χ_2 must independently have period 2π. (In other words, they are defined on a square lattice of side 2π.) This ensures that the constant-ρ surfaces at small ρ are precisely round 3-spheres, with no identifications. The periodicities $\Delta\chi_1 = 2\pi$, $\Delta\chi_2 = 2\pi$ are consistent with the expectation from (3.14).

One might think that there would be another simultaneous degeneration surface at $x = \xi_1 = y$, which could lead to another periodicity restriction. This, however, is not the case. As we already mentioned, $x = \xi_1 = y$ is actually asymptotic infinity. This can be seen clearly if we introduce new coordinates r and θ, defined in terms of x and y by

$$\frac{\sqrt{\xi_1 - y}}{x-y} = \frac{\mu\sqrt{\xi_2-\xi_1}}{\sqrt{1-\xi_1^2}}\,r\,\cos\theta\,, \qquad \frac{\sqrt{x-\xi_1}}{x-y} = \frac{\mu\sqrt{\xi_2-\xi_1}}{\sqrt{1-\xi_1^2}}\,r\,\sin\theta\,. \tag{3.16}$$

As $r \to \infty$, the metric can be seen to approach Minkowski spacetime locally, with

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi_1^2 + \cos^2\theta\,d\phi_2^2)\,. \tag{3.17}$$

Inverting (3.13), we see that the azimuthal coordinates ϕ_1 and ϕ_2 are related to χ_1 and χ_2 by

$$\phi_1 = \frac{1}{m}\,\chi_1\,, \qquad \phi_2 = \chi_2 - \frac{n}{m}\,\chi_1\,. \tag{3.18}$$

Our discussion of the regularity conditions at the double degeneration ($x = \xi_2, y = \xi_1$) showed that χ_1 and χ_2 are periodic on a square lattice of side 2π. It then follows from (3.18) that ϕ_1 and ϕ_2 are periodic on a tilted lattice, in which identifications are made under the two operations

$$\begin{aligned} &1) \qquad \phi_1 \longrightarrow \phi_1\,, \qquad \phi_2 \longrightarrow \phi_2 + 2\pi\,, \\ &2) \qquad \phi_1 \longrightarrow \phi_1 + \frac{2\pi}{m}\,, \qquad \phi_2 \longrightarrow \phi_2 - \frac{2\pi n}{m}\,. \end{aligned} \tag{3.19}$$

These are precisely the identifications that arise for the lens space $L(m;n)$. This can be seen from the definition of $L(m;n)$. One takes $S^3 \subset \mathbb{C}^2$, with complex coordinates (z_1, z_2), and quotients according to

$$(z_1, z_2) \equiv (z_1\, e^{2\pi \mathrm{i}/m}, z_2\, e^{2\pi \mathrm{i} n/m})\,, \tag{3.20}$$

(where m and n are coprime integers with $1 \le n \le m-1$.) Taking $z_1 = \sin\theta\, e^{\mathrm{i}\phi_1}$ and $z_2 = \cos\theta\, e^{-\mathrm{i}\phi_2}$, we see that the lens space $L(m;n)$ is indeed defined by the identifications (3.19).

The horizon of the black hole is located at $y = 0$, and from (2.16), its metric is given by

$$\begin{aligned} ds_H^2 \;=\; & \frac{(x-\xi_1)\Big(\xi_2(1-\xi_1^2) - x(1-\xi_1\xi_2)\Big)}{\mu^2\xi_1(\xi_1-\xi_2)^2 x}\Big(d\phi_1 + \frac{\xi_1(1-\xi_1\xi_2)}{\xi_2\Big((1-\xi_1^2) - x(1-\xi_1\xi_2)\Big)} d\phi_2\Big)^2 \\ & + \frac{(1-\xi_1^2)\xi_2(\xi_2 - x)}{\mu^2(\xi_2-\xi_1)^2 x\Big(\xi_2(1-\xi_1^2) - x(1-\xi_1\xi_2)\Big)} d\phi_2^2 + \frac{dx^2}{4x^2 G(x)}\,. \end{aligned} \tag{3.21}$$

It is easy to verify that this is not an Einstein metric, and it is not homogeneous.

In order to understand the geometry of the event horizon, it is helpful to introduce new azimuthal coordinates $\tilde\phi_1$ and $\tilde\phi_2$, chosen so that the Killing vectors ℓ_1 and ℓ_2 that vanish at $x = \xi_1$ and $x = \xi_2$ are simply given by

$$\ell_1 \equiv \frac{\partial}{\partial\tilde\phi_1}\,, \qquad \ell_2 \equiv \frac{\partial}{\partial\tilde\phi_2}\,. \tag{3.22}$$

These coordinates are related to χ_1 and χ_2, and to ϕ_1 and ϕ_2, by

$$\tilde\phi_1 = \frac{1}{n}\chi_2 = \phi_1 + \frac{1}{n}\phi_2\,, \qquad \tilde\phi_2 = \chi_1 - \frac{m}{n}\chi_2 = -\frac{m}{n}\phi_2\,. \tag{3.23}$$

In terms of $\tilde\phi_1$ and $\tilde\phi_2$, the metric (3.21) on the horizon can conveniently be written in the following two ways:

$$ds_H^2 = \frac{dx^2}{4\mu^2 x^2(\xi_2 - x)(x-\xi_1)} + \frac{(\xi_2 - x)g_1\big(d\tilde\phi_2 + f_1 d\tilde\phi_1\big)^2}{\mu^2 x\xi_2(\xi_2-\xi_1)^2} + \frac{(x-\xi_1)\xi_2}{\mu^2 x g_1}\, d\tilde\phi_1^2\,, \tag{3.24}$$

$$ds_H^2 = \frac{dx^2}{4\mu^2 x^2(\xi_2 - x)(x-\xi_1)} + \frac{(x-\xi_1)g_2\big(d\tilde\phi_1 + f_2 d\tilde\phi_2\big)^2}{\mu^2 x\xi_1(\xi_2-\xi_1)^2} + \frac{(\xi_2-x)\xi_1}{\mu^2 x g_2}\, d\tilde\phi_2^2\,, \tag{3.25}$$

where

$$\begin{aligned} g_1 &= (x-\xi_1)+(\xi_2-x)\xi_1\xi_2\,, \qquad f_1=\frac{(x-\xi_1)(1-\xi_1\xi_2)\sqrt{\xi_2}}{g_1\sqrt{\xi_1}}\,,\\ g_2 &= (\xi_2-x)+(x-\xi_1)\xi_1\xi_2\,, \qquad f_2=\frac{(\xi_2-x)(1-\xi_1\xi_2)\sqrt{\xi_1}}{g_2\sqrt{\xi_2}}\,. \end{aligned} \tag{3.26}$$

We can study the regions in the vicinity of the "north and south poles" at $x = \xi_1$ and $x = \xi_2$ by defining a new "latitude" coordinate ρ_1 such that $x = \xi_1 + \rho_1^2$, or a new latitude coordinate ρ_2 such that $x = \xi_2 - \rho_2^2$. Near $\rho_1 = 0$ at the north pole the metric approaches

$$ds_H^2 \to \frac{1}{\mu^2\xi_1^2(\xi_2-\xi_1)}\Big(d\rho_1^2+\rho_1^2 d\tilde\phi_1^2\Big)+\frac{1}{\mu^2}d\tilde\phi_2^2\,, \tag{3.27}$$

whilst near $\rho_2 = 0$ at the south pole the metric approaches

$$ds_H^2 \to \frac{1}{\mu^2\xi_2^2(\xi_2-\xi_1)}\Big(d\rho_2^2+\rho_2^2 d\tilde\phi_2^2\Big)+\frac{1}{\mu^2}d\tilde\phi_1^2\,. \tag{3.28}$$

As far as degenerations of the local metric are concerned, ds_H^2 exhibits the same essential behaviour as the standard metric on S^3,

$$d\Omega_3^2 = d\theta^2+\sin^2\theta d\tilde\phi_1^2+\cos^2\theta d\tilde\phi_2^2\,. \tag{3.29}$$

The geometric *details* of the actual horizon metric ds_H^2 differ from (3.29) in several respects, but these are all in the form of smooth distortions that do not have any impact on global topological considerations. As we already remarked, the horizon metric is neither Einstein nor homogeneous.

It can be seen from the equations in (3.23) that the relation between $(\tilde\phi_1, \tilde\phi_2)$ and (χ_1, χ_2) is just like the relation (3.18) between (ϕ_1, ϕ_2) and (χ_1, χ_2), except that the roles of the integers m and n are reversed. It follows that if we now repeat, on the horizon, the argument that showed the topology of the r =constant spatial surfaces at large r are lens spaces $L(m; n)$, we will find that the topology of the horizon is the lens space $L(n; m)$. However, as mentioned above, geometrically, the horizon is an inhomogeneously-distorted $L(n; m)$ lens space. In view of the inequalities satisfied by m and n, which are given in (3.11), there are only n inequivalent topologies for the horizons, since the lens spaces $L(n; p)$ and $L(n; p+n)$ are identical. Note that $n = 1$ is an allowed value (see (3.11)), in which case the lens spaces $L(1; m)$ are all topologically just S^3.

It is now a straightforward matter to calculate the area A of the horizon, and hence its entropy $S = \frac14 A$. It is given by

$$\begin{aligned} S &= \tfrac14\int\sqrt{g_3} = \tfrac14\int d\phi_1 d\phi_2\int_{\xi_1}^{\xi_2} dx\frac{(1-\xi_1^2)\sqrt{\xi_2}}{2\mu^3\sqrt{\xi_1}(\xi_2-\xi_1)^2x^2}\\ &= \frac{\pi^2(1-\xi_1\xi_2)}{2\mu^3\xi_1\xi_2(\xi_2-\xi_1)}\,. \end{aligned} \tag{3.30}$$

We may also calculate the surface gravity κ, computed for the timelike Killing vector $K_0 = \partial/\partial t$, and hence obtain the Hawking temperature $T = \kappa/(2\pi)$. It is given by

$$T = \frac{\mu\sqrt{\xi_1 \xi_2}}{2\pi}\,. \tag{3.31}$$

The ADM mass is also easy to calculate, by means of a Komar integral. It is given by

$$M = \frac{3}{32\pi}\int *dK_0 = \frac{3\pi(1-\xi_1\xi_2)}{8\mu^2(\xi_2-\xi_1)\sqrt{\xi_1\xi_2}}\,. \tag{3.32}$$

It should be noted that in this calculation, involving an integration over the boundary $L(m;n)$ lens space at infinity, and also in the calculation of the horizon area, involving an integration over the inhomogeneous lens space $L(n;m)$ at the horizon, one must take care to handle the azimuthal coordinate integrations carefully, paying due regard to the periodicity conditions implied by the lens-space identifications. The general rule is that when a given 3-sphere metric is factored to give the lens space $L(p;q)$, the 3-volume is reduced by a factor of $1/p$.

It is straightforward to verify that the black holes satisfy the first law of thermodynamics, namely

$$dM = T\,dS\,. \tag{3.33}$$

Furthermore, we have

$$M = \tfrac{3}{2} T\,S\,, \tag{3.34}$$

as in the case of the standard Schwarzschild black hole in five dimensions.

Thus we have constructed a large class of $D = 5$ static black holes whose topology is specified by a pair of integers (m, n) lying in the range of (3.11). The metric has three linearly-dependent degenerate spacelike Killing vectors (ℓ_1, ℓ_2, ℓ_3) with unit Euclidean surface gravity. These Killing vectors slump from asymptotic infinity to the horizon. In the horizon, there are only two degenerate Killing vectors (ℓ_1, ℓ_2), giving rise to a geometry of non-homogeneously distorted lens space $L(n;m)$. In the asymptotic region, the (only) two degenerate Killing vectors are (ℓ_1, ℓ_3), and the large-r spatial sections have the geometry of homogeneous lens spaces $L(m;n)$.

It should be emphasised that these results do not contradict results on the uniqueness of higher-dimensional static asymptotically-flat black holes in [17]. Since the spatial sections at large distance in our new solutions have the topology of the $L(m;n)$ lens space, which is the quotient of S^3 by a certain discrete subgroup $\Gamma(m;n)$ of $SO(4)$, it follows that although the curvature tends to zero at infinity the spacetime is not asymptotic to Minkowski spacetime, but, rather, to the quotient Minkowksi/$\Gamma(m;n)$. Thus the conditions assumed in [17], under which uniqueness could be proved, are not satisfied.

One can also, of course, consider a different and considerably simpler static black hole with the same asymptotic geometry Minkowksi/$\Gamma(m;n)$. As was noted in [17], the round S^n in any $D = n+2$ dimensional Schwarzschild-Tangherlini solution can be replaced by an arbitrary Einstein space of the same Ricci curvature. Although the five-dimensional example was not discussed explicitly in [17], one can simply replace S^3 in the five-dimensional Schwarzschild-Tangherlini spacetime by the lens space $L(m;n)$. In this case, unlike our new solutions, the horizon will have the same round $L(p;q)$ lens space geometry as the large$-r$ spatial sections. There are only two zero-length Killing vectors in the whole metric. These factored Schwarzschild-Tangherlini solutions are of cohomogeneity 1, in contrast to our new solutions, which have cohomogeneity 2. For each of the new solutions with asymptotic $L(m;n)$ spatial sections that we have obtained in this paper, there is another, inequivalent, black hole with the same asymptotic structure, obtained instead by simply factoring the S^3 in the Schwarzschild-Tangherlini solution by $\Gamma(m;n)$.

One way to compare the different black-hole metrics is to look at the dimensionless quantity obtained by multiplying the entropy by the cube of the temperature. From (3.10), (3.30) and (3.31) we find

$$S = \frac{1}{16\pi T^3}\,\frac{\sqrt{\xi_1\xi_2}}{n}\,. \tag{3.35}$$

At fixed temperature, therefore, the entropy is *maximised* by the Schwarzschild-Tangherlini spacetime, which corresponds to $m = n = 1$ and $\xi_1\xi_2 = 1$. It is interesting to note that the "factored Schwarzschild-Tangherlini" solution, in which S^3 surfaces are quotiented to give $L(m;n) = S^3/\Gamma(m;n)$, will have a smaller entropy than our new "slumped" black hole with $L(n;m)$ horizon topology. This follows from the fact that the former will have entropy $S = 1/(16m\pi T^3)$, whereas the slumped solution has entropy given by (3.35), which is larger by the factor

$$1 + \frac{\xi_1\,(1-\xi_1\xi_2)}{\xi_2-\xi_1} \ge 1\,. \tag{3.36}$$

One further remark concerns the limit $\xi_1\xi_2 \to 1$, which gives the usual Schwarzschild-Tangherlini metric. It might appear that the mass formula (3.32) is incompatible with this limit, since it vanishes when $\xi_1\xi_2 = 1$. To resolve this apparent paradox, we note that when $\xi_1\xi_2 = 1$, it follows from (3.5) that $\ell_2 = -\partial/\partial\phi_2$, Thus (3.8) can be simply solved by letting $p = 0$ and $m = n = 1$. Then the condition (3.9) no longer holds, and ϕ_1 and ϕ_2 both have independent 2π periods. The solution indeed describes the standard Schwarzschild black hole. However, within our general class of black-hole solutions, taking the limit $\xi_1\xi_2 \to 1$

assumes that the condition (3.9) is still imposed. This corresponds to sending m and n to infinity, while keeping $m/n \to 1$. The resulting metric then describes a Schwarzshild-Tangherlini black hole in which the round S^3 is replaced by $S^3/\Gamma(\infty;\infty)$. This has zero volume, and so the mass would vanish too.

4 Charged Static Black Hole with New Geometry

Having obtained the new static black hole solutions, which exhibit the feature of having an $L(n;m)$ lens-space topology on the horizon, which "slumps" to give $L(m;n)$ lens-space spatial sections at infinity, we can easily construct charged generalisations, by using solution-generating techniques involving Kaluza-Klein reduction and U-duality. We shall consider charged solutions in $D=5$, $N=2$ supergravity coupled to two vector multiplets. This $U(1)^3$ theory can also be obtained as a truncation of the maximal $N=8$ supergravity in $D=5$. The bosonic Lagrangian is given by

$$\mathcal{L} = \sqrt{-g}(R - \tfrac{1}{2}\sum_{i=1}^{3} X_i^{-2}(\partial X_i)^2 - \tfrac{1}{4}\sum_{i=1}^{3} X_i^{-2}(F^i)^2 + \tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda}F^1_{\mu\nu}F^2_{\rho\sigma}A^3_\lambda\,, \qquad (4.1)$$

where $F^i = dA^i$, and X_1, X_2 and X_3 satisfy $X_1X_2X_3 = 1$; they describe two scalar fields.

By a standard solution-generating procedure involving lifting to six dimensions, boosting, reducing and acting with U-duality, our previous neutral static solution can be transformed into a charged one. This new charged black hole is given by

$$\begin{aligned} ds_5^2 &= -(H_1H_2H_3)^{-\frac{2}{3}}\,\frac{y}{x}dt^2 + (H_1H_2H_3)^{\frac{1}{3}}\Big\{\frac{1}{(x-y)^2}\Big[\frac{(1-xy)dx^2}{4G(x)} - \frac{x(1-xy)dy^2}{4yG(y)} \\ &\quad + \frac{xG(x)(d\phi+yd\psi)^2}{(1-xy)} - \frac{xG(y)(d\psi+xd\phi)^2}{(1-xy)}\Big]\Big\}\,, \qquad (4.2) \\ X_i &= H_i^{-1}(H_1H_2H_3)^{1/3}\,, \quad A = (1-H_i^{-1})\coth\beta_i\,dt\,, \quad H_i = 1 + \frac{\sinh^2\beta_i(x-y)}{x}\,, \end{aligned}$$

with $G(\xi)$ having the same form

$$G(\xi) = -\mu^2(\xi-\xi_1)(\xi-\xi_2) \qquad (4.3)$$

as in the previous section.

The global analysis proceeds in the same way as in the static case. We take the roots of $G(\xi)$ to satisfy the inequalities $0<\xi_1<\xi_2$ and $\xi_1\xi_2<1$, and the ranges of x and y is the same as in the static case, namely $\xi_1 \le x \le \xi_2$ and $-\infty \le y \le \xi_1$. Power-law singularities are again avoided outside the horizon, since the functions H_i are positive definite outside the horizon.

The solution describes charged static black holes, which, as in the uncharged case, "slump" from a lens space topology $L(m;n)$ on the spatial sections at infinity to $L(n;m)$ topology at the horizon.

The mass, entropy, charge and their respective potentials can be easily obtained, and are given by

$$
\begin{aligned}
M &= \frac{\pi(3+2(s_1^2+s_2^2+s_3^2))(1-\xi_1\xi_2)}{8\mu^2\sqrt{\xi_1\xi_2}(\xi_2-\xi_1)}\,,\\
S &= \frac{\pi^2 c_1c_2c_3(1-\xi_1\xi_2)}{2\mu^3\xi_1\xi_2(\xi_2-\xi_1)}\,,\qquad T=\frac{\mu\sqrt{\xi_1\xi_2}}{2\pi c_1c_2c_3}\,,\\
Q_i &= \frac{\pi c_i s_i(1-\xi_1\xi_2)}{4\mu^2\sqrt{\xi_1\xi_2}(\xi_2-\xi_1)}\,,\qquad \Phi_i=\frac{s_i}{c_i}\,.
\end{aligned}
\tag{4.4}
$$

where $s_i=\sinh\beta_i$ and $c_i=\cosh\beta_i$. These quantities satisfy the expected thermodynamic relations

$$
dM = TdS+\Phi_i dQ_i\,,\qquad M=\tfrac{3}{2}TS+\Phi_i Q_i\,. \tag{4.5}
$$

The solution can be straightforwardly lifted to six dimensions, where it becomes a dyonic string with a pp-wave propagating along the sixth direction.

The BPS limit, corresponding to $T=0$ and $\Phi_i=1$, with

$$
M=Q_1+Q_2+Q_3\,, \tag{4.6}
$$

can be achieved by letting μ and β_i approach infinity, whilst keep the ratio $q_i=s_i/\mu$ fixed. To implement this limit in the solution, it is necessary to let x approach y, accompanied by an appropriate rescaling to keep the metric from degenerating in this limit. In particular, we make the coordinate transformation (3.16), then set $s_i=q_i\mu$, and take the $\mu\to\infty$ limit. The resulting metric is given by

$$
ds^2=-(H_1H_2H_3)^{-\frac{2}{3}}dt^2+(H_1H_2H_3)^{\frac{1}{3}}(dr^2+r^2d\Omega_3^2)\,, \tag{4.7}
$$

where $H_i=1+q_i^2(1-\xi_1^2)/(\xi_1(\xi_2-\xi_2)r^2)$, and $d\Omega_3^2$ is the metric of the "round" lens space $L(m;n)$ (*i.e.* it is an Einstein metric, obtained by factoring the unit 3-sphere by the discrete subgroup $\Gamma(m;n)$ of $SO(4)$). Thus, the "slumping" feature of the non-extremal solution is lost in the extremal limit. The mass and charge for the extremal solution are simply those obtained from the expressions in (4.4), upon taking the limit. However, the entropy is changed, since now the horizon has the same topology and geometry as the asymptotic spatial metric $d\Omega_3^2$. It would be of interest to see whether one could take a different BPS limit of the non-extremal solution that retained the slumping feature.

5 Global Analysis of the General Solutions

We now turn to an analysis of the full metric (2.8) that we found in section 2.1. In section 2.2, we saw that it admits a limit, described in *Case I*, in which it becomes the standard Myers-Perry black hole with two independent rotation parameters. Actually, the *Case I* limit ostensibly has a further parameter in addition to the mass and the two angular momenta, which one might wish to identify as a five-dimensional NUT charge. However, as discussed in [12, 13], this "NUT charge" is really a trivial parameter, in the sense that it can be removed by means of a coordinate transformation.

In the full metric (2.8), no analogous coordinate transformation can be made, and so the additional parameter that becomes the (trivial) NUT parameter in the *Case I* limit is now instead non-trivial. We shall analyse the global structure of the full metric (2.8) in this section. Our findings are that for suitable choices of the parameters we can obtain metrics that extend smoothly onto spacetime manifolds that have horizons, but are otherwise free of conical singularities. In order to achieve this, it is necessary for the time coordinate to be appropriately periodically identified. In this respect, the situation is reminiscent of the Taub-NUT metric in four dimensions. However, there are significant differences too, which will emerge as our discussion proceeds.

It is convenient to reparameterise the metric function G in (2.9) in the form

$$G(\xi) = \mu^2(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)\,, \tag{5.1}$$

where $a_0 = \mu^2$ and

$$\xi_4 = \frac{1}{\xi_1\xi_2\xi_3}\,. \tag{5.2}$$

The constant μ has dimensions $(\text{length})^{-1}$, whilst ξ_1, ξ_2 and ξ_3 are the three non-trivial dimensionless parameters.

For reasons that will become apparent later, shall restrict the parameters so that

$$\xi_1 < -1 < \xi_2 < 0 < \xi_3 \le \xi_4\,, \qquad \text{and} \qquad \xi_1\xi_2 \le 1\,. \tag{5.3}$$

The coordinates x and y will lie in the ranges

$$\xi_1 \le x \le \xi_2\,, \qquad \xi_2 \le y \le \infty\,. \tag{5.4}$$

The asymptotic region is approached at $x = y = \xi_2$, and the outer and inner horizons are at $y = \xi_3$ and $y = \xi_4$ respectively. The surface of the ergosphere is at $y = 0$. The curvature has power-law singularities at $xy = 1$ and at $y = \infty$. The former does not lie in the spacetime manifold, and the latter lies behind the horizons. For later convenience, we introduce positive parameters

$$\eta_1 \equiv -\xi_1 \qquad \eta_2 \equiv -\xi_2 \,. \tag{5.5}$$

We now analyse the conditions under which there are no conical singularities outside the horizon. To do this, it is useful first to make coordinate transformations as follows. We begin with a redefinition of the time coordinate,

$$t \to t/\mu + (\eta_2 + \eta_2^{-1})\psi - \phi \,, \tag{5.6}$$

and then we introduce new azimuthal angles ϕ_1 and ϕ_2, defined by

$$\begin{aligned}
\phi_1 &= \frac{\mu^2(\eta_1-\eta_2)(\eta_2+\xi_3)(1+\eta_1\eta_2^2\xi_3)}{\eta_1\eta_2^{3/2}\xi_3(1-\eta_2^2)}(\phi-\eta_2\psi)\,,\\
\phi_2 &= \frac{\mu^2(\eta_1-\eta_2)(\eta_2+\xi_3)(1+\eta_1\eta_2^2\xi_3)}{\eta_1\eta_2^{3/2}\xi_3(1-\eta_2^2)}(\psi-\eta_2\phi)\,,
\end{aligned} \tag{5.7}$$

The metric degenerates at $x=-\eta_1$, $x=-\eta_2$ and $y=-\eta_2$. The three corresponding degenerating Killing vectors, normalised to have unit Euclidean surface gravity, are given by

$$\begin{aligned}
x=-\eta_2: &\qquad \ell_1 = \frac{\partial}{\partial\phi_1}\,,\\
y=-\eta_2: &\qquad \ell_2 = \frac{\partial}{\partial\phi_2}\,,\\
x=-\eta_1: &\qquad \ell_3 = -\frac{\eta_1^{3/2}\xi_3(1-\eta_1\eta_2)}{\mu(\eta_1+\xi_3)(1+\eta_1^2\eta_2\xi_3)}\frac{\partial}{\partial t}\\
&\qquad +\frac{\sqrt{\eta_1}(\eta_2+\xi_3)(1+\eta_1\eta_2^2\xi_3)}{\sqrt{\eta_2}(\eta_1+\xi_3)(1-\eta_2^2)(1+\eta_1^2\eta_2\xi_3)}\Big((1-\eta_1\eta_2)\frac{\partial}{\partial\phi_1}+(\eta_1-\eta_2)\frac{\partial}{\partial\phi_2}\Big)\,.
\end{aligned} \tag{5.8}$$

Since each ℓ_i independently generates a 2π translation around its degeneration surface, it follows in particular that the time coordinate t must be periodic; a property of any Taub-NUT solution. Note that the $\partial/\partial t$ term in ℓ_3 is absent if $\eta_1\eta_2=1$; this special case describes the Myers-Perry rotating black hole (and has no time periodicity).

Now let us consider the asymptotic region, located at $x=y=\xi_2=-\eta_2$. We make the coordinate transformation

$$\frac{\sqrt{\xi_2-x}}{y-x} = A\,r\cos\theta\,, \qquad \frac{\sqrt{y-\xi_2}}{y-x} = A\,r\sin\theta\,, \tag{5.9}$$

where

$$A^2 = \frac{\mu^2(\eta_1-\eta_2)(\eta_2+\xi_3)(1+\eta_1\eta_2^2\xi_3)}{\eta_1\eta_2^2\xi_3(1-\eta_2^2)}\,. \tag{5.10}$$

Taking the limit $r \to \infty$, we see that the metric at infinity approaches

$$ds = -dt^2 + dr^2 + r^2(d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2)\,. \tag{5.11}$$

From the form of the metric near infinity, the Komar integrals giving the ADM mass and the angular momenta can be evaluated. We find

$$\begin{aligned} M &= \frac{3\pi\eta_1\eta_2\xi_3(1-\eta_2^2)}{8\mu^2(\eta_1-\eta_2)(\eta_2+\xi_3)(1+\eta_1\eta_2^2\xi_3)}\,, \\ J_{\phi_2} &= \frac{\pi\eta_1^2\eta_2^{3/2}\xi_3^2(1-\eta_2^2)^2}{4\mu^3(\eta_1-\eta_2)^2(\eta_2+\xi_3)^2(1+\eta_1\eta_2^2\xi_3)^2}\,, \qquad J_{\phi_1} = \eta_2\, J_{\phi_2}\,. \end{aligned} \tag{5.12}$$

The outer horizon is at $y = \xi_3$. The asymptotically timelike Killing vector that degenerates there is given by

$$\ell_0 = \frac{\partial}{\partial t} - \frac{\mu(\eta_1-\eta_2)(1+\eta_1\eta_2^2\xi_3)}{\eta_1\sqrt{\eta_2}(1+\eta_2\xi_3)(1-\eta_2^2)}\Big((1+\eta_2\xi_3)\frac{\partial}{\partial\phi_2} - (\eta_2+\xi_3)\frac{\partial}{\partial\phi_1}\Big)\,. \tag{5.13}$$

Calculating the surface gravity, and the volume of the horizon, we obtain the temperature and the entropy, given by

$$\begin{aligned} T &= \frac{\mu(\eta_1+\xi_3)(1-\eta_1\eta_2\xi_3^2)}{2\pi\eta_1\sqrt{\xi_3}(1+\eta_2\xi_3)}\,, \\ S &= \frac{\pi^2(\eta_1\eta_2)^2\xi_3^{3/2}(1+\eta_2\xi_3)(1-\eta_2^2)}{2\mu^3(\eta_1+\xi_3)(\eta_1-\eta_2)(\eta_2+\xi_3)^2(1+\eta_1\eta_2^2\xi_3)^2}\,. \end{aligned} \tag{5.14}$$

The angular velocities on the horizon are given by

$$\begin{aligned} \Omega_{\phi_1} &= \frac{\mu(\eta_1-\eta_2)(\eta_2+\xi_3)(1+\eta_1\eta_2^2\xi_3)}{\eta_1\sqrt{\eta_2}(1+\eta_2\xi_3)(1-\eta_2^2)}\,, \\ \Omega_{\phi_2} &= -\frac{\mu(\eta_1-\eta_2)(1+\eta_1\eta_2^2\xi_3)}{\eta_2\sqrt{\eta_2}(1-\eta_2^2)}\,. \end{aligned} \tag{5.15}$$

The first law of thermodynamics is not satisfied in these solutions in general. In fact the analysis of the thermodynamics of Taub-NUT solutions is notoriously unsettled. However, a special case arises if $\eta_1\eta_2 = 1$. It is easy to verify in this case that we have

$$dM = TdS + \Omega_{\phi_1}dJ_{\phi_1} + \Omega_{\phi_2}dJ_{\phi_2}\,, \qquad M = \tfrac{3}{2}(TS + \Omega_{\phi_1}J_{\phi_1} + \Omega_{\phi_2}J_{\phi_2})\,. \tag{5.16}$$

In fact when $\eta_1\eta_2 = 1$, the metric is nothing but the Myers-Perry rotating black hole, in an unusual coordinate system.

It is interesting to note that the general local metric (2.8) gives rise to the Myers-Perry solution in two very different limiting ways. One way is *via* the limit discussed in *Case I* in section 2.2, and the other is by taking $\eta_1\eta_2 = 1$ in the general metrics. The coordinate transformation that links one to the other is quite complicated, and we have verified their equivalence by studying the relationship between the mass, entropy and angular momenta.

Finally we remark that although the spacetimes we have obtained here in five dimensions are in some respects similar to four-dimensional Taub-NUT spacetimes, in that the time coordinate is periodic, there are also significant differences. In four dimensions, it is a fibration in the time direction at asymptotic infinity that is responsible for imposing the periodicity of the time coordinate. In our five-dimensional solution, on the other hand, the metric approaches Minkowski spacetime locally at infinity, with no fibration in the time direction. Our solution is also very different in structure from the topological soliton "time machines" obtained in [18], where there are no horizons or singularities in the spacetime. By contrast, our spacetimes here describe black objects with horizons and with singularities inside the horizons. Note, further, that the outer horizon of in solution, which is located at $y = \xi_3$, is separated from the velocity of light surface surrounding the time machine at $y = \xi_2 = -\eta_2$.

6 Conclusions

In this paper, we have constructed new stationary cohomogeneity 2 solutions of the vacuum Einstein equations in five dimensions. We obtained it by starting from the five-dimensional rotating black hole, written in the very simple form found in [12], and then making an ansatz that involved generalising certain metric functions in the rotating black hole, and also introducing a conformal factor in a four-dimensional subspace. Our procedure is somewhat analogous to one that was performed in four dimensions in [8]. Our new metric has three non-trivial (dimensionless) parameters, which is one more than the number in the rotating black hole.[2]

We identified three limiting cases of the new metrics that are of particular interest. *Case I* is a limit that gives back the standard rotating black hole, with two independent rotation parameters. *Case II* is a limit that gives the original single-rotation black ring, which was found in [2]. *Case III* is a limit giving rise to a new family of static metrics, with two non-trivial dimensionless parameters.

Having found the local form of the new metrics, we then studied their global structure. For the static metrics obtained in the *Case III* limit, we found that the conditions following

[2]The rotating black hole has three parameters if one counts the mass, and the two angular momenta. But one of these is "trivial" in the sense that it can be absorbed into an overall scaling of the metric.

from requirement of no conical singularities imposes periodicity conditions on the azimuthal coordinates which imply that the horizon has the topology of the lens space $L(n;m)$, where m and n are positive integers satisfying $m \geq n+2 \geq 3$. The lens space $L(n;m)$ is defined as a factoring S^3 by a certain freely-acting discrete subgroup $\Gamma(n;m)$ of the $SO(4)$ isometry group. The black hole horizon is an inhomogeneous distortion of the "round" lens space. By contrast, asymptotically at infinity the spacetime approaches $(\text{Minkowski})_5/\Gamma(m;n)$. This means that the spatial sections at large radius are lens spaces $L(m;n)$. We calculated all the conserved charges and thermodynamic quantities for these lens-space black holes, and showed that the first law of thermodynamics is satisfied. Our solutions demonstrate that black holes with $(\text{Minkowski})_5/\Gamma(m;n)$ asymptotic structure and a given mass are not unique.

We then generalised the static metrics to charged solutions within five-dimensional $N=2$ supergravity coupled to two vector multiplets. Equivalently, they can be viewed as solutions of five-dimensional $N=8$ supergravity, with three independent charges for the $U(1)^3$ gauge fields in the Cartan subalgebra of $SO(6)$. Again we found the same "slumping" feature exhibited by the uncharged black holes, with $L(n;m)$ horizon topology and $L(m;n)$ spatial sections at infinity.

Finally, we investigated the global structure of the general new solutions with three non-trivial dimensionless parameters. For these, we find that (except for limiting cases that reduce to the previous discussion) the avoidance of conical singularities now requires that the time coordinate also be identified periodically. This is reminiscent of the situation in the Taub-NUT metrics in four dimensions. In fact, one can take the view that the general new solutions we have found are the natural five-dimensional analogue of the four-dimensional rotating Taub-NUT metrics. (The general construction of higher-dimensional rotating Taub-NUT metrics in [12, 13] gave only a trivial "NUT parameter" in the special case of five dimensions.)

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A Rotating Black Ring

The black ring solution (2.13) and (2.14), which we obtained as a limit of our general metrics (2.8), was previously obtained in [2]. However, the metric is somewhat simpler in the coordinates and parameterisation used in (2.13). It is useful, for completeness, to present a summary of the global structure and the thermodynamic quantities in the formalism we are using in this paper.

We begin by reparemeterising the constants in such a way that $G(\xi)$ in (2.14) becomes

$$G(\xi) = -\mu^2(\xi - \xi_1)(\xi - \xi_2)(x - \xi_3)\,. \tag{A.1}$$

We can rescale μ purely by means of scalings of the coordinates and parameters ξ_i, without needing to rescale the metric. Since, in this sense, μ is a trivial parameter, we may set $\mu = 1$ without loss of generality.

To describe the ring, we should take the coordinate x to lie in the range $\xi_1 \le x \le \xi_2 < 0$. The coordinate y lies in the range $\xi_2 \le y \le \infty$. The asymptotic region at infinity is located at $x = \xi_2 = y$, and the horizon is at $y = \xi_3 > 0$. The boundary of the "ergo sphere" lies at $y = 0$. There is a curvature singularity at $y = \infty$, which is hidden behind the horizon at $y = \xi_3$. There would also be a curvature singularity at $1 - xy = 0$ outside the horizon, which can be avoided by taking $\xi_1\xi_2 < 1$ and $\xi_1\xi_3 > -1$. For later convenience, we introduce two positive parameters (η_1, η_2), with

$$\xi_1 = -\eta_1^2\,, \qquad \xi_2 = -\eta_2^2\,. \tag{A.2}$$

Having addressed the question of power-law curvature singularities, we must now examine the possible conical singularities at locations where the metric degenerates. First, we note that the Killing vector $\partial/\partial\phi$ is degenerate at both $x = \xi_1 = -\eta_1^2$ and $x = \xi_2 = -\eta_2^2$. Normalising to unit Euclidean surface gravity, the degenerate Killing vectors at each of $x = \xi_1$ and $x = \xi_2$ are, respectively,

$$\ell_1 = \frac{\eta_1}{(\eta_1^2 - \eta_2^2)(\xi_3 + \eta_1^2)}\,\frac{\partial}{\partial\phi}\,, \qquad \ell_2 = \frac{\eta_2}{(\eta_1^2 - \eta_2^2)(\xi_3 + \eta_2^2)}\,\frac{\partial}{\partial\phi}\,. \tag{A.3}$$

Since these must each generate 2π rotations, it follows that the two prefactors must be equal, and hence we must require

$$\xi_3 = \eta_1\eta_2\,. \tag{A.4}$$

There is also a degenerate spacelike Killing vector ℓ_3 at $y = \xi_2$. It is convenient to make a coordinate transformation to remove a $\partial/\partial t$ component from this Killing vector. This is achieved by sending

$$t \to \frac{\sqrt{\eta_2}}{\eta_1^{3/2}} t + \frac{\eta_1^{3/2}}{\sqrt{\eta_2}} \psi \,. \tag{A.5}$$

(We have also scaled the new t variable to give it the canonical normalisation for time at infinity.) Furthermore, we shall define rescaled azimuthal angles ϕ_1 and ϕ_2, given by

$$\phi_1 = (\eta_1 - \eta_2)(\eta_1 + \eta_2)^2\, \psi \,, \qquad \phi_2 = (\eta_1 - \eta_2)(\eta_1 + \eta_2)^2\, \phi \,. \tag{A.6}$$

In terms of ϕ_1 and ϕ_2, the three spacelike Killing vectors discussed above, which degenerate at $x = \xi_1$, $x = \xi_2$, and $y = \xi_3$, are given, when normalised to unit Euclidean surface gravity, by

$$\ell_1 = \frac{\partial}{\partial \phi_2} \,, \qquad \ell_2 = \frac{\partial}{\partial \phi_2} \,, \qquad \ell_3 = \frac{\partial}{\partial \phi_1} \,. \tag{A.7}$$

Thus we see that ϕ_1 and ϕ_2 should both have period 2π.

Note that $g_{\phi_1\phi_1}$ and $g_{\phi_2\phi_2}$ never become negative outside the horizon.

The region near asymptotic infinity can be seen more clearly by introducing coordinates r and θ, defined in terms of x and y by

$$\frac{\sqrt{\xi_2 - x}}{y - x} = \frac{(\eta_1 + \eta_2)\sqrt{\eta_1 - \eta_2}}{\eta_2}\, r\, \cos\theta \,, \qquad \frac{\sqrt{y - \xi_2}}{y - x} = \frac{(\eta_1 + \eta_2)\sqrt{\eta_1 - \eta_2}}{\eta_2}\, r\, \sin\theta \,. \tag{A.8}$$

The metric in the asymptotic region then takes the simple form

$$ds^2 \to -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2) \,. \tag{A.9}$$

The ADM mass and the angular momentum can then be straightforwardly obtained by means of Komar integrals, and are given by

$$M = \frac{3\pi}{8\eta_2(\eta_1^2 - \eta_2^2)(\eta_1 + \eta_2)} \,, \qquad J_{\phi_1} = \frac{\pi \eta_1^{3/2}}{4\eta_2^{3/2}(\eta_1^2 - \eta_2^2)^2(\eta_1 + \eta_2)^2} \,, \qquad J_{\phi_2} = 0 \,. \tag{A.10}$$

The horizon is located at $y = \xi_3$. The geometry is a product of S^2, with coordinates (x, ϕ_2), and S^1, with coordinate ϕ_1. It is straightforward to evaluate the temperature, entropy and angular velocity, leading to

$$T = \frac{\eta_2(\eta_1 + \eta_2)}{2\pi} \,, \qquad S = \frac{\pi^2}{2\eta_2(\eta_1^2 - \eta_2^2)(\eta_1 + \eta_2)^3} \,, \qquad \Omega_{\phi_1} = \frac{\sqrt{\eta_2}(\eta_1^2 - \eta_2^2)}{\sqrt{\eta_1}} \,. \tag{A.11}$$

These quantities satisfy the expected thermodynamic relations

$$dM = TdS + \Omega_{\phi_1} dJ_{\phi_1} \,, \qquad M = \tfrac{3}{2}(TS + \Omega_{\phi_1} J_{\phi_1}) \,. \tag{A.12}$$

B Kaluza-Klein Monopole

In this appendix, we examine the static black hole metrics discussed in section 3 in the limit where $\xi_1 = \xi_2$. Since x in lies in the interval $\xi_1 \le x \le \xi_2$, it follows that we need to blow up the interval when we take such a limit. This can be achieved by means of the coordinate and parameter redefinitions

$$\xi_1 = \xi_0 - \epsilon\,, \qquad \xi_2 = \xi_0 + \epsilon\,, \qquad x = \xi_0 + \epsilon\cos\theta\,, \qquad \phi_2 \to \frac{1}{\epsilon}\phi_2\,, \tag{B.1}$$

in which ϵ is then sent to zero. We then introduce a new radial coordinate r in place of y, and also define new azimuthal coordinates ϕ and ψ, according to

$$y = \xi_0 f\,, \qquad \phi_1 = \tfrac{1}{2}(\phi+\psi)\,, \qquad \phi_2 = \tfrac{1}{2}(\phi-\psi)\,, \tag{B.2}$$

where

$$f = 1 - \frac{\sqrt{1-\xi_0^2}}{2\mu\xi_0\, r}\,. \tag{B.3}$$

The resulting metric then takes the form

$$ds^2 = -f\,dt^2 + \frac{1-\xi_0^2 f}{(1-\xi_0^2)}\Big(\frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big) + \frac{1}{4\mu^2(1-\xi_0^2 f)}(d\psi + \cos\theta d\phi)^2\,. \tag{B.4}$$

This can be recognised as the metric of the Kaluza-Klein monopole in $D = 5$.

C A General Class of Higher-Dimensional Local Metrics

Here, we record some results for a class of Ricci-flat metrics in arbitrary dimensions $D \ge 4$, which we obtained while searching for generalisations of our five-dimensional construction. We find that

$$\begin{aligned} ds &= f\Big(\frac{x^2-y^2}{X}dx^2 + \frac{y^2-x^2}{Y}dy^2 + \frac{X}{x^2-y^2}(d\phi + y^2 d\psi)^2 + \frac{Y}{y^2-x^2}(d\phi + x^2 d\psi)^2\Big) \\ &\quad + (xy)^\gamma dx^\mu dx_\mu \end{aligned} \tag{C.1}$$

is Ricci-flat, where

$$\begin{aligned} f &= \frac{t^{c-1-\frac{1}{2}(D-4)\gamma}}{(\alpha + (xy)^c)^2}\,, \qquad c = \tfrac{1}{2}\sqrt{4 + (D-2)(D-4)\gamma^2}\,, \\ X &= a_0 x^2(1 + a_1 x^c - a_2\alpha x^{-c})\,, \qquad Y = a_0 y^2(1 + a_2 y^c - a_1\alpha y^{-c})\,, \end{aligned} \tag{C.2}$$

and a_0, a_1, a_2, α and γ are constants. There do not appear to be any new and non-trivial regular examples contained within these metrics.

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