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Resolutions of Cones over Einstein-Sasaki Spaces

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ABSTRACT

Recently an explicit resolution of the Calabi-Yau cone over the inhomogeneous five-dimensional Einstein-Sasaki space $Y^{2,1}$ was obtained. It was constructed by specialising the parameters in the BPS limit of recently-discovered Kerr-NUT-AdS metrics in higher dimensions. We study the occurrence of such non-singular resolutions of Calabi-Yau cones in a more general context. Although no further six-dimensional examples arise as resolutions of cones over the L^{pqr} Einstein-Sasaki spaces, we find general classes of non-singular cohomogeneity-2 resolutions of higher-dimensional Einstein-Sasaki spaces. The topologies of the resolved spaces are of the form of an $\mathbb{R}^2$ bundle over a base manifold that is itself an S^2 bundle over an Einstein-Kähler manifold.

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Contents

1 Introduction

The study in string theory of D3-branes located at the apex of a Calabi-Yau cone has provided a powerful motivation in recent years for studying complete and non-singular Calabi-Yau metrics that can be regarded as resolutions of the singular cone metrics. The essential idea is to seek a metric which, at large distance, is asymptotic to the cone metric $ds^2 = dr^2 + r^2 d\bar{s}^2$ over an Einstein-Sasaki space X equipped with metric $d\bar{s}^2$, but which, at small distance, deforms in such a way that the singular apex $r = 0$ of the cone metric is smoothed out into a non-singular endpoint.

Typically, this smoothing out of the apex occurs because the level surfaces at constant radius, which are in general smooth r-dependent deformations of the Einstein-Sasaki space X, degenerate at some small radius $r = r_0$ in a such a way that one obtains a regular "closing off" of the manifold at an origin of the radial coordinate at $r = r_0$. A classic example is the four-dimensional Eguchi-Hanson metric [1], for which the level surfaces are $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$ [2]. The $\mathbb{RP}^3$ surfaces are "squashed" along the $U(1)$ Hopf fibre over S^2 as a function of radius, approaching the round $\mathbb{RP}^3$ at infinity, and degenerating to S^2 at some inner radius r_0 where the Hopf fibre shrinks to zero. This degeneration is like the origin of polar coordinates on the plane, implying that the whole Eguchi-Hanson manifold is an $\mathbb{R}^2$ bundle over S^2. It can be viewed as a resolution of the Calabi-Yau cone over $\mathbb{RP}^3$.

Countless further examples of this kind are known in the literature, including, for instance, metrics on 2-plane bundles over arbitrary Einstein-Kähler base spaces B [3, 4]. They can be viewed as resolved Calabi-Yau cones over Einstein-Sasaki spaces X that are themselves $U(1)$ bundles over the Einstein-Kähler spaces B. Another important example, in six dimensions, is the so-called "deformed conifold," which is an $\mathbb{R}^3$ bundle over S^3 [5]. Further examples include resolved cone metrics with G_2 or Spin(7) holonomy in seven or eight dimensions [6, 7].

In almost all the explicit examples of resolved cones, the metrics have cohomogeneity 1, with level surfaces at constant radius r that are themselves homogeneous spaces. The "shape" of the level surfaces distorts as a function of r, whilst maintaining the homogeneity.

In a recent paper, a six-dimensional example of a smooth resolution of a Calabi-Yau cone was obtained [8] in which the metric has cohomogeneity 2, meaning that the level surfaces are themselves inhomogeneous spaces of cohomogeneity 1. In fact, the example in [8] was obtained by taking a local class of BPS limits of six-dimensional Kerr-NUT-AdS metrics that were obtained in a general construction in [9][1], and showing that for an appropriate specialisation of the parameters, a complete and non-singular metric is obtained. The smooth metric in [8] can be viewed as a resolution of the Calabi-Yau cone over the Einstein-Sasaki space $Y^{2,1}$, which is a special case of the $Y^{p,q}$ class of Einstein-Sasaki spaces constructed in [10].

In fact the BPS limits of the even-dimensional Kerr-NUT-AdS metrics obtained in the general construction in [9] can all be viewed locally as deformations of cones over the $(2n+1)$-dimensional Einstein-Sasaki metrics $L^{p,q,r_1,\ldots,r_{n-1}}$ that were constructed in [11, 12]. In the special case of six dimensions, the general BPS construction in [9] gives cohomogeneity-3 metrics that are cones over the L^{pqr} class of five-dimensional Einstein-Sasaki metrics obtained in [11, 12].

In view of the above observations, it is natural to investigate the possibility of generalising the result in [8], to see whether the classes of BPS limits of Kerr-NUT-AdS metrics considered in [9] contain further examples of smooth resolutions of cones over the $L^{p,q,r_1,\ldots,r_{n-1}}$ Einstein-Sasaki metrics. This forms the subject of the present paper. Our conclusions are that in six dimensions, there are no further examples going beyond the resolution of the cone

[1] The cosmological constant of the Kerr-NUT-AdS metric scales to zero in the process of taking the BPS limit.

over $Y^{2,1}$ that was obtained in [8]. In particular, it does not seem to be possible to obtain resolutions of cones over any of the other L^{pqr} Einstein-Sasaki spaces, within the framework of the local metrics found in [9]. Thus although the local six-dimensional metrics in [9] in general have cohomogeneity 3, it seems that one only obtains complete and non-singular examples under the specialisation to cohomogeneity 2.

However, if we go beyond six dimensions we find large classes of new examples of complete and non-singular metrics on resolved cones over inhomogeneous Einstein-Sasaki spaces. Motivated by our detailed analysis in six dimensions, it is natural to focus attention on the specialisation of the BPS metrics in [9] to cohomogeneity 2. (In general, in $D = 2n$ dimensions, the BPS metrics in [9] have cohomogeneity n.) The special cases of cohomogeneity-2 Kerr-NUT-AdS metrics, and their BPS limits, were obtained previously in [13], and it is in fact more convenient to take these results as the starting point for the investigation of the global regularity issues.

The organisation of this paper is as follows. In section 2, we review the construction of local Ricci-flat Kähler metrics that were obtained in [13] by taking a BPS limit of the cohomogeneity-2 Kerr-NUT-AdS that were also constructed in that paper, giving details that will be useful for our subsequent global analysis. In section 3, we then set up the machinery for studying the circumstances under which these local metrics can extend smoothly onto complete and non-singular manifolds. We do this by using the method that was developed in [11, 12], which involves checking the compatibility between the periodicities of the translations generated by the various Killing vectors whose lengths go to zero on degenerating hypersurfaces. The construction in [13] yields, in the BPS limit, Ricci-flat Kähler metrics of dimension $D = 2n + 4$ built over a $2n$-dimensional Einstein-Kähler base metric $d\Sigma_n^2$. We find that we can obtain complete and non-singular spaces if the Einstein-Kähler base is chosen to be a product of n S^2 factors, or to be $\mathbb{CP}^n$, a product of m S^2 factors and $\mathbb{CP}^{n-m}$, or various other products involving complex projective spaces as factors. The special case with a single S^2 manifold forming the base corresponds to the example discussed in [8].

In section 4, we discuss the topology of the manifolds on which the smooth metrics are defined. They are of the form of an $\mathbb{R}^2$ bundle over a manifold that is itself an S^2 bundle over a the $2n$-dimensional Einstein-Kähler base manifold.

In section 5, we present a discussion of the more general cohomogeneity-n BPS limits of $2n$-dimensional Kerr-NUT-AdS metrics that were obtained in [9]. Although we have not found any further smooth resolutions amongst these more general metrics, it is nonetheless of interest to study them in some detail, since their local interpretation as deformed cones over the $L^{p,q,r_1,\cdots,r_{n-1}}$ Einstein-Sasaki metrics was not considered in [9]. In the process of doing this, we incidentally obtain new, and rather elegant, expressions for the local $L^{p,q,r_1,\cdots,r_{n-1}}$ Einstein-Sasaki metrics that were first constructed in a rather different formalism in [11]. We also present a detailed discussion of the six-dimensional case, showing how a global analysis leads to the conclusion that there are no further smooth resolutions of cones over 5-dimensional Einstein-Sasaki spaces within this class that go beyond the $Y^{2,1}$ example in [8].

The paper ends with conclusions in section 6.

2 The Local Ricci-Flat Kähler Metrics

Our starting point is the class of local Ricci-flat Kähler metrics that were obtained in [13] by taking a BPS limit of the cohomogeneity-2 Kerr-NUT-AdS metrics that were constructed in that paper. (They can also obtained from some Einstein-Kähler metrics given in [14], by taking a limit where they become Ricci flat.) They have dimension $D = 2n + 4$, and are given by

$$\begin{aligned} d\tilde{s}^2 &= \frac{x-y}{4X}dx^2 + \frac{x-y}{4Y}dy^2 + \frac{(x-\alpha)(\alpha-y)}{\alpha}d\Sigma_n^2 \\ &\quad + \frac{X}{x-y}\Big[d\tau + \frac{\alpha-y}{\alpha}(d\psi+A)\Big]^2 + \frac{Y}{x-y}\Big[d\tau - \frac{x-\alpha}{\alpha}(d\psi+A)\Big]^2 \,, \\ X &= x(x-\alpha) + \frac{2\mu}{(x-\alpha)^n}\,, \qquad Y = y(\alpha-y) - \frac{2\nu}{(\alpha-y)^n}\,, \end{aligned} \tag{1}$$

where $d\Sigma_n^2$ is a metric on a $2n$-dimensional Einstein-Kähler space Z, satisfying $R_{ab} = 2(n+1)\,g_{ab}$, with Kähler form $J = \frac{1}{2}dA$.

There is a complete symmetry between the coordinates x and y in the local metric. However, when we consider the circumstances under which we can obtain globally non-singular spaces, we shall find that one of the two coordinates should be interpreted as a non-compact radial variable, ranging from infinity to some specific finite value, while the other coordinate should be interpreted as a compact variable, analogous to a latitude coordinate on a sphere, ranging between two finite values. For definiteness, and without loss of generality, we shall choose to take x as the radial variable, and y as the coordinate ranging over the finite interval. Specifically, we shall take

$$\alpha < x_0 \le x \le \infty\,, \qquad \text{or} \qquad -\infty \le x \le x_0\,, \tag{2}$$

where x_0 is the largest (or, respectively, smallest) real root of $X(x)$, and

$$y_1 \le y \le y_2\,, \tag{3}$$

where y_1 and y_2 are two adjacent real roots of $Y(y)$, satisfying the conditions

$$y_1 < y_2 < \alpha\,. \tag{4}$$

The constant α can be fixed to any specific non-zero value without any loss of generality, and so from now on we shall take

$$\alpha = 1\,. \tag{5}$$

When these conditions are satisfied, the power-law curvature singularities at $x = y$, $x = \alpha$ and $y = \alpha$ can all be avoided, provided that $x_0 > \alpha = 1$ in the case that x_0 is the largest real root of $X(x)$, or that $x_0 < y_1$ in the case that x_0 is the smallest real root of $X(x)$. Note that in this latter case, for which x lies in the range $-\infty \le x \le x_0$, the metric $d\tilde{s}^2$ will be negative definite, and so we should take $ds^2 = -d\tilde{s}^2$ as the positive-definite Ricci-flat metric.

We shall consider cases where the Einstein-Kähler base manifold is taken to be the complex projective space $\mathbb{CP}^n$, or products of S^2 factors, or products of S^2 factors and complex projective spaces. For the subsequent global analysis, it will be useful to present a convenient coordinatisation of the standard Fubini-Study metric on $\mathbb{CP}^n$. In terms of the standard inhomogeneous complex coordinates ζ^i, the Fubini-Study metric and the potential A are given by

$$\begin{aligned} d\Sigma_n^2 &= f^{-1} \sum_{i=1}^n d\bar{\zeta}^i\, d\zeta^i - f^{-2}\, |\omega|^2\,, \\ A &= \tfrac{\mathrm{i}}{2}\, f^{-1}\, (\omega - \bar{\omega})\,, \\ f &= 1 + \sum_{i=1}^n |\zeta^i|^2\,, \qquad \omega = \sum_{i=1}^n \zeta^i\, d\bar{\zeta}^i\,. \end{aligned} \tag{6}$$

We shall find it convenient to parameterise the ζ^i in terms of real coordinates r, μ_i and ϕ_i according to

$$\begin{aligned} \zeta^i &= \mu_i\, \tan\chi\, e^{\mathrm{i}\,\phi_i}\,, \qquad \sum_{i=1}^n \mu_i^2 = 1\,, \\ 0 \le \chi \le \tfrac{1}{2}\pi\,, &\qquad 0 \le \phi_i < 2\pi\,, \qquad 0 \le \mu_i \le 1\,. \end{aligned} \tag{7}$$

in terms of which the $\mathbb{CP}^n$ metric and potential A become

$$
\begin{aligned}
d\Sigma_n^2 &= d\chi^2 + \sin^2\chi\,\Big[\sum_{i=1}^n (d\mu_i^2 + \mu_i^2\, d\phi_i^2) - \sin^2\chi\,\Big(\sum_{i=1}^n \mu_i^2\, d\phi_i\Big)^2\Big]\,,\\
A &= \sin^2\chi \sum_{i=1}^n \mu_i^2\, d\phi_i\,. \qquad (8)
\end{aligned}
$$

The metric is Einstein-Kähler, with $R_{ab} = 2(n+1)\, g_{ab}$.

When considering the case where $d\Sigma_n^2$ is instead the Einstein-Kähler metric on the product of n copies of S^2, the metric and the potential A can be taken to be

$$
\begin{aligned}
d\Sigma_n^2 &= \frac{1}{2(n+1)} \sum_{i=1}^n (d\theta_i^2 + \sin^2\theta_i\, d\phi_i^2)\,,\\
A &= \frac{1}{n+1} \sum_{i=1}^n \cos\theta_i\, d\phi_i\,. \qquad (9)
\end{aligned}
$$

Note that we have scaled the metric so that it has the canonical normalisation $R_{ab} = 2(n+1)\, g_{ab}$, as required in the construction of the Ricci-flat metrics (1).

3 Complete Non-Singular Calabi-Yau Spaces

In this section we shall show how, by choosing the parameters in the local Ricci-flat Kähler metrics (1) appropriately, we can obtain metrics that extend smoothly onto complete and non-singular manifolds. We shall discuss various choices for the Einstein-Kähler base $d\Sigma_n^2$ in the following subsections.

3.1 $\mathbb{CP}^n$ base space

In this case we consider the local Ricci-flat Kähler metrics (1) with $d\Sigma_n^2$ and A chosen to be those of the Fubini-Study metric on $\mathbb{CP}^n$, as given by (8).

For generic values of the coordinates, the metric (1) is non-singular. It becomes singular on "degeneration surfaces," where one or more of the metric coefficients goes to zero. These singularities can occur in the $\mathbb{CP}^n$ base metric (8), where one or more of the latitude coordinates μ_i goes to zero, or where χ goes to 0 or $\frac{1}{2}\pi$. Singularities can also occur in the total metric (1) where the functions $X(x)$ or $Y(y)$ go to zero. Thus these singularities occur at the inner end $x = x_0$ of the range of the radial variable x, and at the two endpoints $y = y_1$ and $y = y_2$ of the "latitude" coordinate y. At all these degeneration surfaces, some angular variable parameterising translations around the surface becomes degenerate.

In order for the local metric to extend smoothly onto a complete and non-singular manifold, it is necessary that at all these degeneration surfaces, the periodicities of the relevant collapsing angular variables be such that the metric locally has the form of a regular origin of polar or hyperspherical polar coordinates. The requirement that the periodicities needed to avoid conical singularities at the various degeneration surfaces should all be compatible is highly non-trivial, and it places severe restrictions on the possible choices of periods and metric parameters. A convenient way to study these compatibility conditions was developed in [11]. It consists of identifying the specific Killing vectors whose lengths go to zero at the various degeneration surfaces, in each case normalising them so that they generate precisely 2π translations around their respective surfaces. There will exist linear relations among the set of Killing vectors so obtained, and compatibility will require that the coefficients in these linear relations must be sets of coprime integers. The task is then to check whether there exist choices of coprime coefficients that are mutually compatible at all the degeneration surfaces, and that are consistent with the disposition of zeros in the metric functions $X(x)$ and $Y(y)$. Lower-dimensional degeneration surfaces, for example where $X(x)$ and $Y(y)$ vanish simultaneously, must also be considered.

In all cases we normalise each Killing vector ℓ so that it generates a 2π translation at its respective degeneration surface where $\ell^2 \equiv \ell^\alpha\, \ell^b\, g_{ab} = 0$. This is achieved by scaling it so that its "Euclidean surface gravity" κ, defined by taking the limit as one approaches the degeneration surface of

$$\kappa^2 = \frac{g^{ab}\, \partial_a(\ell^2)\, \partial_b(\ell^2)}{4\ell^2}\,, \tag{10}$$

is equal to unity.[2] For the case of the metric (1) with $\mathbb{CP}^n$ base, we find the following normalised Killing vectors that vanish on degeneration surfaces:

$$\begin{aligned}
x = x_0: &\qquad \ell_0 = \frac{1}{[1-(n+2)x_0]}\,\frac{\partial}{\partial\psi} + \frac{(x_0-1)}{[1-(n+2)x_0]}\,\frac{\partial}{\partial\tau}\,,\\
y = y_i: &\qquad \ell_i = \frac{1}{[1-(n+2)y_i]}\,\frac{\partial}{\partial\psi} + \frac{(y_i-1)}{[1-(n+2)y_i]}\,\frac{\partial}{\partial\tau}\,, \qquad i=1,2\,,\\
\chi = \tfrac{1}{2}\pi: &\qquad \ell_3 = \frac{\partial}{\partial\psi} - \sum_{i=1}^{n}\frac{\partial}{\partial\phi_i}\,,\\
\mu_i = 0: &\qquad \ell_{i+3} = \frac{\partial}{\partial\phi_i}\,, \qquad 1\le i\le n\,.
\end{aligned} \tag{11}$$

As $|x|$ tends to infinity, the metric (1) asymptotically approaches a cone over a $(2n+3)$-dimensional Einstein-Sasaki space, contained within the class constructed in [11, 12]. (We

[2]The ur example of this kind is the Killing vector $\partial/\partial\phi$ at the origin of polar coordinates in the metric $ds^2 = dr^2 + r^2 d\phi^2$.

shall show this in general in section 4.) Thus before we address the new question of the resolution of the singularity of the cone metric at the innermost value of x, we can first consider the regularity conditions on the level surfaces at constant x. As $|x|$ tends to infinity this will just amount to revisiting the regularity conditions for Einstein-Sasaki spaces considered in [11] (except that for now we are restricting our attention to the special cases where the Einstein-Sasaki spaces have cohomogeneity 1, which were also studied in [15, 16].) An important new feature is to note that the conditions for regularity on the $(2n + 3)$-dimensional level surfaces at fixed x are *independent of the value of* x, and thus regularity at large x ensures regularity on all the non-degenerate level surfaces as x approaches its innermost value $x = x_0$. Finally, we can then address the question of regularity on the degeneration surface at $x = x_0$ itself. The requirements that this be regular and also compatible with the conditions needed for regularity on the surfaces away from $x = x_0$ are highly restrictive, admitting only very few consistent solutions. For example, in six dimensions it is these last requirements which show that only for the case of $Y^{2,1}$ can one obtain a smooth resolution of the cones over $Y^{p,q}$ Einstein-Sasaki spaces [8], within the class of metrics considered here.

Starting, then, by considering the level surfaces away from $x = x_0$, we note from (11) that there must exist a linear relation

$$p\ell_1 + q\ell_2 + \sum_{i=3}^{n+3} r_i\, \ell_i = 0 \qquad (12)$$

among the Killing vectors. As discussed in [11] the coefficients must be rationally related (since otherwise a linear combination could generate a closed periodic translation through an arbitrarily small length, implying a singularity in the manifold), and thus by scaling we can take the coefficients to be coprime integers, p, q, r_i. From the detailed expressions for the Killing vectors in (11), we see from (12) that we must have

$$r_3 = r_4 = \cdots = r_{n+3} = \frac{p+q}{n+1}\,, \qquad (13)$$

and also

$$\frac{p}{q} + \frac{(1-y_2)[1-(n+2)y_1]}{(1-y_1)[1-(n+2)y_2]} = 0\,. \qquad (14)$$

Since y_1 and y_2 are roots of $Y(y) = 0$, where Y is given in (1), we have

$$y_1\,(1-y_1)^{n+1} - y_2\,(1-y_2)^{n+1} = 0\,. \qquad (15)$$

Equations (14) and (15) determine (y_1, y_2).

The above considerations determine the structure of the level surfaces away from the degeneration at the innermost radius $x = x_0$. As we remarked previously, an important point in the analysis above is that the compatibility conditions are independent of x, and thus the conditions for regularity on the level surfaces are the same at arbitrary values of $x \neq x_0$ as they are in the asymptotic region where $|x|$ goes to infinity. Thus the above analysis is essentially equivalent to one given in [11].

We now turn to the consideration of the degeneration at $x = x_0$. From (11) it can be seen that there is a linear relation among the three Killing vectors ℓ_0, ℓ_1 and ℓ_2, which we can express as

$$n_0\,\ell_0 + n_1\,\ell_1 + n_2\,\ell_2 = 0 \tag{16}$$

for coprime integers n_0, n_1 and n_2. Since there exists a lower-dimensional hypersurface on which ℓ_1 and ℓ_0 both have zero length simultaneously (and likewise a hypersurface where ℓ_2 and ℓ_0 have zero length simultaneously), regularity also imposes the stronger requirements that $n_1 = n_2 = 1$. (See [11, 12] for a more detailed exposition of regularity considerations of this type.) From the detailed expressions for the Killing vectors in (11), we see that (16) then implies $n_0 = -2$ and

$$x_0 = \frac{2(n+2)\,y_1\,y_2 - y_1 - y_2}{(n+2)(y_1 + y_2) - 2}\,. \tag{17}$$

Note that in order for y_1, y_2 and x_0 to lie in the appropriate range discussed in section 2, it is necessary that p and q be positive integers.

To summarise, the general linear relation

$$\sum_{i=0}^{n+3} n_i\,\ell_i = 0 \tag{18}$$

for the zero-length Killing vectors imply four inequivalent conditions, namely

$$\begin{aligned}
n_0 = 0: &\qquad n_1 = p\,, \quad n_2 = q\,, \qquad n_i = \frac{p+q}{n+1}\,,\\
n_1 = 1: &\qquad n_0 = 2(n+1)p\,, \quad n_2 = -(n+1)(p-q)\,, \quad n_i = p+q\,, \quad i = 3,4,\ldots n+3\,,\\
n_2 = 1: &\qquad n_0 = 2(n+1)q\,, \quad n_1 = -(n+1)(p-q)\,, \quad n_i = p+q\,, \quad i = 3,4,\ldots n+3\,,\\
n_i = 0: &\qquad n_0 = -2\,, \quad n_1 = 1\,, \quad n_2 = 1\,, \quad i = 3,4,\ldots n+3\,.
\end{aligned} \tag{19}$$

The second and third cases imply the conditions

$$\gcd[2(n+1)p, (n+1)(p-q), p+q] = \gcd[2(n+1)q, (n+1)(p-q), p+q] = p+q\,,\,. \tag{20}$$

where gcd denotes the greatest common divisor.

The condition (20) imply that the pair of integers (p, q) can lie in either of two permitted classes. One can either have all the positive odd integer pairs such that $p+q = 2(n+1)$, or else one can have all the positive integer pairs such that $p + q = n + 1$. Thus for $\mathbb{CP}^0$ (i.e. a zero-dimensional base space in (1)), we can just have $(p, q) = (1, 1)$. This corresponds to the Eguchi-Hanson metric. For $\mathbb{CP}^1$, we can have either $(p, q) = (1, 1)$, corresponding to the cohomogeneity-1 resolution of the cone over the homogeneous space $T^{1,1}/\mathbb{Z}_2$, or else $(p, q) = (1, 3)$. This latter choice gives rise to the resolved cone over $Y^{2,1}$, which was obtained in [8]. For $\mathbb{CP}^2$, we can have $(p, q) = (1, 5)$, $(1, 2)$, or $(3, 3)$; and for $\mathbb{CP}^3$, we can have $(p, q) = (1, 7)$, $(1, 3)$, $(3, 5)$, or $(2, 2)$. In all cases, the solution reduces to cohomogeneity 1 if $p = q$.

For each allowed integer pair (p, q), one then solves equations (14) and (15) for y_1 and y_2, and then equation (17) for x_0.[3] It is then necessary to check that the two roots y_1 and y_2 of $Y(y)$ are indeed adjacent, satisfying $y_1 < y_2 < 1$, and that the root x_0 of $X(x)$ is indeed the largest root, with $x_0 > y_2$ (and with x ranging from $x_0 \le x \le \infty$) or else x_0 is the smallest root, with $x_0 < y_1$ (and with $-\infty \le x \le x_0$). In fact in all the cases we have examined, one finds x_0 is negative, and it is the smallest root. With $-\infty \le x \le x_0$ the metric $d\tilde{s}^2$ in (1) is then negative definite, and so $ds^2 = -d\tilde{s}^2$ is a positive-definite Ricci-flat Kähler metric on a complete and non-singular manifold.

Some explicit examples are as follows. In each case, we shall consider only the non-trivial cohomogeneity-2 examples that arise when $p \ne q$. For the base $\mathbb{CP}^1$, we have only $(p, q) = (1, 3)$, and this implies

$$y_1 = \tfrac{1}{12}(5 - \sqrt{13})\,, \qquad y_2 = \tfrac{1}{12}(11 - \sqrt{13})\,, \qquad x_0 = -\tfrac{1}{3}(1 + \sqrt{13})\,. \tag{21}$$

This case coincides precisely with the six-dimensional resolution of the cone over $Y^{2,1}$ that was found in [8]. Note that here, and in all the other examples, the radial variable x runs from x_0 to $-\infty$, and the positive-definite metric ds^2 is given by (1) with $ds^2 = -d\tilde{s}^2$.

For $\mathbb{CP}^2$, we can have $(p, q) = (1, 5)$ or $(1, 2)$, which imply

$$\begin{aligned} (p,q) = (1,5): &\quad y_1 = 0.0311124\,, \quad y_2 = 0.647818\,, \quad x_0 = -0.72331\,, \\ (p,q) = (1,2): &\quad y_1 = 0.119655\,, \quad y_2 = 0.421351\,, \quad x_0 = -0.839339\,. \end{aligned} \tag{22}$$

(In these higher-dimensional examples the solutions for y_1 and y_2 arise as the roots of cubic or higher-order equations, and we shall just give numerical values in these cases.)

[3] Of course, as can be seen from the expressions for $X(x)$ and $Y(y)$ in (1), once these roots are determined, the constants μ and ν are determined.

For $\mathbb{CP}^3$ we can have:

$$
\begin{aligned}
&(p,q)=(1,7): \qquad y_1=0.00922175\,, \qquad y_2=0.659267\,, \qquad x_0=-0.452677\,,\\
&(p,q)=(1,3): \qquad y_1=0.0477244\,, \qquad y_2=0.459358\,, \qquad x_0=-0.537637\,,\\
&(p,q)=(3,5): \qquad y_1=0.112054\,, \qquad y_2=0.313348\,, \qquad x_0=-0.584858\,. \qquad (23)
\end{aligned}
$$

3.2 $(S^2)^n$ base space

In this and subsequent subsections, we shall consider the cases where $d\Sigma_n^2$ is a direct sum of various $\mathbb{CP}$ metrics. One simple case is to take n 2-spheres S^2. The metric is given by (9). The zero-length Killing vectors are given by

$$
\begin{aligned}
x=x_0: &\qquad \ell_0=\frac{n+1}{[1-(n+2)x_0]}\,\frac{\partial}{\partial\psi}+\frac{x_0-1}{[1-(n+2)x_0]}\,\frac{\partial}{\partial\tau}\,,\\
y=y_1: &\qquad \frac{n+1}{[1-(n+2)y_1]}\,\frac{\partial}{\partial\psi}+\frac{y_1-1}{[1-(n+2)y_1]}\,\frac{\partial}{\partial\tau}\,,\\
y=y_2: &\qquad \frac{n+1}{[1-(n+2)y_2]}\,\frac{\partial}{\partial\psi}+\frac{y_2-1}{[1-(n+2)y_2]}\,\frac{\partial}{\partial\tau}\,,\\
\theta_i=0: &\qquad \ell_{2i+1}=\frac{\partial}{\partial\psi}-\frac{\partial}{\partial\phi_i}\,, \qquad i=1,2,\ldots,n\,,\\
\theta_i=\pi: &\qquad \ell_{2i+2}=\frac{\partial}{\partial\psi}+\frac{\partial}{\partial\phi_i}\,, \qquad i=1,2,\ldots,n\,. \qquad (24)
\end{aligned}
$$

In the asymptotic region where x becomes large, the metric describes a cone over a $(2n+3)$-dimensional Einstein-Sasaki space. The periodicity conditions on the angular coordinates of the Einstein-Sasaki space can be determined by analysing the zero-length Killing vectors $(\ell_1,\ell_2,\ldots\,\ell_{2n+2})$, as in the previous subsection. Since the S^2 factors enter symmetrically, and ℓ_0,ℓ_1,ℓ_2 do not involve the vectors $\partial/\partial\phi_i$, it suffices to check the conditions associated with one of the S^2 factors. Thus we need only check the consistency conditions associated with the Killing vectors $(\ell_0,\ell_1,\ell_2,\ell_3,\ell_4)$. As discussed in the previous case, the linear dependence of these vectors implies that there exists a relation

$$
n_0\,\ell_0+n_1\,\ell_1+n_2\,\ell_2+n_3\,\ell_3+n_4\,\ell_4=0\,, \qquad (25)
$$

for a set of coprime integers n_j.

The conditions associated with the regularity of the level surfaces at constant $x\neq x_0$ are determined by considering the linear relation involving only $(\ell_1,\ell_2,\ell_3,\ell_4)$, and so for this part of the analysis we set $n_0=0$. We shall take the remaining integers in (25) to be given by $(n_1,n_2,n_3,n_4)=(\tilde p,\tilde q,\tilde r\,\tilde s)$. From the expressions for the ℓ_i we can conclude that

$$
\tilde s=\tilde r=\tfrac12(\tilde p+\tilde q)\,, \qquad (26)
$$

and in addition, y_1 and y_2 need to satisfy a condition that is in fact identical to the one that arose for the $\mathbb{CP}^n$ base, namely (14). Furthermore, since y_1 and y_2 are roots of $Y(y) = 0$, they must also satisfy the condition (15), as in the case of the $\mathbb{CP}^n$ base.

Having determined y_1, y_2, and hence ν, in terms of (p, q), the nature of the spaces that form the constant-x level surfaces are fully determined. It remains to investigate the consistency conditions associated with the degeneration surface at $x = x_0$. It is evident from (24) that (ℓ_0, ℓ_1, ℓ_2) are linearly dependent. Taking $n_3 = 0 = n_4$ in (25), we see that $n_1\,\ell_1 + n_2\ell_2 + n_0\ell_0 = 0$ implies $n_0 = -(n_1 + n_2)$. Now, ℓ_0 and ℓ_1 can have zero length simultaneously (at $x = x_0$, $y = y_1$) and similarly so can ℓ_0 and ℓ_2 (at $x = x_0$, $y = y_2$). This implies that we must have $n_1 = 1 = n_2$, and hence $n_0 = -2$.[4] We can also solve for x_0; it is given by (17), which is again the same as in the $\mathbb{CP}^n$ case considered earlier.

Now we have only two non-trivial conditions left, namely

$$\begin{aligned} n_1 = 0: &\qquad n_0 = 4p\,, \qquad n_2 = -2(p-q)\,, \qquad n_3 = n_4 = p + q\,, \\ n_2 = 0: &\qquad n_0 = 4q\,, \qquad n_1 = 2(p-q)\,, \qquad n_3 = n_4 = p + q\,. \end{aligned} \tag{27}$$

This implies that

$$\gcd[4p, p+q, 2(p-q)] = p+q\,, \qquad \gcd[4q, p+q, 2(p-q)] = p+q\,, \tag{28}$$

The only positive integer solutions with $p \le q$ are $(p, q) = (1, 1)$ and $(p, q) = (1, 3)$. The $p = q$ case reduces to previously-considered cohomogeneity-1 examples, which are not of interest to us in this paper.

Thus we see that for each allowed integer pair (p, q), the determination of y_1, y_2 and x_0 depends only on the dimensions of the base space, given by (14), (15) and (17). The allowed pairs (p, q) are determined simply by examining one of the S^2 factors; no further restrictions follow by examining the other S^2 factors.

Some explicit examples of the cohomogeneity-2 metrics with $(p, q) = (1, 3)$ are

$$\begin{aligned} D = 6: &\qquad y_1 = \tfrac{1}{12}(5 - \sqrt{13})\,, \qquad y_2 = \tfrac{1}{12}(11 - \sqrt{13})\,, \qquad x_0 = -\tfrac{1}{3}(1 + \sqrt{13})\,. \\ D = 8: &\qquad y_1 = 0.0696131\,, \qquad y_2 = 0.525812\,, \qquad x_0 = -0.792761\,, \\ D = 10: &\qquad y_1 = 0.0477244\,, \qquad y_2 = 0.459358\,, \qquad x_0 = -0.537637\,. \end{aligned} \tag{29}$$

The example in $D = 6$ is the resolution of the cone over $Y^{2,1}$ that was found in [8]. In all cases y_1 and y_2 are indeed both less than 1, which means there will be no singularity

[4] If instead we chose $n_1 = -n_2 = 1$, and hence $n_0 = 0$, it would force $y_1 = y_2$, which would reduce the metric to cohomogeneity 1, which has been considered previously.

within the interval $y_1 \leq y \leq y_2$. Since x_0 is negative we again, as in the cases with the $\mathbb{CP}^n$ base, take x to range from $-\infty$ to x_0, which ensures that $X(x)$ remains non-negative, and approaches zero only at the degeneration surface at $x = x_0$. Furthermore, the range of the x coordinate does not overlap with $x = 1$ or the range of the y coordinate. Thus we obtain regular positive-definite metrics $ds^2 = -d\tilde{s}^2$ that extend smoothly onto non-singular manifolds.

3.3 $(\mathbb{CP}^m)^s$ base space

The above analysis of n copies of S^2 can be easily generalised to s copies of $\mathbb{CP}^m$, where $n = s\,m$. The formulae to determine y_1, y_2 and x_0 are the same, given by (14), (15) and (17); these depend only on the total dimension $2n$ of the base space. The allowed (p, q) values are the same as those for the single copy of $\mathbb{CP}^m$, namely positive odd integer pairs (p, q) satisfying $p + q = 2(m + 1)$, and positiver integer pairs satisfying $p + q = m + 1$.

3.4 $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2} \times \cdots \times \mathbb{CP}^{m_i}$ base space

If the metric $d\Sigma_n^2$ on the base space is taken to be a direct sum of different $\mathbb{CP}^{m_j}$ metrics, the basic formulae governing the allowed values of y_1, y_2 and x_0 are again given by (14), (15) and (17). However, there is a set of such restrictions on the allowed (p, q) values associated with each of the $\mathbb{CP}^{m_j}$ factors in the base space. The final allowed set of (p, q) pairs is therefore given by the intersection of all the allowed sets associated with each factor. For example, let us consider $S^2 \times \mathbb{CP}^2$. The allowed (p, q) for S^2 are given by $(1, 3), (1, 1)$, and those for $\mathbb{CP}^2$ are $(1, 5), (1, 2), (3, 3)$. There is no intersection, and hence the orbifold singularities cannot be resolved in this case. On the other hand, if we consider $S^2 \times \mathbb{CP}^3$, the (p, q) pairs allowed by $\mathbb{CP}^3$ are $(1, 7), (1, 3), (3, 5), (2, 2)$, and so we have an intersection with the $(1, 3)$ allowed by S^2. It follows that we can obtain a complete metric with base space $S^2 \times \mathbb{CP}^3$, provided that we take $(p, q) = (1, 3)$.

In general, we may consider the case where the $2n$-dimensional base space is the direct product $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2} \times \cdots \times \mathbb{CP}^{m_i}$, with $n = (m_1 + m_2 + \cdots m_i)$. Let $P_j = \{(p, q)\}$, the set of positive odd integer pairs satisfying $p + q = 2(m_j + 1)$ together with the set of positive integer pairs satisfying $p + q = m_j + 1$. We determine the set P_j for each factor $\mathbb{CP}^{m_j}$, for $1 \leq j \leq i$. The final allowed set of pairs (p, q) that gives complete metrics is given by the intersection of all P_j.

This leads to the following conclusion for our metrics: There can be no more than two different types of $\mathbb{CP}^{m_j}$ spaces in the product base space. Thus, the most general allowed

$2n$-dimensional bases space of this type comprises a direct product of s_1 copies of $\mathbb{CP}^{m_1}$ and s_2 copies of $\mathbb{CP}^{m_2}$, with $s_1\, m_1 + s_2\, m_2 = n$. If $m_1 = m_2$, it reduces to a situation that we discussed above. If m_1 and m_2 are not equal, then the allowed m_1 and m_2 are given by $m_2 = 2m_1 + 1$. The allowed (p, q) the odd integer pairs that satisfy $p + q = 2(m_1 + 1)$.

4 Topology

Having established the conditions under which we can obtain complete and non-singular metrics, we may now consider the topology of the manifolds on which they are defined. This can be understood by examining the geometry in the neighbourhood of the degeneration surface at $x = x_0$. This surface is usually referred to as a bolt. Recalling that x ranges from the negative value x_0 to $-\infty$, we define $x - x_0 = -\rho^2$, and so from (1) the metric $ds^2 = -d\tilde{s}^2$ near $\rho = 0$ approaches

$$ds^2 \sim \frac{y - x_0}{1 - (n+2)x_0}\Big(d\rho^2 + \frac{(1 - (n+2)x_0)^2}{(1 - x_0)^2}\rho^2\,(d\tau + \mathcal{A})^2\Big) + ds^2_{\rm bolt}\,, \tag{30}$$

where

$$\begin{aligned} \mathcal{A} &= \Big(\frac{(1-x_0)(1-y)}{y - x_0} + \frac{(1-x_0)^2 Y}{((n+2)x_0 - 1)(y - x_0)^2}\Big)d\widetilde{\psi}\,, \\ ds^2_{\rm bolt} &= \frac{(1-x_0)^2 Y}{y - x_0}(d\widetilde{\psi} + A)^2 + \frac{y - x_0}{4Y}dy^2 + (1 - x_0)(1 - y)d\Sigma_n^2\,, \end{aligned} \tag{31}$$

with $\widetilde{\psi} = \psi + \tau/(1 - x_0)$. It can be seen $\widetilde{\psi}$ must have an appropriate period in order to avoid conical singularities on the bolt, at $y = y_1$ and $y = y_2$. That a single choice of period achieves regularity at both endpoints is automatically guaranteed by virtue of our previous general analysis of the degeneration surfaces. Specifically, it can be seen to follow from (14) and (17); the period is given by

$$\Delta\widetilde{\psi} = \frac{2\pi(p+q)}{(n+1)(q-p)}\,. \tag{32}$$

The coordinates y and $\widetilde{\psi}$ therefore parameterise a 2-sphere, and the bolt B is described as an S^2 bundle over the Einstein-Kähler base space Z whose metric is $d\Sigma_n^2$. The total $(2n+4)$-dimensional space on which the metric $ds^2 = -d\tilde{s}^2$ is defined is an $\mathbb{R}^2$ bundle over the S^2 bundle over the bolt B.

Note that in the case when $p = q$, corresponding to cohomogeneity-1 metrics, the bolt metric becomes homogeneous, and is a direct product of S^2 with the base space Z whose metric is $d\Sigma_n^2$. In the case of $D = 6$, for example, the metric is the resolution of the cone over $T^{11}/\mathbb{Z}_2$ with an $S^2 \times S^2$ bolt, rather than the deformation of the conifold over T^{11} [5] which has an S^3 bolt.

5 Metrics of Higher Cohomogeneity

So far, we have restricted our attention to the cohomogeneity-2 Ricci-flat Kähler metrics that were obtained in [13] as BPS limits of the cohomogeneity-2 Kerr-NUT-AdS metrics that were also constructed in that paper. Subsequently, the construction of Kerr-NUT-AdS metrics was generalised to higher cohomogeneity in [9]. Specifically, in even dimensions $D = 2n$, Kerr-NUT-AdS metrics of cohomogeneity n were constructed, with $(n-1)$ independent NUT parameters in addition to the mass and the $(n-1)$ independent rotation parameters [9]. A BPS limit was also taken, yielding local $D = 2n$ dimensional Ricci-flat Kähler metrics of cohomogeneity n. In this section, we shall study these metrics in more detail, showing, in particular, how they can be viewed as deformations of cones over the classes of Einstein-Sasaki spaces that were constructed in [11, 12]. In the simplest case of $D = 6$, we shall also give a global analysis of the cohomogeneity-3 metrics, showing that the only smooth resolutions of the cones over L^{pqr} Einstein-Sasaki spaces arise in the specialisation to cohomogeneity 2, which leads back the single example with the $Y^{2,1}$ space that was obtained in [8].

5.1 Cohomogeneity-n Calabi-Yau metrics in $D = 2n$

The Ricci-flat Kähler metrics of cohomogeneity n in $D = 2n$ dimensions that were obtained in [9] are given by

$$
\begin{aligned}
ds_{2n}^2 &= \sum_{\mu=1}^{n} \Big[\frac{U_\mu\, dx_\mu^2}{4X_\mu} + \frac{X_\mu}{U_\mu}\Big(\frac{\gamma}{x_\mu} d\tau - \sum_{i=1}^{n-1} \frac{W_i\, d\phi_i}{\alpha_i - x_\mu}\Big)^2\Big]\,,\\
X_\mu &= x_\mu \prod_{i=1}^{n-1}(\alpha_i - x_\mu) - 2\ell_\mu\,, \qquad U_\mu = {\prod}'^{n}_{\nu=1}(x_\nu - x_\mu)\,,\\
W_i &= \prod_{\nu=1}^{n}(\alpha_i - x_\nu)\,, \qquad \gamma = \prod_{\nu=1}^{n} x_\nu\,,
\end{aligned} \tag{33}
$$

where the prime on a product symbol universally indicates that the factor that vanishes is to be omitted.

The n coordinates x_μ enter in a completely symmetrical fashion, and any one of them can be selected to play the role of a radial variable, which takes values in an infinite range from infinity to some finite value at the "centre." For definiteness, we shall choose x_n as the non-compact radial variable, setting $x_n = r^2$. The remaining x_α coordinates with $1 \le \alpha \le n-1$ are analogous to latitude coordinates on a sphere, and they range between adjacent roots of the corresponding polynomials X_α that appear in the metric.

It is easily seen that at large r the metric (33) takes the form

$$ds^2_{2n} \to dr^2 + r^2 ds^2_{2n-1}\,, \tag{34}$$

where ds^2_{2n-1} is an Einstein-Sasaki metric, which is given by

$$ds^2_{2n-1} = \sum_{\beta=1}^{n-1}\Big[\frac{\hat U_\beta}{4X_\beta}\,dx_\beta^2 + \frac{X_\beta}{\hat U_\beta}\Big(\frac{\hat\gamma}{x_\beta}\,d\tau + \sum_{i=1}^{n-1}\frac{\hat W_i\,d\phi_i}{\alpha_i - x_\beta}\Big)^2\Big] + \Big(\hat\gamma\,d\tau - \sum_{i=1}^{n-1}\hat W_i\,d\phi_i\Big)^2\,, \tag{35}$$

where

$$\hat U_\alpha = {\prod}'^{\,n-1}_{\beta=1}(x_\beta - x_\alpha)\,,\qquad \hat W_i = \prod_{\beta=1}^{n-1}(\alpha_i - x_\beta)\,,\qquad \hat\gamma = \prod_{\beta=1}^{n-1} x_\beta\,, \tag{36}$$

and X_α is given by the expression for X_μ in (33) with $\alpha = \mu$ for $1 \le \alpha \le n-1$. It is straightforward to show that the Killing vector

$$K = \frac{1}{c}\,\frac{\partial}{\partial\tau} - \sum_{i=1}^{n-1}\frac{1}{\alpha_i\,B_i}\,\frac{\partial}{\partial\phi_i} \tag{37}$$

has unit length, $|K|^2 = 1$, where the constants c and B_i are given by

$$c = \prod_{i=1}^{n-1}\alpha_i\,,\qquad B_i = {\prod}'^{\,n-1}_{j=1}(\alpha_i - \alpha_j)\,. \tag{38}$$

In fact K is the Reeb vector for the Einstein-Sasaki space. It can be written as $K = \partial/\partial\psi$ by defining new coordinates $(\psi, \hat\phi_i)$ that are related to (τ, ϕ_i) by

$$\tau = \frac{1}{c}\,\psi\,,\qquad \phi_i = \hat\phi_i - \frac{1}{\alpha_i\,B_i}\,\psi\,. \tag{39}$$

The Einstein-Sasaki metric (35) can then be written as

$$ds^2_{2n-1} = (d\psi + \mathcal{A})^2 + d\bar s^2_{2n-2}\,, \tag{40}$$

where $d\bar s^2_{2n-2}$ is an Einstein-Kähler metric, whose Kähler form is given by $J = \frac12 d\mathcal{A}$. This Einstein-Kähler metric can be shown to be equivalent to the one introduced in [11], which arose in the construction of Einstein-Sasaki spaces $L^{p,q,r_1,\ldots,r_n}$ in arbitrary odd dimensions $(2n+1)$.

It is straightforward to show that in terms of the new coordinates defined in (39), the Einstein-Kähler base metric $d\bar s^2_{2n-2}$ can be written as

$$d\bar s^2_{2n-2} = \sum_{\beta=1}^{n-1}\Big[\frac{\hat U_\beta}{4X_\beta}\,dx_\beta^2 + \frac{X_\beta}{\hat U_\beta}\Big(\sum_{i=1}^{n-1}\frac{\hat W_i\,d\hat\phi_i}{\alpha_i - x_\beta}\Big)^2\Big]\,. \tag{41}$$

The Kaluza-Klein potential $\mathcal{A}$ in (40) is given by

$$\mathcal{A} = -\sum_{i=1}^{n-1} \hat{W}_i \, d\hat{\phi}_i \,. \tag{42}$$

The full Ricci-flat Kähler metric (33) itself simplifies considerably if written using the coordinates $(\psi, \hat{\phi}_i)$ introduced in (39), becoming simply

$$ds_{2n}^2 = \sum_{\mu=1}^{n} \Big[\frac{U_\mu \, dx_\mu^2}{4X_\mu} + \frac{X_\mu}{U_\mu}\Big(d\psi - \sum_{i=1}^{n-1} \frac{W_i \, d\hat{\phi}_i}{\alpha_i - x_\mu}\Big)^2\Big] \,. \tag{43}$$

It is useful also to record that the inverse of the metric (43) is given by

$$\Big(\frac{\partial}{\partial s}\Big)^2 = \sum_{\mu=1}^{n} \Big[\frac{4X_\mu}{U_\mu}\Big(\frac{\partial}{\partial x_\mu}\Big)^2 + \frac{S_\mu}{X_\mu U_\mu}\Big(\frac{\partial}{\partial \psi} + \sum_{i=1}^{n-1} \frac{1}{B_i\,(\alpha_i - x_\mu)}\,\frac{\partial}{\partial \hat{\phi}_i}\Big)^2\Big] \tag{44}$$

where we define

$$S_\mu = \prod_{i=1}^{n-1} (\alpha_i - x_\mu)^2 \,. \tag{45}$$

We can now give a general discussion of the Killing vectors that vanish on degenerate surfaces. The general Killing vector can be written as

$$\ell = c_0 \, \frac{\partial}{\partial \psi} + \sum_{i=1}^{n-1} c_i \, \frac{\partial}{\partial \hat{\phi}_i} \,. \tag{46}$$

Let x_μ^0 be a root of the polynomial X_μ; i.e. $\mathrm{X}_\mu(\mathrm{x}_\mu) = 0$ at $x_\mu = x_\mu^0$. We find that the Killing vector (46) vanishes at $x_\mu = x_\mu^0$ if the constants c_i are chosen to be

$$c_i = -\frac{c_0}{(x_\mu^0 - \alpha_i)\, B_i} \,. \tag{47}$$

The surface gravity κ on this degenerate orbit is then given by

$$\kappa^2 = \frac{c_0^2\, [X_\mu'(x_\mu^0)]^2}{S_\mu(x_\mu^0)} \,, \tag{48}$$

where the prime denotes a derivative with respect to the argument. The Killing vector is given its canonical normalisation, meaning that it generates a closed 2π translation, by choosing c_0 so that $\kappa^2 = 1$.

For example, if we use the coordinate x to denote x_μ with $\mu = 1$, then the Killing vector that vanishes at the root $x = x_0$ of the polynomial $X_1(x)$ is given by (46) with

$$c_i = -\frac{c_0}{(x_0 - \alpha_i)\, B_i} \,. \tag{49}$$

The surface gravity is given by

$$\kappa^2 = \frac{c_0^2\,[X_1'(x)]^2}{\prod_{i=1}^{n-1}(x_0-\alpha_i)^2}\,. \tag{50}$$

It is quite involved to give a general discussion of the conditions arising from requiring regularity and compatibility at all the degeneration surfaces for these metrics. In the following subsection we shall examine the case of six dimensions in some detail, and show that in fact the conditions for regularity imply that the parameters in the metric must be specialised so that the cohomogeneity reduces from 3 to 2. This leads back to the situation discussed in the previous section, and thus we shall conclude that in six dimensions the only smooth resolution of a cone over L^{pqr} spaces that lies within the class of metrics (33) is the $Y^{2,1}$ example found in [8].

5.2 Cohomogeneity-3 metrics in $D=6$

Here we consider the specialisation of the discussion in section 5.1 to the case of six dimensions. We shall denote the three coordinates x_μ by x, y and z in this case, choosing z as the radial coordinate, and the two parameters α_i will be denoted by α and β. Thus we need to consider degeneration surfaces at $x=x_1$, $x=x_2$, $y=y_1$, $y=y_2$ and $z=z_0$. The associated normalised Killing vectors that vanish on these surfaces will be labelled $(\ell_0,\ell_1,\ell_2,\ell_3,\ell_4)$ respectively. From the discussion at the end of section (5.1), it follows that these Killing vectors are given by

$$\begin{aligned}
\ell_0 &= \frac{1}{\alpha\beta-2(\alpha+\beta)z_0+3z_0^2}\Big((\alpha-z_0)(\beta-z_0)\frac{\partial}{\partial\psi}+\frac{\beta-z_0}{\beta-\alpha}\frac{\partial}{\partial\hat\phi_1}+\frac{\alpha-z_0}{\alpha-\beta}\frac{\partial}{\partial\hat\phi_2}\Big)\,, \qquad (51)\\
\ell_i &= \frac{1}{\alpha\beta-2(\alpha+\beta)x_i+3x_i^2}\Big((\alpha-x_i)(\beta-x_i)\frac{\partial}{\partial\psi}+\frac{\beta-x_i}{\beta-\alpha}\frac{\partial}{\partial\hat\phi_1}+\frac{\alpha-x_i}{\alpha-\beta}\frac{\partial}{\partial\hat\phi_2}\Big)\,,\ i=1,2\\
\ell_{i+2} &= \frac{1}{\alpha\beta-2(\alpha+\beta)y_i+3y_i^2}\Big((\alpha-y_i)(\beta-y_i)\frac{\partial}{\partial\psi}+\frac{\beta-y_i}{\beta-\alpha}\frac{\partial}{\partial\hat\phi_1}+\frac{\alpha-y_i}{\alpha-\beta}\frac{\partial}{\partial\hat\phi_2}\Big)\,,\ i=1,2
\end{aligned}$$

Note that the Killing vector whose length goes to zero on a given degeneration surface has coefficients that depend only on α, β and the limiting value of the coordinate that parameterises the approach to that specific surface. In particular, this means that the analysis of the regularity of the level surfaces away from the degeneration at $z=z_0$ is independent of z, and therefore the regularity of the level surfaces at general $z\neq z_0$ is assured once the regularity of the asymptotic Einstein-Sasaki space is established. Thus, as in the cohomogeneity-2 metrics we discussed earlier in the paper, the discussion of regularity can be broken into two parts; first analysing the regularity of the non-degenerate level surfaces, and then analysing the regularity at the degeneration surface $z = z_0$.

To be more explicit, at the asymptotic region where $|z|$ goes to infinity, the metric describes a cone over the Einstein-Sasaki metric L^{pqr}. The periodicity conditions for the azimuthal coordinates $\psi, \hat\phi_1, \hat\phi_2$ are given by the analysis of the linearly-related Killing vectors $\ell_1, \ell_2, \ell_3, \ell_4$, namely[5]

$$p\,\ell_1 + q\,\ell_2 + r\,\ell_3 + s\,\ell_4 = 0\,. \tag{52}$$

As mentioned above, these four Killing vectors are independent of z, and so the deformed L^{pqr} level surfaces at constant $z \neq z_0$ will also be regular if (52) holds.

Now, it remains to investigate the degeneration surface at $z = z_0$. To do this, we should therefore include the associated Killing vector ℓ_0 in the linear relation. In general, we shall have

$$n_0\,\ell_0 + n_1\,\ell_1 + n_2\,\ell_2 + n_3\,\ell_3 + n_4\,\ell_4 = 0\,. \tag{53}$$

The vector space is three-dimensional, and so this equation implies three relations. A particularly simple one is obtained by taking the coefficient of $\partial/\partial\psi$ plus $\alpha(\alpha-\beta)/3$ times the coefficient of $\partial/\partial\hat\phi_1$ plus $\beta(\beta-\alpha)/3$ times the coefficient of $\partial/\partial\hat\phi_2$. This implies

$$n_0 + n_1 + n_2 + n_3 + n_4 = 0\,. \tag{54}$$

Since the vector space is only three-dimensional we can consider specialisations of (53) where any one of n_1, n_2, n_3 or n_4 is set to zero. We shall consider two such specialisations.

First, consider the case where $n_4 = 0$. This implies that $n_0 + n_1 + n_2 + n_3 = 0$. Furthermore, it should be noted that ℓ_0, ℓ_1 and ℓ_3 can all have zero length simultaneously (at $z = z_0$, $x = x_1$ and $y = y_1$). Likewise, ℓ_0, ℓ_2 and ℓ_3 can all have zero length simultaneously. This implies (see [11, 12] for further discussion of such multiply-degenerate surfaces) that

$$\gcd(n_0, n_1, n_3) = n_2\,, \qquad \gcd(n_0, n_2, n_3) = n_1\,. \tag{55}$$

Since (n_0, n_1, n_2, n_3) are coprime, this means that $n_1 = n_2 = 1$.

The same argument goes through if we instead choose $n_3 = 0$. Denoting the remaining coprime integers by n_0', n_1', n_2' and n_4' in this case, we must have $n_1' = n_2' = 1$. Now, the set of three relations following from (53) for these two cases can be solved respectively

[5]Note that the coprime integers p, q, r, s we are using here are not necessarily the same as those in [11]. When we speak of the Einstein-Sasaki space L^{pqr}, we are using this as a generic name, rather than implying a specific correlation of parameters.

for (n_1, n_2, n_3) in terms of n_0, and for (n'_1, n'_2, n'_4) in terms of n'_0. Calculating the ratio $(n_1/n_2)/(n'_1/n'_2)$ (which we have just shown must equal 1 for regularity), and subtracting 1, we obtain the simple result

$$(x_1 - x_2)(y_1 - y_2) = 0\,. \tag{56}$$

Thus to obtain regular spaces, we must either have $x_1 = x_2$ or $y_1 = y_2$. This leads back to the cohomogeneity-2 metrics that we have already studied earlier in the paper.[6]

The upshot of the above discussion is that in six dimensions, there are no non-singular cohomogeneity-3 spaces contained within the class (33), and thus the cohomogeneity-2 example with $Y^{2,1}$ level surfaces found in [8] is the only non-singular example beyond cohomogeneity 1. It is of interest to study the situation in higher dimensions.

6 Conclusions

In this paper, we have generalised a recent result obtained in [8], where a smooth resolution of the Calabi-Yau cone over the five-dimensional Einstein-Sasaki space $Y^{2,1}$ was constructed. The construction in [8] involved specialising the parameters in a general class of Ricci-flat Kähler metrics that were obtained in [13, 9] as BPS limits of the Kerr-NUT-AdS metrics that were also constructed in those papers. The same Ricci-flat Kähler metrics also arise as a limit, in which the cosmological constant is sent to zero, of Einstein-Kähler metrics found in [14].

Our generalised analysis showed that we can obtain non-singular resolutions of Calabi-Yau cones over Einstein-Sasaki spaces in all even dimensions $D \geq 6$, by specialising the parameters in the metrics obtained in [13, 9]. In all cases, the topologies of the associated smooth manifolds are of the form of an $\mathbb{R}^2$ bundle over a base space that is itself an S^2 bundle over an Einstein-Kähler manifold Z. We examined in particular the cases where the Einstein-Kähler manifold Z is taken to be a product of complex projective spaces. Details of our results are contained in section 3.

All the non-singular resolutions that we found have cohomogeneity 2, meaning that the level surfaces at constant radius are themselves of cohomogeneity 1. These level surfaces asymptotically approach Einstein-Sasaki spaces at large radius. We also examined the

[6] For example, if $x_1 = x_2$, which means the range of the x coordinate is pinched down to zero, one can obtain a non-singular metric by setting $x_1 = x_0 - \epsilon$, $x_2 = x_0 + \epsilon$, $x = x_0 - \epsilon \cos\theta$ and sending ϵ to zero. The coordinate θ then parameterises the latitude of a (homogeneous) round 2-sphere, and the cohomogeneity of the metric is reduced from 3 to 2.

more general situation of the higher-cohomogeneity Ricci-flat Kähler metrics obtained in [9]. In dimension $D = 2n$, these have cohomogeneity n. We showed that the level surfaces are asymptotic to the general classes of Einstein-Sasaki metrics that were constructed in [11, 12]. However, a detailed analysis of the situation in dimension $D = 6$ shows that although in general these cohomogeneity-3 metrics describe Calabi-Yau cones over the L^{pqr} Einstein-Sasaki metrics, the conditions for the absence of conical singularities can only be satisfied under the specialisation to cohomogeneity 2. This leads to the conclusion that within this class of six-dimensional metrics, only the Calabi-Yau cone over $Y^{2,1}$ admits a non-singular resolution. It is of interest to study the situation in higher dimensions, to see whether the non-singular resolutions we have constructed in this paper are the only ones within the class of metrics found in [9].

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