

Hořava-Witten Stability: eppur si muoveW. Chen,[‡] Z.-W. Chong[‡] G.W. Gibbons[‡], H. Lü[‡] and C.N. Pope^{‡1}

[‡]*George P. & Cynthia W. Mitchell Institute for Fundamental Physics,
Texas A&M University, College Station, TX 77843, USA*

[‡]*DAMTP, Centre for Mathematical Sciences,
Cambridge University, Wilberforce Road, Cambridge CB3 0WA, UK*

ABSTRACT

We construct exact time-dependent solutions of the supergravity equations of motion in which two initially non-singular branes, one with positive and the other with negative tension, move together and annihilate each other in an all-enveloping spacetime singularity. Among our solutions are the Hořava-Witten solution of heterotic M-theory and a Randall-Sundrum I type solution, both of which are supersymmetric, *i.e.* BPS, in the time-independent case. In the absence of branes our solutions are of Kasner type, and the source of instability may ascribed to a failure to stabilise some of the modulus fields of the compactification. It also raises questions about the viability of models based on some sorts of negative tension brane.

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1 Introduction

One of the most popular current models relating M-theory to phenomenology is that of Hořava and Witten [1], in which an eleven-dimensional spacetime is the product of a compact Calabi-Yau manifold with a 5-dimensional spacetime consisting of two parallel 3-branes or domain walls, one with negative tension and one with positive tension. Alternatively one may think of the five-dimensional reduced spacetime as the warped product of Minkowski spacetime with an interval, *i.e.* $\mathbb{E}^{3,1} \times S^1/\mathbb{Z}_2$.

If one performs a generalised dimensional reduction of the eleven-dimensional theory, of the kind introduced in [2] in which the 4-form has non-vanishing flux on a Calabi-Yau internal space, one obtains a five-dimensional supergravity theory [3] which admits an exact static supersymmetric solution of the form

$$\begin{aligned} ds_5^2 &= \tilde{H} (-dt^2 + dx^2) + \tilde{H}^4 d\tilde{y}^2, \\ \tilde{H} &= 1 + \tilde{k} |\tilde{y}|, \quad \phi = -3 \log \tilde{H}, \end{aligned} \tag{1}$$

where \tilde{k} is a constant. The scalar field ϕ characterises the size of the internal Calabi-Yau space. The equations of motion for the metric and ϕ may be consistently obtained from the Lagrangian

$$\mathcal{L}_5 = \sqrt{-g} (R - \frac{1}{2} (\partial\phi)^2 - m^2 e^{2\phi}), \tag{2}$$

with $\tilde{k}^2 = 2m^2/3$, where the exponential potential is the remnant of the 4-form field strength. Note that if the solution is lifted back to $D = 11$, it can be viewed, in the orbifold limit, as an intersection of three equal-charge M5-branes [2].

In the Hořava-Witten picture a second domain wall is introduced, at $y = L$, by taking y to be periodic with period $2L$, such that $y = L$ is identified with $y = -L$. Furthermore, one makes the \mathbb{Z}_2 identification $y \leftrightarrow -y$. This solution has been proposed as a model for our universe [3], and so the question of its stability is of obvious interest.

The Hořava-Witten model based on the solution (1) is not completely physically realistic because it is static, while our expanding universe is time dependent. For that reason, many attempts have been made to incorporate the Hořava-Witten model into a cosmological, so-called brane-world, scenario, in which the positions of the 3-branes or domain walls are not fixed but allowed to move in time. Notable among these attempts are the Ekpyrotic scenario [4], in which the big bang is ascribed to the collision of an external brane with our universe, and the Cyclic model [5], in which the distance between the two branes oscillates in time.

Brane scenario cosmologies, particularly those based on collisions, represent a rather radical departure from previous models, and offer a novel perspective on many long-standing problems and puzzles, but they also present new ones of their own. Understandably there has been a great deal of interest in them. However, much of the recent work has been based on effective four-dimensional theories, typically involving a so-called radion field. Once the theory has been brought to this form it is almost indistinguishable from conventional four-dimensional models involving scalar fields; only the names have been changed. If brane scenarios are to be observationally tested it must be via features that are essentially higher dimensional. In particular, in the case of collision models we need a much better understanding of the higher-dimensional collision dynamics derived from a consistent underlying framework such as M-theory. Since at present we lack a complete theoretical formulation of M-theory, any such further understanding at present must come from incomplete or approximate theories such as low-energy supergravity limits or Dirac-Born-Infeld actions. This paper is concerned with the former approach. For some ideas on topology and signature change using the latter, see [6].

In the light of the comments made above, it is clearly worthwhile to obtain time-dependent solutions of the equations of motion coming from (2), and to relate the analysis of stability to the suggestions of [4] and [5]. In a recent paper [7], exact solutions of the supergravity equations of motion representing colliding D3-branes of Type IIB theory moving in ten spacetime dimensions were obtained.¹ By dimensionally reducing these solutions, a class of five-dimensional time-dependent solutions was obtained for a similar Lagrangian to (2) (but with a different power of the dilaton exponential potential) [7]. The static solutions were supersymmetric, but the time-dependent generalisations represent a positive and a negative tension brane moving towards one another, and leading to the complete disappearance of the universe in a spacetime singularity [7]. In this paper, we shall construct exact solutions of the equations of motion coming from precisely the Lagrangian (2) of the heterotic brane model of [3], with the same type of time-dependent properties. Their existence clearly raises questions about the validity of the general belief that supersymmetric ground states should be stable.

We should say at the outset that in what follows we shall be concerned only with the exact classical supergravity equations of [3], and any effects arising from quantum

¹Time-dependent four-dimensional charged black-hole solutions in a de Sitter background were earlier obtained in [8], and generalisations to higher-dimensional charged black holes with a pure exponential scalar potential were found in [9].

considerations, which can induce potentials for the various massless scalar fields in the theory (see, for example, [10] and [11, 12]), are not taken into account. Clearly, such potentials would modify considerably the discussion that follows.

We shall begin by obtaining a five-dimensional time-dependent domain-wall solution for precisely the Lagrangian (2) that arose in the heterotic brane model of [3]. We shall include a detailed discussion of the nature of the time dependence, and also we shall make analogy with the issue of the stability of ordinary Minkowski spacetime. In subsequent sections we shall obtain more general classes of time-dependent domain-wall solutions in arbitrary dimensions, and we shall also discuss the brane-source terms that arise in all the cases.

2 Time-Dependent Heterotic Brane in $D = 5$

2.1 The local time-dependent solution

As well as admitting the static 3-brane solution (1), we find that the five-dimensional theory described by the Lagrangian (2) also admits a time-dependent 3-brane solution, given by

$$\begin{aligned} ds_5^2 &= H^{1/2} (-dt^2 + d\mathbf{x}^2) + H dy^2, \\ H &= h t + k |y|, \quad \phi = -\frac{3}{2} \log H, \end{aligned} \tag{3}$$

where $k^2 = 8m^2/3$, and h is an arbitrary constant.

Note that if we turn off the time dependence (by setting $h = 0$), the relation to the previous static solution is seen by making a coordinate transformation of the form $y = \tilde{y}^2$. Substituting this into (3), we recover (1), after some simple constant rescalings. If, on the other hand, we set the parameter m in the Lagrangian to zero, the solution describes a Kasner universe.

When we lift the solution back to $D = 11$, the metric becomes

$$ds_{11}^2 = H^{-1/2} (-dt^2 + d\mathbf{x}^2) + H^{1/2} ds_{CY_6}^2 + dy^2, \tag{4}$$

The static solution, in the orbifold limit, can be viewed as an intersection of three equal-charge M5-branes [2]. Turning off the brane charge, the time-dependent metric describes a direct product of a ten-dimensional Kasner universe and a line segment.

2.2 Local static Killing vectors and Killing horizons

Although we have presented the solution (3) in an ostensibly time-dependent form, one might worry that in fact it was not truly time-dependent at all, since locally one can

eliminate the time dependence by means of a coordinate transformation. If we temporarily drop the modulus sign around y in (3), then the coordinate transformation from t and y to \tilde{t} and r given by

$$dt = d\tilde{t} - \frac{hr^{1/2}}{k^2 f} dr, \quad H = r, \quad f = 1 - \frac{h^2 r^{1/2}}{k^2} \quad (5)$$

transforms the solution into the static form

$$\begin{aligned} ds_5^2 &= r^{1/2} \left(-f d\tilde{t}^2 + d\mathbf{x}^2 + r^{1/2} \frac{dr^2}{k^2 f} \right), \\ \phi &= -\frac{3}{2} \log r. \end{aligned} \quad (6)$$

This can be recognised as a black 3-brane, with an horizon at $f = 0$. However, as we shall discuss below, the introduction of the modulus sign on y in (3) changes the conclusion completely.

First, it is useful to continue temporarily with the modulus sign omitted, and to look at the Killing symmetries of the solution. It depends on t and y only though the combination $H = ht + ky$, and so if one adopts H and another combination of t and y , say $v = ht - ky$, as new coordinates, then the solution is independent of v . Thus

$$K = K^\mu \frac{\partial}{\partial x^\mu} = k \frac{\partial}{\partial t} - h \frac{\partial}{\partial y}, \quad (7)$$

is a Killing vector field which lies in the hypersurfaces $H = \text{constant}$, and which has norm

$$g(K, K) \equiv g_{\mu\nu} K^\mu K^\nu = -k^2 H^{\frac{1}{2}} + h^2 H. \quad (8)$$

In other words

$$K(H) \equiv K^\mu \partial_\mu H = 0. \quad (9)$$

Thus, if one moves such that $ht + ky = \text{constant}$, the domain wall appears to be independent of the “time” coordinate v . If $H < \frac{k^4}{h^4}$, then the Killing vector is timelike and v is a genuinely timelike coordinate, but if $H > \frac{k^4}{h^4}$ it is spacelike, and v becomes a spacelike coordinate. On the hypersurface N given by $H = \frac{k^4}{h^4}$, the Killing vector field is null,

$$g(K, K) = g_{\mu\nu} K^\mu K^\nu = 0. \quad (10)$$

Thus the hypersurface N is itself null, and the orbits of K were its null generators. Such hypersurfaces are called Killing horizons by Carter [13].

As mentioned above, our domain-wall solution (3) is in fact genuinely time dependent, despite being locally static, and the time dependence is not a mere coordinate artefact. The

crucial point is the inclusion of the modulus sign in (3), which introduces a physical domain wall at $y = 0$. If we sit on this wall, the metric is explicitly time dependent because the function H is not constant at the wall. In other words the domain wall does not move along the orbits of the Killing vector K .

In fact the hypersurfaces $H = \text{constant}$ have a kink at $y = 0$. The vector K suffers a jump in slope at $y = 0$, and thus the one-parameter group of translations which it generates is discontinuous at $y = 0$.

The fact that the metric is locally static outside a domain wall is a rather general phenomenon. For example in the thin-wall approximation, a four-dimensional domain wall spacetime consists of two pieces of flat Minkowski spacetime inside the timelike hyperboloid

$$x^2 + y^2 + z^2 - t^2 = a^2, \quad (11)$$

which are glued back to back, thus compactifying spacetime. On either side of the domain wall the metric is static, but the domain wall itself is moving with respect to any of the doubly-infinite number of static coordinate systems related by $SO(3,1)$ transformations on either side. However there is no global static coordinate system [14]. The domain wall itself, and the universe containing it, are expanding (or contracting), and in terms of a proper-time parameter τ the expansion is exponential. In other words, the universe is inflating because of the presence of the domain wall. In fact the induced metric on the domain wall is precisely that of 2+1 dimensional de Sitter spacetime,

$$ds^2 = -d\tau^2 + a^2 \cosh^2 \frac{\tau}{a} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (12)$$

Analogously, our 3-brane is also a thin domain wall solution, which is locally static outside the wall. If we could smooth out the kink and replace the $|y|$ profile in (3) by a smooth trough, then the solution would no longer be even locally static.

Having established that our 3-brane solution is indeed genuinely time-dependent, it is appropriate to discuss the nature of its time evolution in more detail. This discussion is closely analogous to the evolution of the time-dependent 3-brane considered in [7].

2.3 Global structure of the time-dependent Hořava-Witten spacetime

In this section we shall discuss the global structure of the time-dependent generalisation of the Hořava-Witten model. This corresponds to taking the time-dependent solution (3) and passing to the case of the S^1/\mathbb{Z}_2 orbifold. Thus we consider a solution of the form (3) for $-L < y < L$. The solution is then extended outside this interval by demanding that it

be periodic with period $2L$. This leads to a negative-tension brane located at $y = 0$ and a positive-tension brane located at $y = L$. The interval $0 \leq y \leq L$ may be thought of as S^1/\mathbb{Z}_2 , where the S^1 occupies $-L \leq y \leq L$ and the \mathbb{Z}_2 action is $y \rightarrow -y$.

Now if, as in [7], h is taken to be negative, H will be positive but decreasing, for all negative values of t and the spacetime occupies the strip $0 \leq y \leq L$ when $t < 0$. Evidently the proper length of the interval is time dependent, and the universe is contracting, but not exponentially. The proper length of the interval is

$$\ell = \int_0^L dy H^{1/2} = \frac{2}{3k} \left[(ht + kL)^{3/2} - (ht)^{3/2} \right]. \quad (13)$$

At large negative t we have

$$\ell \approx (ht)^{1/2} L, \quad (14)$$

and at $t = 0$, the proper length is $\ell = \frac{2}{3} k^{1/2} L^{3/2}$.

For t positive, H vanishes along the straight line

$$-ht = ky. \quad (15)$$

This represents a singularity which starts from the negative tension brane and moves towards the positive tension brane, reaching it at time $t = \frac{kL}{(-h)}$. The spacetime cannot be extended beyond this, because the scalar field ϕ diverges, giving rise to a curvature singularity. This can be seen from the calculation presented in (51) in the appendix; the curvature clearly diverges at $H = 0$. It can also be seen from the metric (4) in $D = 11$, which becomes complex when H is less than zero.

To summarise, the separation of the two 3-branes decreases monotonically as t increases towards zero, but before they actually collide a power-law curvature singularity develops on negative-tension brane at $t = 0$, which spreads out and eventually envelopes the entire spacetime including the positive-tension brane, at time $t = \frac{kL}{(-h)}$.

The norm of the local Killing vector is

$$H^{1/2} (h^2 (ht - ky)^{1/2} - k^2). \quad (16)$$

For large negative t the Killing field is spacelike. For small negative t it becomes timelike near the negative tension brane. For $L < \frac{k^3}{h^2}$, it has become timelike along the entire interval before $t = 0$.

The Einstein conformal frame metric induced on the branes is

$$ds^2 = H(-dt^2 + d\mathbf{x}^2). \quad (17)$$

For the brane at $y = 0$ and negative values of t , the proper time on the brane is $\tau = \frac{2}{3}(ht)^{1/2}t$. The metric becomes

$$ds^2 = -d\tau^2 + a^2(\tau)d\mathbf{x}^2, \quad a(\tau) \propto \tau^{1/3}. \quad (18)$$

The contraction towards the Big Crunch at $\tau = 0$ is power law, with the expected power due to a massless scalar field. A similar contraction takes place on the positive tension brane, simply shifted in time by an amount $\frac{kL}{(-h)}$.

The natural interpretation of the solution is that the collision of the negative tension brane with the positive tension brane brings about the complete annihilation of the universe.

From the $D = 11$ point of view, the interpretation is different in detail but the same in essence. As we can see from (4), the coordinate y itself measures proper distance in $D = 11$. The solution can be viewed as an M5-brane wrapped on the supersymmetric 2-cycles of CY_6 in a Kasner spacetime. If we turn off the brane charge k , the metric is a direct product of a ten-dimensional Kasner universe and a line segment. Thus the coordinate y describes precisely the original Hořava-Witten line-segment of S^1/\mathbb{Z}_2 , except that in the original Hořava-Witten picture, the ten-dimensional spacetime is $(\text{Minkowski})_4 \times CY_6$ instead of a Kasner universe with the spatial sections being $\mathbb{E}^3 \times CY_6$.

When the M5-brane charge is turned on, the wrapped M5-branes are perpendicular to the y direction. The distance between the branes stays fixed in $D = 11$, since y is a proper-distance coordinate. The metric becomes singular when $H = 0$, in which case the volume of the Calabi-Yau manifold shrinks to zero. The metric cannot be extended to $H < 0$ since it would then become complex. If there is only a single brane, so that y is an infinite line, then there are always regions of y , for any given t , for which the metric is well-defined. However in the Hořava-Witten model, where y is a line segment with a brane at each end, the universe is totally annihilated at the time $t = kL/(-h)$, after which H becomes negative for all y in the interval S^1/\mathbb{Z}_2 .

We have shown that the usual static heterotic five-dimensional 3-brane solution found in [3] has a time-dependent generalisation that is highly singular. It describes the subsequent time evolution if one sets the two 3-branes in motion towards each other with a small velocity. It is appropriate therefore to address in more detail the question of whether this represents an inherent instability of the static 3-brane. In fact a useful analogy is to consider first certain singular time-dependent generalisations of the Minkowski metric itself, and the question of whether their existence is indicative of an instability of flat spacetime. We address this question in the following subsection, as a prelude to discussing the stability of the Hořava-Witten spacetime itself in the subsequent subsection.

2.4 Stability versus instability of flat spacetime

It may be useful to begin by recalling some elementary and widely appreciated facts about the stability of flat spacetime. Of course there is no doubt that in theories with fields which can carry only positive energy, flat spacetime is stable against perturbations of finite total energy, such as might be produced in a terrestrial laboratory. Cosmologically however, the situation is different because there is no obvious reason why we should impose a condition of finite total energy, and indeed to do so would seem to violate the so-called Cosmological Principle, which rules out privileged spatial locations in the universe. A perturbation would need to fall off quite sharply away from where it is largest in order to be of finite total energy. In fact if the flat spacetime were spatially compact, for example if the spatial sections were tori, then presumably all perturbations are of finite (but possibly vanishing) total energy. But this raises a problem with stability.

Consider, for example, the exact Kasner solutions of the vacuum Einstein equations,

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \quad (19)$$

where p_1, p_2, p_3 are constants such that

$$p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2. \quad (20)$$

Unless one of the p_i is equal to 1, these metrics have a singularity at $t = 0$. If we set $t = 1 - t'$, then the metric near $t = 1$ starts out looking like a small deformation of the flat metric, with a small homogeneous mode growing linearly with t' . Ultimately, however, non-linear effects take over and the universe ends in a Big Crunch at $t' = 1$, *i.e.* $t = 0$.

This instability is universal in gravity theories, and is closely related to the modulus problem in theories with extra dimensions. Consider, for example, the exact ten-dimensional Ricci-flat metric

$$ds^2 = t^{1/2}(-dt^2 + d\mathbf{x}^2) + t^{1/2}g_{mn}(y)dy^m dy^n, \quad (21)$$

where $g_{mn}(y)$ is a six-dimensional metric on a Calabi-Yau space K . This starts off at $t = 1$ looking like $\mathbb{E}^{3,1} \times K$ with a small perturbation growing linearly in $t' = 1 - t$. However by the time it reaches $t' = 1$, $t = 0$, the solution has evolved to give a spacetime singularity. From the point of view of the four-dimensional reduced theory, the logarithm of the volume of the Calabi-Yau behaves like a massless scalar field – the modulus field which is sometimes thought of as a kind of Goldstone mode for a spontaneously-broken global scaling symmetry. This causes an isotropic expansion or contraction of the three spatial dimensions, with the

scale factor $a(\tau)$ going like $\tau^{\frac{1}{3}}$, which is what one expects for a fluid whose energy density equals its pressure.

It is clear that no argument based on the local energy density of the effective three-dimensional theory, or on the fact that $\mathbb{E}^{3,1} \times K$ is supersymmetric, *i.e.* admits Killing spinors, can eliminate this source of instability. It is intrinsic to the situation being considered, and to the fact that nothing sets the scale of the Calabi-Yau manifold. The same is true of the Kasner type instability of flat space described previously. If one imagines reducing the Kasner solution on all the spatial dimensions, leaving just the time direction, one has three modulus fields or Goldstone modes, corresponding to the three arbitrary length scales. Note further that if we do identify the spatial directions, any global energy or supercharge functional will automatically vanish, because they may be expressed as boundary terms of a space with no boundary. Thus arguments based on Witten identities, *etc.*, cannot be applied in this case.

2.5 Stability of Hořava-Witten spacetime

With the discussion above in mind, we can return to the issue of stability for the brane solutions. The situation is rather similar to the Kasner instability of flat spacetime. Indeed our solution reduces to a Kasner type solution if we set $k = 0$, *i.e.* in the absence of the branes.

As always in physics, whether we say a system is stable or unstable depends upon what boundary conditions we are prepared to allow in the past. In the case of an environmental science like cosmology this is well understood to be problematic. It becomes even more so in brane cosmology since imposing a boundary condition in the past is tantamount to declaring what influences are allowed on our universe from “outer space,” that is in, our case, from higher dimensions. What our calculations clearly show is that if no such restrictions are imposed, then the Hořava-Witten spacetime is certainly unstable. Any argument showing it to be stable must therefore be an argument based on a theory about the initial conditions. In this respect, the parallel with the debate between Catastrophists and Uniformitarians among nineteenth century geologists trying to unravel the history of the earth is rather striking [15].

2.6 Relation to Chamblin-Reall’s work

In this section we shall review the catastrophe awaiting the denizens of a Hořava-Witten brane-world in the manner of Chamblin and Reall [16]. They focus on the locally static

form of the metric and regard the 3-branes as moving in it, rather than using the co-moving description we have adopted. One starts from the full Kruskal-type extension of the locally static manifold. This is constructed by first introducing advanced and retarded Eddington-Finkelstein coordinates (u, v) for the locally-static metric given in (6):

$$dv = k d\tilde{t} + \frac{r^{1/4} dr}{f}, \quad dv = k d\tilde{t} - \frac{r^{1/4} dr}{f}. \quad (22)$$

Next, Kruskal coordinates are defined by

$$V = e^{-\kappa v}, \quad U = e^{\kappa u}. \quad (23)$$

where κ is a constant to be determined shortly.

The metric now takes the form

$$ds^2 = \frac{r^{1/2} f}{k^2 \kappa^2 UV} dU dV + r^{1/2} d\mathbf{x}^2, \quad (24)$$

where r should be regarded as a function of UV . The $SO(1, 1)_0 \equiv \mathbb{R}_+$ symmetry² of the metric is now manifest since it is invariant under the boosts $U \rightarrow \lambda U$, $V \rightarrow \lambda^{-1} V$, whose orbits in the $U - V$ plane are hyperbolae $UV = \text{constant} \Leftrightarrow r \equiv H = \text{constant}$. The thus-far arbitrary constant κ is now chosen so that $\frac{f}{UV}$ is non-zero and analytic in UV on the past and future horizons, which are at $UV = 0 \Leftrightarrow r = \frac{k^4}{h^4}$. This implies that we should take

$$\kappa = \frac{h^4}{4k^4}. \quad (25)$$

(The full Kruskal manifold is illustrated in Fig. 4 below.) Every point represents a copy of three-dimensional Euclidean space \mathbb{E}^3 with coordinates \mathbf{x} . The metric is invariant under the full $O(1, 1)$ group including time reversal and space reflections. It is bounded on the left and right hand sides by the two hyperbolae $r \equiv H = 0$, at which the metric is singular. There are four regions, denoted by I, II, III and IV. In regions II and IV, the orbits of the boosts are spacelike. In region I they are future-directed and timelike, and in region III they are past-directed and timelike. On the two Killing horizons $UV = 0$, the orbits are lightlike. The two horizons cross on the Boyer axis at the origin $U = 0 = V$.

In the time-independent limit, that is $h = 0$, the picture is rather different. The metric is globally static, and there are no horizons. The spacetime is bounded by a single singularity at $r = 0$. The time independent Hořava-Witten spacetime then occupies only a portion of

²The subscript $_0$ denotes the component connected to the identity. The group \mathbb{R}_+ is the group of positive reals under multiplication.

the full static manifold, between two values of r . In Fig. 1 we have sketched the Hořava-Witten interval in co-moving (t, y) coordinates. It extends infinitely far in the positive and negative time directions. Also shown are some representative orbits of the time translation Killing vector field. In Fig. 2 the Hořava-Witten interval is shown lying between two orbits of the static Killing vector field $\frac{\partial}{\partial t}$.

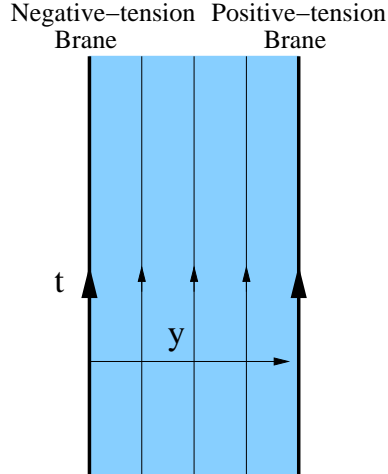


Figure 1: The time-independent Hořava-Witten spacetime in (t, y) coordinates. Also shown are some typical orbits of the timelike Killing vector field $\partial/\partial t$.

In Fig. 3 we have shown the time-dependent Hořava-Witten interval in co-moving coordinates (t, y) . Also indicated are the orbits of the timelike Killing field, which lie in the surfaces $H = \text{constant}$. In the past (*i.e.* in region IV), it is spacelike. In the future (*i.e.* in region I), it is future-directed timelike. These two regions are separated by a null surface – the Killing horizon.

In Fig. 4 we have shown how the Hořava-Witten interval is inserted into the full Kruskal manifold. It lies between two 3-branes, which appear in the Kruskal diagram as two world lines, initially starting in region IV, passing through the past horizon into region I, and coming to an end on the spacetime singularity at $r \equiv H = 0$.

It is interesting that the global Kruskal spacetime picture resembles the situation in de Sitter spacetime, except in that case the origin $r = 0$ is non-singular. The horizons then have more of the character of cosmological horizons than black hole horizons, at least for observers who remain in regions I or III. In particular, if we reverse the direction of time, then the Hořava-Witten interval emerges from a spacetime singularity in the past and then inflates, passing through a future Killing horizon. If we confine our attention to the Hořava-Witten interval however, this Killing horizon is not a future event horizon for denizens of

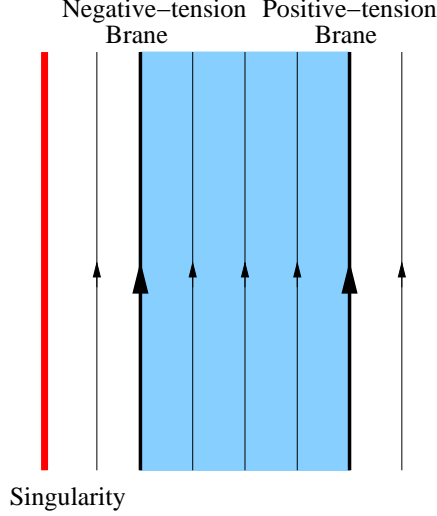


Figure 2: The full globally-static spacetime of the time-independent Hořava-Witten solution. There is a single singularity on the left, at $r = 0$. Some representative orbits of the timelike Killing vector field are indicated. The Hořava-Witten interval lies between two such orbits.

the negative tension brane or indeed of the bulk.

3 General Time-Dependent Domain-Wall Solutions

Here we obtain generalisations of our time-dependent solution (3), to the case of domain-wall solutions in dilaton gravity in arbitrary dimensions.

Let us consider a general case of Einstein-scalar system with an exponential potential:

$$\mathcal{L} = \sqrt{-g} \left(R - \frac{1}{2} (\partial\phi)^2 - 2\Lambda e^{a\phi} \right), \quad (26)$$

where the constant a is parameterised by $a^2 = \Delta + \frac{2(D-1)}{D-2}$ [17]. It is straightforward to verify that the system admits the following solution

$$\begin{aligned} ds^2 &= H^{\frac{8}{(\Delta+4)(D-2)}} \left(-dt^2 + dx^2 + H^{\frac{2\Delta}{\Delta+4}} dy^2 \right), \\ \phi &= -\frac{4a}{\Delta+4} \log H, \quad H = ht + k|y|, \end{aligned} \quad (27)$$

where h is an arbitrary constant and k is determined by

$$k = \frac{1}{2}(\Delta+4) \sqrt{\frac{\Lambda}{\Delta}}. \quad (28)$$

It is interesting to note that for $\Delta > 0$, which is typical in *massive* supergravities, with $\Delta = 4/N$ and N taking integer values from 1 to 8 depending on the dimensions [17,

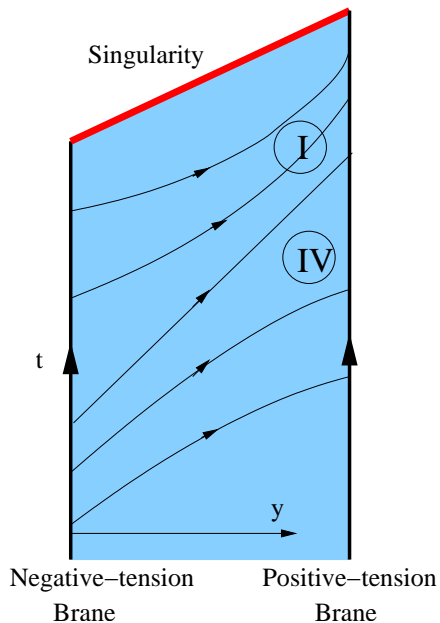


Figure 3: The time-dependent Hořava-Witten spacetime in co-moving (t, y) coordinates. The red line indicating the singularity occurs at $H = 0$. Also indicated are some of the orbits of the Killing vector $\partial/\partial t$. In region I the orbits are timelike and in region IV they are spacelike. These two regions are separated by a Killing horizon on which the orbits are lightlike.

18], Λ is positive. Thus k is real. On the other hand, if $\Delta < 0$, which is typical for *gauged* supergravities, Λ is negative. This again implies that k is real. In particular, for purely Einstein theory with a cosmological constant, we have $\Delta = -2(D-1)/(D-2)$, the cosmological constant is negative, and so k is real.

There are two ways of viewing the above solutions as generalisations of previously-known solutions. One is to show that when $h = 0$, the solutions reduce to standard domain-wall solutions, supported by a purely exponential scalar potential.

The other way of viewing the new solutions as generalisations of old ones is to consider the limit $\Lambda = 0$, under which the solutions become Ricci-flat Kasner metrics with a $(D-2, 1)$ spatial splitting.

Locally, where the modulus sign on y is removed, it is possible to make the coordinate transformation

$$\begin{aligned}
 dt &= d\tilde{t} - \frac{h r^{2\Delta/(\Delta+4)}}{k^2 f} dr, & H &= r, \\
 f &= 1 - \frac{h^2}{k^2} r^{\frac{2\Delta}{\Delta+4}}, & &
 \end{aligned}
 \tag{29}$$

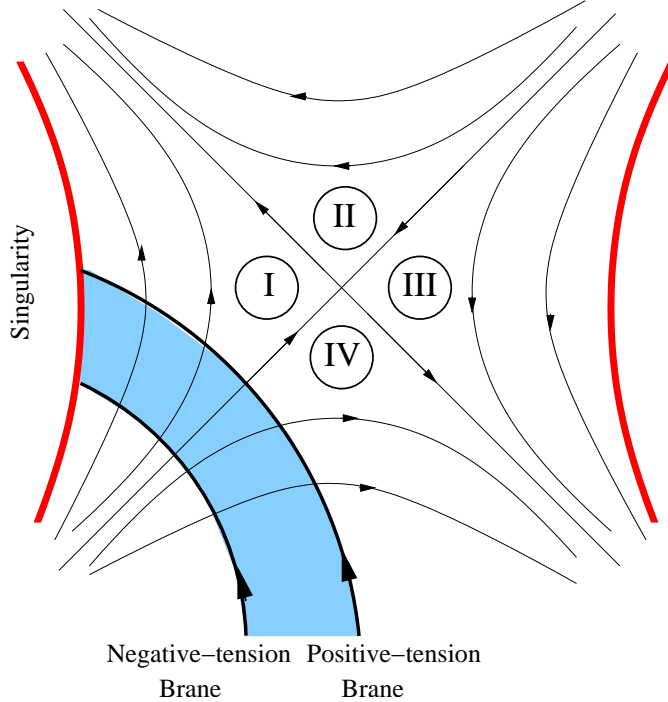


Figure 4: The full Kruskal manifold for the time-dependent Hořava-Witten spacetime. Some typical orbits of the Killing vector $\partial/\partial t$, which are hyperbolae in the Kruskal coordinates U, V , are shown. The orbits are future-directed timelike in region I, past-directed timelike in region III, and spacelike in regions II and IV. Also shown, shaded in blue, is the Hořava-Witten manifold, bounded by the negative and positive tension branes, which move from region IV into region I and then annihilate in the singularity.

under which the solution (27) becomes static, given by

$$\begin{aligned}
 ds^2 &= r^{\frac{8}{(\Delta+4)(D-2)}} \left(-f d\tilde{t}^2 + d\mathbf{x}^2 + r^{\frac{2\Delta}{\Delta+4}} \frac{dr^2}{k^2 f} \right), \\
 \phi &= -\frac{4a}{\Delta+4} \log r,
 \end{aligned} \tag{30}$$

This can be viewed as a static black brane. As we discussed in section 2.2, the mere fact that one can find a local coordinate transformation that renders the metric static does not imply that it is globally static, and in fact again, it is the modulus sign on y in the function H that implies the solutions are genuinely time dependent.

3.1 AdS case

A special case of the above results is when $a = 0$, in which case the scalar-potential term in the Lagrangian (26) becomes a pure cosmological constant. It corresponds to setting

$\Delta = -2(D-1)/(D-2)$. The solution becomes

$$ds^2 = H^{\frac{4}{D-3}} (-dt^2 + d\mathbf{x}^2) + \frac{dy^2}{H^2}, \quad (31)$$

It follows from the curvature calculation (51) that if the solution is static, with $h = 0$, the metric has only a delta-function singularity at the location of the brane. When h is non-vanishing, the metric has also a power-law singularity at $H = 0$.

If one were to remove the modulus sign on y , the metric could be locally transformed using (29) into the AdS black brane:

$$ds^2 = r^{-\frac{4}{D-3}} (-f dt^2 + d\mathbf{x}^2) + \frac{dr^2}{k^2 r^2 f}. \quad (32)$$

Again, it is the inclusion of the modulus sign that renders our solution (31) genuinely time-dependent. Thus we see that even anti-de Sitter spacetime itself is not immune to the Kasner-type instabilities that we have exhibited in the general domain-wall solutions, once the branes with their delta-function sources are introduced. This raises a question about the stability of the Randall-Sundrum scenarios.

4 Brane Sources

To obtain the full description of our solutions, with a source brane action, it is useful first to dualise the cosmological constant to a D -form field strength $F_D = dA_{D-1}$. The full action can be written as

$$\begin{aligned} I = & \int d^D x \sqrt{-g} \left(R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2D!} e^{-a\phi} F_{(D)}^2 \right) \\ & - T \int d^{D-1} \xi \sqrt{-\gamma} \left(\gamma^{ij} \partial_i x^M \partial_j x^N \tilde{g}_{MN} - (D-3) \right) \\ & + \frac{(D-2)}{(D-1)!} \int d^{D-1} \xi \varepsilon^{i_1 \dots i_{D-1}} \partial_{i_1} x^{M_1} \dots \partial_{i_{D-1}} x^{M_{D-1}} A_{M_1 \dots M_{D-1}}, \quad (33) \end{aligned}$$

where $\tilde{g}_{MN} = e^{\frac{a}{D-1}\phi} g_{MN}$ is the metric in the brane frame.³ This frame has the defining property that the Einstein term and the F_D^2 term in the bulk Lagrangian have the same dilaton coupling, and the bulk Lagrangian density takes the form

$$\mathcal{L} = e^{-\frac{1}{2}a\phi(D-2)/(D-1)} \left(\tilde{R} - \frac{1}{2D!} F_{(D)}^2 + \dots \right). \quad (34)$$

Varying the brane-action term in (33) with respect to the metric g_{MN} gives a brane contribution to the energy momentum tensor

$$T_{\text{brane}}^{MN} = T \int d^{D-1} \xi \sqrt{-\gamma} \gamma^{ij} e^{a\phi/(D-1)} \partial_i x^M \partial_j x^N \frac{\delta^D(x - x(\xi))}{\sqrt{-g}}. \quad (35)$$

³See [20] for a discussion of the coupling of the dilaton in the brane action.

Making the static gauge choice $X^\mu = \xi^\mu$ ($0 \leq \mu \leq D - 2$) implies that we have

$$T_{\text{brane}}^{\mu\nu} = T e^{a\phi/(D-1)} \frac{\sqrt{-\gamma}}{\sqrt{-g}} \gamma^{\mu\nu} \delta(y), \quad (36)$$

where $\gamma_{\mu\nu} = \tilde{g}_{\mu\nu} = e^{a\phi/(D-1)} g_{\mu\nu}$. Substituting the solution (27) into this expression yields

$$T_{\mu\nu}^{\text{brane}} = T H^{-(3\Delta+4)/(\Delta+4)} \eta_{\mu\nu} \delta(y). \quad (37)$$

This singular brane source is precisely compatible with the Ricci curvature singularities that we find for the metric in (27), resulting from the discontinuity in the gradient of H at $y = 0$. Specifically, we find that there is a singular term in the Einstein tensor, given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8k}{\Delta + 4} H^{-(3\Delta+4)/(\Delta+4)} \eta_{\mu\nu} \delta(y) + \text{regular terms}. \quad (38)$$

Comparing with $T_{\mu\nu}$, we find that the powers of H match precisely, and so the brane tension T can then be read off as

$$T = -\frac{8k}{\Delta + 4}. \quad (39)$$

This shows that the 3-brane at $y = 0$ has negative tension. If y is assigned period $2L$ and y is identified with $-y$, the second 3-brane at $y = L$ in the resulting S^1/\mathbb{Z}_2 orbifold will have positive tension.

One can also check the dilaton equation of motion, which, including the source term coming from the brane action in (33), becomes

$$\square\phi + \frac{a}{2D!} e^{-a\phi} F_{(D)}^2 = aT \frac{\sqrt{-\gamma}}{\sqrt{-g}} \delta(y). \quad (40)$$

It is straightforward to verify that with the discontinuity in the gradient of ϕ implied by (27), the dilaton equation is indeed satisfied, again with the brane tension T given by (39).

5 Further Time-Dependent Solutions

5.1 Solutions with single-exponential potentials

We can obtain another type of time-dependent brane solution as follows. Consider the extremal static domain-wall solution for the theory (26). By an appropriate choice of the y coordinate it can be written in the conformally-flat frame

$$ds^2 = H^{\frac{2}{(\Delta+2)(D-2)}} \left(-dt^2 + dy^2 + d\mathbf{x}^2 \right), \quad (41)$$

where $H = \tilde{k}y$, with $\tilde{k} = 2(\Delta + 2)k/(\Delta + 4)$. Now we can make a simple Lorentz boost, namely $t \rightarrow ct + sy$ and $s \rightarrow st + cy$, where $c^2 - s^2 = 1$, which implies that H is now given by

$$H = \tilde{k}st + \tilde{k}cy. \quad (42)$$

Of course at this point there is no genuine time-dependence, since it was merely obtained by a (globally-defined) coordinate transformation.

If, however, we introduce a brane, by adding a modulus sign to y and writing

$$H = \tilde{k}st + \tilde{k}c|y|, \quad (43)$$

then the time-dependence is no longer artificial, and the solution describes a moving brane. As usual, one could also extend to the S^1/\mathbb{Z}_2 orbifold in the standard fashion. An analysis of the global structure of these solutions shows that they exhibit the same essential behaviour as the previous examples, with the branes moving towards each other and a power-law singularity developing that eventually engulfs the entire spacetime.

5.2 Scalar potential with an extremum

So far we have considered supergravity domain-wall solutions ($(D - 2)$ -branes) supported by a single exponential potential, which therefore has no extremum. Here, we consider a more general scalar potential that does have an extremum. It is obtained from an S^5 reduction of type IIB supergravity in which the breathing mode is retained, and the 5-form field strength has a non-trivial flux. The relevant scalar Lagrangian is given by [19]

$$\mathcal{L}_5 = \sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2 - V(\phi)), \quad (44)$$

with $V = 8m^2e^{8\alpha\phi} - R_5 e^{16\alpha\phi/5}$ and $\alpha = \sqrt{15}/12$. Here m measures the strength of the 5-form flux and R_5 is the Ricci scalar of the internal S^5 . The scalar potential supports a static domain-wall solution [19]

$$\begin{aligned} ds_5^2 &= (b_1H^{2/7} + b_2H^{5/7})^{-1/2} (-dt^2 + d\mathbf{x}^2) + (b_1H^{2/7} + b_2H^{5/7})^{-2} dy^2, \\ \phi &= -\frac{\sqrt{15}}{7} \log H, \quad H = c + k|y|, \end{aligned} \quad (45)$$

where $b_1^2 = (28m/3k)^2$ and $b_2^2 = 196R_5/45$. The local stability of the solution was recently analysed in [21], where it was shown that subject to certain boundary conditions, the configuration is stable despite the presence of the negative-tension brane.

However, as we have emphasised earlier, it is not clear that one is entitled to impose the kind of energy-localising boundary conditions that are needed in order to argue for

stability, if one is considering brane-world cosmological model. Thus we may again look for a time-dependent generalisation of the static solution, in order to study the stability question from a viewpoint that is more in accordance with the cosmological principle. We find the following time-dependent domain-wall solution:

$$\begin{aligned} ds_5^2 &= (b_2 H^{3/7} + b_1)^{1/2} (ht(H^{3/7} - q) + q)^{1/3} (-dt^2 + d\mathbf{x}^2) \\ &\quad + (b_2 H^{3/7} + b_1)^{-2} (ht(H^{3/7} - q) + q)^{4/3} H^{-8/7} dy^2, \\ \phi &= -\sqrt{\frac{5}{3}} \log \left(ht(H^{3/7} + q) - q \right), \end{aligned} \quad (46)$$

where $q = -b_1/b_2$, and h is an arbitrary constant. The static limit can be achieved by sending $t \rightarrow t + 1/h$ and then sending $h \rightarrow 0$, whereupon the solution reduces to (45).

Making the coordinate transformation $H^{3/7} = \frac{R_5^2}{400} r^4 + q$, the solution (46) can be written as

$$\begin{aligned} ds_5^2 &= (W r^4)^{5/6} \left(W^{-1/2} (-d\tilde{t}^2 + d\tilde{\mathbf{x}}^2) + W^{1/2} dr^2 \right), \\ \phi &= -\sqrt{\frac{5}{3}} \log(h\tilde{t} r^4 + \tilde{q}), \end{aligned} \quad (47)$$

where $W = h\tilde{t} + \tilde{q}/r^4$ and $\tilde{q} = 400q/R_5^2$, $\tilde{x}^\mu = b_2^{1/4} x^\mu$. The r coordinate ranges over an interval $0 < r_1 \leq r \leq r_2$, where r_1 and r_2 are the values corresponding to the brane locations at $y = 0$ and $y = L$ respectively. With h taken to be negative, as usual, we again have the situation that the solution is well-defined for sufficiently negative times \tilde{t} , with the two 3-branes moving towards each other, but as \tilde{t} increases in the positive direction, a time is reached at which $W \leq 0$ for all $r_1 \leq r \leq r_2$, at which point the annihilation of the universe is complete.

Lifting back to $D = 10$, (47) becomes

$$\begin{aligned} ds_5^2 &= \left(h\tilde{t} + \frac{\tilde{q}}{r^4} \right)^{-1/2} (-d\tilde{t}^2 + d\tilde{\mathbf{x}}^2) + \left(h\tilde{t} + \frac{\tilde{q}}{r^4} \right)^{1/2} (dr^2 + r^2 d\Omega_5^2), \\ F_5 &= d^4 \tilde{x} \wedge dW^{-1} + *d^4 x \wedge dW^{-1}. \end{aligned} \quad (48)$$

This solution was obtained in [7], describing a time-dependent D3-brane.

Analogous time-dependent domain-wall solutions can also be found in $D = 4$ and $D = 7$, which can be obtained from S^7 and S^4 reductions of time-dependent M2-branes and M5-branes respectively.

6 Conclusions

In this paper, we have constructed time-dependent solutions of dilaton gravity with an exponential potential, which can be viewed as generalisations of the static domain-wall solutions

of supergravities in various dimensions. Included in these solutions are time-dependent generalisations of the five-dimensional heterotic 3-brane that was proposed in [3] as a model for our universe, and of the AdS 3-brane of the Randall-Sundrum scenarios. The case of principle interest is where the fifth dimension is a line segment, with a positive-tension brane at one end and a negative-tension brane at the other. The time-dependent solution starts out in the distant past in a non-singular regime in which the metric approximates a Kasner model far from the Kasner singularity. The solution evolves to a singularity in which the entire spacetime is annihilated. The existence of this time-dependent solution can be taken as an indication of an inherent classical instability in brane-world models where there are positive-tension and negative-tension branes present. Analogous conclusions can be drawn for brane models of this type in other dimensions, and also in certain more general cases where there is a scalar potential with an extremum. Some support for the idea that Kasner singularities arise in the general case is given by the numerical work reported in [22]. Further support comes from $(2+1)$ -dimensional models [23]. The thermodynamics of negative tension branes has been discussed in [24].

The singular behaviour of solutions may be ascribed to a failure to stabilise some of the modulus fields of the compactification. This is an old problem, and we have little that is new to say about it. By introducing a potential (as for example in [10–12]) the situation should improve. However in the cases we have considered the modulus problem is compounded by the presence of negative tension branes. Such negative tension branes are almost inevitable in any warped compactification [25]. It is by no means obvious that they can be stabilised by additional potentials. Moreover, ideally, one would want not so much a stable static solution, but rather a stable Friedman-Lemaitre expanding solution, and, ideally, it should behave like an “attractor,” as was attempted in the early days of Kaluza-Klein cosmology [26, 27]. In this respect, an unstable static solution might be thought of as an advantage since otherwise the universe could be trapped in limbo in a stable static universe. Unfortunately at present no higher-dimensional field equations using the potentials found in [10] and [11, 12] appear to be available to test this idea.

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APPENDIX

A Curvature of the Domain-Wall Metrics

All the domain-wall metrics that we have been considering in this paper have the general form

$$ds^2 = H^{2\alpha} (-dt^2 + dx^2) + H^{2\beta} dy^2, \quad (49)$$

where α and β are constants. We shall make the obvious choice of orthonormal frame, defining

$$e^0 = H^\alpha dt, \quad e^i = H^\alpha dx^i, \quad e^y = H^\beta dy. \quad (50)$$

The orthonormal frame components of the Riemann tensor are then given by

$$\begin{aligned} R_{0i0j} &= \alpha H^{-2\alpha} \left(\frac{\dot{H}^2}{H^2} - \frac{\ddot{H}}{H} \right) \delta_{ij} + \alpha^2 H^{-2\beta-2} H'^2 \delta_{ij}, \\ R_{0ijy} &= \alpha H^{-\alpha-\beta} \left(\frac{\dot{H}'}{H} - (1+\beta) \frac{\dot{H} H'}{H^2} \right) \delta_{ij}, \\ R_{0y0y} &= \alpha H^{-2\beta} \left(\frac{H''}{H} + (\alpha - \beta - 1) \frac{H'^2}{H^2} \right) - \beta H^{-2\alpha} \left(\frac{\ddot{H}}{H} + (\beta - \alpha - 1) \frac{\dot{H}^2}{H^2} \right), \\ R_{ijkl} &= \alpha^2 \left(H^{-2\alpha} \frac{\dot{H}^2}{H^2} - H^{-2\beta} \frac{H'^2}{H^2} \right) (\delta_{ik} \delta_{j\ell} - \delta_{i\ell} \delta_{jk}), \\ R_{iyjy} &= \alpha \beta H^{-2\alpha-2} \dot{H}^2 \delta_{ij} - \alpha H^{-2\beta} \left(\frac{H''}{H} + (\alpha - \beta - 1) \frac{H'^2}{H^2} \right) \delta_{ij}. \end{aligned} \quad (51)$$

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