KCL-TH-99-47
CTP TAMU-49/99
hep-th/0001169
September 20, 2018

Codimension One Branes

P.S. Howe [1], A. Kaya, E. Sezgin [2†], and P. Sundell [2]

Abstract

We study codimension one branes, i.e. p-branes in $(p+2)$-dimensions, in the superembedding approach for the cases where the worldvolume superspace is embedded in a minimal target superspace with half supersymmetry breaking. This singles out the cases $p = 1, 2, 3, 5, 9$. For $p = 3, 5, 9$ the superembedding geometry naturally involves a fundamental super 2-form potential on the worldvolume whose generalised field strength obeys a constraint deducible from considering an open supermembrane ending on the p-brane. This constraint, together with the embedding constraint, puts the system on-shell for $p = 5$ but overconstrains the 9-brane in $D = 11$ such that the Goldstone superfield is frozen. For $p = 3$ these two constraints give rise to an off-shell linear multiplet on the worldvolume. An alternative formulation of this case is given in which the linear multiplet is dualised to an off-shell scalar multiplet. Actions are constructed for both cases and are shown to give equivalent equations of motion. After gauge fixing a local $Sp(1)$ symmetry associated with shifts in the $Sp(1)_R$ Goldstone modes, we find that the auxiliary fields in the scalar multiplet parametrise a two-sphere. For completeness we also discuss briefly the cases $p = 1, 2$ where the equations of motion (for off-shell multiplets) are obtained from an action principle.

1 Introduction

The superembedding formalism provides a powerful and systematic method for deriving the dynamics of super p-branes and their interactions [1, 2, 3]. In this approach both the worldvolume and target space are superspaces. Restricting the embedding such that the fermionic worldvolume tangent space is a subspace of the fermionic target tangent space gives rise to equations which determine the structure of the worldvolume supermultiplet of the p-brane and which may also determine the dynamics of the brane itself.

There are several types of worldvolume supermultiplets that can arise including scalar multiplets, vector multiplets, tensor multiplets which have 2-form gauge fields with self-dual field strengths, and multiplets with rank 2 or higher antisymmetric tensor gauge fields whose field strengths are not self-dual [3]. Most of these cases have been studied in the superembedding approach, primarily for embeddings with (bosonic) codimension greater than one. In this paper we focus on codimension one embeddings for which the basic embedding constraint gives rise to an unconstrained scalar Goldstone superfield.

We shall consider target spaces with the minimal possible number of supersymmetries and embeddings which preserve half the supersymmetry. This leads to p-branes in $D = p + 2$ dimensions for $p = 1, 2, 3, 5, 9$, as shown in Table 1. The cases of $p = 4, 6, 7, 8$ are excluded since in these cases the minimal worldvolume spinors have the same dimension as the minimal target space spinors and consequently half-supersymmetry breaking is not possible (embedding without supersymmetry breaking would imply that the Goldstone fermions vanish and the Goldstone bosons are constant in a physical gauge).

p	$(D\,\vert\,N)$	$(d\,\vert\,n)$	Worldvolume multiplet
9	$(11\,\vert\,32\,;\,\mathrm{M})$	$(10\,\vert\,16\,;\,\mathrm{MW})$	Frozen
5	$(7\,\vert\,16\,;\,\mathrm{SM})$	$(6\,\vert\,8\,;\,\mathrm{SMW})$	Tensor (on-shell)
3	$(5\,\vert\,8\,;\,\mathrm{PSM})$	$(4\,\vert\,4\,;\,\mathrm{M})$	Linear/Scalar (off-shell)
2	$(4\,\vert\,4\,;\,\mathrm{M})$	$(3\,\vert\,2\,;\,\mathrm{M})$	Scalar (off-shell)
1	$(3\,\vert\,2\,;\,\mathrm{M})$	$(2\,\vert\,1\,;\,\mathrm{MW})$	Scalar (off-shell)

Table 1: p-branes in $D = p + 2$ dimensions. $(D|N)$ and $(d|n)$ denote the superdimensions of the target and worldvolume superspaces, respectively. The spinor types are denoted by M for Majorana, W for Weyl, S for symplectic and P for pseudo.

The basic embedding constraint mentioned above describes a relation between the supervielbeins in the worldvolume and the target space (see [4, 5, 6] for reviews). When applied to the M2 and M5-branes in $D = 11$ and to some of the D-branes in $= 10$, the embedding constraint alone yields all the dynamics. The situation is different for the codimension one embeddings. In that case, the embedding constraint leads to an unconstrained real scalar superfield in a physical gauge. For the cases $p = 1, 2$, this superfield is an off-shell scalar multiplet and the dynamics can be obtained from an action principle [7].

For $p = 3$ an additional constraint is required to obtain an off-shell multiplet and this can in fact be done in two ways resulting in either a linear multiplet [8] or a dual scalar (chiral) multiplet. The required constraint for the linear multiplet can be deduced from considering a membrane ending on the 3-brane and leads to a constraint on a modified 3-form field strength $\mathcal{F}_3$ [9, 10]. For the scalar multiplet one instead has a constraint on a modified 1-form field strength $\mathcal{F}_1$ which arises from considering a particle on the 3-brane. From either multiplet one can then derive the dynamics from an action principle.

In this paper we shall find that the 3-brane in $D = 5$ exhibits novel features related to the existence of a triplet of 3-brane charges in the underlying spacetime superalgebra. Both linear and scalar formulations yield $Sp(1)_R$ covariant Green-Schwarz actions, which are related by worldvolume dualisation of the associated $\mathcal{A}_2$ and $\mathcal{A}_0$ potentials requiring a duality transformation of the spacetime potentials. While $Sp(1)_R$-symmetry is completely broken, the associated Goldstone scalars can be gauged away by a local $Sp(1)$ symmetry. As a result the scalar multiplet auxiliary fields are found to parametrise a two-sphere. In this paper we also study the 3-brane in $D = 6$ since it provides a better understanding of the local $Sp(1)$ symmetry and, moreover, it yields the scalar multiplet formulation of the 3-brane in $D = 5$ upon vertical reduction.

For $p = 5$ the additional $\mathcal{F}_3$-constraint is similar to that of the linear multiplet and indeed occurs in the M5-brane where, however, it follows from the basic embedding condition [11, 12]. This constraint gives the dynamics of the worldvolume $d = 6, (1, 0)$ tensor multiplet directly without the use of an action principle. The dynamics of this brane was obtained in the superembedding formalism in [13] by imposing an additional constraint on the embedding matrix. As we show here, this constraint arises naturally from the geometrical $\mathcal{F}_3$-constraint discussed above. We also give the embedding of the chiral 5-brane theory in a non-chiral theory where the equations of motion follow from an action that involves an unconstrained 2-form potential, upon the imposition of a non-linear self-duality condition [14].

In the case of $p = 9$, the analogous $\mathcal{F}_3$-constraint turns out to be much stronger; in fact, it freezes the worldvolume multiplet degrees of freedom. We expect, however, that suitable modifications of this constraint will lead to either the Horava-Witten type 9-brane [15] or the massive 9-brane [16].

The outline of the paper is as follows: In Section 2, we will describe the embedding constraint and its general consequences that follow from the use of the torsion Bianchi identities. In Section 3, we will discuss the worldvolume super 3-form, its Bianchi identity and the action formalism. In Section 4 we review the cases of 1-brane in $D = 3$ and 2-brane in $D = 4$. In Section 5 we study the aspects 3-brane: the six-dimensional theory, its vertical reduction to scalar 3-brane in $D = 5$ and its dual linear multiplet formulation. In Section 6 we show how to obtain the dynamics of the 5-brane in $D = 7$ from the embedding constraint and an $\mathcal{F}_3$-constraint. The 9-brane in $D = 11$ is studied in Section 7. Further comments on our results, and in particular on the 9-brane, are presented in Section 8. In Appendix A we give our spinor conventions and in Appendix B we collect the results of the dimension half and up analysis of the torsion equations.

2 The Embedding Constraint and the Torsion Equations

A p-brane with n real supersymmetries has a $d = (p+1)$-dimensional worldvolume moving in a D-dimensional spacetime with N real supersymmetries; it is described by the embedding $f : \mathcal{M} \mapsto \underline{\mathcal{M}}$ of a $(d|n)$-dimensional worldvolume supermanifold $\mathcal{M}$ into a $(D|N)$-dimensional spacetime supermanifold $\underline{\mathcal{M}}$. We parametrise $\mathcal{M}$ and $\underline{\mathcal{M}}$ with supercoordinates $z^M = (x^m, \theta^\mu)$, where $m = 0, ..., p$ and $\mu = 1, ..., n$, and $z^{\underline{M}} = (x^{\underline{m}}, \theta^{\underline{\mu}})$, where $\underline{m} = 0, ..., D-1$ and $\underline{\mu} = 1, ..., N$. The embedding matrix $E_A{}^{\underline{A}}$ is defined as

$$E_A{}^{\underline{A}} \;\; \equiv \;\; E_A{}^M \partial_M z^{\underline{M}} E_{\underline{M}}{}^{\underline{A}} \; , \tag{2.1}$$

where $E^A = dz^M E_M{}^A$ and $E^{\underline{A}} = dz^{\underline{M}} E_{\underline{M}}{}^{\underline{A}}$ are supervielbeins in M and $\underline{\mathcal{M}}$ and the 'flat' indices $A = (a, \alpha)$ and $\underline{A} = (\underline{a}, \underline{\alpha})$ has the same ranges as the curved indices. By definition, the worldvolume supervielbein is not an independent worldvolume field, but rather induced by the embedding. (More specifically, the embedding defines normal and tangential subspaces of the spacetime cotangent space and tangent space, respectively. Forming an 'adapted' cotangent frame $(\hat{E}^a, \hat{E}^{a'}, \hat{E}^\alpha, \hat{E}^{\alpha'})$ for the spacetime, such that $f^\star \hat{E}^{A'} = 0$, $A' = (a', \alpha')$, we define an induced worldvolume frame by $E^A \equiv f^\star \hat{E}^A$.)

The basic embedding constraint is

$$E_\alpha{}^{\underline{a}} \;\; = \;\; 0 \; . \tag{2.2}$$

This is a natural geometric condition which states that, at each point on the brane, the odd tangent space of the brane sits inside the odd tangent space of the target space. For a general superembedding preserving half-supersymmetry this constraint leads to on-shell, off-shell or underconstrained worldvolume supermultiplets. It also places constraints on the target space torsion (via integrability) and is intimately related to the characteristic form of kappa-symmetry transformations in the Green-Schwarz formalism.

In the case of codimension one embeddings listed in Table 1 one can deduce that the embedding constraint gives in fact an unconstrained Goldstone superfield. This can be made clear by analysing the constraint at the linearised level. In a flat target space, and in the physical gauge

$$\begin{aligned} x^{\underline{a}} &= (x^a, x^\perp(x,\theta)) \\ \theta^{\underline{\alpha}} &= (\theta^\alpha, \Theta^{\alpha'}(x,\theta)) \; . \end{aligned} \tag{2.3}$$

The embedding matrix (2.1) takes the form

$$\begin{aligned} E_\alpha{}^{\underline{a}} &= (0, D_\alpha \Phi - i(\Gamma^\perp)_{\alpha\beta'}\Theta^{\beta'}) \; , \qquad & E_a{}^{\underline{b}} &= (\delta_a{}^b, \partial_a \Phi) \; , \\ E_\alpha{}^{\underline{\beta}} &= (\delta_\alpha{}^\beta, D_\alpha \Theta^{\beta'}) \; , & E_a{}^{\underline{\beta}} &= (0, \partial_a \Theta^{\beta'}) \; , \end{aligned} \tag{2.4}$$

where $\Phi = x^\perp + \frac{i}{2}\theta^\alpha(\Gamma^\perp)_{\alpha\beta'}\Theta^{\beta'}$ and D_α is the flat worldvolume supercovariant derivative. Thus the linearised form of (2.2) is

$$D_\alpha \Phi \;=\; i(\Gamma^\perp)_{\alpha\beta'}\Theta^{\beta'} \ . \tag{2.5}$$

Since the matrix appearing on the right side is non-degenerate, this equation shows that the scalar superfield Φ is unconstrained, i.e. the content of the equation is simply to give the transverse fermionic superfield $\Theta^{\alpha'}$ in terms of the unconstrained transverse scalar superfield Φ.

The embedding constraint therefore yields an off-shell scalar multiplet in the case of $p = 1, 2$, while it yields an underconstrained multiplet in the case $p = 3, 5$. In the latter two cases it is therefore necessary to impose further constraints, as will be discussed in section 3. In ten-dimensional spacetime an unconstrained superfield Φ (of dimension zero) can be used to describe the $N = 1$, $d = 10$ off-shell supergravity multiplet. In the case of superembedding, however, the superfield Φ (which now has dimension -1) describes an induced geometry. In flat $D = 11$ superspace we will show that the induced geometry is also flat 7.

In the non-linear case, the basic structure of the worldvolume multiplets remain the same as in the linearised case, but there will be a non-trivial geometry induced on the brane. The consequences of the embedding constraint can be studied systematically (i.e. level by level in the engineering dimension) by analysing the pull-back onto the worldvolume of the target space torsion equation $DE^{\underline{A}} = T^{\underline{A}}$:

$$\nabla_A E_B{}^{\underline{C}} - (-1)^{AB}\nabla_B E_A{}^{\underline{C}} + T_{AB}{}^C E_C{}^{\underline{C}} \;=\; (-1)^{A(B+\underline{B})} E_B{}^{\underline{B}} E_A{}^{\underline{A}} T_{\underline{AB}}{}^{\underline{C}} \ , \tag{2.6}$$

where T^A is the induced worldvolume torsion. The covariant derivative D on the target space contains the Lorentz connection one-form $\Omega_{\underline{A}}{}^{\underline{B}}$ and the covariant derivative ∇ in the worldvolume involves the pull-back of this connection and an induced worldvolume connection one-form $\Omega_A{}^B$ taking values in the Lie algebra of the unbroken worldvolume symmetries. In the case of co-dimension embeddings $\Omega_a{}^b$ is $SO(p,1)$ valued.

In order to insert the embedding constraint (2.2) into (2.6) we first parametrise the embedding matrix $E_A{}^{\underline{A}}$ using the vector and spinor representations of the coset element in $SO(D-1,1)/SO(p,1)$ defined by the embedding. Using the freedom in inducing the worldvolume supervielbein E^A we can set

$$E_a{}^{\underline{a}} \;=\; u_a{}^{\underline{a}} \ , \tag{2.7}$$

$$E_\alpha{}^{\underline{\alpha}} \;=\; u_\alpha{}^{\underline{\alpha}} + h_\alpha{}^{\alpha'} u_{\alpha'}{}^{\underline{\alpha}} \ , \tag{2.8}$$

$$E_a{}^{\underline{\alpha}} \;=\; \Lambda_a{}^{\alpha'} u_{\alpha'}{}^{\underline{\alpha}} \ , \tag{2.9}$$

where the primed indices label the normal directions in the super tangent bundle of the target superspace. The matrices $u_\alpha{}^{\underline{\alpha}}$ and $u_{\alpha'}{}^{\underline{\alpha}}$ forms the coset representative in $Spin(D-1,1)/Spin(p,1)$, and the matrix $u_a{}^{\underline{a}}$ together with a vector $u_\perp{}^{\underline{a}}$ make up the corresponding representative in $SO(D-1,1)/SO(p,1)$. In the cases of $p = 3, 5$, where the target space group includes an internal symmetry factor, it will be understood that the meaning of the spinorial u-matrix is modified appropriately. The normal embedding matrix can be chosen to be:

$$\begin{array}{rclcrcl} E_\perp{}^{\underline{b}} &=& u_\perp{}^{\underline{b}} &,& E_\perp{}^{\underline{\beta}} &=& 0 \\ E_{\alpha'}{}^{\underline{a}} &=& 0 &,& E_{\alpha'}{}^{\underline{\beta}} &=& u_{\alpha'}{}^{\underline{\beta}} \end{array} \tag{2.10}$$

The inverses $(E^{-1})_{\underline{A}}{}^{A}$ and $(E^{-1})_{\underline{A}}{}^{A'}$ are given by

$$\begin{array}{rclcrcl} (E^{-1})_{\underline{a}}{}^{b} &=& u_{\underline{a}}{}^{b} &,& (E^{-1})_{\underline{a}}{}^{\beta} &=& 0 \\ (E^{-1})_{\underline{\alpha}}{}^{b} &=& 0 &,& (E^{-1})_{\underline{\alpha}}{}^{\beta} &=& u_{\underline{\alpha}}{}^{\beta} \end{array} . \tag{2.11}$$

and

$$\begin{array}{rclcrcl} (E^{-1})_{\underline{a}}{}^{\perp} &=& u_{\underline{a}}{}^{\perp} &,& (E^{-1})_{\underline{a}}{}^{\beta'} &=& -u_{\underline{a}}{}^{a}\Lambda_{a}{}^{\alpha'} \\ (E^{-1})_{\underline{\alpha}}{}^{\perp} &=& 0 &,& (E^{-1})_{\underline{\alpha}}{}^{\beta'} &=& u_{\underline{\alpha}}{}^{\beta'} - u_{\underline{\alpha}}{}^{\alpha}h_{\alpha}{}^{\beta'} \end{array} \tag{2.12}$$

Notice that the superfields u, $h_{\alpha}{}^{\alpha'}$ and $\Lambda_{a}{}^{\alpha'}$ can be expressed explicitly in terms of the fundamental embedding superfields $z^{\underline{M}}(z^{M})$ leading to a formulation of the embedding as non-linear supersymmetric sigma-model. The parametrisation in (2.9) turns out to be more convenient, however, in the study of the non-linearities of the superembedding.

There are two equivalent ways to induce the worldvolume $SO(p,1)$ connection $\Omega_{A}{}^{B}$. The first one is to fix some of the components of the worldvolume torsion $T_{AB}{}^{C}$ in a convenient form. The second one is to choose $\Omega_{A}{}^{B}$ such that the $SO(D-1,1)$ valued worldvolume covariant derivatives of the Lorentz harmonics defined by

$$X_{A} \equiv (\nabla_{A}u)u^{-1} \tag{2.13}$$

(the contraction is over the underlined index) satisfy

$$\begin{array}{rclcl} X_{A,B}{}^{C} &=& (\nabla_{A}u_{B}{}^{\underline{C}})(u^{-1})_{\underline{C}}{}^{C} &=& 0 , \\ X_{A,B'}{}^{C'} &=& (\nabla_{A}u_{B'}{}^{\underline{C}})(u^{-1})_{\underline{C}}{}^{C'} &=& 0 . \end{array} \tag{2.14}$$

Using this method, which turns out to be more efficient, the worldvolume torsion $T_{AB}{}^{C}$ can be computed in terms of the local composite connection $X_{A,B'}{}^{C}$, the embedding matrix components $h_{\alpha}{}^{\beta'}$ and $\Lambda_{a}{}^{\beta'}$ and the pull-back of the target space torsion. Notice that once the torsion equation (2.6) and the target space torsion Bianchi identity $DT^{\underline{A}} = E^{\underline{B}}R_{\underline{B}}{}^{\underline{A}}$ have been solved, the worldvolume torsion Bianchi identity $\nabla T^{A} = E^{B}R_{B}{}^{A}$ is identically satisfied.

To proceed further, we consider target space superspaces with dimension 0 torsion components

$$T_{\underline{\alpha}\,\underline{\beta}}{}^{\underline{a}} = -i(\Gamma^{\underline{a}})_{\underline{\alpha}\,\underline{\beta}} . \tag{2.15}$$

By systematically analysing the effect of the embedding constraint (2.2) in the torsion equation (2.6), one can then determine all of the components of the induced worldvolume geometry as well as the covariant form of the non-linear worldvolume scalar multiplet. For our main purposes, however, we shall only need the dimension zero results. The results of the higher dimensional components are collected in Appendix B.

The dimension zero component of (2.6), given the embedding condition (2.2), is

$$E_{\alpha}{}^{\underline{\alpha}}E_{\beta}{}^{\underline{\beta}}T_{\underline{\alpha}\,\underline{\beta}}{}^{\underline{c}} = T_{\alpha\beta}{}^{c}E_{c}{}^{\underline{c}} . \tag{2.16}$$

Right-multiplying this equation with the tangential components $(E^{-1})_{\underline{c}}{}^{c}$ of the inverse embedding matrix we find

$$T_{\alpha\beta}{}^{c} \;=\; -i\left((\Gamma^{c})_{\alpha\beta} + h_{\alpha}{}^{\alpha'} h_{\beta}{}^{\beta'} (\Gamma^{c})_{\alpha'\beta'}\right) \;, \tag{2.17}$$

while the normal component yields an algebraic condition on the following component of the spinor-spinor part of the embedding matrix:

$$h_{(\alpha}{}^{\alpha'} (\Gamma^{\perp})_{\beta)\alpha'} \;=\; 0 \;. \tag{2.18}$$

It is convenient to define

$$h_{\alpha\beta} \;\equiv\; h_{\alpha}{}^{\beta'} (\Gamma^{\perp})_{\beta'\beta} \;, \tag{2.19}$$

such that the above two equations can be written as

$$T_{\alpha\beta}{}^{c} \;=\; -i\left((\Gamma^{c})_{\alpha\beta} - h_{\alpha\gamma}(\Gamma^{c})^{\gamma\delta} h_{\delta\beta}\right) \tag{2.20}$$

$$h_{(\alpha\beta)} \;=\; 0 \;. \tag{2.21}$$

3 The $\mathcal{F}$-Constraints and the Action Formula

3.1 Closed Forms and Cartan Integrable Systems

The required new ingredient for $p = 3, 5, 9$ is a constrained, *globally* defined generalised super three-form field strength $\mathcal{F}_3$ (or, alternatively, $\mathcal{F}_1$ in the case $p = 3$). Forms of this type can be introduced in a natural geometrical fashion by considering open superbranes ending on other superbranes as discussed in [9, 10]. As will be explained in Section 3.2, the existence of the $\mathcal{F}_3$-forms follows from the basic requirements of configurations where open membranes (or, for $p = 3$, particles) end on the codimension one branes.

We shall begin, however, by using an alternative method based on the construction of Wess-Zumino forms as the pull-backs of target space Cartan Integrable Systems, i.e. a collection of forms obeying generalised Bianchi identities. For the cases we are considering these systems are:

$$\begin{aligned}
D = 7, 11 \;\; &: \;\; dH_7 = \tfrac{1}{2} H_4 H_4 \;, \quad dH_4 = 0 \;, \\
D = 5 \;\; &: \;\; dH_5^r = H_4^r H_2 \;, \quad dH_4^r = 0 \;, \quad dH_2 = 0 \;, \quad r = 1, 2, 3 \;, \\
D = 4 \;\; &: \;\; dH_4 = 0 \;, \\
D = 3 \;\; &: \;\; dH_3 = 0 \;,
\end{aligned} \tag{3.1}$$

where we have suppressed the wedge product notation. These Bianchi identities arise in the superspace formulation of the relevant target space supergravities (possibly coupled to matter). They take the same form in the limit of flat target superspace as well, although the details of the solutions to them, of course, simplifies in that limit. The triplet of closed five-forms and four-forms for $D=5$ can be viewed as arising from the triplet of closed five-forms in $N=(1,0), D=6$ superspace. These correspond to the triplet of tensorial charges which are allowed in the $N=(1,0), D=6$ super-Poincaré algebra.

The equations (3.1) imply that locally we can write

$$\begin{array}{lllllll} D=7,11 & : & H_7 & = & dC_6+\frac{1}{2}C_3H_4\ , & H_4 & = & dC_3\ , \\ D=5 & : & H_5^r & = & dC_4^r(t)+(1-t)C_1H_4^r+tC_3^rH_2\ , & H_4^r & = & dC_3^r\ , \quad H_2=dC_1\ , \\ D=4 & : & H_4 & = & dC_3\ , \\ D=3 & : & H_3 & = & dC_2\ , \end{array} \tag{3.2}$$

where t is an arbitrary constant and

$$C_4^r(t) \;=\; C_4^r+tC_1C_3^r\ . \tag{3.3}$$

The idea now is to construct the globally well-defined Wess-Zumino form W_{p+2} in the worldvolume, in terms of the pull-backs of these target space superforms. In addition to being globally defined, this form must be closed

$$dW_{p+2} \;=\; 0\ , \tag{3.4}$$

and this implies that locally we can write

$$W_{p+2} \;=\; dZ_{p+1}\ . \tag{3.5}$$

As we shall see in the next section, the form Z_{p+1} gives the Wess-Zumino term in the Green-Schwarz formalism when restricted to the bosonic part of the worldvolume.

For $p=1,2$, the Wess-Zumino forms can be readily constructed as the pull-backs of H_3 and H_4, respectively:

$$\begin{array}{lllll} p=1 & : & W_3 & = & \underline{H}_3\ , \\ p=2 & : & W_4 & = & \underline{H}_4\ , \end{array} \tag{3.6}$$

where the underlining of the target superforms denote their pull-backs to the worldvolume. However, for $p=3,5,9$ we need to introduce a set of globally well-defined forms satisfying the following Bianchi identities

$$d\mathcal{F}_3 \;=\; \underline{H}_4\ , \qquad p=3,5,9\ , \tag{3.7}$$

$$d\mathcal{F}_1 \;=\; \underline{H}_2\ , \qquad p=3\ , \tag{3.8}$$

which implies that locally

$$\mathcal{F}_3 = d\mathcal{A}_2 + \underline{C}_3, \tag{3.9}$$

$$\mathcal{F}_1 = d\mathcal{A}_0 + \underline{C}_1 \ , \tag{3.10}$$

where $\mathcal{A}_0$ and $\mathcal{A}_2$ are worldvolume superforms. Note that for $p = 3$ we take a linear combination of the triplet of closed four-forms H_4^r, i.e.

$$\underline{H}_4 = Q^r \underline{H}_4^r \ ; \qquad \underline{C}_3 = Q^r \underline{C}_3^r \tag{3.11}$$

where $\vec{Q}$ is a three-vector of real constants. These constants are related to the tensorial charge of the $N = 1$, $D = 5$ superalgebra carried by the 3-brane. Equipped with $\mathcal{F}_3$ and $\mathcal{F}_1$, we can construct the appropriate Wess-Zumino forms as follows:

$$\begin{aligned}
p = 3 \ &: \quad W_5 = \underline{H}_5 - (1-t)\underline{H}_2 \mathcal{F}_3 - t\underline{H}_4 \mathcal{F}_1 \ , \\
p = 5 \ &: \quad W_7 = \underline{H}_7 - \tfrac{1}{2}\underline{H}_4 \mathcal{F}_3 \ , \\
p = 9 \ &: \quad W_{11} = \underline{H}_4(- \underline{H}_7 + \tfrac{1}{2}\underline{H}_4 \mathcal{F}_3) \ .
\end{aligned} \tag{3.12}$$

The forms Z_{p+1} defined by $W_{p+2} = dZ_{p+1}$ can be chosen to be

$$\begin{aligned}
p = 1 \ &: \quad Z_2 = \underline{C}_2 \ , \\
p = 2 \ &: \quad Z_3 = \underline{C}_3 \ , \\
p = 3 \ &: \quad Z_4 = \underline{C}_4(t) + (1-t)\underline{C}_1\mathcal{F}_3 + t\underline{C}_3\mathcal{F}_1 \\
p = 5 \ &: \quad Z_6 = \underline{C}_6 + \tfrac{1}{2}\underline{C}_3\mathcal{F}_3 \ , \\
p = 9 \ &: \quad Z_{10} = \underline{C}_3 \left(\underline{H}_7 - \tfrac{1}{2}\mathcal{F}_3\underline{H}_4\right) \ .
\end{aligned} \tag{3.13}$$

Note that for $p = 3$, $\underline{C}_4$ and $\underline{H}_5$ are again given as linear combinations of $\underline{C}_4^r$ and $\underline{H}_5^r$,

$$\underline{H}_5 = Q^r \underline{H}_5^r; \qquad \underline{C}_4 = Q^r \underline{C}_4^r \ . \tag{3.14}$$

where Q^r are the same constants as the ones used in (3.11).

The generalised field strengths obey the following $\mathcal{F}$-constraints [9, 10]:

$$\mathcal{F}_{\alpha AB} = 0 \ , \tag{3.15}$$

$$\mathcal{F}_{\alpha} = 0 \ . \tag{3.16}$$

The derivation of these constraints in the context of open membranes ending on the codimension one branes is briefly explained at the end of next section. Equations (3.15) and (3.16) are consistent with the following constraints on the dimension zero components of the target space superforms

$$D=3,4,7,11 \quad : \quad H_{\underline{a_1\cdots a_k\alpha\beta}} \;=\; -i(\Gamma_{\underline{a_1\cdots a_k}})_{\underline{\alpha\beta}}\ , \tag{3.17}$$

where the relevant values of k can be seen from (3.2), and the spinor types are given in Table 1. Note that for $D=7$, $\underline{\alpha}\to\underline{\alpha}i$ and $(\Gamma_{\underline{a}\cdots})_{\underline{\alpha\beta}}\to(\Gamma_{\underline{a}\cdots})_{\underline{\alpha\beta}}\epsilon_{ij}$. In $D=5$ the spinors are symplectic Majorana and the dimension zero components of H^r_5, H^r_4, H_2 are taken to be

$$D=5\ :\begin{cases} H^r_{\underline{abc\alpha\beta}} &=\; (\Gamma_{\underline{abc}}\,\sigma^r)_{\underline{\alpha\beta}} \\ H^r_{\underline{ab\alpha\beta}} &=\; -i(\Gamma_{\underline{ab}}\,\sigma^r)_{\underline{\alpha\beta}} \\ H_{\underline{\alpha\beta}} &=\; -C_{5\underline{\alpha\beta}} \end{cases} \tag{3.18}$$

where $(\sigma^r)_{ij}$ are the symmetric Pauli matrices. The constraints can be rewritten in a manifestly six-dimensionally covariant form: H_2 is identified with the six-dimensional torsion component $-T^6$ (satisfying (2.15)) and H^r_5 and H^r_4 are combined in a triplet of six-dimensional 5-forms satisfying

$$D=6 \quad : \qquad dH^r \;=\; 0\ , \qquad H^r_{\underline{abc\alpha\beta}} \;=\; (\Gamma_{\underline{abc}}\,\sigma^r)_{\underline{\alpha\beta}}\ . \tag{3.19}$$

The above expressions for the dimension zero components of the H-forms are valid in flat superspace where all other components vanish. In curved superspace, on the other hand, these expressions may require some modifications depending on the supergravity theory under consideration.

3.2 The Action Formula

Since the Wess-Zumino form W_{p+2} is a closed $(p+2)$-form on a manifold which has bosonic dimension $(p+1)$ it follows that it is exact. This is so because the de Rham cohomology of a supermanifold coincides with the de Rham cohomology of its body. Therefore we can always (i.e. for *any* embedding) write

$$W_{p+2} \;=\; dK_{p+1} \tag{3.20}$$

for some *globally* defined $(p+1)$-form K on M. Furthermore, since none of the target space fields or the worldsurface fields has negative dimension, at least for the models under discussion here, it follows that the only non-vanishing component of K is the purely bosonic one, i.e.

$$K_{\alpha A_1\cdots A_p} \;=\; 0\ . \tag{3.21}$$

We remind that once $K_{a_1\cdots a_{p+1}}$ has been obtained from the dimension zero component of (3.20) (with indices $\alpha\beta a_1\cdots a_p$) the remaining dimension half and one components of (3.20) are identically satisfied.

We now define the Green-Schwarz Lagrangian form L_{p+1} to be [7]

$$L_{p+1} \;=\; K_{p+1} - Z_{p+1}, \tag{3.22}$$

with Z_{p+1} defined in (3.5). In view of (3.5) and (3.20), we have

$$dL_{p+1} = 0 \ . \tag{3.23}$$

Under a worldsurface superdiffeomorphism generated by the vector field v one has

$$\delta L_{p+1} \;=\; d i_v L_{p+1} \ . \tag{3.24}$$

Therefore, the action integral

$$S \;=\; \int_{M_0} L^0_{p+1} \ , \tag{3.25}$$

where M_0 is the body of M and where

$$L^0_{p+1} \;=\; dx^{m_{p+1}} \wedge dx^{m_p} \wedge \ldots dx^{m_1} L_{m_1 \ldots m_{p+1}} \ , \tag{3.26}$$

will be invariant under κ-symmetry transformations and diffeomorphisms of M_0, since these transformations are identified with the leading components of v. The vertical bars, that indicate evaluation of a (worldvolume) superfield at $\theta = 0$, will be dropped in the rest of the paper.

In the case of $p = 1, 2, 3$ the worldvolume multiplets are off-shell and which means that the action (3.25) can actually be generalised to a full superspace actions [3].

The super-de-Rham cohomology theorem mentioned above also explains the role of open membranes in deriving the $\mathcal{F}_3$-constraint. We refer the reader to [9, 10] for a rigorous proof. The basic idea, however, is that the cohomology theorem implies that there exists a globally defined three-form K_3 on the membrane obeying $dK_3 = \underline{H}_4$. We then define the three-form $\mathcal{F}_3$ on the codimension one brane by

$$f_1^{p\,\star} \mathcal{F}_3 \;=\; f_1^{2\,\star} K_3 \ , \tag{3.27}$$

where f_1^p and f_1^2 are the embeddings of the membrane boundary in the codimension one brane and the membrane, respectively. By construction $\mathcal{F}_3$ is globally defined and obeys (3.7). If we now assume that the membrane obeys the embedding constraint (2.2) then it can be shown that [9, 10] all involved embeddings obey (2.2). Moreover, it also follows (on dimensional grounds) that K_3 satisfies $i_\xi K_3 = 0$ for any fermionic vector ξ. The $\mathcal{F}$ constraint (3.15) now follows by taking the inner derivative of (3.27) and noting that the embedding condition on f_1^p implies that the push-forward $f^p_{1\star}\xi$ is a fermionic vector that can be varied independently in the fermionic tangent space of the codimension one brane.

4 String in $D = 3$ and membrane in $D = 4$

Let us begin our analysis of codimension one branes by reviewing the string in three dimensions and the membrane in four dimensions. Since results for $p = 1, 2$ are already available in the literature [7] our presentation here will be brief.

Let us consider first the string in $D = 3$. A 2 component $D = 3$ Majorana spinor ψ splits into two Majorana-Weyl spinors ψ_+ and ψ_- on the superstring worldsheet. Let the positive chirality label the tangential direction and the negative chirality label the normal direction. Moreover, let us take the $D = 3$ target space to be flat for simplicity.

The tangential equations (8.19-8.24) given in Appendix B determine the induced torsion component in terms of various quantities. The normal equations (8.25-8.30), also given in Appendix B, yield the following results

$$\begin{aligned} h_{++} &= 0, \\ X_{+,a} &= i\Lambda_{a+}\ , \qquad \nabla_+\Lambda_{a+} = \delta_{+\!\!\!+}^b X_{a,b}\ , \\ X_{[a,b]} &= 0\ , \qquad \nabla_{[a}\Lambda_{b]} = 0\ . \end{aligned} \tag{4.1}$$

These equations describe an $N = (1, 0)$ off-shell scalar multiplet on the string worldsheet. To see this more explicitly, we note that the full content of (4.1) at the linearised level, which can be deduced by using (2.3) and (2.4), is given by

$$\begin{aligned} D_+\Phi &= i\psi\ , \qquad D_+\psi = \delta_{+\!\!\!+}^a\partial_a\Phi\ , \\ X_{a,b} &= \partial_a\partial_b\Phi\ , \qquad \Lambda_a = \partial_a\psi\ , \end{aligned} \tag{4.2}$$

where $\psi := \Theta'^{-}$.

The Green-Schwarz action can be obtained by using (3.22), (3.25) and (3.26). In flat target space, the only non-vanishing component of H_3 is given in (3.17). Next, recall the definitions

$$W_3 = \underline{H}_3 = \begin{cases} dK_2 & \text{globally} \\ d\underline{C}_2 & \text{locally} \end{cases} \tag{4.3}$$

From (3.20) and (3.21) one finds that the only non-vanishing component of K_2 is

$$K_{ab} = -\epsilon_{ab}\ . \tag{4.4}$$

Using the above results in (3.22) yields the action

$$S_2 = \int_{M_o}\frac{1}{2}E^aE^b(\epsilon_{ba} + \underline{C}_{ba}) = \frac{1}{2}\int_{M_o}\frac{1}{2}d^2x(\sqrt{-\det g} + \epsilon^{mn}\underline{C}_{mn})\ , \tag{4.5}$$

where M_o is the body of the worldvolume superspace and where the induced metric g_{mn} is given in terms of the supersymmetric line element $\mathcal{E}_m{}^{\underline{a}} = e_m{}^a E_a{}^{\underline{a}}|$ by

$$g_{mn} = e_m{}^a e_n{}^b\eta_{ab} = \mathcal{E}_m{}^{\underline{a}}\mathcal{E}_n{}^{\underline{b}}\eta_{\underline{ab}}\ . \tag{4.6}$$

We now turn to the membrane in $D = 4$. Again, we consider flat target superspace for simplicity. In this case all the Y- and Z-tensors vanish and from (8.19), (8.25) and (8.26) and we readily find

$$
\begin{aligned}
h_{\alpha\beta} &= hC_{\alpha\beta} \ , \\
\nabla_\alpha h &= -i\frac{1+h^2}{2}(\gamma^a)_\alpha{}^\delta \Lambda_{a\delta} \ , \\
T_{\alpha\beta}{}^a &= -i(1+h^2)(\gamma^a)_{\alpha\beta} \ .
\end{aligned}
\tag{4.7}
$$

This system describes an $N = 1$ off-shell scalar multiplet in $d = 3$. The field h plays the role of an auxiliary field. An action is needed to set $h = 0$ which leads to the Dirac equation $(\gamma^a \Lambda_a)_\alpha = 0$ and the remaining equations of motion [7]. Indeed, from (3.6) and (3.17) one finds that the only non-vanishing component of the superform K_3 defined by $W_4 = dK_3$ is [7]

$$
K_{abc} = \epsilon_{abc} K \ , \qquad K = \frac{1-h^2}{1+h^2} \ . \tag{4.8}
$$

and

$$
\nabla_\alpha K = \frac{2ih}{1+h^2} (\gamma^a \Lambda_a)_\alpha \ . \tag{4.9}
$$

As was explained in Section 3.2, this result is consistent with (4.8) without any condition on Λ. A Lagrangian can be constructed by using the formula (3.22), which yields the result [7]

$$
\mathcal{L} = \sqrt{-\det g} \left(\frac{1-h^2}{1+h^2} \right) - \frac{1}{6} \epsilon^{mnp} \underline{C}_{mnp} \ , \tag{4.10}
$$

where g is again the standard GS induced metric given in (4.6). The only difference from the usual GS Lagrangian is the presence of the auxiliary field h. However, the equation of motion for this field is purely algebraic and can be used to set $h = 0$. We thus recover the standard GS action.

5 3-brane in $D = 5$

As we have remarked previously there are two formulations of the 3-brane in $D = 5$: the first has a worldvolume linear multiplet and correspondingly a 3-form field strength $\mathcal{F}_3$, while the second has a worldvolume scalar multiplet with a 1-form field strength $\mathcal{F}_1$. In the five-dimensional context the first is perhaps more natural since it has only one physical scalar corresponding to the transverse direction. The scalar multiplet, on the other hand, which has two physical scalars is more naturally formulated in a six-dimensional context. The 3-brane in $D = 6$ was discovered some time ago [17], and here we shall begin by presenting its covariant formulation with auxiliary fields, and then demonstrate that how its vertical reduction gives the scalar formulation of the 3-brane in $D = 5$. We shall then obtain the linear multiplet formulation in the GS formalism by dualising the leading component $\mathcal{F}_a|$ of $\mathcal{F}_1$. Finally we shall give the linear multiplet formulation directly as a superembedding.

5.1 A Useful Digression: The 3-brane in $D = 6$

The odd-odd part of the embedding matrix $E_\alpha{}^{\underline{a}}$ requires slightly more careful treatment for the three-brane because of the presence of an internal symmetry group in the target space, and because we want to split the target spinor index (which is symplectic Majorana-Weyl) into two four-dimensional Majorana spinor indices. As discussed in section 2, the spinor-spinor part of the embedding matrix can be written

$$E_\alpha{}^{\underline{\alpha}} = \frac{1}{\sqrt{2}}\left[(u\otimes v_1)_\alpha{}^{\underline{\alpha}} + (\gamma^5 u\otimes v_2)_\alpha{}^{\underline{\alpha}}\right] + \frac{1}{\sqrt{2}} h_\alpha{}^\beta \left[(\gamma^5 u\otimes v_1)_\beta{}^{\underline{\alpha}} - (u\otimes v_2)_\beta{}^{\underline{\alpha}}\right] \qquad (5.1)$$

The notation here is as follows: u is a 4×4 matrix belonging to the six-dimensional spin group $Spin(1,5)$, (v_1, v_2) are both two-component objects which together make up an element of $Sp(1)$ (so when $\underline{\alpha}\to\underline{\alpha}i$ one replaces $v_1\to v_1{}^i$, $v_2\to v_2{}^i$, where $v_I{}^i v_J{}^j\epsilon_{ij} = \epsilon_{IJ}$) and the underlined spinor index $\underline{\alpha}$, running from 1 to 8, is a combined six-dimensional Majorana-Weyl spinor $\otimes$ internal $Sp(1)$ doublet index.

The linearised analysis of the embedding constraint $E_\alpha{}^{\underline{a}} = 0$ along the lines explained in Section 2, in this case gives the constraint

$$D_\alpha\Phi^{a'} = i\left(\Gamma^{a'}\right)_{\alpha\beta'}\Theta^{\beta'}\ , \qquad a' = 1,2\ , \qquad (5.2)$$

where $\Phi^{a'} = x^{a'} + \frac{i}{2}\theta^\alpha(\Gamma^{a'})_{\alpha\beta'}\Theta^{\beta'}$ is the Goldstone superfield which describes an off-shell $N=1$, $d=4$ scalar supermultiplet.

In the non-linear case, the dimension zero worldvolume torsion is given by the standard form (2.20). In this case, $h_{\alpha\beta}$ satisfies (2.21), which follows from the projection of (2.16) on the fifth direction, and

$$\left(h\gamma^5\right)_{(\alpha\beta)} = 0\ , \qquad (5.3)$$

which follows from the projection on the sixth direction (notice that the matrix $v_I{}^i$ drops from these calculations since the target space torsion contains the $Sp(1)$ invariant ϵ_{ij}). The solution to (2.21) and (5.3) is

$$h_{\alpha\beta} = AC_{\alpha\beta} + i\,B(\gamma^5)_{\alpha\beta} = \tfrac{1}{2}(1+\gamma^5)z + \tfrac{1}{2}(1-\gamma^5)\bar{z}\ , \qquad z = A + iB\ , \qquad (5.4)$$

where A and B are real worldvolume superfields. Thus, in addition to the Goldstone superfield associated with the breaking of the super-translation group, we have so far introduced the three scalar superfields $v_I{}^j$ parameterizing $Sp(1)$ and the two additional superfields A and B.

Naively, one would expect that the leading components of A and B are related to the auxiliary fields of the Goldstone superfield, and that the leading components of $v_I{}^i$ are the Goldstone superfields associated with the complete breaking of the R-symmetry group $Sp(1)$. However, as we shall see below, there exists an extra local $Sp(1)$ symmetry in the spinor-spinor part of the embedding matrix, which enables us to either gauge away $v_I{}^i$, or equivalently, to gauge away the fields A, B and a scalar in $v_I{}^i$ associated with a $U(1)$ subgroup of $Sp(1)$. In both cases, the remaining two superfields will indeed be related to the auxiliary fields of the Goldstone superfield and as we shall see they parametrise the two-sphere $Sp(1)/U(1)$.

To exhibit the extra symmetry we first use a combined super-Weyl and transverse $SO(2)$ transformation λ (generated by 1 and $i\gamma^5$) of the induced worldvolume supersechsbein to replace the spinor-spinor component of the embedding matrix by the following equivalent parametrisation

$$E_\alpha{}^{\underline{\alpha}} = \tfrac{1}{\sqrt{2}} h^+{}_\alpha{}^\beta \left[(u \otimes v_1)_\beta{}^{\underline{\alpha}} + (\gamma^5 u \otimes v_2)_\beta{}^{\underline{\alpha}} \right] + \tfrac{1}{\sqrt{2}} h^-{}_\alpha{}^\beta \left[(\gamma^5 u \otimes v_1)_\beta{}^{\underline{\alpha}} - (u \otimes v_2)_\beta{}^{\underline{\alpha}} \right] , \quad (5.5)$$

where $h^\pm = (h^1 \pm h^2)/\sqrt{2}$ and

$$h^i = A^i + i\gamma^5 B^i = \tfrac{1}{2}(1+\gamma^5) z^i + \tfrac{1}{2}(1-\gamma^5)\bar{z}^i , \qquad z^i = A^i + iB^i . \quad (5.6)$$

Defining the worldvolume chiral projections $E_\pm = \frac{1}{2}(1 \pm \gamma^5)E$, equation (5.5) can be written in the following manifestly $Sp(1)$ invariant form

$$E_+ = z^i v_i \otimes u_+ , \qquad E_- = \bar{z}^i v_i \otimes u_- . \quad (5.7)$$

The freedom in inducing the worldvolume supersechsbein, i.e. $E \sim \lambda E$, implies the following equivalence relation between embeddings

$$(z^1, z^2) \sim \lambda\, (z^1, z^2) , \qquad \lambda \in \mathbf{C} , \quad \lambda \neq 0 . \quad (5.8)$$

Notice that λ is a local superfield. Hence inequivalent embedding matrices are parametrised by maps to CP^1. The parametrisation in (5.1) and (5.4) corresponds to the representative

$$(z, 1) = (z^-/z^+, 1) , \quad (5.9)$$

where

$$\begin{pmatrix} z^- \\ z^+ \end{pmatrix} = \Omega \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} , \qquad \Omega = (1 - i\sigma^2)/\sqrt{2} . \quad (5.10)$$

In this parametrisation, the local $Sp(1)$ transformation $v \to g\,v$ acts on z by the Möbius transformation $z \to (az+b)/(cz+d)$ such that

$$(cz+d)E_+\left(g\,v\,, \frac{az+b}{cz+d}\right) = E_+(v\,, z) , \quad (5.11)$$

where $a = \bar{d}$ and $b = -\bar{c}$ make up the $Sp(1)$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Omega (g^{-1})^T \Omega^{-1} . \quad (5.12)$$

Thus we can fix $v = 1$, in which case CP^1 is parametrised by z, or $z = 0$, in which case $CP^1 = Sp(1)/U(1)$ (where $U(1)$ is the stability subgroup that leaves $z = 0$ invariant) is parametrised by the coset representative

$$v \;=\; \Omega(L^{-1})^T\Omega^{-1} \;, \tag{5.13}$$

where

$$L \;=\; \frac{1}{\sqrt{1+|\phi|^2}}\begin{pmatrix} 1 & \phi \\ -\phi^* & 1 \end{pmatrix} \;. \tag{5.14}$$

The right action of $U(1)$ on v is generated by σ^1 and its right action on L is generated by $\Omega\sigma^1\Omega^{-1} = -\sigma^3$. The two gauge choices are related by $\phi = -z$.

Let us first consider the $v = 1$ gauge. In this case, the dimension zero worldvolume torsion is given by

$$T_{\alpha\beta}{}^c \;=\; -i(1+\bar{z}z)(\gamma^c)_{\alpha\beta} \;. \tag{5.15}$$

As mentioned in Section 3.1, there exists an $Sp(1)$ triplet of closed five-forms H_5^r in the $N = (1,0)$, $D = 6$ superspace. Its dimension zero components are given by (3.19). To construct a κ-symmetric action for the off-shell scalar (chiral) supermultiplet along the lines described in Section 3.2, we take the Wess-Zumino form in (3.20) to be $W_5 = Q^r \underline{H}_5^r$, where Q^r is the tensorial charge of the $N = (1,0)$, $D = 6$ superalgebra carried by the 3-brane. The resulting kinetic term is given by

$$K_{abcd} \;=\; \epsilon_{abcd}K \;, \qquad K \;=\; \frac{Q_1(1-\bar{z}z) + q\bar{z} + \bar{q}z}{1+\bar{z}z} \;, \tag{5.16}$$

where we have defined

$$q \;\equiv\; q_1 + iq_2 \;=\; Q_3 + iQ_2 \;. \tag{5.17}$$

Let us now consider the $z = 0$ gauge instead. The dimension zero worldvolume torsion in this case is given by

$$T_{\alpha\beta}{}^c \;=\; -i(\gamma^c)_{\alpha\beta} \;, \tag{5.18}$$

Note that the v's drop out of this expression by virtue of the fact that ϵ_{ij} is $Sp(1)$ invariant. The kinetic term is now found to be

$$K \;=\; \tfrac{1}{2}Q^r \mathrm{tr}\,(\sigma^1 v\,\sigma^r\,v^{-1}) \;=\; \tfrac{1}{2}Q^r \mathrm{tr}\,(L^{-1}\sigma^3 L\Omega^{-1}\sigma^r\Omega) \;. \tag{5.19}$$

Introducing the $Sp(1)$ generators $[T_i, T_j] = \epsilon_{ijk}T_k$, we define so called C and S functions by

$$L^{-1}T_3L \;=\; CT_3 + \bar{S}T_+ + ST_- \;, \tag{5.20}$$

where $T_\pm = \frac{1}{2}(T_1 \pm iT_2)$. In terms of these functions

$$K \;=\; Q_1C - q\bar{S} - \bar{q}S \;. \tag{5.21}$$

The C and S functions are easily found from (5.20) to be

$$C = \frac{1-|\phi|^2}{1+|\phi|^2} , \qquad S = \frac{2\phi}{1+\bar{\phi}\phi} , \tag{5.22}$$

which together with $\phi = -z$ shows the equivalence between (5.21) and the kinetic term (5.16) derived in the $v = 1$ gauge.

The complete action is therefore given by

$$\mathcal{L} = \frac{Q_1(1-\bar{z}z) + q\bar{z} + \bar{q}z}{1+\bar{z}z}\sqrt{-\det g} - \tfrac{1}{4!}\epsilon^{mnpq}\vec{Q}\cdot\underline{\vec{C}}_{mnpq} , \tag{5.23}$$

where g_{mn} is the induced metric in (4.6). The Wess-Zumino term breaks the local $Sp(1)$ to a local $U(1)$ with generator $Q^r(\sigma^r)_{ij}$. When this $U(1)$ coincides with the local right action on the coset, i.e. when $[Q^r\sigma^r, \sigma^1] = 0$, the resulting single $U(1)$ is a local symmetry of the action. This condition implies $q = 0$, as can also be seen directly from (5.23) by inspection.

In order to give a global description of the CP^1 manifold we cover the upper hemisphere with coordinate z and the lower hemisphere with $1/z$. Therefore, we must check that the action is invariant on the overlap region, modulo possible transformations of other fields in the theory (including the target space fields). We find that the action (5.23) is invariant under the following combined transformations:

$$z \to \frac{1}{z} , \qquad (Q_1, Q_2, Q_3) \to (-Q_1, Q_2, -Q_3) , \qquad (C_4^1, C_4^2, C_4^3) \to (-C_4^1, C_4^2, -C_4^3) . \tag{5.24}$$

Consistency of the theory therefore requires the target space theory containing C_4^r to be self-dual under $(C_4^1, C_4^2, C_4^3) \to (-C_4^1, C_4^2, -C_4^3)$.

The action (5.23) remains invariant under simultaneous $Sp(1)$ rotations of the 3-brane charge $\vec{Q}$ and the spacetime 4-form potentials $\vec{C}_4$. In order to obtain a manifestly $Sp(1)$ covariant action we need to first eliminate the auxiliary fields. In order to eliminate the auxiliary fields through their field equations we observe the following useful properties of the C and S functions:

$$dC = \epsilon_{ij}V_iS_j , \qquad dS_i = \epsilon_{ij}V_jC , \tag{5.25}$$

where d is the exterior derivative on CP^1 and i, j its $SO(2)$ tangent space indices, V^i are the basis one-forms and $S = S_1 + iS_2$. The field equation then reads:

$$Q_1S_i - q_iz = 0 . \tag{5.26}$$

Substituting back into the action (5.23) we find the following manifestly $Sp(1)$ covariant form of the action:

$$\mathcal{L} = |\vec{Q}|\sqrt{-\det g} - \tfrac{1}{4!}\epsilon^{mnpq}\vec{Q}\cdot\underline{\vec{C}}_{mnpq} . \tag{5.27}$$

5.2 3-brane in $D = 5$ via vertical reduction from $D = 6$

The construction of the 3-brane in $D = 5$ can proceed either by directly solving the embedding and $\mathcal{F}_1$ constraints obtained by vertical reduction of the six-dimensional embedding constraint and directly constructing the GS action from the reduced Wess-Zumino term, or, equivalently, by vertical reduction of the six-dimensional embedding matrix and Green-Schwarz action.

We begin with the first approach in which we vertically reduce the six-dimensional embedding constraint. To this end, we first separate off E^6 and identify

$$\mathcal{F}_1 \equiv -\underline{E}^6 \, , \qquad H_2 \equiv -T^6 \, . \tag{5.28}$$

The pull-back of the six-dimensional torsion identity $dE^6 = T^6$ (since $\hat{\omega}_{\underline{\hat{a}}}{}^6 = 0$) then automatically gives the $\mathcal{F}$ Bianchi identity (3.9). Furthermore, the six-dimensional embedding condition $\hat{E}_\alpha{}^{\underline{\hat{a}}} = 0$ implies the five-dimensional embedding and $\mathcal{F}_1$-constraints:

$$E_\alpha{}^{\underline{a}} = 0 \, , \qquad \mathcal{F}_\alpha = 0 \, . \tag{5.29}$$

Making the definition

$$\hat{H}_5 \equiv H_5 + E^6 H_4 \, , \tag{5.30}$$

it follows that the six-dimensional Wess-Zumino term $\hat{W}_5 = \underline{\hat{H}}_5$ reduces to the W_5 given in (3.12) for $t = 1$. The five-dimensional constraints (5.29) together with the action formalism discussed in Section 3.2, provide a basis for a self-contained formulation of the scalar multiplet of the 3-brane in five dimensions.

Without loss of generality we can parametrise the spinor-spinor part of the embedding matrix as

$$E_\alpha{}^{\underline{\alpha}} = \frac{1}{\sqrt{2}} \left[(u \otimes v_1)_\alpha{}^{\underline{\alpha}} + (\gamma^5 u \otimes v_2)_\alpha{}^{\underline{\alpha}} \right] + \frac{1}{\sqrt{2}} h_\alpha{}^\beta \left[(\gamma^5 u \otimes v_1)_\beta{}^{\underline{\alpha}} - (u \otimes v_2)_\beta{}^{\underline{\alpha}} \right] \tag{5.31}$$

where $\underline{\alpha}$ is a five-dimensional symplectic Majorana index. In flat target space, the dimension zero torsion equations (2.20-2.21) yield

$$\begin{aligned} h_{\alpha\beta} &= AC_{\alpha\beta} + iB(\gamma^5)_{\alpha\beta} + ih_a(\gamma^a\gamma^5)_{\alpha\beta} \, , \qquad z = A + iB \\ T_{\alpha\beta}{}^a &= -im^a{}_b(\gamma^b)_{\alpha\beta} + 2[(A\gamma^5 + iB)\gamma^{ab}h_b]_{\alpha\beta} \, , \\ m_{ab} &= \eta_{ab}(1 + |z|^2 + h^2) - 2h_a h_b \, , \end{aligned} \tag{5.32}$$

where $h^2 := h^a h_a$. Just as in the six-dimensional case, the v's are not constrained by the torsion equations, and there exists an extra $Sp(1)$ symmetry in the spinor-spinor part of the embedding matrix enabling us to gauge away three of the five scalar superfields in $(v_I{}^i, A, B)$ leaving two superfields related to the two auxiliary fields of the scalar multiplet. For $h_{\alpha\beta}$ given by (5.32), the infinitesimal local $Sp(1)$ transformations take the form:

$$
\begin{aligned}
\delta E_{+} &= (iN_1 + (N_2 + iN3)z)\, E_{+} - i(N_2 - iN_3)\gamma^a h_a E_{-}\ , \\
\delta v_I{}^j &= iN^r (\sigma^r)_I{}^J v_J{}^j\ , \\
\delta z &= (N_2 - iN_3)(1+h^2) + 2iN_1 z + (N_2 + iN_3) z^2\ , \\
\delta h_a &= 2\mathrm{Re}\left[(N_2 + iN_3)z\right]\, h_a\ ,
\end{aligned}
\tag{5.33}
$$

where $E_{\pm} = \frac{1}{2}(1 \pm \gamma^5)E$ and N^r are three real superfields. As we shall see below, (5.33) is in agreement with the vertical reduction of the Möbius symmetry found in the six-dimensional case.

As discussed in Section 2, the embedding constraint leaves the worldvolume multiplet under-constrained and therefore we need to make use of the $\mathcal{F}_1$ constraint $\mathcal{F}_\alpha = 0$ subject to the appropriate Bianchi identity $d\mathcal{F}_1 = \underline{H}_2$ to obtain the off-shell scalar multiplet. At dimension zero this reads

$$
T_{\alpha\beta}{}^c \mathcal{F}_c \;=\; E_\alpha{}^{\underline{\alpha}} E_\beta{}^{\underline{\beta}} H_{\underline{\alpha\beta}}\ . \tag{5.34}
$$

Tracing this equation with γ^d (the γ^{de} trace is identically obeyed) yields

$$
m_a{}^b \mathcal{F}_b \;=\; -2h_a\ , \tag{5.35}
$$

with m_{ab} given in (5.32). Inverting we find

$$
\mathcal{F}_a \;=\; \frac{-2h_a}{1+|z|^2-h^2}\ . \tag{5.36}
$$

Analysing also the dimension half components of the $\mathcal{F}_1$ Bianchi identity[1] one finds that the worldvolume multiplet is an off-shell scalar multiplet whose components are two scalars and a spinor, given by the leading components of the transverse supercoordinates of the superembedding and the leading component of the 0-form potential, and two auxiliary fields given by the leading component of z.

We construct the action in the usual way. With $W_5^{(S)}$ given by (3.12) for $t=1$, the dimension zero component of $W_5^{(S)} = dK_4^{(S)}$ reads

$$
T_{\alpha\beta}{}^d K^{(S)} \epsilon_{dabc} \;=\; \vec{Q}\cdot \underline{\vec{H}}_{abc\alpha\beta} - 3\mathcal{F}_{[a}\vec{Q}\cdot \underline{\vec{H}}_{bc]\alpha\beta}\ , \tag{5.37}
$$

where we have set $K^{(S)}_{abcd} = \epsilon_{abcd}K^{(S)}$ and the background is specified in (3.18) and $\vec{Q}$ is the brane charge. Tracing with $(\gamma^e)^{\alpha\beta}$ we find

$$
K^{(S)} \;=\; \frac{Q_1(1-|z|^2+h^2) + q\bar{z} + \bar{q}z}{1+|z|^2-h^2}\ . \tag{5.38}
$$

[1] A convenient trick [8] is to replace this problem by the equivalent problem of finding the conditions on the torsion such that the only globally well-defined solution to $d\Omega_2 = 0$ is $\Omega_2 = 0$.

As mentioned before all the remaining components of $W = dK$ (including the higher dimensional ones) will then be identically satisfied. To eliminate h_a in favor of $\mathcal{F}_a$ in (5.38), we use the following redefinition of the auxiliary fields:

$$\hat{z} \;=\; \frac{z}{2|z|^2}\left(\sqrt{(1+h^2-|z|^2)^2+4|z|^2}-1-h^2+|z|^2\right) \;. \tag{5.39}$$

Actually, as we shall see in the next section, $\hat{z}$ is identical to the six-dimensional auxiliary field in the $v = 1$ gauge discussed earlier. Inserting (5.39) in (5.38) we obtain the kinetic term in Born-Infeld form with an auxiliary field dependent prefactor as follows:

$$K^{(S)} \;=\; \frac{Q_1(1-|\hat{z}|^2)+q\bar{\hat{z}}+\bar{q}\hat{z}}{1+|\hat{z}|^2}\sqrt{-\det(\eta_{ab}+\mathcal{F}_a\mathcal{F}_b)} \;, \tag{5.40}$$

The corresponding GS Lagrangian is

$$\begin{aligned}\mathcal{L}^{(S)} \;=&\; \frac{Q_1(1-|\hat{z}|^2)+q\bar{\hat{z}}+\bar{q}\hat{z}}{1+|\hat{z}|^2}\sqrt{-\det g}\sqrt{\det(\delta_a^b+\mathcal{F}_a\mathcal{F}^b)}\\ &-\frac{1}{4!}\epsilon^{mnpq}\vec{Q}\cdot(\underline{\vec{C}}_{mnpq}+4\underline{\vec{C}}_{mnp}\mathcal{F}_q) \;,\end{aligned} \tag{5.41}$$

where, as usual, g is the standard induced metric on the bosonic worldvolume and the 4-form potential is evaluated at $t = 1$ in (3.3).

We now turn to the second approach to obtain the 3-brane action in $D = 5$ by vertical reduction of the 3-brane action in six-dimensions. The key object object in this approach is the spinor-spinor part of the embedding matrix. To study its vertical reduction, we start by parametrising the element $\hat{u}_\alpha{}^{\hat{\underline{\alpha}}}$ of $Spin(5,1)/Spin(3,1)$ in terms of an element $u_\alpha{}^{\underline{\alpha}}$ in $Spin(4,1)/Spin(3,1)$ and a worldvolume superfield K_a corresponding to the coset generator Γ^{a6}:

$$\hat{u}^{\underline{\alpha}} \;=\; \frac{1}{\sqrt{1+K^2}}(1+i\gamma^a K_a)u^{\underline{\alpha}} \;, \tag{5.42}$$

where the six-dimensional symplectic Majorana-Weyl index on the left side has been identified with the five-dimensional symplectic Majorana index on the right side. Since the vertical reduction does not affect the $Sp(1)$ group elements, we have

$$\hat{v}_I{}^j \;=\; v_I{}^j \;. \tag{5.43}$$

We next write

$$\hat{E}_\alpha{}^{\underline{\alpha}} \;=\; M_\alpha{}^\beta E_\beta{}^{\underline{\alpha}} \;, \tag{5.44}$$

where $M_\alpha{}^\beta$ is an invertible matrix which can be chosen such that the five-dimensional $E_\alpha{}^{\underline{\alpha}}$ assumes the canonical form given in (5.31). Inserting the definitions made in (5.42) and the six-dimensional embedding matrix $\hat{E}_\alpha{}^{\underline{\alpha}}$ given in (5.1) (with $\hat{h}_{\alpha\beta}$ given by (5.4)) into (5.44) one finds that

$$
\begin{aligned}
M &= \frac{1}{\sqrt{1+K^2}}\left(1 - i(\hat{A} + i\gamma^5\hat{B})\gamma^a\gamma^5 K_a\right) \ , \\
h_{\alpha\beta} &= A + i\gamma^5 B + i\gamma^a\gamma^5 h_a \ ,
\end{aligned}
\tag{5.45}
$$

where

$$
\begin{aligned}
K_a &= \frac{1-|\hat{z}|^2K^2}{1+|\hat{z}|^2}\, h_a \ \equiv \ U h_a \ , \\
\hat{z} &= \frac{1-|\hat{z}|^2K^2}{1+K^2}\, z \ \equiv \ V z \ .
\end{aligned}
\tag{5.46}
$$

From these equations we obtain the solution:

$$
\begin{aligned}
U &= \frac{1}{2h^2}\left(\Delta \pm (1+|z|^2-h^2)\right) \ , \\
V &= \frac{1}{2|z|^2}\left(\Delta \pm (1+h^2-|z|^2)\right) \ , \\
\Delta &\equiv \sqrt{(1+|z|^2-h^2)^2+4h^2} \ = \ \sqrt{(1-|z|^2+h^2)^2+4|z|^2} \ .
\end{aligned}
\tag{5.47}
$$

It is also useful to note the characteristic equations:

$$
\begin{aligned}
1+h^2U^2 &= \Delta U \ , \qquad 1-h^2U^2 \ = \ (1+|z|^2-h^2)U \ , \\
1+|z|^2V^2 &= \Delta V \ , \qquad 1-|z|^2V^2 \ = \ (1+h^2-|z|^2)V \ .
\end{aligned}
\tag{5.48}
$$

The local $Sp(1)$ transformations inherited from the six-dimensional theory can now be shown to reproduce the infinitesimal version of these transformations given in (5.33).

To reduce the vector-vector part of the embedding matrix, we first write the vectorial counterpart of (5.42) as

$$
\hat{u}_{\hat{a}}{}^{\underline{\hat{a}}} \ = \ \Lambda_{\hat{a}}{}^{\hat{b}} u_{\hat{b}}{}^{\underline{\hat{a}}} \ , \tag{5.49}
$$

where the index $\hat{a} \to (a,5,6)$, and $\hat{u}$ belongs to $SO(5,1)/SO(3,1)$ and u to $SO(4,1)/SO(3,1)$ (embedded in $SO(5,1)$ such that $u_{\hat{a}}{}^6 = \delta^6_{\hat{a}}$). The basic spinor identity

$$
\hat{u}_\alpha{}^{\underline{\alpha}} \hat{u}_\beta{}^{\underline{\beta}} (\Gamma^{\underline{\hat{a}}} C_6)_{\underline{\alpha}\,\underline{\beta}} \ = \ (\Gamma^{\hat{b}} C_6)_{\alpha\beta} \hat{u}_{\hat{b}}{}^{\underline{\hat{a}}} \tag{5.50}
$$

and the Dirac matrices $\Gamma^{\hat{a}} C_6 \to (\gamma^a C_4, C_4, i\gamma^5)$ implies that

$$\Lambda_{\hat{a}\hat{b}} = \frac{1}{1+K^2}\begin{pmatrix} (1+K^2)\eta_{ab} - 2K_aK_b & 0 & 2K_b \\ 0 & 1+K^2 & 0 \\ -2K_a & 0 & 1-K^2 \end{pmatrix} . \tag{5.51}$$

Assuming that the vector-vector part of the embedding matrix is given by the canonical expression (2.7) (in both five and six dimensions) it follows that

$$\hat{E}_a{}^{\underline{a}} = \Lambda_a{}^b E_b{}^{\underline{a}} , \qquad \hat{E}_a{}^6 = \Lambda_a{}^6 . \tag{5.52}$$

Notice that Λ_{ab} is invertible but not orthogonal. Furthermore, the basic embedding constraint imply that $\underline{E}^{\underline{a}} = \hat{E}^a \hat{E}_a{}^{\underline{a}} = E^a E_a{}^{\underline{a}}$, where $\hat{E}^a$ and E^a are the induced co-vector frames on the worldvolume in the six- and five-dimensional formulations, respectively. Comparing with (5.52) we find that

$$E^a = \hat{E}^b \Lambda_b{}^a . \tag{5.53}$$

Defining the components of $\mathcal{F} = -\underline{E}^6$ with respect to the frame E^a by

$$\mathcal{F} \equiv E^a \mathcal{F}_a , \tag{5.54}$$

and using (5.52) and (5.53) we find that

$$\mathcal{F}_a = -\frac{2}{1-K^2} K_a . \tag{5.55}$$

From (5.46) and (5.48) it follows that this equation is equivalent to (5.35).

Thus, we conclude that the vertical reduction of the six-dimensional Green-Schwarz action (5.23) is equivalent to the five-dimensional action (5.41) found by direct construction. The appearance of the Born-Infeld factor in the first term is a straightforward consequence of the reduction of the induced metric (using the relation given in (4.6)):

$$\hat{g}_{mn} = \hat{E}_m{}^{\hat{\underline{a}}} \hat{E}_{n\hat{\underline{b}}} = \hat{E}_m{}^a \hat{E}_n{}^b \left(\hat{E}_a{}^{\underline{a}} \hat{E}_{b\underline{a}} + \hat{E}_a{}^6 \hat{E}_{b6} \right) = g_{mn} + \mathcal{F}_m \mathcal{F}_n , \tag{5.56}$$

and the reduced Wess-Zumino term corresponds to the choice $t = 1$ in (3.13).

5.3 Dualisation of the 3-brane in $D = 5$ and the linear multiplet

We shall now dualise the scalar potential $\mathcal{A}_0$ and recover the linear multiplet version of the theory. To this end, we eliminate the auxiliary field z using its equation of motion and obtain the simpler Lagrangian

$$\mathcal{L}^{(S)} = |\vec{Q}|\sqrt{-\det g}\sqrt{\det(\delta_a^b + \mathcal{F}_a\mathcal{F}^b)} - \frac{1}{4!}\epsilon^{mnpq}\vec{Q} \cdot (\underline{\vec{C}}(1)_{mnpq} + 4\underline{\vec{C}}_{mnp}\mathcal{F}_q) , \tag{5.57}$$

where we recall the definition of $C_4^r(1)$ given in (3.3). To dualise $\mathcal{A}_0$, we first relax the $\mathcal{F}_1$ Bianchi identity and add a Lagrange multiplier term as follows

$$\mathcal{L}^{(S)'} \;=\; \mathcal{L}^{(S)} - \mathcal{A}_2(d\mathcal{F}_1 - \underline{H}_2)\ . \tag{5.58}$$

Notice that under the background gauge transformation

$$\delta C_3 \;=\; d\Lambda_2\ , \qquad \delta C_4 \;=\; -H_2\Lambda_2\ , \qquad \delta\mathcal{F}_1 \;=\; 0\ , \tag{5.59}$$

the variation of the scalar Lagrangian is $\delta\mathcal{L}^{(S)} = -\underline{\Lambda}_2(d\mathcal{F}_1 - \underline{H}_2)$. Therefore the Lagrangian $\mathcal{L}^{(S)'}$ is invariant under (5.59) provided

$$\delta A_2 \;=\; -\underline{\Lambda}_2\ . \tag{5.60}$$

$\mathcal{F}_1$ can now be treated as an independent worldvolume field that can be integrated out. This is achieved by using the algebraic field equation for $\mathcal{F}_1$ which can be put into the form

$$\mathcal{F}_a \;=\; -\frac{1}{|\vec{Q}|}\sqrt{1+\mathcal{F}^2}\,\mathcal{H}_a\ , \tag{5.61}$$

where $\mathcal{H}_1$ is the Hodge dual of the gauge-invariant three-form field strength $\mathcal{F}_3 = d\mathcal{A}_2 + \vec{Q}\cdot\underline{\vec{C}}_3$. Substituting this result into the Lagrangian $\mathcal{L}^{(S)'}$ and using $C_4^r(1) = C_4^r + C_1C_3^r$ we obtain the dualised Lagrangian

$$\mathcal{L}^{(S)'} \;=\; \sqrt{-\det g}\,\sqrt{\det(|\vec{Q}|^2\delta_a{}^b - \mathcal{H}_a\mathcal{H}^b)} \;-\; \frac{1}{4!}\epsilon^{mnpq}\vec{Q}\cdot(\underline{\vec{C}}_{mnpq} + 4\underline{\vec{C}}_m\mathcal{F}_{npq})\ , \tag{5.62}$$

where the 4-form field strength in the Wess-Zumino term corresponds to the value $t=0$ in (3.3). The dualisation could also have been obtained on-shell by using the relation (5.61) to map the $\mathcal{F}_1$ Bianchi identity into the $\mathcal{A}_2$ field equation and the $\mathcal{A}_0$ field equation into the $\mathcal{F}_3$ Bianchi identity.

5.4 Direct construction of the dual 3-brane in $D = 5$

The dualised Lagrangian $\mathcal{L}^{(S)'}$ found in the previous section can also be obtained directly from the superembedding formalism. To achieve this we impose the standard embedding constraint $E_\alpha{}^{\underline{a}} = 0$, but we replace the $\mathcal{F}_1$ constraint by the dual $\mathcal{F}_3$ constraint (3.15) subject to the appropriate Bianchi identity $d\mathcal{F}_3 = \underline{H}_4$, and obtain the off-shell linear multiplet. Thus, we start with the form of the spinor-spinor part of the embedding matrix given in (5.31), and use the local $Sp(1)$ symmetry (5.33) to set the $Sp(1)$ valued superfields $v_I{}^i = \delta_I{}^i$. At dimension 0 the $\mathcal{F}_3$ Bianchi reads

$$T_{\alpha\beta}{}^c\mathcal{F}_{abc} \;=\; E_\alpha{}^{\underline{\alpha}}E_\beta{}^{\underline{\beta}}E_a{}^{\underline{a}}E_b{}^{\underline{b}}\,\vec{Q}\cdot\vec{H}_{\underline{ab\alpha\beta}} \tag{5.63}$$

where the background value of H_4 is given in (3.18) and $\vec{Q}$ is the brane charge. This equation, being symmetric on the spinor indices, can then be traced with either γ^d or γ^{de}. This leads to the following results:

$$
\begin{aligned}
\mathcal{F}_{abc} &= \frac{2Q_1}{1-|z|^2+h^2}\epsilon_{abcd}h^d \ , \\
z &= \tfrac{1}{2}(1-|z|^2+h^2)\, q \ ,
\end{aligned} \tag{5.64}
$$

where h_a and z are defined in (5.32). Thus z is no longer an independent (auxiliary) field and h_a is the non-linear dual of the field strength of the 2-form potential. Thus the worldvolume multiplet is a linear multiplet whose components are a scalar and spinor given by the leading components of the transverse coordinates of the superembedding and the leading component of the 2-form potential.

We construct the action following the recipe given in Section 3.2. The appropriate $W_5^{(L)}$ is given by(3.12) for $t=1$. With $W_5^{(L)} = dK_4^{(L)}$ we find that the only non-vanishing component of $K_4^{(L)}$ is $K_{abcd}^{(L)} = \epsilon_{abcd}K^{(L)}$, where

$$
K^{(L)} = Q_1 \frac{1+|z|^2-h^2}{1-|z|^2+h^2} \ . \tag{5.65}
$$

In terms of the Hodge dual $\mathcal{H}_a$ of $\mathcal{F}_{abc}$ defined by

$$
\mathcal{F}_{abc} = \epsilon_{abcd}\mathcal{H}^d \ , \tag{5.66}
$$

the kinetic $K^{(L)}$ is given by the Born-Infeld style Lagrangian

$$
K^{(L)} = \sqrt{-\det(|\vec{Q}|^2\eta_{ab} - \mathcal{H}_a\mathcal{H}_b)} \ , \tag{5.67}
$$

and the total Green-Schwarz Lagrangian is given by (5.62) found by the dualisation procedure.

6 5-brane in 7-dimensions

In flat $D=7$ target space equation (8.25), which is the consequence of the dimension 0 torsion Bianchi identity in the normal directions, implies that

$$
h_{\alpha\beta}^{ij} = \epsilon^{ij}(\gamma^{abc})_{\alpha\beta}h_{abc} + u_a^{ij}(\gamma^a)_{\alpha\beta} \ . \tag{6.1}
$$

The dimension 0 torsion Bianchi identity in the tangential direction, on the other hand, yields the result

$$
T_{\alpha i,\beta j}{}^c = -i\epsilon_{ij}M^c{}_d\,(\gamma^d)_{\alpha\beta} - iM_{ij}^{c,def}\,(\gamma_{def})_{\alpha\beta} \ , \tag{6.2}
$$

where

$$
\begin{aligned}
M_{ab} &= \eta_{ab} - 72k_{ab} + \tfrac{1}{2}\eta_{ab}u^2 - (u\cdot u)_{ab} \ , \\
M_{ij}^{d,abc} &= 12h^{d[ab}\,u_{ij}^{c]+} + \eta^{d[a}\,u_{ik}^{b}\,u^{c]+}{}_{\ell j}\,\epsilon^{k\ell} \ ,
\end{aligned} \tag{6.3}
$$

where $[abc]_+$ denotes the self-dual projection, and

$$
\begin{aligned}
k_a{}^b &:= h_{acd}\, h^{bcd} \ , \\
u^2 &:= u_a^{ij} u_{ij}^a \ , \qquad (u\cdot u)_{ab} \ := \ u_a^{ij} u_{bij} \ .
\end{aligned} \tag{6.4}
$$

One can show that the analysis of the dimension 1/2 and higher torsion Bianchi identities do not put the system on shell, and that the superfields h_{abc} and u_a^{ij} remain underconstrained. In [13], the superfield u_a^{ij} was set equal to zero by hand, and consequently, the on-shell system describing the $(1,0)$ tensor multiplet on the worldvolume was obtained. This procedure is not altogether satisfactory from a geometrical point of view. Here we will show that the introduction of super 3-form field strength $\mathcal{F}_3$ defined in (3.9) subject to the constraint (3.15) indeed leads to the equation $u_a^{ij} = 0$. As discussed earlier, $\mathcal{F}_3$ is a natural geometrical ingredient for constructing the necessary Wess-Zumino term and the constraint it satisfies can be understood from geometrical considerations involving a superstring ending on the superfivebrane.

At dimension 0, the $\mathcal{F}$-Bianchi identity (3.7) implies that

$$
M_a{}^d\, \mathcal{F}_{bcd} \ = \ 24 h_{abc} \ , \tag{6.5}
$$

$$
M_{ij}^{f,abc}\, \mathcal{F}_{def} \ = \ -2u_{ij}^{[a}\, \delta_{de}^{bc]_+} \ . \tag{6.6}
$$

Some useful definitions and relations are

$$
\begin{aligned}
(\gamma_{abc})_{\alpha\beta} &= -\tfrac{1}{6}\, \epsilon_{abcdef}\, (\gamma^{def})_{\alpha\beta} \ , \qquad h_{abc} \ = \ \tfrac{1}{6}\, \epsilon_{abcdef}\, h^{def} \ , \\
k_a{}^d\, h_{bcd} &= k_{[a}{}^d\, h_{bc]d} \ = \ -\tfrac{1}{6}\, \epsilon_{abc}{}^{def}\, k_d{}^g\, h_{efg} \ , \\
h_{abe}\, h^{cde} &= \delta_{[a}^{[c}\, k_{b]}{}^{d]} \ , \qquad k_{ac}\, k^{bc} \ = \ \tfrac{1}{6}\delta_a^b\, \mathrm{tr}\, k^2 \ , \qquad \eta^{ab} k_{ab} \ = \ 0 \ .
\end{aligned} \tag{6.7}
$$

We shall now show that u_{ij}^a vanishes as a consequence of (6.5) and (6.6). We begin by rewriting (6.5) (assuming that M_{ab} is invertible) as

$$
\mathcal{F}_{abc} \ = \ 24 (M^{-1})_a{}^d\, h_{bcd} \ . \tag{6.8}
$$

Multiplying this equation with suitable M matrices twice yields

$$
M_{(a}{}^d\, h_{b)cd} \ = \ 0 \ . \tag{6.9}
$$

Noting that $m_{(a}{}^d\, h_{b)cd} = 0$, which can be deduced from the lemmas listed in (6.7), we find from (6.9) that

$$
(u^2)_{(a}{}^d\, h_{b)cd} \ = \ 0 \ . \tag{6.10}
$$

Next, we turn to the analysis of (6.6). Using (6.8) in (6.6) and multiplying with a suitable M, and symmetrizing with respect to a pair of indices in which the left hand side is manifestly antisymmetric and therefore drops out, we arrive at

$$u_{ij}^{[a} M_{(d}{}^{b} \delta_{e)}^{c]+} \;=\; 0 \;. \tag{6.11}$$

Using equations (6.10) and (6.11) one can show, after some algebra, that

$$\begin{aligned} (u^2)_{ab} &= A k_{ab} + B \eta_{ab} && (6.12)\\ M_{ab} &= (1+2B)\eta_{ab} - (A+72) k_{ab} \;, && (6.13) \end{aligned}$$

where $A = \text{tr }(u^2 k)/\text{tr }(k^2)$ and $B = \text{tr }(u^2)/6$. Furthermore, either A and B are both zero, in which case u_{ij}^a itself must be zero, or $A = -72$ and B is undetermined. However, a further property of the matrix $(u^2)_{ab}$ that it has maximal rank three as a consequence of its definition. This implies that

$$(u^2)_{[a_1}{}^{[b_1} \cdots (u^2)_{a_4]}{}^{b_4]} \;=\; 0 \;. \tag{6.14}$$

This equation can be used to show that, if $A = -72$, then $B = \pm A\sqrt{\text{tr }(k^2)/6}$. This implies that $(u^2)_{ab}$ is proportional to a projection operator,

$$(u^2)_{ab} \;=\; \pm A\sqrt{\text{tr }(k^2)/6}(1 \pm \bar{k}) \;, \tag{6.15}$$

where $\sqrt{\text{tr }(k^2)/6}\bar{k} = k$. The matrix $\bar{k}$ therefore square to the identity and is traceless. Bringing it to a canonical form, one can then show the positivity properties of $(u^2)_{ab}$ rule out the solution (6.15) so that, finally

$$u_{ij}^a \;=\; 0 \;. \tag{6.16}$$

Using this result in the dimension 1/2 and 1 torsion Bianchi identities, one finds the following field equations:

$$\begin{aligned} m^{ab}(\gamma_b)^{\alpha\beta}\,\Lambda_{bj} &= 0 \;,\\ m^{ac}\, m_c{}^b\, X_{ab}{}^{\perp} &= \tfrac{i}{4}\, m^{ab}\,(\gamma_a)^{\alpha\beta}\, U_{b,\alpha\beta} \;,\\ m^{ab}\,\nabla_a h_{bcd} &= -\tfrac{i}{48}\, m^{ab}\,(\gamma_{cd}\gamma_b)^{\alpha\beta}\; U_{a,\alpha\beta} \;, \end{aligned} \tag{6.17}$$

where $X_{a,b}{}^{\perp}$ was defined in (2.13), and where

$$\begin{aligned} m_a{}^b &= \delta_a^b - 72 h_{acd} h^{bcd} \;,\\ U_{a,\alpha\beta} &= \Big[h_{cde}(\gamma^{cde}\gamma^b)_\alpha{}^\gamma\, \Lambda_{a\gamma i}\Lambda_{b\beta j}\,\epsilon^{ij} + \tfrac{1}{2}(\alpha \;\leftrightarrow \beta) \Big] \;. \end{aligned} \tag{6.18}$$

These equations are equivalent to those of [13] up to field redefinitions.

One can also construct an action for an *unconstrained* 2-form potential $\mathcal{A}_2$ such that its field equation is equivalent to the Bianchi identity $d\mathcal{F}_3 = \underline{H}_4$ upon the imposition of a non-linear self-duality condition. Since the 4-form and 7-form field strengths in $D = 7$ and $D = 11$ obey formally equivalent Cartan integrable systems, the result found [14] for the 5-brane in $D = 11$ carries over essentially unchanged to the 5-brane in $D = 7$ (see also [18] for a review). The resulting for the 5-brane in $D = 7$ is

$$\mathcal{L} \;=\; \frac{1}{2}\sqrt{-\det g}\mathcal{K} - \frac{1}{6!}\epsilon^{mnpqrs}\left(\underline{C}_{mnpqrs} + 10\underline{C}_{mnp}\mathcal{F}_{qrs}\right) \,, \tag{6.19}$$

where the kinetic term is given by

$$\mathcal{K} \;=\; \sqrt{1 + \tfrac{1}{12}\mathcal{F}^2 + \tfrac{1}{288}(\mathcal{F}^2)^2 - \tfrac{1}{96}\mathcal{F}_{abc}\mathcal{F}^{bcd}\mathcal{F}_{def}\mathcal{F}^{efa}} \,, \tag{6.20}$$

and the Wess-Zumino term corresponds to the Wess-Zumino form W_7 given in (3.12). Varying the unconstrained 2-form potential $\mathcal{A}_2$ one finds the second order tensor field equation

$$d\left(\star\frac{\partial\mathcal{K}}{\partial\mathcal{F}_3}\right) \;=\; \underline{H}_4 \,. \tag{6.21}$$

The self-duality condition interchanging this field equation and the $\mathcal{F}_3$ Bianchi identity therefore must read

$$\star\mathcal{F}_3 \;=\; \frac{\partial\mathcal{K}}{\partial\mathcal{F}_3} \,. \tag{6.22}$$

For the particular form of $\mathcal{K}$ in (6.20) the self-duality condition is consistent (in the sense that $\star^2 = 1$). In fact, (6.22) is equivalent to (6.8), and one can show that the second order tensor field equation (6.21) and the $z^{\underline{M}}$ field equations following from (6.19) are equivalent to the field equations (6.17) obtained from the superembedding upon imposition of (6.22). The κ transformation $\delta z^{\underline{M}} = \kappa^{\underline{M}}$ and $\delta\mathcal{A}_3 = i_\kappa\underline{C}_3$ vanishes provided (6.22) holds. The κ invariance can be understood from the embedding formalism by solving for K_6 from (3.20). One then finds that the only non-vanishing component is $K_{abcdef} = \epsilon_{abcdef}K$ with

$$K \;=\; \sqrt{1 + \tfrac{1}{24}\mathcal{F}^{abc}\mathcal{F}_{abc}} \,. \tag{6.23}$$

This expression can be shown to equal $\mathcal{K}$ given in (6.20) upon imposition of (6.22). The action formula (3.25) cannot be applied, however, to derive the tensor field equations, since the multiplet is on-shell *and* since there are on-shell constraints on the tensor field strength at dimension zero which affect the functional form of K_6.

7 9-brane in 11-dimensions

We begin with the analysis of (2.16) which yields

$$
\begin{aligned}
h_{\alpha\beta} &= h_{abc}(\gamma^{abc})_{\alpha\beta} \ , \\
T_{\alpha\beta}{}^a &= m^a{}_b(\gamma^b)_{\alpha\beta} + m^a{}_{b_1\cdots b_5}(\gamma^{b_1\cdots b_5})_{\alpha\beta} \ , \\
m_{ab} &= (1+6h^2)\eta_{ab} - 12k_{ab} \ , \\
m_{a,b_1\cdots b_5} &= 6h_{a[b_1b_2}\, h_{b_3b_4b_5]+} - 9\eta_{a[b_1}h_{b_2b_3c}\, h^c{}_{b_4b_5]+} \ ,
\end{aligned} \tag{7.1}
$$

where $[abcdef]_+$ denotes the self-dual projection, $h^2 := h_{abc}h^{abc}$ and

$$
k_{ab} \ := \ h_{acd}\, h^{bcd} \ . \tag{7.2}
$$

At this stage we have an underconstrained system described by a single unconstrained Goldstone superfield in $N=1$, $d=10$ superspace.

Let us begin by assuming that the 9-brane intersects an ordinary open membrane in $D=11$. As explained in section open, this implies the $\mathcal{F}$-Bianchi identity (3.7) subject to the constraint (3.15). At dimension zero this leads to the equations

$$
m_a{}^d\mathcal{F}_{dbc} \ = \ -12h_{abc} \ , \tag{7.3}
$$

$$
m^{c,b_1\cdots b_5}\mathcal{F}_{cab} \ = \ 2h^{[b_1b_2b_3}\delta^{b_4b_5]+}_{a\,b} \ . \tag{7.4}
$$

The manipulation of (7.3), in the same fashion as described in the previous section in the analysis of (6.5), gives

$$
k_{(a}{}^d\, h_{b)cd} \ = \ 0 \ . \tag{7.5}
$$

On the other hand, substituting the expression for $\mathcal{F}$ obtained from (7.3) into (7.4), and multiplying once with a suitable m-matrix and symmetrizing in a pair of indices such that the left hand sides drops out, we find

$$
h^{[abc}\, m_{(f}{}^d\, \delta^{e]+}_{g)} \ = \ 0 \ . \tag{7.6}
$$

Multiplying with h_{abc} and then tracing two of the remaining free indices gives, in matrix notation for k_{ab},

$$
k^2 - (\mathrm{tr}\, k)\, k \ = \ 0 \ , \tag{7.7}
$$

which implies that the matrix k has rank 1 and therefore can be written in the form

$$
k_{ab} \ = \ u_a u_b \tag{7.8}
$$

for some vector u^a. Using this in (7.5) then leads to the result

$$h_{abc} \;=\; 0 \;. \tag{7.9}$$

This result can be shown to freeze the Goldstone superfield thereby leading to only global degrees of freedom. To see this, we proceed by analysing the constraints at the linearised level. We begin by writing (2.5) as

$$D_\alpha \Phi \;=\; \Psi_\alpha \;, \tag{7.10}$$

where we have defined $\Psi_\alpha := i(\Gamma^\perp)_{\alpha\beta'}\Theta^{\beta'}$. The linearised form of (2.8), on the other hand, gives

$$h_{\alpha\beta} \;=\; D_{[\alpha}\Psi_{\beta]} \;. \tag{7.11}$$

Therefore, using $h_{\alpha\beta} = 0$, together with (7.10) and the superalgebra obeyed by the supercovariant derivatives, we find

$$D_\alpha D_\beta \Phi \;=\; \tfrac{i}{2}(\gamma^a)_{\alpha\beta}\,\partial_a \Phi \;. \tag{7.12}$$

Applying a third spinorial derivative and making repeated use of (7.10) and (7.12) yields

$$\begin{aligned} D_\delta D_\beta D_\alpha \Phi \;&=\; \frac{i}{2}(\Gamma^a)_{\beta\alpha}\,\partial_a \Psi_\delta \\ &=\; -D_\beta D_\delta D_\alpha \Phi + i(\gamma^a)_{\delta\beta}\,\partial_a \Psi_\alpha \\ &=\; -\frac{i}{2}(\gamma^a)_{\delta\alpha}\partial_a\,\Psi_\beta + i(\gamma^a)_{\delta\beta}\,\partial_a \Psi_\alpha \;. \end{aligned} \tag{7.13}$$

Comparing the first and the third lines one obtains

$$(\gamma^a)_{\delta\beta}\,\partial_a \Psi_\alpha \;=\; \tfrac{1}{2}(\gamma^a)_{\alpha\beta}\,\partial_a \Psi_\delta + \tfrac{1}{2}(\gamma^a)_{\alpha\delta}\,\partial_a \Psi_\beta \;. \tag{7.14}$$

Contracting the last equation with $(\gamma_c)^{\gamma\delta}$ and then with δ^β_γ gives

$$15\,\partial_c \Psi_\alpha \;=\; (\gamma_c{}^d)^\delta{}_\alpha\,\partial_d \Psi_\delta \;. \tag{7.15}$$

The contraction of (7.15) with $(\gamma^c)^{\beta\alpha}$ implies the Dirac equation

$$(\gamma^c)^{\beta\alpha}\partial_c \Psi_\alpha \;=\; 0 \;, \tag{7.16}$$

while contracting with $(\Gamma_d)^{\beta\alpha}$ gives

$$(\Gamma_d)^{\beta\alpha}\partial_c \Psi_\alpha + (\gamma_c)^{\beta\alpha}\partial_d \Psi_\alpha \;=\; 0 \;. \tag{7.17}$$

Further contraction of the last equation with $(\gamma_a)_{\delta\beta}$ implies

$$\eta_{ad}\partial_c \Psi_\delta + (\Gamma_{ad})_\delta{}^\alpha \partial_c \Psi_\alpha + \eta_{ac}\partial_d \Psi_\delta + (\Gamma_{ac})_\delta{}^\alpha \partial_d \Psi_\alpha \;=\; 0 \;. \tag{7.18}$$

Contracting the last equation with η^{ad} and using the Dirac equation we find

$$\partial_c \Psi_\delta \;=\; 0 \;. \tag{7.19}$$

Thus Ψ must be a constant showing that the fermionic degrees of freedom of the brane are frozen. To demonstrate a similar result for the scalar we start from

$$D_\gamma D_\delta D_\beta D_\alpha \Phi \;=\; 0 \tag{7.20}$$

which follows from (7.13) and (7.19). Symmetrizing in $(\gamma\delta)$ and $(\beta\alpha)$ and using the supersymmetry algebra of the supercovariant derivatives we obtain

$$(\gamma^a)_{\gamma\delta}(\gamma^b)_{\beta\alpha}\partial_a\partial_b\Phi \;=\; 0 \;, \tag{7.21}$$

which in turn implies

$$\partial_a\partial_b\Phi \;=\; 0 \;. \tag{7.22}$$

This represents a flat plane moving with a constant velocity. Notice that it is a non-trivial fact that the brane allows for a zero-mode spacetime momentum.

8 Conclusions

In this paper we have shown systematically how the embedding constraint together with a constrained field strength $\mathcal{F}$ in the worldvolume provide a full description of the worldvolume multiplet for branes with codimension one for which the standard embedding constraint generically leads to an underconstrained system. The constrained field strength arises in the context of open membranes (or particles for $p = 3$) ending on the codimension one branes.

For the 3-brane in five dimensions we have seen that there are two possible choices for $\mathcal{F}$, a 1-form or a 3-form. The 3-form option gives rise to a worldvolume $N = 1, d = 4$ linear multiplet which is off-shell; this is therefore the L-brane with 4 worldvolume supersymmetries. The dualisation of the linear multiplet gives an worldvolume $N = 1, d = 4$ scalar multiplet based on the 1-form option. This multiplet is off-shell in contrast to the L-branes with 8 supersymmetries where dualisation of the antisymmetric tensor gauge field to a scalar leads to an on-shell hypermultiplet [7]. We have also shown that the scalar version of the 3-brane in five dimensions can be obtained by vertical reduction of a 3-brane in six dimensions. It is worth noting that, from this perspective, the five-dimensional $\mathcal{F}_1$ constraint emerges as a component of the standard embedding condition for a six-dimensional target.

The 3-brane completely breaks the $Sp(1)_R$ symmetry of the $N = 1$, $D = 5$ supersymmetry algebra (the same situation arises for the 3-brane in $N = (1, 0)$, $D = 6$). The resulting three Goldstone scalars parametrise the group element v appearing in the spinor-spinor part of the embedding matrix given in eq. (5.31). The local $Sp(1)$ transformation $v \to gv$, however, turns out to be an invariance of the embedding, as we saw by writing the embedding matrix on the manifestly $Sp(1)$ invariant form (5.7). In the scalar case we found that the auxiliary fields parametrise a two-sphere. Upon elimination of the auxiliary fields we then found an $Sp(1)$ covariant Green-Schwarz action, which we dualised to obtain the linear multiplet formulation.

For the 5-brane in seven dimensions we have shown that the $\mathcal{F}_3$-constraint restricts the unconstrained scalar superfield determined by the embedding constraint to be an on-shell $N = (1,0), d = 6$ tensor multiplet. The resulting equations of motion are in agreement with those derived earlier in [13] where they were obtained by imposing an additional torsion constraint by hand. We also have constructed the 5-brane action analogous to the one given for the 5-brane in $D = 11$ [14].

Finally, we have seen that the $\mathcal{F}_3$-constraint for a 9-brane in eleven dimensions severely restricts the worldvolume multiplet in such a way that the degrees of freedom are frozen out such that the only remaining degree of freedom is a spacetime momentum.

This suggests a connection with the Horava-Witten picture of a 9-brane as a boundary of the $D = 11$ spacetime [15]. However, the vector multiplets which arise on the boundary need to be included in the superembedding formalism. To this end, it would be interesting to investigate the consequences of modifying the $\mathcal{F}_3$ constraint by including an $g_{YM}^{1/2}\mathrm{tr}\ (F^2)$ term to its Bianchi identity.

Finally, we note that there is another kind of 9-brane, known as the M9-brane, which we have not considered in this paper. This is a wrapped domain wall solution of massive $D = 11$ supergravity in the sense of [19], and it is closely related to an $M2$-brane in which a $U(1)$ isometry direction in the target space is gauged. It would be useful to find the fully nonlinear and supersymmetric action for this brane, which is still lacking, within the framework of the superembedding formalism.

Appendix A

In this appendix we give our conventions for the Dirac matrices that are relevant to co-dimension one embeddings studied in this paper. We also describe the splitting of the target space spinor space into the tangential and normal directions appropriate to these embeddings.

$\mathbf{D=3\to d=2}$

A 2 component Majorana spinor ψ in $D=3$ splits as

$$\psi_{\underline{\alpha}} \quad\to\quad \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{8.1}$$

where $+$ labels the tangent direction and $-$ labels the normal direction in the superembedding. The $D=3$ gamma matrices $\Gamma^a C$ split as

$$\begin{aligned}
&\Gamma^a_{++} = \delta^a_+ \ , \qquad \Gamma^a_{--} = \delta^a_- \ , \qquad \Gamma^a_{+-} = \Gamma^a_{-+} = 0 \ , \quad a=0,1 \ , \\
&\Gamma^3_{++} = \Gamma^3_{--} = 0 \ , \qquad \Gamma^3_{+-} = \Gamma^3_{-+} = 1 \ .
\end{aligned} \tag{8.2}$$

We also define $\delta^a_{\mp} = \frac{1}{2}(1,1)$

$\mathbf{D=4\to d=3}$

A 4-component Majorana spinor $\psi_{\underline{\alpha}}$ in $D=4$ splits as

$$\psi_{\underline{\alpha}} \quad\to\quad \begin{pmatrix} \psi_\alpha \\ \psi_{\alpha'} \end{pmatrix} \tag{8.3}$$

where α labels a 2-component Majorana spinor in the tangent direction and α' labels another 2-component Majorana spinor, this one being in the normal direction. The latter spinor is in a representation of the worldvolume Lorentz group which is equivalent to that carried by the first spinor. The Γ-matrices split as

$$(\Gamma^a)_{\underline{\alpha}}{}^{\underline{\beta}} \;=\; \begin{pmatrix} \gamma^a & 0 \\ 0 & \gamma^a \end{pmatrix} \qquad (\Gamma^4)_{\underline{\alpha}}{}^{\underline{\beta}} \;=\; \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad C_{\underline{\alpha}\underline{\beta}} \;=\; \begin{pmatrix} C_3 & 0 \\ 0 & C_3 \end{pmatrix} \tag{8.4}$$

where C_3 is the standard charge conjugation matrix in three dimensions and $\Gamma^{\underline{a}}C_4$ are symmetric.

$\mathbf{D=6\to d=4}$

As discussed in detail in Section 5.1, the splitting of an $Sp(1)$ symplectic Majorana-Weyl spinor Ψ_i in $D=6$ involves the R-symmetry doublet index i. Such a spinor consist of two Dirac spinors obeying

$$
\begin{aligned}
\Psi^i_{c6} &= -\epsilon^{ij}\Psi_j\ , \\
\Gamma^7\Psi_i &= \Psi_i\ ,
\end{aligned}
\tag{8.5}
$$

where $\Psi^i_{c6} = C_6\bar{\Psi}^{i\,T}$ is the charge-conjugated spinor, $\Gamma^{\underline{abcdef}} = \epsilon^{\underline{abcdef}}\Gamma^7$ and $\gamma^{abcd} = i\epsilon^{abcd}$ with $\epsilon^{012356} = \epsilon^{0123} = 1$. The Dirac matrices $\Gamma^{\underline{a}}$ in $D = 6$ can be written in terms of the $d = 4$ Dirac matrices as

$$
\begin{aligned}
\Gamma^a &= \begin{pmatrix} 0 & \gamma^a\gamma^5 \\ -\gamma^a\gamma^5 & 0 \end{pmatrix}\ , \qquad \Gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \Gamma^6 = \begin{pmatrix} 0 & i\gamma^5 \\ -i\gamma^5 & 0 \end{pmatrix}\ , \\
C_6 &= \begin{pmatrix} 0 & -C_4 \\ C_4 & 0 \end{pmatrix}\ ,
\end{aligned}
\tag{8.6}
$$

where $a = 0, 1, 2, 3$ and C_6 is symmetric and $\Gamma^{\underline{a}}C_6$ are anti-symmetric. The symplectic Majorana-Weyl condition (8.5) translates into the following reality condition on the Majorana conjugate spinors in $d = 4$:

$$
\Psi^i_{c4} = \gamma^5\epsilon^{ij}\Psi_j\ .
\tag{8.7}
$$

Since (8.7) implies that $\left(\gamma^5\Psi^i\right)_{c4} = -\epsilon^{ij}\Psi_j$ we can define two $d = 4$ Majorana spinors as follows:

$$
\begin{aligned}
\Psi_+ \equiv P^i_+\Psi_i &= \frac{1}{\sqrt{2}}\left(\Psi_1 + \gamma^5\Psi_2\right)\ , \\
\Psi_- \equiv P^i_-\Psi_i &= \frac{1}{\sqrt{2}}\left(-\Psi_2 + \gamma^5\Psi_1\right)\ .
\end{aligned}
\tag{8.8}
$$

Note that in the cases of $D = 5, 6, 7$ where the target space spinors are $Sp(1)$ symplectic Majorana spinors, we shall use a notation where $\underline{\alpha}$ denotes a composite Lorentz spinor and $Sp(1)$ doublet index. Thus, for instance the symmetric composite Dirac matrices $(\Gamma^{\underline{a}}C)_{\underline{\alpha\beta}}$ splits into an anti-symmetric Dirac matrix $(\Gamma^{\underline{a}}C)_{\underline{\alpha\beta}}$ times ϵ_{ij}.

$\mathbf{D = 5 \to d = 4}$

An $Sp(1)$ Majorana-Weyl spinor in $D = 5$ reduces to an $Sp(1)$ symplectic Majorana spinor Ψ_i in $D = 5$ and splits essentially in the same way as described in (8.8). This expression is exactly applicable if we choose the anti-symmetric charge conjugation matrix in $D = 5$ to be

$$
C_5 = \gamma^5 C_4\ .
\tag{8.9}
$$

The five-dimensional symplectic Majorana condition then reads:

$$
\Psi^i_{c5} = \epsilon^{ij}\Psi_j\ .
\tag{8.10}
$$

We take the $D = 5$ Dirac matrices (without the $Sp(1)$ doublet indices) to be $\Gamma^{\underline{a}} = (\gamma^a, \gamma^5)$. Note that $\Gamma^{\underline{a}} C_5$ are symmetric.

$\mathbf{D = 7 \to d = 6}$

An $Sp(1)$ symplectic Majorana spinor in $D = 7$ decomposes into two independent $Sp(1)$ symplectic Majorana-Weyl spinors of opposite chiralities in $d = 6$. Using chiral notation in which the spinor indices which indicate the chiralities and cannot be raised or lowered, we write

$$\psi_{\underline{\alpha}} \quad \to \quad \begin{pmatrix} \psi_\alpha \\ \psi_{\alpha'} \end{pmatrix} \equiv \begin{pmatrix} \lambda_{\alpha i} \\ \lambda_i^\alpha \end{pmatrix} . \tag{8.11}$$

where the upper and lower spinor indices $\alpha = 1, ..., 4$ represent left- and right-handedness. The spinors $\lambda_{\alpha i}$ and λ_i^α are independent of each other. Using the chirally projected γ-matrices in $d = 6$, we can choose the $D = 7$ Γ-matrices as

$$\begin{aligned} (\Gamma^a)_{\underline{\alpha}}{}^{\underline{\beta}} &= \delta_i{}^j \begin{pmatrix} 0 & (\gamma^a)_{\alpha\beta} \\ (\gamma^a)^{\alpha\beta} & 0 \end{pmatrix} , \\ (\Gamma^7)_{\underline{\alpha}}{}^{\underline{\beta}} &= \delta_i{}^j \begin{pmatrix} \delta_\alpha^\beta & 0 \\ 0 & -\delta_\beta^\alpha \end{pmatrix} . \end{aligned} \tag{8.12}$$

The anti-symmetric charge-conjugation matrix is

$$C_{\underline{\alpha}\,\underline{\beta}} = \epsilon_{ij} \begin{pmatrix} 0 & \delta_\alpha^\beta \\ \delta_\beta^\alpha & 0 \end{pmatrix} . \tag{8.13}$$

Note that the Dirac $\Gamma^{\underline{a}} C_7$ (without the $Sp(1)$ doublet index) are anti-symmetric. Spinor indices in $D = 7$ are raised and lowered by the charge conjugation matrix so that

$$\begin{aligned} (\Gamma^a)_{\underline{\alpha}\,\underline{\beta}} &= \epsilon_{ij} \begin{pmatrix} (\gamma^a)_{\alpha\beta} & 0 \\ 0 & (\gamma^a)^{\alpha\beta} \end{pmatrix} , \\ (\Gamma^7)_{\underline{\alpha}\,\underline{\beta}} &= \epsilon_{ij} \begin{pmatrix} 0 & \delta_\alpha^\beta \\ -\delta_\beta^\alpha & 0 \end{pmatrix} , \end{aligned} \tag{8.14}$$

$\mathbf{D = 11 \to d = 10}$

A Majorana spinor in $D = 11$ decomposes into two independent Majorana-Weyl spinors of opposite chiralities in $d = 10$. Using chiral notation, we write

$$\psi_{\underline{\alpha}} \quad \to \quad \begin{pmatrix} \psi_\alpha \\ \psi_{\alpha'} \end{pmatrix} \equiv \begin{pmatrix} \lambda_\alpha \\ \lambda^\alpha \end{pmatrix} , \tag{8.15}$$

where the upper and lower spinor indices $\alpha = 1, ..., 16$ represent left- and right-handedness. The spinors λ_α and λ^α are independent of each other. Using the chirally projected γ-matrices in $d = 10$, we can choose the $D = 11$ Γ-matrices as

$$
\begin{aligned}
(\Gamma^a)_{\underline{\alpha}}{}^{\underline{\beta}} &= \begin{pmatrix} 0 & (\gamma^a)_{\alpha\beta} \\ (\gamma^a)^{\alpha\beta} & 0 \end{pmatrix} , \\
(\Gamma^{11})_{\underline{\alpha}}{}^{\underline{\beta}} &= \begin{pmatrix} \delta_\alpha^\beta & 0 \\ 0 & -\delta^\alpha_\beta \end{pmatrix} .
\end{aligned} \tag{8.16}
$$

The charge-conjugation matrix is

$$
C_{\underline{\alpha}\,\underline{\beta}} = \begin{pmatrix} 0 & \delta_\alpha^\beta \\ -\delta^\alpha_\beta & 0 \end{pmatrix} \tag{8.17}
$$

and $\Gamma^{\underline{a}}C$ are symmetric and given by

$$
\begin{aligned}
(\Gamma^a)_{\underline{\alpha}\,\underline{\beta}} &= \begin{pmatrix} -(\gamma^a)_{\alpha\beta} & 0 \\ 0 & (\gamma^a)^{\alpha\beta} \end{pmatrix} , \\
(\Gamma^7)_{\underline{\alpha}\,\underline{\beta}} &= \begin{pmatrix} 0 & \delta_\alpha^\beta \\ \delta^\alpha_\beta & 0 \end{pmatrix} .
\end{aligned} \tag{8.18}
$$

Appendix B

In this appendix we give the results of solving the torsion equations (2.6). We first split these equations into tangential and normal components by contracting with $(E^{-1})_{\underline{C}}{}^C$ and $(E^{-1})_{\underline{C}}{}^{C'}$, respectively. The result is:

Tangential Projections:

- *dimension* 0*:*

$$
T_{\alpha\beta}{}^c = -i\left\{(\Gamma^c)_{\alpha\beta} - h_{\alpha\gamma}(\Gamma^c)^{\gamma\delta}h_{\delta\beta}\right\} \tag{8.19}
$$

- *dimension* $\frac{1}{2}$*:*

$$
T_{\alpha\beta}{}^\gamma = -ih_{\delta(\alpha}(\Gamma^a)^{\delta\gamma}\Lambda_{a\beta)} + Y_{\alpha\beta}{}^\gamma + h_{\delta(\alpha}(\Gamma^a)^{\delta\gamma}Y_{a\beta)}{}^\perp \, , \tag{8.20}
$$

$$
T_{a\beta}{}^c = -i\Lambda_{a\gamma}(\Gamma^c h)^\gamma{}_\beta + Y_{a\beta}{}^c \, . \tag{8.21}
$$

- *dimension* 1*:*

$$T_{a\beta}{}^{\gamma} \;=\; \frac{1}{2}\left\{h_{\beta\delta}(\Gamma^b)^{\delta\gamma}X_{a,b}{}^{\perp}+i\Lambda_{a\delta}(\Gamma^b)^{\delta\gamma}\Lambda_{b\beta}\right\}+Y_{a\beta}{}^{\gamma}-\frac{1}{2}\Lambda_{a\delta}(\Gamma^b)^{\delta\gamma}Y_{b\beta}{}^{\perp}\ , \tag{8.22}$$

$$T_{ab}{}^{c} \;=\; -i\Lambda_{a\alpha}(\Gamma^c)^{\alpha\beta}\Lambda_{b\beta}+Y_{ab}{}^{c}\ . \tag{8.23}$$

- *dimension* $\frac{3}{2}$*:*

$$T_{ab}{}^{\gamma} \;=\; -\Lambda_{[a\delta}X_{b],c}{}^{\perp}(\Gamma^c)^{\delta\gamma}+Y_{ab}{}^{\gamma}\ . \tag{8.24}$$

Normal Projections:

- *dimension* 0*:*

$$h_{(\alpha\beta)} \;=\; 0\ , \tag{8.25}$$

- *dimension* $\frac{1}{2}$*:*

$$\nabla_{(\alpha}h_{\beta)\gamma} \;=\; \frac{1}{2}T_{\gamma(\alpha}{}^{a}\Lambda_{a\beta)}-\frac{1}{2}T_{\alpha\beta}{}^{a}\Lambda_{a\gamma}+Z_{\alpha\beta,\gamma}\ , \tag{8.26}$$

$$X_{\alpha,b}{}^{\perp} \;=\; i\Lambda_{b\alpha}-Y_{b\alpha}{}^{\perp}\ . \tag{8.27}$$

- *dimension* 1*:*

$$\begin{aligned}\nabla_{\alpha}\Lambda_{a\beta} \;=\;& \nabla_a h_{\alpha\beta}+\frac{i}{2}T_{\alpha\beta}{}^{b}X_{a,b}{}^{\perp}\\ &-i\left\{\Lambda_{a\gamma}(\Gamma^b)^{\gamma\delta}h_{\delta\alpha}\Lambda_{b\beta}+\frac{1}{2}\Lambda_{a\gamma}(\Gamma^b)^{\gamma\delta}h_{\delta\beta}\Lambda_{b\alpha}\right\}+Z_{a,\alpha\beta}\ ,\end{aligned} \tag{8.28}$$

$$X_{[a,b]}{}^{\perp} \;=\; \frac{1}{2}Y_{ab}{}^{D}\ . \tag{8.29}$$

- *dimension* $\frac{3}{2}$*:*

$$\nabla_{[a}\Lambda_{b]\alpha} \;=\; \frac{1}{2}\left\{\Lambda_{[a\beta}X_{b],c}{}^{\perp}(\Gamma^c)^{\beta\gamma}h_{\gamma\alpha}+i\Lambda_{a\beta}(\Gamma^c)^{\beta\gamma}\Lambda_{b\gamma}\Lambda_{c\alpha}\right\}+Z_{ab,\alpha}\ . \tag{8.30}$$

where we have collected the contributions from curved target space backgrounds in the quantities $Y_{AB}{}^{C}$ and $Y_{AB}{}^{C'}$ and $Z_{AB,C}$ (which vanish in flat target space) defined by

$$\begin{aligned}Y_{AB}{}^{C} \;\equiv\;& (-1)^{A(B+\underline{B})}E_B{}^{\underline{B}}E_A{}^{\underline{A}}T_{\underline{AB}}{}^{\underline{C}}(u^{-1})_{\underline{C}}{}^{C}-(-1)^{A(B+1)}E_B{}^{\underline{\beta}}E_A{}^{\underline{\alpha}}T_{\underline{\alpha}\,\underline{\beta}}{}^{\underline{c}}(u^{-1})_{\underline{c}}{}^{C}\\ Y_{AB}{}^{C'} \;\equiv\;& (-1)^{A(B+\underline{B})}E_B{}^{\underline{B}}E_A{}^{\underline{A}}T_{\underline{AB}}{}^{\underline{C}}(u^{-1})_{\underline{C}}{}^{C'}-(-1)^{A(B+1)}E_B{}^{\underline{\beta}}E_A{}^{\underline{\alpha}}T_{\underline{\alpha}\,\underline{\beta}}{}^{\underline{c}}(u^{-1})_{\underline{c}}{}^{C'}\end{aligned} \tag{8.31}$$

and

$$
\begin{aligned}
Z_{\alpha\beta,\gamma} &\equiv \frac{1}{2}Y_{a(\alpha}{}^{\perp}T_{\beta)\gamma}{}^{a} - \frac{1}{2}Y_{\alpha\beta}{}^{\delta}h_{\delta\gamma} + \frac{1}{2}Y_{\alpha\beta,\gamma} \ , \\
Z_{a,\alpha\beta} &\equiv -\frac{1}{2}Y_{b\alpha}{}^{\perp}(\Gamma^{b})^{\gamma\delta}\Lambda_{a\gamma}h_{\delta\beta} + Y_{a\alpha}{}^{b}\Lambda_{b\beta} + Y_{a\alpha}{}^{\delta}h_{\delta\beta} - Y_{a\alpha,\beta} \ , \\
Z_{ab,\alpha} &\equiv \frac{1}{2}Y_{ab,\alpha} - \frac{1}{2}Y_{ab}{}^{c}\Lambda_{c\alpha} - \frac{1}{2}Y_{ab}{}^{\gamma}h_{\gamma\alpha} \ .
\end{aligned} \tag{8.32}
$$

It is also convenient to absorb all the perpendicular Dirac matrices by defining the following objects (whose spinor indices can then be treated as usual):

$$
\begin{aligned}
h_{\alpha\beta} &\equiv h_{\alpha}{}^{\beta'}(\Gamma^{\perp})_{\beta'\beta} \ , \\
\Lambda_{a\alpha} &\equiv \Lambda_{a}{}^{\alpha'}(\Gamma^{\perp})_{\alpha'\alpha} \ , \\
X_{\alpha,\beta\gamma} &\equiv X_{\alpha,\beta}{}^{\gamma'}(\Gamma^{\perp})_{\gamma'\gamma} \ = \ \frac{1}{2}(\Gamma^{a})_{\beta\gamma}X_{\alpha,a}{}^{\perp} \ , \\
X_{\alpha,}{}^{\beta\gamma} &\equiv (\Gamma^{\perp})^{\beta\gamma'}X_{\alpha,\gamma'}{}^{\gamma} \ = \ -\frac{1}{2}(\Gamma^{a})^{\beta\gamma}X_{\alpha,a}{}^{\perp} \ , \\
Y_{AB,\gamma} &\equiv Y_{AB}{}^{\delta'}(\Gamma^{\perp})_{\delta'\gamma} \ .
\end{aligned} \tag{8.33}
$$

The tangential equations (8.19-8.24) determine the worldvolume torsion components in terms of the pull-backs of the target space torsion components, the composite local connection $X_{a,b}{}^{\perp}$ and the superfields $\Lambda_{a\alpha}$ and $h_{\alpha\beta}$.

Let us now examine the normal equations. At dimension zero we have the algebraic equation (8.25) which plays an important role in determining the worldvolume supermultiplets and in solving the remaining normal equations. At dimension half, eq. (8.26) relates the 'hooked' component of the spinorial derivative to the same structure in $Z_{\alpha\beta,\gamma}$, while the totally antisymmetric part $\nabla_{[\alpha}h_{\beta\gamma]}$ remains undetermined (except in the cases of $p=1,2$, when this component vanishes identically). Eq. (8.26) also imposes the integrability condition $Z_{(\alpha\beta,\gamma)}=0$ on a curved target space background. Eq. (8.27) relates the local composite connection $X_{\alpha,a}{}^{\perp}$ to the superembedding matrix element $\Lambda_{a\alpha}$. At dimension one and three half one can then demonstrate that eq.(8.28) and eq. (8.30) are identically satisfied. Finally the dimension one equation (8.29) is identically satisfied. In fact, this equation and eq. (8.30) are non-linear versions of the linear integrability conditions $\partial_{[a}\partial_{b]}X^{c'}=0$ and $\partial_{[a}\partial_{b]}\Theta^{\beta'}=0$. In summary, the torsion equation leads to an unconstrained transverse Goldstone superfield with supercovariant component expansion given by $\Theta_{\beta}|$, $h_{\alpha\beta}|$, $\nabla_{[\alpha}h_{\beta\gamma]}|$ up to order θ^3.

References

[1] D. Sorokin, V. Tkach and D.V. Volkov, *Superparticles, twistors and Siegel symmetry* , Mod. Phys. Lett. **A4** (1989) 901.

[2] I.A. Bandos, D. Sorokin, M. Tonin, P. Pasti and D. Volkov, *Superstrings and supermembranes in the doubly supersymmetric geometrical approach* , Nucl. Phys. **B446** (1995) 79.

[3] P.S. Howe and E. Sezgin, *Superbranes*, Phys. Lett. **B390** (1997) 133.

[4] P.S. Howe, E. Sezgin and P.C. West, *Aspects of superembeddings*, in *Supersymmetry and Quantum Field Theory*, eds. J. Wess and V.P. Akulov (Springer 1998), hep-th/9705093.

[5] E. Sezgin, *Topics in M-theory*, hep-th/9809204.

[6] D. Sorokin, *Superbranes and superembeddings*, hep-th/9906142.

[7] P.S. Howe, O. Raetzel and E. Sezgin, *On brane actions and superembeddings*, hep-th/9804051.

[8] P.S. Howe, O. Raetzel, I. Rudychev and E. Sezgin, *L-branes*, Class. Quant. Grav. **16** (1999) 705, hep-th/9810081.

[9] C.S. Chu and E. Sezgin, *M-Fivebrane from the open supermembrane*, JHEP **12** (1997) 001, hep-th/9710223.

[10] C.S. Chu, P.S. Howe, E. Sezgin and P.C. West, *Open superbranes*, Phys. Lett. **B429** (1998) 273, hep-th/9803041.

[11] P.S. Howe and E. Sezgin, *D=11,p=5*, Phys. Lett. **B394** (1997) 62, hep-th/9611008.

[12] P.S. Howe, E. Sezgin and P.C. West, *Covariant field equations of the M-theory five-brane*, Phys. Lett. **399B** (1997) 49, hep-th/9702008.

[13] T. Adawi, M. Cederwall, U. Gran, M. Holm and B.E.W. Nilsson, *Superembeddings, non-Linear supersymmetry and 5-branes*, Int. J. Mod. Phys. **A13** (1998) 4691, hep-th/9711203.

[14] M. Cederwall, B.E.W. Nilsson and P. Sundell, *An action for the super-5-brane in D=11 supergravity*, JHEP 9804 (1998) 007.

[15] P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. **B460** 506, hep-th/9510209.

[16] E. Bergshoeff, J. P. van der Schaar, *On M9-branes*, Class. Quant. Grav. **16** (1999) 23, hep-th/9806069.

[17] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, Phys. Lett. **189B** (1987) 75.

[18] E. Sezgin and P. Sundell, *Aspects of the M5-Brane*, hep-th/9902171.

[19] E. Bergshoeff, Y. Lozano and T. Ortin, *Massive branes*, Nucl. Phys. **B518** 363, hep-th/9712115.