

# Hitchin Equation, Singularity, and $N = 2$ Superconformal Field Theories

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ABSTRACT: We argue that Hitchin's equation determines not only the low energy effective theory but also describes the UV theory of four dimensional  $N = 2$  superconformal field theories when we compactify six dimensional  $A_N(0,2)$  theory on a punctured Riemann surface. We study singular solutions to Hitchin's equation and the Higgs field of equation has a simple pole at the punctures; We show that the massless theory is associated with Higgs field whose residue is a nilpotent element; We identify the flavor symmetry associated with the puncture by studying the singularity of closure of the moduli space of solutions with the appropriate boundary conditions. For mass-deformed theory the residue of the Higgs field is a semi-simple element, we identify the semi-simple element by arguing that the moduli space of solutions of mass-deformed theory must be a deformation of the closure of the moduli space of massless theory. We also study the Seiberg-Witten curve by identifying it as the spectral curve of the Hitchin's system. The results are all in agreement with Gaiotto's results derived from studying the Seiberg-Witten curve of four dimensional quiver gauge theory.

KEYWORDS: Hitchin system, Singularities,  $N = 2$  Superconformal field theory.

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## 1. Introduction

$D = 6$  is the maximal dimension in which we can formulate a superconformal field theory (SCFT). Six dimensional  $(0, 2)$  superconformal field theory has the famous ADE classification. The compactification of this six dimensional theory on a Riemann surface provides a lot of insights on four dimensional conformal field theory [1]. For instance, if we compactify  $A_{N-1}$  theory on a smooth torus, the  $SL(2, Z)$  duality invariance of four dimensional  $N = 4$   $SU(N)$  gauge theory is directly related to  $SL(2, Z)$  modular group of the torus.

We may wonder if we can also find a six dimensional description of four dimensional  $N = 2$  superconformal field theory; In analogy with  $N = 4$  theory, the duality of four dimensional field theory can be interpreted geometrically as the property of Riemann surface on which we compactify the six dimensional theory. Motivated by earlier work on  $N = 2$   $S$  duality [2], Gaiotto [3] provided a six dimensional framework to understand  $S$  duality of four dimensional  $N = 2$  scale invariant theory. Here we need to turn on codimension two defects [4] of six dimensional theory. These defects are labeled by Yang tableaux from which we can also read the flavor symmetries of four dimensional theory.

Gauge couplings of four dimensional theories are interpreted as the complex structure of this punctured Riemann surface. The  $S$  duality of gauge theory is realized as the conformal

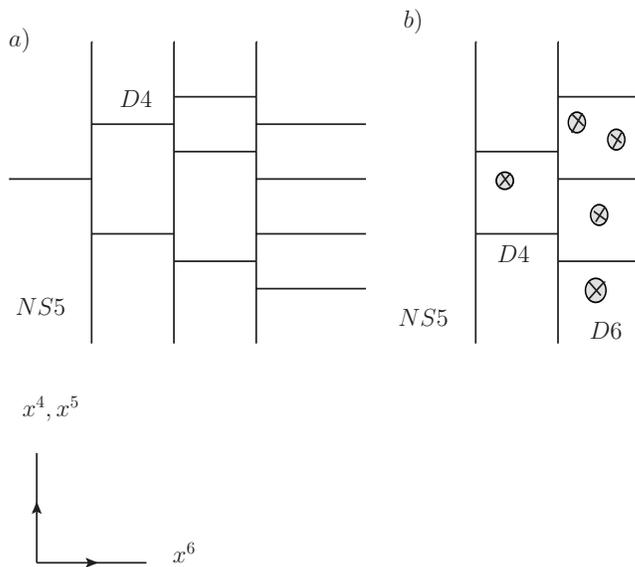
mapping group of the complex structure space. Different weakly coupled four dimensional theories are described as the different degeneration limits of punctured Riemann surface.

There are more information encoded in those punctures. The punctures are labeled by a Yang tableaux of total boxes  $N$  and we can read four dimensional flavor symmetry associated with it. The Seiberg-Witten curve [5, 6] is also described by a subspace in the cotangent bundle of this Riemann surface, it is entirely determined by the information encoded in the Yang tableaux. After the description of this idea in [3], there are a lot of developments along this idea to understand  $N = 2$  SCFT [7, 8, 9, 10, 11, 12, 13]; See also the interesting relation between four dimensional theory and two dimensional conformal field theory [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

Gaiotto discovered above description by using brane construction [35] and explored the Seiberg-Witten curve. It is important to know whether we can have a truly six dimensional description, namely, we want to know what is the dynamical equation living on Riemann surface when we compactify six dimensional theory on a Riemann surface. In this paper, we argue that it is Hitchin's equation which governs the dynamics of this compactification.

Hitchin's equation has appeared before as a description of certain  $N = 2$  low energy effective theory [36]. That is based on observation about the relation between Seiberg-Witten theory and integrable system. In this paper, we will show that Hitchin's equation not only describes low energy effective theory but also describes the four dimensional UV theory for a large class of  $N = 2$  SCFT. We study singular solution to Hitchin's equation and the Higgs field of solution has simple pole at singularity; We show that the massless theory is associated with the Higgs field whose residue is a nilpotent element; We identify the flavor symmetry associated with the puncture by studying moduli space of solutions with appropriate boundary conditions. For mass-deformed theory the residue of the Higgs field is a semi-simple element, we identify the semi-simple element by arguing that the moduli space of solutions of mass-deformed theory must be a deformation of closure of the moduli space of massless theory. We also study the Seiberg-Witten curve by identifying it as the spectral curve of the Hitchin's system. The results are all in agreement with Gaiotto's results derived from studying the Seiberg-Witten curve of four dimensional quiver gauge theory.

This paper is organized as follows: In section 2, we review the connection between six dimensional theory  $(0, 2)$  theory and four dimensional  $N = 2$  superconformal field theory. In section 3, we study string duality of certain brane configuration engineering  $N = 2$  theory and argue that Hitchin's equation can be used to study a large class  $N = 2$  SCFT; In section 4, we study Hitchin's equation for  $SU(2)$  gauge group and prove that it is the correct description of IR and UV behavior of four dimensional  $N = 2$   $SU(2)$  quiver theory; In section 5, we generalize  $SU(2)$  results to  $SU(n)$  case; In section 6, we give the conclusion and discuss some future directions. In appendix I, we give some mathematical introduction to Nahm's equation and discuss the isomorphism between the moduli space of solutions and the adjoint orbit of  $SL_n$  lie algebra.



**Figure 1:** (a) A Type IIA NS5-D4 brane configuration which gives four dimensional  $N = 2$  superconformal field theory, there are semi-infinite D4 branes on both ends which provide the fundamental hypermultiplets; (b) We use  $D6$  branes instead of semi-infinite D4 branes to provide the fundamental hypermultiplets.

## 2. Six dimensional $(0, 2)$ $A_N$ theory and Four dimensional $N = 2$ SCFT

We can construct a large class of four dimensional  $N = 2$  superconformal field theories by using Type IIA brane configurations. The NS5 branes which extend in the direction  $x^0, x^1, x^2, x^3, x^4, x^5$ , are sitting at  $x^7, x^8, x^9 = 0$  and at the arbitrary value of  $x^6$ . The  $x^6$  position is only well defined classically. The D4 branes are stretched between the fivebranes and their world volume is in  $x^0, x^1, x^2, x^3$  direction; These D4 branes have finite length in  $x^6$  direction. We also have  $D6$  branes which extend in the direction  $x^0, x^1, x^2, x^3, x^7, x^8, x^9$ . Two typical brane configurations are depicted in Figure 1.

There are two different ways to introduce fundamental hypermultiplets to the gauge groups at both ends: we can either attach semi-infinite D4 branes as in Figure 1a) or add  $D6$  branes as in Figure 1b). In this section, we only consider brane configurations with  $D6$  branes. Let's consider a brane configuration with  $n + 1$  NS5 branes and a total of  $k_\alpha$  D4 branes stretched between  $\alpha$ th and  $(\alpha + 1)$ th NS5 brane, the gauge group is  $\prod_{\alpha=1}^n SU(k_\alpha)$ , and there are bifundamental hypermultiplets transforming in the representation  $(k_\alpha, \bar{k}_{\alpha+1})$ ; To make the theory conformal, we need to add  $d_\alpha$  fundamental hypermultiplets to  $SU(k_\alpha)$  gauge group using  $D6$  branes.  $d_\alpha$  is given by

$$d_\alpha = 2k_\alpha - k_{\alpha+1} - k_{\alpha-1}, \quad (2.1)$$

where we understand that  $k_0 = k_{n+1} = 0$ .

The Seiberg-Witten curve for this theory is derived by lifting the above configuration to M theory. The  $D6$  branes are described by Taub-NUT space. NS5 – D4 brane configu-

rations become a single M5 brane embedded in D6 branes background. Define coordinate  $v = x^4 + ix^5$  and polynomials:

$$J_s = \prod_{a=i_{s-1}+1}^{i_s} (v - e_a), \quad (2.2)$$

where  $1 \leq s \leq n$  and  $d_\alpha = i_\alpha - i_{\alpha-1}$ ,  $e_a$  is the constant which represents the position of D6 brane. The Seiberg-Witten curve is

$$\begin{aligned} & y^{n+1} + g_1(v)y^n + g_2(v)J_1(v)y^{n-1} + g_3(v)J_1(v)^2J_2(v)y^{n-2} \\ & + \dots + g_\alpha \prod_{s=1}^{\alpha-1} J_s^{\alpha-s} y^{n+1-\alpha} + \dots + f \prod_{s=1}^n J_s^{n+1-s} = 0, \end{aligned} \quad (2.3)$$

here  $g_\alpha$  is a degree  $k_\alpha$  polynomial of variable  $v$ . The Seiberg-Witten differential is given by  $\lambda = \frac{vdt}{t}$ .

The gauge couplings are determined by  $x^6$  positions of the NS5 branes. If the beta functions for all the gauge groups vanish, the asymptotic behaviors of the roots of Seiberg-Witten curve regarded as a polynomial in  $y$  determine the gauge couplings. In large  $v$  limit, the roots are  $y \sim \lambda_i v^{k_1}$ , where  $\lambda_i$  are the roots of the polynomial equation:

$$x^{n+1} + h_1 x^n + h_2 x^{n-1} + \dots + h_n x + f = 0. \quad (2.4)$$

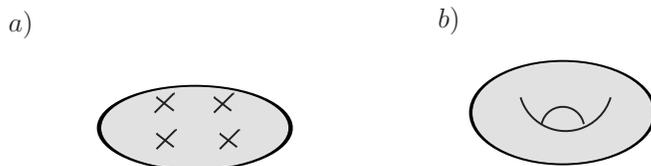
In  $x$  plane, there are  $n + 3$  distinguished points, namely  $0, \infty$ , and  $\lambda_i$ . The choice of  $\lambda_i$  determines the asymptotic distances in coordinate  $x^6$  between fivebranes and hence the gauge coupling constants. The gauge coupling space is the complex structure moduli of the sphere with  $n + 3$  marked points among which  $0, \infty$  are distinguished. Denote the moduli space as  $M_{0,n+3;2}$ , the fundamental group  $\pi_1(M_{0,n+3;2})$  is interpreted as the duality group.

There are two Riemann surfaces in describing the theory: one is used to determine bare gauge couplings and the other determine the low energy effective theory, see Figure 2 for an illustration. Is there any connection between them? Another question is: Can we get four dimensional theory by compactifying certain higher dimensional theory on this Riemann surface with marked points so that  $S$  duality is manifest?

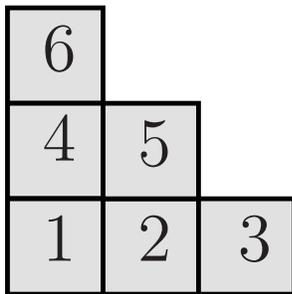
Gaiotto [3] found the connection between these two Riemann surfaces by transforming the Seiberg-Witten curve to the following form

$$x^n = \sum_{i=2}^n \phi_i(z) x^{n-i}, \quad (2.5)$$

here  $\phi_i(z)$  is a degree  $i$  meromorphic differential on the punctured Riemann surface which we use to determine the gauge couplings.  $\phi_i(z)$  has poles at those punctures  $0, \infty, \lambda_i$ . The parameters of these degree  $i$  differential are interpreted as four dimensional field theory operators with scaling dimension  $i$ . The Seiberg-Witten differential is  $\lambda = zdx$ . Put it in another way, the Seiberg-Witten curve is a subspace in the cotangent bundle  $T^*\Sigma$ , where  $\Sigma$  is our punctured Riemann surface.



**Figure 2:** (a) Riemann surface with punctures whose complex structure moduli determines the four dimensional gauge couplings, we take  $SU(2)$  with four fundamentals as an example; (b) Seiberg-Witten curve which determines the low energy effective action, in fact we have a family of these curves which are parameterized by Coulomb branch parameters, it may be degenerate for certain parameter.

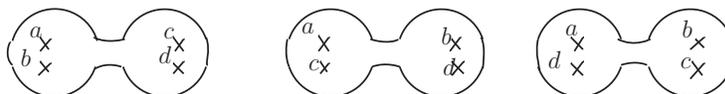


**Figure 3:** A Young tableaux with total of boxes 6, the order of poles of the meromorphic differential  $\phi_i$  are:  $p_1 = 1 - 1 = 0, p_2 = 2 - 1 = 1, p_3 = 3 - 1 = 2, p_4 = 4 - 2 = 2, p_5 = 5 - 2 = 3, p_6 = 6 - 3 = 3$ ; the flavor symmetry associated with this puncture is  $U(1)$ .

To answer the second question, Gaiotto proposed that four dimensional  $N = 2$  SCFT can be derived by compactifying six dimensional  $(0, 2)$   $A_{N-1}$  theory on this punctured Riemann surface. The gauge coupling constants of four dimensional theory only depend on the complex structure of Riemann surface, this means that the  $S$  duality group is identified with the conformal mapping group of complex structure space. We need to turn on defects on the punctures when we do the compactification. With this interpretation, all information about four dimensional theory is encoded in this punctured Riemann surface. In particular, the global symmetry is encoded in the description of the puncture. Gaiotto showed that the punctures are labeled by a partition of  $N$  and can be described by the Young tableaux. A example is shown in Figure 3. If the Young tableaux has  $l_h$  columns with height  $h$ , then the flavor symmetry associated with this puncture is

$$S(\sum_h U(l_h)). \tag{2.6}$$

It is remarkable that this same Young tableaux determine the Seiberg-Witten curve as well. The order of pole of  $\phi_i$  at the puncture labeled by a Young tableaux is  $p_i = i - s_i$ ,



**Figure 4:** The various weakly coupled limit of  $SU(2)$  theory with four fundamental matter. The narrow strip denotes the weakly coupled  $SU(2)$  gauge group. The punctures are associated with flavor symmetry  $SU(2)$ .

where  $s_i$  is the height of  $i$ th box in the Young tableaux, the number of parameters of this differential can be calculated by using Riemann-Roch theorem.

The above results are mainly derived by studying the Seiberg-Witten curve. It is really interesting to find a direct six dimensional description. However, there is no lagrangian description of  $(0, 2)$   $A_N$  theory, we really don't know what is the dynamical equation which governs the compactification. It is the purpose of this paper to provide such a description. Hopefully, we can also get the low energy effective theory from this equation. It will be proven in later sections that this can be done.

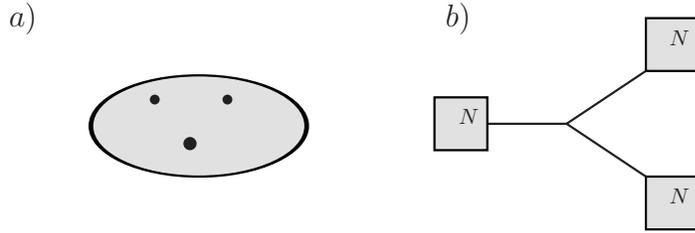
By realizing the four dimensional theory as compactification of six dimensional theory,  $S$  duality of the four dimensional theory is transparent since the compactification only depends on the complex structure of the punctured Riemann surface, and four dimensional theory is invariant under the conformal mapping group of complex structure space of the punctured surface. It is also becoming clear that the different weakly coupled four dimensional theories at the cusp of the coupling space are realized as the different degeneration limits of the punctured Riemann surface. We show an example of  $SU(2)$  theory with four fundamental fields in Figure 4.

We can construct more general four dimensional  $N = 2$  SCFT by using the sphere with three full punctures (the flavor symmetry with this puncture is  $SU(N)$ ). This theory has no gauge couplings and has no lagrangian description for  $N > 2$ ; We call this theory as  $T_N$ , it is shown in Figure 5. Using this build block  $T_N$ , we can construct a large class of generalized quiver gauge theories which don't have conventional lagrangian descriptions, see an example in Figure 6; A  $S$  dual generalized quiver gauge theory of Figure 6 is shown in Figure 7, the quiver corresponds to a different degeneration limit of genus two Riemann surface.

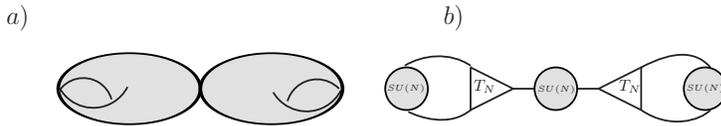
### 3. Hitchin's equation

In this section, we will use brane configuration and string dualities to find Hitchin's equation which is used to describe the Coulomb branch of  $N = 2$  SCFT.

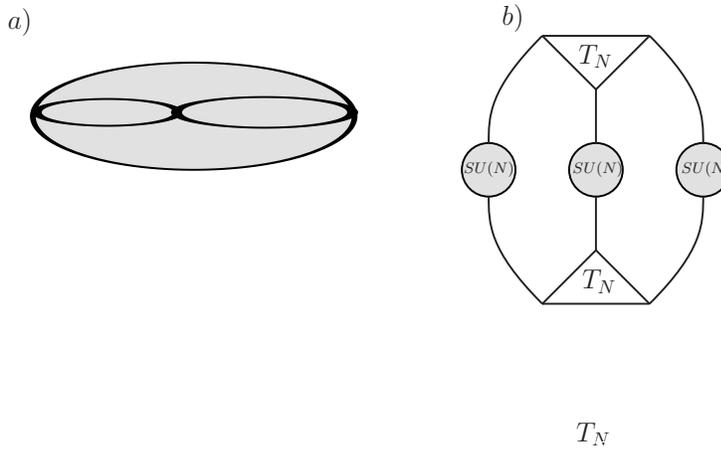
The property of Coulomb branch of  $N = 2$  theory is described by Seiberg-Witten curve. The Coulomb branch of the theory is a special Kahler manifold of complex dimension  $r$ , where  $r$  is the dimension of Cartan algebra of gauge group. The moduli space  $U$  is topologically  $C^r$ . The special Kahler metric is determined by a function: prepotential. This function is multivalued and Seiberg-Witten considered a more invariant description:



**Figure 5:** a) The sphere with three full punctures; b) The graph representation of the theory derived from compactification on a).



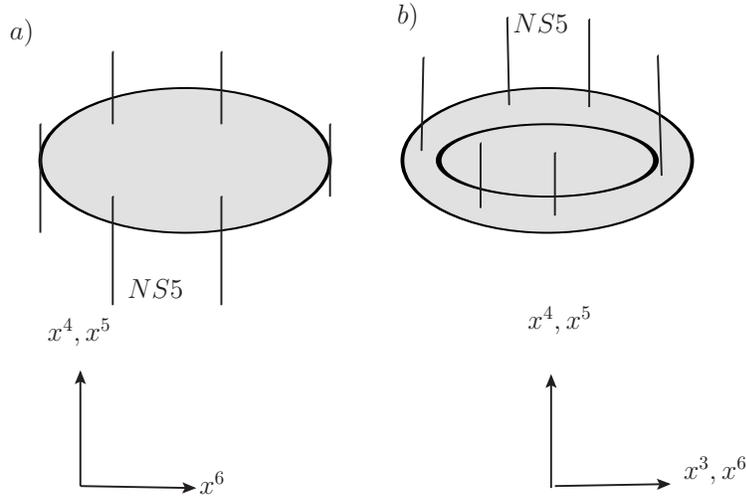
**Figure 6:** a) Genus two Riemann surface without punctures and one of its degeneration limit; there are three nodes and we identify them with gauge groups; note the nodes are equivalent with our previous representation by long necks; b) A generalized quiver description with a six dimensional compactification on a).



**Figure 7:** a) Another degeneration limit of genus two Riemann surface without punctures. b) A generalized quiver corresponding to a).

Seiberg-Witten fibration. Consider a fibration  $\pi : X \rightarrow U$ , where  $X$  is a complex manifold with dimension  $2r$ , and the fibers are Abelian varieties  $A_r$  of dimension  $r$  (in other words,  $A_r$  is a complex Riemann surface whose first homology class has dimension  $2r$ , and we have a  $(1,1)$  form  $t$  which is positive and has integral periods). We also need a holomorphic  $(2,0)$  form  $\omega$  on  $X$  whose restriction to the fibers of  $\pi$  is zero.

The metric on moduli space is calculated by taking a  $(r+1, r+1)$  form  $t^{r+1} \wedge \omega \wedge \bar{\omega}$  and



**Figure 8:** a) The electric brane configuration of elliptic model; b) The magnetic brane configuration of elliptic model.

integrating it over the fibers of  $\pi$ . The  $\omega$  is taken nondegenerate away from the singular fiber. We also need to specify the coupling constant of the low energy theory; these coupling constants are determined by the complex structure constant of the fibers.

On the other hand, we can think of  $X$  as a complex symplectic manifold which is identified with the complex phase space for a mechanical system. Since the restriction of  $\omega$  on the fibre is zero, this is an integrable system. The coordinates on  $U$  are seen as the coordinate, while the coordinates on the fibre are thought of as the conjugate momentum. Thus any  $N = 2$  system corresponds to a certain complex integrable system of classical mechanics.

It is useful to make a direct connection with these two approaches. Kapustin made this connection for a certain  $N = 2$  model [37], we will review his derivation below. We start with the elliptic model studied by Witten [35]. The brane configuration is almost the same as we described in last section with the difference that we take  $x^6$  to be compact, see Figure 8a). The low energy field theory on this system is a four dimensional  $N = 2$  SCFT, it is called elliptic model. The gauge group is  $SU(k)^n \times U(1)$ . The solution of the low energy theory is also solved by using M theory.

Now we compactify further coordinate  $x^3$ , the theory becomes effectively a three dimensional theory. The Coulomb branch of the low energy theory is described as a hyperkahler manifold  $X$  with a distinguished complex structure in which it looks like a fibration  $\pi : X \rightarrow C^r$  with fibers being abelian varieties  $A_r$  with complex dimension  $r$  [38]. We first do a T duality on  $x^3$  and then do a Type IIB S duality and finally another T duality on  $x^3$  coordinate, We finally come back to Type IIA configuration. The NS5 brane becomes a IIB NS5 branes under first T duality; S duality turns it into IIB D5 brane, and the second T duality turns it into IIA D4 brane located at fixed  $x^3, x^6, x^7, x^8, x^9$ , we call it  $D4'$  brane; The original  $D4$  branes are not changed. The whole brane configuration

becomes  $D4 - D4'$  system: D4 brane wrapped on  $x^3, x^6$  torus and  $D4'$  is sitting at a fixed point of the torus. The gauge theory on D4 branes is a  $U(k)$  theory with fundamentals coming from the string stretched between the D4 branes and  $D4'$  branes. The Coulomb branch of the original theory is matched to Higgs branch of the dual theory which does not receive quantum corrections and can be calculated classically. The theory on D4 branes are five dimensional Super Yang-Mills (SYM) theory and  $D4'$  branes are codimension two impurities on this five dimensional theory.

We now make a connection with Gaiotto's description. The three dimensional theory in the dual system is interpreted as five dimensional SYM theory compactified on a torus with impurities, those impurities are coming from  $D4'$  branes. Now let's lift it to M theory. There is another compact coordinate  $x^{10}$ , and the original D4 branes are M5 branes wrapped on a three torus  $T^3$ . In the dual description, we have M5 branes which from D4 branes wrapped on  $x^3, x^6, x^{10}$  and M5 branes from  $D4'$  branes are sitting at a point of the torus  $x^3, x^6$ . The effective theory can be interpreted as M5 branes compactified on a three dimensional manifold: We first compactify  $n$  M5 branes on a circle and then compactify it on a punctured torus, we get a three dimensional theory. Interesting things happen if we change the order of compactification, we first compactify it on a punctured torus, and then on a circle, we can get the same three dimensional theory. We also assume the description of the theory on punctured torus is not changed. Now we take the circle to infinity and get back to a four dimensional theory, and we get the picture discovered by Gaiotto: Four dimensional  $N = 2$  theory can be described as  $n$  M5 branes compactified on a punctured torus, see Figure 8b). The second series of the compactification can be interpreted as regarding  $x^3$  as the M theory cycle.

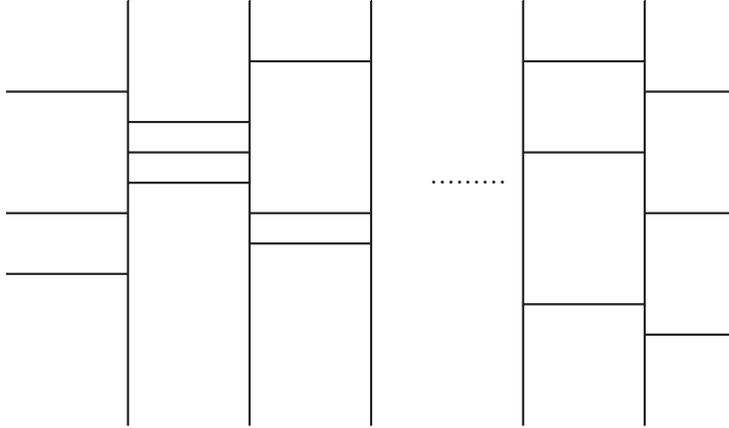
Now the Coulomb branch of the original theory is mapped to Higgs branch of the five dimensional theory on a torus with punctures. The moduli space of Higgs branch is described by the Hitchin's equation [39]:

$$F_{z\bar{z}} - [\Phi_z, \Phi_z^+] = 0 \quad (3.1)$$

$$\bar{D}\Phi_z = \frac{-\pi}{RL} \sum_{\alpha=1}^k \delta^2(z - z_\alpha) \text{diag}(m_\alpha, -M, \dots, -M). \quad (3.2)$$

There are source terms coming from the  $D4'$  impurities. The connection between the description of Hitchin's equation in dual picture and Seiberg-Witten curve in original picture is that the Seiberg-Witten curve is described by the spectral curve of the moduli space to Hitchin's equation, see the early attempt of using Hitchin's equation to derive Seiberg-Witten curve [36]. Hitchin's equation is an integrable system and this answers the initial question to make a connection between Seiberg-Witten curve and integrable system. We hope to generalize this to a large class of  $N = 2$  SCFT, see [40] for extensive study for the use of the Hitchin's equation on  $N = 2$  wall crossing.

Now we have two different descriptions of the same four dimensional gauge theory, in the original description, gauge groups and matter contents are explicitly described, we can define and study all kinds of observable in this description by using conventional field theory technics. The dual description is realized as the six dimensional theory compactified



**Figure 9:** The brane configuration for  $N = 2$ ,  $\prod_{\alpha=1}^n SU_{\alpha}(3)$  superconformal field theory.

on punctured Riemann surface, the gauge group description is obscure but the  $S$  duality property is clear.  $S$  duality is realized as the fundamental group of complex structure moduli space of the punctured torus. The Seiberg-Witten curve is also easily written by exploring the Hitchin's equation.

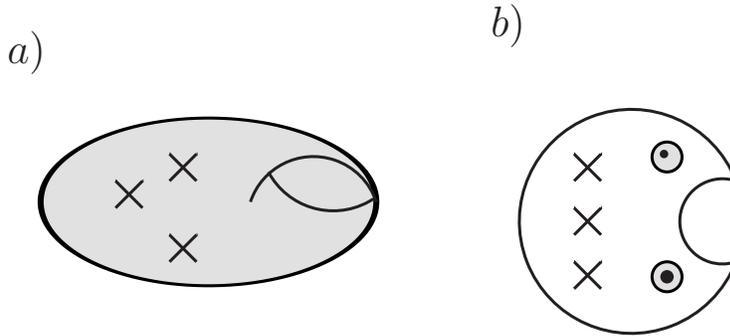
We can also give this kind of description to the  $A_N$  type conformal quiver. We first consider the simplest case with  $n$   $SU(N)$  gauge groups. We have bifundamental hypermultiplets between the adjacent gauge groups and we need to add  $N$  fundamental hypermultiplets at the both ends to make the whole theory conformal, see Figure 9 for a brane configuration. This model can be derived from the elliptic model with  $n + 1$   $SU(N)$  gauge group: we decouple one of the  $SU(N)$  gauge group of the elliptic model and it becomes the linear quiver we are interested in. The elliptic model is described as the six dimensional  $(0, 2)$  theory compactified on a torus with  $n + 1$  punctures. The decoupling of one gauge group means a complete degeneration of the torus, and we are left a sphere with five punctures, the extra two punctures come from the degeneration of the torus. We put those two extra punctures at  $0, \infty$ . This is exactly the compact Riemann surface to describe the linear quiver, see Figure 10 for illustration.

It is conceivable that Coulomb branch of the linear quiver is still described by Hitchin's equation with the source terms coming from the punctures, the difference here is that the source terms at  $0, \infty$  are different from other punctures. The Seiberg-Witten curve is described by the spectral curve of the Hitchin system.

This conjecture may be seen from the Seiberg-Witten curve. The linear curve is described in Figure 9, and the Seiberg-Witten curve of the massless theory is [35]

$$v^N t^{n+1} + g_1(v)t^n + \dots + g_{\alpha}t^{n+1-\alpha} \dots + v^N = 0, \quad (3.3)$$

where  $g_{\alpha}(v) = v^N + c_2 V^{N-2} + \dots c_N$ . It is easy to see that coefficients of  $t^k$  have the same order in  $v$ , so this curve can be described as the spectral curve. We can find the appropriate boundary conditions on the puncture of the Hitchin's equation by mapping the above curve



**Figure 10:** a) Torus with 3 punctures; b) After degeneration, we have a sphere with 5 punctures, the two punctures coming from the degeneration of torus are different from other punctures.

to the spectral curve of the Hitchin system [40]. When we turn on the mass, it can be shown similarly that there is a Hitchin system description of the Seiberg-Witten curve.

For the general linear quiver gauge theory we described in section II, it is not obvious that we can still use Hitchin's equation and write the Seiberg-Witten curve as the spectral curve. However, we can also give a heuristic argument that this is possible by transforming Seiberg-Witten curve. The Seiberg-Witten curve does not depend on the  $x^6$  position of the  $D6$  branes so we can move the  $D6$  branes to  $x^6 = \infty, -\infty$ . There is Hanany-Witten effect [41]: when we move the  $D6$  branes across the NS5 branes,  $D4$  branes will be created, and the initial configuration is equivalent to a brane configuration without the  $D6$  branes, the number of  $D4$  branes are different now. See Figure 11 for an example. The brane configuration with  $D6$  branes moved to infinity is exactly the same as the linear quiver we just studied in Figure 9, so we conclude that we can still use Hitchin's equation to describe the Coulomb branch of the general linear superconformal quiver. The boundary conditions at the  $0, \infty$  are different here, since the semi-infinite branes are attached to the  $D6$  branes sitting at  $\infty$ , this might give us different boundary conditions at  $0$  and  $\infty$  from the ones we just studied.

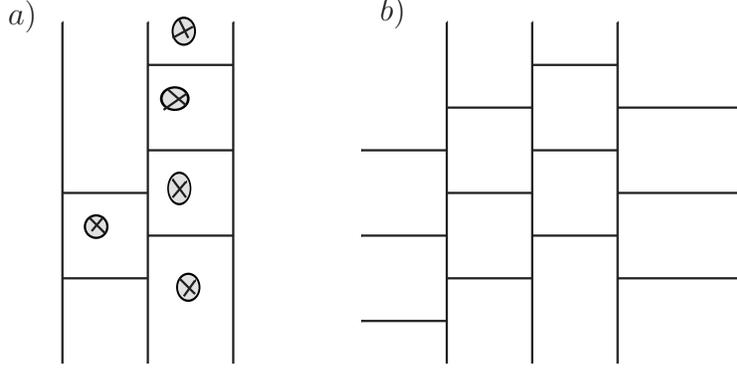
The Hanany-Witten effect may be seen by doing some transformation on Seiberg-Witten curve. The Seiberg-Witten curve is

$$\begin{aligned}
 & y^{n+1} + g_1(v)y^n + g_2(v)J_1(v)y^{n-1} + g_3(v)J_1(v)^2J_2(v)y^{n-2} \\
 & + \dots + g_\alpha \prod_{s=1}^{\alpha-1} J_s^{\alpha-s} y^{n+1-\alpha} + \dots + f \prod_{s=1}^n J_s^{n+1-s} = 0.
 \end{aligned} \tag{3.4}$$

We study the massless theory for simplicity, so we put all the  $D6$  branes at  $v = 0$ . The number of fundamental hypermultiplets are given by  $d_\alpha = 2k_\alpha - k_{\alpha-1} - k_{\alpha+1} \geq 0$ , and we have the following relation on the rank of the gauge group:

$$k_1 \leq k_2 \dots = k_r = \dots k_s \geq k_{s+1} \dots \geq k_n, \tag{3.5}$$

we define  $N = k_r = \dots k_s$ . The total number of  $D6$  branes is given by  $\sum_\alpha d_\alpha = k_n + k_1$ . We need to redefine the  $y$  coordinate so that the coefficient of  $y'^{n+1}$  is the same as the constant



**Figure 11:** a) Brane configurations with  $D6$  branes sitting between the NS5 branes; b) Equivalent brane configuration when we move all the  $D6$  branes to the infinity.

term. Define  $y = v^a y'$ , then the coefficient of  $y'^{n+1}$  is  $v^{an+a}$ . Examine the constant term:

$$\prod_{s=1}^n J_s^{n+1-s} = v^{d_1 n} v^{d_2(n-1)} \dots v^{d_n} = v^{(\sum_{i=1}^n d_i)n - \sum_{i=2}^n (i-1)d_i} = v^{(k_1+k_n)n + (k_1 - nk_n)} = v^{nk_1+k_1}. \quad (3.6)$$

$\sum_{i=1}^n d_i = k_1 + k_n$  and  $\sum_{i=2}^n (i-1)d_i = -k_1 + nk_n$  are used. We conclude that  $a = k_1$ . Substituting  $y = y' v^{k_1}$ , the coefficient before  $y'^{n+1-\alpha}$  is

$$c_\alpha = g_\alpha(v) v^{d_1(\alpha-1)} v^{d_2(\alpha-2)} \dots v^{d_{\alpha-1}} v^{k_1 n + k_1 - k_1 \alpha}. \quad (3.7)$$

Calculating the exponent carefully, one finds

$$c_\alpha = g_\alpha v^{(n+1)k_1 - k_\alpha}. \quad (3.8)$$

Recall that  $g_\alpha$  is degree  $k_\alpha$  polynomial in  $v$ , this means that all  $c_\alpha$  has the same order  $(n+1)k_1$ , we can have a spectral curve description! The maximal value of  $k_\alpha$  is  $N$ , we can factorize out  $v^{(n+1)k_1 - N}$  for each coefficient and we are left with a Seiberg-Witten curve with the form:

$$v^N y'^{n+1} + \sum_{\alpha=1}^n g'_\alpha(v) y'^{n+1-\alpha} + v^N = 0, \quad (3.9)$$

where  $g'_\alpha$  is a degree  $N$  polynomial in  $v$ , this is exactly the same form as the Seiberg-Witten curve of the linear quiver shown in Figure 9, so we can use Hitchin's equation to describe the general linear superconformal quiver!

There is a more general way in which we can see the emergence of the Hitchin's equation. Consider six dimensional  $(0, 2)$   $A_{N-1}$  SCFT compactified on a punctured Riemann surface  $\Sigma$  (the following analysis is also true for a Riemann surface without singularity), we get a four dimensional  $N = 2$  SCFT. To preserve some supersymmetry on the curved manifold, we actually need to twist the six dimensional theory [3]. We can further compactify four dimensional theory on a torus  $T^2$ , and the two dimensional theory is a sigma model.

The first step of compactification is hard to study since there is no lagrangian description of the six dimensional theory. Things become clear if we do the compactification in reverse order in the same spirit as we did in previous analysis on  $D^4$  brane system. We first compactify on  $T^2$ , and then further on a punctured Riemann surface down to two dimension. We obtain four dimensional  $N = 4$  theory with gauge group  $SU(N)$  when we compactify the theory on  $T^2$ , we then compactify the theory on a two dimensional Riemann surface  $\Sigma$  and get a sigma model on two dimension. To preserve some supersymmetry, we also need to twist  $N = 4$  theory, there are different kinds of twist we can make. The twist which is relevant for our purpose is the so called GL twist [44], it turns out that Hitchin's equation is the equation for the BPS condition. The theory can also be extended to the case that the fields have singularities [13, 45, 46]. In the language of Geometric Langlands program, the case with simple pole is called tame ramification and the case with higher order pole is called wild ramification. By comparing two different kinds of compactification, we may conjecture that Hitchin's equation is the BPS equation governing the compactification of six dimensional theory on a Riemann surface (with or without singularities). In this paper, we only consider the Hitchin's equation without singularity and with simple singularity and leave the wild singularity for future analysis.

#### 4. $A_1$ theory

In last section, we conjecture that it is Hitchin's equation which is relevant when we compactify six dimensional  $(0,2)$  theory on a punctured Riemann surface. We will show in this section that Hitchin's equation provides an description of both UV theory and IR theory for four dimensional SCFT.

We first analyze four dimensional  $N = 2$   $SU(2)$  SCFT. These theories can be described as the six dimensional  $(0,2)$   $A_1$  theories compactified on punctured Riemann surface. As we described in section 2, Gaiotto proposed that the Seiberg-Witten curve has the form

$$x^2 = \phi_2(z), \tag{4.1}$$

where  $z$  is the coordinate on punctured Riemann surface. For massless theory,  $\phi_2(z)$  has simple pole at various punctures, and near the puncture, say  $z = 0$ ,

$$\phi_2(z) = \frac{c}{z}. \tag{4.2}$$

The Seiberg-Witten curve for massive theory is also of the form (4.1), but the degree 2 differential near the puncture now takes the form [3]:

$$\phi_2(z) = \frac{q}{z^2} + \frac{c}{z}. \tag{4.3}$$

As we discussed in last section, the four dimensional gauge theory is controlled by Hitchin equation on a Riemann surface with possible source terms. In order to understand the four dimensional  $N = 2$  SCFT, we need to solve the Hitchin's equation with the added source terms. It turns out that for  $N = 2$  SCFT we reviewed in section 2, 3, the fields should have the simple singularities at various punctures.

## 4.1 Singular Solutions to Hitchin's Equation

In previous sections, we see from brane construction that there are added source terms on the right-hand side of Hitchin's equation due to other branes intersecting at the puncture; These source terms induce the singularity to the solution of Hitchin's equation. An alternative point of view is to study the singular solution of the Hitchin's equation without the source terms, and this will tell us enough information about the singularity. The same problem has been studied by Witten and Gukov in the physics approach to Geometric Langlands problem for the tame ramification case (see extensive study in [4, 45]), we give a review below for the necessary information we need.

The Hitchin's equation for  $SU(2)$  gauge group [42, 43] is

$$\begin{aligned} F_A - \phi \wedge \phi &= 0 \\ d_A \phi &= 0, \quad d_A * \phi = 0, \end{aligned} \tag{4.4}$$

here  $A$  is a connection for a  $SU(2)$  bundle on Riemann surface and  $\phi$  is a one form taking value on adjoint bundle and we call it Higgs field;  $d_A$  is the familiar covariant derivative. We first consider the solution without any singularity, the moduli space of solutions is a hyperkahler manifold with three complex structures  $I, J, K$ , and it has a hyper-kahler quotient description. In complex structure I, a solution of Hitchin's equation on a Riemann surface describes a Higgs bundle, that is a pair  $(E, \phi)$ , where  $E$  is a holomorphic G-bundle and  $\phi$  is a holomorphic section of  $K_C \otimes ad(E)$  (here  $K_C$  is the canonical bundle on  $C$ ). The Higgs bundle is constructed as follows: we interpret the  $(0, 1)$  part of the covariant derivative  $d_A$  as a  $\bar{\partial}_A$  operator that gives the bundle  $E$  a holomorphic structure. We denote  $\Phi$  as the  $(1, 0)$  part of the Higgs field  $\phi$ , and Hitchin's equation implies that  $\phi$  is holomorphic.

If we study the solution of Hitchin's equation with singularity at the origin, the moduli space of solutions still has hyper-kahler structure. In complex structure I, the solution describes a Higgs bundle but the Higgs field has a pole at the origin. We only consider local behavior of the solution and leave the global property for future study. We choose local holomorphic coordinate  $z = re^{i\theta}$  around the singularity. We only consider regular singularity here (the solution has simple pole at  $r = 0$ ), irregular singularity is important when we study asymptotically free theory. We consider the superconformal theory, so we need to find conformal invariant solutions to Hitchin's equation. The most general scale-invariant and rotation-invariant solution is

$$\begin{aligned} A &= a(r)d\theta + f(r)\frac{dr}{r} \\ \phi &= b(r)\frac{dr}{r} - c(r)d\theta. \end{aligned} \tag{4.5}$$

$f(r)$  can be set to zero by a gauge transformation and after introducing a new variable  $s = -\ln r$ , Hitchin's equation becomes Nahm's equations:

$$\begin{aligned} \frac{da}{ds} &= [b, c] \\ \frac{db}{ds} &= [c, a] \end{aligned}$$

$$\frac{dc}{ds} = [a, b]. \quad (4.6)$$

The most general conformal invariant solutions are derived by setting  $a, b, c$  to constant  $\alpha, \beta, \gamma$  of the Lie algebra of  $SU(2)$ , and they must commute and we can conjugate them to lie algebra of a maximal torus of  $SU(2)$ . The resulting solution is

$$\begin{aligned} A &= \alpha d\theta + \dots \\ \phi &= \beta \frac{dr}{r} - \gamma d\theta + \dots \end{aligned} \quad (4.7)$$

We ignore possible terms which are less singular than the terms presented above.

We also want to know the behavior of the solution when we take  $\alpha, \beta, \gamma \rightarrow 0$ . One may think that there is no singularity at all. This is not the case if we note that we may have less singular terms to the equation. When  $\alpha, \beta, \gamma \rightarrow 0$ , those less singular terms play dominant role.

Indeed, we do have less singular solution to Hitchin's equation, the Nahm's equations can be solved by:

$$a = -\frac{t_1}{s + 1/f}, \quad b = -\frac{t_2}{s + 1/f}, \quad c = -\frac{t_3}{s + 1/f}, \quad (4.8)$$

where  $s = -\ln r$  and  $[t_1, t_2] = t_3$  and cyclic permutation thereof which are the usual commutation relations for  $SU(2)$  lie algebra. A convenient basis for  $SU(2)$  is

$$e_1 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \quad (4.9)$$

The choice of  $f$  spoils conformal invariance, but it is not natural to make a choice, since then the derivative of  $A$  and  $\phi$  with respect to  $f$  is square-integrable. So this solution with  $f$  allowed to fluctuate is conformal invariant. The advantage of including this parameter is that when  $f = \infty$ , we get the trivial solution. Combined the previous discussion, the second type of solution can be thought of the zero limit of the first type.

The fact that the second type of solution is a limit of the first type of solution can also be seen by studying moduli space of Hitchin's equation in complex structure  $J$ . In complex structure  $J$ , solution of Hitchin's equation describes a flat  $SL(2, C)$  bundle. It is important to study the monodromy of the flat connection. Define complex-valued flat connection  $\mathcal{A} = A + i\phi$  taking value in  $SL(2, C)$ , the monodromy is

$$U = P \exp\left(-\int_l \mathcal{A}\right), \quad (4.10)$$

where  $l$  is the contour surrounding the singularity. This monodromy characterizes the singular behavior of the solution. The curvature  $\mathcal{F}$  defined as  $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$  is equal to zero due to Hitchin's equation, so the monodromy calculates as above is independent of the contour we choose. Define  $\zeta = \alpha - i\gamma$ , the monodromy for our first set of solutions (4.7) is

$$U = \exp(-2\pi\zeta). \quad (4.11)$$

The monodromy of another solution (4.8) is

$$U' = \exp(-2\pi(t_1 - it_3)/(s_1 + 1/f)). \quad (4.12)$$

The conjugacy class of this matrix is independent of  $s_1$  due to the property that  $(t_1 - it_3)$  can be taken as up triangular form. We choose a basis in which  $t_2 = e_1$ ,  $t_1 = e_3$ , and  $t_3 = e_2$ .

Indeed, what's relevant is the conjugacy class for the monodromy. Let's denote the conjugacy class for  $U$  as  $C_\zeta$ , and the conjugacy class for  $U'$  is a union of  $C_0$  and  $C'$ , where  $C_0$  is the conjugacy class for identity and  $C'$  is the conjugacy class for unipotent orbit. It can be shown that as  $\zeta \rightarrow 0$ ,  $C_\zeta$  approaches the union of  $C_0$  and  $C'$ . This also indicates that the second set of solutions is a limit of the first set of solutions.

With this relation, we might think that the conjugate class of  $C'$  is associated with massless  $N = 2$  SCFT while  $C_\zeta$  is associated with mass-deformed theory. When we compactify six dimensional  $A_1$  theory on punctured Riemann surface, Hitchin's equation is the dynamical equation we need to solve, and the information of four dimensional  $N = 2$  gauge theory is encoded in the solutions to Hitchin's equation. Moreover, we will prove that we can read the flavor symmetry and tail of the quiver gauge theory from the solutions; and the Seiberg-Witten curve is the spectral curve of the Hitchin system. The UV and IR information are both encoded in the same system. In the following parts of this section, we will confirm this conjecture.

## 4.2 Massless Theory and Mass deformed theory

Some group theoretical definitions are useful for our later use. An element of a complex Lie group is called semisimple if it can be diagonalized (or conjugated to a maximal torus). The conjugate class of this element is called semisimple accordingly and it is closed. This element can be expressed as  $U = \exp(\tilde{n})$ , where  $\tilde{n}$  is the semisimple element of the lie algebra ( $\tilde{n}$  is diagonalizable). In contrast, an element  $U'$  is called unipotent if in any finite representation, it takes the form  $U' = \exp(n)$ , where  $n$  is a nilpotent element of the lie algebra. From now on, we will work on  $sl(2, c)$  lie algebra. Let's discuss the conjugacy classes of  $sl_2$  lie algebra to give a concrete idea about the concepts we just discussed. For each  $X \in sl_2$ , we can form the conjugacy class ( sometimes we call them orbit)

$$\mathcal{O}_X = \{A.X.A^{-1} | A \in GL_2\}. \quad (4.13)$$

Using  $tr(AB) = tr(BA)$ , the  $GL_2$  conjugate of an element is also in  $sl_2$ .

The semisimple elements are

$$X(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (4.14)$$

where  $\lambda = \Lambda/(0)$ , and  $\Lambda = \{C/\{\lambda \sim -\lambda\}\}$ . We also have two nilpotent elements

$$Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.15)$$



**Figure 12:** (a) Young tableaux of nilpotent element  $Y_1$ ; (b) Young tableaux of nilpotent element  $Y_2$ .

$Y_2$  is special since it is both semisimple and nilpotent. We call  $Y_1$  as regular nilpotent element and its orbit regular nilpotent orbit. Those nilpotent orbit may be labeled as Young tableaux as in Figure 12.

We can prove that the partition of  $sl_2$  algebra has the form, see [47]:

$$sl_2 = \bigcup_{\lambda} \mathcal{O}_{X(\lambda)} \bigcup \mathcal{O}_{Y_1} \bigcup \mathcal{O}_{Y_2}. \quad (4.16)$$

There are infinite number of semisimple conjugacy classes and we only have two nilpotent conjugacy classes. We will prove that four dimensional massless  $N = 2$  SCFT theory is associated with the nilpotent class of the lie algebra and mass-deformed theory is associated with the semisimple class.

Let's go back to Hitchin's equation. In complex structure I, the moduli space depends on the complex structure of the Riemann surface  $\Sigma$ , since we want to identify the complex structure as the gauge coupling constants, and the low energy effective theory depends on those parameters, we will work on complex structure I. In this complex structure, there is famous Hitchin's fibration which we will identify as the Seiberg-Witten fibration.

The spectral curve for Hitchin system [43] is

$$\det(x - \Phi(z)) = 0, \quad (4.17)$$

where  $\Phi$  is the  $(1, 0)$  part of the Higgs field, and the natural differential  $\lambda = xdz$  is identified with the Seiberg-Witten differential.  $\Phi(z)$  is a degree one differential on the Riemann surface, so when we expand the determinant, the coefficient of  $x^{n-i}$  is a degree  $i$  differential on Riemann surface. It is easy to see that the characteristic polynomial only depends on the conjugate class of  $\Phi$ . This spectral curve is conjectured to be the Seiberg-Witten curve of the  $N = 2$  system. We will analyze the behavior of spectral curve near the singular point  $z = 0$ . Let's first analyze the solution (4.8)( $f$  is not taken as 0), it seems that there is no simple pole for  $\Phi$ , since it is less singular than  $\frac{1}{r}$ . However, whether there is a simple pole for  $\phi$  depends on the local trivialization of the holomorphic bundle. The  $(0, 1)$  part of the gauge field and  $(1, 0)$  part of the Higgs field are :

$$A_{\bar{\omega}} d\bar{\omega} = \frac{d\bar{\omega}}{2(\omega + \bar{\omega})} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.18)$$

$$\Phi d\omega = -i \frac{d\omega}{\omega} \frac{\omega}{(\omega + \bar{\omega})} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (4.19)$$

We choose  $t_1 = e_1$ ,  $t_2 = e_2$ ,  $t_3 = e_3$ , and  $w = \ln z$  and  $s = -\ln r = -\frac{1}{2}(\omega + \bar{\omega})$ ; we take  $f = \infty$  here.

The  $\bar{\partial}_A$  operator is

$$\bar{\partial}_A = d\bar{\omega} \left( \frac{\partial}{\partial \bar{\omega}} + A_{\bar{\omega}} \right) = d\bar{\omega} \left( \frac{\partial}{\partial \bar{\omega}} + \frac{1}{2(\omega + \bar{\omega})} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \quad (4.20)$$

We can do a local trivialization to make this operator into standard form:

$$\bar{\partial}_A = f \bar{\partial} f^{-1}, \quad (4.21)$$

where  $\bar{\partial} = d\bar{\omega} \partial / \partial \bar{\omega}$  is the standard  $\bar{\partial}$  operator, and

$$f = \begin{pmatrix} \left( \frac{\omega + \bar{\omega}}{\omega} \right)^{-1/2} & 0 \\ 0 & \left( \frac{\omega + \bar{\omega}}{\omega} \right)^{1/2} \end{pmatrix}. \quad (4.22)$$

With this trivialization, the Higgs field becomes

$$f^{-1} \Phi f = -i \frac{d\omega}{\omega} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (4.23)$$

We conclude that the Higgs field has a simple pole at the singularity, and its residue is proportional to the nilpotent element. We can also add the regular constant terms to the solution, and the full solution is

$$\Phi(\omega) d\omega = \begin{pmatrix} a & \frac{1}{\omega} + b \\ c & -a \end{pmatrix} d\omega + \omega(0) d\omega. \quad (4.24)$$

Now let's analyze the spectral curve associated with this solution. Expand the determinant, we get the equation:

$$x^2 = Tr \Phi^2. \quad (4.25)$$

Calculate the trace and take the singular part, we have the explicit form

$$x^2 = \frac{c}{\omega}. \quad (4.26)$$

This is exactly the Seiberg-Witten curve for the  $SU(2)$  theory around one of the singularity labeled by the Young tableaux in Figure 12a). The coefficient of the pole comes from the regular term.

Consider the other solution (4.7), the holomorphic part of the Higgs field is

$$\Phi = \frac{1}{2}(\beta + i\gamma) \frac{dz}{z}. \quad (4.27)$$

We also need to include less regular terms; In proper basis, the Higgs field has the form

$$\Phi = \begin{pmatrix} \frac{q}{z} & \frac{1}{z} \\ 0 & -\frac{q}{z} \end{pmatrix} dz + \mathcal{O}(0) dz. \quad (4.28)$$

It can be checked that the residue of the Higgs field is conjugate with the regular semisimple element:

$$\begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}. \quad (4.29)$$

Include the constant regular terms, Higgs field takes the following form:

$$\Phi(z)dz = \begin{pmatrix} a + \frac{q}{z} & \frac{1}{z} + b \\ c & -\frac{q}{z} - a \end{pmatrix} dz + \mathcal{O}(z)dz. \quad (4.30)$$

The spectral curve around this singularity becomes

$$x^2 = \frac{2q^2}{z^2} + \frac{d}{z}. \quad (4.31)$$

This is exactly same as the Seiberg-Witten curve for  $SU(2)$  theory around the singularity. The parameter for  $\frac{1}{z^2}$  term comes from the parameter for the most singular part which is fixed, while the parameter for  $\frac{1}{z}$  term comes from the regular term.

### 4.3 Flavor Symmetry

To understand what is the flavor symmetry associated with the massless theory, we need to study the local singularity type of moduli space of Hitchin's equation. The reason we study the singularity type in moduli space and identify the flavor symmetry as the singularity type may be explained by further compactifying our four dimensional theory to three dimensions. The moduli space of Hitchin's equation now has a physical meaning: it is the target space of the coulomb branch of three dimensional theory [38]. The flavor symmetry of four dimensional theory appears as the singularity type in the coulomb branch of three dimensional theory (see also the discussion in [50]). There are some subtleties about  $U(1)$  factors though, we will discuss this in next section since it is not relevant in this section.

Let's examine again the solution (4.8).  $f$  takes values in  $R^+ = [0, \infty)$ . There is a limit for  $f \rightarrow 0$ , namely the trivial solution  $a = b = c = 0$ . We can also pick an element  $R \in SO(3)$ , and generalize that solution to

$$\begin{aligned} a &= -\frac{1}{s + f^{-1}} R t_1 R^{-1} \\ b &= -\frac{1}{s + f^{-1}} R t_2 R^{-1} \\ c &= -\frac{1}{s + f^{-1}} R t_3 R^{-1}. \end{aligned}$$

So the parameter space of this family is  $R^+ \times SO(3) = \frac{C^2}{Z_2}$ , there is a singularity at the origin which corresponds to the trivial solution (when we consider only the non-trivial solution, the parameter space is  $C^2/Z_2 - \{0\}$ ). It is well known that the space  $C^2/Z_2$  is described by the equation

$$a^2 + bc = 0. \quad (4.32)$$

There is a  $A_1$  singularity at the origin, so we identify the flavor symmetry as  $SU(2)$ . In fact, this space has a nature hyper-Kahler structure. We can understand this result from group theory. In fact, Kronheimer has found an isomorphism between the moduli space of solution to Nahm's equation with the closure of nilpotent orbit [48]. The interested reader can find more information on appendix I. The nilpotent orbit of  $sl(2, c)$  consists of matrix with the condition  $\det(x) = 0$ , write a matrix in the form

$$x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \quad (4.33)$$

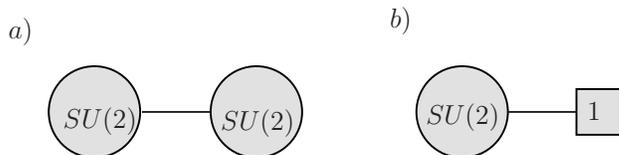
Calculate the determinant, we have the equation  $a^2 + bc = 0$ , this is exactly the space we found before; we can also recognize that the identity element is at the origin of this space as we need to set  $a = b = c = 0$  simultaneously. The whole space is the closure of regular nilpotent orbit. The closure of the regular nilpotent element contains identity orbit and regular nilpotent orbit; The identity orbit is at the boundary of the closure and is of the codimension two. The closure has a rational singularity at the identity orbit which is at the origin of the closure. The appearance of the codimension two singularity is a generic phenomenon for the geometry of closure of nilpotent orbit. This fact is important when we discuss the flavor symmetry for  $A_{N-1}$  theory.

For the solution (4.7), We also need to study the moduli space of solution of Nahm's equations with the constraints that the solution is asymptotically as (4.7) (see appendix I for more precise explanation). To understand what this space is, we also need to apply the group theoretical result. Kronheimer also found an isomorphism between the moduli space with the regular semi-simple orbit [49] (see discussion in appendix I) . In  $sl(2, c)$ , the regular semisimple orbit is characterized as  $\det(x) = q^2$ ; Express the matrix in the equation (4.33), we find the equation  $a^2 + bc = q^2$ , this is the deformation of the  $A_1$  space we discussed previously! From the four dimensional  $N = 2$  field theory point of view, the solution (4.7) corresponds to the mass-deformed theory. The dimension of the nilpotent orbit and the semi-simple orbit are the same, namely, complex dimension two; this means that a generic element in nilpotent orbit can be deformed to a generic semisimple element.

It is time now to describe what kind of quiver tail in four dimension in which we can find the corresponding flavor symmetry. There are two types of tail in four dimensional  $SU(2)$  quiver which give the  $SU(2)$  flavor symmetry. One type of the matter hypermultiplet is the bifundamental fields between two  $SU(2)$  gauge group, and the other type of tail is a  $SU(2)$  quiver with one fundamental. They are shown in Figure 13. For the second type of quiver tail, we may read the form from the Young tableaux associated with the nilpotent element. The rule is that: the rank of first gauge group is the number of boxes  $r_1$  in the first row, the second gauge group has the rank  $r_2 - r_1$ , etc. we have one single  $SU(2)$  gauge group for the regular nilpotent element; we then add fundamentals to make the theory conformal.

#### 4.4 Summary

Up to now, we only consider the nilpotent element  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ , one may ask what happened



**Figure 13:** (a) Bifundamental hypermultiplet between two adjacent  $SU(2)$  gauge groups gives  $SU(2)$  flavor symmetry. (b) A fundamental hypermultiplet of a  $SU(2)$  gauge group also gives  $SU(2)$  flavor symmetry.

to the other nilpotent element  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ . We can calculate the spectral curve associated with this solution and find that there is no pole in  $Tr(\Phi)^2$ . We might want to ask what is the flavor symmetry associated with this solution, as we learn from the regular nilpotent element, we need to find the semi-simple element which has the same dimension as the identity nilpotent element. The identity orbit is zero dimensional, and the only semi-simple orbit with this dimension is the identity orbit itself, this means that the identity orbit is rigid, it is against deformation to any other orbits, we therefore claim that there is no flavor symmetry associated with this solution. According to our rule of identifying the four dimensional quiver tail by using Young tableaux, we may associate a quiver tail with a gauge group with rank one and another gauge group with rank 2; namely, the quiver tail has the form  $SU(2) \times SU(1)$ ; the  $SU(1)$  gauge group might appear bizarre, but it has brane interpretation, see discussion in [7]. It is easy to see that there is no flavor symmetry associated with this tail.

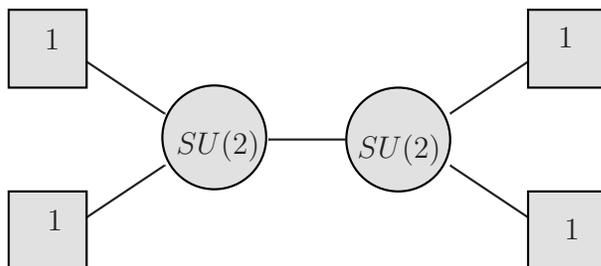
We only consider one singularity up to know, in general,  $n$  singularities can be allowed.  $\Phi$  is a degree 1 differential on Riemann surface, so  $Tr(\Phi)^2$  is a degree two differential on Riemann surface. For massless theory, the pole structure at the singularity is the same as local analysis. The degree  $d$  differential  $\Phi_d$  with the prescribed singularity has the dimension

$$\text{moduli of } \phi_d = \sum_{\text{punctures}} p_d^{(i)} + 3g + 1 - 2d, \quad (4.34)$$

where  $p_d^{(i)}$  is the pole structure of the  $i$ th puncture,  $g$  is the genus of the Riemann surface. The moduli space of the complex structure of the punctured space is  $3g + n - 3$ , so we have  $3g + n - 3$  gauge groups. Since there is one  $SU(2)$  flavor symmetry for each puncture, the theory has a total of  $n$   $SU(2)$  flavor symmetry.

We can construct four dimensional  $N = 2$  quiver gauge theory by using the building blocks associated with one  $SU(2)$  flavor symmetry. For instance, if  $g = 0, n = 5$ , we have two gauge groups and 5  $SU(2)$  flavor symmetries. The quiver gauge theory in one S-dual frame is shown in Figure 14.

The number of moduli of degree two differential has dimension 2; The Seiberg-Witten



**Figure 14:** One four dimensional quiver diagram with a six dimensional  $A_1$  theory compactified on a sphere with 5 punctures, we have two gauge groups and five  $SU(2)$  flavor symmetries.

differential for this theory is

$$x^2 = Tr(\Phi)^2 = \frac{a_1 z + a_2}{\sum_{i=1}^5 (z - z_i)}. \quad (4.35)$$

where  $a_1$  and  $a_2$  are identified with the dimension *two* Coulomb branch parameters for two gauge groups.

Here is a short summary of this section. We have done local analysis of solution to Hitchin's equation, and four dimensional  $SU(2)$  quiver gauge theory can be constructed from the singular solution:

1)The four dimensional theory is determined by six dimensional  $(0, 2)$   $A_1$  theory on the genus  $g$  Riemann surface with punctures  $n$ . The number of gauge groups are the dimension of complex structure moduli space for the punctured Riemann surface, and the number reads  $3g - 3 + n$ .

2)Hitchin's equation on Riemann surface is the dynamical equation we need to consider. The massless theory is associated with the singular solution among which Higgs field has only simple pole and the residue at simple pole is a regular nilpotent element of  $sl_2$  algebra, the mass deformed theory is associated with Higgs field with only simple at the punctures and the residue is a semi-simple element.

3)The flavor symmetry is read from the singularity of the closure of the nilpotent orbit. For  $SU(2)$  theory, there is a  $A_1$  singularity for the closure of the regular nilpotent orbit, and the flavor symmetry is identified as  $SU(2)$ . The semisimple orbit is the deformation of the nilpotent orbit.

4) The IR behavior which is encoded in Seiberg-Witten curve is described by the spectral curve of Hitchin system. For massless theory, the spectral curve is expressed as in the form  $x^2 = Tr(\Phi)^2$ , where  $Tr(\Phi)^2 = \frac{c}{z}$  near the puncture. For massive theory, the Seiberg-Witten curve also have the form  $x^2 = Tr(\Phi)^2$  but  $Tr(\Phi)^2 = \frac{q^2}{z^2} + \frac{c}{z}$ .

## 5. $A_{N-1}$ theory

We can generalize the  $SU(2)$  results to  $SU(N)$  theory. From what we learn about  $SU(2)$  theory, what's important is the singular solution to Hitchin's equation with gauge group

$SU(N)$ . we conjecture that for massless theory, the holomorphic part of the Higgs field has simple pole at singularity and the residue is a nilpotent element of the  $sl_n$  algebra; for mass-deformed theory, the holomorphic part of the Higgs field also has simple pole at singularity but the residue now is a semisimple element of  $sl_n$  algebra.

### 5.1 Some Mathematical Backgrounds

We first give a short introduction to relevant mathematical results on lie algebra structure, an readable book for physicists is [47]. Since the pole of holomorphic part of the Higgs field is taking value in  $sl_n$ , we need to consider the structure of  $sl_n$  instead of  $su(n)$ .

If  $G$  is a reductive group over  $C$ ,  $g$  its Lie algebra, we study the adjoint action of  $G$  on  $g$ :

$$\mathcal{O}_X := G_{ad}.X = \{\phi(X)|\phi \in G_{ad}\}. \quad (5.1)$$

The orbits of this action are the conjugacy classes or adjoint orbits. I

A semisimple element  $U$  of the lie algebra is an element which can be diagonalizable, an nilpotent element  $U'$  is an element satisfying the relation  $U'^n = 0$ , where  $n$  is an integer. A conjugacy class  $\mathcal{O}_X$  is semisimple if and only if  $\mathcal{O}_X = \mathcal{O}_U$ ; while a conjugacy class  $\mathcal{O}_X$  is nilpotent if and only if  $\mathcal{O}_X = \mathcal{O}_{U'}$ .

We first define what is called a regular semisimple element in Lie algebra. The characteristic polynomial of a matrix  $X$  in  $sl_n$  is

$$\Omega(X) = \det(t - X). \quad (5.2)$$

We can expand it as

$$\Omega(X) = \sum_{0 \leq i \leq m} (-1)^i p_i(X) t^{n-i}. \quad (5.3)$$

$p_1$  is zero since  $trX = 0$ . A semisimple element is called regular semisimple if  $p_l \neq 0, l \geq 2$ . In particular, this means that the diagonal elements are all different. For  $sl_2$  case, we only have the regular semi-simple orbit while for  $sl_n$  case other options are possible.

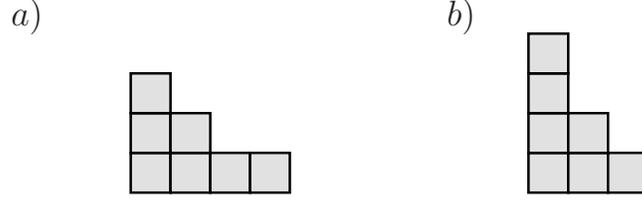
There are infinite number of semisimple conjugacy classes and we have only finite number of nilpotent conjugacy classes in  $sl_n$  algebra. The nilpotent elements of the  $sl_n$  lie algebra are labeled by partitions of  $n$  and can be put into standard form. Introduce a partition of  $n$  satisfy the conditions:

$$d_1 \geq d_2 \geq \dots \geq d_k > 0 \quad \text{and} \quad d_1 + d_2 + \dots + d_k = n. \quad (5.4)$$

We label this partition as  $d = [d_1, d_2, \dots, d_k]$ . We can construct Young tableaux associated with this partition as shown in Figure 15a). We can also construct a dual partition  $d^t$  of  $d$ . The first row of  $d^t$  is the first column of  $d$ , and the second row of  $d^t$  is the second column of  $d$ , and so on. There is another characterization for the dual partition: the parts of  $d^t$  is given by the following formula:

$$s_i = \{j | d_j \geq i\}, \quad (5.5)$$

$s_i$  equals the maximal index  $j$  so that  $d_j \geq i$ . We also draw a Young tableaux of the dual partition in Figure 15b).



**Figure 15:** (a) Young tableaux of one partition  $[4, 3, 1]$  of  $sl_3$ ; (b) The Young tableaux for transpose partition  $[3, 2, 1, 1]$  of (a).

Each nilpotent element is labeled by a partition of  $n$ . It can be put into a form using only Jordan block. The Jordan block is defined as: given a positive integer  $i$ , we construct the  $i \times i$  matrix

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (5.6)$$

This matrix is called the elementary Jordan block of type  $i$ .

Now the nilpotent element of partition  $d$  has the following form:

$$n = \begin{pmatrix} J_{d_1} & 0 & 0 & \dots & 0 \\ 0 & J_{d_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & J_{d_k} \end{pmatrix}, \quad (5.7)$$

where  $J_{d_i}$  is the Jordan block with dimension  $d_i$ . The dimension of this nilpotent orbits is given by

$$\dim(\mathcal{O}_X) = n^2 - \sum_i s_i^2 = n^2 - 1 - \left(\sum_i s_i^2 - 1\right). \quad (5.8)$$

Using the formula  $\dim(\mathcal{O}_X) = \dim(g) - \dim(g^X)$ , we have  $\dim(g^X) = (\sum_i s_i^2 - 1)$  and  $g^X$  is the centralizer of  $X$ , namely the set of elements of lie algebra which commute with  $g$ . The maximal dimension occurs when the partition is  $d = [n]$ , and we call it principal orbit; when  $n = [2, 1, 1, \dots, 1]$ , the nilpotent orbit has the minimal dimension, we call it minimal orbit.

## 5.2 Massless Theory and Singular Solutions to Hitchin's Equation

After introducing those mathematical results, let's go back to Hitchin's equation and try to find singular solutions to the equation so that the holomorphic part of the Higgs field has

simple pole at the singularity and the residue is an  $sl_n$  nilpotent element. In  $sl_2$  case, such a solution is found 4.8; we can construct a similar solution if we can find a  $sl_2$  subalgebra which contains a nilpotent element. This can be done by establishing a homomorphism between  $sl_2$  and  $sl_n$  which involves a nilpotent element of the  $sl_n$  algebra.

We introduce a different basis for  $sl_2$  lie algebra:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (5.9)$$

In this basis the nilpotent element is given by  $X$ , they satisfy the commutation relation:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad \text{and} \quad [X, Y] = H. \quad (5.10)$$

For an integer  $r \geq 0$ , we can define a map

$$\rho_r : sl_2 \rightarrow sl_{r+1}, \quad (5.11)$$

via

$$\begin{aligned} \rho_r(H) &= \begin{pmatrix} r & 0 & 0 & \dots & 0 & 0 \\ 0 & r-2 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -r+2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -r \end{pmatrix}, \\ \rho_r(X) &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \\ \rho_r(Y) &= \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mu_1 & 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & \mu_r & 0 \end{pmatrix}, \end{aligned} \quad (5.12)$$

where  $\mu_i = i(r+1-i)$  for  $1 \leq i \leq r$ . The homomorphism for the nilpotent element labeled by  $d$  is

$$\Phi_d : sl_2 \rightarrow sl_n, \quad \text{via} \quad \Phi_d = \bigoplus_{1 \leq i \leq k} \rho_{d_i-1}. \quad (5.13)$$

We can also find the commutator in  $SL_n$  of this homomorphism. Assume the nilpotent element associated with this homomorphism has the partition  $d = [d_1, d_2, \dots, d_k]$ , let  $r_i = |\{j|d_j = i\}|$ , namely,  $r_i$  is the number of rows with parts  $i$ . The commutant is given by

$$G_{commu} = S\left(\prod_i (GL_{r_i})\right). \quad (5.14)$$

Using this homomorphism, we can construct the singular solution: replacing  $t_1, t_2, t_3$  by  $\Phi_d\{t_1, t_2, t_3\}$ . The holomorphic part of the Higgs field has a simple pole at the singularity and the residue is the nilpotent element labeled by  $d$ . We want to associate these kind of solutions with four dimensional massless  $N = 2$  SCFT. The first thing we find is that we recover the result discovered by Gaiotto: The singularity is labeled by partition of  $N$  for  $N = 2$   $SU(n)$  SCFT.

We next study the behavior of spectral curve near the singularity to further confirm our conjecture. We first consider the solution associated with the partition  $[2, 1, \dots, 1]$  which is the minimal orbit. We need to add the regular term to the solution so that the holomorphic part of the Higgs field is a regular semisimple element and looks like

$$\Phi(z)dz = \begin{pmatrix} * (\frac{1}{z} + *) & * & \dots & * & * \\ * & * & * & \dots & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & * & \dots & * & * \\ * & * & * & \dots & * & * \end{pmatrix} dz + \mathcal{O}(z)dz, \quad (5.15)$$

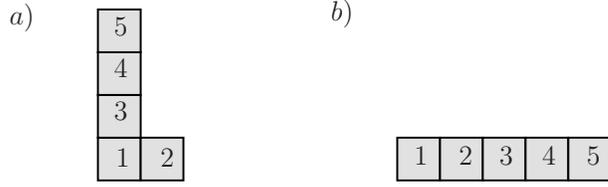
where  $*$  is the generic numbers so that this matrix is regular semisimple. We calculate the determinant and expand it as a polynomial in  $x$ :

$$\det(x - \Phi(z)) = \sum_{i=2} (-1)^i p_i(z) x^{n-i}. \quad (5.16)$$

The coefficient  $p_1$  is zero since the matrix is traceless,  $p_i, i \geq 2$  has simple pole at  $z = 0$  as we can see from calculating the determinant. Let's recall the rule of calculating the determinant: each term in determinant is derived by selecting numbers from the matrix, the rule is that there is only one item selected from one row and one column, we multiple those  $n$  selected terms.

For  $p_l x^{n-l}$  term in our determinant, we select  $n - l$  diagonal elements from the  $\{(x - \Phi)_{22} \dots (x - \Phi)_{nn}\}$ , we also select  $\frac{1}{z} + *$  term from first row and a proper constant term from the second row of  $(x - \phi(z))$ . We see that the coefficient  $p_l$  is of order  $\frac{1}{z}$ . The result can be summarized from the corresponding Young tableaux if we label the boxes as in Figure 16a), the pole of the coefficient is given by  $p_i = i - s_i$ , where  $s_i$  is the height of the  $i$ th box.

Next let's consider the solution labeled by the partition  $[n]$ , the matrix  $\Omega = (x - \Phi(z))dz$



**Figure 16:** a) Young tableaux with partition  $[2, 1, 1, 1]$ , the order of poles are  $p_1 = 1 - 1 = 0, p_2 = 2 - 1 = 1, p_3 = 3 - 2 = 1, p_4 = 4 - 3 = 1, p_5 = 5 - 4 = 1$ ; b) Young tableaux with partition  $[5]$ , the order of poles are  $p_1 = 1 - 1 = 0, p_2 = 2 - 1 = 1, p_3 = 3 - 1 = 2, p_4 = 4 - 1 = 3, p_5 = 5 - 1 = 4$

(including the constant regular term)

$$(x - \Phi(z))dz = \begin{pmatrix} (x + *) (\frac{1}{z} + *) & * & \dots & * & * \\ * & (x + *) (\frac{1}{z} + *) & \dots & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & * & \dots & (x + *) (\frac{1}{z} + *) \\ * & * & * & \dots & * & (x + *) \end{pmatrix} dz + \mathcal{O}(z)dz. \quad (5.17)$$

Calculate the characteristic polynomial of this matrix and leave only the singular terms in  $z$ , we find that  $p_i$  has pole of order  $(i - 1)$ . To show this, we simply expand the determinant and find the most singular term for the coefficient. For term  $p_i x^{n-i}$ , we select  $(i - 1)$   $(\frac{1}{z} + *)$  terms just above the diagonal terms, and then select the remaining  $n - i$  diagonal terms, this is the maximal pole we can get at  $z = 0$ . The order of pole can be read from the Young tableaux, namely  $p_i = i - s_i$ , see Figure 16b).

For general partition, the matrix  $(x - \Phi(z))$  has the form:

$$(x - \Phi(z))dz = \begin{pmatrix} I_{d_1} & * & * & \dots & * \\ * & I_{d_2} & * & \dots & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & * & \dots & I_{d_k} \end{pmatrix} dz + \mathcal{O}(z)dz. \quad (5.18)$$

where  $I_{d_i}$  takes the form

$$I_{d_i} = \begin{pmatrix} x + * (\frac{1}{z} + *) & * & \dots & * \\ * & x + * (\frac{1}{z} + *) & \dots & * \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot (\frac{1}{z} + *) \\ * & * & * & \dots & x + * \end{pmatrix} \quad (5.19)$$

The orders of pole for the coefficients  $p_i, 2 \leq i \leq d_1$  are calculated as follows: we choose the diagonal terms from the other blocks except the first block  $I_{d_1}$ , then we do the same analysis on the first block as we do on the partition  $[n]$ ; the order of pole is given by  $i - 1$  when  $i \leq d_1$ . To calculate term  $p_{d_1+1}x^{n-d_1-1}$ , We select  $d_1 - 1$  terms of form  $\frac{1}{z}$  and a constant term from first block  $I_{d_1}$ ; We can not choose another  $\frac{1}{z}$  term, since if we choose a  $\frac{1}{z}$  term say coming from the first row from the second block, we can not choose the two diagonal terms adjacent to it in calculating the determinant, the maximal order of  $x$  we can get is  $n - d_1 - 2$ . Therefore, the order of pole is  $d_1 - 1$ , or  $d_1 + 1 - 2$ . The order of poles for other terms  $p_i, d_1 < i \leq d_2$  is given by  $i - 2$ . We do the same analysis when we jump from  $d_i$  to  $d_{i+1}$ , in general, the order of pole is read from the Young tableaux and given by  $i - s_i$ , where  $s_i$  is the height of the  $i$ th box. This exactly matches the result of Gaiotto [3], see the discussion in section 2.

### 5.3 Flavor symmetry

Next, we want to analyze the flavor symmetry associated with singularity with nilpotent residue. We first analyze the principal nilpotent orbit. According to our previous discussion on  $A_1$  theory, we can understand the flavor symmetry from the singularity of the moduli space of solution with the prescribed boundary behavior. There is an isomorphism between the moduli space of solutions and nilpotent orbit. In the present case, We need to study the geometry of the principal nilpotent orbit or more precisely we need to consider the closure of this orbit. Let's denote this orbit as  $C_{reg}$ , and its closure  $\bar{C}_{reg}$  is the set of all nilpotent elements. The boundary  $\partial C_{reg} := \bar{C}_{reg} - C_{reg}$  is the closure of the conjugacy class  $C_{(n-1,1)}$ , where  $d = [n - 1, 1]$  is the partition of this nilpotent class. This class is called the subregular nilpotent conjugacy class and will be denoted by  $C_{subreg}$ . From our formula counting the dimension (5.8), we have  $codim_{\bar{C}_{reg}} C_{subreg} = 2$ . It might be useful for our understanding if we recall in  $sl_2$  case, the principal nilpotent orbit is  $O_{Y_1}$ , and its closure contains  $O_{Y_2}$  which is the subregular nilpotent class and has codimension two in the closure of  $O_{Y_1}$ .

The following result is due to Brieskorn[51]: The singularity of  $\bar{C}_{reg}$  in  $C_{subreg}$  is smoothly equivalent to the simple surface singularity  $A_{n-1}$ :

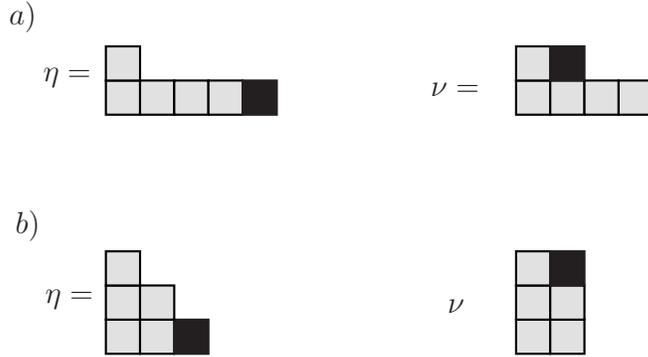
$$Sing(\bar{C}_{reg}, C_{subreg}) = A_{n-1}. \quad (5.20)$$

As usual  $A_{n-1}$  denotes the isolated singularity given by the equation  $x^n + y^2 + z^2 = 0$ . So we can identify the flavor symmetry associated with this solution as  $SU(n)$ .

Next, let's consider the flavor symmetry associated with the minimal nilpotent element. Let's denote the conjugacy class of this element as  $C_{min} = C_{(2,1,\dots,1)}$ , it has dimension  $dim C_{min} = 2(n - 1)$ . We have  $\bar{C}_{min} = C_{min} \cup \{0\}$ , where  $\{0\}$  is the identity orbit. We have the following result about the closure:  $\bar{C}_{min}$  has a isolated rational singularity in zero.

To understand this singularity, we can describe a resolution of the singularity in  $\bar{C}_{min}$ . Let  $P \subset GL_n$  be the stabilizer of the line  $e_1$ ,  $e_1 := (1, 0, \dots, 0)$ , and denote by  $n$  the nilradical of the parabolic subalgebra  $Lie P$  of the all the nilpotent orbits. Then  $GL_n/P \cong P^{n-1}$  and the associated vector bundle

$$GL_n(k) \times n \rightarrow GL_n/P, \quad (5.21)$$



**Figure 17:** a)  $\nu$  is derived from  $\eta$  by moving a box up to the next row; b)  $\nu$  is derived from  $\eta$  by moving a box up to the next column

is the cotangent bundle. Furthermore  $n \subset \bar{C}_{min}$  and the canonical map

$$\phi : GL_n \times n \rightarrow \bar{C}_{min}, \quad (5.22)$$

induced by  $(g, A) \rightarrow gAg^{-1}$  is a resolution of singularities (i.e. is proper and birational) with  $\phi^{-1}(0) = \text{zero section of the cotangent bundle}$ . This means that we obtain the singularity  $\bar{C}_{min}$  by "collapsing" the cotangent bundle of  $P^{n-1}$ . We call this singularity by  $a_{n-1}$ :

$$\text{Sing}(\bar{C}_{min}, 0) = a_{n-1}. \quad (5.23)$$

We claim that the flavor symmetry associated with this singularity is  $U(1)$  with only one exception  $a_1$ . Since  $a_1 = A_1$ , in that case, the flavor symmetry is enhanced to  $SU(2)$ .

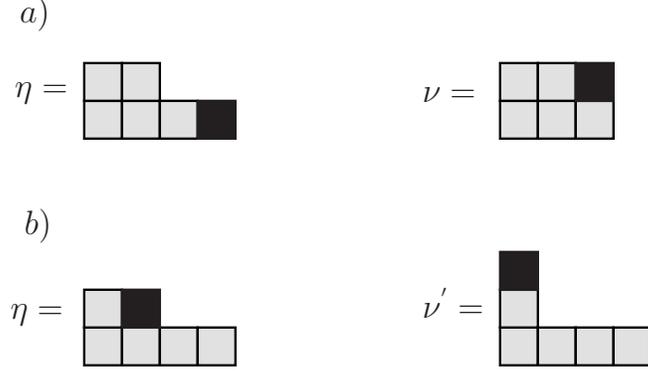
To understand the flavor symmetry associated to the general partition  $d = [d_1 \dots d_k]$ , we can go along the same line as the principal orbit and minimal orbit with some complication. We follow [52] to illustrate the main point. First, we define an order relation among the nilpotent orbits. Given two partitions  $\eta = (p_1, p_2, \dots, p_s)$  and  $\nu = (q_1, q_2, \dots, q_t)$  of  $n$ , we say  $\eta \geq \nu$  if

$$\sum_{i=1}^j p_i \geq \sum_{i=1}^j q_i \quad \text{for all } j. \quad (5.24)$$

If  $\eta > \nu$  and no partition is in between them (i.e.  $\eta$  and  $\nu$  are adjacent in the ordering), then the Young tableaux of  $\nu$  is obtained from  $\eta$  by moving one box up either to the next row or to the next column, see Figure 17 for the illustration. There might be more than one adjacent partition to  $\eta$ , see an example in Figure 18. Formally, the adjacent pair  $\eta = (p_1, p_2, \dots, p_s)$  and  $\nu = (q_1, q_2, \dots, q_t)$  can be expressed as the following two types:

I) If we need to move up a box to next row of  $\eta$ , then there is an integer  $i$ , such that  $q_k = p_k$  for  $k \neq i, i+1$  and  $q_i = p_i - 1 \geq q_{i+1} = p_{i+1} + 1$ .

II) If we need to move up a box to next column of  $\eta$ , then there are integers  $i, j$  such that  $p_k = q_k$  for  $k \neq i, j$  and  $q_i = p_i - 1 = q_j = p_j + 1$ .



**Figure 18:** We can find different adjacent Young diagram by moving different boxes. a)  $\nu$  is derived from  $\eta$  by moving a box up to the next row; b)  $\nu$  is derived from  $\eta$  by moving a different box up to the next row.

Given two partitions  $\eta$  and  $\nu$  of  $n$ ,  $\eta \geq \nu$  if and only if  $\bar{C}_\eta \supseteq C_\nu$ . We call a degeneration  $C_\nu \subseteq \bar{C}_\eta$  minimal if  $C_\nu$  is open in  $\bar{C}_\eta - C_\eta$ , i.e. if  $C_\nu \neq C_\eta$  and there is no conjugacy class  $C$  such that  $\bar{C}_\nu \subseteq \bar{C} \subseteq \bar{C}_\eta$ . This means that  $\eta$  and  $\nu$  are adjacent.

Let  $C_\nu \subset \bar{C}_\eta$  be a minimal degeneration, we have one of the following two cases:

I)  $\text{codim}_{\bar{C}_\eta} C_\nu = 2$  and the Young tableaux of  $\nu$  is obtained from  $\eta$  by moving one box up to the next row.

II)  $\text{codim}_{\bar{C}_\eta} C_\nu = 2r$  and the Young tableaux of  $\nu$  is obtained from  $\eta$  by moving one box up to the next column. If the box is moved from the  $i$ -th row to the  $j$ -th row, then  $r = j - i$ .

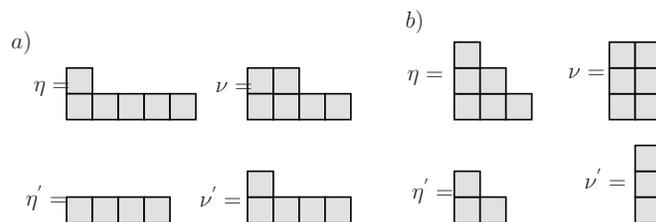
We can find the singularity of  $\bar{C}_\eta$  its minimal degeneration  $C_\nu$  by exploring the reduction relation. Let  $C_\nu \subseteq \bar{C}_\eta$  be a degeneration of nilpotent conjugacy classes and assume the first  $r$  rows and the first columns of  $\eta$  and  $\nu$  coincide. Denote by  $\eta'$  and  $\nu'$  the Young diagrams obtained from  $\eta$  and  $\nu$  by erasing these rows and columns, then  $C_{\nu'} \subset \bar{C}_{\eta'}$ ,

$$\text{codim}_{\bar{C}_{\eta'}} C_{\nu'} = \text{codim}_{\bar{C}_\eta} C_\nu \quad \text{and} \quad \text{Sing}(\bar{C}_{\eta'}, C_{\nu'}) = \text{Sing}(\bar{C}_\eta, C_\nu) \quad (5.25)$$

Using the reduction formula, We can determine the singularity for closure of any nilpotent orbit on a codimension two orbit which lies in the closure. Let  $C' \subseteq \bar{C}$  be a minimal degeneration of nilpotent conjugacy classes, the singularity of  $\bar{C}$  in  $C'$  is either of type  $A_m$  or is of type  $a_l$ . More accurately,

$$\begin{aligned} \text{Sing}(\bar{C}, C') &= A_{m-1} \quad \text{for some } m < n \text{ if } \text{codim}_{\bar{C}} C' = 2 \\ \text{Sing}(\bar{C}, C') &= a_l \quad \text{if } \text{codim}_{\bar{C}} C' = 2l \geq 2 \end{aligned} \quad (5.26)$$

We give a heuristic proof of above theorem and in the process of the proof we will see how to determine  $m$  and  $l$ . Let  $\eta$  and  $\nu$  be the associated partitions to  $C$  and  $C'$ ,  $\eta = (p_1, p_2, \dots, p_s)$ . If  $\nu$  is derived from  $\eta$  by moving up a box to next row, then  $\nu = (p_1, \dots, p_i - 1, p_{i+1} + 1, \dots, p_s)$  for some  $i$ .  $\eta$  and  $\nu$  have the same  $i - 1$  rows and the same first



**Figure 19:** a)  $\nu$  is derived from  $\eta$  by moving a box up to the next row, after erasing the first same set of rows and the first same set of columns, we find the singularity is  $A_4$ . b)  $\nu$  is derived from  $\eta$  by moving a different box up to the next row, we find the singularity is  $a_2$

$p_{i+1}$  columns by examining their partitions; After erasing these same rows and columns, the Young diagram of  $\eta'$  has the total boxes  $m = p_i - p_{i+1}$ , and it is of the type  $\eta' = [m]$ , and the Young diagram of  $\nu'$  also has total boxes  $m$  and it is of the type  $\nu' = [m - 1, 1]$ , the following relation holds

$$\text{Sing}(\bar{C}, C') = \text{Sing}(\bar{C}_{(m)}, C_{(m-1,1)}) = A_{m-1} \quad (5.27)$$

One example of this type is given in Figure 19a).

If  $\nu$  is derived from  $\eta$  by moving a box to next column, then  $\nu = (p_1, \dots, p_i - 1, p_{i+1}, \dots, p_j + 1, p_{j+1}, \dots, p_s)$ , for some  $i < j$  with  $p_i - 1 = p_{i+1} = \dots = p_j + 1$ . The first  $i - 1$  rows of  $\eta$  and  $\nu$  are the same, and the first  $p_j$  columns of them are the same. After erasing these same rows and columns,  $\eta'$  has total boxes  $j - i + 1$  and has the partition  $\eta' = [2, 1, 1, \dots, 1]$ , and  $\nu'$  has the same total of boxes and has the partition  $\nu' = [1, 1, 1, \dots, 1]$ . Now the singularity is determined as

$$\text{Sing}(\bar{C}, C') = \text{Sing}(\bar{C}_{(2,1,\dots,1)}, 0) = a_{j-i} \quad (5.28)$$

and  $\text{codim}_{\bar{C}} C' = 2(j - i)$ . One example of this type is given in Figure 19b).

In fact, the minimal singularities of can be read from Young tableaux of the orbit itself. The rule is stated in the above proof process. As we noted before, it is possible for a nilpotent orbit to have more than one minimal degeneration. We give a table of  $sl_6$  which illustrates the singularities for the minimal degeneration in Figure 20.

Now we can state what kind of flavor symmetries are associated with a singular solution of Hitchin's equation with nilpotent residue:

The flavor symmetries associated with a nilpotent orbit are determined by the singularity of its closure on its minimal degeneration, i.e, if the singularity is of type  $A_{m-1}$ , we have  $SU(m)$  flavor symmetry; If the singularity is of type  $a_m, m > 1$ , the flavor symmetry is  $U(1)$ ; If the singularity is  $a_1$ , the flavor symmetry is enhanced to  $SU(2)$ . If there are more than one degeneration, the flavor symmetry are from the product of all the singularities.

Let's state some examples from  $sl_6$  lie algebra to illustrate the idea. We can read the flavor symmetries from Figure 20 for all the nilpotent elements. To get the real flavor symmetry for the four dimensional theory, there are subtleties about  $U(1)$  factors. Since we study the singularity of the three dimensional coulomb branch, the information about

$SL_6$	$\eta$	Dimension	Flavor symmetry
	(6)	30	$SU(6)$
	(5, 1)	28	$SU(4)$
	(4, 2)	26	$SU(2) \times SU(2)$
	(4, 1, 1)	24	$SU(3)$
	(3, 3)	24	$SU(3)$
	(3, 2, 1)	22	$U(1) \times U(1)$
	(2, 2, 2)	18	$SU(2)$
	(3, 1, 1, 1)	18	$SU(2)$
	(2, 2, 1, 1)	16	$U(1)$
	(2, 1 <sup>4</sup> )	10	$U(1)$
	(1 <sup>6</sup> )	0	None

**Figure 20:** The singularity of closure of nilpotent orbit on its minimal degeneration of  $sl_6$  lie algebra.

$U(1)$  flavor symmetry of four dimensional theory is usually lost. For the partition  $[n]$  and  $[2, 1, \dots, 1]$ , there is no ambiguity, the flavor symmetry is  $SU(n)$  and  $U(1)$  respectively. For other generic partition, if the singularity type is from the degeneration by moving the box from the highest row, we do not add a  $U(1)$  factor, otherwise, we need to include a  $U(1)$  factor. This rule is also true for partition  $[n]$ . One can check that this really recovers the flavor symmetry described by Gaiotto. The reason why the  $U(1)$  symmetry is not included in this case is that for the fundamentals attached on the gauge group  $SU(N)$ , the  $U(1)$  factor is denoted by the simple puncture.

It is the same as in  $SU(2)$  case to determine what kind of basic building block we can derive for a given singularity. If the nilpotent element associated with the singularity has the partition  $d = [d_1, d_2, \dots, d_k]$ , the tail has gauge group  $SU(d_1) \times SU(d_2 + d_1) \dots \times SU(N)$ , and we add fundamental hypermultiplets to gauge group to make the theory conformal. When we have the minimal orbit, namely the partition  $d = [2, 1, \dots, 1]$ , the bifundamentals can be also represented by this orbit.

## 5.4 Mass Deformed Theory

In this subsection, we are going to study what kind of singular solutions correspond to mass-deformed theory. Let's recall what we learned about  $SU(2)$  theory. The massless theory is associated with Higgs field whose residue is a nilpotent element  $Y_1$  labeled by the partition  $d = [2]$ ; what is actually important is the moduli space of solutions with appropriate boundary conditions so that the residue is living in the conjugacy class of  $Y_1$ . There is an isomorphism between this moduli space and nilpotent orbit itself. On the other hand, the mass deformed theory is described by a solution to Hitchin's equation so that the residue of the Higgs field is a semisimple element (which is also regular for  $su(2)$ ). We

also are concerned about the moduli space of solutions and there is also an isomorphism between the space of solutions and the semi-simple orbit itself. Both nilpotent orbit and semi-simple orbit are hyper-Kähler manifolds and the closure of nilpotent orbit is singular and semi-simple orbit can be thought of the deformation of nilpotent orbit. The basic requirement for this understanding is that they must have the same complex dimensions.

Generalizing above considerations of  $SU(2)$  to  $SU(N)$ , we need to find certain kinds of solutions of Nahm's equation whose moduli space is a deformation of the moduli space of solutions we are studying in the last subsection. In general, given a triple  $(\tau_1, \tau_2, \tau_3)$ , let  $\sigma_1, \sigma_2, \sigma_3$  be elements of  $\mathfrak{g}$  which commute with  $\tau_j$  and which satisfy the  $\mathfrak{su}(2)$  relations, a solution to the equation is

$$a = \tau_1 + \frac{\sigma_1}{2s}, \quad b = \tau_2 + \frac{\sigma_2}{2s}, \quad a = \tau_1 + \frac{\sigma_3}{2s}, \quad s \rightarrow \infty. \quad (5.29)$$

These conditions means that the residue of the Higgs field takes value in  $\tau_2 + i\tau_3 + \sigma^c$ , where  $\sigma^c$  is the nilpotent element we can get from  $\mathfrak{su}(2)$  algebra  $\sigma_1, \sigma_2, \sigma_3$ . There is a one-to-one correspondence between the solution space with this boundary conditions and the adjoint orbit which contains  $\tau_2 + i\tau_3 + \sigma^c$  (see appendix I for more details), it is also proved that this space is a hyper-Kähler manifold.

Since the nilpotent orbit is identified with the massless theory, so we are led to think that the mass-deformed theory corresponds to semisimple orbit. The question is to identify the semi-simple orbit, we will call those semi-simple orbits as the mass-deformed orbits. The closure of the nilpotent orbit is singular and we can think of the mass-deformed orbit as the deformation of the closure. The necessary condition for this is that the mass-deformed orbit has the same dimension as the closure of the nilpotent orbit.

The dimension for a nilpotent orbit is given by (5.8). The following lemma can be used to calculate dimension of a semi-simple orbit:

Let  $\mathfrak{g}$  be a reductive lie algebra and  $X$  is element in a semisimple orbit, its centralizer  $\mathfrak{g}^X$  is reductive and there exists a Cartan subalgebra  $\mathfrak{h}$  containing  $X$ . If  $\Phi$  denotes the roots for the pair  $(\mathfrak{g}, \mathfrak{h})$ , then  $\mathfrak{g}^X = \mathfrak{h} \oplus \sum_{\alpha \in \Phi_X} \mathfrak{g}_\alpha$ , where  $\Phi_X = \{\alpha \in \Phi | \alpha(X) = 0\}$ .

We study  $\mathfrak{sl}_3$  as an example to show how to use the above lemma to calculate the dimension of a semisimple orbit. The traceless diagonal matrices in  $\mathfrak{sl}_3$ , denoted as  $\mathfrak{h}$ , form a three dimensional Cartan subalgebra. For each  $1 \leq i \leq 3$ , define a linear functional in the dual space  $\mathfrak{h}^*$  by

$$e_i \left( \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \right) = h_i. \quad (5.30)$$

The standard choices of positive and simple roots are

$$\Phi^+ = \{e_i - e_j | 1 \leq i < j \leq 3\} \text{ and } \Delta = \{e_i - e_{i+1} | 1 \leq i \leq 2\}. \quad (5.31)$$

Consider the following matrices

$$X_1 = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & -(m_1 + m_2) \end{pmatrix}, \quad X_2 = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & -2m_1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.32)$$

We now describe how to calculate dimension of semi-simple orbit  $\mathcal{O}_{X_k}$  for  $1 \leq k \leq 3$ .

For case  $X_1$ , since  $\alpha(X_1) \neq 0$  for any simple roots  $\alpha \in \Delta$ ,  $g^{X_1}$  is a Cartan subalgebra using our lemma. The dimension for  $X_1$  is

$$\dim(\mathcal{O}_{X_1}) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}^{X_1}) = 8 - 2 = 6. \quad (5.33)$$

For case  $X_2$ ,  $\alpha(X_2) = 0$  if and only if  $\alpha = \pm(e_1 - e_2)$ , so  $\mathfrak{g}^{X_2} = \mathfrak{h} \oplus \mathfrak{g}_{e_1 - e_2} \oplus \mathfrak{g}_{e_2 - e_1}$  and  $\dim(\mathcal{O}_{X_2}) = 8 - 4 = 4$ .

For case  $X_3$ ,  $\Phi_{X_3} = \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\}$ , then  $\dim(\mathfrak{g}^{X_3}) = 8$ , and  $\dim(\mathcal{O}_{X_3}) = 0$ .

Let's study the semi-simple orbits for general  $sl_n$  algebra. We want to study semi-simple elements labeled by a partition  $d = [d_1, d_2, \dots, d_k]$  of  $n$ . It has the form  $X_d = \text{diag}(m_1, \dots, m_1, m_2, \dots, m_2, \dots, m_k, \dots, m_k)$ , where the first  $d_1$  diagonal terms have the same value, etc. It is interesting that we can also label semi-simple orbits by partitions of  $n$ . The dimension of the orbit  $\mathcal{O}_{X_d}$  can be calculated by using the lemma we introduced above.

Let  $h$  be traceless diagonal  $n \times n$  matrices; Define the linear functional  $e_i \in \mathfrak{h}^*$  by  $e_i(H) = i^{\text{th}}$  diagonal entry of  $H$ , here  $1 \leq i \leq n$ . The root system is  $\{e_i - e_j | 1 \leq i, j \leq n, i \neq j\}$  in this representation. Elements in  $\phi(X_d)$  from first block are

$$\Phi^1(X_d) = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \dots, \pm(e_1 - e_{d_1}), \pm(e_2 - e_3), \pm(e_2 - e_4), \dots, \pm(e_2 - e_{d_1}) \dots \pm(e_{d_1-1} - e_{d_1})\} \quad (5.34)$$

For the other block we can similarly find the other roots which satisfy the condition  $\alpha(X_d) = 0$

The dimension of the centralizer of  $X_d$  is

$$\dim(\mathfrak{g}^{X_d}) = n - 1 + 2(d_1 - 1 + d_1 - 2 + \dots + 1) + 2(d_2 - 1 + d_2 - 2 + \dots + 1) + \dots + (2d_k - 1 + \dots + 1), \quad (5.35)$$

sum them up, we have

$$\dim(\mathfrak{g}^{X_d}) = n - 1 + \sum_i^k d_i(d_i - 1) = n - 1 + \sum_i^k d_i^2 - n = \sum_i^k d_i^2 - 1. \quad (5.36)$$

where the condition  $\sum_i^k d_i = n$  is used. The dimension of the semisimple orbit is

$$\dim(\mathcal{O}_{X_d}) = n^2 - 1 - \sum_i^k d_i^2 + 1 = n^2 - \sum_i^k d_i^2. \quad (5.37)$$

Recall the dimension (5.8) of a nilpotent orbit with the partition  $d_1$ :

$$\dim(\mathcal{O}_{X_{d_1}}) = n^2 - \sum_i s_i^2. \quad (5.38)$$

Where  $s^i$  is the rows of the dual partition of  $d_1$ .

Comparing the dimension of the nilpotent orbits and dimension of the semi-simple orbits, we have the following observation of the mass-deformed theory for a puncture labeled by the partition  $d$ :

The mass deformed theory is described by the singular solution of Hitchin's equation; The Higgs field has simple pole at the singularity and whose residue is a semisimple element with the form labeled by the dual partition  $d^t = [d_1^t, d_2^t, \dots, d_k^t]$  of  $d$ :

$$\Phi(z)dz = \frac{dz}{z} \text{diag}(m_1, \dots, m_1, m_2, \dots, m_2, \dots, m_k, \dots, m_k) + \dots, \quad (5.39)$$

where  $\Phi$  has  $d_1$   $m_1$  eigenvalues,  $d_2$   $m_2$  eigenvalues and so on. The Seiberg- Witten curve is the spectral curve of the Hitchin's system.

We can read flavor symmetry from the dual partition  $d^t$ . For this partition, we can also define a  $sl_2$  homomorphism and the commutant of this homomorphism in  $sl_n$  is given by

$$G_{commu} = S\left(\prod_i (GL_{r_i})\right), \quad (5.40)$$

where  $r_i$  is the number of rows of  $d_i$  with boxes  $i$ . This number  $r_i$  is also the number of columns of  $d$  with heights  $i$ . Using its real form, we identify this as the flavor symmetry associated with the puncture and this agrees with Gaiotto's result. See [7] for the relevant discussion.

## 6. Conclusion

In this paper, we argue that it is the Hitchin's equation which determines not only IR limit but also the UV theory of the four dimensional  $N = 2$  superconformal field theories. We study the local singular solution of Hitchin's equation so that the Higgs field has a simple pole at the singularity. We show that the massless theory is associated with the solution so that Higgs field has nilpotent residue and the mass deformed theory is associated with the semi-simple residue; The moduli space of solutions of the mass deformed theory can be thought of as the deformation of the moduli space of the massless theory. The Seiberg-Witten curve which determines the IR behavior is given by the spectral curve of Hitchin's equation.

It is interesting to extend our analysis to six dimensional  $D_N$  theory compactified on a Riemann surface. The four dimensional theory can be derived by adding O4 planes to the brane configurations we considered in this paper [53, 54]. The flavor symmetry and the Young tableaux classification is given by Tachikawa [7]. In that paper, the author shows that there are two different kinds of tails which are related to  $USp$  and  $SO$  group. We might need to study Hitchin's equation with  $SO(2n)$  group, we have almost the same group structure associated with the nilpotent orbits and semisimple orbits. However, it is puzzling why we need  $USp$  and  $SO$  groups at different type of singularities.

Similar analysis can be done on six dimensional  $E_N$  theory on punctured Riemann surface. We don't have any brane configuration so we don't have any four dimensional understanding about the field theory. However, following the same analysis from this paper, we hope we can learn something about this type of theories.

In this paper, we only carry the local analysis of solutions to Hitchin's equation, it is interesting to extend the analysis to consider the global constraint. we also only study

the solutions with simple poles at the singularity. We can also studied the solutions with higher order singularities. This type of solutions can be also used to study the SCFT and it can be used to represent the asymptotically free theory. We leave this for the future study.

Nahm's equation plays a fundamental role in our study. It is also used in an essential way to study boundary conditions for  $N = 4$  theory by Gaiotto and Witten [55, 56]. In those papers, Nahm's equation is important to get three dimensional mirror symmetry [50]. Since the moduli space of Hitchin's equation is closed related to coulomb branch of three dimensional theory, we may wonder whether we can find new three dimensional mirror pair by compactifying four dimensional  $N = 2$  theory down to three dimension. This is indeed the case [57].

Finally, Hitchin's equation is related to the KZ(Knizhnik-Zamolodchikov) equation of the two dimensional conformal field theory, this shed the light about why *AGT* [14] relation is possible, we hope we can learn more about the relation between four dimensional SCFT and two dimensional CFT by studying Hitchin's equation.

## Acknowledgments

This research was supported in part by the Mitchell-Heep chair in High Energy Physics (CMC) and by the DOE grant DEFG03-95-Er-40917.

## Appendix I

The Hitchin's equation defined on a Riemann sruface is

$$\begin{aligned} F - \phi \wedge \phi &= 0 \\ d_A \phi &= 0, \quad d_A * \phi = 0, \end{aligned} \tag{6.1}$$

where  $d_A$  is the covariant derivative and  $*$  is the Hodge star operator. Define the local coordinates  $z = re^{i\theta}$ , We are looking for the rotational invariant solution with the form:

$$\begin{aligned} A &= a(r)d\theta + f(r)\frac{dr}{r} \\ \phi &= b(r)\frac{dr}{r} - c(r)d\theta, \end{aligned} \tag{6.2}$$

with the functions  $a, b, c, f$  which take values in lie algebra  $g$  of  $G$ . Though  $f$  can be gauged away, we keep it for later use. The space of solutions with appropriate boundary conditions has a hyper-Kahler structure. One way to understand this fact is to think of the functions  $(f, a, b, c)$  as giving a map from the open unit interval to the quarternions  $H \cong R^4$ , tensored with  $g$ . The hyper-Kahler structure comes from the hyper-Kahler structure on  $H$ . In one complex structure,  $b + ic$  is the complex structure parameters and  $f - ia$  is the kahler structure parameters. The others can be obtained by applying an  $SO(3)$  rotation to the triple  $(a, b, c)$ .

If we consider a local region around the singularity  $r \in [0, 1]$ , and we define another coordinate  $s = -\ln r$  and in this new coordinate the range is  $s \in [0, \infty)$ . Define  $D/Ds = d/ds + [f, \cdot]$ , Hitchin's equation becomes

$$\begin{aligned}\frac{Da}{Ds} &= [b, c] \\ \frac{Db}{Ds} &= [c, a] \\ \frac{Dc}{Ds} &= [a, b],\end{aligned}\tag{6.3}$$

when  $f = 0$ , the above equation becomes the Nahm's equation. The above equations is invariant under gauge transformations:

$$\begin{aligned}f &\rightarrow Ad(g)(f) - \frac{dg}{ds}g^{-1} \\ a &\rightarrow Ad(g)a, \quad b \rightarrow Ad(g)b, \quad c \rightarrow Ad(g)c.\end{aligned}\tag{6.4}$$

where  $g : [0, \infty) \rightarrow G$  is any path. Introducing the complex variables

$$\alpha(s) = \frac{1}{2}(f(s) + ia(s)), \quad \beta(s) = \frac{1}{2}(b(s) + ic(s)),\tag{6.5}$$

the Nahm's equation can be rewritten as one 'real' equation and one 'complex' equation:

$$\frac{d}{ds}(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0,\tag{6.6}$$

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0.\tag{6.7}$$

### Boundary Conditions, Isomorphism between Moduli Space of Solutions and Nilpotent Orbits

We are studying the Nahm's equation with appropriate boundary conditions. Let  $\rho : su(2) \rightarrow g$  be a Lie algebra homomorphism and write

$$H = \rho \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.\tag{6.8}$$

Let's denote  $M(\rho)$  as the space of smooth solutions  $\alpha(s), \beta(s) : [0, \infty) \rightarrow g^c$  which satisfy the following boundary conditions:

i)

$$2s\alpha(s) \rightarrow Ad(g)H, \quad s\beta(s) \rightarrow Ad(g_0)Y, \quad s \rightarrow \infty,\tag{6.9}$$

with some  $g_0 \in g$ .

ii)

$$2s\alpha(s) \rightarrow 0, \quad s\beta(s) \rightarrow 0, \quad s \rightarrow 0.\tag{6.10}$$

It is proven by Kronheimer [48] that  $M(\rho)$  has a hyper-kahler structure and is isomorphic to the nilpotent orbit associated with  $Y$ .

## Isomorphism between Moduli Space of Solutions and Semisimple orbits

Let's first define a triple  $(\tau_1, \tau_2, \tau_3)$  which lives in the Cartan subalgebra ( $(\tau_1, \tau_2, \tau_3)$  are denoted as  $(\alpha, \beta, \gamma)$  in our main discussion). We first assume that the triple is regular in the sense that the union of the centralizer  $C(\tau_1) \cup C(\tau_2) \cup C(\tau_3)$  is a Cartan subalgebra.

let  $M(\tau_1, \tau_2, \tau_3)$  be the space of solutions  $(a(s), b(s), c(s))$ , satisfying the boundary condition

$$\lim_{s \rightarrow \infty} a(s) = Ad(g_0)\tau_1, \quad \lim_{s \rightarrow \infty} b(s) = Ad(g_0)\tau_2, \quad \lim_{s \rightarrow \infty} c(s) = Ad(g_0)\tau_3. \quad (6.11)$$

for some  $g_0 \in g$ . This space has a hyper-kahler structure.

If we further require  $(\tau_2, \tau_3)$  be a regular pair, and suppose that  $(a(s), b(s), c(s))$  is a solution of Nahm's equation (we set  $f = 0$  at this step) belonging to  $M(\tau_1, \tau_2, \tau_3)$ . If we set  $\sigma_1 = a(0), \sigma_2 = b(0), \sigma_3 = c(0)$ , we have the following statement:

*The element  $\sigma_2 + i\sigma_3 \in g^c$  belongs to the same complex adjoint orbit as  $\tau_2 + i\tau_3$ .*

We can prove above statement by using equation (6.7). Consider the solution  $(\alpha, \beta)$  of that equation which is obtained from  $(a(s), b(s), c(s))$  by setting  $f(s) = 0$ , the equation implies that the path  $2\beta(s)$  lies entirely within a single adjoint orbit of the complex group. let  $\mathcal{O}(2\beta)$  denote this orbit, and let  $\mathcal{O}(\sigma)$  and  $\mathcal{O}(\tau)$  be the orbits which contain  $(\sigma_2 + i\sigma_3)$  and  $\tau_2 + i\tau_3$  respectively. The boundary conditions are

$$\begin{aligned} 2\beta(s) &\rightarrow \tau_2 + i\tau_3, \quad \text{as } s \rightarrow \infty \\ 2\beta(s) &\rightarrow \sigma_2 + i\sigma_3, \quad \text{as } s \rightarrow 0, \end{aligned} \quad (6.12)$$

so the closure of  $\mathcal{O}(2\beta)$  contain  $\mathcal{O}(\sigma)$  and  $\mathcal{O}(\tau)$ , but  $\mathcal{O}(\tau)$  is a regular semisimple orbit, which means that it is closed and is contained in the closure of no other orbit. It follows that  $\mathcal{O}(2\beta) = \mathcal{O}(\sigma) = \mathcal{O}(\tau)$ .

From this point, we denote  $\mathcal{O}$  the adjoint orbit which contains  $\tau_2 + i\tau_3$ . The above statement provides us with a map

$$f : M(\tau_1, \tau_2, \tau_3) \rightarrow \mathcal{O} \quad (a(s), b(s), c(s)) \rightarrow (b(0) + c(0)), \quad (6.13)$$

Kronheimer has proved that this is a bijection [49].

For a non-regular triple, the boundary conditions can be different. Let  $\tau_1, \tau_2, \tau_3$  be non-regular elements of the Cartan algebra  $h$ , and let  $\sigma_1, \sigma_2, \sigma_3$  be elements of  $g$  which commute with  $\tau_j$  and which satisfy the  $su(2)$  relations. A solution of the equations is

$$a = \tau_1 + \frac{\sigma_1}{2s}, \quad b = \tau_2 + \frac{\sigma_2}{2s}, \quad c = \tau_3 + \frac{\sigma_3}{2s}. \quad (6.14)$$

and this can be used as the boundary conditions for the non-regular triple. The residue of the Higgs field is  $\tau_2 + i\tau_3 + \sigma$ , where  $\sigma$  is the nilpotent element in the centralizer of  $(\tau_1, \tau_2, \tau_3)$ .

We have following theorem proved in [58]: Given  $(\tau_2, \tau_3)$  and  $\sigma$  is a nilpotent element which lives in the centralizer of  $(\tau_2, \tau_3)$ , the moduli space of solutions with the above boundary conditions is isomorphic to the adjoint orbit  $\tau_2 + i\tau_3 + \sigma$ , and it has a family of hyper-Kahler structures. The family is parameterized by  $\tau_1$  such that the centralizer of the pair  $(\tau_2, \tau_3)$  and the triple  $(\tau_1, \tau_2, \tau_3)$  coincide. For the proof and more technical details, see [58].

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