CONVECTED STRESS/STRAIN MEASURES IN THE PHYSICAL FRAME

A Thesis

by

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ABSTRACT

An oblique, Cartesian coordinate system arises from the geometry affiliated with a **QR** decomposition of the deformation gradient **F**, wherein **Q** is an orthogonal matrix and **R** is an upper-triangular matrix. We analyzed the deformation of a cube into a parallelepiped, whose convected coordinates are oblique. Components for the metric tensor and its dual evaluated in this convected coordinate system are established for any state of deformation. Strains and strain rates are defined and quantified in terms of these metrics and their rates. Quotient laws are constructed that govern how vector and tensor fields map between the convected coordinate system and the rectangular Cartesian coordinate systems, where boundary value problems are typically solved.

We also derived a set of thermodynamically admissible stress-strain pairs that are quantified in terms of physical components from a convected stress and velocity gradient, with elastic models being presented. This model supports two modes of deformation: elongation and shear, which is distinguished by the pure- and simple-shear responses.

Then, we start out in an oblique convected coordinate system and finish up in an orthonormal coordinate system. This reversing process results in working with convected tensor fields that are evaluated in a rectangular Cartesian, coordinate system at the current time, which are 'physical' by construction. We also compared the classical approach with our proposed approach.

DEDICATION

To my parents and my spouse, without whom none of this would be possible.

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1. INTRODUCTION AND LITERATURE REVIEW

The ability of a convected framework to simplify constitutive equations, especially for soft solids, should make it attractive for scholars. A convected coordinate system that is attached to the material particles and deforms with the body is suitable for analyses that engage large deformations.

Symmetric and non-symmetric conditions effect the constraints of constitutive equations. These constraints play a fundamental role in the construction of functions associated with tensor and scalar variables, and in the experimental observation of constitutive equations. It was in Arthur Lodge's 1951 paper entitled "On the use of convected coordinate systems in the mechanics of continuous media" published in the *Proceedings of the Cambridge Philosophical Society* where he introduced body fields—a formalism he made precise in his 1974 book on *Body Tensor Fields in Continuum Mechanics* [1]. He showed that a connection exists between convected space-tensor fields and body-tensor fields, viz., their components are equivalent at that instant when their coordinate axes become coincident [2]. It is at this juncture where we construct our analysis using convected space-coordinate systems derived from the geometry of a parallelepiped generated from a Gram-Schmidt decomposition of the deformation gradient.

Practical applications that employ convected coordinate systems have been restricted to the study of certain well-established boundary value problems, e.g., rheological experiments [3, 1]. The challenge has been to quantify the relevant tensor fields for any arbitrary state of deformation, e.g., as they would appear in a finite element analysis. The objective of this dissertation is to resolve this long-standing challenge based on our hypothesis that is used in our analysis: *Deformation is homogeneous at a mass point*.

The coordinate axes of this system are tangents to an embedded curvilinear system whose origin is located at some mass point of interest. In the orthonormal coordinate system associated with a Gram-Schmidt decomposition of **F**, the 1 coordinate direction remains tangent to the 1 material curve at the mass point whereat **F** is evaluated; plus, the 12 coordinate plane remains tangent to the 12 material surface at this mass point.

In 2012, Srinivasa [4] introduced a Gram-Schmidt (**QR**) decomposition of the deformation gradient **F**, instead of using the polar decomposition for **F** commonly used by the mechanics community, and derived some results. Among the various features that this triangular decomposition possesses is a coordinate system that is 'nearly' embedded within the material of interest. It is nearly embedded in that a truly embedded coordinate system would be curvilinear with each coordinate curve being comprised of the same set of material particles over time, e.g., see [5, 6, 7, 3, 8].

Souchet [9] derived a lower-triangular factorization of the deformation gradient, which he used to obtain compatibility conditions. His study is the one most similar to Srinivasa's. Rosakis [10] arrived at an upper-triangular deconstruction of the deformation gradient, absent of an in-plane shear contribution. He studied spatially discontinuous deformations separated by a common material surface. Guo *et al*. [11] also obtained an upper-triangular deformation gradient. They studied composite laminates. The uniqueness and existence of such deconstructions have been addressed by Rosakis [10] and Lembo [12], as they apply to **F**; and, in the more general setting, in textbooks on linear algebra, as they apply to the Cholesky factorization of a positive-definite matrix.

We first derive the covariant and contravariant base-vectors, metrics, strains, and their differential rates in this convected framework, in the sense of Lodge [2, 13, 14, 6, 15, 1, 16, 17, 18]. They are constructed from the geometry that describes Laplace stretch. Laplace stretch is the upper-triangular contribution (the **R** in **QR**) in a Gram-Schmidt decomposition of the deformation gradient **F**.

Then, we start out in the oblique convected coordinate system and finish up in an orthonormal coordinate system. This reversing process results in working with convected tensor fields that are evaluated in a rectangular, Cartesian, coordinate system at the current time, which are 'physical' by construction.

The real advantage of this work is rooted in working with a coordinate system with physical components, and working with stress components that are evaluated in this rectangular, Cartesian, coordinate system. In short, choosing our convected coordinate system to be rectangular Cartesian at the current time allows one to work with convected tensor components that are 'physical' by construction.

Working within a convected coordinate system is often challenging, especially when it is oblique, so selecting 'physical components' instead of 'convected components' as one's measures for stress and strain goes a long way towards providing a user-friendly theory.

Physical components transform tensor components evaluated in a curvilinear coordinate system in such a way that their values possess characteristics as if they had been quantified in a rectangular, Cartesian, coordinate system. Historically, the transformation rules that govern physical components have not obeyed the transformation rules that govern tensor components [19]. However, an upper-triangular decomposition of the deformation gradient has lead to a convected coordinate system that warps out of an orthonormal coordinate system, a warping caused by motion [20].

The hyperelastic material is a category of constitutive equations in elasticity. The relationship between stress and strain for this ideally elastic material comes from strain-energy density function. We shall consider hyperelastic material models whose strain-energy function can be written as a function of unitary invariants. The sets of invariants that one finds in the mechanic's literature pertain to symmetric matrices, e.g., [21, 22, 23]. We focus our attention on sets of invariants that pertain to the deformation gradient $\mathbf{F}$ and the right Cauchy-Green deformation tensor $\mathbf{C}$ in the Lagrangian coordinate system as a classical approach, and that pertain to an upper-triangular matrix, like Laplace stretch $\widetilde{\mathbf{F}}$, and left Cauchy-Green deformation tensor $\widetilde{\mathbf{B}}$ evaluated in the physical coordinate system as our approach, and we shall discuss their utility when constructing constitutive equations. We narrowed down our analyses for membranes with a spatial dimension of two.

Note: Fields defined and quantified in a coordinate system that convects with the motion of a body are observer indifferent by their very construction.

Note: An 'attribute', in the terminology of Criscione [24], is any scalar-valued field that describes a physical phenomenon of tensorial quality that is neither a tensor component nor a tensor invariant.

2. ANALYSIS IN CONVECTED COORDINATE SYSTEM*

2.1 Kinematics and Coordinate System

The elongations a, b and shear γ are quantified through a **QR** (Gram-Schmidt) factorization of a Jacobian matrix **F** associated with deformation gradient **F**, wherein **Q** is a proper orthogonal matrix in that $\mathbf{Q}^{\mathsf{T}} = \mathbf{Q}^{-1}$ with $\det \mathbf{Q} = +1$, and where **R** is denoted as $\widetilde{\mathbf{F}}$ (what Freed *et al.* [25] call the Laplace stretch). The Laplace stretch populates an upper-triangular matrix with components acquired from a Cholesky factorization of the Green [26] deformation tensor $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F} = \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{F}}$, namely [4]

$$
\begin{aligned}
\widetilde{F}_{11} &= \sqrt{C_{11}} & \widetilde{F}_{12} &= \frac{C_{12}}{\widetilde{F}_{11}} \\
\widetilde{F}_{21} &= 0 & \widetilde{F}_{22} &= \sqrt{C_{22} - \widetilde{F}_{12}^{2}}
\end{aligned}
\tag{2.1}
$$

which populate the Laplace stretch $\widetilde{\mathbf{F}}$ evaluated in basis $\{\vec{\mathbf{e}}_i\}$ through its matrix components

$$
\widetilde{\mathbf{F}} = \begin{bmatrix} \widetilde{F}_{11} & \widetilde{F}_{12} \\ 0 & \widetilde{F}_{22} \end{bmatrix} \quad \text{with} \quad \widetilde{\mathbf{F}}^{-1} = \begin{bmatrix} 1/\widetilde{F}_{11} & -\widetilde{F}_{12}/\widetilde{F}_{11}\,\widetilde{F}_{22} \\ 0 & 1/\widetilde{F}_{22} \end{bmatrix} \tag{2.2}
$$

or alternatively [27]

$$
\widetilde{\mathbf{F}} = \begin{bmatrix} a & a\gamma \\ 0 & b \end{bmatrix} \quad \text{with} \quad \widetilde{\mathbf{F}}^{-1} = \begin{bmatrix} 1/a & -\gamma/b \\ 0 & 1/b \end{bmatrix} \tag{2.3}
$$

which decompose into extension $\mathbf{\Lambda}$ and shear $\mathbf{\Gamma}$ contributions described by

$$
\widetilde{\mathbf{F}} = \mathbf{\Lambda}\mathbf{\Gamma} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{extension } \mathbf{\Lambda}} \underbrace{\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}}_{\text{shear } \mathbf{\Gamma}} \tag{2.4}
$$

Elements of the shear and extension matrices have a physical interpretation (see Fig. 2.1) based

[1]*Part of this chapter is reprinted with permission from "On the use of convected coordinate systems in the mechanics of continuous media derived from a QR factorization of F" by Alan D. Freed and Shahla Zamani, 2018. International Journal of Engineering Science, 127, 145-161, Copyright [2018] by Elsevier.

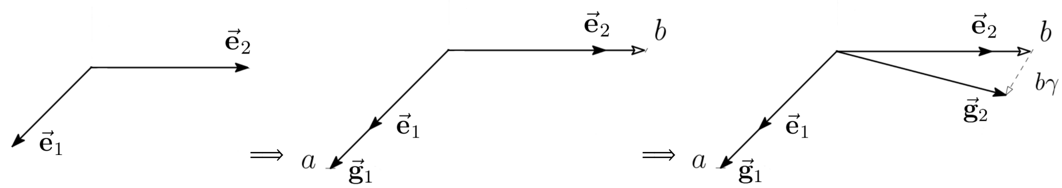

Figure 2.1: Deformation of physical coordinates into oblique coordinates

upon the following definitions;

$$a := \widetilde{F}_{11} = \sqrt{C_{11}}, \quad b := \widetilde{F}_{22} = \sqrt{C_{22} - C_{12}^2/C_{11}}, \quad \gamma := C_{12}/C_{11} \tag{2.5}$$

where elongation ratios a and b must be positive due to the conservation of mass, while shear γ may be of either sign. It is in this physical coordinate system with orthogonal base vectors $\{\vec{\mathbf{e}}_i\}$ where properties a, b, γ can be measured uniquely and unambiguously! In Fig. 2.1, a set of covariant base vectors $\{\vec{\mathbf{g}}_i\}$ are introduced that convect with the motion of a deformation. They are selected to be congruent with the physical base vectors $\{\vec{\mathbf{e}}_i\}$ at some reference time t_0, from which they then deform into oblique coordinates at current time t[20].

The orthogonal rotational contribution $\mathbf{Q}$ is considered to be

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.6}$$

where θ is the angle of rotation and can be described in terms of deformation gradient components [28]

$$\theta = \begin{cases} -\sin^{-1}(F_{21}/a) \\ \sin^{-1}\left((F_{12} - \gamma\, F_{11})/b\right) \end{cases} \tag{2.7}$$

One can use directly the components of the deformation gradient to define a, b and γ via

$$a := \sqrt{F_{11}^2 + F_{21}^2}, \quad b := \frac{F_{11}F_{22} - F_{12}F_{21}}{\sqrt{F_{11}^2 + F_{21}^2}}, \quad \gamma := \frac{F_{11}F_{12} + F_{21}F_{22}}{F_{11}^2 + F_{21}^2} \tag{2.8}$$

hence the rotation becomes

$$\mathbf{Q} = \begin{bmatrix} \frac{F_{11}}{\sqrt{F_{11}^2+F_{21}^2}} & -\frac{F_{21}}{\sqrt{F_{11}^2+F_{21}^2}} \\ \frac{F_{21}}{\sqrt{F_{11}^2+F_{21}^2}} & \frac{F_{11}}{\sqrt{F_{11}^2+F_{21}^2}} \end{bmatrix} \tag{2.9}$$

and the deformation gradient $\mathbf{F} = \mathbf{Q}\widetilde{\mathbf{F}}^{-1}$ is given by

$$\mathbf{F} = \begin{bmatrix} a\cos\theta & b\sin\theta + a\gamma\cos\theta \\ -a\sin\theta & b\cos\theta - a\gamma\sin\theta \end{bmatrix}. \tag{2.10}$$

By writing the two column vectors of $\mathbf{F}$ as $\{\vec{\mathbf{f}}_i\}$, $i = 1, 2$, and the two column vectors of $\mathbf{Q}$ as $\{\vec{\mathbf{e}}_i\}$, $i = 1, 2$, then according to Gram-Schmidt method these column vectors are related to each other through

$$\begin{aligned} \vec{\mathbf{e}}_1 &= \frac{\vec{\mathbf{f}}_1}{\left\|\vec{\mathbf{f}}_i\right\|} & \vec{\mathbf{e}}_1 &= \frac{\vec{\mathbf{f}}_1}{a} \\ \vec{\mathbf{e}}_2 &= \frac{\vec{\mathbf{f}}_2 - (\vec{\mathbf{f}}_2 \cdot \vec{\mathbf{e}}_1)\vec{\mathbf{e}}_1}{\left\|\vec{\mathbf{f}}_2 - (\vec{\mathbf{f}}_2 \cdot \vec{\mathbf{e}}_1)\vec{\mathbf{e}}_1\right\|} & \vec{\mathbf{e}}_2 &= \frac{\vec{\mathbf{f}}_2 - \gamma\vec{\mathbf{f}}_1}{b} \end{aligned} \tag{2.11}$$

The physical set of base vectors $[\{\vec{\mathbf{e}}_1\}\{\vec{\mathbf{e}}_2\}]$ rotate from a reference set of base vectors $[\{\vec{\imath}\}\{\vec{\jmath}\}]$ according to $[\{\vec{\mathbf{e}}_1\}\{\vec{\mathbf{e}}_2\}] = [\{\vec{\imath}\}\{\vec{\jmath}\}]\mathbf{Q}$, where $\mathbf{Q}$ comes from a $\mathbf{QR}$ decomposition of $\mathbf{F}$, specifically $\mathbf{Q} = \mathbf{F}\widetilde{\mathbf{F}}^{-1}$. Both of these coordinate frames are orthonormal.

The orthogonal base vectors $(\vec{\imath}, \vec{\jmath})$ describe a two-dimensional, Euclidean, point space in which our body of interest is embedded. The directions associated with these base vectors are considered to be fixed in the space. Which base vector associates with which direction is obtained by two relabelings. One is a mapping of a right-handed coordinate system into another right-handed coordinate system (identity map), and the other is a mapping a right-handed coordinate system into left-handed coordinate system. We determine an optimal Jacobian $[\hat{\mathbf{F}}]$ obtained by $[\hat{\mathbf{F}}] = [\mathbf{P}][\mathbf{F}][\mathbf{P}]^T$

in $(\hat{\imath}, \hat{\jmath})$ coordinate directions, wherein the orthogonal matrix $[\mathbf{P}]$ selects the Jacobian with the most prevailing upper-triangular matrix by rearranging the coordinate criterias, and hence, the rotation $[\mathbf{Q}]$ should be minimized. Therefore, the Green deformation $[\hat{\mathbf{C}}]$ expected in terms of an optimal Jacobian $[\hat{\mathbf{F}}]$ should be used instead of $[\mathbf{C}]$ found in Eq. (2.1) to compute the Laplace stretch. By considering the identity map, the coordinate criteria have no changes

$$[\mathbf{P}_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad (\vec{\imath}, \vec{\jmath}) \mapsto (\hat{\imath}, \hat{\jmath}) = (\vec{\imath}, \vec{\jmath}) \tag{2.12}$$

however, the second relabeling makes changes in the coordinate system

$$[\mathbf{P}_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Longrightarrow \quad (\vec{\imath}, \vec{\jmath}) \mapsto (\hat{\imath}, \hat{\jmath}) = (\vec{\jmath}, \vec{\imath}) \tag{2.13}$$

where $\det \mathbf{P}_0 = 1$ and $\det \mathbf{P}_1 = -1$. There is no rotation in these mapping. The identity map on the deformation gradient $\mathbf{F}$ associated with coordinate system $(\vec{\imath}, \vec{\jmath})$ imply an optimal Jacobian $[\hat{\mathbf{F}}]$ such that

$$[\mathbf{P}_0\mathbf{F}\mathbf{P}_0^T] = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \quad \Longrightarrow \quad [\hat{\mathbf{F}}] = \begin{bmatrix} \hat{F_{11}} & \hat{F_{12}} \\ \hat{F_{21}} & \hat{F_{22}} \end{bmatrix} \tag{2.14}$$

and $\mathbf{P}_1$ transforms the criterias of F_{ij} such that

$$[\mathbf{P}_1\mathbf{F}\mathbf{P}_1^T] = \begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix} \quad \Longrightarrow \quad [\hat{\mathbf{F}}] = \begin{bmatrix} \hat{F_{11}} & \hat{F_{12}} \\ \hat{F_{21}} & \hat{F_{22}} \end{bmatrix} \tag{2.15}$$

The way we assign directions to a coordinate system is done to produce an optimal upper-triangular optimal Jacobian matrix $[\hat{\mathbf{F}}]$, and thereby accompany the geometric interpretation discussed by Freed & Srinivasa [27]. This is achieved by pivoting the rows and columns when solving linear systems of equations to get more robust implementations, while there is no change in the problem. The relative strengths of the transverse shears that cut across the 1 and 2 directions are described by

$$\mathcal{F}_1 = \frac{|F_{21}|}{F_{11}} \tag{2.16a}$$

$$\mathcal{F}_2 = \frac{|F_{12}|}{F_{22}} \tag{2.16b}$$

and are obtained from the Jacobian of deformation [$\mathbf{F}$] associated with $(\vec{\imath}, \vec{\jmath})$. According to Alg. 1

Algorithm 1 Coordinate Pivoting

```
if F1 ≤ F2 then
    (î, ĵ) = (i⃗, j⃗)
else
    (î, ĵ) = (j⃗, i⃗)
end if
```

the 1-direction comes from the minimal transverse shear, and the 2-direction is obtained from coordinate orthogonality.

2.2 Quotient Law related to Convected Coordinate system

Quotient laws determine how the components of vector and tensor fields map from one coordinate system into another coordinate system [7]. They are linear transformations; *they are not tensor equations* [1]. They provide a connection between two otherwise separate configurations (manifolds).

Usage of the word 'push' implies moving a field forward in configurations along a path of: Lagrangian $\mapsto$ convected $\mapsto$ physical $\mapsto$ Eulerian. Usage of the word 'pull' implies moving a field along this same path of configurations, but in the reverse direction: Lagrangian $\leftarrow\!\shortmid$ convected $\leftarrow\!\shortmid$ physical $\leftarrow\!\shortmid$ Eulerian.

How our Jacobians describe the mapping of a tangent vector, for example, between these various configurations is illustrated in Fig. 2.2. There are four configurations that one can work with in a convected analysis: Lagrangian, Eulerian, physical and convected. The mapping of vector

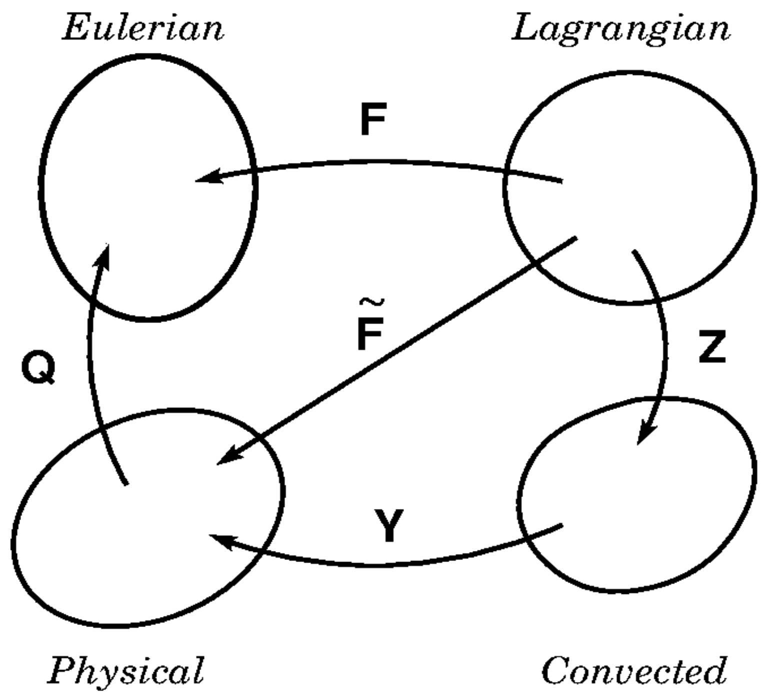

Figure 2.2: Mapping between various configurations

and tensor fields between these configurations is handled through their appropriate Jacobians [29, 20]. The deformation gradient **F** maps Lagrangian tangent vectors into Eulerian tangent vectors. The Laplace stretch $\widetilde{\mathbf{F}}$ maps Lagrangian tangent vectors into their counterparts in the physical frame of reference. Jacobian **Z** maps Lagrangian tangent vectors into convected tangent vectors, viz., into reference vectors for the convected state. Jacobian **Y** maps tangent vectors from the convected description into tangent vectors for the physical description. While **Q** rotates vectors from the physical frame of reference into Eulerian vectors. The inverse transpose of these Jacobians establish the maps for normal vectors. Mapping a convected vector or tensor field into the physical configuration transforms its 'convected components' into 'physical components'; hence, its name: physical configuration.

2.2.1 Field Transfer Between Convected And Lagrangian

To push a Lagrangian field quantified in coordinate system $(\vec{\imath}, \vec{\jmath})$ into a convected field quantified in coordinate system $\{\vec{\mathbf{g}}_i\}$ or, vice versa, to pull a convected field back into the Lagrangian frame, one must first construct a quotient law that governs this particular type of field transfer. We begin with the fact that Laplace stretch $\widetilde{\mathbf{F}}$ is a gradient, which itself is a product of gradients in that

$$\widetilde{F}^i_j = \frac{\partial \vec{x}^i}{\partial X^j} = \frac{\partial \vec{x}^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial X^j} \quad \text{or} \quad \widetilde{\mathbf{F}} = \mathbf{YZ} \quad \text{so that} \quad \mathbf{F} = \mathbf{QYZ} \tag{2.17}$$

wherein Lagrangian coordinates X^i exist in basis $[\{\vec{\imath}\}\{\vec{\jmath}\}]$, Physical coordinates $\vec{x}^i$ exist in basis $[\{\vec{\mathbf{e}}_1\}\{\vec{\mathbf{e}}_2\}]$, and convected coordinates ξ^k exist in basis $[\{\vec{\mathbf{g}}_1\}\{\vec{\mathbf{g}}_2\}]$. Like $\mathbf{Y}$, $\mathbf{Z}$ is a Jacobian matrix pertaining to a coordinate transformation, this time between the convected $[\{\vec{\mathbf{g}}_1\}\{\vec{\mathbf{g}}_2\}]$ and Lagrangian $[\{\vec{\imath}\}\{\vec{\jmath}\}]$ coordinate frames. It too appears in four forms

$$\mathbf{Z} := \left[\frac{\partial \xi^r}{\partial X^c}\right], \quad \mathbf{Z}^{-1} = \left[\frac{\partial X^r}{\partial \xi^c}\right], \quad \mathbf{Z}^{\mathsf{T}} = \left[\frac{\partial \xi^c}{\partial X^r}\right], \quad \mathbf{Z}^{-\mathsf{T}} = \left[\frac{\partial X^c}{\partial \xi^r}\right] \tag{2.18}$$

it follows that $\mathbf{Z} = \mathbf{Y}^{-1}\widetilde{\mathbf{F}} = \mathbf{\Lambda}^{-1}\mathbf{\Gamma}^{-1}\mathbf{\Lambda\Gamma}$, while $\mathbf{Z}^{-1} = \widetilde{\mathbf{F}}^{-1}\mathbf{Y} = \mathbf{\Gamma}^{-1}\mathbf{\Lambda}^{-1}\mathbf{\Gamma\Lambda}$. Transposes $\mathbf{Z}^{\mathsf{T}} = \widetilde{\mathbf{F}}^{\mathsf{T}}\mathbf{Y}^{-\mathsf{T}} = \mathbf{\Gamma}^{\mathsf{T}}\mathbf{\Lambda}\mathbf{\Gamma}^{-\mathsf{T}}\mathbf{\Lambda}^{-1}$ and $\mathbf{Z}^{-\mathsf{T}} = \mathbf{Y}^{\mathsf{T}}\widetilde{\mathbf{F}}^{-\mathsf{T}} = \mathbf{\Lambda}\mathbf{\Gamma}^{\mathsf{T}}\mathbf{\Lambda}^{-1}\mathbf{\Gamma}^{-\mathsf{T}}$ populate as lower-triangular matrices. Jacobian $\mathbf{Z}$ maps tangent vectors from the Lagrangian frame into the convected frame, while Jacobian $\mathbf{Z}^{-\mathsf{T}}$ maps normal vectors from the Lagrangian frame into the convected frame. Their inverses reverse the direction of these maps. In an absence of all shearing, i.e., whenever $\gamma = 0$, or in the presence of uniform elongation, i.e., whenever $a = b$, the $\mathbf{Z}$ Jacobian reduces to $\mathbf{I}$. Under these conditions, matrices $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ commute.

From this strategy, covariant vectors $\mathbf{w}$ in basis $[\{\vec{\imath}\}\{\vec{\jmath}\}]$ and $\boldsymbol{\omega}$ in basis $[\{\vec{\mathbf{g}}^1\}\{\vec{\mathbf{g}}^2\}]$ have maps that pull $\mathbf{w} \hookleftarrow \boldsymbol{\omega}$ and push $\mathbf{w} \mapsto \boldsymbol{\omega}$ as

$$\mathbf{w} = \mathbf{Z}^{\mathsf{T}}\boldsymbol{\omega} \qquad\qquad \boldsymbol{\omega} = \mathbf{Z}^{-\mathsf{T}}\mathbf{w}, \tag{2.19a}$$

contravariant vectors $\mathbf{w}$ in basis $[\{\vec{\imath}\}\{\vec{\jmath}\}]$ and $\boldsymbol{\omega}$ in basis $[\{\vec{\mathbf{g}}_1\}\{\vec{\mathbf{g}}_2\}]$ have maps that pull $\mathbf{w} \hookleftarrow \boldsymbol{\omega}$ and push $\mathbf{w} \mapsto \boldsymbol{\omega}$ as

$$\mathbf{w} = \mathbf{Z}^{-1}\boldsymbol{\omega} \qquad\qquad \boldsymbol{\omega} = \mathbf{Z}\mathbf{w}, \tag{2.19b}$$

covariant tensors $\mathbf{W}$ in basis $[\{\vec{\imath}\}\{\vec{\jmath}\}]$ and $\mathbf{\Omega}$ in basis $[\{\vec{\mathbf{g}}^1\}\{\vec{\mathbf{g}}^2\}]$ have maps that pull $\mathbf{W} \leftarrow\!\dashv \mathbf{\Omega}$ and push $\mathbf{W} \mapsto \mathbf{\Omega}$ as

$$\mathbf{W} = \mathbf{Z}^{\mathsf{T}}\mathbf{\Omega}\mathbf{Z} \qquad \mathbf{\Omega} = \mathbf{Z}^{-\mathsf{T}}\mathbf{W}\mathbf{Z}^{-1}, \tag{2.19c}$$

contravariant tensors $\mathbf{W}$ in basis $[\{\vec{\imath}\}\{\vec{\jmath}\}]$ and $\mathbf{\Omega}$ in basis$[\{\vec{\mathbf{g}}_1\}\{\vec{\mathbf{g}}_2\}]$ have maps that pull $\mathbf{W} \leftarrow\!\dashv \mathbf{\Omega}$ and push $\mathbf{W} \mapsto \mathbf{\Omega}$ as

$$\mathbf{W} = \mathbf{Z}^{-1}\mathbf{\Omega}\mathbf{Z}^{-\mathsf{T}} \qquad \mathbf{\Omega} = \mathbf{Z}\mathbf{W}\mathbf{Z}^{\mathsf{T}}, \tag{2.19d}$$

and mixed (right covariant) tensors $\mathbf{W}$ in basis $[\{\vec{\imath}\}\{\vec{\jmath}\}]$ and $\mathbf{\Omega}$ in basis $[\{\vec{\mathbf{g}}_1\}\{\vec{\mathbf{g}}^2\}]$ have maps that pull $\mathbf{W} \leftarrow\!\dashv \mathbf{\Omega}$ and push $\mathbf{W} \mapsto \mathbf{\Omega}$ as

$$\mathbf{W} = \mathbf{Z}^{-1}\mathbf{\Omega}\mathbf{Z} \qquad \mathbf{\Omega} = \mathbf{Z}\mathbf{W}\mathbf{Z}^{-1}. \tag{2.19e}$$

Note that this Jacobian determinant, $\det \mathbf{Z}$, has a value of one, so the above quotient laws pertain to all vectors and tensors, irrespective of weight.

2.2.2 Field Transfer Between Eulerian And Physical Fields

According to our quotient law, to map an Eulerian field into a physical field one must rotate the Eulerian field into the physical coordinate system $\{\vec{\mathbf{e}}_i\}$. Or, vice versa, one must push the field out of the physical coordinate system $\{\vec{\mathbf{e}}_i\}$ and finish by rotating that result into the Eulerian coordinate system $\{\hat{\mathbf{e}}_i\}$.

Putting this strategy to work, applying decomposition $\mathbf{F} = \mathbf{Q}\widetilde{\mathbf{F}} = \mathbf{Q}\mathbf{Y}\mathbf{Z}$, covariant and contravariant vectors $\mathbf{w}$ in basis $[\{\hat{\mathbf{e}}_1\}\{\hat{\mathbf{e}}_2\}]$ and ω in basis $[\{\vec{\mathbf{e}}_1\}\{\vec{\mathbf{e}}_2\}]$ have maps that push $\omega \mapsto \mathbf{w}$ and pull $\mathbf{w} \mapsto \omega$ as

$$\mathbf{w} = \mathbf{Q}\,\omega \tag{2.20a}$$

$$\omega = \mathbf{Q}^{\mathsf{T}}\mathbf{w} \tag{2.20b}$$

covariant and contravariant tensors $\mathbf{W}$ in basis $[\{\hat{\mathbf{e}}_1\}\{\hat{\mathbf{e}}_2\}]$ and $\boldsymbol{\Omega}$ in basis $[\{\vec{\mathbf{e}}_1\}\{\vec{\mathbf{e}}_2\}]$ have maps that push $\boldsymbol{\Omega} \mapsto \mathbf{W}$ and pull $\mathbf{W} \mapsto \boldsymbol{\Omega}$ as

$$\mathbf{W} = \mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^{\mathsf{T}} \tag{2.20c}$$

$$\boldsymbol{\Omega} = \mathbf{Q}^{\mathsf{T}}\mathbf{W}\mathbf{Q} \tag{2.20d}$$

where these scenarios in these case statements are direct maps between the physical and Eulerian frames.

2.3 Field Transfer Between Convected And Physical Fields

The transfer of vector or tensor fields between the convected $\{\vec{\mathbf{g}}_i\}$ and physical $\{\vec{\mathbf{e}}_i\}$ coordinate systems is governed by its Jacobian (or coordinate gradient) whose inverse exists. It appears in one of four forms

$$\mathbf{Y} := \left[\frac{\partial \vec{x}^r}{\partial \xi^c}\right], \quad \mathbf{Y}^{-1} = \left[\frac{\partial \xi^r}{\partial \vec{x}^c}\right], \quad \mathbf{Y}^{\mathsf{T}} = \left[\frac{\partial \vec{x}^c}{\partial \xi^r}\right], \quad \mathbf{Y}^{-\mathsf{T}} = \left[\frac{\partial \xi^c}{\partial \vec{x}^r}\right] \tag{2.21}$$

where r and c are the row and column indices, with coordinates $\vec{x}^i$ locating a mass point in coordinate system $\{\vec{\mathbf{e}}_i\}$, while coordinates ξ^i locate the same mass point in coordinate system $\{\vec{\mathbf{g}}_i\}$. In terms of the fundamental modes of Laplace stretch, viz., $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$, the various forms for this Jacobian become

$$\mathbf{Y} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}, \quad \mathbf{Y}^{-1} = \boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}^{-1}, \quad \mathbf{Y}^{\mathsf{T}} = \boldsymbol{\Lambda}\boldsymbol{\Gamma}^{\mathsf{T}}, \quad \mathbf{Y}^{-\mathsf{T}} = \boldsymbol{\Gamma}^{-\mathsf{T}}\boldsymbol{\Lambda}^{-1} \tag{2.22}$$

recalling that $\widetilde{\mathbf{F}} = \mathbf{\Lambda}\mathbf{\Gamma}$, where

$$\mathbf{Y} = \begin{bmatrix} a & b\gamma \\ 0 & b \end{bmatrix} \quad \text{and} \quad \mathbf{Y}^{-1} = \begin{bmatrix} 1/a & -\gamma/a \\ 0 & 1/b \end{bmatrix} \tag{2.23}$$

whose transposes $\mathbf{Y}^{\mathsf{T}}$ and $\mathbf{Y}^{-\mathsf{T}}$ populate as lower-triangular matrices. Matrices $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ do not commute, and as such, $\mathbf{Y} = \mathbf{\Gamma}\mathbf{\Lambda}$ is distinct from $\widetilde{\mathbf{F}} = \mathbf{\Lambda}\mathbf{\Gamma}$. Jacobian $\mathbf{Y}$ maps tangent vectors from the oblique convected frame into the orthonormal physical frame, while Jacobian $\mathbf{Y}^{-\mathsf{T}}$ maps normal vectors from the oblique convected frame into the orthonormal physical frame. Jacobians $\mathbf{Y}^{-1}$ and $\mathbf{Y}^{\mathsf{T}}$ run these maps in the reverse direction.

Given this set of Jacobian matrices, all covariant vectors $\widetilde{\mathbf{w}}$ in basis $[\{\vec{\mathbf{e}}^1\}\{\vec{\mathbf{e}}^2\}]$ and $\boldsymbol{\omega}$ in basis $[\{\vec{\mathbf{g}}^1\}\{\vec{\mathbf{g}}^2\}]$ push $\boldsymbol{\omega} \mapsto \widetilde{\mathbf{w}}$ and pull $\boldsymbol{\omega} \mapsfrom \widetilde{\mathbf{w}}$ via

$$\widetilde{\mathbf{w}} = \mathbf{Y}^{-\mathsf{T}}\boldsymbol{\omega} \qquad \text{and} \qquad \boldsymbol{\omega} = \mathbf{Y}^{\mathsf{T}}\widetilde{\mathbf{w}}, \tag{2.24a}$$

all contravariant vectors $\widetilde{\mathbf{w}}$ in basis $[\{\vec{\mathbf{e}}_1\}\{\vec{\mathbf{e}}_2\}]$ and $\boldsymbol{\omega}$ in basis $[\{\vec{\mathbf{g}}_1\}\{\vec{\mathbf{g}}_2\}]$ push $\boldsymbol{\omega} \mapsto \widetilde{\mathbf{w}}$ and pull $\boldsymbol{\omega} \mapsfrom \widetilde{\mathbf{w}}$ via

$$\widetilde{\mathbf{w}} = \mathbf{Y}\boldsymbol{\omega} \qquad \text{and} \qquad \boldsymbol{\omega} = \mathbf{Y}^{-1}\widetilde{\mathbf{w}}, \tag{2.24b}$$

all covariant tensors $\widetilde{\mathbf{W}}$ in basis $[\{\vec{\mathbf{e}}^1\}\{\vec{\mathbf{e}}^2\}]$ and $\mathbf{\Omega}$ in basis $[\{\vec{\mathbf{g}}^1\}\{\vec{\mathbf{g}}^2\}]$ push $\mathbf{\Omega} \mapsto \widetilde{\mathbf{W}}$ and pull $\mathbf{\Omega} \mapsfrom \widetilde{\mathbf{W}}$ via

$$\widetilde{\mathbf{W}} = \mathbf{Y}^{-\mathsf{T}}\mathbf{\Omega}\mathbf{Y}^{-1} \qquad \text{and} \qquad \mathbf{\Omega} = \mathbf{Y}^{\mathsf{T}}\widetilde{\mathbf{W}}\mathbf{Y}, \tag{2.24c}$$

all contravariant tensors $\widetilde{\mathbf{W}}$ in basis $[\{\vec{\mathbf{e}}_1\}\{\vec{\mathbf{e}}_2\}]$ and $\mathbf{\Omega}$ in basis $[\{\vec{\mathbf{g}}_1\}\{\vec{\mathbf{g}}_2\}]$ push $\mathbf{\Omega} \mapsto \widetilde{\mathbf{W}}$ and pull $\mathbf{\Omega} \mapsfrom \widetilde{\mathbf{W}}$ via

$$\widetilde{\mathbf{W}} = \mathbf{Y}\mathbf{\Omega}\mathbf{Y}^{\mathsf{T}} \qquad \text{and} \qquad \mathbf{\Omega} = \mathbf{Y}^{-1}\widetilde{\mathbf{W}}\mathbf{Y}^{-\mathsf{T}}, \tag{2.24d}$$

and all mixed (right covariant) tensors $\widetilde{\mathbf{W}}$ in basis $[\{\vec{\mathbf{e}}_1\}\{\vec{\mathbf{e}}^2\}]$ and $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}$ in basis $[\{\vec{\mathbf{g}}_1\}\{\vec{\mathbf{g}}^2\}]$ push $\boldsymbol{\Omega} \mapsto \widetilde{\mathbf{W}}$ and pull $\boldsymbol{\Omega} \hookleftarrow \widetilde{\mathbf{W}}$ via

$$\widetilde{\mathbf{W}} = \mathbf{Y}\boldsymbol{\Omega}\mathbf{Y}^{-1} \qquad \text{and} \qquad \boldsymbol{\Omega} = \mathbf{Y}^{-1}\widetilde{\mathbf{W}}\mathbf{Y}. \tag{2.24e}$$

These maps are for absolute vector and tensor fields, since the Jacobian determinant $\det \mathbf{Y} = ab$ plays no role, see Oldroyd [5] and Sokolnikoff [7]. Absolute fields are sufficient for our needs. Left-covariant mixed tensors exist, but are not needed here; consequently, they are not introduced.

2.3.1 Derivatives

A time derivative $\circ$ taken in the convected coordinate system $\{\vec{\mathbf{g}}_i\}$ pushes forward as a Lie derivative in the Physical coordinate system $\{\vec{\mathbf{e}}_i\}$. To quantify these Lie derivatives requires the introduction of a convected velocity gradient for the physical coordinate frame $\{\vec{\mathbf{e}}_i\}$ that we denote as

$$\dot{\mathbf{H}} := \dot{H}^i_j \, \vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}^j \qquad \text{with} \qquad \dot{H}^i_j = \frac{\partial \dot{\vec{x}}^i}{\partial \xi^k} \frac{\partial \xi^k}{\partial \vec{x}^j} \tag{2.25a}$$

or, alternatively, in terms of the Jacobian $\mathbf{Y}$ and its decomposition $\boldsymbol{\Gamma\Lambda}$,

$$\dot{\mathbf{H}} = \dot{\mathbf{Y}} \cdot \mathbf{Y}^{-1} \qquad \text{or} \qquad \dot{\mathbf{H}} = \dot{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Gamma}^{-1} + \boldsymbol{\Gamma}(\dot{\boldsymbol{\Lambda}} \cdot \boldsymbol{\Lambda}^{-1})\boldsymbol{\Gamma}^{-1} \tag{2.25b}$$

whose components populate an upper-triangular matrix.

With a velocity gradient $\dot{\mathbf{H}}$ defined over $\{\vec{\mathbf{e}}_i\}$ now in hand, the temporal derivative of a covariant vector defined in $\{\vec{\mathbf{g}}_i\}$ pushes forward into the physical frame $\{\vec{\mathbf{e}}_i\}$ as $\dot{\omega} \mapsto \mathring{\widetilde{\mathbf{w}}}$, given that $\omega \mapsto \widetilde{\mathbf{w}}$, whose Lie derivative is defined by

$$\mathring{\widetilde{\mathbf{w}}} = \mathbf{Y}^{-\mathsf{T}} \cdot \dot{\omega} \qquad\qquad \widetilde{\mathbf{w}} := \mathring{\widetilde{\mathbf{w}}} + \dot{\mathbf{H}}^{\mathsf{T}} \cdot \widetilde{\mathbf{w}}, \tag{2.26a}$$

the derivative of a contravariant vector pushes as $\dot{\boldsymbol{\omega}} \mapsto \overset{\circ}{\widetilde{\mathbf{w}}}$, given that $\boldsymbol{\omega} \mapsto \widetilde{\mathbf{w}}$, whose Lei derivative is defined by

$$\overset{\circ}{\widetilde{\mathbf{w}}} = \mathbf{Y} \cdot \dot{\boldsymbol{\omega}} \qquad \overset{\triangledown}{\widetilde{\mathbf{w}}} := \overset{\circ}{\widetilde{\mathbf{w}}} - \dot{\mathbf{H}} \cdot \widetilde{\mathbf{w}}, \tag{2.26b}$$

the derivative of a covariant tensor pushes as $\dot{\boldsymbol{\Omega}} \mapsto \overset{\circ}{\widetilde{\mathbf{W}}}$, given that $\boldsymbol{\Omega} \mapsto \widetilde{\mathbf{W}}$, whose Lie derivative is defined by

$$\overset{\circ}{\widetilde{\mathbf{W}}} = \mathbf{Y}^{-\mathsf{T}} \cdot \dot{\boldsymbol{\Omega}} \cdot \mathbf{Y}^{-1} \qquad \overset{\triangledown}{\widetilde{\mathbf{W}}} := \overset{\circ}{\widetilde{\mathbf{W}}} + \dot{\mathbf{H}}^{\mathsf{T}} \cdot \widetilde{\mathbf{W}} + \widetilde{\mathbf{W}} \cdot \dot{\mathbf{H}}, \tag{2.26c}$$

the derivative of a contravariant tensor pushes as $\dot{\boldsymbol{\Omega}} \mapsto \overset{\circ}{\widetilde{\mathbf{W}}}$, given that $\boldsymbol{\Omega} \mapsto \widetilde{\mathbf{W}}$, whose Lie derivative is defined by

$$\overset{\circ}{\widetilde{\mathbf{W}}} = \mathbf{Y} \cdot \dot{\boldsymbol{\Omega}} \cdot \mathbf{Y}^{\mathsf{T}} \qquad \overset{\triangledown}{\widetilde{\mathbf{W}}} := \overset{\circ}{\widetilde{\mathbf{W}}} - \dot{\mathbf{H}} \cdot \widetilde{\mathbf{W}} - \widetilde{\mathbf{W}} \cdot \dot{\mathbf{H}}^{\mathsf{T}}, \tag{2.26d}$$

and the derivative of a mixed tensor pushes as $\dot{\boldsymbol{\Omega}} \mapsto \overset{\circ}{\widetilde{\mathbf{W}}}$, given that $\boldsymbol{\Omega} \mapsto \widetilde{\mathbf{W}}$, whose Lie derivative is defined by

$$\overset{\circ}{\widetilde{\mathbf{W}}} = \mathbf{Y} \cdot \dot{\boldsymbol{\Omega}} \cdot \mathbf{Y}^{-1} \qquad \overset{\triangledown}{\widetilde{\mathbf{W}}} := \overset{\circ}{\widetilde{\mathbf{W}}} - \dot{\mathbf{H}} \cdot \widetilde{\mathbf{W}} + \widetilde{\mathbf{W}} \cdot \dot{\mathbf{H}} \tag{2.26e}$$

wherein $\dot{\mathbf{H}}^{\mathsf{T}}$ is taken to mean $\mathbf{Y}^{-\mathsf{T}} \cdot \dot{\mathbf{Y}}^{\mathsf{T}} = \boldsymbol{\Gamma}^{-\mathsf{T}} \cdot \dot{\boldsymbol{\Gamma}}^{\mathsf{T}} + \boldsymbol{\Gamma}^{-\mathsf{T}}(\dot{\boldsymbol{\Lambda}} \cdot \boldsymbol{\Lambda}^{-1})\boldsymbol{\Gamma}^{\mathsf{T}}$ whose components populate a lower-triangular matrix. The pull versions of these maps follow straightaway; consequently, they are not presented.

2.4 Convected Base Vectors

We derived a Jacobian that maps a set of rectangular, Cartesian, base vectors into a set of oblique, Cartesian, base vectors. These oblique vectors convect with the motion of a deformation within a neighborhood surrounding any particle $\mathcal{P}$ whose deformation gradient $\mathbf{F}(\mathcal{P})$ exists. Specifically, the set of covariant base vectors $\{\vec{\mathbf{g}}_i\}$ arise from the map

$$\{\vec{\mathbf{g}}_i\} = \mathbf{\Lambda}\mathbf{\Gamma}^{\mathsf{T}}\{\vec{\mathbf{e}}_i\}, \tag{2.27}$$

or

$$\begin{Bmatrix} \vec{\mathbf{g}}_1 \\ \vec{\mathbf{g}}_2 \end{Bmatrix} = \begin{bmatrix} a & 0 \\ b\gamma & b \end{bmatrix} \begin{Bmatrix} \vec{\mathbf{e}}_1 \\ \vec{\mathbf{e}}_2 \end{Bmatrix} \tag{2.28}$$

that invert as

$$\{\vec{\mathbf{e}}_i\} = \mathbf{\Gamma}^{-\mathsf{T}}\mathbf{\Lambda}^{-1}\{\vec{\mathbf{g}}_i\} \tag{2.29}$$

or

$$\begin{Bmatrix} \vec{\mathbf{e}}_1 \\ \vec{\mathbf{e}}_2 \end{Bmatrix} = \begin{bmatrix} 1/a & 0 \\ -\gamma/a & 1/b \end{bmatrix} \begin{Bmatrix} \vec{\mathbf{g}}_1 \\ \vec{\mathbf{g}}_2 \end{Bmatrix}. \tag{2.30}$$

Their duals $\{\vec{\mathbf{g}}_i\}$, a set of contravariant base vectors $\{\vec{\mathbf{g}}^i\}$, arise from the map

$$\{\vec{\mathbf{g}}^i\} = \mathbf{\Lambda}^{-1}\mathbf{\Gamma}^{-1}\{\vec{\mathbf{e}}^i\} \tag{2.31}$$

or

$$\begin{Bmatrix} \vec{\mathbf{g}}^1 \\ \vec{\mathbf{g}}^2 \end{Bmatrix} = \begin{bmatrix} 1/a & -\gamma/a \\ 0 & 1/b \end{bmatrix} \begin{Bmatrix} \vec{\mathbf{e}}^1 \\ \vec{\mathbf{e}}^2 \end{Bmatrix} \tag{2.32}$$

which invert as

$$\{\vec{\mathbf{e}}^i\} = \mathbf{\Gamma}\mathbf{\Lambda}\{\vec{\mathbf{g}}^i\} \tag{2.33}$$

or

$$\begin{Bmatrix} \vec{\mathbf{e}}^1 \\ \vec{\mathbf{e}}^2 \end{Bmatrix} = \begin{bmatrix} a & b\gamma \\ 0 & b \end{bmatrix} \begin{Bmatrix} \vec{\mathbf{g}}^1 \\ \vec{\mathbf{g}}^2 \end{Bmatrix} \tag{2.34}$$

Note that $\vec{\mathbf{e}}_i \equiv \vec{\mathbf{e}}^i$, because they share a common, rectangular, Cartesian, coordinate system; however, $\vec{\mathbf{g}}_i \neq \vec{\mathbf{g}}^i$, but they do obey $\vec{\mathbf{g}}^i \cdot \vec{\mathbf{g}}_j = \delta^i_j$ where δ^i_j is the Kronecker delta.

The relative area associated with this set of convected base vectors is

$$\frac{A_{\{\vec{\mathbf{g}}_i\}}}{A_{\{\vec{\mathbf{e}}_i\}}} = \det \mathbf{F} = \det \mathbf{F}^{-1} = ab \tag{2.35}$$

in the above formulæ, a, b are two positive elongations and γ is a shear.

2.5 Convected Metrics Associated With Coordinate Basis $\{\vec{\mathbf{g}}^i\}$

The Riemannian metric tensor $\mathbf{g}$ associated with the convected coordinate system describes the geometry of Laplace stretch in a deformed body where the coordinate system is oblique Cartesian. These coordinates are tangents to the curvilinear coordinates (located at the mass point where the deformation gradient $\mathbf{F}$ is quantified) from which Lodge [2, 13, 14, 6, 15, 1, 16, 17, 18] constructed his body metric tensor.

The ability to work with an oblique, Cartesian, coordinate system instead of having to work with a general, curvilinear, coordinate system, in accordance with our hypothesis, affords a practical utility to the convected tensor analysis presented herein. The convected metric can be quantified given any state of deformation, a feature that has been absent until now. The metric tensor $\mathbf{g} = g_{ij}\, \vec{\mathbf{g}}^i \otimes \vec{\mathbf{g}}^j$ has symmetric covariant components $g_{ij} := \vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = \vec{\mathbf{g}}_j \cdot \vec{\mathbf{g}}_i = g_{ji}$ where

$$g_{ij} = \begin{bmatrix} a^2 & ab\gamma \\ ab\gamma & b^2(1+\gamma^2) \end{bmatrix} \tag{2.36}$$

that we [20] derived from Eq. (2.28). The dual to this metric tensor $\mathbf{g}^{-1} = g^{ij}\, \vec{\mathbf{g}}_i \otimes \vec{\mathbf{g}}_j$ has symmetric contravariant components $g^{ij} := \vec{\mathbf{g}}^i \cdot \vec{\mathbf{g}}^j = \vec{\mathbf{g}}^j \cdot \vec{\mathbf{g}}^i = g^{ji}$ where

$$g^{ij} = \begin{bmatrix} \big(1+\gamma^2+(\beta-\alpha\gamma)^2\big)/a^2 & -\big(\gamma(1+\alpha^2)-\alpha\beta\big)/ab \\ -\big(\gamma(1+\alpha^2)-\alpha\beta\big)/ab & (1+\alpha^2)/b^2 \end{bmatrix} \tag{2.37}$$

These convected metrics obey $\mathbf{g}^{-1}\mathbf{g} = \boldsymbol{\delta}$. Furthermore, in some reference state associated with time t_0, one has $\mathbf{g}_0 \equiv \mathbf{g}_0^{-1} = \boldsymbol{\delta}$ whereat $a_0 = b_0 = 1$ and $\gamma_0 = 0$.

In terms of the Jacobian $\mathbf{Y}$ (the matrix present in Eq. 2.34) introduced by the authors [20], the convected metric $\mathbf{g} = \mathbf{Y}^{\mathsf{T}}\mathbf{Y}$ and its dual $\mathbf{g}^{-1} = \mathbf{Y}^{-1}\mathbf{Y}^{-\mathsf{T}}$ have components described in convected bases $\{\vec{\mathbf{g}}^i\}$ and $\{\vec{\mathbf{g}}_i\}$, respectively. Their analogs in classical mechanics are the Lagrangian deformation tensors of Green [26] $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$ and Cauchy [30] $\mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-\mathsf{T}}$.

2.6 Convected Strain Associated With Coordinate Basis $\{\vec{\mathbf{g}}^i\}$

Strain, in the sense of Green [26], is a difference between two Riemannian metrics, i.e., it is a measure of the change in geometry or change in shape between two otherwise arbitrary states. Typically, a reference configuration is selected that has the geometric interpretation of a cube, i.e., with shears of zero and elongation ratios of one, whose metrics we denote as $\mathbf{g}_0$ and $\mathbf{g}_0^{-1}$.

From our analysis of a cube being transformed into a parallelepiped, the convected, covariant, strain tensor of Lodge [6, 1] $\boldsymbol{\varepsilon} = \varepsilon_{ij}\, \vec{\mathbf{g}}^i \otimes \vec{\mathbf{g}}^j$, when quantified via the Laplace stretch $\widetilde{\mathbf{F}}$ arising from a Gram-Schmidt decomposition of the deformation gradient $\mathbf{F}$, has components

$$\boldsymbol{\varepsilon} := \tfrac{1}{2}(\mathbf{g} - \mathbf{g}_0) = \frac{1}{2}\begin{bmatrix} a^2 - 1 & ab\gamma \\ ab\gamma & b^2(1+\gamma^2) - 1 \end{bmatrix} \tag{2.38}$$

while the convected contravariant strain tensor has components

$$\boldsymbol{\mathcal{E}} := \tfrac{1}{2}(\mathbf{g}_0^{-1} - \mathbf{g}^{-1}) = \frac{1}{2}\begin{bmatrix} \big(a^2 - 1 - \gamma^2 - (\beta - \alpha\gamma)^2\big)/a^2 & \big(\gamma(1+\alpha^2) - \alpha\beta\big)/ab \\ \big(\gamma(1+\alpha^2) - \alpha\beta\big)/ab & (b^2 - 1 - \alpha^2)/b^2 \end{bmatrix} \tag{2.39}$$

wherein $\mathbf{g}_0 = \delta_{ij}\, \vec{\mathbf{g}}_0^i \otimes \vec{\mathbf{g}}_0^j$ and $\mathbf{g}_0^{-1} = \delta^{ij}\, \vec{\mathbf{g}}_{0,i} \otimes \vec{\mathbf{g}}_{0,j}$ are the metrics that are defined the geometric interpretation of a cube in reference configuration with shears of zero and elongation ratios of one.

2.7 Convected Stress Associated With Coordinate Basis $\{\vec{\mathbf{g}}^i\}$

We [20] decomposed Laplace stretch $\widetilde{\mathbf{F}}$ into a product of two gradients, viz., $\widetilde{\mathbf{F}} = \mathbf{YZ}$. In a Lagrangian frame of reference, Jacobian $\mathbf{F}$ describing the motion. The deformation gradient decomposes as $\mathbf{F} = \mathbf{QYZ}$. Here tensor $\mathbf{Q}$ is proper orthogonal (viz., $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathsf{T}}$ with $\det \mathbf{Q} = +1$) while tensors $\widetilde{\mathbf{F}}$, $\mathbf{Y}$ and $\mathbf{Z}$ are upper-triangular.

The Jacobian matrices in this decomposition for Laplace stretch $\widetilde{\mathbf{F}} = \mathbf{YZ}$, including inverses

$\widetilde{\mathbf{F}}^{-1} = \mathbf{Z}^{-1}\mathbf{Y}^{-1}$, can be written in terms of the physical measures for Laplace stretch a, b, γ defined in Eq. (2.5) as [20]

$$\mathbf{Y} = \begin{bmatrix} a & b\gamma \\ 0 & b \end{bmatrix}, \quad \mathbf{Y}^{-1} = \begin{bmatrix} 1/a & -\gamma/a \\ 0 & 1/b \end{bmatrix} \tag{2.40}$$

and

$$\mathbf{Z} = \begin{bmatrix} 1 & \frac{a-b}{a}\,\gamma \\ 0 & 1 \end{bmatrix}, \mathbf{Z}^{-1} = \begin{bmatrix} 1 & -\frac{a-b}{a}\,\gamma \\ 0 & 1 \end{bmatrix} \tag{2.41}$$

where $\det \mathbf{Z} = 1$ and, from the conservation of mass, $\det \mathbf{Y} = \det \widetilde{\mathbf{F}} = \det \mathbf{F} = ab$. Jacobians $\mathbf{Y}$ and $\mathbf{Z}$ map fields in-to and out-of the convected coordinate system. Specifically, they transform vector and tensor components between the convected coordinate system and its two, neighboring, orthonormal, coordinate systems (in a push/pull sense, see Ref. [20, App. A]), viz., the physical and the Lagrangian coordinate systems. The Lagrangian and physical coordinate systems are rectangular Cartesian. The convected coordinate system is oblique Cartesian [20].

Here the Eulerian components of $\mathbf{L}$ and $\mathbf{T}$, the velocity gradient and Cauchy stress, are taken to be evaluated in an orthonormal coordinate system. Consequently, their covariant, contravariant and mixed tensor components are equivalent, i.e., $L_{ij} \equiv L^{ij} \equiv L^i_j$ and $T_{ij} \equiv T^{ij} \equiv T^i_j$ because the base vectors and their duals are co-linear with one another [7]. In contrast, the base vectors for the convected coordinate system and their duals are not co-linear with one another. The convected basis $\{\vec{\mathbf{g}}_i\}$ is not orthonormal; it is oblique [20].

We map the mixed components of tensors $\mathbf{L}$ and $\mathbf{T}$, first pulling their Eulerian components back into their associated Lagrangian components, and then pushing these Lagrangian components forward into their convected components. The reason for using mixed tensor components is because Eulerian components L^i_j for the velocity gradient map into mixed components $\mathrm{d}\eta^i_j$ for a convected velocity gradient that are upper-triangular, which in turn vastly simplifies the expression for stress power used in the construction of constitutive equations; hence, the motivation. The property of triangularity would be lost if either covariant or contravariant components had been selected.

The mechanical power caused by stressing a deformable body, i.e., Eq. (4.16), can also be expressed as

$$\dot{W} = \mathrm{tr}(\boldsymbol{\sigma}\,\dot{\boldsymbol{\eta}}) = \sigma^i_j\,\dot{\eta}^j_i = \hat{\sigma}^i_j\,\dot{\hat{\eta}}^j_i \tag{2.42}$$

wherein

$$\boldsymbol{\sigma} = \hat{\sigma}^i_j\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}^j \quad \text{with} \quad \hat{\sigma}^i_j := [Q^{-1}]^i_k T^k_\ell Q^\ell_j \tag{2.43}$$

where $\sigma^i_j = [Y^{-1}]^i_k \hat{\sigma}^k_\ell Y^\ell_j$ maps the physical components $\hat{\sigma}^i_j$ into the convected components σ^i_j of stress $\boldsymbol{\sigma} = \sigma^i_j\,\vec{\mathbf{g}}_i \otimes \vec{\mathbf{g}}^j$, while Cauchy stress $\mathbf{T} = T^i_j\,\vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}^j$ follows from the physical components of stress $\hat{\sigma}^i_j$ through a rotation described by $T^i_j = Q^i_k \hat{\sigma}^k_\ell [Q^{-1}]^\ell_j$. Furthermore

$$\dot{\boldsymbol{\eta}} = \dot{\hat{\eta}}^i_j\,\vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}^j \quad \text{with} \quad \dot{\hat{\eta}}^i_j := \dot{Y}^i_k\,[Y^{-1}]^k_j + Y^i_k\,\dot{Z}^k_\ell\,[Z^{-1}]^\ell_n [Y^{-1}]^n_j \tag{2.44}$$

where $\dot{\eta}^i_j = [Y^{-1}]^i_k\,\dot{\hat{\eta}}^k_\ell\,Y^\ell_j$ maps the physical components $\dot{\hat{\eta}}^j_i$ into the convected components $\dot{\eta}^j_i$ of velocity gradient $\dot{\boldsymbol{\eta}} = \dot{\eta}^i_j\,\vec{\mathbf{g}}_i \otimes \vec{\mathbf{g}}^j$. In terms of the physical components of convected tensors $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\eta}}$, the work expended by stressing a deformable body can be expressed as

$$\dot{W} = \hat{\sigma}^i_j\,\dot{\hat{\eta}}^j_i = \mathrm{tr}\left(\begin{bmatrix} \hat{\sigma}^1_1 & \hat{\sigma}^1_2 \\ \hat{\sigma}^2_1 & \hat{\sigma}^2_2 \end{bmatrix} \begin{bmatrix} \dot{a}/a & a\,\dot{\gamma}/b \\ 0 & \dot{b}/b \end{bmatrix}\right) \tag{2.45}$$

Proof: Pulling back the mixed-component representations of the Eulerian Cauchy-stress and velocity-gradient tensors into their Lagrangian counterparts implies that $\dot{W} = \mathrm{tr}(\mathbf{TL})$ becomes $\dot{W} = \mathrm{tr}(\mathbf{F}^{-1}\mathbf{TF} \cdot \mathbf{F}^{-1}\mathbf{LF}) = \mathrm{tr}(\mathbf{F}^{-1}\mathbf{TF} \cdot \mathbf{F}^{-1}\dot{\mathbf{F}})$, cf. Oldroyd [5]. Pushing these Lagrangian fields forward into the convected frame gives $\dot{W} = \mathrm{tr}(\mathbf{ZF}^{-1}\mathbf{TFZ}^{-1} \cdot \mathbf{ZF}^{-1}\,\dot{\mathbf{F}}\,\mathbf{Z}^{-1})$, cf. Freed & Zamani [20, App. A]. A **QR** decomposition of **F**, viz., $\mathbf{Q}\widetilde{\mathbf{F}}$ [4], and the further decomposition of Laplace stretch $\widetilde{\mathbf{F}}$ into product **YZ** [20] gives $\mathbf{F} = \mathbf{QYZ}$ so that $\dot{W} = \mathrm{tr}[\mathbf{Y}^{-1}\mathbf{Q}^{-1}\mathbf{TQY} \cdot (\mathbf{Y}^{-1}\mathbf{Q}^{-1}\,\dot{\mathbf{Q}}\,\mathbf{Y} + \mathbf{Y}^{-1}\,\dot{\mathbf{Y}} + \dot{\mathbf{Z}}\,\mathbf{Z}^{-1})]$. The first of the above three traces is zero, because it is a trace between a symmetric stress **T** and a skew-symmetric spin $\dot{\mathbf{Q}}\,\mathbf{Q}^{-1}$; specifically, from the properties of the trace, $\mathrm{tr}[\mathbf{Y}^{-1}\mathbf{Q}^{-1}\mathbf{TQY} \cdot \mathbf{Y}^{-1}\mathbf{Q}^{-1}\,\dot{\mathbf{Q}}\,\mathbf{Y}) = \mathrm{tr}(\mathbf{T} \cdot \dot{\mathbf{Q}}\,\mathbf{Q}^{-1}) = 0$. This leaves $\dot{W} = \mathrm{tr}[\mathbf{Y}^{-1}\mathbf{Q}^{-1}\mathbf{TQY} \cdot (\mathbf{Y}^{-1}\,\dot{\mathbf{Y}} + \dot{\mathbf{Z}}\,\mathbf{Z}^{-1})]$, which we write as $\dot{W} = \mathrm{tr}(\boldsymbol{\sigma}\,\dot{\boldsymbol{\eta}})$, whose components are evaluated in the convected coordinate system $\{\vec{\mathbf{g}}_i\}$.

This rearranges to $\dot{W} = \mathrm{tr}[\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q}\cdot(\dot{\mathbf{Y}}\,\mathbf{Y}^{-1}+\mathbf{Y}\,\dot{\mathbf{Z}}\,\mathbf{Z}^{-1}\mathbf{Y}^{-1})]$ where now the components are evaluated in the physical coordinate system $\{\vec{\mathbf{e}}_i\}$, which is rectangular Cartesian. Furthermore, as Jacobian $\mathbf{Y}$ establishes this mapping, it implicates that these are in fact physical components, because the metric for this convected coordinate system is $\mathbf{g} = \mathbf{Y}^{\mathsf{T}}\mathbf{Y}$ with covariant base vectors that transform according to $\{\vec{\mathbf{g}}_1\,\vec{\mathbf{g}}_2\}^{\mathsf{T}} = \mathbf{Y}^{\mathsf{T}}\{\vec{\mathbf{e}}_1\,\vec{\mathbf{e}}_2\}^{\mathsf{T}}$ and contravariant base vectors that go as $\{\vec{\mathbf{g}}^1\,\vec{\mathbf{g}}^2\}^{\mathsf{T}} = \mathbf{Y}^{-1}\{\vec{\mathbf{e}}^1\,\vec{\mathbf{e}}^2\}^{\mathsf{T}}$ [20]. In short, $\dot{W} = \mathrm{tr}(\hat{\boldsymbol{\sigma}} \cdot \dot{\mathbf{Y}}\,\mathbf{Y}^{-1}) + \mathrm{tr}(\boldsymbol{\sigma} \cdot \dot{\mathbf{Z}}\,\mathbf{Z}^{-1})$ where the first trace is evaluated in $\{\vec{\mathbf{e}}_i\}$ and the second is evaluated in $\{\vec{\mathbf{g}}_i\}$, which when pushed into the physical coordinate system $\{\vec{\mathbf{e}}_i\}$ allows one to rewrite $\dot{W}$ as $\hat{\sigma}^i_j\,\dot{\hat{\eta}}^j_i$.

A hypothesis is put forward that, in effect, replace having to work with a contraction between a frame-indifferent stress tensor and a frame-indifferent strain-rate tensor. Instead, one works with a sum of scalar products between paired elements taken from sets of observer-indifferent attributes for stress and strain rate. This hypothesis produces two distinct modes of deformation. Material anisotropy is not addressed here.

Note: The hypothesis introduces three extensive variables $\{\varepsilon_1, \varepsilon_2, \gamma\}$ and three intensive variables $\{\sigma_1, \sigma_2, \tau\}$. These variables are stress and strain 'attributes' in the terminology of Criscione [24].

Note: In the hypothesis, the three strain-rates $\{\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\gamma}\}$ are exact differentials; consequently, their strains $\{\varepsilon_1, \varepsilon_2, \gamma\}$ are two-state fields, independent of the motion traversed between states. This is a tacit requirement from thermodynamics: *The work needed to go from one equilibrium state to a neighboring equilibrium state does not depend upon the path traversed* [31, 32].

2.7.1 Two-Mode Theory

Hypothesis: Trace $\mathrm{tr}(\boldsymbol{\sigma}\,\dot{\boldsymbol{\eta}}) = \hat{\sigma}^i_j\,\dot{\hat{\eta}}^j_i$ establishes stress power $\dot{W}$ in terms of physical components describing a convected stress tensor and a convected velocity-gradient tensor. This frame-indifferent thermodynamic property can be decomposed into a set of three, conjugate, stress-strain pairs:

$$\dot{W} = \hat{\sigma}^i_j\,\dot{\hat{\eta}}^j_i = \sum_{i=1}^{2}(\sigma_i\,\dot{\varepsilon}_i) + \tau\,\dot{\gamma} \tag{2.46}$$

where $\{\sigma_1, \sigma_2, \tau\}$ defines a set of three, intensive, scalar-valued stresses whose thermodynamic conjugates $\{\varepsilon_1, \varepsilon_2, \gamma\}$ describe a set of three, extensive, scalar-valued strains that, as of yet, are unspecified. By specifying strains, say, Eq. (2.46) enables one to uniquely establish their conjugate stresses.

Note: From a Lagrangian perspective, a reference configuration κ would be chosen so that, typically, $a_0 = b_0 = 1$ and $\gamma_0 = 0$ with their current values a, b and γ being response functions. From an Eulerian perspective, a reference configuration κ would be chosen so that, typically, $a = b = 1$ and $\gamma = 0$ with their reference values a_0, b_0 and γ_0 being response functions, cf. Lodge [6, 1]. From a Lagrangian perspective, a cube deforms into a parallelepiped, while from an Eulerian perspective, a parallelepiped deforms into a cube which is going to be discussed in the next chapter.

Note: This hypothesis supposes there are two separate modes of straining. There are two elongation strains $\varepsilon_i \in \{\varepsilon_1, \varepsilon_2\}$, and there is a shear strain γ. Conjugate to these strains, there are two normal stresses $\sigma_i \in \{\sigma_1, \sigma_2\}$, and there is a shear stress $\{\tau\}$. Interpreting the elongational attributes is straightforward. Interpreting the shear attributes is less obvious. Strain γ shear the 1-2 plane. Stress τ act on shear plane 1-2, which is oriented orthogonal to the shear plane.

3. ANALYSIS IN ORTHONORMAL COORDINATE SYSTEM

3.1 Convected Metrics Associated With Coordinate Basis $\{\vec{e}^{\,i}\}$

In the previous chapter, a set of malleable base vectors $\{\vec{\mathbf{g}}_i\}$ were introduced that convect with the motion of a deformation, as illustrated in Fig. 2.1. These base vectors were selected to be congruent with a physical set of base vectors $\{\vec{\mathbf{e}}_i\}$ at some reference time t_0, from which they would then deform into an oblique set of base vectors $\{\vec{\mathbf{g}}_i\}$ at current time t.

Reversing this process of base vector motion, we now choose to start out at time t_0 with an oblique set of base vectors $\{\vec{\mathbf{g}}_i\}$ that finish up at current time t as an orthonormal set of base vectors $\{\vec{\mathbf{e}}_i\}$, as illustrated in Fig. 3.1. Here the set of base vectors $\{\vec{\mathbf{g}}_i\}$ are selected to be congruent with the physical base vectors $\{\vec{\mathbf{e}}_i\}$ at current time t, which is a rectangular Cartesian coordinate system, having been warped from an oblique shape associated with some reference time t_0. This runs the deformation of Fig. 2.1 in reverse. This has the desirable outcome of allowing one to work with convected tensor fields that are evaluated in a rectangular, Cartesian, coordinate system at current time t. The real advantage is in working with a stress tensor whose components are evaluated in this rectangular, Cartesian, coordinate system. In short, by choosing our convected coordinate system to be rectangular Cartesian at current time allows one to work with convected tensor components that are 'physical' by construction, cf. McConnell [33], Oldroyd [5], Lodge [2, 6], Truesdell [19], etc.

The convected metrics for an orthonormal frame $\boldsymbol{\beta} = \beta_{ij}\,\vec{\mathbf{e}}^i \otimes \vec{\mathbf{e}}^j$ has components $\beta_{ij} := \vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j$, which according to Eq.(2.30) can be described by

$$\boldsymbol{\beta} := \begin{bmatrix} 1/a^2 & -\gamma/a^2 \\ -\gamma/a^2 & 1/b^2 + \gamma^2/a^2 \end{bmatrix} \tag{3.1}$$

however the dual $\boldsymbol{\beta}^{-1} = \beta^{ij}\,\vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}_j$ has components $\beta^{ij} := \vec{\mathbf{e}}^i \cdot \vec{\mathbf{e}}^j$, and according to Eq.(2.34)

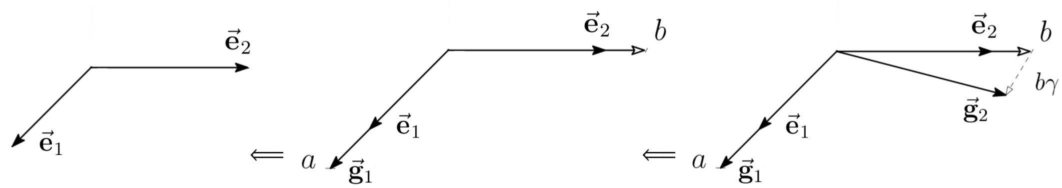

Figure 3.1: Deformation of convected coordinates into orthonormal coordinates.

populate a matrix with elements

$$\boldsymbol{\beta}^{-1} := \begin{bmatrix} a^2 + b^2\gamma^2 & \gamma b^2 \\ \gamma b^2 & b^2 \end{bmatrix} \tag{3.2}$$

when expressed in terms of the Jacobian $\mathbf{Y}$, they can take the form of

$$\boldsymbol{\beta} = \mathbf{Y}^{-\mathsf{T}}\mathbf{Y}^{-1} \quad \textit{and} \quad \boldsymbol{\beta}^{-1} = \mathbf{Y}\mathbf{Y}^{\mathsf{T}}. \tag{3.3}$$

wherein Jacobian $\mathbf{Y}$ maps tangent vectors from the oblique convected frame into the orthonormal physical frame.

3.2 Convected Strains Associated With Coordinate Basis $\{\vec{\boldsymbol{e}}_i\}$

There are two, quadratic, strain measures with geometric significance in 2 space: A change in length of line, i.e., $\frac{1}{2}(\mathbf{g} - \mathbf{g}_0)$, cf. Lodge [6, 1]. A change in area of surface, i.e., $\frac{1}{2}(\det \mathbf{g} - \det \mathbf{g}_0)$. These strain measures are expressed here as convected tensor fields that when pushed forward into the rectangular physical frame $\{\vec{\mathbf{e}}_i\}$ become

$$\frac{(\mathrm{d}s)^2 - (\mathrm{d}S)^2}{2\,(\mathrm{d}s)^2} = \frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}s} \cdot \boldsymbol{\varepsilon} \frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}s} \qquad \boldsymbol{\varepsilon} := \tfrac{1}{2}\big(\mathbf{I} - \boldsymbol{\beta}\big) \tag{3.4a}$$

$$\frac{(\mathrm{d}a)^2 - (\mathrm{d}A)^2}{2\,(\mathrm{d}a)^2} = e \qquad e := \frac{1}{2}\left(\frac{a^2b^2 - 1}{a^2b^2}\right) \tag{3.4b}$$

where $\boldsymbol{\varepsilon} = \varepsilon_{ij}\,\vec{\mathbf{e}}^i \otimes \vec{\mathbf{e}}^j$ or, simply, $\boldsymbol{\varepsilon} = \varepsilon_{ij}\,\vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}_j$ because $\vec{\mathbf{e}}^i \equiv \vec{\mathbf{e}}_i$. Here $\mathrm{d}s$ and $\mathrm{d}S$ are the distances separating two neighboring particles at times t and t_0 connected by a vector $\mathrm{d}\vec{\mathbf{x}} = \mathrm{d}x_i\,\vec{\mathbf{e}}_i$ of infinitesimal length $\mathrm{d}s$ evaluated at time t, $\mathrm{d}a$ and $\mathrm{d}A$ are the areas of a material surface at times t and t_0 whose unit normal is $\vec{\mathbf{n}} = n_i\,\vec{\mathbf{e}}_i$ at time t. Here $\det\boldsymbol{\beta} = 1/a^2b^2$. Strain $\boldsymbol{\varepsilon}$ is the convected analog to Almansi [34] strain.

3.2.1 Rates of Convected Fields

Because the convected coordinate system selected here is rectangular Cartesian at current time t, the time derivative of a convected field residing in frame $\{\vec{\mathbf{e}}_i\}$ is just the material derivative of that field. For the strains listed in Eq. (3.4), their rates are

$$\dot{\boldsymbol{\varepsilon}} = \tfrac{1}{2}\,\boldsymbol{\beta}\dot{\boldsymbol{\beta}}^{-1}\boldsymbol{\beta} \tag{3.5a}$$

$$\dot{e} = \frac{1}{2a^2b^2}\mathrm{tr}\big(\boldsymbol{\beta}\dot{\boldsymbol{\beta}}^{-1}\big) \tag{3.5b}$$

because $\boldsymbol{\beta} = \mathbf{Y}\mathbf{Y}^{\mathsf{T}}$ and $\mathbf{Y} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}$, it follows that $\dot{\boldsymbol{\beta}} = \dot{\mathbf{Y}}\mathbf{Y}^{\mathsf{T}} + \mathbf{Y}\dot{\mathbf{Y}}^{\mathsf{T}}$ wherein $\dot{\mathbf{Y}} = \dot{\boldsymbol{\Gamma}}\boldsymbol{\Lambda} + \boldsymbol{\Gamma}\dot{\boldsymbol{\Lambda}}$ with Iwasawa matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$ and their rates $\dot{\boldsymbol{\Gamma}}$ and $\dot{\boldsymbol{\Lambda}}$ being given by

$$\Gamma_{ij} = \begin{bmatrix} 1 & \gamma \\ & 1 \end{bmatrix},\ \dot{\Gamma}_{ij} = \begin{bmatrix} 0 & \dot{\gamma} \\ & 0 \end{bmatrix},\ \Lambda_{ij} = \begin{bmatrix} a & \\ & b \end{bmatrix},\ \dot{\Lambda}_{ij} = \begin{bmatrix} \dot{a} & \\ & \dot{b} \end{bmatrix} \tag{3.6}$$

where finite difference approximations for the elongation rates $\dot{a}, \dot{b}$ and the shear rate $\dot{\gamma}$ will be discussed in the next chapter.

3.3 Convected Stretch Associated With Coordinate Basis $\{\vec{e}_i\}$

We define convected stretch tensor $\boldsymbol{\kappa}$ in the rectangular Cartesian frame in such a way that $\boldsymbol{\beta} = \boldsymbol{\kappa}^{-T}\boldsymbol{\kappa}^{-1}$ so that by considering $\boldsymbol{\kappa}^{-1}$ as $\mathbb{k}$, it can take of the form $\boldsymbol{\beta} = \mathbb{k}^T\mathbb{k}$. By applying the Gram-Schmidt factorization of this convected stretch, we can define $\boldsymbol{\beta} = \widetilde{\mathbb{k}}^T\widetilde{\mathbb{k}}$, where $\widetilde{\mathbb{k}}$ is the triangular convected stretch tensor. The components of $\widetilde{\mathbb{k}}$ can be obtained from the Cholesky factorization of the convected metric $\boldsymbol{\beta}$ as follows:

$$\widetilde{\mathbb{k}}_{11} = \sqrt{\beta_{11}} \quad \widetilde{\mathbb{k}}_{12} = \frac{\beta_{12}}{\widetilde{\mathbb{k}}_{11}} \quad \widetilde{\mathbb{k}}_{21} = 0 \quad \widetilde{\mathbb{k}}_{22} = \sqrt{\beta_{22} - \widetilde{\mathbb{k}}_{21}^{2}} \tag{3.7}$$

hence

$$\widetilde{\mathbb{k}} := \begin{bmatrix} \frac{1}{a} & -\frac{\gamma}{a} \\ 0 & \frac{1}{b} \end{bmatrix} \tag{3.8}$$

therefore, the triangular convected stretch components become

$$\widetilde{\boldsymbol{\kappa}} = \widetilde{\mathbb{k}}^{-1} := \begin{bmatrix} a & b\gamma \\ 0 & b \end{bmatrix} \tag{3.9}$$

it has the same tensor field as Jacobian $\mathbf{Y}$.

4. STRESS POWER

4.1 Unitary Invariants

A symmetric 2×2 matrix **A** has two characteristic numbers (λ_1, λ_2), that provide the invariants of matrix **A** once it is transformed under any non-singular transformation. By restricting the transformation to be unitary, other considerable invariants appear. There are two invariants that arise from the characteristic equation, and one more unitary invariant in the general case[35].

These unitary invariants are used in a class of operators which turn out a W-algebra of invariants in the Kaplansky's terminology[36]. According to his idea, if an appropriate set of invariants could be found, the unitary equivalence problem is solvable for homogeneous operators.

The complete set of unitary invariants for any square matrix was an unsolved problem for a long time. Specht [37] obtained a collection from $\{tr[I(\mathbf{A}, \mathbf{A}^{\mathsf{T}})] \mid I(x, y) \in W\}$ that can be considered as a complete set of unitary invariants for any square matrix **A**.

4.1.1 Unitary Invariants of Murnagahan's Theory

Definition. Any $n \times n$ real matrix **U** whose transpose equals its reciprocal, i.e., $\mathbf{U}^{\mathsf{T}} = \mathbf{U}^{-1}$, is said to be an unitary matrix, a.k.a. an orthogonal matrix. Consider a 2×2 matrix **M** with invariants $\mathcal{I}(\mathbf{M}) = \{I_1(\mathbf{M}), I_2(\mathbf{M}), I_3(\mathbf{M}) : \ \mathbf{M} \in \mathbf{R}^{2\times 2}\}$ whose elements are defined by

$$I_1 := \operatorname{tr}\mathbf{M} \qquad I_2 := \operatorname{tr}\mathbf{M}^2 \qquad I_3 := \operatorname{tr}\mathbf{M}^{\mathsf{T}}\mathbf{M} \tag{4.1}$$

then:

Corollary. (Murnagahan [35]) If **A** and **B** are 2×2 real matrices that obey $\mathcal{I}(\mathbf{A}) = \mathcal{I}(\mathbf{B})$ via Eq. (4.1), then **A** and **B** are unitary equivalent; consequently, there exists an unique 2×2 real orthogonal matrix **U** such that $\mathbf{U}\mathbf{A}\mathbf{U}^{\mathsf{T}} = \mathbf{B}$.

Note. Whenever matrices **A** and **B** are symmetric, then the set $\mathcal{I}$ of three unitary invariants listed in Eq. (4.1) reduces to the two moment invariants used by the mechanics community today, because

$I_3 = I_2$ from symmetry.

4.1.2 Symmetric and Skew-Symmetric Invariants of Zheng's Theory

Corollary. (Zheng [38]) Any 2×2 matrix, say $\mathbf{M}$ can be decomposed into a sum of its symmetric and skew-symmetric parts, say $\mathbf{Y} := sym\,\mathbf{M}$ and $\mathbf{K} := skew\,\mathbf{M}$, such that $\mathbf{M} = \mathbf{Y} + \mathbf{K}$ for which $\mathbf{M}$ has three independent invariants $\mathcal{I}(\mathbf{M}) = \{I_1(\mathbf{M}), I_2(\mathbf{M}), I_3(\mathbf{M}) : \ \mathbf{M} \in R^{2\times 2}\}$ whose elements are defined in term of $\mathbf{Y}$ and $\mathbf{K}$ as follows

$$I_1 := \operatorname{tr}\mathbf{Y} \qquad I_2 := \operatorname{tr}\mathbf{Y}^2 \qquad I_3 := \operatorname{tr}\mathbf{K}^2 \tag{4.2}$$

4.2 Stress Response Function

In mathematics, a function is said to be convex if the drawn line segment between any two points of its graph lies either above or on it. Considering X to be a convex set in a vector space, f could be defined as a convex function if [39]

$$\forall x_1, x_2 \in X, \forall t \in [0,1]: \qquad f(tx_1 + (1-t)x_2 \leq tf(x_1) + (1-t)f(x_2) \tag{4.3}$$

where $f : X \implies R$. The concept of polyconvexity is the generalization of concept of convexity for functions that are defined on a space of matrices. Matrix $\mathbf{A}$ is a polyconvex function if $\mathbf{A} \implies f(\mathbf{A})$ can be identified as a convex function [40].

The free energy function ensuring to have the minimum principle invariants for transverse isotropic material in finite strain depends upon the notion of polyconvexity [41]. The polyconvexity of a stored energy results in the associated acoustic tensor function being elliptic for all deformations. It focuses on the presence of at least one energy minimizing the deformation gradient. Many nonlinear elastic materials, especially the Mooney-Rivlin and incompressible materials [42], satisfy the hypotheses described by Ball [43].

Definition. (Ball [43]) Stored-energy function $W(\mathbf{A})$ is said to be polyconvex if

$$W(\mathbf{A}) = \hat{W}(\mathbf{A}, \det \mathbf{A}) \tag{4.4}$$

with $\hat{W}(\mathbf{A}, det\mathbf{A})$ being convex. The first term in the right hand side describes the deformation of line, and the second term describes the deformation of surface.

Later Schröder & Neff [44] proposed an additive polyconvex functions for a more restrictive class of energy densities.

Definition. (Schröder & Neff [44]) $W(\mathbf{A})$ is said to be polyconvex if

$$W(\mathbf{A}) = \hat{W}_1(\mathbf{A}) + \hat{W}_2(\det \mathbf{A}) \tag{4.5}$$

wherein $\hat{W}_i$, $i = 1, 2$, are convex in their associated variable.

Let $W(\mathbf{A})$ be polyconvex, then $W(\mathbf{A})$ is elliptic. And let $W(\mathbf{A})$ be sufficiently smooth, then rank-one convexity and ellipticity are equivalent.

With the assumption of a free-energy function being a function of matrix $\mathbf{A}$, the stress definition can take of the form

$$\mathbf{T} = \partial_A \hat{W}(\mathbf{A}) \tag{4.6}$$

since the free energy is a function of invariants, $W(\mathbf{A}) = \hat{W}(I_1, I_2, I_3)$, one can achieve the stress function using chain rule

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{A}} = \left[\frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial \mathbf{A}} + \frac{\partial W}{\partial I_2} \frac{\partial I_2}{\partial \mathbf{A}} + \frac{\partial W}{\partial I_3} \frac{\partial I_3}{\partial \mathbf{A}} \right] \tag{4.7}$$

where $\frac{\partial I_i}{\partial \mathbf{A}}$ is the tensor generator for the general state of free energy function with the description of material symmetry, and $\frac{\partial W}{\partial I_i}$ is the I_i material response function.

4.3 Velocity Gradients

The Eulerian velocity gradient is defined by $\mathbf{L} := \dot{\mathbf{F}}\,\mathbf{F}^{-1}$ where $\dot{\mathbf{F}}$ is the material derivative of $\mathbf{F}$. When this definition is used into the decomposition $\mathbf{F} = \mathbf{Q}\,\widetilde{\mathbf{F}}$, the velocity gradient become

$$\mathbf{L} = \boldsymbol{\Omega} + \mathbf{Q}\,\widetilde{\mathbf{L}}\,\mathbf{Q}^{\mathsf{T}} \tag{4.8}$$

with $\boldsymbol{\Omega} := \dot{\mathbf{Q}}\,\mathbf{Q}^{\mathsf{T}}$ specifying the spin (rate of rotation) of frame $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ about $(\hat{\imath}, \hat{\jmath})$, and it is readily quantified by $\boldsymbol{\Omega} = \mathbf{K} - \widetilde{\mathbf{K}}$ [28]. The vorticities are defined by $\mathbf{K} := \frac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathsf{T}})$ and $\widetilde{\mathbf{K}} := \frac{1}{2}(\widetilde{\mathbf{L}} - \widetilde{\mathbf{L}}^{\mathsf{T}})$. Alternatively, one can describe $\widetilde{\mathbf{L}}$ by $\widetilde{\mathbf{L}} := \dot{\boldsymbol{\Lambda}} \cdot \boldsymbol{\Lambda}^{-1} + \boldsymbol{\Lambda}(\dot{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Gamma}^{-1})\boldsymbol{\Lambda}^{-1}$. Wherein $\dot{\boldsymbol{\Lambda}}$ and $\dot{\boldsymbol{\Gamma}}$ have the components

$$\dot{\boldsymbol{\Lambda}} = \begin{bmatrix} \dot{a} & 0 \\ 0 & \dot{b} \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{\Gamma}} = \begin{bmatrix} 0 & \dot{\gamma} \\ 0 & 0 \end{bmatrix} \tag{4.9}$$

Proof: We know that the deformation gradient $\mathbf{F} = \mathbf{Q}\,\widetilde{\mathbf{F}}$, wherein $\widetilde{\mathbf{F}} = \boldsymbol{\Lambda}\,\boldsymbol{\Gamma}$. On the other hand the velocity gradient $\dot{\mathbf{L}} = \dot{\mathbf{F}}\,\mathbf{F}^{-1}$. By taking the derivative of deformation gradient and substituting all into the velocity gradient, it becomes $\mathbf{L} = \left[\dot{\mathbf{Q}}\,\boldsymbol{\Lambda}\,\boldsymbol{\Gamma} + \mathbf{Q}\,(\dot{\boldsymbol{\Lambda}}\,\boldsymbol{\Gamma} + \boldsymbol{\Lambda}\,\dot{\boldsymbol{\Gamma}})\right]\,\boldsymbol{\Gamma}^{-1}\,\boldsymbol{\Lambda}^{-1}\,\mathbf{Q}^{-1}$, that gives $\mathbf{L} = \dot{\mathbf{Q}}\,\mathbf{Q}^{-1} + \mathbf{Q}\,\left[\dot{\boldsymbol{\Lambda}}\,\boldsymbol{\Lambda}^{-1} + \boldsymbol{\Lambda}\,\dot{\boldsymbol{\Gamma}}\,\boldsymbol{\Gamma}^{-1}\,\boldsymbol{\Lambda}^{-1}\right]\,\mathbf{Q}^{-1}$. Now we can define $\dot{\boldsymbol{\Lambda}}\,\boldsymbol{\Lambda}^{-1} + \boldsymbol{\Lambda}\,\dot{\boldsymbol{\Gamma}}\,\boldsymbol{\Gamma}^{-1}\,\boldsymbol{\Lambda}^{-1}$ as $\widetilde{\mathbf{L}}$, which makes the velocity gradient as $\mathbf{L} = \dot{\mathbf{Q}}\,\mathbf{Q}^{-1} + \mathbf{Q}\,\widetilde{\mathbf{L}}\,\mathbf{Q}^{-1}$.

The material derivative of Laplace stretch can be approximated by using the finite difference formulæ. Let the physical components for Laplace stretch at step n be denoted by

$$\widetilde{\mathbf{F}}_n = \boldsymbol{\Lambda}_n \boldsymbol{\Gamma}_n = \begin{bmatrix} \widetilde{\mathrm{F}}^{1}_{1(n)} & \widetilde{\mathrm{F}}^{1}_{2(n)} \\ 0 & \widetilde{\mathrm{F}}^{2}_{2(n)} \end{bmatrix} = \begin{bmatrix} a_n & a_n \gamma_n \\ 0 & b_n \end{bmatrix} \tag{4.10}$$

with analogous components assigned to steps $n - 1$ and $n + 1$, as required. The differential change at step n for each physical component in the Laplace stretch matrix is obtained via the product rule as

$$\begin{aligned} \dot{\tilde{\text{F}}}{}^1_{1(n)} &= \dot{a}_n & \dot{\tilde{\text{F}}}{}^1_{2(n)} &= \gamma_n \dot{a}_n + a_n \dot{\gamma}_n \\ \dot{\tilde{\text{F}}}{}^2_{1(n)} &= 0 & \dot{\tilde{\text{F}}}{}^2_{2(n)} &= \dot{b}_n \end{aligned} \tag{4.11}$$

with analogous components being assigned at other steps, e.g., at step $n + 1$.

In a typical numerical application, one would be given the deformation gradient at the beginning and end of a time step of size h, say, denoted here as $\mathbf{F}_n$ and $\mathbf{F}_{n+1}$, and whose affiliated Laplace stretch $\widetilde{\mathbf{F}}_n$ and $\widetilde{\mathbf{F}}_{n+1}$ come from Eq. (2.1). With this information, finite difference formulæ can be constructed to acquire estimates for the differential change in the physical components established in Eqs. (4.9 & 4.11).

The forward difference formula for Laplace stretch $\dot{\widetilde{\mathbf{F}}}_n = \frac{\widetilde{\mathbf{F}}_{n+1} - \widetilde{\mathbf{F}}_n}{h} + O(h)$ gives

$$\begin{aligned} \dot{a}_n &\approx \frac{a_{n+1} - a_n}{h} \\ \dot{b}_n &\approx \frac{b_{n+1} - b_n}{h} \\ \dot{\gamma}_n &\approx \frac{a_{n+1}}{a_n} \left(\frac{\gamma_{n+1} - \gamma_n}{h} \right) \end{aligned} \tag{4.12}$$

while the backward difference formula $\dot{\widetilde{\mathbf{F}}}_{n+1} = \frac{\widetilde{\mathbf{F}}_{n+1} - \widetilde{\mathbf{F}}_n}{h} + O(h)$ gives

$$\begin{aligned} \dot{a}_{n+1} &\approx \frac{a_{n+1} - a_n}{h} \\ \dot{b}_{n+1} &\approx \frac{b_{n+1} - b_n}{h} \\ \dot{\gamma}_{n+1} &\approx \frac{a_n}{a_{n+1}} \left(\frac{\gamma_{n+1} - \gamma_n}{h} \right) \end{aligned} \tag{4.13}$$

with there being a distinction in how the shear rates are approximated.

Equations (4.12 & 4.13) are first-order approximations for these derivatives. Second-order approximations can be established whenever $n > 0$ and when the step size for step $[n, n + 1]$ equals the step size for step $[n - 1, n]$, where state $n = 0$ associates with an initial condition. The central difference formula for Laplace stretch $\dot{\tilde{\mathbf{F}}}_n = \frac{\tilde{\mathbf{F}}_{n+1} - \tilde{\mathbf{F}}_{n-1}}{2h} + O(h^2)$ gives

$$\begin{aligned}
\dot{a}_n &\approx \frac{a_{n+1} - a_{n-1}}{2h} \\
\dot{b}_n &\approx \frac{b_{n+1} - b_{n-1}}{2h} \\
\dot{\gamma}_n &\approx \frac{a_{n+1}}{a_n}\left(\frac{\gamma_{n+1} - \gamma_n}{2h}\right) + \frac{a_{n-1}}{a_n}\left(\frac{\gamma_n - \gamma_{n-1}}{2h}\right)
\end{aligned} \tag{4.14}$$

while the backward difference formula $\dot{\tilde{\mathbf{F}}}_{n+1} = \frac{3\tilde{\mathbf{F}}_{n+1} - 4\tilde{\mathbf{F}}_n + \tilde{\mathbf{F}}_{n-1}}{2h} + O(h^2)$ gives

$$\begin{aligned}
\dot{a}_{n+1} &\approx \frac{3a_{n+1} - 4a_n + a_{n-1}}{2h} \\
\dot{b}_{n+1} &\approx \frac{3b_{n+1} - 4b_n + b_{n-1}}{2h} \\
\dot{\gamma}_{n+1} &\approx \frac{2a_n}{a_{n+1}}\left(\frac{\gamma_{n+1} - \gamma_n}{h}\right) - \frac{a_{n-1}}{a_{n+1}}\left(\frac{\gamma_{n+1} - \gamma_{n-1}}{2h}\right)
\end{aligned} \tag{4.15}$$

which require that values for state $n - 1$ be stored.

With values for $\dot{a}$, $\dot{b}$ known, one can populate $\dot{\mathbf{\Lambda}}$ in Eq. (4.9). Likewise, with values for $\dot{\gamma}$ known, one can populate $\dot{\mathbf{\Gamma}}$ in Eq. (4.9).

4.4 Stress Power Defined in the Physical Coordinate System

The mechanical power exerted upon a material particle, caused by stressing a deformable body, is a frame-indifferent [45, 46] physical property described by [23]

$$\dot{W} = \mathrm{tr}(\mathbf{TL}) = T^{ij} L_{ji} = T_{ij} L^{ji} = T^i_j L^j_i \tag{4.16}$$

where $\mathbf{T}$ is the symmetric Cauchy stress, $\mathbf{L} := \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$ is the non-symmetric velocity gradient, and $\dot{\mathbf{F}}$ representing a material derivative of the deformation gradient $\mathbf{F}$. The velocity gradient $\mathbf{L}$ and Cauchy stress $\mathbf{T}$ are Eulerian tensor fields.

Herein the Eulerian components of $\mathbf{T}$ and $\mathbf{L}$ are considered to be evaluated in an orthonormal coordinate system. Consequently, their covariant and contravariant tensor components are equivalent, i.e., $T^{ij} \equiv T^i_j \equiv T_{ij}$ and $L^{ij} \equiv L^i_j \equiv L_{ij}$. This follows because the selected base vectors and their duals are co-linear with one another. For our purposes, it is advantageous to pull these Eulerian components back into the physical coordinate system, by using quotient laws.

The mechanical power caused by stressing a deformable body, when evaluated in a physical coordinate system with base vectors $\{\vec{\mathbf{e}}_i\}$, becomes

$$\dot{\tilde{W}} = \mathrm{tr}(\widetilde{\mathbf{T}}\widetilde{\mathbf{L}}) \tag{4.17}$$

wherein

$$\widetilde{\mathbf{T}} = \mathbf{Q}^{-1}\,\mathbf{T}\,\mathbf{Q} \quad \text{and} \quad \widetilde{\mathbf{L}} = \mathbf{Q}^{-1}\,(\mathbf{L} - \mathbf{\Omega})\,\mathbf{Q}. \tag{4.18}$$

According to quotient law, $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{L}}$ could be obtained by pulling backs the Cauchy stress and velocity gradient into the physical coordinate system through $\mathbf{Q}$, hence

$$\dot{\tilde{W}} = \mathrm{tr}(\widetilde{\mathbf{T}}\widetilde{\mathbf{L}}) = \mathrm{tr}(\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q}\cdot\mathbf{Q}^{-1}\,(\mathbf{L} - \mathbf{\Omega})\,\mathbf{Q}) = \mathrm{tr}(\mathbf{Q}^{-1}\mathbf{T}\mathbf{L}\mathbf{Q}) = \mathrm{tr}(\mathbf{T}\mathbf{L}) = \dot{W} \tag{4.19}$$

because $\mathrm{tr}(\mathbf{Q}^{-1}\,\mathbf{T}\,\mathbf{\Omega}\,\mathbf{Q}) = 0$, since $\mathbf{T}$ is symmetric and $\mathbf{\Omega}$ is skew symmetric.

5. CLASSICAL APPROACH AND PROPOSED METHOD

5.1 Classical Approach

Here we focus our attention on sets of invariants that pertain to **F** and the constructed right-Cauchy Green tensor $\mathbf{C} = \mathbf{F}^{\top}\,\mathbf{F}$ that are defined in the Lagrangian coordinate system.

5.1.1 Unitary Invariants of Murnagahan's Theory for Deformation Gradient F

For a real, deformation gradient matrix **F** with components

$$\mathbf{F} = \begin{bmatrix} a\cos\theta & b\sin\theta + a\gamma\cos\theta \\ -a\sin\theta & b\cos\theta - a\gamma\sin\theta \end{bmatrix}$$

wherein $a, b \in R_+$ and $\gamma \in R$, the three unitary invariants listed in Eq. (4.1) are

$$\begin{aligned} I_1(\mathbf{F}) &= (a+b)\cos\theta - a\gamma\sin\theta \\ I_2(\mathbf{F}) &= (a^2+b^2)\cos^2\theta + (a^2\gamma^2 - 2ab)\sin^2\theta - 2(ab\gamma + a^2\gamma)\sin\theta\cos\theta \\ I_3(\mathbf{F}) &= a^2 + b^2 + a^2\,\gamma^2 \end{aligned} \tag{5.1}$$

while the two invariants I and II of **F** that come from the Cayley-Hamilton theorem are

$$I(\mathbf{F}) = (a+b)\cos\theta - a\gamma\sin\theta \quad \text{and} \quad II(\mathbf{F}) = ab$$
$$\text{and} \quad I = I_1 \quad \text{and} \quad II = \frac{1}{2}\left((I_1)^2 - I_2\right).$$

In the reference state where $a = b = 1$ and $\gamma = 0$, $I_1(\mathbf{F}) = I_2(\mathbf{F}) = I_3(\mathbf{F}) = 2$.

5.1.2 Unitary Invariants of Murnagahan's Theory for Right Cauchy-Green Tensor C

Starting from deformation gradient matrix **F**, the right Cauchy-Green tensor **C** has components

$$\mathbf{C} = \begin{bmatrix} a^2 & a^2\,\gamma \\ a^2\,\gamma & a^2\,\gamma^2 + b^2 \end{bmatrix} \tag{5.2}$$

wherein $a, b \in R_+$ and $\gamma \in R$. The three unitary invariants listed in Eq. (4.1) are

$$
\begin{aligned}
I_1(\mathbf{C}) &= a^2 + b^2 + a^2\gamma^2 \\
I_2(\mathbf{C}) &= (b^2 + a^2\gamma^2)^2 + 2(a^2\gamma)^2 + a^4 \\
I_3(\mathbf{C}) &= I_2(\mathbf{C})
\end{aligned} \tag{5.3}
$$

while the two invariants I and II of $\mathbf{C}$ that come from the Cayley-Hamilton theorem are

$$I(\mathbf{C}) = a^2 + b^2 + a^2\gamma^2 \quad \text{and} \quad II(\mathbf{C}) = a^2 b^2$$

with

$$I = I_1 \quad \text{and} \quad II = \frac{1}{2}\left((I_1)^2 - I_2\right).$$

In a reference state where $a = b = 1$ and $\gamma = 0$, $I_1(\mathbf{C}) = I_2(\mathbf{C}) = I_3(\mathbf{C}) = 2$.

5.1.3 Symmetric and Skew-Symmetric Invariants of Zheng's Theory for Deformation Gradient F

For deformation gradient matrix, $\mathbf{F}$ with components

$$\mathbf{F} = \begin{bmatrix} a\cos\theta & b\sin\theta + a\gamma\cos\theta \\ -a\sin\theta & b\cos\theta - a\gamma\sin\theta \end{bmatrix}$$

while its symmetric and skew-symmetric parts are

$$
\begin{aligned}
\mathbf{Y} &= \frac{1}{2}\begin{bmatrix} 2a\cos\theta & (b-a)\sin\theta + a\gamma\cos\theta \\ (b-a)\sin\theta + a\gamma\cos\theta & 2b\cos\theta - 2a\gamma\sin\theta \end{bmatrix} \\
\mathbf{K} &= \frac{1}{2}\begin{bmatrix} 0 & (b+a)\sin\theta + a\gamma\cos\theta \\ -(b+a)\sin\theta - a\gamma\cos\theta & 0 \end{bmatrix}
\end{aligned} \tag{5.4}
$$

wherein $a, b \in R_+$ and $z \in R$, the three unitary invariants listed in Eq. (4.2) are

$$
\begin{aligned}
I_1(\mathbf{F}) &= (a+b)\cos\theta - a\gamma\sin\theta \\
I_2(\mathbf{F}) &= a^2\cos^2\theta + \frac{1}{2}\left[(a-b)\sin\theta - a\gamma\cos\theta\right]^2 + (a\gamma\sin\theta - ab\cos\theta)^2 \\
I_3(\mathbf{F}) &= -\frac{1}{2}\left[(a+b)\sin\theta + a\gamma\cos\theta\right]^2
\end{aligned}
\tag{5.5}
$$

while considering the trace properties $\operatorname{tr}\mathbf{F} = (\operatorname{tr}\mathbf{Y} + \operatorname{tr}\mathbf{K})$, the Cayley-Hamilton invariants of $\mathbf{F}$ are contains the invariants of $\mathbf{Y}$ and $\mathbf{K}$. Therefore

$$
\begin{aligned}
I(\mathbf{Y}) &= (a+b)\cos\theta - a\gamma\sin\theta \\
II(\mathbf{Y}) &= \frac{1}{2}\Big[((a+ab)\cos\theta - a\gamma\sin\theta)^2 - a^2\cos^2\theta - \frac{1}{2}((a-b)\sin\theta - a\gamma\cos\theta)^2 \\
&\quad - (a\gamma\sin\theta - ab\cos\theta)^2\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
I(\mathbf{K}) &= 0 \\
II(\mathbf{K}) &= \frac{1}{4}\left[(a+b)\sin\theta + a\gamma\cos\theta\right]^2
\end{aligned}
$$

hence the invariants of $\widetilde{\mathbf{F}}$ that come from the Cayley-Hamilton theorem are

$$
I(\mathbf{F}) = I(\mathbf{Y}) \quad , \quad II(\mathbf{F}) = II(\mathbf{Y}) \quad , \quad III(\mathbf{F}) = III(\mathbf{K})
$$

in a reference state where $a = b = 1$ and $\gamma = 0$, $I_1(\mathbf{F}) = I_2(\mathbf{F}) = 2$ and $I_3(\mathbf{F}) = 0$.

5.1.4 Stress Response Function

The material objectivity and unitary equivalent condition

$$
\mathrm{f}(\mathbf{F}) = \mathrm{f}(\mathbf{QF}) \qquad\qquad \mathrm{f}(\mathbf{C}) = \mathrm{f}(\mathbf{QF^TFQ^T}) \tag{5.6a}
$$

result in

$$W(\mathbf{F}) = \hat{W}_1(\mathbf{F}) + \hat{W}_2(det\mathbf{F}) \tag{5.7a}$$

$$W(\mathbf{C}) = \hat{W}_1(\mathbf{C}) + \hat{W}_2(det\mathbf{C}) \tag{5.7b}$$

with each term having to satisfy conditions of invariance and polyconvexity.

5.1.4.1 Stress Response Function of Murnagahan's Theory for Deformation Gradient **F**

With an assumption of the free-energy function being a function of the deformation gradient **F**, the 1st Piola-Kirchhoff stress tensor can be calculated for a hyperelastic material as $\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^{\mathsf{T}}$. Hence, using chain rule leads one to derive the 1st Piola-Kirchhoff stress in terms of its individual principle invariants as

$$\mathbf{P} = \left[\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}}\right]\mathbf{F}^{\mathsf{T}} \tag{5.8}$$

where the derivatives of the invariants of **F** are

$$\frac{\partial I_1}{\partial \mathbf{F}} = \frac{\partial I_1}{\partial F_{ij}} = \frac{\partial F_{kk}}{\partial F_{ij}} = \delta_{ij} \tag{5.9a}$$

$$\frac{\partial I_2}{\partial \mathbf{F}} = \frac{\partial I_2}{\partial F_{ij}} = \frac{\partial (F_{kl}F_{lk})}{\partial F_{ij}} = \delta_{ki}\delta_{lj}F_{lk} + F_{kl}\delta_{li}\delta_{kj} = F_{ji} + F_{ji} = 2\mathbf{F}^{\mathsf{T}} \tag{5.9b}$$

$$\frac{\partial I_3}{\partial \mathbf{F}} = \frac{\partial I_3}{\partial F_{ij}} = \frac{\partial (F_{kl}F_{kl})}{\partial F_{ij}} = \delta_{ki}\delta_{lj}F_{kl} + F_{kl}\delta_{ki}\delta_{lj} = F_{ij} + F_{ij} = 2\mathbf{F} \tag{5.9c}$$

hence, Eq.(5.8) becomes

$$P_{ij} = \frac{\partial W}{\partial F_{ik}}F_{jk} = \left[\frac{\partial W}{\partial I_1}\delta_{ik} + 2\frac{\partial W}{\partial I_2}F_{ki} + 2\frac{\partial W}{\partial I_3}F_{ik}\right]F_{jk}. \tag{5.10}$$

5.1.4.2 Stress Response Function of Murnagahan's Theory for Right Cauchy-Green Strain Tensor **C**

With an assumption of the free-energy function being a function of right Cauchy-Green strain tensor, we need to work with the second Piola-Kirchhoff stress tensor, with the expression of

$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}}$. It takes on the following form when expressed in terms of the individual principle invariants via the chain rule

$$\mathbf{S} = 2\left[\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{C}}\right] \tag{5.11}$$

Where the derivatives of the invariants of **C** are

$$\frac{\partial I_1}{\partial \mathbf{C}} = \frac{\partial I_1}{\partial C_{ij}} = \frac{\partial C_{kk}}{\partial C_{ij}} = \delta_{ij} \tag{5.12a}$$

$$\frac{\partial I_2}{\partial \mathbf{C}} = \frac{\partial I_2}{\partial C_{ij}} = \frac{\partial (C_{kl}C_{lk})}{\partial C_{ij}} = \delta_{ki}\delta_{lj}C_{lk} + C_{kl}\delta_{li}\delta_{kj} = C_{ji} + C_{ji} = 2\mathbf{C}^{\mathsf{T}} \tag{5.12b}$$

$$\frac{\partial I_3}{\partial \mathbf{C}} = \frac{\partial I_3}{\partial C_{ij}} = \frac{\partial (C_{kl}C_{kl})}{\partial C_{ij}} = \delta_{ki}\delta_{lj}C_{kl} + C_{kl}\delta_{ki}\delta_{lj} = C_{ij} + C_{ij} = 2\mathbf{C} \tag{5.12c}$$

hence, Eq.(5.11) becomes

$$S_{ij} = 2\left[\frac{\partial W}{\partial I_1}\delta_{ij} + 2\frac{\partial W}{\partial I_2}C_{ji} + 2\frac{\partial W}{\partial I_3}C_{ij}\right] = 2\left[\frac{\partial W}{\partial I_1}\delta_{ij} + 2\left(\frac{\partial W}{\partial I_2} + \frac{\partial W}{\partial I_3}\right)C_{ij}\right] \tag{5.13}$$

because **C** is symmetric.

5.1.4.3 Stress Response Function of Zheng's Theory for Deformation Gradient **F**

With an assumption of the free-energy function being a function of the deformation gradient **F**, hence using chain rule leads to derive the 1st Piola-Kirchhoff stress tensor in terms of individual principle invariants as

$$\mathbf{P} = \left[\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}}\right]\mathbf{F}^{\mathsf{T}} \tag{5.14}$$

and then the derivatives of the invariants of **F** are

$$\frac{\partial I_1}{\partial \mathbf{F}} = \frac{\partial I_1}{\partial F_{ij}} = \frac{1}{2}\,\frac{\partial (F_{kk} + F_{ll})}{\partial F_{ij}} = \frac{1}{2}\,\delta_{ij} \tag{5.15a}$$

$$\frac{\partial I_2}{\partial \mathbf{F}} = \frac{\partial I_2}{\partial F_{ij}} = \frac{1}{4}\,\frac{\partial\left[F_{kl}F_{lk} + F_{kl}F_{lk} + F_{lk}F_{lk} + F_{kl}F_{kl}\right]}{\partial F_{ij}} = 2\,Y_{ij} = 2\,\mathbf{Y} \tag{5.15b}$$

$$\frac{\partial I_3}{\partial \mathbf{F}} = \frac{\partial I_3}{\partial F_{ij}} = \frac{1}{4}\,\frac{\partial\left[F_{kl}F_{lk} + F_{kl}F_{lk} - F_{lk}F_{lk} - F_{kl}F_{kl}\right]}{\partial F_{ij}} = -2\,K_{ij} = -2\,\mathbf{K} \tag{5.15c}$$

(5.15d)

hence, Eq.(5.14) becomes

$$P_{ij} = \frac{\partial W}{\partial F_{ik}}\,F_{jk} = \left[\frac{\partial W}{\partial I_1}\,\delta_{ik} + 2\,\frac{\partial W}{\partial I_2}\,Y_{ki} + 2\,\frac{\partial W}{\partial I_3}\,K_{ik}\right] F_{jk} \tag{5.16}$$

5.1.5 Velocity Gradients

The Eulerian velocity gradient, i.e., Eq. (4.8), can be expressed as

$$\mathbf{L} = \begin{bmatrix} \frac{\dot a}{a}\,\cos^2\theta + \frac{\sin\theta}{b}\,\left(a\,\dot\gamma\,\cos\theta + \dot b\,\sin\theta\right) & \dot\theta - \frac{\dot a}{a}\,\cos\theta\,\sin\theta + \frac{\cos\theta}{b}\,\left(a\,\dot\gamma\,\cos\theta + \dot b\,\sin\theta\right) \\ -\dot\theta - \frac{\dot a}{a}\,\cos\theta\,\sin\theta + \frac{\cos\theta}{b}\,\left(-a\,\dot\gamma\,\cos\theta + \dot b\,\sin\theta\right) & \frac{\dot a}{a}\,\sin^2\theta + \frac{\cos\theta}{b}\,\left(-a\,\dot\gamma\,\sin\theta + \dot b\,\cos\theta\right) \end{bmatrix} \tag{5.17}$$

wherein $\dot\theta$ can be obtained from Eq. (2.7). To find the Lagrangian velocity gradient **V** we need to pull it back from the Eulerian coordinate system to the Lagrangian coordinate system. Hence, the Lagrangian velocity gradient can take of the form

$$\mathbf{V} = \mathbf{F}^{-1}\,\mathbf{L}\,\mathbf{F} \tag{5.18}$$

wherein **F** is the deformation gradient.

5.1.6 Stress Power

With an assumption of the 1st Piola-Kirchhoff stress being a function of the deformation gradient **F**, and adopting Murnagahan's invariants, i.e., Eq. (5.10), one can find the mechanical power caused by stressing a deformable body as follows

$$\dot{W} = P_{ij}\, V_{ji} = \left[\frac{\partial W}{\partial I_1}\, \delta_{ik} + 2\, \frac{\partial W}{\partial I_2}\, F_{ki} + 2\, \frac{\partial W}{\partial I_3}\, F_{ik} \right] F_{jk}\, V_{ji} \tag{5.19}$$

With an assumption of second Piola-Kirchhoff stress being a function of right Cauchy-Green strain tensor $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\, \mathbf{F}$ and Murnagahan's invariants, i.e., Eq. (5.13), the mechanical power leads to

$$\dot{W} = S_{ij}\, V_{ji} = 2 \left[\frac{\partial W}{\partial I_1}\, \delta_{ij} + 2 \left(\frac{\partial W}{\partial I_2} + \frac{\partial W}{\partial I_3} \right) C_{ij} \right] V_{ji} \tag{5.20}$$

With an assumption of the 1st Piola-Kirchhoff stress being a function of the deformation gradient **F**, and adopting Zheng's invariants, i.e., Eq. (5.16), the mechanical power becomes

$$\dot{W} = P_{ij}\, V_{ji} = \left[\frac{\partial W}{\partial I_1}\, \delta_{ik} + 2\, \frac{\partial W}{\partial I_2}\, Y_{ki} + 2\, \frac{\partial W}{\partial I_3}\, K_{ik} \right] F_{jk}\, V_{ji} \tag{5.21}$$

5.2 Proposed Method

Here we focus our attention on the set of invariants that pertain to an upper-triangular matrix, like Laplace stretch $\widetilde{\mathbf{F}}$, and its symmetric left Cauchy-Green tensor $\widetilde{\mathbf{B}} = \widetilde{\mathbf{F}}\widetilde{\mathbf{F}}^{\mathsf{T}}$ and their utility when constructing the constitutive equations.

5.2.1 Unitary Invariants of Murnagahan's Theory for Laplace Stretch $\widetilde{\mathbf{F}}$

For a real, upper-triangular matrix $\widetilde{\mathbf{F}}$ with components

$$\widetilde{\mathbf{F}} = \begin{bmatrix} a & a\gamma \\ 0 & b \end{bmatrix}$$

wherein $a, b \in R_+$ and $\gamma \in R$, the three unitary invariants listed in Eq. (4.1) are

$$I_1(\widetilde{\mathbf{F}}) = a + b, \quad I_2(\widetilde{\mathbf{F}}) = a^2 + b^2, \quad I_3(\widetilde{\mathbf{F}}) = a^2 + b^2 + a^2\gamma^2 \tag{5.22}$$

while the two invariants I and II of $\widetilde{\mathbf{F}}$ that come from the Cayley-Hamilton theorem are

$$I(\widetilde{\mathbf{F}}) = a + b \quad \text{and} \quad II(\widetilde{\mathbf{F}}) = ab$$

with

$$I = I_1 \quad \text{and} \quad II = \frac{1}{2}\left(I_1^2 - I_2\right)$$

which are independent of the off-axis term γ, hence, we see the need for three invariants. In a reference state where $a = b = 1$ and $\gamma = 0$, $I_1(\widetilde{\mathbf{F}}) = I_2(\widetilde{\mathbf{F}}) = I_3(\widetilde{\mathbf{F}}) = 2$.

5.2.2 Unitary Invariants of Murnagahan's Theory for Left Cauchy Green-tensor $\widetilde{\mathbf{B}}$

For a real, Laplace stretch matrix $\widetilde{\mathbf{F}}$ which result in the left Cauchy-Green tensor $\widetilde{\mathbf{B}} = \widetilde{\mathbf{F}}\,\widetilde{\mathbf{F}}^{\mathsf{T}}$ with components

$$\widetilde{\mathbf{B}} = \begin{bmatrix} a^2 + a^2\gamma^2 & ab\gamma \\ ab\gamma & b^2 \end{bmatrix}$$

wherein $a, b \in R_+$ and $\gamma \in R$, the three unitary invariants listed in Eq. (4.1) are

$$\begin{aligned} I_1(\widetilde{\mathbf{B}}) &= a^2 + b^2 + a^2\gamma^2 \\ I_2(\widetilde{\mathbf{B}}) &= (a^2 + a^2\gamma^2)^2 + 2(ab\gamma)^2 + b^4 \\ I_3(\widetilde{\mathbf{B}}) &= I_2(\widetilde{\mathbf{B}}) \end{aligned} \tag{5.23}$$

while the two invariants I and II of $\widetilde{\mathbf{B}}$ that come from the Cayley-Hamilton theorem are

$$I(\widetilde{\mathbf{B}}) = a^2 + b^2 + a^2\gamma^2 \quad \text{and} \quad II(\widetilde{\mathbf{B}}) = a^2 b^2$$

with

$$I = I_1 \quad \text{and} \quad II = \frac{1}{2}\left(I_1^2 - I_2\right).$$

In a reference state where $a = b = 1$ and $\gamma = 0$, $I_1(\widetilde{\mathbf{B}}) = I_2(\widetilde{\mathbf{B}}) = I_3(\widetilde{\mathbf{B}}) = 2$.

5.2.3 Symmetric and Skew-Symmetric Invariants of Zheng's Theory for Triangular Matrix $\widetilde{\mathbf{F}}$

For a real, upper-triangular, 2×2 matrix $\widetilde{\mathbf{F}}$ with components

$$\widetilde{\mathbf{F}} = \begin{bmatrix} a & a\gamma \\ 0 & b \end{bmatrix}$$

its symmetric and skew-symmetric parts are

$$\widetilde{\mathbf{Y}} = \begin{bmatrix} a & \frac{a\gamma}{2} \\ \frac{a\gamma}{2} & b \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{K}} = \begin{bmatrix} 0 & \frac{a\gamma}{2} \\ -\frac{a\gamma}{2} & 0 \end{bmatrix} \tag{5.24}$$

wherein $a, b \in R_+$ and $z \in R$. The three unitary invariants listed in Eq. (4.2) are

$$I_1(\widetilde{\mathbf{F}}) = a + b, \quad I_2(\widetilde{\mathbf{F}}) = a^2 + b^2 + \frac{a^2\gamma^2}{2}, \quad I_3(\widetilde{\mathbf{F}}) = -\frac{a^2\gamma^2}{2} \tag{5.25}$$

while considering the trace properties $\operatorname{tr}\widetilde{\mathbf{F}} = (\operatorname{tr}\widetilde{\mathbf{Y}} + \operatorname{tr}\widetilde{\mathbf{K}})$, the Cayley-Hamilton invariants of $\widetilde{\mathbf{F}}$ are contains the invariants of $\widetilde{\mathbf{Y}}$ and $\widetilde{\mathbf{K}}$. Therefore

$$I(\widetilde{\mathbf{Y}}) = a + b \quad , \quad II(\widetilde{\mathbf{Y}}) = ab - \frac{a^2\gamma^2}{4}$$

and

$$I(\widetilde{\mathbf{K}}) = 0 \quad , \quad II(\widetilde{\mathbf{K}}) = \frac{a^2\gamma^2}{4}$$

hence the invariants of $\widetilde{\mathbf{F}}$ that come from the Cayley-Hamilton theorem are

$$I(\widetilde{\mathbf{F}}) = I(\widetilde{\mathbf{Y}}) = a + b \quad , \quad II(\widetilde{\mathbf{F}}) = II(\widetilde{\mathbf{Y}}) = ab - \frac{a^2\gamma^2}{4} \quad , \quad III(\widetilde{\mathbf{F}}) = III(\widetilde{\mathbf{K}}) = \frac{a^2\gamma^2}{4}$$

In a reference state where $a = b = 1$ and $\gamma = 0$, $I_1(\widetilde{\mathbf{F}}) = I_2(\widetilde{\mathbf{F}}) = 2$.

5.2.4 Stress Response Function

The conditions of material objectivity and unitary equivalence require

$$\mathrm{f}(\widetilde{\mathbf{F}}) = \mathrm{f}(\mathbf{Q}\widetilde{\mathbf{F}}\mathbf{Q}^T) \qquad \mathrm{f}(\widetilde{\mathbf{F}}\widetilde{\mathbf{F}}^T) = \mathrm{f}(\mathbf{Q}\widetilde{\mathbf{F}}\widetilde{\mathbf{F}}^T\mathbf{Q}^T) \tag{5.26a}$$

resulting in

$$W(\widetilde{\mathbf{F}}) = \hat{W}_1(\widetilde{\mathbf{F}}) + \hat{W}_2(\det\widetilde{\mathbf{F}}) \tag{5.27a}$$

$$W(\widetilde{\mathbf{B}}) = \hat{W}_1(\widetilde{\mathbf{B}}) + \hat{W}_2(\det\widetilde{\mathbf{B}}) \tag{5.27b}$$

where each term has to satisfy conditions of invariance and polyconvexity conditions.

5.2.4.1 Stress Response Function of Murnagahan's Theory for Laplace Stretch $\widetilde{\mathbf{F}}$

In order to obtain an explicit expression for the stress, we will use the fact that

$$\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{D}} = \rho\dot{W} \tag{5.28}$$

wherein $\widetilde{\mathbf{D}}$ is the symmetric part of physical velocity gradient. By noting that W is a function of $\widetilde{\mathbf{F}}$, we obtain

$$\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{D}} = \rho\frac{\partial W}{\partial\widetilde{\mathbf{F}}} \cdot \dot{\widetilde{\mathbf{F}}} = \rho\frac{\partial W}{\partial\widetilde{\mathbf{F}}} \cdot \widetilde{\mathbf{L}}\,\widetilde{\mathbf{F}} \tag{5.29}$$

since $tr(\widetilde{\mathbf{T}}\,\widetilde{\mathbf{D}}) = \frac{1}{2}\,tr(\widetilde{\mathbf{T}}\,\widetilde{\mathbf{L}} + \widetilde{\mathbf{T}}\,\widetilde{\mathbf{L}}^{\mathrm{T}})$, one can find $tr(\widetilde{\mathbf{T}}\,\widetilde{\mathbf{D}}) = tr(\widetilde{\mathbf{T}}\,\widetilde{\mathbf{L}})$. The above equations have to be satisfied for all $\widetilde{\mathbf{L}}$, hence

$$\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{L}} = \rho\frac{\partial W}{\partial\widetilde{\mathbf{F}}}\widetilde{\mathbf{F}}^{\mathrm{T}} \cdot \widetilde{\mathbf{L}} \tag{5.30}$$

therefore the rotated Cauchy stress tensor $\widetilde{\mathbf{T}}$ can be expressed as

$$\widetilde{\mathbf{T}} = \rho \frac{\partial W}{\partial \widetilde{\mathbf{F}}} \widetilde{\mathbf{F}}^{\mathrm{T}} \tag{5.31}$$

Hence, using the chain rule leads to Cauchy stress being described in terms of individual unitary invariants as

$$\widetilde{\mathbf{T}} = \rho \left[\frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial \widetilde{\mathbf{F}}} + \frac{\partial W}{\partial I_2} \frac{\partial I_2}{\partial \widetilde{\mathbf{F}}} + \frac{\partial W}{\partial I_3} \frac{\partial I_3}{\partial \widetilde{\mathbf{F}}} \right] \widetilde{\mathbf{F}}^{\mathrm{T}} \tag{5.32}$$

wherein the derivatives of the invariants of $\widetilde{\mathbf{F}}$ are

$$\frac{\partial I_1}{\partial \widetilde{\mathbf{F}}} = \frac{\partial I_1}{\partial \widetilde{F}_{ij}} = \frac{\partial B_{kk}}{\partial \widetilde{F}_{ij}} = \delta_{ij} \tag{5.33a}$$

$$\frac{\partial I_2}{\partial \widetilde{\mathbf{F}}} = \frac{\partial I_2}{\partial \widetilde{F}_{ij}} = \frac{\partial (\widetilde{F}_{kl} \widetilde{F}_{lk})}{\partial \widetilde{F}_{ij}} = \delta_{ki}\delta_{lj}\widetilde{F}_{lk} + \widetilde{F}_{kl}\delta_{li}\delta_{kj} = \widetilde{F}_{ji} + \widetilde{F}_{ji} = 2\widetilde{\mathbf{F}}^{\mathsf{T}} \tag{5.33b}$$

$$\frac{\partial I_3}{\partial \widetilde{\mathbf{F}}} = \frac{\partial I_3}{\partial \widetilde{F}_{ij}} = \frac{\partial (\widetilde{F}_{kl} \widetilde{F}_{kl})}{\partial \widetilde{F}_{ij}} = \delta_{ki}\delta_{lj}\widetilde{F}_{kl} + \widetilde{F}_{kl}\delta_{ki}\delta_{lj} = \widetilde{F}_{ij} + \widetilde{F}_{ij} = 2\widetilde{\mathbf{F}} \tag{5.33c}$$

hence, Eq.(5.32) becomes

$$\widetilde{T}_{ij} = \rho \left[\frac{\partial W}{\partial I_1} \delta_{ik} + 2 \frac{\partial W}{\partial I_2} \widetilde{F}_{ki} + 2 \frac{\partial W}{\partial I_3} \widetilde{F}_{ik} \right] \widetilde{F}_{jk}. \tag{5.34}$$

5.2.4.2 Stress Response Function of Murnagahan's Theory for Left-Cauchy Green Strain Tensor $\widetilde{\mathbf{B}}$

Considering the stress to be a function of $\widetilde{\mathbf{B}}$, specifically

$$\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{D}} = \rho \frac{\partial W}{\partial \widetilde{\mathbf{B}}} \cdot \dot{\widetilde{\mathbf{B}}}. \tag{5.35}$$

We know that $\widetilde{\mathbf{B}} = \widetilde{\mathbf{F}}\widetilde{\mathbf{F}}^{\mathsf{T}}$, and its derivative is $\dot{\widetilde{\mathbf{B}}} = \dot{\widetilde{\mathbf{F}}}\widetilde{\mathbf{F}}^{\mathsf{T}} + \widetilde{\mathbf{F}}\dot{\widetilde{\mathbf{F}}}^{\mathsf{T}}$; therefore, we use these expressions, and Eq.(5.35) becomes

$$\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{D}} = \rho \frac{\partial W}{\partial \widetilde{\mathbf{B}}} \cdot (\dot{\widetilde{\mathbf{F}}}\widetilde{\mathbf{F}}^{\mathsf{T}} + \widetilde{\mathbf{F}}\dot{\widetilde{\mathbf{F}}}^{\mathsf{T}}) \tag{5.36}$$

on the other hand, by considering the expression for velocity gradient $\dot{\widetilde{\mathbf{F}}}^{\mathsf{T}} = \widetilde{\mathbf{F}}^{\mathsf{T}}\,\widetilde{\mathbf{L}}^{\mathsf{T}}$ we have

$$\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{D}} = \rho\,\frac{\partial W}{\partial \widetilde{\mathbf{B}}} \cdot\ (\widetilde{\mathbf{L}}\,\widetilde{\mathbf{B}} + \widetilde{\mathbf{B}}\,\widetilde{\mathbf{L}}^{\mathsf{T}}). \tag{5.37}$$

Since $\widetilde{\mathbf{D}}$ is the symmetric part of velocity gradient, it becomes

$$\widetilde{\mathbf{T}} \cdot \frac{1}{2}\,(\widetilde{\mathbf{L}} + \widetilde{\mathbf{L}}^{\mathsf{T}}) = \rho\,\frac{\partial W}{\partial \widetilde{\mathbf{B}}} \cdot\ \widetilde{\mathbf{B}}\,(\widetilde{\mathbf{L}} + \widetilde{\mathbf{L}}^{\mathsf{T}}) \tag{5.38}$$

by using the property of trace into the above equation we will get the following expression for the Cauchy stress to have a free-energy function of $\widetilde{\mathbf{B}}$

$$\widetilde{\mathbf{T}} = 2\,\rho\,\widetilde{\mathbf{B}}\,\frac{\partial W}{\partial \widetilde{\mathbf{B}}} \tag{5.39}$$

hence using chain rule leads to derive the Cauchy stress in terms of individual principle invariants as

$$\widetilde{\mathbf{T}} = 2\,\rho\,\widetilde{\mathbf{B}}\,\left[\frac{\partial W}{\partial I_1}\,\frac{\partial I_1}{\partial \widetilde{\mathbf{B}}} + \frac{\partial W}{\partial I_2}\,\frac{\partial I_2}{\partial \widetilde{\mathbf{B}}} + \frac{\partial W}{\partial I_3}\,\frac{\partial I_3}{\partial \widetilde{\mathbf{B}}}\right] \tag{5.40}$$

and then the derivatives of the invariants of $\widetilde{\mathbf{B}}$ are

$$\frac{\partial I_1}{\partial \widetilde{\mathbf{B}}} = \frac{\partial I_1}{\partial \widetilde{B}_{ij}} = \frac{\partial B_{kk}}{\partial \widetilde{B}_{ij}} = \delta_{ij} \tag{5.41a}$$

$$\frac{\partial I_2}{\partial \widetilde{\mathbf{B}}} = \frac{\partial I_2}{\partial \widetilde{B}_{ij}} = \frac{\partial(\widetilde{B}_{kl}\widetilde{B}_{lk})}{\partial \widetilde{B}_{ij}} = \delta_{ki}\delta_{lj}\widetilde{B}_{lk} + \widetilde{B}_{kl}\delta_{li}\delta_{kj} = \widetilde{B}_{ji} + \widetilde{B}_{ji} = 2\widetilde{\mathbf{B}}^{\mathsf{T}} \tag{5.41b}$$

$$\frac{\partial I_3}{\partial \widetilde{\mathbf{B}}} = \frac{\partial I_3}{\partial \widetilde{B}_{ij}} = \frac{\partial(\widetilde{B}_{kl}\widetilde{B}_{kl})}{\partial \widetilde{B}_{ij}} = \delta_{ki}\delta_{lj}\widetilde{B}_{kl} + \widetilde{B}_{kl}\delta_{ki}\delta_{lj} = \widetilde{B}_{ij} + \widetilde{B}_{ij} = 2\widetilde{\mathbf{B}} \tag{5.41c}$$

hence, Eq.(5.40) becomes

$$\widetilde{T}_{ij} = 2\,\rho\,\widetilde{B}_{ik}\,\left[\frac{\partial W}{\partial I_1}\,\delta_{kj} + 2\,\frac{\partial W}{\partial I_2}\,\widetilde{B}_{jk} + 2\,\frac{\partial W}{\partial I_3}\,\widetilde{B}_{kj}\right] \tag{5.42}$$

5.2.4.3 Stress Response Function of Zheng's Theory for Laplace Stretch $\widetilde{\mathbf{F}}$

Assuming the free energy function to be a function of upper-triangular Laplace stretch $\widetilde{\mathbf{F}}$, the Cauchy stress tensor can be expressed as

$$\widetilde{\mathbf{T}} = \rho \, \frac{\partial W}{\partial \widetilde{\mathbf{F}}} \, \widetilde{\mathbf{F}}^{\mathrm{T}} \tag{5.43}$$

hence the rotated Cauchy stress is expressed in terms of individual principle invariants as

$$\widetilde{\mathbf{T}} = \rho \, \left[\frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial \widetilde{\mathbf{F}}} + \frac{\partial W}{\partial I_2} \frac{\partial I_2}{\partial \widetilde{\mathbf{F}}} + \frac{\partial W}{\partial I_3} \frac{\partial I_3}{\partial \widetilde{\mathbf{F}}} \right] \widetilde{\mathbf{F}}^{\mathrm{T}} \tag{5.44}$$

The derivatives of the invariants of $\widetilde{\mathbf{F}}$ imply that

$$\frac{\partial I_1}{\partial \widetilde{\mathbf{F}}} = \frac{\partial I_1}{\partial \widetilde{F}_{ij}} = \frac{1}{2} \, \frac{\partial (\widetilde{F}_{kk} + \widetilde{F}_{ll})}{\partial \widetilde{F}_{ij}} = \frac{1}{2} \, \delta_{ij} \tag{5.45a}$$

$$\frac{\partial I_2}{\partial \widetilde{\mathbf{F}}} = \frac{\partial I_2}{\partial \widetilde{F}_{ij}} = \frac{1}{4} \, \frac{\partial \left[\widetilde{F}_{kl}\widetilde{F}_{lk} + \widetilde{F}_{kl}\widetilde{F}_{lk} + \widetilde{F}_{lk}\widetilde{F}_{lk} + \widetilde{F}_{kl}\widetilde{F}_{kl} \right]}{\partial \widetilde{F}_{ij}} = 2 \, \widetilde{\mathbf{Y}} \tag{5.45b}$$

$$\frac{\partial I_3}{\partial \widetilde{\mathbf{F}}} = \frac{\partial I_3}{\partial \widetilde{F}_{ij}} = \frac{1}{4} \, \frac{\partial \left[\widetilde{F}_{kl}\widetilde{F}_{lk} + \widetilde{F}_{kl}\widetilde{F}_{lk} - \widetilde{F}_{lk}\widetilde{F}_{lk} - \widetilde{F}_{kl}\widetilde{F}_{kl} \right]}{\partial \widetilde{F}_{ij}} = -2 \, \widetilde{\mathbf{K}} \tag{5.45c}$$

(5.45d)

With an assumption of the free-energy function being a function of the upper-triangular Laplace stretch $\widetilde{\mathbf{F}}$, the rotated Cauchy stress, written in terms of individual principle invariants, leads to

$$\widetilde{T}_{ij} = \rho \, \left[\frac{1}{2} \, \frac{\partial W}{\partial I_1} \, \delta_{ik} + 2 \, \frac{\partial W}{\partial I_2} \, \widetilde{Y}_{ik} - 2 \, \frac{\partial W}{\partial I_3} \, \widetilde{K}_{ik} \right] \, \widetilde{F}_{kj} \tag{5.46}$$

5.2.5 Velocity Gradients

The rotated velocity gradient, i.e., $\widetilde{\mathbf{L}} := \dot{\boldsymbol{\Lambda}} \cdot \boldsymbol{\Lambda}^{-1} + \boldsymbol{\Lambda}(\dot{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Gamma}^{-1})\boldsymbol{\Lambda}^{-1}$ can be expressed as

$$\widetilde{\mathbf{L}} = \begin{bmatrix} \frac{\dot{a}}{a} & \frac{a\,\dot{\gamma}}{b} \\ 0 & \frac{\dot{b}}{b} \end{bmatrix}. \tag{5.47}$$

5.2.6 Stress Power

With an assumption of the rotated Cauchy stress being a function of the upper-triangular Laplace stretch $\widetilde{\mathbf{F}}$, and applying Murnagahan's invariants, i.e., Eq. (5.34), one can find the mechanical power caused by stressing a deformable body as follows

$$\dot{W} = \widetilde{T}_{ij}\,\widetilde{L}_{ji} = \rho\left[\frac{\partial W}{\partial I_1}\,\delta_{ik} + 2\,\frac{\partial W}{\partial I_2}\,\widetilde{F}_{ki} + 2\,\frac{\partial W}{\partial I_3}\,\widetilde{F}_{ik}\right]\widetilde{F}_{jk}\,\widetilde{L}_{ji} \tag{5.48}$$

With an assumption of the rotated Cauchy stress being a function of the left Cauchy-Green strain tensor $\widetilde{\mathbf{B}}$, and applying Murnagahan's invariants, i.e., Eq. (5.42), the mechanical power leads to

$$\dot{W} = \widetilde{T}_{ij}\,\widetilde{L}_{ji} = 2\,\rho\,\widetilde{B}_{ik}\left[\frac{\partial W}{\partial I_1}\,\delta_{kj} + 2\,\frac{\partial W}{\partial I_2}\,\widetilde{B}_{jk} + 2\,\frac{\partial W}{\partial I_3}\,\widetilde{B}_{kj}\right]\widetilde{L}_{ji} \tag{5.49}$$

With an assumption of the rotated Cauchy stress being a function of the upper-triangular Laplace stretch $\widetilde{\mathbf{F}}$, and applying Zheng's invariants, i.e., Eq. (5.46), the mechanical power becomes

$$\dot{W} = \widetilde{T}_{ij}\,\widetilde{L}_{ji} = \rho\left[\frac{1}{2}\,\frac{\partial W}{\partial I_1}\,\delta_{ik} + 2\,\frac{\partial W}{\partial I_2}\,\widetilde{Y}_{ik} - 2\,\frac{\partial W}{\partial I_3}\,\widetilde{K}_{ik}\right]\widetilde{F}_{kj}\,\widetilde{L}_{ji} \tag{5.50}$$

How the stress power in the Lagrangian and physical coordinate systems relate, can be found through the quotient law, pushing forward from the Lagrangian coordinate system into the physical coordinate system via $\widetilde{\mathbf{F}}$. Considering $\widetilde{\mathbf{T}}$ as a function of $\widetilde{\mathbf{F}}$, their relationships are

$$\widetilde{\mathbf{T}} = \frac{1}{\det\widetilde{\mathbf{F}}}\,\mathbf{Q}^{-1}\,\mathbf{P}\,\widetilde{\mathbf{F}}^{\mathsf{T}} \quad \text{and} \quad \widetilde{\mathbf{L}} = \widetilde{\mathbf{F}}\,\mathbf{V}\,\widetilde{\mathbf{F}}^{-1} \tag{5.51}$$

wherein **P** is the 1st Piola-Kirchhoff stress. Considering $\widetilde{\mathbf{T}}$ as a function of $\widetilde{\mathbf{B}}$ results in

$$\widetilde{\mathbf{T}} = \frac{1}{\det \widetilde{\mathbf{F}}} \widetilde{\mathbf{F}} \mathbf{S} \widetilde{\mathbf{F}}^{\mathsf{T}} \quad \text{and} \quad \widetilde{\mathbf{L}} = \widetilde{\mathbf{F}} \mathbf{V} \widetilde{\mathbf{F}}^{-1} \tag{5.52}$$

wherein **S** is the second Piola-Kirchhoff stress. Therefore, the mechanical power caused by stressing a deformable body, when evaluated in a physical coordinate system with base vectors $\{\vec{\mathbf{e}}_i\}$ transform the concept of mechanical work into the different coordinate systems.

6. CONCLUSION

6.1 Summary

In this work, we derived components for a convected metric tensor and its inverse. They are described in an oblique, Cartesian, coordinate system whose axes are tangents to a basis of curvilinear coordinate axes originating at some particle of interest in a deforming body. Strains and strain rates are constructed in terms of these metrics that are quantified in this locally convected coordinate system. Quotient laws, and their associated Jacobians of transformation are derived that map vector and tensor fields from this convected coordinate system in-to and out-of the Lagrangian and physical coordinate systems.

Then, we start with an oblique convected coordinate system that finishes as an orthonormal coordinate system. This reversing process results in working with convected tensor fields that are evaluated in a rectangular, Cartesian, coordinate system at the current time whose components are 'physical' by construction. The real advantages of this work are rooted in working with a coordinate system when convected tensor fields have physical components, and working with stress components that are evaluated in this rectangular, Cartesian, coordinate system. Therefore, we have obtained two convected deformation tensors, viz., $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^{-1}$, and two strain tensors, viz., $\boldsymbol{\varepsilon}$ and e, both quantified in a rectangular, Cartesian, physical coordinate system. They and their rates have physical components when expressed in this $\mathbf{QR}$ rotated frame $\{\vec{\mathbf{e}}_i\}$ at the current time.

We constructed the constitutive equation for stress in this rectangular Cartesian frame by using the unitary invariants that we obtained from theories in the literature. The deformation gradient $\mathbf{F}$ and right Cauchy-Green strain tensor $\mathbf{C}$ are discussed in the Lagrangian coordinate system as a classical approach. Then we used the Laplace stretch $\widetilde{\mathbf{F}}$ and left Cauchy-Green like convected strain tensor $\widetilde{\mathbf{B}}$ to find the stress in the physical coordinate system defined in our approach.

6.2 Future Work

In the following, we present some of the possible topics of research as future work based upon the current study;

- Developing the finite element model to carry out the numerical analysis of nonlinear behavior based upon the developed formulation.
- Extending the analysis to a three-dimensional problem and study the emerged three-dimensional terms in the analysis.
- Developing a geometrically nonlinear model using various measures of strain and descriptions of motion.
- Construction of bio-materials constitutive equations based upon their strain energy functions.

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