SOME RESULTS ON BI-FREE PROBABILITY

A Dissertation

by

WONHEE NA

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Chair of Committee,	Kenneth Dykema
Committee Members,	David Kerr
	James Long
	Roger Smith
Head of Department,	Emil Straube

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ABSTRACT

This study consists of two projects on bi-free probability. In the first project, a bi-free central limit distribution is investigated. We find the principal function of the completely non-normal operator $l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*)$ on a subspace of the full Fock space $\mathcal{F}(\mathcal{H})$ which arises from a bi-free central limit distribution. By the fact that the principal function of a pure hyponormal operator with trace class self-commutator is an extension of the Fredholm index of the operator, we find the essential spectrum of this operator. In the second part, we examine the reduced bi-free product C*-algebra generated by two pairs of commuting self-adjoint projections. In particular, we partially describe how to find the bi-free product states and the corresponding C*-algebra given by the GNS construction for a generic distribution of the projections. We prove some general results analogous to Voiculescu's partial R- and S-transforms by using combinatorial techniques on bi-free setting.

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1. INTRODUCTION AND PRELIMINARIES

Free probability theory is initiated by Voiculescu in the 80s in order to solve certain operator algebra problems. Free independence (or freeness) is an analogue of the classical independence and free probability theory has evolved into a close parallel to basic probability theory. Also, it brings together many different fields of mathematics, for example, operator algebras, random matrix theory, and combinatorics.

In 2013, Voiculescu introduced a notion of bi-free independence as a generalization of freeness in a non-commutative probability space. He considered two-faced pairs of non-commutative random variables and the moments for such a combined system of left and right variables. In [15, 16, 17], the essential properties and theorems for bi-free independence are discussed, including additive and multiplicative bi-free convolutions, two-variables partial transforms, and the bi-free central limit theorem. The combinatorial constructions and proofs for those results are presented in [3, 4, 12, 13].

In my dissertation, I intend to further develop bi-free probability theory. The dissertation has four chapters including the introduction and preliminaries in Chapter 1. In Chapter 2, we investigate a bi-free central limit distribution which is an analogue of a semicircular distribution in free probability theory. We find the Pincus principal function of a certain seminormal operator which arises from a central limit distribution and the essential spectrum of the operator as an application. The next two chapters, Chapter 3 and 4, are devoted to discuss the reduced bi-free product C*-algebra generated by two two-faced pairs of commuting projections. For certain combinations of bi-free pairs of non-commutative random variables, we find their ordered joint moments and cumulant series through combinatorial techniques.

1.1 Free probability

A *non-commutative probability space* is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional with $\phi(1) = 1$. A non-commutative probability space $(\mathcal{A}, \phi)$

is called a *C*-probability space*, if in addition, $\mathcal{A}$ is a C*-algebra and ϕ is a state. The *joint distribution* of random variables $a_1, \dots, a_n$ in $\mathcal{A}$ is the linear functional $\mu : \mathbb{C}\langle X_1, \dots, X_n\rangle \to \mathbb{C}$ given by

$$\mu(P) \ = \ \phi(P(a_1, \dots, a_n)), \quad P \in \mathbb{C}\langle X_1, \dots, X_n\rangle$$

where $\mathbb{C}\langle X_1, \dots, X_n\rangle$ is the algebra of complex polylnomials in the non-commuting variables $X_1, \dots, X_n$. If $(\mathcal{A}, \phi)$ is a C*-probability space and $a \in \mathcal{A}$ is normal, i.e., $aa^* = a^*a$, then the distribution of the random variable a is given by a probability measure ν supported on the spectrum of a by

$$\mu_a(P(X)) \ = \ \int_{\sigma(a)} P(t)\, d\nu.$$

We will often write μ_a instead of ν.

Definition 1.1.1. A family of unital subalgebras $(\mathcal{A}_k)_{k\in K}$ in a non-commutative probability space $(\mathcal{A}, \phi)$ is *freely independent* if $\phi(a_1 \cdots a_n) = 0$ whenever $a_i \in \mathcal{A}_{k_i}$ with $k_i \neq k_{i+1}$ and $\phi(a_i) = 0$ for all $1 \leq i \leq n$.

We define the notion of a full Fock space which will be useful for the future arguments.

Definition 1.1.2. Let $\mathcal{H}$ be a complex Hilbert space. Then the *full Fock space* on $\mathcal{H}$ is

$$\mathcal{F}(\mathcal{H}) \ = \ \mathbb{C}\Omega \oplus \bigoplus_{n\geq 1} \mathcal{H}^{\otimes n}$$

where Ω is called the *vacuum vector* and has norm one. The *vacuum expectation* is defined as $\phi_\Omega(\cdot) = \langle \cdot\, \Omega, \Omega\rangle$ on $\mathcal{B}(\mathcal{F}(\mathcal{H}))$. For an element ξ in $\mathcal{H}$, the *left creation operator* $l(\xi) \in \mathcal{B}(\mathcal{F}(\mathcal{H}))$ is given by

$$l(\xi)\Omega \ = \ \xi$$
$$l(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) \ = \ \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$$

for all $n \geq 1$ and $\xi_1, \cdots, \xi_n \in \mathcal{H}$. The adjoint $l(\xi)^*$ of $l(\xi)$ is called the *left annihilation operator*.

The *right creation operator* $r(\xi) \in \mathcal{B}(\mathcal{F}(\mathcal{H}))$ is determined by

$$\begin{aligned} r(\xi)\Omega &= \xi \\ r(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) &= \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi \end{aligned}$$

for all $n \geq 1$ and $\xi_1, \cdots, \xi_n \in \mathcal{H}$. Its adjoint $r(\xi)^*$ is called the *right annihilation operator*.

Under the above notations, let $(e_i)_{i \in I}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$, and let $l_i = l(e_i)$ and $C^*(l_i)$ denote the unital subalgebra generated by l_i for $i \in I$. Then the family $(C^*(l_i))_{i \in I}$ is free in $(\mathcal{B}(\mathcal{F}(\mathcal{H})), \phi_\Omega)$.

The free cumulants were introduced by Speicher to understand free independence using a combinatorial approach.

Definition 1.1.3. Let S be a finite totatlly ordered set. A *partition* of the set S is a set $\pi = \{V_1, ..., V_n\}$ of pairwise disjoint, non-empty sebsets of S such that $S = \cup_{i=1}^n V_i$. We call $V_1, ..., V_n$ *blocks* of the partition π. For two elements $a, b \in S$, we write $a \sim_\pi b$ if a and b are contained in the same block of π.

The set of all partitions of the set $\{1, ..., n\}$ is denoted by $\mathcal{P}(n)$. A paritition $\pi \in \mathcal{P}(n)$ is called *non-crossing* if whenever $1 \leq a_1 < b_1 < a_2 < b_2 \leq n$ are such that $a_1 \sim_\pi a_2$ and $b_1 \sim_\pi b_2$, we have $b_1 \sim_\pi a_2$. We denote the set of all non-crossing partitions of $\{1, \ldots, n\}$ by $NC(n)$. With the usual refinement order, let 0_n denote the minimal element of $NC(n)$ and let 1_n denote the maximal element of $NC(n)$. Let $NC'(n)$ denote the set of all non-crossing partitions π in $\mathcal{P}(n)$ such that the singleton set $\{1\}$ is a block of π.

Definition 1.1.4. Let $\pi \in \mathcal{P}(n)$ and $S \subseteq \{1, \ldots, n\}$. We say S *splits* π if for each block $V \in \pi$, we have either $V \subseteq S$ or $V \subseteq S^c$.

Definition 1.1.5. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. The *free cumulants* are a family of multilinear functionals $\kappa_\pi : \mathcal{A}^n \to \mathbb{C}$ determined recursively by the *moment-cumulant* *formula*

$$\phi(a_1 \cdots a_n) \;=\; \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \ldots, a_n)$$

and $\kappa_\pi(a_1, \ldots, a_n) := \prod_{V \in \pi} \kappa_V((a_1, \ldots, a_n)|_V)$, where the product is taken over all the blocks of π and $\kappa_V((a_1, \ldots, a_n)|_V) = \kappa_k(a_{i_1}, \ldots, a_{i_k})$ for $V = \{i_1 < \cdots < i_k\}$. We use the notation $\kappa_k := \kappa_{1_k}$.

The next theorem states that freeness is equivalent to vanishing of mixed cumulants.

Theorem 1.1.6 ([11])**.** *Let $(\mathcal{A}_i)_{i \in I}$ be a family of unital subalgebras of a non-commutative probability space $(\mathcal{A}, \phi)$ and let $(\kappa_n)_{n \in \mathbb{N}}$ be the corresponding free cumulants. Then $(\mathcal{A}_i)_{i \in I}$ are freely independent if and only if $\kappa_n(a_1, \ldots, a_n) = 0$ whenever $a_j \in \mathcal{A}_{i_j}$ and there exist $i_j \neq i_k$ for $1 \leq j,\ k \leq n$.*

For $\pi, \sigma \in NC(n)$ with $\pi \leq \sigma$, the interval $[\pi, \sigma]$ denotes the set $\{\rho \in NC(n) \mid \pi \leq \rho \leq \sigma\}$ and the interval has the canonical factorization of the form $NC(1)^{k_1} \times \cdots NC(n)^{k_n}$ where $k_j \geq 0$. The incidence algebra on the lattice of $NC(n)$, denoted by $I(NC)$, is the algebra of all complex-valued functions on $\cup_{n \geq 1}\{(\pi, \sigma) \mid \pi, \sigma \in NC(n) \text{ and } \pi \leq \sigma\}$, equipped with a pointwise addition, a scalar multiplication, and a convolution product defined by

$$(f_1 * f_2)(\pi, \sigma) = \sum_{\rho \in [\pi, \sigma]} f_1(\pi, \rho) f_2(\rho, \sigma).$$

A function $f \in I(NC)$ is said to be *multiplicative* if $f(\pi, \sigma) = f(0_1, 1_1)^{k_1} \cdots f(0_n, 1_n)^{k_n}$ whenever the interval $[\pi, \sigma]$ is factorized by $[\pi, \sigma] \cong NC(1)^{k_1} \times \cdots NC(n)^{k_n}$. Note that a multiplicative funtion $f \in I(NC)$ is completely determined by the sequence $(f(0_n, 1_n))_{n \geq 1}$. Let $\mathcal{M}$ be the set of all multiplicative functions, and denote the set of all multiplicative functions f with $f(0_1, 1_1) = 1$ by $\mathcal{M}_1$. Note that the convlution of two multiplicative functions is multiplicative.

Definition 1.1.7. Let $\pi \in NC(n)$. The *Kreweras complement* $K(\pi)$ of π is defined to be the biggest element among $\sigma \in NC(\bar{1}, ..., \bar{n})$ such that $\pi \cup \sigma \in NC(1, \bar{1}, ..., n, \bar{n})$.

If $f_1, f_2 \in \mathcal{M}$, then one can verify that

$$(f_1 * f_2)(0_n, 1_n) \;=\; \sum_{\pi \in NC(n)} f_1(0_n, \pi) f_2(0_n, K(\pi)).$$

For $f_1, f_2 \in \mathcal{M}_1$, the *pinched-convolution* $f_1 \breve{*} f_2$ of f_1 and f_2 is defined by

$$(f_1 \breve{*} f_2)(0_n, 1_n) \;=\; \sum_{\pi \in NC'(n)} f_1(0_n, \pi) f_2(0_n, K(\pi)). \tag{1.1}$$

where $f_1 \breve{*} f_2 \in \mathcal{M}_1$. Notice that the pinched-convolution $f_1 \breve{*} f_2$ is obtained from the convolution by pinching out the terms in $NC(n) \backslash NC'(n)$. In [10], it is demonstrated that for freely independent random variables $a_1, a_2 \in \mathcal{A}$, if $f_1, f_2 \in \mathcal{M}$ are the multiplicative functions associated to the cumulants of a_1, a_2, respectively, that is, $f_1(0_n, 1_n) = \kappa_n(a_1)$ and $f_2(0_n, 1_n) = \kappa_n(a_2)$ for all $n \geq 1$, then we have $(f_1 * f_2)(0_n, 1_n) = \kappa_n(a_1 a_2) = \kappa_n(a_2 a_1)$. Moreover, $f_1(0_n, \pi) = \kappa_\pi(a_1)$ (respectively, $f_2(0_n, \pi) = \kappa_\pi(a_2)$) is satisfied for all $\pi \in NC(n)$.

For every $f \in \mathcal{M}$, we define a formal power series ϕ_f by

$$\phi_f(z) \;=\; \sum_{n=1}^{\infty} f(0_n, 1_n) z^n. \tag{1.2}$$

In [10], it is proved that $\phi_{f_1}(\phi_{f_1 \breve{*} f_2}(z)) \;=\; \phi_{f_1 * f_2}(z)$ for every $f_1, f_2 \in \mathcal{M}_1$, and therefore we have

$$\phi_{f_1 \breve{*} f_2}\left(\phi^{\langle -1 \rangle}_{f_1 * f_2}(z)\right) = \phi^{\langle -1 \rangle}_{f_1}(z). \tag{1.3}$$

Before finishing this section, we will recall the definitions of free transforms and their equalities which will be useful in Chapter 3. Let $(\mathcal{A}, \phi)$ be a C*-non-commutative probability space and let $a \in \mathcal{A}$. Let $\kappa_n(a)$ denote the n-th free cumulant of a. As a formal power series, the *Cauchy transform* of a is

$$G_a(z) \;=\; \phi((zI - a)^{-1}) \;=\; \frac{1}{z} \sum_{n \geq 0} \phi(a^n) z^{-n},$$

and the *R-transform* is defined by

$$R_a(z) \;=\; \sum_{n\geq 0} \kappa_{n+1}(a) z^n.$$

To use the combinatorial arguments, we need the following analogues of the moment and cumulant series. The *moment series* of a is

$$M_a(z) \;=\; 1 + \sum_{n\geq 1} \phi(a^n) z^n,$$

and the *cumulant series* of a is

$$C_a(z) \;=\; 1 + \sum_{n\geq 1} \kappa_n(a) z^n.$$

The *S-transform* of a is the power series given by

$$S_a(z) \;=\; \frac{1+z}{z}\, \chi_a(z),$$

where $\chi_a(z)$ is the formal power series inverse of $M_a(z) - 1$ under composition, so that

$$M_a(\chi_a(z)) \;=\; z + 1. \tag{1.4}$$

Recall the relations between the above series as follows.

$$C_a\left(z \cdot M_a(z)\right) \;=\; M_a(z) \tag{1.5}$$

$$M_a\left(\frac{z}{C_a(z)}\right) \;=\; C_a(z) \tag{1.6}$$

1.2 Bi-free probability

1.2.1 Bi-freeness and examples

We recall the basics about a free product construction to define bi-free independence. Given a family of Hilbert spaces with specified unit vectors, $\mathcal{H}_i = \mathbb{C}\xi_i \oplus \mathcal{H}_i^\circ$ for $i \in I$, the *Hilbert space free product* $(\mathcal{H}, \mathcal{H}^\circ, \xi) = *_{i\in I}(\mathcal{H}_i, \mathcal{H}_i^\circ, \xi_i)$ is defined by

$$\mathcal{H} \ = \ \mathbb{C}\xi \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{i_j\neq i_{j+1}} \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ\right) \ = \ \mathbb{C}\xi \oplus \mathcal{H}^\circ$$

where $\|\xi\| = 1$ and the direct sums are orthogonal. On $\mathcal{B}(\mathcal{H})$, the vector state ϕ_ξ corresponding to the specified unit vector ξ is defined by

$$\phi_\xi(T) = \langle T\xi, \xi\rangle, \quad T \in \mathcal{B}(\mathcal{H}).$$

For each $i \in I$, there exist unitary operators V_i and W_i such that

$$V_i : \ \mathcal{H}_i \otimes \left(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{\substack{i\neq i_1 \\ i_j\neq i_{j+1}}} \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ\right)\right) \ \to \ \mathcal{H}$$

is defined by

$$\xi_i \otimes \xi \to \xi$$

$$\mathcal{H}_i^\circ \otimes \xi \to \mathcal{H}_i^\circ$$

$$\xi_i \otimes (\mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ) \to \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ$$

$$\mathcal{H}_i^\circ \otimes (\mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ) \to \mathcal{H}_i^\circ \otimes \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ$$

and similarly,

$$W_i : \left(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \left(\bigoplus_{\substack{i\neq i_n \\ i_j \neq i_{j+1}}} \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ \right) \right) \otimes \mathcal{H}_i \to \mathcal{H}$$

is defined by

$$\xi \otimes \xi_i \to \xi$$

$$\xi \otimes \mathcal{H}_i^\circ \to \mathcal{H}_i^\circ$$

$$(\mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ) \otimes \xi_i \to \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ$$

$$(\mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ) \otimes \mathcal{H}_i^\circ \to \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ \otimes \mathcal{H}_i^\circ.$$

For $T \in \mathcal{B}(\mathcal{H}_i)$, we define the left and right operators, $\lambda_i(T)$ and $\rho_i(T)$, on $\mathcal{H}$ by

$$\lambda_i(T) \;=\; V_i(T \otimes I)V_i^{-1} \;\in\; \mathcal{B}(\mathcal{H}) \tag{1.7}$$

$$\rho_i(T) \;=\; W_i(I \otimes T)W_i^{-1} \;\in\; \mathcal{B}(\mathcal{H}). \tag{1.8}$$

For each $i \in I$, we refer to λ_i and ρ_i as left and right representations of $\mathcal{B}(\mathcal{H}_i)$ on $\mathcal{B}(\mathcal{H})$.

A two-faced pair of non-commutative random variables in $(\mathcal{A}, \phi)$ is an ordered pair (b, c) of random variables in $\mathcal{A}$. We refer to b as *left* and c as *right* variables. Using the free product construction of Hilbert spaces, we can define a bi-free independence of two-faced pairs.

Definition 1.2.1 ([15])**.** A family $((b_i, c_i))_{i\in I}$ of two-faced pairs in $(\mathcal{A}, \phi)$ is said to be *bi-freely independent* (abbreviated *bi-free*) if there exists a family of Hilbert spaces with specified unit vectors $((\mathcal{H}_i, \mathcal{H}_i^\circ, \xi_i))_{i\in I}$ and unital homomorphisms $l_i : \mathbb{C}\langle X_i \rangle \to \mathcal{B}(\mathcal{H}_i)$ and $r_i : \mathbb{C}\langle Y_i \rangle \to \mathcal{B}(\mathcal{H}_i)$, $i \in I$, such that the joint distribution of the family $((b_i, c_i))_{i\in I}$ with respect to ϕ is equal to the joint distribution of the family of pairs $((\pi(X_i), \pi(Y_i)))_{i\in I}$ in $(\mathcal{B}(\mathcal{H}), \phi_\xi)$, where $(\mathcal{H}, \mathcal{H}^\circ, \xi) = *_{i\in I}(\mathcal{H}_i, \mathcal{H}_i^\circ, \xi_i)$ and $\pi : \mathbb{C}\langle X_i, Y_i | i \in I \rangle \to \mathcal{B}(\mathcal{H})$ with $\pi(X_i) = \lambda_i \circ l_i(X_i)$ and $\pi(Y_i) = \rho_i \circ r_i(Y_i)$.

Example 1.2.2. In Definition 1.1.2, we defined left and right creation and annihilation operators on the full Fock space. For a complex Hilbert space $\mathcal{H}$ with orthonormal basis $(e_i)_{i\in I}$, let $\mathcal{F}(\mathcal{H})$ be the full Fock space on which left and right creation operators $l_i = l(e_i)$ and $r_i = r(e_i)$ with their adjoints $l_i^*,\ r_i^*$. Then the family of two-faced families $((l_i, l_i^*), (r_i, r_i^*))_{i\in I}$ is bi-free in $(\mathcal{B}(\mathcal{H}), \phi_\Omega)$, where $\phi_\Omega(\cdot) = \langle \cdot\Omega, \Omega\rangle$.

Definition 1.2.3. Let $n \in \mathbb{N}$ and let $\chi : \{1, \ldots, n\} \to \{l, r\}$ be a map such that $\chi^{-1}(l) = \{i_1 < \cdots < i_k\}$ and $\chi^{-1}(r) = \{i_{k+1} > \cdots > i_n\}$. Define a permutation s_χ on $\{1, \ldots, n\}$ by $s_\chi(j) = i_j$. A partition $\pi \in \mathcal{P}(n)$ is said to be *bi-non-crossing* with respect to χ if the partition $s_\chi^{-1} \cdot \pi$ is non-crossing, i.e., $s_\chi^{-1} \cdot \pi \in NC(n)$. We denote the set of all bi-non-crossing partitions with respect to χ by $BNC(\chi)$. Let 1_χ and 0_χ denote the maximal and minimal elements in $BNC(\chi)$, respectively.

For $n \geq 1$ and given a map $\chi : \{1, \ldots, n\} \to \{l, r\}$, a multilinear functional $\kappa_\chi : \mathcal{A}^n \to \mathbb{C}$ is uniquely determined by the moment-cumulant relation

$$\phi(a_1 \cdots a_n) \ = \sum_{\pi \in \mathrm{BNC}(\chi)} \left(\prod_{V \in \pi} \kappa_{\chi|_V}((a_1, \ldots, a_n)|_V) \right) \ \text{ for } \ a_1, \cdots, a_n \in \mathcal{A}$$

where the product is over all blocks V of π. These κ_χ's are called the (l, r)*-cumulant functionals* (or, *bi-free cumulants*) of $(\mathcal{A}, \phi)$. As in the free case, we use the notations

$$\kappa_\pi(a_1, \ldots, a_n) \ = \ \prod_{V \in \pi} \kappa_{\chi|_V}((a_1, \ldots, a_n)|_V).$$

Let $n, m \geq 0$. Consider a map $\chi_{n,m} : \{1, \ldots, n + m\} \to \{l, r\}$ such that $\chi_{n,m}(k) = l$ if $1 \leq k \leq n$ and $\chi_{n,m}(k) = r$ if $n + 1 \leq k \leq n + m$. For notational purpose, we will refer to this map as $\chi_{n,m} : \{1_l, \ldots, n_l, 1_r, \ldots, m_r\} \to \{l, r\}$ with $i_l \mapsto l$ for $1 \leq k \leq n$ and $j_r \mapsto r$ for $1 \leq k \leq m$. In this particular case, we shall denote $BNC(\chi_{n,m})$ by $BNC(n, m)$, $1_{\chi_{n,m}}$ by $1_{n,m}$, and $\kappa_{1_{n,m}}$ by $\kappa_{n,m}$, for $n, m \geq 1$. We also write $\kappa_{n,m}(a, b)$ for $\kappa_{1_{n,m}}(\underbrace{a, \ldots, a}_{\text{n}}, \underbrace{b, \ldots, b}_{\text{m}})$ for $a, b \in \mathcal{A}$. Note that $\kappa_{n,0}(a, b) = \kappa_n(a)$ and $\kappa_{0,m}(a, b) = \kappa_m(b)$.

As in the free case, it is proved in [4] that bi-free independence is equivalent to vanishing of mixed cumulants. Let $((b_i, c_i))_{i\in I}$ be a family of two-faced pairs in $(\mathcal{A}, \phi)$. Then $((b_i, c_i))_{i\in I}$ are bi-freely independent if and only if

$$\kappa_\chi(a_1, \ldots, a_n) \ = \ 0$$

whenever the map $\epsilon : \{1, \ldots, n\} \to I$ satisfying $a_j \in \{b_{\epsilon(j)}, c_{\epsilon(j)}\}$ is non-constant. Note that κ_χ has the cumulant property; that is, if $z_1 = (b_1, c_1)$ and $z_2 = (b_2, c_2)$ are bi-free, then $\kappa_\chi(z_1 + z_2) \ = \kappa_\chi(z_1) + \kappa_\chi(z_2)$.

For some $1 \leq m \leq n$, let $\chi : \{1, \ldots, m\} \to \{l, r\}$ be a map, and let $k(0) := 0 < k(1) < \cdots < k(m) := n$. Define a map $\hat{\chi} : \{1, \ldots, n\} \to \{l, r\}$ by $\hat{\chi}(i) = \chi(k(j))$ for all $i \in \{k(j-1)+1, \ldots, k(j)\}$ where $1 \leq i \leq n$ and $1 \leq j \leq m$. For each partition π in $\mathrm{BNC}(\chi)$, there exists a corresponding partition $\hat{\pi}$ in $\mathrm{BNC}(\hat{\chi})$ by replacing each node i of π by the block $\{k(j-1)+1, \ldots, k(j)\}$ where $i \in \{k(j-1)+1, \ldots, k(j)\}$. Then we can easily see that $\widehat{1_\chi} = 1_{\hat{\chi}}$ and $\widehat{0_\chi} = \bigcup\limits_{1 \leq j \leq m} \{\{k(j-1)+1, \ldots, k(j)\}\}$. For any given two partitions π and σ in $\mathrm{BNC}(\chi)$, let $\pi \vee \sigma$ denote the smallest partition in $\mathrm{BNC}(\chi)$ greater than both π and σ. Then we have the following result on bi-free cumulants for products of random variables which will be very useful in Chapter 3.

Theorem 1.2.4 (Theorem 9.1.5 of [3])**.** *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and $\{a_i\}_{i=1}^n \in \mathcal{A}$. Under the above notations, we have*

$$\kappa_{1_\chi}(a_1 \cdots a_{k(1)}, a_{k(1)+1} \cdots a_{k(2)}, \ldots, a_{k(m-1)+1} \cdots a_{k(m)}) = \sum_{\substack{\pi \in \mathrm{BNC}(\hat{\chi}) \\ \pi \vee \widehat{0_\chi} = 1_{\hat{\chi}}}} \kappa_\pi(a_1, \ldots, a_n). \tag{1.9}$$

1.2.2 Partial bi-free transforms

Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and (a, b) be a two-faced pair in $\mathcal{A}$. The the *two-variable Green's function* is the power series

$$G_{a,b}(z, w) \ = \ \phi((z-a)^{-1}(w-b)^{-1}) \ = \ \frac{1}{zw} + \sum_{\substack{m,n \geq 0 \\ m+n \geq 1}} \phi(a^m b^n) z^{-m-1} w^{-n-1}$$

and the *partial bi-free R-transform* is the generating series

$$R_{a,b}(z,w) \;=\; \sum_{\substack{m,n\geq 0\\ m+n\geq 1}} \kappa_{m,n}(a,b) z^m w^n.$$

Theorem 1.2.5 (Theorem 2.4 of [16]). *We have the equality of germs of holomorphic functions near* $(0,0) \in \mathbb{C}^2$,

$$R_{a,b}(z,w) \;=\; 1 + zR_a(z) + wR_b(w) - \frac{zw}{G_{(a,b)}\left(\frac{1}{z} + R_a(z), \frac{1}{w} + R_b(w)\right)}$$

where $R_a(z)$ *and* $R_b(w)$ *are one variable R-transforms.*

The *moment series* of a two-faced pair (a,b) is a power series

$$M_{a,b}(z,w) \;=\; 1 + \sum_{\substack{n,m\geq 0\\ n+m\geq 1}} \phi(a^n b^m) z^n w^m$$

and the *cumulant series* of (a,b) is

$$C_{a,b}(z,w) \;=\; 1 + \sum_{\substack{n,m\geq 0\\ n+m\geq 1}} \kappa_{n,m}(a,b) z^n w^m \;=\; 1 + R_{a,b}(z,w).$$

For computational convenience, define the power series $K_{a,b}$ of the form

$$K_{a,b}(z,w) \;=\; \sum_{n,m\geq 1} \kappa_{n,m}(a,b) z^n w^m.$$

Theorem 2.1 in [17] shows that if (a,b) is a two-faced pair of non-commutative random variables in $(\mathcal{A},\phi)$ with $\phi(a),\phi(b) \neq 0$, the two-variables partial bi-free S-transform as a holomorphic function of $(z,w) \in (\mathbb{C}\setminus\{0\})^2$ when z,w are near 0 is of the form

$$S_{a,b}(z,w) \;=\; \frac{(z+1)(w+1)}{zw}\left(1 - \frac{1+z+w}{M_{a,b}(\chi_a(z),\chi_b(w))}\right). \tag{1.10}$$

Theorem 1.2.6 (Theorem 7.2.4 of [13]). *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and let $a, b \in \mathcal{A}$ be arbitrary elements. Then,*

$$M_a(z) + M_b(w) \;=\; \frac{M_a(z)M_b(w)}{M_{a,b}(z,w)} + C_{a,b}(zM_a(z), wM_b(w)).$$

Using Theorem 1.2.6, we can obtain

$$K_{a,b}(zM_a(z), wM_b(w)) \;=\; 1 - \frac{M_a(z)M_b(w)}{M_{a,b}(z,w)}. \tag{1.11}$$

Indeed, for $a, b \in \mathcal{A}$,

$$\begin{aligned}
&K_{a,b}(zM_a(z), wM_b(w)) \\
&= \; M_a(z) + M_b(w) - \frac{M_a(z)M_b(w)}{M_{a,b}(z,w)} - C_a(zM_a(z)) - C_b(wM_b(w)) + 1 \\
&= \; M_a(z) + M_b(w) - \frac{M_a(z)M_b(w)}{M_{a,b}(z,w)} - M_a(z) - M_b(w) + 1 \\
&= \; 1 - \frac{M_a(z)M_b(w)}{M_{a,b}(z,w)}
\end{aligned}$$

where the second equality is by (1.5).

1.2.3 Central limit theorem

In free probability theory, Voiculescu proved the existence of central limit distributions.

Theorem 1.2.7 ([18]). *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, and let $(a_i)_{i=1}^{\infty}$ be a family of free random variables in $\mathcal{A}$ such that*

(i) $\phi(a_i) = 0$ *for all* $i \geq 1$*,*

(ii) $\sup_{i \geq 1} |\phi(a_i^k)| < \infty$ *for all* $k \geq 2$*,*

(iii) $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \phi(a_i^2) = r^2/4 > 0$.

Then letting $s_n = \frac{a_1+\cdots+a_n}{\sqrt{n}}$, the sequence $(s_n)_{n=1}^{\infty}$ converges in distribution to the semicircular distribution γ_r, given by

$$\gamma_r(X^n) = \frac{2}{\pi r^2}\int_{-r}^{r} t^n\sqrt{r^2-t^2}dt.$$

In [15], the bi-free version of central limit theorem is also shown by Voiculescu.

Theorem 1.2.8 ([15])**.** *A two-faced pair $z := (z_l, z_r)$ has a bi-free central limit distribution if and only if $\kappa_\chi(z) = 0$ whenever $\chi : \{1, ..., n\} \to \{l, r\}$ with $n = 1$ or $n \geq 3$.*

Theorem 1.2.9. *Let $(z^{(n)})_{n\in\mathbb{N}} = ((z_l^{(n)}, z_r^{(n)}))_{n\in\mathbb{N}}$ be a bi-free sequence of two-faced pairs in $(\mathcal{A}, \phi)$ such that*

(i) $\varphi(z_l^{(n)}) = \phi(z_r^{(n)}) = 0$,

(ii) $sup_{n\geq1}|\phi(z_{i_1}^{(n)}\cdots z_{i_m}^{(n)})| < \infty$ *for every* $i_1, ..., i_m \in \{l, r\}$,

(iii) $\lim_{N\to\infty}\frac{\sum_{n=1}^{N}\phi(z_i^{(n)}z_j^{(n)})}{N} = C_{ij}$ *for every* $i, j \in \{l, r\}$.

Then, letting $S_N = (\frac{\sum_{n=1}^{N} z_l^{(n)}}{\sqrt{N}}, \frac{\sum_{n=1}^{N} z_r^{(n)}}{\sqrt{N}})$, we have $\lim_{N\to\infty}\mu_{S_N}(P) = \gamma_C(P)$ for all $P \in \mathbb{C}\langle X_l, X_r\rangle$, where γ_C is the bi-free central limit distribution with $\gamma_C(X_iX_j) = C_{ij}$ for every $i, j \in \{l, r\}$.

Theorem 1.2.10. *For each matrix $C = (C_{ij})_{i,j\in\{l,r\}}$ with complex number entries, there is exactly one bi-free central limit distribution $\gamma_C : \mathbb{C}\langle X_l, X_r\rangle \to \mathbb{C}$ so that*

$$\gamma_C(X_iX_j) = C_{ij} \text{ for each } i, j \in \{l, r\}.$$

For each matrix C, there exist vectors $v_l, v_l', v_r, v_r' \in \mathcal{H}$ such that $C_{ij} = \langle v_j, v_i'\rangle$ for each $i, j \in \{l, r\}$, and for every such choice, letting

$$z_l = l(v_l) + l^*(v_l') \qquad \text{and} \qquad z_r = r(v_r) + r^*(v_r'),$$

the pair (z_l, z_r) has the bi-free central limit distribution γ_C.

2. PRINCIPAL FUNCTIONS FOR BI-FREE CENTRAL LIMIT DISTRIBUTIONS *

2.1 Principal function of a completely non-normal operator

Let T be a completely non–normal operator on a Hilbert space $\mathcal{H}$ with self-commutator $T^*T - TT^* = -2C$ where C is trace class. Set $U = \frac{1}{2}(T + T^*)$ and $V = -\frac{1}{2}i(T - T^*)$. Consider the unital C*-algebra generated by C in $\mathcal{B}(\mathcal{H})$. This C*-algebra is isometrically isomorphic to $C(\sigma(C))$, the complex valued continuous functions on $\sigma(C)$, by the Gelfand-Naimark theorem. Consider a function on $\sigma(C)$ defined by

$$t \mapsto \begin{cases} -i\sqrt{-t}, & t < 0 \\ 0, & t = 0 \\ \sqrt{t}, & t > 0 \end{cases}$$

and then there exists the unique element $\hat{C}$ in the C*-algebra corresponding to this function by the Gelfand transform. Note that $\hat{C}^2 = C$ and $\hat{C}\hat{C}^* = \hat{C}^*\hat{C} = |C|$.

The *determining function* of the operator T is defined to be

$$E(l, s) = I + \frac{1}{i}\hat{C}(V - l)^{-1}(U - s)^{-1}\hat{C}$$

for $l \in \mathbb{C} \setminus \sigma(V)$ and $s \in \mathbb{C} \setminus \sigma(U)$. Then $E(l, s)$, for each fixed l and s, is an invertible element in the C*-algebra generated by T and I. Since $\det(I + AB) = \det(I + BA)$ when A is compact with AB and BA in trace class, we have

$$\begin{aligned} \det E(l, s) &= \det\left(I + \frac{1}{i}C(V - l)^{-1}(U - s)^{-1}\right) \\ &= \det\left((V - l)(U - s)(V - l)^{-1}(U - s)^{-1}\right). \qquad (2.1) \end{aligned}$$

*Reprinted with permission from [7].

The *principal function* g is defined in [2] to be the element of $L_1(\mathbb{R}^2)$ such that

$$\det E(l,s) = \exp\left(\frac{1}{2\pi i}\iint g(\delta,\gamma)\frac{d\delta}{\delta - l}\frac{d\gamma}{\gamma - s}\right). \tag{2.2}$$

It is known that $\operatorname{supp}(g)$ is contained in $\{(\delta,\gamma)\in\mathbb{R}^2 \mid \gamma + i\delta \in \sigma(T)\}$. Moreover, it is a complete unitary invariant for T if C has one dimensional range; that is, two completely non-normal operators T and T' are unitarily equivalent if and only if their principal functions agree, assuming each of T and T' has a self-commutator with one dimensional range. In Theorem 8.1 of [2], it is proved that

$$g(\delta,\gamma) = \operatorname{ind}(T - (\gamma + i\delta))$$

if $\gamma + i\delta$ is not in the essential spectrum $\sigma_e(T)$. This result implies that the principal function g of T is an extension of the Fredholm index of $T - z$ to the whole plane. However, it is not the typical situation that g assumes only integer values on the plane; indeed the map $T \mapsto g$ is onto, namely (see [1]), any summable function on $\mathbb{R}^2$ with compact support is the principal function of a completely non-normal operator with a trace class self-commutator.

2.2 The principal functions of certain operators

Definition 2.2.1. An *implemented non-commutative probability space* is a triple $(\mathcal{A},\phi,P)$ where $(\mathcal{A},\phi)$ is a non-commutative probability space and $P = P^2 \in \mathcal{A}$ is an idempotent so that

$$PaP = \phi(a)P \text{ for all } a \in \mathcal{A}.$$

An *implemented C^*-probability space* $(\mathcal{A},\phi,P)$ will satisfy additional requirements that $(\mathcal{A},\phi)$ is a C^*-probability space and that $P = P^*$. If a two-faced family $((z_i)_{i\in I},(z_j)_{j\in J})$ in an implemented non-commutative probability space $(\mathcal{A},\phi,P)$ satisfies that

$$[z_i, z_j] = \lambda_{i,j}P \text{ for some } \lambda_{i,j} \in \mathbb{C}, i \in I, j \in J,$$

then the family $((z_i)_{i\in I}, (z_j)_{j\in J})$ is called a *system with rank* ≤ 1 *commutation* where $(\lambda_{i,j})_{i\in I, j\in J}$ is the coefficient matrix of the system.

Remark 2.2.2. The bi-free two-faced system in Theorem 1.2.10 is an example of rank ≤ 1 commutation. Indeed, $(\mathcal{B}(\mathcal{F}(\mathcal{H})), \phi_\Omega, P)$ is an implemented C^*-probability space where ϕ_Ω is the vacuum expectation and P is a projection on $\mathbb{C}\Omega$. We have $[z_l, z_r] = (\langle v_r, v_l'\rangle - \langle v_l, v_r'\rangle)P$.

Let $\mathcal{H}$ be a Hilbert space and $v_1, v_2 \in \mathcal{H}$. We consider the operator T on $\mathcal{F}(\mathcal{H})$ given by

$$T = X_1 + iX_2, \qquad \text{with} \quad X_1 = l(v_1) + l(v_1)^*, \quad X_2 = r(v_2) + r(v_2)^*.$$

This arises from the bi-free central limit distribution and was described in Example 3.10 of [16]. We have $[X_1, X_2] = 2i(\mathrm{Im}\langle v_2, v_1\rangle)P$ in the implemented C^*-probability space $(\mathcal{B}(\mathcal{F}(\mathcal{H})), \phi_\Omega, P)$, so that

$$[T^*, T] \; = \; -4(\mathrm{Im}\langle v_2, v_1\rangle)P. \tag{2.3}$$

Both the spectrum and the essential spectrum of X_1 on $\mathcal{F}(\mathcal{H})$ equal $[-2\|v_1\|, 2\|v_1\|]$ and those of X_2 equal $[-2\|v_2\|, 2\|v_2\|]$. By the following easy lemma, which is well known but whose proof we include for convenience, the spectrum of the operator $T = X_1 + iX_2$ on $\mathcal{F}(\mathcal{H})$ is contained in $[-2\|v_1\|, 2\|v_1\|] + i[-2\|v_2\|, 2\|v_2\|]$. Throughout this chapter, we are interested in non-normal operators T, so we assume that $\mathrm{Im}\langle v_2, v_1\rangle$ is non-zero.

Lemma 2.2.3. *If A and B are self-adjoint with $\sigma(A) \subseteq [r_1, r_2]$ and $\sigma(B) \subseteq [t_1, t_2]$, then $\sigma(A + iB) \subseteq [r_1, r_2] + i[t_1, t_2]$.*

Proof. If $A_1 = A_1^*$, $B_1 \geq 0$, and B_1 is invertible, then $A_1 + iB_1 = B_1^{\frac{1}{2}} \left(B_1^{-\frac{1}{2}} A_1 B_1^{-\frac{1}{2}} + i\right) B_1^{\frac{1}{2}}$ is invertible since $B_1^{-\frac{1}{2}} A_1 B_1^{-\frac{1}{2}}$ is self-adjoint. Suppose $a + ib \notin [r_1, r_2] + i[t_1, t_2]$. Then either $a < r_1$ or $a > r_2$ or $b < t_1$ or $b > t_2$. If $b < t_1$, then $A + iB - (a + ib) = (A - a) + i(B - b)$ and $B - b \geq 0$ is invertible. So $a + ib \notin \sigma(A + iB)$. If $b > t_2$, then $a + ib - (A + iB) = (a - A) + i(b - B)$ and $b - B \geq 0$ is invertible, so that $a + ib \notin \sigma(A + iB)$. Since $(A - a) + i(B - b) = i((B - b) - i(A - a))$, we can easily show that $A + iB - (a + ib)$ is invertible for each case of $a < r_1$ and $a > r_2$. Therefore,

$\sigma(A+iB) \subseteq [r_1, r_2] + i[t_1, t_2]$. $\square$

The operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal*, if its self-commutator $T^*T - TT^*$ is positive. Furthermore, if there is no reducing subspace of T, the restriction of T to which is normal, then T is said to be *pure hyponormal* or *completely non-normal hyponormal*.

Theorem 2.2.4 (Theorem 2.1.3 of [9])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator with $[T^*, T] = D$. Then there is a unique orthogonal decomposition $\mathcal{H} = \mathcal{H}_p(T) \oplus \mathcal{H}_n(T)$ where $\mathcal{H}_p(T)$ and $\mathcal{H}_n(T)$ are reducing subspaces for T, such that*

(i) $T_p = T|_{\mathcal{H}_p(T)}$ *is pure hyponormal,*

(ii) $T_n = T|_{\mathcal{H}_n(T)}$ *is normal.*

Moreover,

$$\begin{aligned}\mathcal{H}_p(T) &= \bigvee\{T^{*k}T^l D(\mathcal{H}) \mid k, l \in \mathbb{N}\}\\ \mathcal{H}_n(T) &= \{\zeta \in \mathcal{H} \mid DT^{*l}T^k\zeta = 0 \textit{ for every } k, l \in \mathbb{N}\}.\end{aligned}$$

As we can see in (2.3), if $\mathrm{Im}\langle v_2, v_1\rangle \leq 0$ (or ≥ 0), then $T = X_1 + iX_2$ is a hyponormal operator (or cohyponormal, respectively) on $\mathcal{F}(\mathcal{H})$. By Theorem 2.2.4, the pure parts $\mathcal{H}_p(T)$ and $\mathcal{H}_p(T^*)$ of T and T^* are equal to $\overline{\mathrm{alg}(T, T^*, 1)\Omega}$.

Assuming that $\mathrm{Im}\langle v_2, v_1\rangle \leq 0$, if v_2 is a scalar mutiple of v_1, then $\mathrm{alg}(T, T^*, 1)\Omega$ is dense in $\mathcal{F}(\mathbb{C}v_1)$ so that T is pure hyponormal on $\mathcal{F}(\mathbb{C}v_1)$. However, if v_2 is not a scalar multiple of v_1, then T is not a pure hyponormal operator on $\mathcal{F}(\mathbb{C}v_1 + \mathbb{C}v_2)$, that is, there exists a nontrivial reducing subspace $\mathcal{N}$ of T in $\mathcal{F}(\mathbb{C}v_1 + \mathbb{C}v_2)$ such that $T|_{\mathcal{N}}$ is normal. For, suppose that u is a unit vector which is orthogonal to v_1 in $\mathbb{C}v_1 + \mathbb{C}v_2$ and $v_2 = cv_1 + du$ where $c, d \in \mathbb{C}$ are non-zero. Since v_2 and $\frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u$ are orthogonal to each other, for each $m, n \in \mathbb{N}$,

$$(l(v_1) + l(v_1)^*)^m \left(u \otimes \left(\frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u\right)\right) \in \text{span}\left\{v_1^{\otimes k} \otimes u \otimes \left(\frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u\right) \,\middle|\, k \in \mathbb{N}\right\}$$

and

$$(r(v_2)+r(v_2)^*)^n\left(u\otimes\left(\frac{c}{|c|^2}v_1-\frac{d}{|d|^2}u\right)\right)\in\text{span}\left\{u\otimes\left(\frac{c}{|c|^2}v_1-\frac{d}{|d|^2}u\right)\otimes v_2^{\otimes k}\;\middle|\;k\in\mathbb{N}\right\}.$$

Since $\operatorname{alg}(T,T^*,1)=\operatorname{alg}(X_1,X_2,1)$ and $[X_1,X_2]=2i(\operatorname{Im}\langle v_2,v_1\rangle)P$,

$$\begin{aligned}\mathcal{N}:=&\bigvee\left\{(r(v_2)+r(v_2)^*)^n(l(v_1)+l(v_1)^*)^m\left(u\otimes\left(\frac{c}{|c|^2}v_1-\frac{d}{|d|^2}u\right)\right)\;\middle|\;m,n\in\mathbb{N}\right\}\\ =&\bigvee\left\{v_1^{\otimes m}\otimes u\otimes\left(\frac{c}{|c|^2}v_1-\frac{d}{|d|^2}u\right)\otimes v_2^{\otimes n}\;\middle|\;m,n\in\mathbb{N}\right\}.\end{aligned}$$

is a nontrivial reducing subspace of T in $\mathcal{F}(\mathbb{C}v_1+\mathbb{C}v_2)$ which is orthogonal to $\mathbb{C}\Omega$. Clearly, the restrictions of $l(v_1)+l(v_1)^*$ and $r(v_2)+r(v_2)^*$ to $\mathcal{N}$ commute, so the restriction of T to $\mathcal{N}$ is normal. Now we will characterize the pure part $\overline{\operatorname{alg}(T,T^*,1)\Omega}$ of T in $\mathcal{F}(\mathcal{H})$ when v_1 and v_2 are linearly independent.

Proposition 2.2.5. *Let $T=l(v_1)+l(v_1)^*+i(r(v_2)+r(v_2)^*)$ with v_1 and v_2 linearly independent, and let u and w be non-zero vectors in $\mathbb{C}v_1+\mathbb{C}v_2$ with $u\perp v_1$ and $w\perp v_2$. For each $n\in\mathbb{N}$, let A_n be the span of length n tensor products in $\mathcal{F}(\mathbb{C}v_1+\mathbb{C}v_2)$. Then*

$$\overline{\operatorname{alg}(T,T^*,1)\Omega}\;=\;\mathbb{C}\Omega\oplus\bigoplus_{n\in\mathbb{N}}(A_n\cap\operatorname{alg}(T,T^*,1)\Omega)\tag{2.4}$$

and for every $n\in\mathbb{N}$,

$$B_n:=\;\{v_1^{\otimes n},v_1^{\otimes n-1}\otimes u,v_1^{\otimes n-2}\otimes u\otimes v_2,\cdots,v_1\otimes u\otimes v_2^{\otimes n-2},u\otimes v_2^{\otimes n-1}\}\tag{2.5}$$

and

$$B_n':=\;\{v_2^{\otimes n},w\otimes v_2^{\otimes n-1},v_1\otimes w\otimes v_2^{\otimes n-2},\cdots,v_1^{\otimes n-2}\otimes w\otimes v_2,v_1^{\otimes n-1}\otimes w\}\tag{2.6}$$

are orthogonal bases of $A_n \cap \operatorname{alg}(T, T^*, 1)\Omega$. Furthermore, we have the obvious isomorphisms

$$\begin{aligned}\overline{\operatorname{alg}(T, T^*, 1)\Omega} &\cong \mathcal{F}(\mathbb{C}v_1) \oplus (\mathcal{F}(\mathbb{C}v_1) \otimes u \otimes \mathcal{F}(\mathbb{C}v_2)) \\ &\cong \mathcal{F}(\mathbb{C}v_2) \oplus (\mathcal{F}(\mathbb{C}v_1) \otimes w \otimes \mathcal{F}(\mathbb{C}v_2)).\end{aligned}$$

Proof. We will prove by induction on n that B_n is an orthogonal basis for $A_n \cap \operatorname{alg}(T, T^*, 1)$. This is clear for $n = 1$. For $n = 2$, consider the orthogonal basis of A_2

$$Z_2 = \{v_1^{\otimes 2}, v_1 \otimes u, u \otimes v_2, u \otimes w\}$$

containing B_2. Here, $B_2 = \{v_1^{\otimes 2}, v_1 \otimes u, u \otimes v_2\} \subseteq \operatorname{alg}(T, T^*, 1)\Omega$ and $Z_2 \setminus B_2 = \{u \otimes w\} \subseteq (\operatorname{alg}(T, T^*, 1)\Omega)^\perp$ as we saw in the above argument describing $\mathcal{N}$. Now the assertion is proved for $n = 2$. Consider another orthogonal basis of A_2, $Z_2' = \{v_2^{\otimes 2}, v_2 \otimes w, w \otimes v_2, w^{\otimes 2}\}$. Then $Z_3 := \{v_1 \otimes Z_2\} \cup \{u \otimes Z_2'\}$ is an orthogonal basis of A_3. Since $\operatorname{alg}(T, T^*, 1)\Omega$ is a reducing subspace of T, $v_1 \otimes B_2 \subseteq \operatorname{alg}(T, T^*, 1)\Omega$ and $v_1 \otimes \{Z_2 \setminus B_2\} \subseteq (\operatorname{alg}(T, T^*, 1)\Omega)^\perp$. In $u \otimes Z_2'$, only $u \otimes v_2^{\otimes 2}$ is contained in $\operatorname{alg}(T, T^*, 1)\Omega$ because every tensor product in $\mathcal{F}(\mathbb{C}v_1 + \mathbb{C}v_2)$ which starts with u and ends with w belongs to $\mathcal{N}$ and is therefore orthogonal to $\operatorname{alg}(T, T^*, 1)\Omega$; moreover $u \otimes w \otimes v_2 = (r(v_2) + r(v_2)^*)(u \otimes w) \in (\operatorname{alg}(T, T^*, 1)\Omega)^\perp$. Hence, $B_3 = \{v_1 \otimes B_2\} \cup \{u \otimes v_2^{\otimes 2}\}$ is contained in $\operatorname{alg}(T, T^*, 1)\Omega$ and $Z_3 \setminus B_3$ is contained in $(\operatorname{alg}(T, T^*, 1)\Omega)^\perp$. Thus the assertion holds for $n = 3$.

The induction step for general n proceeds similarly. For each $n \in \mathbb{N}$, construct an orthogonal basis Z_n for A_n as follows.

$$Z_n = \{v_1^n\} \cup \left(\bigcup_{1 \le j \le n} \{v_1^{n-j} \otimes u \otimes Z_{j-1}'\} \right),$$

where Z_k' is the set of all length k tensor products in $\mathcal{F}(\mathcal{H})$ whose components consist of v_2 and w. The induction hypothesis is that B_j is an orthogonal basis of $A_j \cap \operatorname{alg}(T, T^*, 1)\Omega$ and $Z_j \setminus B_j$ is orthogonal to $\operatorname{alg}(T, T^*, 1)\Omega$ for each $1 \le j \le n$. Then $Z_{n+1} = \{v_1 \otimes Z_n\} \cup \{u \otimes Z_n'\}$ and it is an orthogonal basis of A_{n+1}. Since $\operatorname{alg}(T,T^*,1)\Omega$ is a reducing subspace of T and is invariant under $l(v_1)+l(v_1)^*$, we have $v_1\otimes B_n=\{v_1^{\otimes n+1}, v_1^{\otimes n}\otimes u, v_1^{\otimes n-1}\otimes u\otimes v_2,\cdots, v_1\otimes u\otimes v_2^{\otimes n-1}\}\subseteq \operatorname{alg}(T,T^*,1)\Omega$ and $v_1\otimes\{Z_n\setminus B_n\}\subseteq (\operatorname{alg}(T,T^*,1)\Omega)^{\perp}$. In $u\otimes Z_n'$, only $u\otimes v_2^{\otimes n}$ is contained in $\operatorname{alg}(T,T^*,1)\Omega$ and the other elements are orthogonal to $\operatorname{alg}(T,T^*,1)\Omega$ by the induction hypothesis. Therefore, $B_{n+1}=\{v_1\otimes B_n\}\cup\{u\otimes v_2^{\otimes n}\}$ is an orthogonal basis for $A_{n+1}\cap \operatorname{alg}(T,T^*,1)\Omega$ and $Z_{n+1}\setminus B_{n+1}$ is an orthogonal basis for $A_{n+1}\cap(\operatorname{alg}(T,T^*,1)\Omega)^{\perp}$. Thus, for every $n\in\mathbb{N}$, B_n is an orthogonal basis for the set of all length n tensor products in $\operatorname{alg}(T,T^*,1)\Omega$. This finishes the proof by induction.

The proof that for $n\in\mathbb{N}$, B_n' is also an orthogonal basis for $A_n\cap\operatorname{alg}(T,T^*,1)\Omega$ follows similarly by induction on n, using the invariance of $\operatorname{alg}(T,T^*,1)\Omega$ under $r(v_2)+r(v_2)^*$ rather than $l(v_1)+l(v_1)^*$. The equality (2.4) follows by the above proofs. □

Before we further investigate the operator $T=X_1+iX_2$ having v_1 and v_2 linearly independent, we will take a look at the case when the vectors v_1 and v_2 are linearly dependent. We will refer to the following result.

Theorem 2.2.6 ([5]). *If T is a hyponormal operator on $\mathcal{H}$, then $C^*(T)$ is generated by the unilateral shift if and only if T is unitarily equivalent to S, where S satisfies the conditions*

(i) *S is irreducible,*

(ii) *self-commutator S^*S-SS^* is compact,*

(iii) *$\sigma_e(S)$ is a simple closed curve,*

(iv) *$\sigma(S)$ is the closure of V, where V is the bounded component of $\mathbb{C}\backslash\sigma_e(S)$,*

(v) $\operatorname{ind}(S-\lambda)=-1$ *for* $\lambda\in\sigma(S)\backslash\sigma_e(S)$.

Example 2.2.7. Let $v_1=\alpha\cdot v_2$, where $\alpha\in\mathbb{C}$, $\operatorname{Im}\alpha\neq 0$, and $\|v_2\|=1$. Let T be given by

$$\begin{aligned} T &= l(v_1)+l(v_1)^*+i(r(v_2)+r(v_2)^*) \\ &= (\alpha l(v_2)+ir(v_2))+(\bar{\alpha}l(v_2)^*+ir(v_2)^*) \end{aligned}$$

on $\mathcal{F}(\mathbb{C})$. Then,

$$T(\Omega) \;=\; (\alpha+i)v_2$$

and for each $n \in \mathbb{N}$,

$$T(v_2^{\otimes n}) \;=\; (\alpha+i)v_2^{\otimes n+1} + (\bar{\alpha}+i)v_2^{\otimes n-1}.$$

Therefore,

$$T \;=\; (\alpha+i)U + (\bar{\alpha}+i)U^*,$$

where U is the unilateral shift on $\mathcal{F}(\mathbb{C})$. If $\alpha = i$, then $T = 2iU$ and $[T^*, T] = 4P$ so that T is a hyponormal operator. If $\alpha = -i$, then $T = 2iU^*$ and $[T^*, T] = -4P$, so T is cohyponormal.

Since the image of the unilateral shift U in the Calkin algebra is a normal operator, by the functional calculus, we have

$$\begin{aligned}
\sigma_e((\alpha+i)U + (\bar{\alpha}+i)U^*) \;&=\; \{(\alpha+i)t + (\bar{\alpha}+i)\bar{t} \mid t \in \sigma_e(U)\} \\
&=\; \{\alpha t + \overline{\alpha t} + i(t+\bar{t}) \mid t \in \mathbb{T}\}.
\end{aligned}$$

This curve is the solution set of

$$x^2 + |\alpha|^2 y^2 - 2(\operatorname{Re}\alpha)xy \;=\; 4(\operatorname{Im}\alpha)^2 \tag{2.7}$$

in the xy-plane, which is an ellipse centered at the origin. So the essential spectrum of T is a simple closed curve. Let V_0 be the bounded component of $\mathbb{C}\backslash\sigma_e(T)$. Then by Theorem 2.2.6, we have

$$\sigma(T) \;=\; \overline{V_0},$$

and for $\lambda \in \sigma(T)\backslash\sigma_e(T)$,

$$\operatorname{ind}(T-\lambda) \;=\; \begin{cases} -1, & \operatorname{Im}\alpha > 0 \\ 1, & \operatorname{Im}\alpha < 0. \end{cases}$$

Thus, the principal function is the characteristic function of the interior of the ellipse (2.7) when $\operatorname{Im}\alpha < 0$, and is the negative of this when $\operatorname{Im}\alpha > 0$.

In the rest of this chapter, we consider the pure part of $T = X_1 + iX_2$ acting on $\overline{\operatorname{alg}(T, T^*, 1)\Omega}$ where $X_1 = l(v_1) + l(v_1)^*$ and $X_2 = r(v_2) + r(v_2)^*$. So T is a completely non-normal operator.

Now we will find a formula for the principal function of T when v_1 and v_2 are linearly independent. For this, we will use equation (2.2), so we will first establish a formula for $\det E(l, s)$ of T. Suppose $l \in \mathbb{C} \setminus \sigma(X_2)$ and $s \in \mathbb{C} \setminus \sigma(X_1)$. From (2.1), we have

$$\begin{aligned}
\det E(l,s) &= \det((X_2 - l)(X_1 - s)(X_2 - l)^{-1}(X_1 - s)^{-1}) \\
&= \det(((X_1 - s)(X_2 - l) - 2\mathrm{Im}\langle v_2, v_1\rangle iP)(X_2 - l)^{-1}(X_1 - s)^{-1}) \\
&= \det(1 - 2\mathrm{Im}\langle v_2, v_1\rangle iP(X_2 - l)^{-1}(X_1 - s)^{-1}) \\
&= \det(1 - 2\mathrm{Im}\langle v_2, v_1\rangle iP^2(X_2 - l)^{-1}(X_1 - s)^{-1}) \\
&= \det(1 - 2\mathrm{Im}\langle v_2, v_1\rangle iP(X_2 - l)^{-1}(X_1 - s)^{-1}P) \\
&= \det(1 - 2\mathrm{Im}\langle v_2, v_1\rangle i \cdot \phi_\Omega((X_2 - l)^{-1}(X_1 - s)^{-1})P) \\
&= 1 - 2\mathrm{Im}\langle v_2, v_1\rangle i \cdot \phi_\Omega((X_2 - l)^{-1}(X_1 - s)^{-1}) \\
&= 1 - 2\mathrm{Im}\langle v_2, v_1\rangle i \cdot \overline{\phi_\Omega((\bar{s} - X_1)^{-1}(\bar{l} - X_2)^{-1})} \\
&= 1 - 2\mathrm{Im}\langle v_2, v_1\rangle i \cdot \overline{G_{(X_1,X_2)}(\bar{s}, \bar{l})} \qquad (2.8)
\end{aligned}$$

where $G_{(X_1,X_2)}(z, w) = \phi((z - X_1)^{-1}(w - X_2)^{-1})$. Note that $G_{(X_1,X_2)}(z, w)$ is the germ of a holomorphic function near (∞, ∞) in $\mathbb{C}_\infty \times \mathbb{C}_\infty$ (see [16]). For the given two-faced pair (X_1, X_2), the definition of the partial bi-free R-transform and Lemma 7.2 of [15] give

$$\begin{aligned}
R_{(X_1,X_2)}(z,w) &= R_{2,0}(X_1, X_2) + R_{0,2}(X_1, X_2) + R_{1,1}(X_1, X_2) \\
&= \phi(X_1^2)z^2 + \phi(X_2^2)w^2 + \phi(X_1X_2)zw \\
&= \|v_1\|^2 z^2 + \|v_2\|^2 w^2 + \langle v_2, v_1\rangle zw. \qquad (2.9)
\end{aligned}$$

From the formula for the partial bi-free R-transform in Theorem 1.2.5, we also have

$$R_{(X_1,X_2)}(z,w) \;=\; 1+\|v_1\|^2z^2+\|v_2\|^2w^2-\frac{zw}{G_{(X_1,X_2)}(\frac{1}{z}+\|v_1\|^2z,\frac{1}{w}+\|v_2\|^2w)}. \tag{2.10}$$

Denoting

$$t_1=\frac{1}{z}+\|v_1\|^2z \quad \text{and} \quad t_2=\frac{1}{w}+\|v_2\|^2w,$$

for $z,w\in\mathbb{C}\setminus\{0\}$ close to 0, we have

$$z=\frac{t_1-\sqrt{t_1^2-4\|v_1\|^2}}{2\|v_1\|^2} \quad \text{and} \quad w=\frac{t_2-\sqrt{t_2^2-4\|v_2\|^2}}{2\|v_2\|^2},$$

where the branches of the square roots are $\sqrt{t_1^2-4\|v_1\|^2}\approx t_1$ and $\sqrt{t_2^2-4\|v_2\|^2}\approx t_2$ for $|t_1|$ and $|t_2|$ large. From the formulas (2.9) and (2.10), we get

$$\begin{aligned} G_{(X_1,X_2)}(t_1,t_2) \;&=\; \frac{zw}{1-\langle v_2,v_1\rangle zw} \\ &=\; \frac{(t_1-\sqrt{t_1^2-4\|v_1\|^2})(t_2-\sqrt{t_2^2-4\|v_2\|^2})}{4\|v_1\|^2\|v_2\|^2-\langle v_2,v_1\rangle(t_1-\sqrt{t_1^2-4\|v_1\|^2})(t_2-\sqrt{t_2^2-4\|v_2\|^2})}, \end{aligned} \tag{2.11}$$

where $|t_1|$ and $|t_2|$ are large.

Let

$$q(t) \;=\; \frac{t-\sqrt{t^2-4}}{2} \qquad t\in\mathbb{C}\setminus[-2,2]. \tag{2.12}$$

The function $z\mapsto z+\frac{1}{z}$ sends the punctured unit disk $\{z \mid 0<|z|<1\}$ biholomorphically onto $\mathbb{C}\setminus[-2,2]$. The function q is its inverse with respect to composition. We deduce that the identity $q(t)=\overline{q(\bar{t})}$ holds for all $t\in\mathbb{C}\setminus[-2,2]$.

By (2.8) and (2.11), for $|l|$ and $|s|$ large, we have

$$\begin{aligned} &\det E(l,s) \\ &=\; 1-2(\mathrm{Im}\langle v_2,v_1\rangle)i\left(\overline{\frac{(\bar{s}-\sqrt{\bar{s}^2-4\|v_1\|^2})(\bar{l}-\sqrt{\bar{l}^2-4\|v_2\|^2})}{4\|v_1\|^2\|v_2\|^2-\langle v_2,v_1\rangle(\bar{s}-\sqrt{\bar{s}^2-4\|v_1\|^2})(\bar{l}-\sqrt{\bar{l}^2-4\|v_2\|^2})}}\right) \end{aligned}$$

$$= 1 + \frac{2(\mathrm{Im}\langle v_1, v_2\rangle)i(s - \sqrt{s^2 - 4\|v_1\|^2})(l - \sqrt{l^2 - 4\|v_2\|^2})}{4\|v_1\|^2\|v_2\|^2 - \langle v_1, v_2\rangle(s - \sqrt{s^2 - 4\|v_1\|^2})(l - \sqrt{l^2 - 4\|v_2\|^2})}$$

$$= \frac{4\|v_1\|^2\|v_2\|^2 - \overline{\langle v_1, v_2\rangle}(s - \sqrt{s^2 - 4\|v_1\|^2})(l - \sqrt{l^2 - 4\|v_2\|^2})}{4\|v_1\|^2\|v_2\|^2 - \langle v_1, v_2\rangle(s - \sqrt{s^2 - 4\|v_1\|^2})(l - \sqrt{l^2 - 4\|v_2\|^2})}$$

$$= \frac{1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} q\left(\frac{s}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}{1 - \frac{\alpha}{\|v_1\|\|v_2\|} q\left(\frac{s}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)} \tag{2.13}$$

where $\alpha = \langle v_1, v_2\rangle$, and for the second equality, we have used

$$\overline{(\bar{s} - \sqrt{\bar{s}^2 - 4\|v_1\|^2})} = 2\|v_1\| \, \overline{q\left(\frac{\bar{s}}{\|v_1\|}\right)} = 2\|v_1\| \, q\left(\frac{s}{\|v_1\|}\right) = s - \sqrt{s^2 - 4\|v_1\|^2}$$

and

$$\overline{(\bar{l} - \sqrt{\bar{l}^2 - 4\|v_2\|^2})} = 2\|v_2\| \, \overline{q\left(\frac{\bar{l}}{\|v_2\|}\right)} = 2\|v_2\| \, q\left(\frac{l}{\|v_2\|}\right) = l - \sqrt{l^2 - 4\|v_2\|^2}.$$

Since v_1 and v_2 are linearly independent, we have $|\alpha| < \|v_1\|\|v_2\|$. Since $\left|q\left(\frac{s}{\|v_1\|}\right)\right| < 1$ and $\left|q\left(\frac{l}{\|v_2\|}\right)\right| < 1$ for $s \in \mathbb{C}\backslash[-2\|v_1\|, 2\|v_1\|]$ and $l \in \mathbb{C}\backslash[-2\|v_2\|, 2\|v_2\|]$, the numerator and denominator in (2.13) do not vanish for such s and l. So the right-hand side of (2.13) is a holomorphic function there. Since by definition in (2.8), $\det E(l, s)$ is holomorphic on $(\mathbb{C}_\infty\backslash\sigma(X_2)) \times (\mathbb{C}_\infty\backslash\sigma(X_1))$, it follows from the analytic continuation that the formula of $\det E(l, s)$ in (2.13) holds for all $s \in \mathbb{C} \setminus [-2\|v_1\|, 2\|v_1\|]$ and $l \in \mathbb{C} \setminus [-2\|v_2\|, 2\|v_2\|]$.

In the rest of this section, we find the formula of the principal function $g(\delta, \gamma)$ of T by using the formula (2.13). The principal function g was defined on $\mathbb{R}^2$ by

$$\det E(l, s) = \exp\left(\frac{1}{2\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}} g(\delta, \gamma) \, \frac{d\delta}{\delta - l} \frac{d\gamma}{\gamma - s}\right)$$

and $\mathrm{supp}(g) \subseteq \{(\delta, \gamma) \in \mathbb{R}^2 \mid \gamma + i\delta \in \sigma(T)\}$. To find the principal function of T, consider the function f defined by

$$f(l,\gamma) \ = \ \int_{\mathbb{R}} g(\delta,\gamma)\,\frac{d\delta}{\delta - l}.$$

for $l \in \mathbb{C}\backslash\sigma(X_2)$ and $\gamma \in \mathbb{R}$. Fixing $\gamma \in \mathbb{R}$, $f(l,\gamma)$ is a holomorphic function for $l \in \mathbb{C} \setminus \sigma(X_2)$. From (2.13) and the definition of $g(\delta,\gamma)$, we have

$$\int_{\mathbb{R}} f(l,\gamma)\,\frac{d\gamma}{\gamma - s} = (2\pi i)\ \mathrm{Log}\left(\frac{1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} q\left(\frac{s}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}{1 - \frac{\alpha}{\|v_1\|\|v_2\|} q\left(\frac{s}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}\right), \tag{2.14}$$

where $s \in \mathbb{C}_\infty\backslash\sigma(X_1)$ and $l \in \mathbb{C}_\infty\backslash\sigma(X_2)$. Now we will find the function $f(l,\gamma)$ by using the Stieltjes inversion formula. We defined the function $q(t)$ for $t \in \mathbb{C} \setminus [-2,2]$ in (2.12).

Lemma 2.2.8. *If* $t_0 \in [-2,2]$*, then*

$$\lim_{\epsilon \searrow 0} q(t_0 + i\epsilon) \ = \ \frac{t_0 - i\sqrt{4 - t_0^2}}{2}.$$

Proof. For $t_0 \in (-2,2)$,

$$\begin{aligned}
\lim_{\epsilon \searrow 0} q(t_0 + i\epsilon) \ &= \ \lim_{\epsilon \searrow 0} \frac{t_0 + i\epsilon - \sqrt{(t_0 + i\epsilon)^2 - 4}}{2} \\
&= \ \lim_{\epsilon \searrow 0} \frac{t_0 + i\epsilon - \sqrt{-(4 + \epsilon^2 - t_0^2) + 2i\epsilon t_0}}{2} \\
&= \ \frac{t_0 - i\sqrt{4 - t_0^2}}{2}.
\end{aligned}$$

For, when ϵ is large and positive, the branch of a square root is such that $\sqrt{-(4 + \epsilon^2 - t_0^2) + 2i\epsilon t_0} \approx t_0 + i\epsilon$. So $\lim_{\epsilon \searrow 0} \sqrt{-(4 + \epsilon^2 - t_0^2) + 2i\epsilon t_0} = i\sqrt{4 - t_0^2}$. $\square$

Define a function $\zeta(t)$ for $t \in [-2,2]$ by

$$\zeta(t) \ = \ \frac{t - i\sqrt{4 - t^2}}{2}.$$

Then $\zeta(t) \in \mathbb{T}$ for $t \in [-2,2]$, where $\mathbb{T}$ is a unit circle in $\mathbb{C}$. By Lemma 2.2.8, the limit of $q(t+i\epsilon)$ goes to $\zeta(t)$ as $\epsilon \searrow 0$, where $t \in [-2,2]$. Then we have for $\gamma \in [-2\|v_1\|, 2\|v_1\|]$,

$$\lim_{\epsilon \searrow 0} q\left(\frac{\gamma}{\|v_1\|} + i\frac{\epsilon}{\|v_1\|}\right) = \frac{\frac{\gamma}{\|v_1\|} - i\sqrt{4 - \left(\frac{\gamma}{\|v_1\|}\right)^2}}{2} = \zeta\left(\frac{\gamma}{\|v_1\|}\right) \in \mathbb{T}. \tag{2.15}$$

Fix $l \in \mathbb{R} \setminus [-2\|v_2\|, 2\|v_2\|]$. Since clearly $f(l,\gamma) = 0$ for $\gamma \in \mathbb{R} \setminus \sigma(X_1)$, we suppose $\gamma \in \sigma(X_1)$. Using (2.14), the Stieltjes inversion formula, and (2.15), we have

$$\begin{aligned}
&f(l,\gamma) \\
&= \frac{1}{\pi} \lim_{\epsilon \searrow 0} \operatorname{Im}\left((2\pi i) \operatorname{Log}\left(\frac{1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} q\left(\frac{\gamma}{\|v_1\|} + i\frac{\epsilon}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}{1 - \frac{\alpha}{\|v_1\|\|v_2\|} q\left(\frac{\gamma}{\|v_1\|} + i\frac{\epsilon}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}\right)\right) \\
&= 2\log\left|\frac{1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}{1 - \frac{\alpha}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}\right| \\
&= \log \frac{\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right)\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|}\overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right)}{\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right)\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right)} \\
&= \operatorname{Log}\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) + \operatorname{Log}\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|}\overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right) \\
&\quad - \operatorname{Log}\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) - \operatorname{Log}\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right)
\end{aligned} \tag{2.16}$$

where Log is the principal branch of the logarithm. This equality holds where $\gamma \in \sigma(X_1)$ and $l \in \mathbb{R} \setminus [-2\|v_2\|, 2\|v_2\|]$.

Fix $\gamma \in \sigma(X_1)$. Since each expression appearing as an argument of Log, above, remains in the disk of radius 1 centered at 1 for $l \in \mathbb{C} \setminus [-2\|v_2\|, 2\|v_2\|]$. So the expression (2.16) is holomorphic on $\mathbb{C}\backslash[-2\|v_2\|, 2\|v_2\|]$. The equality (2.16) was derived for $l \in \mathbb{R}\backslash[-2\|v_2\|, 2\|v_2\|]$, but as defined, $f(l,\gamma)$ is holomorphic in $\mathbb{C} \setminus [-2\|v_2\|, 2\|v_2\|]$. By the analytic continuation, we have

$$\begin{aligned}
&\int_{\mathbb{R}} g(\delta,\gamma)\, \frac{d\delta}{\delta - l} \\
&= \ \mathrm{Log}\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) + \mathrm{Log}\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|}\overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right) \\
&\quad - \mathrm{Log}\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) - \mathrm{Log}\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right)
\end{aligned}$$

for $\gamma \in \sigma(X_1)$ and $l \in \mathbb{C}\backslash\sigma(X_2)$. Now we will apply the Stieltjes inversion formula to $f(l,\gamma)$ in order to recover the principal function $g(\delta,\gamma)$ of T. Since $\lim_{\epsilon \searrow 0} q\left(\frac{\delta}{\|v_2\|} + i\frac{\epsilon}{\|v_2\|}\right) = \zeta\left(\frac{\delta}{\|v_2\|}\right)$ for $\delta \in \sigma(X_2)$ as in (2.15), it follows that

$$\begin{aligned}
&g(\delta,\gamma) \\
&= \ \frac{1}{\pi} \lim_{\epsilon \searrow 0} \ \mathrm{Im}\ f(\delta + i\epsilon, \gamma) \\
&= \ \frac{1}{\pi}\Bigg(\mathrm{Arg}\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right)\zeta\left(\frac{\delta}{\|v_2\|}\right)\right) + \mathrm{Arg}\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|}\overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)}\zeta\left(\frac{\delta}{\|v_2\|}\right)\right) \\
&\quad - \mathrm{Arg}\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|}\zeta\left(\frac{\gamma}{\|v_1\|}\right)\zeta\left(\frac{\delta}{\|v_2\|}\right)\right) - \mathrm{Arg}\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}\overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)}\zeta\left(\frac{\delta}{\|v_2\|}\right)\right)\Bigg).
\end{aligned}$$

Set $\frac{\alpha}{\|v_1\|\|v_2\|} = re^{i\phi}$ for $0 < r < 1$, $\ \zeta\left(\frac{\gamma}{\|v_1\|}\right) = e^{i\theta_1}$, and $\ \zeta\left(\frac{\delta}{\|v_2\|}\right) = e^{i\theta_2}$. Then we get

$$\begin{aligned}
g(\delta,\gamma) \ &= \ \frac{1}{\pi}\Big(\mathrm{Arg}\left(1 - re^{i(-\phi+\theta_1+\theta_2)}\right) + \mathrm{Arg}\left(1 - re^{i(\phi-\theta_1+\theta_2)}\right) \\
&\qquad\qquad - \mathrm{Arg}\left(1 - re^{i(\phi+\theta_1+\theta_2)}\right) - \mathrm{Arg}\left(1 - re^{i(-\phi-\theta_1+\theta_2)}\right)\Big) \\
&= \ \frac{1}{\pi}\left(\arctan\left(\frac{-r\sin(-\phi + (\theta_1+\theta_2))}{1 - r\cos(-\phi + (\theta_1+\theta_2)}\right) + \arctan\left(\frac{-r\sin(\phi - (\theta_1 - \theta_2))}{1 - r\cos(\phi - (\theta_1 - \theta_2))}\right)\right.
\end{aligned}$$

$$- \arctan\left(\frac{-r\sin(\phi + (\theta_1 + \theta_2))}{1 - r\cos(\phi + (\theta_1 + \theta_2))}\right) - \arctan\left(\frac{-r\sin(-\phi - (\theta_1 - \theta_2))}{1 - rcos(-\phi - (\theta_1 - \theta_2))}\right)\Bigg)$$

$$= \frac{1}{\pi}\Bigg(\arctan\left(\frac{r\sin(\phi - (\theta_1 + \theta_2))}{1 - r\cos(\phi - (\theta_1 + \theta_2))}\right) + \arctan\left(\frac{r\sin(\phi + (\theta_1 + \theta_2))}{1 - r\cos(\phi + (\theta_1 + \theta_2))}\right)$$

$$- \arctan\left(\frac{r\sin(\phi - (\theta_1 - \theta_2))}{1 - r\cos(\phi - (\theta_1 - \theta_2))}\right) - \arctan\left(\frac{r\sin(\phi + (\theta_1 - \theta_2))}{1 - r\cos(\phi + (\theta_1 - \theta_2))}\right)\Bigg). \quad (2.17)$$

2.3 On the essential spectrum

As an application, we determine the essential spectrum of the operator T whose principal function we found in Section 2.2. We will use the following, which follows from Theorem 8.1 of [2].

Theorem 2.3.1 ([2])**.** *Suppose T is an operator on a Hilbert space $\mathcal{H}$ with self-commutator $T^*T - TT^*$ in trace class. For $\gamma + i\delta$ not in the essential spectrum of T,*

$$g(\delta, \gamma) = \operatorname{ind}(T - (\gamma + i\delta)),$$

where $g(\delta, \gamma)$ is the principal function for T.

Lemma 2.3.2. *Let $0 < r < 1$ and let*

$$h(r, \phi, \theta_1, \theta_2)$$
$$= \frac{1}{\pi}\Bigg(\arctan\left(\frac{r\sin(\phi - (\theta_1 + \theta_2))}{1 - r\cos(\phi - (\theta_1 + \theta_2))}\right) + \arctan\left(\frac{r\sin(\phi + (\theta_1 + \theta_2))}{1 - r\cos(\phi + (\theta_1 + \theta_2))}\right)$$
$$- \arctan\left(\frac{r\sin(\phi - (\theta_1 - \theta_2))}{1 - r\cos(\phi - (\theta_1 - \theta_2))}\right) - \arctan\left(\frac{r\sin(\phi + (\theta_1 - \theta_2))}{1 - r\cos(\phi + (\theta_1 - \theta_2))}\right)\Bigg).$$

(i) If $\phi = 0$ or $\phi = \pi$, then $h(r, \phi, \theta_1, \theta_2) = 0$.

(ii) If $0 < \phi < \pi$, then for all $\theta_1, \theta_2 \in [-\pi, 0]$, we have

$$-\frac{2}{\pi}\arctan\left(\frac{2r\sin\phi}{1 - r^2}\right) \le h(r, \phi, \theta_1, \theta_2) \le 0,$$

with equality holding on the left when $\theta_1 = \theta_2 = -\frac{\pi}{2}$ and equality holding on the right only when $\theta_1 \in \{-\pi, 0\}$ or $\theta_2 \in \{-\pi, 0\}$.

(iii) If $\pi < \phi < 2\pi$, then for all $\theta_1, \theta_2 \in [-\pi, 0]$, we have

$$0 \;\leq\; h(r,\phi,\theta_1,\theta_2) \;\leq\; -\frac{2}{\pi}\arctan\left(\frac{2r\sin\phi}{1-r^2}\right),$$

with equality holding on the right when $\theta_1 = \theta_2 = -\frac{\pi}{2}$ and equality holding on the left only when $\theta_1 \in \{-\pi, 0\}$ or $\theta_2 \in \{-\pi, 0\}$.

Proof. Part (i) is clear and we may assume $\phi \in (0,\pi) \cup (\pi, 2\pi)$.

Let $\nu = \theta_1 + \theta_2$ and $\mu = \theta_1 - \theta_2$. Then we are interested in the function

$$\begin{aligned}\tilde{h}(r,\phi,\nu,\mu) \;=\; \frac{1}{\pi}\Bigg(&\arctan\left(\frac{r\sin(\phi-\nu)}{1-r\cos(\phi-\nu)}\right) + \arctan\left(\frac{r\sin(\phi+\nu)}{1-r\cos(\phi+\nu)}\right)\\ &- \arctan\left(\frac{r\sin(\phi-\mu)}{1-r\cos(\phi-\mu)}\right) - \arctan\left(\frac{r\sin(\phi+\mu)}{1-r\cos(\phi+\mu)}\right)\Bigg),\end{aligned}$$

where

$$-2\pi \;\leq\; \nu \;\leq\; 0 \tag{2.18}$$

$$-\min(-\nu, 2\pi+\nu) \;\leq\; \mu \;\leq\; \min(-\nu, 2\pi+\nu). \tag{2.19}$$

In particular, we always have $|\mu| \leq \pi$. Note that the boundaries of the region described by (2.18) and (2.19) correspond to $\theta_1 \in \{-\pi, 0\}$ or $\theta_2 \in \{-\pi, 0\}$, where the function h vanishes.

An extreme point of $\tilde{h}$ not on the boundary can occur only where

$$\frac{\partial\tilde{h}}{d\nu} \;=\; \frac{\partial\tilde{h}}{d\mu} \;=\; 0.$$

We compute

$$\frac{d}{dx}\arctan\left(\frac{r\sin(x)}{1-r\cos(x)}\right) \;=\; \frac{r(\cos(x)-r)}{1-2r\cos(x)+r^2}.$$

We also compute

$$\frac{d}{dc}\left(\frac{c-r}{1-2rc+r^2}\right) \;=\; \frac{1-r^2}{(1-2rc+r^2)^2} \;>\; 0,$$

so the function

$$c \;\mapsto\; \frac{r(c-r)}{1-2rc+r^2}$$

is strictly increasing on $[-1,1]$. Therefore,

$$\begin{aligned}\frac{\partial\tilde{h}}{d\nu} \;=\; \frac{d}{d\nu}\left(\arctan\left(\frac{r\sin(\phi-\nu)}{1-r\cos(\phi-\nu)}\right)+\arctan\left(\frac{r\sin(\phi+\nu)}{1-r\cos(\phi+\nu)}\right)\right)\\ =\; \frac{-r(\cos(\phi-\nu)-r)}{1-2r\cos(\phi-\nu)+r^2}+\frac{r(\cos(\phi+\nu)-r)}{1-2r\cos(\phi+\nu)+r^2}\end{aligned}$$

vanishes if and only if $\cos(\phi-\nu)=\cos(\phi+\nu)$, which in turn occurs if and only if either $\nu\in\pi\mathbb{Z}$ or $\phi\in\pi\mathbb{Z}$. We assumed $\phi\notin\pi\mathbb{Z}$. If $\nu\in\{-2\pi,0\}$, then ν is on the boundary of the interval (2.18), so the only possibility that is not on the boundary of the region is $\nu=-\pi$.

Arguing as above, $\frac{\partial\tilde{h}}{d\mu}=0$ if and only if $\cos(\phi-\mu)=\cos(\phi+\mu)$. Avoiding the boundary, this leaves only $\mu=0$. We conclude that the only extreme point of $\tilde{h}$ not on the boundary occurs at $(\nu,\mu)=(-\pi,0)$, i.e., at $(\theta_1,\theta_2)=(-\frac{\pi}{2},-\frac{\pi}{2})$, and the value of $\tilde{h}$ there is

$$-\,\frac{2}{\pi}\;\left(\arctan\left(\frac{r\sin\phi}{1-r\cos\phi}\right)+\arctan\left(\frac{r\sin\phi}{1+r\cos\phi}\right)\right). \tag{2.20}$$

We have the identity, for $\alpha,\beta\in\mathbb{R}$,

$$\arctan(\alpha)+\arctan(\beta)\;\in\;\arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)+\pi\mathbb{Z}.$$

Letting

$$\alpha=\frac{r\sin\phi}{1-r\cos\phi}\quad\text{ and }\quad\beta=\frac{r\sin\phi}{1+r\cos\phi},$$

since

$$0\;<\;\alpha\beta\;=\;\frac{r^2\sin^2\phi}{1-r^2+r^2\sin^2\phi}\;<\;1,$$

we find that the quantity (2.20) equals

$$-\frac{2}{\pi}\arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right) = -\frac{2}{\pi}\arctan\left(\frac{2r\sin\phi}{1-r^2}\right). \tag{2.21}$$

We already observed that on the boundaries of the region described by (2.18)–(2.19), the function $\tilde{h}$ vanishes and we just showed that the only extreme value not on the boundary is (2.21), which is attained when $\theta_1 = \theta_2 = -\frac{\pi}{2}$. In particular, $\tilde{h}$ is never vanishing on the interior of the region. This completes the proof of (ii) and (iii). $\square$

Theorem 2.3.3. *Let $T = l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*)$ with v_1 and v_2 linearly independent and $\mathrm{Im}\langle v_1, v_2\rangle \neq 0$. Then the essential spectrum $\sigma_e(T)$ of T is the closed rectangle*

$$\{\gamma + i\delta \in \mathbb{C} \mid |\gamma| \leq 2\|v_1\| \textit{ and } |\delta| \leq 2\|v_2\|\}, \tag{2.22}$$

which equals the spectrum $\sigma(T)$ of T.

Proof. By Lemma 2.2.3, we have that $\sigma(T)$ is contained in the rectangle (2.22). For $\gamma \in \sigma(X_1)$ and $\delta \in \sigma(X_2)$, we have the formula of the principal function $g(\delta, \gamma)$ in (2.17). By Lemma 2.3.2, $-1 < g(\delta,\gamma) \leq 0$ if $\mathrm{Im}\langle v_1, v_2\rangle > 0$, and $0 \leq g(\delta,\gamma) < 1$ if $\mathrm{Im}\langle v_1, v_2\rangle < 0$. The equality $g(\delta,\gamma) = 0$ holds only when $\gamma \in \{2\|v_1\|, -2\|v_1\|\}$ or $\delta \in \{2\|v_2\|, -2\|v_2\|\}$, i.e., when γ and δ are on the boundary of the rectangle (2.22). So the function $g(\delta,\gamma)$ does not assume any integer value on the interior of the rectangle. But, by Theorem 2.3.1, if $\gamma + i\delta \notin \sigma_e(T)$, then $g(\delta,\gamma) = \mathrm{ind}(T - (\gamma + i\delta))$. So the whole interior of the rectangle is included in the essential spectrum of T. Since $\sigma_e(T)$ is closed in $\mathbb{C}$ and is contained in $\sigma(T)$, we have $\sigma_e(T)$ equals the rectangle (2.22). $\square$

Proposition 2.3.4 ([8])**.** *Suppose that T has a compact self-commutator $T^*T - TT^*$ on a Hilbert space $\mathcal{H}$ and $\mathrm{ind}(T-\lambda) = 0$ for all $\lambda \in \mathbb{C} \setminus \sigma_e(T)$. Then T is of the form $N + K$ where N is normal and K is compact.*

Corollary 2.3.5. *The operator $T = l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*)$ with linearly independent*

v_1 *and* v_2 *and* $\mathrm{Im}\langle v_1, v_2\rangle \neq 0$ *is normal plus compact.*

Example 2.3.6. Let v_2 and u be orthogonal vectors in a Hilbert space $\mathcal{H}$ with $\|v_2\| = \|u\| = 1$, and let $\alpha = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \in \mathbb{C}$. Set v_1 in $\mathcal{H}$ by $v_1 = \alpha v_2 + u$. Suppose that T is a bounded operator on the full Fock space $\mathcal{F}(\mathcal{H})$ defined by $T = l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*)$. Then, $[T^*, T] = -2\sqrt{2}P$ and it is an one-dimensional projection on $\mathcal{F}(\mathcal{H})$. So, by restricting T^* to its pure part $\overline{\mathrm{alg}(T, T^*, 1)\Omega}$, T^* is a completely non-normal hyponormal operator.

We can find the principal function $g(\delta, \gamma)$ of T by the formula (2.17). For each pair (δ, γ) such that $|\delta| \leq 2$ and $|\gamma| \leq 2\sqrt{2}$, we have

$$\begin{aligned} g(\delta,\gamma) \;=\; & \frac{1}{\pi}\left(\mathrm{Arg}\left(1-\left(\frac{1}{2}+\frac{1}{2}i\right)\zeta\left(\frac{\gamma}{\sqrt{2}}\right)\zeta(\delta)\right)+\mathrm{Arg}\left(1-\left(\frac{1}{2}-\frac{1}{2}i\right)\overline{\zeta\left(\frac{\gamma}{\sqrt{2}}\right)}\zeta(\delta)\right)\right. \\ & \left. -\,\mathrm{Arg}\left(1-\left(\frac{1}{2}-\frac{1}{2}i\right)\zeta\left(\frac{\gamma}{\sqrt{2}}\right)\zeta(\delta)\right)-\mathrm{Arg}\left(1-\left(\frac{1}{2}+\frac{1}{2}i\right)\overline{\zeta\left(\frac{\gamma}{\sqrt{2}}\right)}\zeta(\delta))\right)\right), \end{aligned}$$

where $\zeta(t) = \frac{t - i\sqrt{4-t^2}}{2}$ for $t \in [-2, 2]$. Since $\mathrm{Im}\langle v_1, v_2\rangle < 0$, we have $0 \leq g(\delta, \gamma) < 1$ for all $(\delta, \gamma) \in \mathbb{R}^2$. By Lemma 2.3.2, $g(\delta, \gamma)$ is vanishing only when (δ, γ) is on the boundary of the rectangle $\{(\delta, \gamma) \in \mathbb{R}^2 \mid |\gamma| \leq 2\sqrt{2} \text{ and } |\delta| \leq 2\}$. Therefore, $\sigma(T) = \sigma_e(T) = \{\gamma + i\delta \in \mathbb{C} \mid |\gamma| \leq 2\sqrt{2} \text{ and } |\delta| \leq 2\}$. See Figure 2.1.

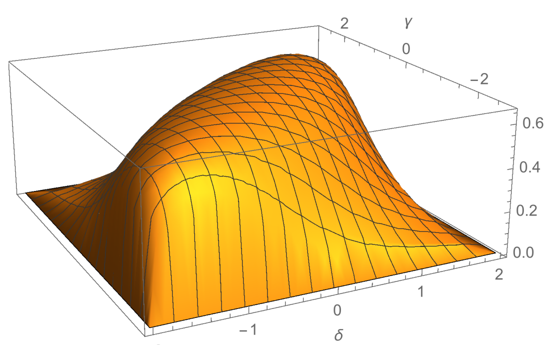

Figure 2.1: The principal function $g(\delta, \gamma)$ of T where $v_1 = \alpha v_2 + u$, $\alpha = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, $u \perp v_1$ and $\|v_2\| = \|u\| = 1$.

3.1 Multiplicative convolution of bi-free two-faced families

In this section, we will discuss bi-free cumulants for certain combinations for random variables. Let (a_1, b_1) and (a_2, b_2) be bi-free two-faced pairs in a non-commutative probability space $(\mathcal{A}, \phi)$ with $\phi(a_i) = \phi(b_i) = 1$ for $i = 1, 2$. Furthermore, let f_i (respectively g_i) denote the multiplicative functions associated to the cumulants of a_i (respectively b_i) defined by $f_i(0_n, 1_n) = \kappa_n(a_i)$ (respectively $g_i(0_n, 1_n) = \kappa_n(b_i)$), where 0_n and 1_n are the minimal and maximal elements in $NC(n)$ and $i = 1, 2$.

Lemma 3.1.1. *Under the above assumptions, as formal power series,*

(i) $$\sum_{n,m\geq 1} \kappa_{n,m}(\underbrace{a_1a_2, \ldots, a_1a_2}_{n}, b_2, \underbrace{b_1b_2, \ldots, b_1b_2}_{m-1})z^n w^{m-1} = \frac{K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z), \phi_{g_2\check{*}g_1}(w))}{\phi_{g_2\check{*}g_1}(w)}$$

(ii) $$\sum_{n,m\geq 1} \kappa_{n,m}(a_2, \underbrace{a_1a_2, \ldots, a_1a_2}_{n-1}, \underbrace{b_1b_2, \ldots, b_1b_2}_{m})z^{n-1} w^{m} = \frac{K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z), \phi_{g_2\check{*}g_1}(w))}{\phi_{f_2\check{*}f_1}(z)}$$

(iii) $$\sum_{n,m\geq 1} \kappa_{n,m}(a_2, \underbrace{a_1a_2, \ldots, a_1a_2}_{n-1}, b_2, \underbrace{b_1b_2, \ldots, b_1b_2}_{m-1})z^{n-1} w^{m-1}$$
$$= \frac{K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z), \phi_{g_2\check{*}g_1}(w))}{\phi_{f_2\check{*}f_1}(z)\phi_{g_2\check{*}g_1}(w)}$$

Proof. We will prove the equality (i). Let $n, m \geq 1$. By Theorem 1.2.4,

$$\kappa_{n,m}(a_1a_2, \ldots, a_1a_2, b_2, b_1b_2, \ldots, b_1b_2)$$
$$= \sum_{\substack{\pi\in\mathrm{BNC}(2n,2m)\\ \pi\vee\sigma_{n,m}=1_{2n,2m}}} \kappa_\pi(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, 1_{\mathcal{A}}, \underbrace{b_2, b_1, b_2, \ldots, b_1, b_2}_{b_2 \text{ occurs } m \text{ times}})$$

where $\sigma_{n,m} := \{\{(2k-1)_l, (2k)_l\}\}_{k=1}^n \cup \{\{(2k-1)_r, (2k)_r\}\}_{k=1}^m \in \mathrm{BNC}(2n, 2m)$.
Let $\pi \in \mathrm{BNC}(2n, 2m)$ and $\pi\vee\sigma_{n,m} = 1_{2n,2m}$. Since (a_1, b_1) and (a_2, b_2) are bi-freely independent,

the cumulant $\kappa_\pi(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, 1_{\mathcal{A}}, \underbrace{b_2, b_1, b_2, \ldots, b_1, b_2}_{b_2 \text{ occurs } m \text{ times}})$ vanishes when the partition π contains either a non-singleton block including $1_{\mathcal{A}}$ or a block with both $(2k)_i$ and $(2k'-1)_j$ for some $i, j \in \{l, r\}$ and k, k'. So we will only consider a partition π which does not include such blocks.

Since $\pi \vee \sigma_{n,m} = 1_{2n,2m}$, the partition π must contain at least one block V which has both left and right indices. Let V_π be the block V_π such that $\min(V_\pi)$ is smallest among all blocks containing both left and right indices, where $\min(V_\pi)$ denotes the minimum index of the block V_π; that is, if $V_\pi := \{(u_1)_l, \ldots, (u_s)_l \mid u_1 < \cdots < u_s\} \cup \{(v_1)_r, \ldots, (v_t)_r \mid v_1 < \cdots < v_t\}$ for $s, t \geq 1$, then $\min(V_\pi) = \min\{u_1, v_1\}$.

Assume that v_1 is odd. Since the block V_π is not a singleton, we cannot have $v_1 = 1$, so $v_1 \geq 3$. But, by choice of V_π, a set $\{1_r, 2_r, \ldots, (v_1-1)_r\}$ splits the partition π, and it also splits the partition $\pi \vee \sigma_{n,m}$ contrary to hypothesis. Therefore v_1 must be even. We claim that $v_1 = 2$. Indeed, suppose $v_1 \geq 4$ and denote the smallest right index in the block of π containing $(v_1 - 1)_r$ by x_r. Since π connects only evens to evens and odds to odds, the set $\{1_r, 2_r, \ldots, x_r - 1\}$ splits π, and hence it also splits $\pi \vee \sigma_{n,m}$, which is contrary to hypothesis. Note that both left and right indices of V_π are even numbers. Moreover, we can easily see that $u_s = 2n$ and $v_t = 2m$. For, otherwise, the set $\{(u_s+1)_l, \ldots, (2n)_l\} \cup \{(v_t+1)_r, \ldots, (2m)_r\}$ splits $\pi \vee \sigma_{n,m}$, which is contrary to hypothesis. Therefore, we have

$$V_\pi \;=\; \{(2k_i)_l\}_{i=1}^s \cup \{(2k'_j)_r\}_{j=1}^t \text{ with } 1 \leq k_1 < \cdots < k_s = n \;\text{ and }\; 1 = k'_1 < \cdots < k'_t = m.$$

Let $k_0 = k'_0 = 0$ and set

$$s_i \;=\; k_{i+1} - k_i \quad\text{ and }\quad t_j \;=\; k'_{j+1} - k'_j$$

for $0 \leq i \leq s-1$ and $0 \leq j \leq t-1$. Define sub-partitions $\pi_{l,i}$ (respectively, $\pi_{r,j}$) of π corresponding to the set $\{(2l_i+1)_l, \ldots, (2l_{i+1}-1)_l\}$ (respectively, $\{(2r_j+1)_r, \ldots, (2r_{j+1}-1)_r\}$); that is,

$$\pi_{l,i} \;=\; \pi|_{\{(2k_i+1)_l, \ldots, (2k_{i+1}-1)_l\}} \quad\text{ and }\quad \pi_{r,j} \;=\; \pi|_{\{(2k'_j+1)_r, \ldots, (2k'_{j+1}-1)_r\}}$$

where $0 \le i \le s-1$ and $0 \le j \le t-1$. Note that $\pi_{l,0} = \{\{1_l\}\}$ and $\pi_{r,0} = \{\{1_r\}\}$. Then $\pi = \{V_\pi\} \cup \left(\cup_{i=0}^{s-1} \pi_{l,i}\right) \cup \left(\cup_{j=0}^{t-1} \pi_{r,j}\right)$. Define new partitions $\pi'_{l,i}$ (respectively, $\pi'_{r,j}$) by adding a singleton block $\{(2k_{i+1})_l\}$ (respectively $\{(2k'_{j+1})_r\}$) on a partition $\pi_{l,i}$ (respectively $\pi_{r,j}$), i.e.,

$$\pi'_{l,i} = \pi_{l,i} \cup \{\{(2k_{i+1})_l\}\} \quad \text{and} \quad \pi'_{r,j} = \pi_{r,j} \cup \{\{(2k'_{j+1})_r\}\}.$$

Consider their restrictions to even and odd indices, denoted by

$$\begin{aligned}
\pi'_{l,i}|_E &:= \pi'_{l,i}|_{\{(2l_i+2)_l,(2l_i+4)_l,\ldots,(2l_{i+1})_l\}} \in NC'(s_i) \\
\pi'_{l,i}|_O &:= \pi'_{l,i}|_{\{(2l_i+1)_l,(2l_i+3)_l,\ldots,(2l_{i+1}-1)_l\}} \in NC(s_i) \\
\pi'_{r,j}|_E &:= \pi'_{r,j}|_{\{(2r_j+2)_r,(2r_j+4)_r,\ldots,(2r_{j+1})_r\}} \in NC'(t_j) \\
\pi'_{r,j}|_O &:= \pi'_{r,j}|_{\{(2r_j+1)_r,(2r_j+3)_r,\ldots,(2r_{j+1}-1)_r\}} \in NC(t_i)
\end{aligned}$$

where $0 \le i \le s-1$ and $0 \le j \le t-1$. Since $\pi \vee \sigma_{n,m} = 1_{2n,2m}$, it follows that the sub-partition $\pi'_{l,i}|_O$ (respectively $\pi'_{r,j}|_O$) is the Kreweras complement of $\pi'_{l,i}|_E$ (respectively $\pi'_{r,j}|_E$); that is,

$$\pi'_{l,i}|_O = K(\pi'_{l,i}|_E) \quad \text{and} \quad \pi'_{r,j}|_O = K(\pi'_{r,j}|_E)$$

where $K(\cdot)$ is the Kreweras complement. For $\pi \in \mathrm{BNC}(2n, 2m)$ satisfying $\pi \vee \sigma_{n,m} = 1_{2n,2m}$, we have

$$\begin{aligned}
&\kappa_\pi(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, 1_{\mathcal{A}}, \underbrace{b_2, b_1, b_2, \ldots, b_1, b_2}_{b_2 \text{ occurs } m \text{ times}}) \\
&= \kappa_{s,t}(a_2, b_2) \cdot \prod_{i=0}^{s-1} \kappa_{\pi_{l,i}}(a_1, a_2, a_1, \ldots, a_2, a_1) \cdot \kappa_{\pi_{r,0}}(1_{\mathcal{A}}) \cdot \prod_{j=1}^{t-1} \kappa_{\pi_{r,j}}(b_1, b_2, b_1, \ldots, b_2, b_1) \\
&= \kappa_{s,t}(a_2, b_2) \cdot \prod_{i=0}^{s-1} \kappa_{\pi'_{l,i}}(a_1, a_2, a_1, \ldots, a_2, a_1, a_2) \cdot \prod_{j=1}^{t-1} \kappa_{\pi'_{r,j}}(b_1, b_2, b_1, \ldots, b_2, b_1, b_2)
\end{aligned}$$

$$= \kappa_{s,t}(a_2,b_2)\cdot \prod_{i=0}^{s-1}\Big(\kappa_{\pi'_{l,i}|_E}(a_2)\cdot\kappa_{\pi'_{l,i}|_O}(a_1)\Big)\cdot\prod_{j=1}^{t-1}\Big(\kappa_{\pi'_{r,j}|_E}(b_2)\cdot\kappa_{\pi'_{r,j}|_O}(b_1)\Big)$$

$$= \kappa_{s,t}(a_2,b_2)\cdot \prod_{i=0}^{s-1}\big(f_2(0_{s_i},\pi'_{l,i}|_E)\cdot f_1(0_{s_i},K(\pi'_{l,i}|_E))\big)\cdot\prod_{j=1}^{t-1}\big(g_2(0_{t_j},\pi'_{r,j}|_E)\cdot g_1(0_{t_j},K(\pi'_{r,j}|_E))\big)$$

where $\pi'_{r,j}|_E \in NC'(s_i)$ and $\pi'_{r,j}|_E \in NC'(t_j)$. Recall that $f_1, g_1, f_2,$ and g_2 are the multiplicative functions associated to the cumulants of $a_1, b_1, a_2,$ and b_2, respectively. The sum of $\kappa_\rho(a_1, a_2, a_1, \ldots, a_2, 1_{\mathcal{A}}, b_2, b_1, \ldots, b_2)z^n w^{m-1}$ over all partitions $\rho \in \mathrm{BNC}(2n, 2m)$ satisfying $V_\rho = V_\pi$ and $\rho \vee \sigma_{n,m} = 1_{2n,2m}$ is equal to

$$\kappa_{s,t}(a_2,b_2)\ \cdot\ \prod_{i=0}^{s-1}(f_2\breve{*}f_1)(0_{s_i},1_{s_i})z^{s_i}\ \cdot\ \prod_{j=1}^{t-1}(g_2\breve{*}g_1)(0_{t_j},1_{t_j})w^{t_j}$$

where $s_i \geq 1$ and $t_j \geq 1$. By summing this over all possible V_π and using (1.1), we obtain

$$\sum_{s,t\geq 1}\kappa_{s,t}(a_2,b_2)\ \cdot\ \prod_{i=0}^{s-1}\Bigg(\sum_{s_i\geq 1}(f_2\breve{*}f_1)(0_{s_i},1_{s_i})z^{s_i}\Bigg)\ \cdot\ \prod_{j=1}^{t-1}\Bigg(\sum_{t_j\geq 1}(g_2\breve{*}g_1)(0_{t_j},1_{t_j})w^{t_j}\Bigg)$$
$$= \sum_{s,t\geq 1}\kappa_{s,t}(a_2,b_2)\,(\phi_{f_2\breve{*}f_1}(z))^s\,(\phi_{g_2\breve{*}g_1}(w))^{t-1},$$

which proves the equality (i). Similar proofs can be done for equalities (ii) and (iii). □

Lemma 3.1.2. *Let $n, m \geq 1$. Under the above assumptions,*

(i) $\kappa_{n,m}(\underbrace{a_2a_1,\ldots,a_2a_1}_{n},\underbrace{b_2b_1,\ldots,b_2b_1}_{m-1},b_2) = \kappa_{n,m}(\underbrace{a_1a_2,\ldots,a_1a_2}_{n},b_2,\underbrace{b_1b_2,\ldots,b_1b_2}_{m-1})$

(ii) $\kappa_{n,m}(\underbrace{a_2a_1,\ldots,a_2a_1}_{n-1},a_2,\underbrace{b_2b_1,\ldots,b_2b_1}_{m}) = \kappa_{n,m}(a_2,\underbrace{a_1a_2,\ldots,a_1a_2}_{n-1},\underbrace{b_1b_2,\ldots,b_1b_2}_{m})$

Proof. By symmetry, we only prove the equality (i). Theorem 1.2.4 implies

$$\kappa_{n,m}(\underbrace{a_2a_1,\ldots,a_2a_1}_{n},\underbrace{b_2b_1,\ldots,b_2b_1}_{m-1},b_2)$$
$$= \sum_{\substack{\pi\in\mathrm{BNC}(2n,2m)\\ \pi\vee\sigma_{n,m}=1_{2n,2m}}}\kappa_\pi(\underbrace{a_2,a_1,\ldots,a_2,a_1}_{a_2 \text{ occurs } n \text{ times}},\underbrace{b_2,b_1\ldots,b_2,b_1,b_2}_{b_2 \text{ occurs } m \text{ times}},1_{\mathcal{A}})$$

where $\sigma_{n,m} = \{\{(2k-1)_l, (2k)_l\}\}_{k=1}^{n} \cup \{\{(2k-1)_r, (2k)_r\}\}_{k=1}^{m} \in \mathrm{BNC}(2n, 2m)$. By flipping the order of indices for left and right variables, we can easily see that there uniquely exists $\rho \in \mathrm{BNC}(2n, 2m)$ such that $\rho \vee \sigma_{n,m} = 1_{2n,2m}$ and

$$\kappa_\pi(\underbrace{a_2, a_1, \ldots, a_2, a_1}_{a_2 \text{ occurs } n \text{ times}}, \underbrace{b_2, b_1 \ldots, b_2, b_1, b_2}_{b_2 \text{ occurs } m \text{ times}}, 1_{\mathcal{A}}) = \kappa_\rho(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, 1_{\mathcal{A}}, \underbrace{b_2, b_1, \ldots, b_2, b_1, b_2}_{b_2 \text{ occurs } m \text{ times}}).$$

With the corresponding partition ρ, we have

$$\begin{aligned}
&\kappa_{n,m}(\underbrace{a_2a_1, \ldots, a_2a_1}_{n}, \underbrace{b_2b_1, \ldots, b_2b_1}_{m-1}, b_2) \\
&= \sum_{\substack{\rho \in \mathrm{BNC}(2n,2m) \\ \rho \vee \sigma_{n,m} = 1_{2n,2m}}} \kappa_\rho(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, 1_{\mathcal{A}}, \underbrace{b_2, b_1, \ldots, b_2, b_1, b_2}_{b_2 \text{ occurs } m \text{ times}}) \\
&= \kappa_{n,m}(\underbrace{a_1a_2, \ldots, a_1a_2}_{n}, b_2, \underbrace{b_1b_2, \ldots, b_1b_2}_{m-1})
\end{aligned}$$

for each $n, m \geq 1$. $\square$

The symmetries among cumulants shown in Lemma 3.1.2 give the following corollary.

Corollary 3.1.3. *Under the above assumptions and notation, as formal power series,*

(i) $\displaystyle\sum_{n,m \geq 1} \kappa_{n,m}(\underbrace{a_2a_1, \ldots, a_2a_1}_{n}, \underbrace{b_2b_1, \ldots, b_2b_1}_{m-1}, b_2) z^n w^{m-1} = \frac{K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w))}{\phi_{g_2 \check{*} g_1}(w)}$

(ii) $\displaystyle\sum_{n,m \geq 1} \kappa_{n,m}(\underbrace{a_2a_1, \ldots, a_2a_1}_{n-1}, a_2, \underbrace{b_2b_1, \ldots, b_2b_1}_{m}) z^{n-1} w^m = \frac{K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w))}{\phi_{f_2 \check{*} f_1}(z)}.$

In Lemma 3.1.1, we found the formulas for ordered joint cumulant series for the combinations $(a_1a_2, \ldots, a_1a_2, b_2, b_1b_2, \ldots, b_1b_2)$ and $(a_2, a_1a_2, \ldots, a_1a_2, b_1b_2, \ldots, b_1b_2)$. The following lemma shows a relationship between bi-free cumulant and moment series.

Lemma 3.1.4. *Let a_1, b_1 be random variables in $(\mathcal{A}, \phi)$. Then,*

$$\sum_{n=0}^{\infty} \kappa_{n,1}(a_1, b_1) z^n \;=\; \frac{1}{C_{a_1}(z)} \cdot \sum_{n=0}^{\infty} \phi(a_1^n b_1) \left(\frac{z}{C_{a_1}(z)} \right)^n \tag{3.1}$$

$$\sum_{m=0}^{\infty} \kappa_{1,m}(a_1, b_1) w^m \;=\; \frac{1}{C_{b_1}(w)} \cdot \sum_{m=0}^{\infty} \phi(a_1 b_1^m) \left(\frac{w}{C_{b_1}(w)} \right)^m \tag{3.2}$$

Proof. Let $a_1, b_1 \in \mathcal{A}$. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi(a_1^n b_1) z^n \;&=\; \sum_{n=0}^{\infty} \sum_{\pi \in BNC(n,1)} \kappa_\pi(a_1, \ldots, a_1, b_1) z^n \\
&=\; \sum_{n=0}^{\infty} \sum_{r=1}^{n+1} \kappa_{r-1,1}(a_1, b_1) z^{r-1} \cdot \left(\sum_{\substack{p_1, \ldots, p_r \geq 0 \\ p_1 + \cdots + p_r = n+1-r}} \prod_{i=1}^{r} \Bigg(\sum_{\pi_i \in NC(p_i)} \kappa_{\pi_i}(a_1) z^{p_i} \Bigg) \right) \\
&=\; \sum_{r=1}^{\infty} \kappa_{r-1,1}(a_1, b_1) z^{r-1} \cdot \prod_{i=1}^{r} \left(\sum_{p=0}^{\infty} \phi(a_1^p) z^p \right) \\
&=\; \sum_{r=1}^{\infty} \kappa_{r-1,1}(a_1, b_1) z^{r-1} \cdot M_{a_1}(z)^r \\
&=\; M_{a_1}(z) \cdot \sum_{r=0}^{\infty} \kappa_{r,1}(a_1, b_1) \left(z M_{a_1}(z) \right)^r ,
\end{aligned}$$

which implies

$$\sum_{r=0}^{\infty} \kappa_{r,1}(a_1, b_1) \left(z M_{a_1}(z) \right)^r = \frac{1}{M_{a_1}(z)} \cdot \sum_{n=0}^{\infty} \phi(a_1^n b_1) z^n .$$

By substituting $\frac{z}{C_{a_1}(z)}$ for z in this equation and using the equality (1.6), we obtain

$$\sum_{r=0}^{\infty} \kappa_{r,1}(a_1, b_1) z^r \;=\; \frac{1}{C_{a_1}(z)} \cdot \sum_{n=0}^{\infty} \phi(a_1^n b_1) \left(\frac{z}{C_{a_1}(z)} \right)^n .$$

By symmetry, we can easily prove the second equality. $\square$

3.2 Moment series of certain pairs

Let $(\mathcal{A}, \phi)$ be a C*-non-commutative probability space. Let (a, b) and (p, q) be bi-free two-faced pairs in $(\mathcal{A}, \phi)$ such that p and q are non-trivial self-adjoint projections, i.e., $p^2 = p = p^*$ and $q^2 = q = q^*$, and let $\phi(a), \phi(b), \phi(p), \phi(q) \neq 0$. Note that by freeness of left algebras and right algebras in bi-free pairs, the random variables a and p are freely independent, and so are b and q. Moreover, $\phi(ap) = \phi(a)\phi(p)$ and $\phi(bq) = \phi(b)\phi(q)$.

For notational convenience, we define

$$k \;=\; 1 - \frac{\phi(ab)}{\phi(a)\phi(b)} \qquad \text{and} \qquad l \;=\; 1 - \frac{\phi(pq)}{\phi(p)\phi(q)} \tag{3.3}$$

where $k \leq 1$ and $l \leq 1$.

Theorem 3.2.1. *Under the above notation and assumptions,*

$$\begin{aligned} M_{pap,qbq}(\chi_{ap}(z), \chi_{bq}(w)) \;=\; 1 - \phi(pq) + \left(1 - \frac{\phi(pq)}{\phi(p)}\right) z + \left(1 - \frac{\phi(pq)}{\phi(q)}\right) w \\ + (1-l)\; M_{ap,bq}(\chi_{ap}(z), \chi_{bq}(w)) \cdot \frac{(z+\phi(p))(w+\phi(q))}{(z+1)(w+1) \;-\; l \cdot zw} \end{aligned}$$

for $(z, w) \in (\mathbb{C} \setminus \{0\})^2$ *near* $(0, 0)$.

Under the above notation and assumptions, we denote the normalized random variables of a and p by $\bar{a}$ and $\bar{p}$, that is, $\bar{a} = \frac{a}{\phi(a)}$ and $\bar{p} = \frac{p}{\phi(p)}$. Let f_1 and f_2 be the multiplicative functions with respect to the cumulants of $\bar{a}$ and $\bar{p}$, repectively. Then we can obtain

$$\phi_{f_2 \check{*} f_1}(\phi(ap) z M_{ap}(z)) \;=\; \frac{\phi(p)(M_{ap}(z) - 1)M_{ap}(z)}{M_{ap}(z) - 1 + \phi(p)}. \tag{3.4}$$

Indeed, by definition, $M_{ap}(z) = M_{\bar{a}\bar{p}}(\phi(ap)z)$ and

$$\begin{aligned}
\phi_{f_2} \circ \phi_{f_2 \check{*} f_1}(\phi(ap)z \cdot M_{ap}(z)) &= \phi_{f_2 * f_1}(\phi(ap)z \cdot M_{\bar{a}\bar{p}}(\phi(ap)z)) \\
&= C_{\bar{a}\bar{p}}(\phi(ap)z \cdot M_{\bar{a}\bar{p}}(\phi(ap)z)) - 1 \\
&= M_{\bar{a}\bar{p}}(\phi(ap)z) - 1 \\
&= M_{ap}(z) - 1, \qquad (3.5)
\end{aligned}$$

where for the first and third equalities, we have used (1.3) and (1.5), respectively. From (3.5), we get $\phi_{f_2 \check{*} f_1}(\phi(ap)zM_{ap}(z)) = \phi_{f_2}^{\langle -1 \rangle}(M_{ap}(z) - 1)$. Note that $\phi_{f_2}(z) = C_{\bar{p}}(z) - 1 = C_p\left(\frac{z}{\phi(p)}\right) - 1$ by definition. Since p is a projection, we can easily show $(C_p(z) - 1)^{\langle -1 \rangle} = \frac{z(z+1)}{z+\phi(p)}$, and it follows that $\phi_{f_2}^{\langle -1 \rangle}(z) = \frac{\phi(p)z(z+1)}{z+\phi(p)}$, which implies (3.4). Furthermore, from the equality (3.5), it is trivial that

$$C_{\bar{p}}\left(\phi_{f_2 \check{*} f_1}(\phi(ap)zM_{ap}(z))\right) = M_{ap}(z). \qquad (3.6)$$

Proposition 3.2.2. *Under the above assumptions given in this section,*

(i) $\sum_{n=1}^{\infty} \phi((pap)^n q)z^n = \sum_{n=1}^{\infty} \phi((pa)^n q)z^n = \sum_{n=1}^{\infty} \phi((ap)^n q)z^n = \frac{\phi(pq)}{\phi(p)} \cdot (M_{ap}(z) - 1)$

(ii) $\sum_{m=1}^{\infty} \phi(p(qbq)^m)w^m = \sum_{m=1}^{\infty} \phi(p(qb)^m)w^m = \sum_{m=1}^{\infty} \phi(p(bq)^m)w^m = \frac{\phi(pq)}{\phi(q)} \cdot (M_{bq}(w) - 1)$

Proof. Let $\bar{a},\ \bar{b},\ \bar{p}$, and $\bar{q}$ be the normalized elements of $a,\ b,\ p$, and q in $(\mathcal{A}, \phi)$. By using Lemma 3.1.3 and 3.1.4, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi((pa)^n q)\ z^n &= M_{pa}(z) \cdot \sum_{n=0}^{\infty} \kappa_{n,1}(pa, q)(zM_{pa}(z))^n \\
&= M_{pa}(z)\ \phi(q) \cdot \sum_{n=0}^{\infty} \kappa_{n,1}(\bar{p}\bar{a}, \bar{q})(\phi(pa)zM_{pa}(z))^n \\
&= M_{pa}(z)\ \phi(q) \cdot \sum_{n=0}^{\infty} \kappa_{n,1}(\bar{p}, \bar{q})\left(\phi_{f_2 \check{*} f_1}(\phi(pa)zM_{pa}(z))\right)^n
\end{aligned}$$

$$= \frac{M_{pa}(z)\ \phi(q)}{C_{\bar{p}}(\phi_{f_2 \check{*} f_1}(\phi(pa)zM_{pa}(z)))} \cdot \sum_{n=0}^{\infty} \phi(\bar{p}^n \bar{q}) \left(\frac{\phi_{f_2 \check{*} f_1}(\phi(pa)zM_{pa}(z))}{C_{\bar{p}}(\phi_{f_2 \check{*} f_1}(\phi(pa)zM_{pa}(z)))} \right)^n . \tag{3.7}$$

By using equalities (3.4) and (3.6), the last equation in (3.7) is equal to

$$\phi(q) \ + \ \frac{\phi(pq)}{\phi(q)} \cdot (M_{ap}(z) - 1),$$

and then we obtain $\sum_{n=1}^{\infty} \ \phi((pa)^n q)\ z^n = \frac{\phi(pq)}{\phi(q)} \cdot (M_{ap}(z) - 1)$. Note that $M_{ap}(z) = M_{pa}(z)$ by the freeness of a and p. Since $\sum_{n=1}^{\infty} \phi((ap)^n q) z^n = M_{ap}(z) \cdot \sum_{n=0}^{\infty} \kappa_{n,1}(ap, q)(zM_{ap}(z))^n$ and $\kappa_{n,1}(ap, q) = \kappa_{n,1}(pa, q)$ for $n \geq 0$, it follows that $\sum_{n=1}^{\infty} \phi((pa)^n q) z^n = \sum_{n=1}^{\infty} \phi((ap)^n q) z^n = \frac{\phi(pq)}{\phi(p)} \cdot (M_{ap}(z) - 1)$. Its equality to $\sum_{n=1}^{\infty} \phi((pap)^n q) z^n$ can be proved by using the techniques in the proof of Theorem 3.2.1 to derive the formula for $M_{pap,qbq}(\chi_{ap}(z), \chi_{bq}(w))$. □

Now we will prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Let $n, m \geq 1$. By definition of (l, r)-cumulants, we have

$$\begin{aligned}
\phi((pap)^n (qbq)^m) \ &= \ \phi\left((1p \cdot ap \cdots ap) \cdot (1q \cdot bq \cdots bq)\right) \\
&= \sum_{\pi \in \mathrm{BNC}(n+1,m+1)} \kappa_\pi(\underbrace{1p, ap, \ldots, ap}_{n+1}, \underbrace{1q, bq, \ldots, bq}_{m+1}) \\
&= \sum_{\pi \in \mathrm{BNC}_{vs}(n+1,m+1)} \kappa_\pi(\underbrace{1p, ap, \ldots, ap}_{n+1}, \underbrace{1q, bq, \ldots, bq}_{m+1}) \\
&\quad + \sum_{\pi \in \mathrm{BNC}_{vs}(n+1,m+1)^c} \kappa_\pi(\underbrace{1p, ap, \ldots, ap}_{n+1}, \underbrace{1q, bq, \ldots, bq}_{m+1}),
\end{aligned} \tag{3.8}$$

where $\mathrm{BNC}_{vs}(n+1, m+1)$ consists of all left-right split bi-noncrossing partitions $\pi \in \mathrm{BNC}(n+1, m+1)$; that is, either $V \subseteq \chi_{n+1,m+1}^{-1}(l)$ or $V \subseteq \chi_{n+1,m+1}^{-1}(r)$ for every block V of $\pi \in \mathrm{BNC}(n+1, m+1)$. $\mathrm{BNC}_{vs}(n+1, m+1)^c$ denotes $\mathrm{BNC}(n+1, m+1) \setminus \mathrm{BNC}_{vs}(n+1, m+1)$. For $\pi \in \mathrm{BNC}_{vs}(n+1, m+1)$, let $\pi_l = \pi|_{\{1_l, \ldots, (n+1)_l\}}$ and $\pi_r = \pi|_{\{1_r, \ldots, (m+1)_r\}}$ denote the sets of left and right indexed blocks of π. For the sum over all partitions in $\mathrm{BNC}_{vs}(n+1,m+1)$ in (3.8), we have

$$\begin{aligned}
&\sum_{\pi\in BNC_{vs}(n+1,m+1)} \kappa_\pi(1p,ap,\ldots,ap,1q,bq,\ldots,bq) \\
&\quad= \left(\sum_{\pi_l\in NC(n+1)} \kappa_{\pi_l}(1p,ap,\ldots,ap)\right)\cdot\left(\sum_{\pi_r\in NC(m+1)} \kappa_{\pi_r}(1q,bq,\ldots,bq)\right) \\
&\quad= \phi\left((pap)^n\right)\phi((qbq)^m). \qquad (3.9)
\end{aligned}$$

Denoting the sum over all partitions in $\mathrm{BNC}(n+1,m+1)^c$ in (3.8) by $\psi(z,w)$, the moment series $M_{pap,qbq}(z,w)$ can be written as

$$\begin{aligned}
M_{pap,qbq}(z,w) &= 1+\sum_{n\geq 1}\phi((pap)^n)z^n+\sum_{m\geq 1}\phi((qbq)^m)w^m+\sum_{n,m\geq 1}\phi((pap)^n(qbq)^m)z^nw^m \\
&= (1+\sum_{n\geq 1}\phi((pap)^n)z^n)(1+\sum_{m\geq 1}\phi((qbq)^m)w^m)+\psi(z,w) \\
&= M_{ap}(z)\cdot M_{bq}(w)+\psi(z,w). \qquad (3.10)
\end{aligned}$$

We will now consider the sum $\psi(z,w)$. If $\pi\in\mathrm{BNC}_{vs}(n+1,m+1)^c$, then there exists a block V of π which includes both left and right indices so that $V_l := V\cap\{1_l,\ldots,(n+1)_l\}\neq\emptyset$ and $V_r := V\cap\{1_r,\ldots,(m+1)_r\}\neq\emptyset$. Define V_π by the block of π containing both left and right indices such that $\min(V_\pi)$ is smallest among all partitions containing both left and right indices. Rearrange the sum $\psi(z,w)$ by choosing integers $s\in\{1,\ldots,n+1\}$ and $t\in\{1,\ldots,m+1\}$ and blocks V satisfying $|V_l|=s$ and $|V_r|=t$. Given such V, summing over $\pi\in\mathrm{BNC}_{vs}(n+1,m+1)^c$

such that $V_\pi = V$, we obtain

$$\psi(z,w) \;=\; \sum_{t\geq 1}\sum_{s\geq 1}\left(\sum_{V=V_l\cup V_r}\left(\sum_{\substack{\pi\in \mathrm{BNC}_{vs}(n+1,m+1)^c\\ V_\pi=V}} \kappa_\pi(1p,\underbrace{ap,\ldots,ap}_{n},1q,\underbrace{bq,\ldots,bq}_{m})\right)\right).$$

Let Θ denote the set of all blocks containing both left and right indicies. We devide up the sum $\psi(z,w)$ into four parts based on the types of blocks in Θ. For each block $V\in\Theta$, denote the left and right parts by V_l and V_r. Depending on the values of $\min(V_l)$ and $\min(V_r)$, define four disjoint subsets $\Theta_1,\ \Theta_2,\ \Theta_3$, and Θ_4 of Θ by

$$\begin{aligned}
\Theta_1 \;&=\; \{\, V\in\Theta \;|\; \min(V_l)>1 \ \text{ and } \ \min(V_r)>1\}\\
\Theta_2 \;&=\; \{\, V\in\Theta \;|\; \min(V_l)>1 \ \text{ and } \ \min(V_r)=1\}\\
\Theta_3 \;&=\; \{\, V\in\Theta \;|\; \min(V_l)=1 \ \text{ and } \ \min(V_r)>1\}\\
\Theta_4 \;&=\; \{\, V\in\Theta \;|\; \min(V_l)=1 \ \text{ and } \ \min(V_r)=1\}.
\end{aligned}$$

Note that $\Theta=\Theta_1\cup\Theta_2\cup\Theta_3\cup\Theta_4$. For each set Θ_i, define a partial sum $\psi_i(z,w)$ of $\psi(z,w)$ by

$$\psi_i(z,w) \;=\; \sum_{s,t\geq 1}\left(\sum_{V\in\Theta_i}\left(\sum_{\substack{\pi\in \mathrm{BNC}_{vs}(n+1,m+1)^c\\ V_\pi=V,\ n,m\geq 1}} \kappa_\pi(1p,ap,\ldots,ap,1q,bq,\ldots,bq)\; z^n w^m\right)\right) \quad (3.11)$$

where $1\leq i\leq 4$. Then we have $\psi(z,w) \;=\; \psi_1(z,w)+\psi_2(z,w)+\psi_3(z,w)+\psi_4(z,w)$.

We will define several notations in order to discuss each term of $\psi_i(z,w)$ for $1\leq i\leq 4$. Assume that $\pi\in \mathrm{BNC}_{vs}(n+1,m+1)^c$ and $V_\pi=\{(u_1)_l,\ldots,(u_s)_l,(v_1)_r,\ldots,(v_t)_r \mid u_1< \cdots u_s,\ v_1<\cdots<v_t\}$. where $s,t\geq 1$. Let $\pi_{l,k}$ for $0\leq k\leq s-1$ (respectively, $\pi_{r,k}$ for $0\leq k\leq t-1$) be the set of all blocks of π whose indices are belong to $\{(u_k+1)_l,\ldots,(u_{k+1}-1)_l\}$ (respectively, $\{(v_k+1)_r,\ldots,(v_{k+1}-1)_r\}$), where $u_0=v_0=0$. Let $\tilde\pi$ denote the set of all blocks of π whose indices are contained in $\{(u_s+1)_l,\ldots,(n+1)_l,(v_t+1)_r,\ldots,(m+1)_r\}$. Then we

have the disjoint union of

$$\pi \;=\; \{V_\pi\} \;\cup\; \left(\bigcup_{k=0}^{s-1} \pi_{l,k}\right) \;\cup\; \left(\bigcup_{k=0}^{t-1} \pi_{r,k}\right) \;\cup\; \tilde{\pi}.$$

Let i_k for $0 \le k \le s-1$ (respectively, j_k for $0 \le k \le t-1$) be the size of the block $\pi_{l,k}$ (respectively, $\pi_{r,k}$), and let $i_s = n - u_s + 1$ and $j_t = m - v_t + 1$ so that $i_s + j_t = |\tilde{\pi}|$. In the following four lemmas, we will find ψ_i for $1 \le i \le 4$.

Lemma 3.2.3. *Under the above notation and assumptions, we have*

$$\psi_1(z,w) \;=\; \frac{(M_{ap}(z) - 1 + \phi(p))(M_{bq}(w) - 1 + \phi(q))}{M_{ap}(z)M_{bq}(w)} \cdot (M_{ap,bq}(z,w) - M_{ap}(z)M_{bq}(w)).$$

Proof. Fix $n, m \ge 1$. If $\pi \in BNC_{vs}(n+1, m+1)^c$ and $V_\pi \in \Theta_1$, then we have $V_\pi = \{(u_1)_l, \ldots, (u_s)_l, (v_1)_r, \ldots, (v_t)_r \mid 1 < u_1 < \cdots < u_s \le n+1,\ 1 < v_1 < \cdots < v_t \le m+1\}$ for some $s, t \ge 1$. Then, $\kappa_\pi(1p, ap, \ldots, ap, 1q, bq, \ldots, bq)$ is equal to

$$\kappa_{\pi_{l,0}}(1p, \underbrace{ap, \ldots, ap}_{i_0 - 1})\, \kappa_{\pi_{r,0}}(1q, \underbrace{bq, \ldots, bq}_{j_0 - 1})\, \kappa_{s,t}(ap, bq) \cdot \left(\prod_{k=1}^{s-1} \kappa_{\pi_{l,k}}(ap)\right) \left(\prod_{k=1}^{t-1} \kappa_{\pi_{r,k}}(bq)\right) \cdot \kappa_{\tilde{\pi}}(ap, bq).$$

Recall that $\psi_1(z,w)$ is defined by

$$\psi_1(z,w) \;=\; \sum_{s,t\ge1} \left(\sum_{V\in\Theta_1} \left(\sum_{\substack{\pi\in \mathrm{BNC}_{vs}(n+1,m+1)^c \\ V_\pi = V,\ n,m\ge1}} \kappa_\pi(1p, ap, \ldots, ap, 1q, bq, \ldots, bq)\; z^n w^m \right)\right).$$

Given V_π, summing the terms of $\kappa_\rho(1p, ap, \ldots, ap, 1q, bq, \ldots, bq)\; z^n w^m$ over all $\rho \in BNC_{vs}(n+1, m+1)^c$ satisfying $V_\rho = V_\pi$, we obtain

$$\left(\sum_{\rho_{l,0}\in NC(i_0)} \kappa_{\rho_{l,0}}(1p, ap, \ldots, ap) z^{i_0-1}\right)\left(\sum_{\rho_{r,0}\in NC(j_0)} \kappa_{\rho_{r,0}}(1q, bq, \ldots, bq) w^{j_0-1}\right)\kappa_{s,t}(ap,bq) z^s w^t$$

$$\cdot \sum_{\substack{i_1,\ldots,i_s\geq 0\\ j_1,\ldots,j_t\geq 0}} \left(\prod_{k=1}^{s-1}\phi((ap)^{i_k})z^{i_k}\right)\left(\prod_{k=1}^{t-1}\phi((bq)^{j_k})w^{j_k}\right)\phi((ap)^{i_s}(bq)^{j_t})z^{i_s}w^{j_t}.$$

For fixed s and t, summing over the above equation over all blocks $V \in \Theta_1$ satisfying $|V_l| = s$, $|V_r| = t$, and $V_\pi = V$, we get

$$\sum_{\substack{V\in\Theta_1\\ |V_l|=s,\ |V_r|=t}} \left(\sum_{\substack{\pi\in BNC_{vs}(n+1,m+1)\\ V_\pi=V,\ n,m\geq 1}} \kappa_\pi(1p, ap, \ldots, ap, 1q, bq, \ldots, bq)\ z^n w^m\right) \tag{3.12}$$

$$= \left(\phi(p)\phi(q) + \sum_{i_0,j_0\geq 2}\phi((pap)^{i_0-1})\phi((qbq)^{j_0-1})z^{i_0-1}w^{j_0-1} + \sum_{i_0\geq 2}\phi((pap)^{i_0-1})\phi(q)z^{i_0-1}\right.$$

$$\left. + \sum_{j_0\geq 2}\phi(p)\phi((qbq)^{j_0-1})w^{j_0-1}\right)\cdot \kappa_{s,t}(ap,bq)z^s w^t \cdot (M_{ap}(z))^{s-1}(M_{bq}(w))^{t-1} M_{ap,bq}(z,w)$$

$$= \frac{(M_{ap}(z)-1+\phi(p))(M_{bq}(w)-1+\phi(q))}{M_{ap}(z)M_{bq}(w)}\cdot M_{ap,bq}(z,w)\cdot \kappa_{s,t}(ap,bq)(zM_{ap}(z))^s(wM_{bq}(w))^t.$$

Summing the formula in (3.12) over all $s, t \geq 1$, we obtain

$$\frac{(M_{ap}(z)-1+\phi(p))(M_{bq}(w)-1+\phi(q))}{M_{ap}(z)M_{bq}(w)}\cdot M_{ap,bq}(z,w)\cdot K_{ap,bq}(zM_{ap}(z), wM_{bq}(w)).$$

By (1.11), we have the equality $K_{ap,bq}(zM_{ap}(z), wM_{bq}(w)) = 1 - \frac{M_{ap}(z)M_{bq}(w)}{M_{ap,bq}(z,w)}$, and therefore

$$\psi_1(z,w) = \frac{(M_{ap}(z)-1+\phi(p))(M_{bq}(w)-1+\phi(q))}{M_{ap}(z)M_{bq}(w)}\cdot (M_{ap,bq}(z,w) - M_{ap}(z)M_{bq}(w)).$$

$\square$

In a similar way, we can derive expressions for $\psi_2,\ \psi_3$, and ψ_4. Recall that $l = 1 - \frac{\phi(pq)}{\phi(p)\phi(q)}$.

Lemma 3.2.4. *Under the above notation and assumptions, we have*

$$\begin{aligned}\psi_2(z,w) \ &= \ \frac{l\,(M_{ap}(z)-1+\phi(p))(M_{ap}(z)-1)}{M_{ap}(z)}\\ &\qquad \cdot \left(\phi(q) \ + \ \frac{M_{bq}(w)-1+\phi(q)}{M_{bq}(w)} \cdot \frac{M_{ap,bq}(z,w)}{l\,(M_{ap}(z)-1)(M_{bq}(w)-1)-M_{ap}(z)M_{bq}(w)}\right).\end{aligned}$$

Lemma 3.2.5. *Under the above notation and assumptions, we have*

$$\begin{aligned}\psi_3(z,w) \ &= \ \frac{l\,(M_{bq}(w)-1+\phi(q))(M_{bq}(w)-1)}{M_{bq}(w)}\\ &\qquad \cdot \left(\phi(p) \ + \ \frac{M_{ap}(z)-1+\phi(p)}{M_{ap}(z)} \cdot \frac{M_{ap,bq}(z,w)}{l\,(M_{ap}(z)-1)(M_{bq}(w)-1)-M_{ap}(z)M_{bq}(w)}\right).\end{aligned}$$

Lemma 3.2.6. *Under the above notation and assumptions, we have*

$$\begin{aligned}\psi_4(z,w) \ = \ l \cdot \Bigg(&-\phi(p)\phi(q) \ + \ \frac{\phi(q)(M_{ap}(z)-1+\phi(p))}{M_{ap}(z)} \ + \ \frac{\phi(p)(M_{bq}(w)-1+\phi(q))}{M_{bq}(w)}\\ &+ \ \frac{M_{ap}(z)-1+\phi(p)}{M_{ap}(z)} \cdot \frac{M_{bq}(w)-1+\phi(q)}{M_{bq}(w)}\\ &\cdot \frac{M_{ap,bq}(z,w)}{l\,(M_{ap}(z)-1)(M_{bq}(w)-1)-M_{ap}(z)M_{bq}(w)}\Bigg).\end{aligned}$$

We will go back to the proof of Theorem 3.2.1. Since $\psi(z,w)$ is defined to be the sum of $\psi_i(z,w)$ for $1 \le i \le 4$, by using the above four lemmas, $\psi(z,w)$ is of the form

$$\begin{aligned}\psi(z,w) \ &= \ -\left(M_{ap}(z)-1+\frac{\phi(pq)}{\phi(q)}\right)\left(M_{bq}(w)-1+\frac{\phi(pq)}{\phi(p)}\right) \ - \ \phi(pq)\,l\\ &\quad - \ (1-l)\ M_{ap,bq}(z,w) \cdot \frac{(M_{ap}(z)-1+\phi(p))(M_{bq}(w)-1+\phi(q))}{l\,(M_{ap}(z)-1)(M_{bq}(w)-1)-M_{ap}(z)M_{bq}(w)}. \qquad (3.13)\end{aligned}$$

Recall that $M_{pap,qbq}(z,w) \ = \ M_{ap}(z)M_{bq}(w) + \psi(z,w)$ in (3.10) and $\chi_a(z) \ = \ (M_a(z)-1)^{\langle -1 \rangle}$.

Substituting $\chi_{ap}(z)$ and $\chi_{bq}(w)$, respectively, for z and w in (3.13), it follows that

$$\begin{aligned}
&M_{pap,qbq}(\chi_{ap}(z),\chi_{bq}(w))\\
&= (z+1)(w+1)+\psi(\chi_{ap}(z),\chi_{bq}(w))\\
&= (z+1)(w+1)-\left(z+\frac{\phi(pq)}{\phi(q)}\right)\left(w+\frac{\phi(pq)}{\phi(p)}\right)-\phi(pq)\cdot l\\
&\qquad\qquad\qquad + (1-l)\, M_{ap,bq}(\chi_{ap}(z),\chi_{bq}(w))\cdot\frac{(z+\phi(p))(w+\phi(q))}{(z+1)(w+1)-l\cdot zw}\\
&= 1-\phi(pq)+\left(1-\frac{\phi(pq)}{\phi(p)}\right)z+\left(1-\frac{\phi(pq)}{\phi(q)}\right)w\\
&\qquad\qquad\qquad + (1-l)\, M_{ap,bq}(\chi_{ap}(z),\chi_{bq}(w))\cdot\frac{(z+\phi(p))(w+\phi(q))}{(z+1)(w+1)-l\cdot zw},
\end{aligned}$$

which completes the proof. □

With the similar techniques as in Theorem 3.2.1, we can find the moment series $M_{pap,qb}$ and $M_{pap,bq}$ of pairs (pap,qb) and (pap,bq) in $(\mathcal{A},\phi)$.

Theorem 3.2.7. *Under the above notation and assumptions,*

$$\begin{aligned}
&M_{pap,qb}(\chi_{ap}(z),\chi_{bq}(w))\\
&\qquad = 1-\phi(p)+\left(1-\frac{\phi(pq)}{\phi(q)}\right)w+M_{pa,qb}(\chi_{ap}(z),\chi_{bq}(w))\cdot\frac{(z+\phi(p))((1-l)w+1)}{(z+1)(w+1)-l\cdot zw}
\end{aligned}$$

on $(\mathbb{C}\setminus\{0\})^2$ *near* $(0,0)$. *Furthermore, we have*

$$M_{pap,qb}(\chi_{ap}(z),\chi_{bq}(w)) = M_{pap,bq}(\chi_{ap}(z),\chi_{bq}(w)).$$

For the rest of this section, we will assume that a and b are non-trivial self-adjoint projections in $(\mathcal{A},\phi)$ as well. So we are considering bi-free two-faced pairs (a,b) and (p,q) in a C*-non-commutative probability space $(\mathcal{A},\phi)$ such that a, b, p, and q are projections and $\phi(a)$, $\phi(b)$, $\phi(p)$, $\phi(q)$

are neither 0 nor 1.

Under these assumptions, we can find $M_{ap,bq}(\chi_{ap}(z), \chi_{bq}(w))$. By definition of the bi-free partial S-transform, we have

$$S_{ap,bq}(z,w) \;=\; \frac{z+1}{z} \cdot \frac{w+1}{w} \cdot \left(1 - \frac{1+z+w}{M_{ap,bq}(\chi_{ap}(z), \chi_{bq}(w))}\right)$$

and $S_{ap,bq}(z,w) = S_{a,b}(z,w) \cdot S_{p,q}(z,w)$ by bi-freeness of (a,b) and (p,q). Then it follows that

$$M_{ap,bq}(\chi_{ap}(z), \chi_{bq}(w)) \;=\; \frac{1+z+w}{1 - \frac{(z+1)(w+1)}{zw}\left(1 - \frac{1+z+w}{M_{a,b}(\chi_a(z),\chi_b(w))}\right)\left(1 - \frac{1+z+w}{M_{p,q}(\chi_p(z),\chi_q(w))}\right)}. \tag{3.14}$$

For self-adjoint projections a and b, we can easily find $M_{a,b}(\chi_a(z), \chi_b(w))$. Indeed, recall some equalities of free and bi-free transforms

$$\chi_a(z) = (M_a(z) - 1)^{\langle -1 \rangle} = \frac{z}{z + \phi(a)} \quad \text{and} \quad \chi_b(w) = (M_b(w) - 1)^{\langle -1 \rangle} = \frac{w}{w + \phi(b)},$$

and

$$M_{a,b}(z,w) \;=\; 1 + \frac{\phi(a)z}{1-z} + \frac{\phi(b)w}{1-w} + \frac{\phi(ab)zw}{(1-z)(1-w)}.$$

Substituting $\chi_a(z)$ and $\chi_b(w)$ for z and w in the equality of $M_{a,b}(z,w)$ above, we obtain

$$M_{a,b}(\chi_a(z), \chi_b(w)) \;=\; 1 + z + w + \frac{\phi(ab)}{\phi(a)\phi(b)} \cdot zw.$$

Similarly, we can find $M_{p,q}(\chi_p(z), \chi_q(w))$. Therefore, substituting those formulas for $M_{a,b}(\chi_a(z), \chi_b(w))$ and $M_{p,q}(\chi_p(z), \chi_q(w))$ in (3.14), we get

$$M_{ap,bq}(\chi_{ap}(z), \chi_{bq}(w)) \;=\; \frac{((1+z)(1+w) - k \cdot zw) \cdot ((1+z)(1+w) - l \cdot zw)}{(1+z)(1+w) - kl \cdot zw}$$

where $k = 1 - \frac{\phi(ab)}{\phi(a)\phi(b)}$ and $l = 1 - \frac{\phi(pq)}{\phi(p)\phi(q)}$.

In Theorem 3.2.1 and 3.2.7, we represent the joint moment series $M_{pap,qbq}$ and $M_{pap,qb}$ using

$M_{ap,bq}$, when (a, b) and (p, q) are bi-free with only p and q projections. Assuming further that a and b are also projections, we can obtain the following result.

Theorem 3.2.8. *Let (a, b) and (p, q) be bi-free two-faced pairs in a C*-non-commutative probability space $(\mathcal{A}, \phi)$ such that a, b, p, and q are non-trivial self-adjoint projections. Then*

(i) $M_{pa,qbq}(\chi_{ap}(z), \chi_{bq}(w)) = 1 + z + w + (1-l)zw \cdot \left(1 - \dfrac{k(w + \phi(q))((1-l)z+1)}{(z+1)(w+1) - kl \cdot zw}\right)$

(ii) $M_{pap,qb}(\chi_{ap}(z), \chi_{bq}(w)) = 1 + z + w + (1-l)zw \cdot \left(1 - \dfrac{k(z + \phi(p))((1-l)w+1)}{(z+1)(w+1) - kl \cdot zw}\right)$

(iii) $M_{pap,qbq}(\chi_{ap}(z), \chi_{bq}(w)) = 1 + z + w + (1-l)zw \cdot \left(1 - \dfrac{k(1-l)(z + \phi(p))(w + \phi(q))}{(z+1)(w+1) - kl \cdot zw}\right)$

where $(z, w) \in (\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$.

4. BI-FREE PRODUCTS OF C*-ALGEBRAS

4.1 Reduced free product C*-algebras

The reduced free product was introduced by Voiculescu in [14]. For a given set I and each $i \in I$, $(\mathcal{A}_i, \phi_i)$ is a C*-non-commutative probability space whose GNS representation is faithful. Then there is a unique C*-non-commutative probability space $(\mathcal{A}, \phi)$ with unital embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}$ such that

(i) $\phi|_{\mathcal{A}_i} = \phi_i$

(ii) $(\mathcal{A}_i)_{i \in I}$ is free in $(\mathcal{A}, \phi)$

(iii) $\mathcal{A}$ is the C*-algebra generated by $\cup_{i \in I} \mathcal{A}_i$

(iv) the GNS representation of $\mathcal{A}$ associated to ϕ is faithful.

We denote the *reduced free product C*-algebra* by

$$(\mathcal{A}, \phi) = *_{i \in I} (\mathcal{A}_i, \phi_i)$$

and call ϕ the *free product state*. We will examine the most transparent notrivial free product, namely, the reduced free product of two two-dimensional C*-algebras.

Proposition 4.1.1. *There exits a universal, unital C*-algebra, $\mathfrak{A}$, on two self-adjoint projections P and Q. This means that whenever $\mathcal{B}$ is a unital C*-algebra containing self-adjoint projections p and q, there is a unique $*$-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}$ such that $\pi(1) = 1, \pi(P) = p$, and $\pi(Q) = q$. Moreover, we have*

$$\mathfrak{A} \cong \{f : [0,1] \to M_2(\mathbb{C}) \mid f \textit{ continuous}, f(0), f(1) \textit{ diagonal}\}$$

with the functions $P,\ Q \in \mathfrak{A}$ given by

$$P(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad Q(s) = \begin{bmatrix} s & \sqrt{s(1-s)} \\ \sqrt{s(1-s)} & 1-s \end{bmatrix}.$$

where $s \in [0,1]$.

Theorem 4.1.2 ([6]). *Let p, q be nontrivial self-adjoint projections with $\mu_1 = \phi(p)$ and $\mu_2 = \phi(q)$. Let*

$$(D, \tau) \ = \ (\underset{\mu_1}{\overset{p}{\mathbb{C}}} \oplus \underset{1-\mu_1}{\overset{1-p}{\mathbb{C}}}) * (\underset{\mu_2}{\overset{q}{\mathbb{C}}} \oplus \underset{1-\mu_2}{\overset{1-q}{\mathbb{C}}}).$$

be the C-algebra reduced free product with $\frac{1}{2} \leq \mu_2 \leq \mu_1 < 1$.*

(i) If $\mu_1 = \mu_2 = \frac{1}{2}$, then

$$D \ = \ \{f : [0,1] \to M_2(\mathbb{C}) \mid f \text{ continuous}, f(0), f(1) \text{ diagonal}\} \tag{4.1}$$

(ii) If $\mu_1 = \mu_2 > \frac{1}{2}$, then

$$D \ = \ \{f : [0,\beta] \to M_2(\mathbb{C}) \mid f \text{ continuous}, f(0) \text{ diagonal}\} \oplus \underset{2\mu_1-1}{\overset{p\wedge q}{\mathbb{C}}} \tag{4.2}$$

(iii) If $\mu_1 > \mu_2 \geq \frac{1}{2}$, then

$$D \ = \ \underset{\mu_1-\mu_2}{\overset{p\wedge(1-q)}{\mathbb{C}}} \oplus \quad C([\alpha,\beta]) \otimes M_2(\mathbb{C}) \quad \oplus \underset{\mu_1+\mu_2-1}{\overset{p\wedge q}{\mathbb{C}}} \tag{4.3}$$

where $\alpha, \beta = \mu_1 + \mu_2 - 2\mu_1\mu_2 \pm \sqrt{4\mu_1\mu_2(1-\mu_1)(1-\mu_2)}$.

4.2 Basic examples of bi-free product C*-algebras

Let P, Q, A, and B be four self-adjoint projections with a commutative condition $[A, B] = [A, Q] = [P, B] = [P, Q] = 0$. Then the universal, unital C*-algebra on those four projections is

$\mathfrak{A} \otimes \mathfrak{A}$, where $\mathfrak{A}$ is a universal C*-algebra on two self-adjoint projections as shown in proposition 4.1.1. The universal C*-algebra $\mathfrak{A} \otimes \mathfrak{A}$ is of the form as below.

$$\begin{aligned}\mathfrak{A} \otimes \mathfrak{A} \;\cong\; \{f : [0,1]^2 \to M_4(\mathbb{C}) \mid f \text{ continuous},\; f(0,t), f(1,t) \in D_2(\mathbb{C}) \otimes M_2(\mathbb{C}),\ \text{and} \\ f(s,0), f(s,1) \in M_2(\mathbb{C}) \otimes D_2(\mathbb{C}) \text{ for all } s,\ t \in [0,1]\}\end{aligned}$$

with the projections given by the functions in $\mathfrak{A} \otimes \mathfrak{A}$ as follows.

$$P(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \mathrm{I} \qquad\qquad Q(t) = \mathrm{I} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A(s) = \begin{bmatrix} s & \sqrt{s(1-s)} \\ \sqrt{s(1-s)} & 1-s \end{bmatrix} \otimes \mathrm{I} \qquad B(t) = \mathrm{I} \otimes \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{bmatrix}$$

where $s, t \in [0,1]$. A general state ψ on $\mathfrak{A} \otimes \mathfrak{A}$ is the restriction of a state on the C*-algebra $\mathcal{C}([0,1]^2,\ M_4(\mathbb{C}))$ which is of the form, for every $f : [0,1]^2 \to M_4(\mathbb{C})$ in $\mathfrak{A} \otimes \mathfrak{A}$,

$$\psi(f) \;=\; \sum_{1 \le i,j \le 4} \int_{[0,1]^2} f(s,t)_{ij}\, d\mu_{ij}(s,t) \tag{4.4}$$

where μ_{ij} are complex measures on $[0,1]^2$. Note that the measures μ_{ii} are positive and $\mu_{ij} = \overline{\mu_{ji}}$ for all $1 \le i, j \le 4$ because ψ is a state.

Assume that (a, b) and (p, q) are bi-free pairs of commuting projections in a non-commutative C*-probability space $(\mathcal{A}, \phi)$, where $[a, b] = [p, q] = 0$. Under this assumption, we will recall the definition of bi-free independence and define the reduced bi-free product of C^*-algebras and the bi-free product state. Let $C^*(a, b)$ (respectively $C^*(p, q)$) denote the C*-subalgebra of $\mathcal{A}$ generated by a and b (respectively p and q), and let ϕ_1 (respectively ϕ_2) be the restriction of ϕ to $C^*(a, b)$ (respectively $C^*(p, q)$). The GNS construction applied to $C^*(a, b)$ with ϕ_1 gives rise to a $*$-representation π_1 on a Hilbert space $\mathcal{H}_1$ with a specified unit vector ξ_1, and for $C^*(p, q)$ with

ϕ_2 we have $(\pi_2, \mathcal{H}_2, \xi_2)$. For $\mathcal{H}_i = \mathbb{C}\xi_i \oplus \mathcal{H}_i^\circ$ with $i = 1, 2$, consider the free product Hilbert space with specified unit vector, $(\mathcal{H}, \xi) = *_{i=1,2}(\mathcal{H}_i, \xi_i)$, and on $B(\mathcal{H})$ consider the bi-free state, $**_{i=1,2}\, \phi_i(\cdot) = \langle \cdot \xi, \xi \rangle$. As defined in (1.7) and (1.8), we have the left and right $*$-representations λ_i and ρ_i from $B(\mathcal{H}_i)$ to $B(\mathcal{H})$ for $i = 1, 2$.

Let D be the unital C*-subalgebra of $B(\mathcal{H})$ generated by two pairs of projections, $(\lambda_1 \circ \pi_1(a),\ \rho_1 \circ \pi_1(b))$ and $(\lambda_2 \circ \pi_2(p),\ \rho_2 \circ \pi_2(q))$, and let $\tau := \langle \cdot \xi, \xi \rangle$ be the bi-free state restricted on D. In this section, we will find the image of GNS representation of D associated to the state τ which may be called the *reduced bi-free product C*-algebra*. It follows

$$C([0,1]^2, M_4(\mathbb{C})) \supset \mathfrak{A} \otimes \mathfrak{A} \ \xrightarrow{\pi}\ D \ \xrightarrow{\tau}\ \mathbb{C} \tag{4.5}$$

where π is the $*$-representation onto D given by $\pi(A) = \lambda_1 \circ \pi_1(a),\ \pi(B) = \rho_1 \circ \pi_1(b),\ \pi(P) = \lambda_2 \circ \pi_2(p)$, and $\pi(Q) = \rho_2 \circ \pi_2(q)$. The image of the GNS representation of $\mathfrak{A} \otimes \mathfrak{A}$ associated to $\tau \circ \pi$ is isomorphic to the image of the GNS representation of D associated to τ. Thus we want to deside the bi-free state $\tau \circ \pi$ on $\mathfrak{A} \otimes \mathfrak{A}$. For notational purpose, denote $\tau \circ \pi$ by ψ. By symmetry, we assume

$$0 < \phi(a) \leq \phi(p) \leq \frac{1}{2} \quad \text{and} \quad 0 < \phi(b) \leq \phi(q) \leq \frac{1}{2}.$$

By the general form of states on $\mathfrak{A} \otimes \mathfrak{A}$ given in (4.4), there exists a matrix of complex measures $(\mu_{ij})_{1 \leq i,j \leq 4}$ such that

$$\begin{aligned} H_{11}(z, w) \ &:= \ \phi\left(p(z - pap)^{-1}(w - qbq)^{-1}q\right) \\ &= \ \tau \circ \pi\left(P(z \cdot \mathrm{I} - PAP)^{-1}(w \cdot \mathrm{I} - QBQ)^{-1}Q\right) \\ &= \ \iint_{[0,1]^2} \frac{1}{(z-s)(w-t)}\, d\mu_{11}(s,t). \end{aligned} \tag{4.6}$$

Similarly, we have

$$\begin{aligned} H_{12}(z,w) &:= \phi\left(p(z-pap)^{-1}(w-qbq)^{-1}qb(1-q)\right) \\ &= \iint_{[0,1]^2} \frac{\sqrt{t(1-t)}}{(z-s)(w-t)}\, d\mu_{12}(s,t) \end{aligned} \tag{4.7}$$

$$\begin{aligned} H_{13}(z,w) &:= \phi\left((z-pap)^{-1}pa(1-p)(w-qbq)^{-1}q\right) \\ &= \iint_{[0,1]^2} \frac{\sqrt{s(1-s)}}{(z-s)(w-t)}\, d\mu_{13}(s,t) \end{aligned} \tag{4.8}$$

$$\begin{aligned} H_{14}(z,w) &:= \phi\left((z-pap)^{-1}pa(1-p)(w-qbq)^{-1}qb(1-q)\right) \\ &= \iint_{[0,1]^2} \frac{\sqrt{s(1-s)t(1-t)}}{(z-s)(w-t)}\, d\mu_{14}(s,t). \end{aligned} \tag{4.9}$$

To obtain the measures μ_{1j} for $1 \leq j \leq 4$, we will first derive the expressions for H_{11}, H_{12}, H_{13}, and H_{14}. The form of the Cauchy transform of the distribution of the product of two projections is well known ([18]), and we recall it below.

$$G_{ap}(z) = \frac{1}{z} + \frac{z-(\phi(a)+\phi(p)) - \sqrt{(z-\alpha_1)(z-\beta_1)}}{2(1-z)z} \tag{4.10}$$

$$G_{bq}(w) = \frac{1}{w} + \frac{w-(\phi(b)+\phi(q)) - \sqrt{(w-\alpha_2)(w-\beta_2)}}{2(1-w)w} \tag{4.11}$$

where

$$\begin{aligned} \alpha_1, \beta_1 &= \phi(a)+\phi(p)-2\phi(a)\phi(p) \pm \sqrt{4\phi(a)\phi(p)(1-\phi(a))(1-\phi(p))} \\ \alpha_2, \beta_2 &= \phi(b)+\phi(q)-2\phi(b)\phi(q) \pm \sqrt{4\phi(b)\phi(q)(1-\phi(b))(1-\phi(q))}. \end{aligned}$$

Note that $0 \leq \alpha_i < \beta_i \leq 1$ for $i = 1, 2$. Remark that $\alpha_1 = 0$ if and only if $\phi(a) = \phi(p)$, and $\beta_1 = 1$ if and only if $\phi(a) + \phi(p) = 1$, that is, $\phi(a) = \phi(p) = \frac{1}{2}$. The analogous statements hold for α_2 and β_2.

For $z \in \mathbb{C} \setminus [\alpha_1, \beta_1]$ and $w \in \mathbb{C} \setminus [\alpha_2, \beta_2]$, define

$$\eta_{ap}(z) = \frac{G_{ap}(z)}{G_{ap}(z) - \frac{1}{z}} \quad \text{and} \quad \eta_{bq}(w) = \frac{G_{bq}(w)}{G_{bq}(w) - \frac{1}{w}}.$$

Lemma 4.2.1. *The functions η_{ap} and η_{bq} are biholomorphic onto $\mathbb{C} \setminus \left(\sqrt{\frac{(1-\phi(a))(1-\phi(p))}{\phi(a)\phi(p)}} \cdot \mathbb{D}_1\right)$ and $\mathbb{C} \setminus \left(\sqrt{\frac{(1-\phi(b))(1-\phi(q))}{\phi(b)\phi(q)}} \cdot \mathbb{D}_1\right)$, respectively, where $\mathbb{D}_1$ denotes the closed unit disk on $\mathbb{C}$.*

Proof. By symmetry we will only prove that the function η_{ap} is biholomorphic from $\mathbb{C} \setminus [\alpha_1, \beta_1]$ onto $\mathbb{C} \setminus \left(\sqrt{\frac{(1-\phi(a))(1-\phi(p))}{\phi(a)\phi(p)}} \cdot \mathbb{D}_1\right)$. Note that substituting the formula for $G_{ap}(z)$, we have

$$\eta_{ap}(z) = \frac{-2 + \phi(a) + \phi(p) + z + \sqrt{(z-\alpha_1)(z-\beta_1)}}{\phi(a) + \phi(p) - z + \sqrt{(z-\alpha_1)(z-\beta_1)}}.$$

Define a function $g(x)$ by

$$g(x) \ = \ \frac{(1 + \phi(a)(x-1))(1 + \phi(p)(x-1))}{x}$$

for $x \in \mathbb{C} \setminus \left(\sqrt{\frac{(1-\phi(a))(1-\phi(p))}{\phi(a)\phi(p)}} \cdot \mathbb{D}_1\right)$. Combining the functions g and η_{ap}, we have

$$\begin{aligned}
\eta_{ap}\left(g(x)\right) \ &= \ \frac{-2 + \phi(a) + \phi(p) + g(x) + \sqrt{(g(x)-\alpha_1)(g(x)-\beta_1)}}{\phi(a) + \phi(p) - g(x) + \sqrt{(g(x)-\alpha_1)(g(x)-\beta_1)}} \\
&= \ \frac{\left(-2 + \phi(a) + \phi(p)\right) x + g(x) \cdot x + x\sqrt{\left(\frac{\phi(a)\phi(p)x^2 - (1-\phi(a))(1-\phi(p))}{x}\right)^2}}{\left(\phi(a) + \phi(p)\right) x - g(x) \cdot x + x\sqrt{\left(\frac{\phi(a)\phi(p)x^2 - (1-\phi(a))(1-\phi(p))}{x}\right)^2}} \\
&= \ \frac{2\phi(a)\phi(p)x^2 - 2(1 - \phi(a) - \phi(p) + \phi(a)\phi(p))x}{2\phi(a)\phi(p)x - 2(1-\phi(a))(1-\phi(p))} \\
&= \ x
\end{aligned}$$

where the branch of a square root is specified by the approximation $\sqrt{\left(\frac{\phi(a)\phi(p)x^2 - (1-\phi(a))(1-\phi(p))}{x}\right)^2} \approx \frac{\phi(a)\phi(p)x^2 - (1-\phi(a))(1-\phi(p))}{x}$ for large $|x|$. Solving the equation $g(x) = z$ for x gives $x = \frac{1}{2\phi(a)\phi(p)} \cdot$

$\left(z - \phi(a) - \phi(p) + 2\phi(a)\phi(p) \pm \ i\sqrt{(z-\alpha_1)(\beta_1 - z)}\right)$, and so when $z \in [\alpha_1, \beta_1]$, we have $|x| = \sqrt{\frac{(1-\phi(a))(1-\phi(p))}{\phi(a)\phi(p)}}$. Thus, the function g is the inverse of η_{ap} with respect to composition, which completes the proof. □

Now we will find H_{11}, H_{12}, H_{13}, and H_{14} in the following lemmas. Recall that

$$k \ = \ 1 - \frac{\phi(ab)}{\phi(a)\phi(b)} \qquad \text{and} \qquad l \ = \ 1 - \frac{\phi(pq)}{\phi(p)\phi(q)}.$$

Lemma 4.2.2. *For* $z, w \in \mathbb{C} \setminus [0,1]$,

$$H_{11}(z,w) \ = \ (1-l)\left(G_{ap}(z) - \frac{1-\phi(p)}{z}\right)\left(G_{bq}(w) - \frac{1-\phi(q)}{w}\right) \cdot \frac{\eta_{ap}(z)\cdot\eta_{bq}(w) - k}{\eta_{ap}(z)\cdot\eta_{bq}(w) - kl}$$

Proof. From (4.6), we have

$$\begin{aligned}
H_{11}(z,w) \ &= \ \phi(p(z-pap)^{-1}(w-qbq)^{-1}q) \\
&= \ \phi\left(p\left(\sum_{n\geq 0}(pap)^n z^{-n-1}\right)\left(\sum_{m\geq 0}(qbq)^m w^{-m-1}\right)q\right) \\
&= \ G_{pap,qbq}(z,w) - \left(1 - \frac{\phi(pq)}{\phi(p)}\right)\cdot\frac{G_{ap}(z)}{w} - \left(1 - \frac{\phi(pq)}{\phi(q)}\right)\cdot\frac{G_{bq}(w)}{z} \\
&\qquad + \left(1 + \phi(pq)\left(1 - \frac{1}{\phi(p)} - \frac{1}{\phi(q)}\right)\right)\cdot\frac{1}{zw} \quad (4.12)
\end{aligned}$$

for $z, w \in \mathbb{C} \setminus [0,1]$. Recall that $\chi_{ap}(z)$ and $\chi_{bq}(w)$ are formal power series inverses of $M_{ap}(z) - 1$ and $M_{bq}(w) - 1$, respectively, so we have the following equalities.

$$G_{ap}\left(\frac{1}{\chi_{ap}(z)}\right) \ = \ M_{ap}(\chi_{ap}(z))\cdot\chi_{ap}(z) \ = \ (z+1)\cdot\chi_{ap}(z)$$

$$G_{bq}\left(\frac{1}{\chi_{bq}(w)}\right) \ = \ M_{bq}(\chi_{bq}(w))\cdot\chi_{bq}(w) \ = \ (w+1)\cdot\chi_{bq}(w).$$

By definition of the two-variable Cauchy transform, we have

$$G_{pap,qbq}\left(\frac{1}{\chi_{ap}(z)}, \frac{1}{\chi_{bq}(w)}\right) \ = \ \chi_{ap}(z) \cdot \chi_{bq}(w) \cdot M_{pap,qbq}\left(\chi_{ap}(z), \chi_{bq}(w)\right),$$

and it is shown in Theorem 3.2.8 that

$$M_{pap,qbq}(\chi_{ap}(z), \chi_{bq}(w)) \ = \ 1 + z + w + (1-l)zw \cdot \left(1 - \frac{k(1-l)(z+\phi(p))(w+\phi(q))}{(z+1)(w+1) - kl \cdot zw}\right).$$

Then it follows from (4.12) and the above equalities that

$$\begin{aligned}
&H_{11}\left(\frac{1}{\chi_{ap}(z)}, \frac{1}{\chi_{bq}(w)}\right) \\
&= \ \chi_{ap}(z) \cdot \chi_{bq}(w) \cdot \Bigg(M_{pap,qbq}\left(\chi_{ap}(z), \chi_{bq}(w)\right) - \left(1 - \frac{\phi(pq)}{\phi(p)}\right)(z+1) \\
&\qquad\qquad - \left(1 - \frac{\phi(pq)}{\phi(q)}\right)(w+1) + 1 + \phi(pq)\left(1 - \frac{1}{\phi(p)} - \frac{1}{\phi(q)}\right)\Bigg) \\
&= \ \chi_{ap}(z) \cdot \chi_{bq}(w) \cdot \frac{(1-l)(z+\phi(p))(w+\phi(q)) \cdot ((z+1)(w+1) - k \cdot zw)}{(z+1)(w+1) - kl \cdot zw}. \qquad (4.13)
\end{aligned}$$

Note that the inverses of $\frac{1}{\chi_{ap}(z)}$ and $\frac{1}{\chi_{bq}(w)}$ are $z \cdot G_{ap}(z) - 1$ and $w \cdot G_{bq}(w) - 1$, respectively. By substituting these inverses for z and w in the last equation in (4.13), we have

$$H_{11}(z,w) \ = \ (1-l)\left(G_{ap}(z) - \frac{1-\phi(p)}{z}\right)\left(G_{bq}(w) - \frac{1-\phi(q)}{w}\right) \cdot \frac{\eta_{ap}(z) \cdot \eta_{bq}(w) - k}{\eta_{ap}(z) \cdot \eta_{bq}(w) - kl}.$$

□

Lemma 4.2.3. *For* $z, w \in \mathbb{C} \setminus [0,1]$,

$$H_{12}(z,w) \ = \ k(1-l)((1-l)\phi(q) - 1) \cdot \left(G_{ap}(z) - \frac{1-\phi(p)}{z}\right) \cdot \frac{1}{\eta_{ap}(z) \cdot \eta_{bq}(w) - kl}$$

Proof. By (4.7), we have

$$\begin{aligned}
H_{12}(z,w) &= \phi(p(z-pap)^{-1}(w-qbq)^{-1}qb(1-q)) \\
&= \phi\left(p\left(\sum_{n\geq 0}(pap)^n z^{-n-1}\right)\left(\sum_{m\geq 0}(qbq)^m w^{-m-1}\right)qb(1-q)\right) \\
&= w\cdot\big(G_{pap,qb}(z,w)-G_{pap,qbq}(z,w)\big) \\
&\qquad\qquad + \frac{1}{z}\cdot\left(\sum_{m\geq 1}\phi(p(qb)^m)w^{-m}-\sum_{m\geq 1}\phi(p(qbq)^m)w^{-m}\right) \\
&= w\cdot\big(G_{pap,qb}(z,w)-G_{pap,qbq}(z,w)\big) \qquad (4.14)
\end{aligned}$$

for $z,w\in\mathbb{C}\setminus[0,1]$. Note that Proposition 3.2.2 is used for the last equality of (4.14). By using Theorem 3.2.8, we obtain

$$\begin{aligned}
H_{12}\left(\frac{1}{\chi_{ap}(z)},\frac{1}{\chi_{bq}(w)}\right) &= \chi_{ap}(z)\cdot\Big(M_{pap,qb}(\chi_{ap}(z),\chi_{bq}(w))-M_{pap,qbq}(\chi_{ap}(z),\chi_{bq}(w))\Big) \\
&= \chi_{ap}(z)\cdot\frac{k(1-l)((1-l)\phi(q)-1)\cdot(z+\phi(p))zw}{(z+1)(w+1)-kl\cdot zw}.
\end{aligned}$$

Plugging in $z\cdot G_{ap}(z)-1$ and $w\cdot G_{bq}(w)-1$ for z and w, it follows that

$$H_{12}(z,w) = k(1-l)((1-l)\phi(q)-1)\cdot\left(G_{ap}(z)-\frac{1-\phi(p)}{z}\right)\cdot\frac{1}{\eta_{ap}(z)\cdot\eta_{bq}(w)-kl}.$$

□

By symmetry, we can easily find H_{13}.

Lemma 4.2.4. *For* $z,w\in\mathbb{C}\setminus[0,1]$,

$$H_{13}(z,w) = k(1-l)((1-l)\phi(p)-1)\cdot\left(G_{bq}(w)-\frac{1-\phi(q)}{w}\right)\cdot\frac{1}{\eta_{ap}(z)\cdot\eta_{bq}(w)-kl}.$$

Lemma 4.2.5. *For* $z, w \in \mathbb{C} \setminus [0,1]$,

$$H_{14}(z,w) \;=\; \frac{k(1-l)(l\phi(p)\phi(q) - (1-\phi(p))(1-\phi(q)))}{\eta_{ap}(z)\eta_{bq}(w) - kl}.$$

Proof. From (4.9), we have

$$\begin{aligned} H_{14}(z,w) &= \phi\left((z-pap)^{-1}pa(1-p)(w-qbq)^{-1}qb(1-q)\right) \\ &= \phi\left(\left(\sum_{n\geq 0}(pap)^n z^{-n-1}\right)pa(1-p)\left(\sum_{m\geq 0}(qbq)^m w^{-m-1}\right)qb(1-q)\right) \\ &= zw \cdot (G_{pa,qb}(z,w) + G_{pap,qbq}(z,w) - G_{pa,qbq}(z,w) - G_{pap,qb}(z,w)) \end{aligned}$$

for $z, w \in \mathbb{C} \setminus [0,1]$. It follows from Theorem 3.2.8 that

$$\begin{aligned} H_{14}\left(\frac{1}{\chi_{ap}(z)}, \frac{1}{\chi_{bq}(w)}\right) &= M_{pa,qb}\left(\chi_{ap}(z), \chi_{bq}(w)\right) + M_{pap,qbq}\left(\chi_{ap}(z), \chi_{bq}(w)\right) \\ &\quad - M_{pa,qbq}\left(\chi_{ap}(z), \chi_{bq}(w)\right) - M_{pap,qb}\left(\chi_{ap}(z), \chi_{bq}(w)\right) \\ &= \frac{k(1-l)(l\phi(p)\phi(q) - (1-\phi(p))(1-\phi(q)))zw}{(z+1)(w+1) - klzw}. \end{aligned}$$

Substituting $z \cdot G_{ap}(z) - 1$ and $w \cdot G_{bq}(w) - 1$ for z and w, we obtain

$$H_{14}(z,w) \;=\; \frac{k(1-l)(l\phi(p)\phi(q) - (1-\phi(p))(1-\phi(q)))}{\eta_{ap}(z)\eta_{bq}(w) - kl}.$$

□

Remark 4.2.6. Consider the fomulas of H_{1j} for $1 \leq j \leq 4$ in Lemma 4.2.2 - 4.2.5. If $l = 1$ (or equivalently, $\phi(pq) = 0$), then all of H_{11}, H_{12}, H_{13}, and H_{14} vanish. If $k = 0$ (or equivalently, random variables a and b are classically independent in $(\mathcal{A}, \phi)$), then H_{12}, H_{13}, and H_{14} vanish. The condition, $\phi(p(1-q)) = 0$, $\phi((1-p)q) = 0$, and $\phi((1-p)(1-q)) = 0$, respectively implies that $H_{12} = 0$, $H_{13} = 0$, and $H_{14} = 0$. Note that if $H_{1j} = 0$, then the corresponding measure μ_{1j}

is vanishing by the definition in (4.6), (4.7), (4.8), and (4.9). For now we are not considering these special cases, so we will assume the following conditions in this section, which implies that H_{1j} is not vanishing for $1 \leq j \leq 4$.

(i) $\phi(pq), \phi(p(1-q)), \phi((1-p)q)$, and $\phi((1-p)(1-q))$ do not vanish.

(ii) $\phi(ab) \neq \phi(a)\phi(b)$ and $\phi(pq) \neq \phi(p)\phi(q)$, that is, both (a,b) and (p,q) are not pairs of classically independent random variables in $(\mathcal{A}, \phi)$.

We would like to recover the measures μ_{1j} by applying the Stieltjes inversion formula to $H_{1j}(z,w)$ for $1 \leq j \leq 4$. Before discussing that, it should be verified that $H_{1j}(z,w)$ is well defined for $z, w \in \mathbb{C} \setminus [0,1]$; that is, the denominators of $H_{1j}(z,w)$ do not vanish on $(\mathbb{C} \setminus [0,1])^2$. By the equalities in Lemma 4.2.2 - 4.2.5, it suffices to show that $\eta_{ap}(z)\eta_{bq}(w) - kl \neq 0$ for $z, w \in \mathbb{C} \setminus [0,1]$. Since a, b, p, and q are self-adjoint projections with $[a,b] = [p,q] = 0$, we have $0 \leq \phi(a \wedge b) \leq \phi(ab) \leq \min(\phi(a), \phi(b))$ and $0 \leq \phi(p \wedge q) \leq \phi(pq) \leq \min(\phi(p), \phi(q))$. Then, by the definition of k and l, we have

$$\max\left(1 - \frac{1}{\phi(a)},\ 1 - \frac{1}{\phi(b)}\right) \leq k \leq 1 \quad \text{and} \quad \max\left(1 - \frac{1}{\phi(p)},\ 1 - \frac{1}{\phi(q)}\right) \leq l \leq 1.$$

Since we assumed that $\phi(a) \leq \phi(p) \leq \frac{1}{2}$ and $\phi(b) \leq \phi(q) \leq \frac{1}{2}$, it follows that

$$|k \cdot l| \leq \sqrt{\frac{(1-\phi(a))(1-\phi(b))(1-\phi(p))(1-\phi(q))}{\phi(a)\phi(b)\phi(p)\phi(q)}}. \tag{4.15}$$

In Lemma 4.2.1, it is shown that for $z \in \mathbb{C} \setminus [\alpha_1, \beta_1]$ and $w \in \mathbb{C} \setminus [\alpha_2, \beta_2]$,

$$|\eta_{ap}(z)| > \sqrt{\frac{(1-\phi(a))(1-\phi(p))}{\phi(a)\phi(p)}} \quad \text{and} \quad |\eta_{bq}(w)| > \sqrt{\frac{(1-\phi(b))(1-\phi(q))}{\phi(b)\phi(q)}}. \tag{4.16}$$

The inequalities (4.15) and (4.16) imply that $|k \cdot l|$ is strictly less than $|\eta_{ap}(z) \cdot \eta_{bq}(w)|$, and therefore $\eta_{ap}(z) \cdot \eta_{bq}(w) - k \cdot l$ does not vanish for $z, w \in \mathbb{C} \setminus [0,1]$.

In order to find the measures μ_{1j} for $1 \leq j \leq 4$, we will use the following limit.

Lemma 4.2.7. *If $s_0 \in [\alpha_1, \beta_1]$, then*

$$\lim_{\epsilon \searrow 0} \eta_{ap}(s_0 + i\epsilon) = \frac{s_0 - \phi(a) - \phi(p) + 2\phi(a)\phi(p) + i\sqrt{(s_0 - \alpha_1)(\beta_1 - s_0)}}{2\phi(a)\phi(p)}.$$

Proof. Let $s_0 \in [\alpha_1, \beta_1]$. Then we have

$$\begin{aligned}
&\lim_{\epsilon \searrow 0} \eta_{ap}(s_0 + i\epsilon) \\
&= \lim_{\epsilon \searrow 0} \frac{-2 + \phi(a) + \phi(p) + s_0 + i\epsilon + \sqrt{(s_0 + i\epsilon - \alpha_1)(s_0 + i\epsilon - \beta_1)}}{\phi(a) + \phi(p) - s_0 - i\epsilon + \sqrt{(s_0 + i\epsilon - \alpha_1)(s_0 + i\epsilon - \beta_1)}} \\
&= \lim_{\epsilon \searrow 0} \frac{-2 + \phi(a) + \phi(p) + s_0 + i\epsilon + \sqrt{-(\epsilon^2 + (s - \alpha_1)(\beta_1 - s)) - i\epsilon(2s - \alpha_1 - \beta_1)}}{\phi(a) + \phi(p) - s_0 - i\epsilon + \sqrt{-(\epsilon^2 + (s - \alpha_1)(\beta_1 - s)) - i\epsilon(2s - \alpha_1 - \beta_1)}} \\
&= \frac{-2 + \phi(a) + \phi(p) + s_0 + i\sqrt{(s_0 - \alpha_1)(\beta_1 - s_0)}}{\phi(a) + \phi(p) - s_0 + i\sqrt{(s_0 - \alpha_1)(\beta_1 - s_0)}} \\
&= \frac{s_0 - \phi(a) - \phi(p) + 2\phi(a)\phi(p) + i\sqrt{(s_0 - \alpha_1)(\beta_1 - s_0)}}{2\phi(a)\phi(p)}
\end{aligned} \tag{4.17}$$

For the third equality in (4.17), when ϵ is large and positive, the branch of a square root is specified by the approximation $\sqrt{-(\epsilon^2 + (s_0 - \alpha_1)(\beta_1 - s_0)) - i\epsilon(2s_0 - \alpha_1 - \beta_1)} \approx \left(-s_0 + \frac{\alpha_1 + \beta_1}{2}\right) + i\epsilon$. Then we have $\lim_{\epsilon \searrow 0} \sqrt{(s_0 + i\epsilon - \alpha_1)(s_0 + i\epsilon - \beta_1)} = i\sqrt{(s_0 - \alpha_1)(\beta_1 - s_0)}$. $\square$

For $s \in [\alpha_1, \beta_1]$ and $t \in [\alpha_2, \beta_2]$, denote the limits of $\eta_{ap}(s + i\epsilon)$ and $\eta_{bq}(t + i\epsilon)$ as $\epsilon > 0$ goes to 0 by

$$f_{ap}(s) := \frac{s - \phi(a) - \phi(p) + 2\phi(a)\phi(p) + i\sqrt{(s - \alpha_1)(\beta_1 - s)}}{2\phi(a)\phi(p)} \tag{4.18}$$

$$f_{bq}(t) := \frac{t - \phi(b) - \phi(q) + 2\phi(b)\phi(q) + i\sqrt{(t - \alpha_2)(\beta_2 - t)}}{2\phi(b)\phi(q)}. \tag{4.19}$$

Then $f_{ap}(s) \in \sqrt{\frac{(1-\phi(a))(1-\phi(p))}{\phi(a)\phi(p)}} \cdot \mathbb{T}$ and $f_{bq}(t) \in \sqrt{\frac{(1-\phi(b))(1-\phi(q))}{\phi(b)\phi(q)}} \cdot \mathbb{T}$ by Lemma 4.2.1 and Lemma 4.2.7, where $\mathbb{T}$ is the unit circle on $\mathbb{C}$.

Lemma 4.2.8.

$$\mu_{11} \ = \ c_1 \cdot \delta_{(0,0)} + (\sigma_1 \times \delta_0) + (\delta_0 \times \zeta_1) + \omega_1$$

where c_1 is a non-negative real number, the measures σ_1 and ζ_1 are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ whose supports are $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$, respectively, and ω_1 is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$ whose support is equal to $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$.

Note that formulas for c_1 and the densities of the measures σ_1, ζ_1, and ω_1 are given in the proof below.

Proof. By the equation (4.6) and Lemma 4.2.2, we have

$$\begin{aligned} H_{11}(z, w) \ &= \ (1-l)\left(G_{ap}(z) - \frac{1-\phi(p)}{z}\right)\left(G_{bq}(w) - \frac{1-\phi(q)}{w}\right) \cdot \frac{\eta_{ap}(z) \cdot \eta_{bq}(w) - k}{\eta_{ap}(z) \cdot \eta_{bq}(w) - kl} \\ &= \ \iint_{[0,1]^2} \frac{1}{(z-s)(w-t)} \, d\mu_{11}(s,t) \end{aligned}$$

for $z, w \in \mathbb{C} \setminus [0,1]$. Let θ be the pushforward of the measure μ_{11} under the coordinate projection onto the first coordinate. By the disintegration theorem, there exists a family, $(\nu_s)_{s \in [0,1]}$, of probability measures on $[0,1]$ such that

$$\iint_{[0,1]^2} \frac{1}{(z-s)(w-t)} \ d\mu_{11}(s,t) \ = \ \int_{[0,1]} \frac{1}{z-s} \left(\int_{[0,1]} \frac{1}{w-t} \, d\nu_s(t)\right) d\theta(s).$$

For each $s \in [0,1]$, let

$$h(s, w) \ := \ \int_{[0,1]} \frac{1}{w-t} \, d\nu_s(t), \tag{4.20}$$

so that by definition we have

$$\lim_{|w| \to \infty} w \cdot h(s, w) = 1. \tag{4.21}$$

Fix $w \in \mathbb{R} \setminus [0,1]$. Let γ_w denote the measure satisfying $\frac{d\gamma_w}{d\theta}(s) = h(s,w)$, which implies that

$$\int_{[0,1]} \frac{1}{z-s}\, d\gamma_w(s) \;=\; H_{11}(z,w). \tag{4.22}$$

Recall that $G_{ap}(z) - \frac{1-\phi(p)}{z} = \frac{z(2\phi(p)-1)+\phi(a)-\phi(p)+\sqrt{(z-\alpha_1)(z-\beta_1)}}{2(z-1)z}$. Given the equality (4.22), it follows that γ_w has point masses where H_{11} has simple poles and the mass equals the residue there, and γ_w is absolutely continuous with respect to Lebesgue measure where $H_{11}(\cdot,w)$ has nonzero imaginary part on the real axis. Note that $\lim_{z\to 1}\sqrt{(z-\alpha_1)(z-\beta_1)} = \sqrt{(1-\alpha_1)(1-\beta_1)} = 1-\phi(a)-\phi(p)$ and $\lim_{z\to 0}\sqrt{(z-\alpha_1)(z-\beta_1)} = -\sqrt{\alpha_1\beta_1} = \phi(a)-\phi(p)$. Since $\lim_{z\to 1}(z-1)\cdot H_{11}(z,w) = 0$, $H_{11}(\cdot,w)$ has a removable singularity at $z=1$ so that the measure γ_w has no atom at $s=1$. Since H_{11} has a simple pole at $z=0$ with residue

$$\lim_{z\to 0} z\cdot H_{11}(z,w) \;=\; (1-l)(\phi(p)-\phi(a))\cdot\left(G_{bq}(w) - \frac{1-\phi(q)}{w}\right)\cdot\frac{\left(1-\frac{1}{\phi(a)}\right)\eta_{bq}(w)-k}{\left(1-\frac{1}{\phi(a)}\right)\eta_{bq}(w)-kl},$$

the measure γ_w has a point mass equal to the above residue at $s=0$. Since $|\eta_{bq}(w)|$ goes to infinity as $|w|\to\infty$, the equality (4.21) implies that

$$\theta(\{0\}) \;=\; \lim_{|w|\to\infty} w\cdot h(0,w)\theta(\{0\}) \;=\; (1-l)(\phi(p)-\phi(a))\phi(q), \tag{4.23}$$

and hence

$$h(0,w) \;=\; \frac{1}{\phi(q)}\cdot\left(G_{bq} - \frac{1-\phi(q)}{w}\right)\cdot\frac{\left(1-\frac{1}{\phi(a)}\right)\eta_{bq}(w)-k}{\left(1-\frac{1}{\phi(a)}\right)\eta_{bq}(w)-kl}. \tag{4.24}$$

For $s \in [\alpha_1,\beta_1]$, applying the Stieltjes inversion formula to (4.22), we have $\frac{d\gamma_w}{d\lambda^{(1)}}(s) = -\frac{1}{\pi}\cdot \lim_{\epsilon\searrow 0}\operatorname{Im} H_{11}(s+i\epsilon,w) = \frac{(1-l)\sqrt{(s-\alpha_1)(\beta_1-s)}}{2\pi(1-s)s}\cdot\left(G_{bq}(w)-\frac{1-\phi(q)}{w}\right)\cdot\left(1-\frac{f_{ap}(s)\eta_{bq}(w)-k}{f_{ap}(s)\eta_{bq}(w)-kl}\right)$. It follows from (4.21) that

$$\frac{d\theta}{d\lambda^{(1)}}(s) \;=\; \lim_{|w|\to\infty} w\cdot\frac{d\gamma_w}{d\lambda^{(1)}}(s) \;=\; \frac{\phi(q)(1-l)\sqrt{(s-\alpha_1)(\beta_1-s)}}{2\pi(1-s)s} \tag{4.25}$$

and

$$h(s,w) \;=\; \frac{1}{\phi(q)}\left(G_{bq}(w)-\frac{1-\phi(q)}{w}\right)\left(1-\frac{f_{ap}(s)\eta_{bq}(w)-k}{f_{ap}(s)\eta_{bq}(w)-kl}\right). \tag{4.26}$$

Both equalities (4.24) and (4.26) are derived for $w \in \mathbb{R} \setminus [0,1]$. However the expressions on the right-hand sides of the equalities are holomorphic functions for $w \in \mathbb{C} \setminus [0,1]$, and as defined in (4.20), the function $h(s,\cdot)$ is holomorphic on $\mathbb{C}\setminus[0,1]$ for every $s \in [0,1]$. By analytic continuation, we have

$$\int_{[0,1]} \frac{1}{w-t}\, d\nu_0(t) \;=\; h(0,w) \;=\; \frac{1}{\phi(q)} \cdot \left(G_{bq}-\frac{1-\phi(q)}{w}\right) \cdot \frac{\left(1-\frac{1}{\phi(a)}\right)\eta_{bq}(w)-k}{\left(1-\frac{1}{\phi(a)}\right)\eta_{bq}(w)-kl} \tag{4.27}$$

and

$$\begin{aligned} &\int_{[0,1]} \frac{1}{w-t}\, d\nu_s(t) \;=\; h(s,w) \\ &=\; \frac{1}{\phi(q)}\left(G_{bq}(w)-\frac{1-\phi(q)}{w}\right)\left(1-\frac{k(1-l)\left(\eta_{bq}(w)\left(1-\frac{1}{\phi(p)}\right)\left(1-\frac{s}{\phi(a)}\right)-kl\right)}{\left(f_{ap}(s)\eta_{bq}(w)-kl\right)\left(\overline{f_{ap}(s)}\eta_{bq}(w)-kl\right)}\right) \end{aligned} \tag{4.28}$$

where $w \in \mathbb{C} \setminus [0,1]$ and $s \in [\alpha_1, \beta_1]$. Consider the equality (4.27) first. Notice that $h(0,w)$ has a removable singularity at $w=1$, and it has a simple pole at $w=0$ with residue

$$\lim_{w\to 0} w \cdot h(0,w) \;=\; \frac{\phi(q)-\phi(b)}{\phi(q)} \cdot \frac{\left(1-\frac{1}{\phi(a)}\right)\left(1-\frac{1}{\phi(b)}\right)-k}{\left(1-\frac{1}{\phi(a)}\right)\left(1-\frac{1}{\phi(b)}\right)-kl} \tag{4.29}$$

which is equal to the point mass, $\nu_0(\{0\})$, of ν_0 at $t=0$. By the Stieltjes inversion formula, we have

$$\begin{aligned} \frac{d\nu_0}{d\lambda^{(1)}}(t) \;&=\; -\frac{1}{\pi} \lim_{\epsilon\searrow 0} \operatorname{Im} h(0,t+i\epsilon) \\ &=\; \frac{\sqrt{(t-\alpha_2)(\beta_2-t)}}{2\pi\phi(q)(1-t)t} \cdot \left(1-\frac{k(1-l)\left(\left(1-\frac{1}{\phi(a)}\right)\left(1-\frac{1}{\phi(q)}\right)\left(1-\frac{t}{\phi(b)}\right)-kl\right)}{\left|\left(1-\frac{1}{\phi(a)}\right)f_{bq}(t)-kl\right|^2}\right) \end{aligned} \tag{4.30}$$

for $t \in [\alpha_2, \beta_2]$. Then, $\nu_0 = \nu_0(\{0\}) \cdot \delta_{(0,0)} + \nu_0|_{[\alpha_2,\beta_2]}$, where $\nu_0|_{[\alpha_2,\beta_2]}$ is absolutely continuous with respect to Lebesgue measure having a support equal to $[\alpha_2, \beta_2]$. Now consider (4.28). Notice that $h(s,w)$ has a removable singularity at $w = 1$, and it has a simple pole at $w = 0$ with residue

$$\lim_{w\to 0} w \cdot h(s,w) \;=\; \frac{\phi(q)-\phi(b)}{\phi(q)} \cdot \left(1 - \frac{k(1-l)\left(\left(1-\frac{s}{\phi(a)}\right)\left(1-\frac{1}{\phi(b)}\right)\left(1-\frac{1}{\phi(p)}\right)-kl\right)}{\left|\left(1-\frac{1}{\phi(b)}\right)f_{ap}(s)-kl\right|^2}\right)$$

which is equal to the point mass, $\nu_s(\{0\})$, of ν_s at $t = 0$. For $t \in [\alpha_2, \beta_2]$,

$$\begin{aligned}\frac{d\nu_s}{d\lambda^{(1)}}(t) \;&=\; -\frac{1}{\pi}\lim_{\epsilon\searrow 0} \operatorname{Im} h(s, t+i\epsilon) \\ &=\; \frac{\sqrt{(t-\alpha_2)(\beta_2-t)}}{2\phi(q)\pi(1-t)t} \cdot \left(1 + \frac{k(1-l)\cdot L_1(s,t)}{\left|(f_{ap}(s)f_{bq}(t)-kl)\left(\overline{f_{ap}(s)}f_{bq}(t)-kl\right)\right|^2}\right),\end{aligned}$$

where

$$\begin{aligned}L_1(s,t) \;=\; kl&\left(\frac{s\cdot d_1(s)}{\phi(a)\phi(p)}\left(1-\frac{1}{\phi(q)}\right)+\frac{t\cdot d_2(t)}{\phi(b)\phi(q)}\left(1-\frac{1}{\phi(p)}\right)-\frac{st}{\phi(a)\phi(b)}\right) \\ &-\;\left(\left(1-\frac{1}{\phi(p)}\right)\left(1-\frac{1}{\phi(q)}\right)-kl\right)d_1(s)d_2(t) \quad (4.31)\end{aligned}$$

and

$$d_1(s) = \left(1-\frac{s}{\phi(a)}\right)\left(1-\frac{1}{\phi(b)}\right)\left(1-\frac{1}{\phi(p)}\right)-kl \quad (4.32)$$

$$d_2(t) = \left(1-\frac{1}{\phi(a)}\right)\left(1-\frac{t}{\phi(b)}\right)\left(1-\frac{1}{\phi(q)}\right)-kl. \quad (4.33)$$

Then we have $\nu_s = \nu_s(\{0\}) \cdot \delta_{(s,0)} + \nu_s|_{[\alpha_2,\beta_2]}$, where $\nu_s|_{[\alpha_2,\beta_2]}$ is a measure, absolutely continuous with respect to Lebesgue measure with its support equal to $[\alpha_2, \beta_2]$. Then we can conclude that

$$\mu_{11} = c_1 \cdot \delta_{(0,0)} + (\sigma_1 \times \delta_0) + (\delta_0 \times \zeta_1) + \omega_1,$$

where

$$
\begin{aligned}
c_1 &= \theta(\{0\}) \cdot \nu_0(\{0\}) \\
&= (1-l)(\phi(p)-\phi(a))(\phi(q)-\phi(b)) \cdot \frac{\left(1-\frac{1}{\phi(a)}\right)\left(1-\frac{1}{\phi(b)}\right)-k}{\left(1-\frac{1}{\phi(a)}\right)\left(1-\frac{1}{\phi(b)}\right)-kl}, \\
\frac{d\sigma_1}{d\lambda^{(1)}}(s) &= \frac{d\theta}{d\lambda^{(1)}}(s) \cdot \nu_s(\{0\}) \\
&= (1-l)(\phi(q)-\phi(b)) \cdot 1_{[\alpha_1,\beta_1]}(s) \\
&\qquad \cdot \frac{\sqrt{(s-\alpha_1)(\beta_1-s)}}{2\pi(1-s)s} \cdot \left(1 - \frac{k(1-l)d_1(s)}{\left|\left(1-\frac{1}{\phi(b)}\right) f_{ap}(s) - kl\right|^2}\right), \\
\frac{d\zeta_1}{d\lambda^{(1)}}(t) &= \theta(\{0\}) \cdot \frac{d\nu_0}{d\lambda^{(1)}}(t) \\
&= (1-l)(\phi(p)-\phi(a)) \cdot 1_{[\alpha_2,\beta_2]}(t) \\
&\qquad \cdot \frac{\sqrt{(t-\alpha_2)(\beta_2-t)}}{2\pi(1-t)t} \cdot \left(1 - \frac{k(1-l)d_2(t)}{\left|\left(1-\frac{1}{\phi(a)}\right) f_{bq}(t) - kl\right|^2}\right), \\
\frac{d\omega_1}{d\lambda^{(2)}}(s,t) &= \frac{d\theta}{d\lambda^{(1)}}(s) \cdot \frac{d\nu_s}{d\lambda^{(1)}}(t) \\
&= \frac{(1-l)\cdot 1_{[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]}(s,t)}{4\pi^2(1-s)s(1-t)t} \cdot \left(1 + \frac{k(1-l)L_1(s,t)}{\left|(f_{ap}(s)f_{bq}(t)-kl)\left(\overline{f_{ap}(s)}f_{bq}(t)-kl\right)\right|^2}\right).
\end{aligned}
$$

Note that L_1, d_1, and d_2 are defined in (4.31) - (4.33). □

Similar techniques will apply to describe the other measures μ_{ij} for $1 \leq i, j \leq 4$, and these can be used to identify the reduced bi-free product C*-algebra.

5. SUMMARY

We have described the Pincus principal function of a certain operator arising from a bi-free central limit distribution. We have also shown the relations between the ordered joint moment and cumulant series for some combinations of bi-free two-faced pairs of random variables in a non-commutative probability space. Using these relations, we have discussed how to derive the reduced bi-free product C*-algebra generated by two bi-freely independent pairs of commuting projections in the generic case.

Further work in this direction could show whether the bi-free state on the image of GNS representation of the free product C*-algebra is faithful or not. Another interesting extension might be to find the bi-free product C*-algebra for two-faced pairs of non-commuting projections and their bi-free state.

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