ON CONSISTENT KALUZA-KLEIN-PAULI REDUCTIONS

A Dissertation

by

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ABSTRACT

Kaluza-Klein dimensional reduction is an indispensable ingredient of the theoretical physics, since, M-theory and superstring theories are consistent in eleven and ten dimensions and thus, to make a connection to our four-dimensional space-time physics, it is crucial to use this mechanism. Dimensional reduction on a general coset space such as a sphere, introduced by Pauli, is more subtle that that of a group manifold reduction, introduced by DeWitt, including the circle reduction of Kaluza and Klein. While there is a group-theoretic argument for the consistency of the latter, there is no such an argument for the former, hence, besides the exceptional cases, all Pauli reductions may be inconsistent.

We study an uplift ansatz for two specific truncations of gauged STU supergravity. This theory itself is an important truncation of the renowned $N = 8$, gauged $SO(8)$ supergravity in four dimensions. We consider two truncations of the former theory, named as $3+1$ and $2+2$, due to the way of truncations of their gauge fields. We find the uplift ansätze for the metric and the four-form field strength in these cases.

We consider two theories and explore the possibility of their consistent Pauli S^2 reductions. First, minimal supergravity in five dimensions, and second, the Salam-Sezgin theory. We use the Hopf reduction technique in both cases, and by that, we show while it is not possible to perform a consistent reduction of the former, there is a consistent Pauli reduction of the latter, and by this construction, we can recover the result of Gibbons-Pope in 2003. In other words, we can provide a group theoretical argument for their work. To make the latter case happen, we find a new higher dimensional origin for the Salam-Sezgin theory, at least in the bosonic sector.

DEDICATION

To my mother, my father,

To Sophia,

To Sahar.

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1. INTRODUCTION

1.1 Historical remarks

One of the most important and deepest unanswered questions in the history of science is finding a unique scheme where all forces of nature can be combined to the elegant theory with a beautiful underlying mathematical structure. The quest for such a “Final Theory” has been initiated mostly by Albert Einstein, especially when he eventually obtained the final form of general relativity in November 1915. Right after that, he started a long journey of investigation to find a unification of the gravity in form of his general relativity and Maxwell’s electromagnetism, the only known forces at that time. The exploration for such a theory has not been successful during his life time. However, several brilliant ideas emerged either by himself or by other physicists.

One of the most striking ideas aiming a unification of all forces introduced by Nordström [1] in 1914, even one year before the completion of general relativity, known as the dimensional reduction. However this work had been ignored by community for long time. Five years later, Theodor Kaluza, inspired by a work of Hermann Weyl in 1918, found the same scheme. The main idea of the dimensional reduction is as follows. Considering pure gravity in five space-time dimensions and assuming the fifth dimension is compactified on a small circle, one can obtain gravity and electromagnetism along with a scalar field and an infinite tower of massive fields in four dimensions. A natural unification of gravity and electromagnetism has been achieved! Kaluza sent the draft of his paper to Einstein, and while Einstein found out his theory “startling”, he did not submit the paper to the Prussian Academy for two years [2].

Nordström had developed his own version of gravity, know as scalar gravity, and hence, his dimensional reduction is different with that of Kaluza. He wrote the higher dimensional Maxwell vector potential as a combination of the four dimensional vector potential, corresponding to electromagnetic vector potential, and a scalar field describes his scalar gravity. It is remarkable that he introduced exactly the same ansatz as it is written today for a vector field. Therefore, his five-dimensional Lagrangian is a pure Maxwell one and after dimensional reduction he obtained four-dimensional Maxwell electromagnetism and a kinetic scalar term for "gravity".

Kaluza in his published work [3] in 1921, like Nordström assumed "cylinder condition" , meaning that all fields of the four-dimensional theory are independent of the fifth dimension [2]. In other words, he implicitly discarded an infinite tower of massive fields. The crucial issue of the *consistency* then shall be arisen here, i.e. whether a truncation of massive fields is consistent or not. As we will see later, in the special case of the circle reduction, the case where they studied, this consistency is guaranteed by a simple group-theoretical argument, however, for most of "non-trivial" dimensional reductions, such a simple test does not exist. Moreover, Kaluza considered just the linearized level of equations of motion, therefore all of the complications may arise as a result of considering the highly non-linear nature of the Einstein equations, shall not be addressed by this consideration.

Klein in 1926 [4] improved the program by considering the full non-linear theory, and by the assumption of the cylinder condition, he could find out pure Einstein gravity in five dimensions gives rise to Einstein gravity, Maxwell electromagnetism and a scalar in four dimensions. Of course, the latter field is not a desirable choice for him since there was no known scalar field at that time. Therefore, he assumed the g_{55} component of the metric which is proportional to the scalar field, is constant, and hence the scalar field can be discarded. If just the Lagrangian would be considered, there is no inconsistency in discarding the scalar field has been arisen. However, the modern criterion for the consistency of the reduction ansatz, pioneered by the work of Duff, Nilsson, Pope and Warner in 1984 [5], is based on the equations of motion rather than the Lagrangian.

A consistent dimensional reduction is defined as follows. An initial $(D+n)$-dimensional theory can be compactified on a compact n-dimensional internal manifold. One can find a Fourier expansion of the D-dimensional theory in terms of the internal manifold harmonics, however, retaining all of an infinite tower of fields is not essentially interested. Henceforth, one needs to truncate an infinite number of fields and retain just finite number of them, named *ansatz*. Especially, one needs to keep the bosonic fields which express the isometry group of the internal manifold. The criterion for the consistency of an ansatz is as follows. Upon inserting the ansatz in the higher dimensional equations of motion, due to the conspiracies between several fields, the ones depend upon the internal manifold cancel out of the equations and the remaining equations shall be those of the D-dimensional space-time. Specifically, the retaining fields should not be the sources for discarding ones. In that sense, discarding the scalar field, as Klein considered, is not consistent, since vanishing R_{55} in five dimensions implies the Maxwell field strength is a source term for the scalar field and discarding the latter should be accompanied by discarding the former, i.e. no more unification shall be achieved.

Now, the question is, finding a general mechanism to construct the consistent reductions. This is, in general, a very non-trivial question. In fact, besides reductions which their consistency is guaranteed by a group-theoretic argument, there is no known litmus test to check the consistency of a reduction. The only way is calculating the higher dimensional equations of motion as we have just emphasized.

1.2 DeWitt and Pauli reductions

One may classify general dimensional reductions in two categories of DeWitt (group manifold) and Pauli (coset) reductions. The former was studied first by Bryce DeWitt in 1963 [6] and then named DeWitt reduction in [7]. The internal space in this case is a group manifold. Then one can construct fields which are invariant under the left (or right) action of the group. In other words, one can easily show $U^{-1}\, dU$, where U is an element of the group manifold, is invariant under the transformation $U \to GU$, where G is a global rigid element of the group. Therefore, $U^{-1}\, dU$ is invariant under the left action of the group, or the latter is a singlet under the left action of the group G_L. Then if one writes $U^{-1}\, dU = \sigma_a\, T^a$, where T^a are the generators of the algebra, hence σ_a may be labeled as left-invariant one-forms. Similarly, $dU\, U^{-1}$ is invariant under the right action of the group, i.e. $U \to UG$, and it is a singlet under G_R

Having said the above, let us consider the group manifold reduction. Assume an ansatz is written in terms of the left-invariant fields (or equivalently the right-invariant ones). In other words, one may retain the singlets under the left action of the isometry group, G_L and discard all other fields. Then, one can divide all higher dimensional equations of motion, which are written in terms of the lower dimensional fields, into two parts: a part includes all terms which are singlets of the group G_L, and the other part involves non-singlet ones. Since the product of all singlets is again a singlet of the group, by truncating all non-singlet fields to zero, all equations of the latter part actually are trivially satisfied. There is no danger for surviving fields (i.e. singlets under G_L) to be sources for vanishing ones (i.e. non-singlets under G_L). This is why the consistency of a reduction on a group manifold G is guaranteed by a group-theoretic argument. Note that since a circle S^1 or more general a torus T^n are both group manifolds ($U(1)$ and $U(1)^n$ respectively), then the consistency of Kaluza and Klein original example of a circle S^1, or a torus T^n reductions are guaranteed by a group-theoretic argument.

The first coset reduction was investigated by Wolfgang Pauli in an unpublished work in 1953 [2]. He unsuccessfully, attempted to find an S^2 reduction of pure Einstein gravity in six dimensions, while keeping the non-Abelian $SO(3)$ Yang-Mills fields. It is clear now, to obtain a consistent S^2 reduction of six-dimensional theory, one needs to start from a supergravity theory, instead of a pure gravity which Pauli considered. The coset reduction was named Pauli reduction in [7].

In contrast to the DeWitt case, there is no group-theoretical argument applicable to the Pauli reduction. As a matter of fact, almost all Pauli reductions are *inconsistent*, however there are exceptional cases where the Pauli reductions are consistent. Hence, the important question becomes finding a deeper understanding of why such "miraculous" reductions exist. The main examples of consistent Pauli reductions include of the renowned S^7 reduction of eleven-dimensional supergravity of deWit and Nicolai [11] and later [12], S^4 reduction of eleven-dimensional supergravity [13], [14] and [15], the Pauli consistent reduction of type IIB supergravity on S^5 in [16], and the Pauli reduction of the bosonic string on a group manifold G where the the full isometry group of $G \times G$ is retained in [17].

One may ask since a DeWitt reduction can be constructed in an algorithmic method, why one has to consider the Pauli reductions. First of all, since the isometry group of the bosonic gauge fields of the lower-dimensional theory is the same as that of the internal manifold, hence to obtain a specific isometry group, a Pauli reduction needs a higher dimensional theory with smaller dimensions in compare with a DeWitt one. In other words, assuming the lower d-dimensional theory has a gauge group of G, in the higher D-dimensional theory, $D = d + \dim G - \dim H$ in the Pauli reduction on the coset G/H, whereas $D = d + \dim G$ in the DeWitt reduction on the group manifold G. Hence this is somewhat more "economical" way of obtaining the Yang-Mills in the lower-dimensional theory. Secondly, the mere existence of these reductions motivates us to investigate their underlying mathematical structure and it may lead us to new aspects, such as the generalized geometry, which will be addressed briefly later.

1.3 Three revivals of the dimensional reduction

The dimensional reduction had not played a central role in the theoretical physics after its birth in 1910s until 1970s. However, it has been revived in 1970's and 1980's because of the constructions of supergravities in dimensions higher than four and especially the fact that superstring theories are consistent in ten space-time dimensions. Then to connect these theories to our four-dimensional space-time, the dimensional reduction is the best tool and technique. Therefore, in early 1980s, this subject was extensively studied and interesting results were obtained [5, 11, 18–22]. This is the first revival of the program.

The second revival of the dimensional reduction started in late 1997 when Juan Maldacena made a conjecture about a correspondence between a gravity theory in an anti-de Sitter space and a conformal field theory on its boundary [23–25]. The prominent example of this correspondence is type IIB superstring theory on $AdS_5 \times S^5$ and $N = 4$ super Yang-Mills on its four dimensional boundary. Motivated by this example, there was a notable amount of researches conducted to understand the non-linear structure of a reduction of a higher dimensional theory on a general coset space like sphere [27–30]. For instance, the Pauli S^4 reduction of eleven-dimensional supergravity was constructed in [13, 14].

The third revival of this program has been started in past few years. There has been an intense research program known as generalized geometry, where it aims to understand the different dualities of string theory better. This aim can be achieved by assuming the internal manifold has the same dimensions as number of the adjoint representation of the underlying symmetry group of the theory. Furthermore, by imposing the section constraint on the theory, it makes the hidden symmetries of a theory manifest. Within this framework, there is a generalized Scherk-Schwarz [19] mechanism, where it is possible to have a systematic way of construction of the Kaluza-Klein ansatz. [31,32] However, there are two issues related to this program. Firstly, the ansatz it is found by this method is not very practical and one needs to find out a more feasible ansatz, and more importantly, it is not obvious how to incorporate the fermions in this program. Therefore, without inclusion of the fermions, the Kaluza-Klein ansatz will not address the crucial concept of the supersymmetry. There are several examples of consistent Kaluza-Klein construction of a bosonic sector of a theory, however, the construction becomes inconsistent after inclusion of the fermions. In fact, in our alternative M-theory origin of the Salam-Sezgin theory which will be addressed shortly, one observes that while it is somewhat an easy task to construct the bosonic truncations, it is very difficult to find a consistent truncations of the fermionic fields.

1.4 Dissertation outline

This dissertation organizes as follows. In chapter 2, we review the Kaluza-Klein theory by a toy example of the Klein-Gordon scalar field. Also, a circle reduction shall be addressed in this chapter. The bosonic Lagrangian and equations of motion will be discussed and it will be shown that discarding the dilaton field results from the circle reduction is inconsistent with retaining the Maxwell field.

In chapter 3, we will find the embedding of two specific truncations of the STU supergravity in eleven dimensions. The latter theory is a maximal Abelian (i.e. $U(1)^4$) sub-group of the renowned four-dimensional $N = 8$, $SO(8)$ gauge supergravity, and has an essential role in study of black holes in four dimensions. We will study two different truncations of this theory, i.e. $3 + 1$ and $2 + 2$, where besides a graviton, two gauge fields and a dilaton and an axion survive in these two scenarios. We will find out the embedding eleven-dimensional metric and four-form field strength in the above mentioned truncations.

In chapter 4, we will raise a question of the possibility of consistent Pauli S^2 reduction of minimal five-dimensional supergravity. The motivation for this study comes from the similarity between the latter theory's bosonic Lagrangian and that of eleven-dimensional supergravity. Hence, since there are consistent Pauli S^4 and S^7 reductions of eleven-dimensional supergravity, one may conjecture a consistent Pauli S^2 reduction of five-dimensional minimal supergravity exists. We will show, using the Hopf reduction technique, there is no such a consistent Pauli reduction in this case.

The Einstein-Maxwell $N = (1,0)$ six-dimensional theory, known as the Salam-Sezgin theory [36], will be subject of two chapters 5 and 6. The latter theory has a supersymmetric four-dimensional $Minkowski_4 \times S^2$ vacuum solution, then one may wonder about the possibility of consistent S^2 Pauli reduction of it. This reduction found in 2003 by Gibbons and Pope [38], however, one does not have an understanding of why this reduction works. Using the Hopf reduction technique, we have been able to find a group-theoretic understanding of this reduction. For this purpose, the Salam-Sezgin theory should be derived from a seven-dimensional theory. This has been done in [39], however, the Kaluza-Klein vector potential results from reduction of the metric has been set to zero in that work, meaning that the Dirac monopole on S^2 has been vanishing, thus one cannot recover S^3 as a $U(1)$ Hopf fibration over S^2 and the the Hopf fibration technique fails. Therefore, one needs to find an alternative origin for the Salam-Sezgin theory. We will present an alternative reduction in chapter 5. The seven-dimensional theory is an $N = 2$, $SO(4)$ supergravity with some exotic signs in the bosonic Lagrangian. Although the supersymmetry transformations maintain and the entire Lagrangian is invariant under them, but, the reality condition of this theory is somewhat obscured. We expect , however, this problem will be resolved and there will be an "exotic" half maximal supergravity with an $SO(4)$ gauging in seven dimensions. We will not present our calculation about the fermionic truncation which yields to the Salam-Sezgin theory.

In chapter 6, we will show the possibility of obtaining the bosonic sector of the Salam-Sezgin theory from a half maximal supergravity with an $SO(2,2)$ gauging in seven dimensions, where the Kaluza-Klein vector potential has an active role. Therefore, we will be able to follow the Hopf fibration technique to recover the Gibbons-Pope result of the Pauli S^2 reduction of the Salam-

Sezgin theory.

Finally, we will conclude in chapter 7.

2. REVIEW OF THE KALUZA-KLEIN REDUCTION

In this chapter, we first introduce the concept of the dimensional reduction by considering a scalar field satisfying the Klein-Gordon equation in a higher dimension and then, by performing a circle reduction, we find out an infinite tower of massive fields in a lower dimension.

Furthermore, we study the metric ansazt and the gauge potential ansatz result from the Kaluza-Klein circle reduction starting from pure gravity in a higher dimension.

2.1 The Klein-Gordon case

Consider a massless scalar field ϕ in $D+1$ dimensions. One may divide the coordinates to a D-dimensional space-time, denoted by x, and a component, say z, which will be compactified on a circle with a radius of L . The Klein-Gordon equation in $D+1$ dimensions reads

$$\hat{\Box}\hat{\phi}(x,z) = 0\,, \tag{2.1}$$

where we put hat on the higher dimensional quantities to distinguish them from the lower dimensional ones. Now, one may perform a Fourier transformation along the z component and write

$$\hat{\phi}(x,z) = \sum_{n\in\mathbb{Z}} \phi_n(x)\, e^{\frac{inz}{L}}\,. \tag{2.2}$$

Therefore, assuming the metric is flat, the massless Klein-Gordon equation implies

$$\hat{\Box}\hat{\phi}(x,z) = \sum_{n\in\mathbb{Z}} e^{\frac{inz}{L}}\,(\Box\phi_n(x) - \tfrac{n^2}{L^2}\,\phi_n(x)) = 0\,. \tag{2.3}$$

Since these modes are linearly independent, then one can conclude

$$\Box\phi_n(x) - \tfrac{n^2}{L^2}\,\phi_n(x) = 0\,. \tag{2.4}$$

In other words, in the lower dimension, one has an infinite number of scalar fields which all of them are massive with mass of $\frac{n^2}{L^2}$ except the $n = 0$ case, i.e. the massless mode.

This is a common feature of the Kaluza-Klein reduction. One has to truncate an infinite tower of massive fields, however, in a simple case we have seen above, it can be done very easily, but in general, one has to consider a consistent truncation. As we have emphasized in chapter 1, in case of a DeWitt reduction, one can perform a consistent truncation, but, in case of a Pauli reduction, the problem becomes overwhelmingly harder.

2.2 The Kaluza-Klein circle reduction

In this section, we consider a circle Kaluza-Klein reduction, and we present its metric ansatz, and find out the spin connection components. Also, we present the reduction ansazt for a p-form vector potential. One may put hat on the higher dimensional fields, to make them distinguishable from the lower dimensional ones.

2.2.1 The metric ansatz

Assume the higher dimensional theory is Einstein gravity, i.e. $\hat{\mathcal{L}}_{D+1} = \hat{R}\hat{*}\mathbb{1}$. The standard metric ansatz reads [26]

$$d\hat{s}^2_{D+1} = e^{2\alpha\varphi}\, ds^2_D + e^{2\beta\varphi}\,(dz + \mathcal{A}_{(1)})^2\,, \tag{2.5}$$

where φ is a “breathing mode”. The constants α and β will be determined later. Using the vielbein formalism, the obvious choice is

$$\hat{e}^a = e^{\alpha\varphi}\, e^a, \qquad \hat{e}^z = e^{\beta\varphi}\,(dz + \mathcal{A}_{(1)})\,. \tag{2.6}$$

The next step is finding the spin connection. One may derive it by using $d\hat{e}^A = -\hat{\omega}^A{}_B \wedge \hat{e}^B$, where the torsion has been assumed to be vanishing. Here $M, N, P, ...$ $(A, B, C, ...)$ denote the curved (flat) higher dimensional indices respectively. While, in the lower dimension, we use $\mu, \nu, \rho, ..., \zeta$ $(a, b, c, ..., z)$ for the curved (flat) indices respectively. Let the spin connection components be the following

$$\hat{\omega}_{ABC} = \hat{\omega}_{A[BC]}, \qquad \hat{\omega}_{BC} = \hat{\omega}_{ABC}\,\hat{e}^{A}, \qquad \hat{\omega}_{C} = \eta^{AB}\,\hat{\omega}_{ABC}. \tag{2.7}$$

Then the components of the higher dimensional spin connection read

$$\begin{aligned} &\hat{\omega}_{abc} = e^{-\alpha\varphi}\left(\omega_{abc} + 2\alpha\,\eta_{a[b}\,\partial_{c]}\,\varphi\right), \qquad \hat{\omega}_{abz} = -\hat{\omega}_{azb} = -\hat{\omega}_{zab} = \tfrac{1}{2}e^{(\beta-2\alpha)\varphi}\,\mathcal{F}_{ab}\,, \\ &\hat{\omega}_{zaz} = -\beta\,e^{-\alpha\varphi}\,\partial_a\varphi\,, \qquad \hat{\omega}_{a} = e^{-\alpha\varphi}\left(\omega_a + (\alpha(D-1)+\beta)\,\partial_a\varphi\right), \qquad \hat{\omega}_z = 0\,. \end{aligned} \tag{2.8}$$

According to eqn (3.14) of [7], one can write the Ricci scalar as follows

$$\hat{R} = \hat{\omega}_{ABC}\,\hat{\omega}^{CAB} + \hat{\omega}_{A}\,\hat{\omega}^{A}\,, \tag{2.9}$$

Then by this consideration, and using (2.8), Ricci scalar becomes as follows

$$\begin{aligned} \hat{R} = \hat{\omega}_{ABC}\,\hat{\omega}^{CAB} + \hat{\omega}_{A}\,\hat{\omega}^{A} = e^{-2\alpha\varphi}\big[R + \big(-\alpha^2(D-1) - \beta^2 + (\alpha(D-1)+\beta)^2\big) \\ \times\partial_a\,\varphi\,\partial^a\,\varphi - \tfrac{1}{4}e^{2(\beta-\alpha)\,\varphi}\,\mathcal{F}^2_{(2)}\big]\,, \end{aligned} \tag{2.10}$$

where $\mathcal{F}^2_{(2)} = \mathcal{F}_{ab}\,\mathcal{F}^{ab}$.

Taking into account $\hat{e} = e^{(\alpha D+\beta)\,\varphi}\,e$, one can write the Lagrangian in the lower dimension. Now constants α and β can be found by demanding that there is no scalar pre-factor for the Einstein-Hilbert term and also, the scalar field has a canonical normalized kinetic term. Thus we have

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}\,, \qquad \beta = -(D-2)\,\alpha\,. \tag{2.11}$$

With the above relation for β, one can write the following expression for the spin connection components

$$\hat{\omega}_{abc} = e^{-\alpha\varphi}\,(\omega_{abc} + 2\alpha\,\eta_{a[b}\,\partial_{c]}\,\varphi)\,, \qquad \hat{\omega}_{abz} = -\hat{\omega}_{azb} = -\hat{\omega}_{zab} = \tfrac{1}{2}e^{-\alpha D\varphi}\,\mathcal{F}_{ab}\,,$$
$$\hat{\omega}_{zaz} = \alpha\,(D-2)\,e^{-\alpha\varphi}\,\partial_a\varphi\,, \qquad \hat{\omega}_a = e^{-\alpha\varphi}\,(\omega_a + \alpha\,\partial_a\varphi), \qquad \hat{\omega}_z = 0\,, \tag{2.12}$$

where $\alpha^2 = \frac{1}{2(D-1)(D-2)}$.

Having obtained these constants, it is instructive to write the Ricci tensor components

$$\begin{aligned}\hat{R}_{ab} &= e^{-2\alpha\,\varphi}\left(R_{ab} - \tfrac{1}{2}\partial_a\varphi\,\partial_b\varphi - \tfrac{1}{2}e^{-2(D-1)\alpha\varphi}\,\mathcal{F}^2_{ab} - \alpha\,\eta_{ab}\,\Box\varphi\right),\\ \hat{R}_{az} &= \hat{R}_{zb} = \tfrac{1}{2}e^{(D-3)\,\alpha\varphi}\,\nabla^b\!\left(e^{-2(D-1)\,\alpha\varphi}\,\mathcal{F}_{ab}\right),\\ \hat{R}_{zz} &= e^{-2\alpha\,\varphi}\left((D-2)\alpha\,\Box\varphi + \tfrac{1}{4}e^{-2(D-1)\,\alpha\varphi}\,\mathcal{F}^2\right).\end{aligned} \tag{2.13}$$

There is an intricate point about using the relation (2.9) for finding the Ricci scalar. If one, instead were used the Ricci tensor components to find the Ricci scalar, one obtains

$$\begin{aligned}\hat{R} = \eta^{ab}\,\hat{R}_{ab} + \eta^{zz}\,\hat{R}_{zz} &= e^{-2\alpha\varphi}\left[R + \left(-\alpha^2(D-1) - \beta^2 + (\alpha(D-1)+\beta)^2\right)\right.\\ &\left.\times\partial_a\,\varphi\,\partial^a\,\varphi - \alpha D\,\Box\varphi - \tfrac{1}{4}e^{2(\beta-\alpha)\,\varphi}\,\mathcal{F}^2_{(2)}\right].\end{aligned} \tag{2.14}$$

Note that, the difference between (2.10) and (2.14) is a term which involves $D\,\Box\varphi$, and since this is a total derivative term and shall not contribute in equations of motion, this difference may be neglected. [1]

Beginning with the Einstein equations in the higher dimension in (2.13), one can find the Einstein, scalar and the Maxwell equations in the lower dimension

$$\begin{aligned}R_{\mu\nu} &= \tfrac{1}{2}\partial_\mu\varphi\,\partial_\nu\varphi + \tfrac{1}{2}e^{-2\alpha(D-1)\,\varphi}\,\mathcal{F}^2_{\mu\nu} - \tfrac{1}{4(D-1)}\,e^{-2\alpha(D-1)\,\varphi}\,\mathcal{F}^2\,g_{\mu\nu}\\ \Box\,\varphi &= -\tfrac{\alpha(D-1)}{2}\,e^{-2\alpha(D-1)\,\varphi}\,\mathcal{F}^2\,, \qquad d(e^{-2\alpha(D-1)\,\varphi}\,*\mathcal{F}_{(2)}) = 0\,,\end{aligned} \tag{2.15}$$

[1] Note since $\alpha D + \beta = 2\alpha$, and hence $\hat{e} = e^{2\alpha\,\varphi}\,e$, then the scalar pre-factor $e^{-2\alpha\varphi}$ appears in the Ricci scalar cancels out with this prefactor from $\hat{e}$, and therefore $D\,\Box\varphi$ appears in the final lower-dimensional Lagrangian without any scalar pre-factor.

where $\mathcal{F}^2_{\mu\nu} = \mathcal{F}_{\mu\rho}\,\mathcal{F}_\nu{}^\rho$.

Having found the lower dimensional equations of motions, one can obtain the D-dimensional Lagrangian, which is the same as the one which may be derived from (2.10)

$$\mathcal{L}_D = R * \mathbb{1} - \tfrac{1}{2} * d\,\varphi \wedge\, d\varphi - \tfrac{1}{2} e^{-2\alpha(D-1)\,\varphi} * \mathcal{F}_{(2)} \,\wedge \mathcal{F}_{(2)}\,. \tag{2.16}$$

Now, from the scalar equation of motion, it is clear that setting the scalar to zero should be accompanied by setting the Maxwell field to zero, otherwise, the latter is a source for the former. This is one of the most common type of inconsistencies which may occur in the kaluza-Klein reduction and truncation. Actually, the original proposal of Klein to keep the graviton and Maxwell field and discard the scalar field, dilaton, is the first example of the inconsistent truncation.

2.2.2 The vector potential ansatz

Now, we consider the reduction ansatz for a gauge potential which usually appears in the Lagrangian. Assuming the fields are independent of the compact coordinate z, the natural reduction ansatz for a general p-form potential is [2]

$$\hat{A}_{(p)} = A_{(p)} + A_{(p-1)} \wedge dz\,. \tag{2.17}$$

Hence for the field strength $\hat{F}_{(p+1)} = d\hat{A}_{(p)}$, the reduction ansatz becomes

$$\begin{aligned} \hat{F}_{(p+1)} &= dA_{(p)} + dA_{(p-1)}\, \wedge dz = dA_{(p)} - dA_{(p-1)} \wedge \mathcal{A}_{(1)} + dA_{(p-1)}\, \wedge (\mathcal{A}_{(1)} + dz) \\ &= F_{(p+1)} + F_{(p)} \wedge \big(dz + \mathcal{A}_{(1)}\big)\,. \end{aligned} \tag{2.18}$$

Then, one can find the following lower-dimensional relations between the vector potentials and

[2] Actually according to [2], Nordström [1] in 1914, five years before Kaluza suggested his idea, had used the exact same ansatz for the gauge potential!

the field strengths

$$F_{(p+1)} = dA_{(p)} - dA_{(p-1)} \wedge \mathcal{A}_{(1)}\,, \qquad F_{(p)} = dA_{(p-1)}\,. \tag{2.19}$$

For the case of $p = 1$, then one may write

$$\begin{aligned} &\hat{A}_{(1)} = A_{(1)} + \chi\, dz\,, \qquad \hat{F}_{(2)} = d\hat{A}_{(1)} = F_{(2)} + d\chi \wedge (dz + \mathcal{A}_{(1)})\,, \\ &F_{(2)} = dA_{(1)} - d\chi \wedge \mathcal{A}_{(1)} \end{aligned} \tag{2.20}$$

where scalar χ is normally called ‘axion’.

The preliminary relations we presented in this chapter, will help us for the calculations which we will perform in the rest of the dissertation.

3. THE EMBEDDING OF TRUNCATED GAUGED STU SUPERGRAVITIES IN 11 DIMENSIONS [1]

3.1 Introduction

Eleven-dimensional supergravity is a fundamental theory for two reasons. Firstly, it is a low energy limit of M-theory, which itself is yet the best candidate for a consistent realization of the quantum theory of gravity and also the unification of all forces of nature. Secondly, it is the unique theory which describes supergravity in the highest possible dimension. One can obtain maximal four-dimensional gauged supergravity, $N = 8$ and $SO(8)$ of de Wit and Nicolai [10], by the compactification of eleven-dimensional supergravity on S^7. This is the most famous example of consistent Kaluza-Klein-Pauli reductions and it was studied extensively in 1980s [20, 22], and the partial consistency of the reduction was proved by de Wit and Nicolai in 1987 [11]. Although, they found an eleven-dimensional uplift ansatz for the metric, but they did not provide the full uplift ansätze for all components of the four-form field strength $F_{(4)}$ at that time. Nonetheless, more recently, they presented the complete ansätze for all components of this field and have completed their earlier proof [12] . However, this result is somewhat complicated, it is possible to obtain some truncations of the maximal $SO(8)$ supergravity. One of the notable truncations is so-called gauged STU supergravity. In this $N = 2$ theory, besides the graviton, one retains the maximum abelian subgroup of the $SO(8)$ gauged group, i.e. $U(1)^4$ gauge bosons and also, three dilatons and three axions of the original theory in the bosonic sector. The three dilatons and axions belong to 35_v and 35_c of original $SO(8)$ theory respectively. Gauged STU theory is particularly intriguing since almost all four-dimensional black hole solutions can be characterized by this theory. Hence, to obtain an embedding ansatz for a specific black hole solution in eleven dimensions, one may need to find that of gauged STU supergravity. This motivation yields to an exploration of the uplift ansatz for this case.

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We consider further truncations of gauged STU supergravity as follows. The bosonic sector comprises a graviton, two (instead of four in gauged STU case) gauge fields, a dilatonic and an axionic scalar fields. We study two different possibilities of this truncation. In the first case, named $3+1$ in [33], one sets three gauge potentials equal, while all axions as well as all dilatons are considered to be equal. In the second scenario, named $2+2$ in [33], one sets four gauge fields of gauged STU theory pairwise equal. Also, one sets two axions and two dilatons to zero, while the third axion and dilaton are kept. Even though the ansätze for the metric and four-form for the latter case were found a while ago in [34], we find the full ansätze for the former case for the first time.

The metric ansatz for both of the above truncations, already presented in a more general case of gauged STU supergravity in [35]. Therefore, finding the uplift ansatz for the metric is a straightforward task. To obtain the metric ansatz, one shall use appropriate parametrizations for both cases. We show the ansatz matches with the previous results of [34] and [49], with considering the relevant truncations and rescalings.

The four-form ansatz for the original maximal $SO(8)$ theory had not been known before the work of de Wit and Nicolai in [12] in 2013. However, for special truncations of this theory, such as the cases in [34] and [49] the ansatz was found. For $3+1$ and $2+2$ truncations, we found the four-form ansatz for former case for the first time by the method we will describe shortly, but for the latter case, the ansatz already has been presented in [34]. We obtained the ansatz in $3+1$ case by writing a trial ansatz for $A'_{(3)}$, where $dA'_{(3)}$ is the only unknown part of the four-form field strength. Considering the eleven-dimensional equations for four-form field, i.e. $d*F_{(4)} = -\frac{1}{2} * F_{(4)} \wedge F_{(4)}$, one can obtain several different equations which the trial ansatz should satisfy. Then, using Mathematica for solving these related equations, we could successfully find the complete ansatz for the 4-form.

Shortly after our calculation, the full uplifting ansatz for gauged STU supergravity was found and presented in sections 2, 3, 4 and 5 of [33] based on what de Wit and Nicolai had found in [12]. Now, finding the ansatz is a straightforward calculation, and we will present different steps for obtaining it in this chapter. It matches exactly with what we already had found by a trial ansatz. In addition to this, one can employ the field strength ansatz for general gauged STU supergravity to find out that of the $2+2$ case and recover the result in [34].

Let us begin by specifying the two truncations we mentioned above. It is standard to use $SL(2,\mathbb{R})$ parametrization for the scalar fields of gauged STU theory rather than an $SO(2,1)$ one. In that sense, dilaton/axion pairs (φ_i, χ_i) given by

$$e^{\varphi_i} = \cosh\lambda_i + \sinh\lambda_i\, \cos\sigma_i\,, \qquad \chi_i\, e^{\varphi_i} = \sinh\lambda_i\, \sin\sigma_i\,, \tag{3.1}$$

Now, two truncations are as follows

$$\begin{aligned}
& & &\lambda_1 = \lambda \qquad \sigma_1 = \sigma\,, \qquad \lambda_2 = \lambda_3 = \sigma_2 = \sigma_3 = 0\,,\\
&2+2: & &\varphi_1 = \varphi \qquad \chi_1 = \chi\,, \qquad \varphi_2 = \varphi_3 = \chi_2 = \chi_3 = 0\,,\\
& & &A^1_\mu = A^2_\mu = A_\mu\,, \qquad A^3_\mu = A^4_\mu = \widetilde{A}_\mu\,,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
& & &\lambda_1 = \lambda_2 = \lambda_3 = \lambda\,, \qquad \sigma_1 = \sigma_2 = \sigma_3 = \sigma\,,\\
&3+1: & &\varphi_1 = \varphi_2 = \varphi_3 = \varphi\,, \qquad \chi_1 = \chi_2 = \chi_3 = \chi\,,\\
& & &A^1_\mu = \widetilde{A}_\mu\,, \qquad A^2_\mu = A^3_\mu = A^4_\mu = A_\mu\,.
\end{aligned} \tag{3.3}$$

In the rest of this chapter, we will find the metric and four-form field strength ansätze, and also the bosonic Lagrangian for these two cases.

3.2 $3+1$ Truncation of gauged STU supergravity

One may introduce the following re-parametrizations for μ^i, where $\mu^i\mu^i = 1$

$$\mu_1 = \cos\xi\,, \qquad \mu_a = \nu_a\, \sin\xi\,, \quad a = 2,3,4\,, \qquad \sum_a \nu_a^2 = 1\,. \tag{3.4}$$

We define $c = \cos\xi$ and $s = \sin\xi$, since we have used them frequently.

3.2.1 The embedding of the metric

3.2.1.1 The $\mathbb{CP}^2$ geometry

Consider $\sum_a \nu_a^2 = 1$ where $a = 2, 3, 4$. Now, define complex variables z_a as $z_a = \nu_a\, e^{\mathrm{i}\,\phi_a}$. The unit Fubini-Study metric on $\mathbb{CP}^2$ reads

$$d\Sigma_2^2 = \sum_a dz_a\, d\bar{z}_a - |\sum_a \bar{z}_a dz_a|^2\,. \tag{3.5}$$

Hence, using the above definition, one can obtain the following relation for the metric

$$d\Sigma_2^2 = \sum_a d\nu_a^2 + \nu_a^2\, d\phi_a^2 - \big(\sum_a \nu_a^2\, d\phi_a\big)^2\,. \tag{3.6}$$

The Kähler form on $\mathbb{CP}^2$ can be written as follows

$$J = \tfrac{1}{2} dB\,, \qquad \text{and} \qquad d\psi + B = \sum_a \nu_a^2\, d\phi_a\,. \tag{3.7}$$

The unit metric for a five-sphere is

$$d\Omega_5^2 = \sum_a d\nu_a^2 + \nu_a^2\, d\phi_a^2\,. \tag{3.8}$$

Then, the Fubini-Study metric shall be

$$d\Sigma_2^2 = d\Omega_5^2 - (d\psi + B)^2\,. \tag{3.9}$$

3.2.1.2 Obtaining the metric ansatz in terms of $\mathbb{CP}^2$ geometry

The metric ansazt for the embedding of general gauged STU spergravity in eleven dimensions was found quite long time ago in [35]. Hence finding the $3+1$ specialization of the metric is a straightforward task. To do so, one may start from functions Y_i and $\widetilde{Y}_i$ introduced in terms of the axions and dilatons in eqn (11) of [35]. According to our specialization, they read

$$Y_i \equiv Y = e^{\frac{\varphi}{2}}\,, \qquad \widetilde{Y}_i \equiv \widetilde{Y} = (1+\chi^2\, e^{2\varphi})^{\frac{1}{2}}\, e^{-\frac{\varphi}{2}}\,, \qquad b_i \equiv b = \chi\, e^{\varphi}\,, \quad i=1,2,3\,. \tag{3.10}$$

The next step is finding Z_i introduced in eqn (20) of [35]. Therefore, one can find

$$Z_1 = \mu_1^2\,(1-\widetilde{Y}^4) + \widetilde{Y}^4 = c^2 + s^2\, e^{-2\varphi}\,(1+b^2)^2\,, \tag{3.11}$$
$$Z_a = \mu_a^2\,(1-Y^2\widetilde{Y}^2) + Y^2\widetilde{Y}^2 + \mu_1^2\, Y^2(Y^2-\widetilde{Y}^2) = -s^2\, b^2\nu_a^2 + \beta\,, \qquad a=2,3,4\,,$$

where

$$\beta = Y^2\,(Y^2\, c^2 + \widetilde{Y}^2\, s^2) = e^{2\varphi}\, c^2 + (1+b^2)\, s^2\,, \tag{3.12}$$

Also, one needs to find the function Ξ defined by eqn (21) of [35], which reads, in our special case, as follows

$$\Xi = Y^2\,(Y^2\,\mu_1^2 + \widetilde{Y}^2\,(1-\mu_1^2))^2 = e^{-\varphi}\,\beta^2\,. \tag{3.13}$$

Now, having obtained all of the above relations, the metric ansazt can be derived from eqn (28) of [35]. It is a tedious procedure and we shall show the result in few steps. First, one can write down directly from eqn (28) the following relation

$$\begin{aligned} d\hat{s}_{11}^2 &= \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\,\Xi^{-\frac{2}{3}}\Big\{ Z_1\,(s^2 d\xi^2 + c^2\, d\phi_1^2) + \sum_a Z_a\left[(c\,\nu_a\, d\xi + s\, d\nu_a)^2 + s^2\,\nu_a^2\, d\phi_a^2\right] \\ &\quad + 2b^2\left[c^2 s^2\, d\phi_1 \sum_a \nu_a^2\, d\phi_a - s^4\left(\nu_2^2\, d\phi_2\,\nu_3^2\, d\phi_3 + \nu_2^2\, d\phi_2\,\nu_4^2\, d\phi_4 + \nu_3^2\, d\phi_3\,\nu_4^2\, d\phi_4\right)\right] \\ &\quad + b^2\big(3\mu_1^2\, d\mu_1^2 + \sum_a \mu_a^2\, d\mu_a^2 + 2\sum_a \mu_1\, d\mu_1\,\mu_a\, d\mu_a\big)\Big\}\,. \end{aligned} \tag{3.14}$$

This is an ungauged metric and the gauging can be simply recovered by

$$d\phi_i \longrightarrow d\phi_i - g\, A^i_{(1)}\,, \tag{3.15}$$

where in general $A^i_{(1)}$ is four $U(1)^4$ gauge potentials of gauged STU supergravity. However, in the special case of $3+1$ truncation, one should truncate the gauge potentials according to (3.3). For simplicity, one may consider the ungauged metric and recover the gauge potentials in the last step.

Now, inserting Z_a from (3.11), and making use of the relation

$$3\mu_1^2\, d\mu_1^2 + \sum_a \mu_a^2\, d\mu_a^2 + 2\mu_1\, d\mu_1\, \mu_a\, d\mu_a = \sum_a c^2 s^2\, d\xi^2 (1+\nu_a^4) + s^4\, \nu_a^2 d\nu_a^2 + 2cs^3\, \nu_a^3\, d\nu_a\, d\xi\,,$$

one can write down

$$\begin{aligned} d\hat{s}_{11}^2 &= \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\,\Xi^{-\frac{2}{3}}\,\sum_a \Big\{ Z_1\,(s^2\, d\xi^2 + c^2\, d\phi_1^2) + \beta\,(c^2\, d\xi^2 + s^2\, d\nu_a^2 + s^2\, \nu_a^2\, d\phi_a^2) \\ &+ b^2 \Big[- c^2 s^2 \nu_a^4 d\xi^2 - s^4\, \nu_a^2 d\nu_a^2 - 2cs^3\, \nu_a^3 d\nu_a\, d\xi - s^4 \nu_a^4\, d\phi_a^2 + 2c^2 s^2 d\phi_1\,(d\psi + B) \\ &- s^4\,\big((d\psi+B)^2 - \nu_a^4\, d\phi_a^2\big) + c^2 s^2\, d\xi^2 (1+\nu_a^4) + s^4\, \nu_a^2 d\nu_a^2 + 2cs^3\, \nu_a^3\, d\nu_a\, d\xi \Big] \Big\}\,. \end{aligned}$$

Using (3.8), one can make a further simplification as follows

$$\begin{aligned} d\hat{s}_{11}^2 &= \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\,\Xi^{-\frac{2}{3}}\,\Big\{ Z_1\,(s^2\, d\xi^2 + c^2\, d\phi_1^2) + \beta\,(c^2\, d\xi^2 + s^2\, d\Omega_5^2) \\ &\quad + b^2\,\big(2c^2 s^2\, d\phi_1\,(d\psi+B) - s^4\,(d\psi+B)^2 + c^2 s^2\, d\xi^2\big) \Big\} \\ &= \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\,\Xi^{-\frac{2}{3}}\,\Big\{ d\xi^2\,(Z_1\, s^2 + \beta c^2 + b^2 c^2 s^2) + \beta\, s^2\,(d\Sigma_2^2 + (d\psi+B)^2) \\ &\quad + Z_1\, c^2\, d\phi_1^2 + 2\, b^2 c^2 s^2\, d\phi_1\,(d\psi+B) - s^4 b^2\,(d\psi+B)^2 \Big\}\,. \end{aligned} \tag{3.16}$$

Now, one can use relations for Z_1 and β in (3.11) and (3.12) to write down the following expression

$$Z_1\, s^2 + \beta c^2 + b^2 c^2 s^2 = e^{-2\varphi}\,\beta^2\,. \tag{3.17}$$

The next step is completing the square in terms involving $d\psi + B$, and writing

$$\begin{aligned} d\hat{s}_{11}^2 &= \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\,\Xi^{-\frac{2}{3}}\,\Big\{ e^{-2\varphi}\,\beta^2\, d\xi^2 + \beta\, s^2\, d\Sigma_2^2 + Z_1\, c^2\, d\phi_1^2 \\ &\quad + \gamma s^2 \big[(d\psi+B)^2 + \tfrac{2b^2c^2}{\gamma}\,(d\psi+B)\, d\phi_1\big] \Big\} = \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\,\Xi^{-\frac{2}{3}} \\ &\quad \times \Big\{ e^{-2\varphi}\,\beta^2\, d\xi^2 + \beta\, s^2\, d\Sigma_2^2 + \gamma s^2 \big[(d\psi+B) + \tfrac{b^2c^2}{\gamma}\, d\phi_1\big]^2 + e^{-2\varphi}\,\tfrac{c^2\,\beta^2}{\gamma}\, d\phi_1^2 \Big\}\,, \end{aligned} \tag{3.18}$$

where we have used the following result for the coefficient of $d\phi_1^2$

$$Z_1\, c^2 - b^4 c^4 s^2\, \gamma^{-1} = e^{-2\varphi}\, \tfrac{c^2\, \beta^2}{\gamma}\,, \tag{3.19}$$

where $\gamma = Y^4\, c^2 + s^2$.

The last step is retrieving the gauge potentials according to (3.15) and our $3{+}1$ specialization in (3.3). Note that since $d\psi + B = \sum_a \nu_a^2\, d\phi_a$, one can observe the inclusion of the gauge potentials leads to

$$d\psi + B \longrightarrow d\psi + B - g\, A_{(1)}\,, \qquad d\phi_1 \longrightarrow d\phi_1 - g\, \widetilde{A}_{(1)}\,. \tag{3.20}$$

Hence, the metric, as it was presented in eqn (6.22) of [33] shall be

$$\begin{aligned} d\hat{s}_{11}^2 = \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\, \Xi^{-\frac{2}{3}} \Big[& \frac{\beta^2}{Y^4}\, d\xi^2 + \gamma\, s^2 \Big((d\psi + B - gA_{(1)}) + \frac{b^2\, c^2}{\gamma}\, (d\phi_1 - g\widetilde{A}_{(1)}) \Big)^2 \\ & + \beta\, s^2\, d\Sigma_2^2 + \frac{\beta^2\, c^2}{\gamma\, Y^4}\, (d\phi_1 - g\widetilde{A}_{(1)})^2 \Big]\,. \end{aligned} \tag{3.21}$$

3.2.2 The embedding of the four-form

The full ansatz for the four-form field strength can be written as

$$\hat{F}_{(4)} = -2gU\, \epsilon_{(4)} + \hat{G}_{(4)} + d\hat{A}'_{(3)} + \hat{F}''_{(4)}\,, \tag{3.22}$$

where U, $\hat{A}'_{(3)}$, $\hat{F}''_{(4)}$ and $\hat{G}_{(4)}$ are given by equations (5.4), (5.5), (5.6) and (5.8) of [33] respectively. Note that all of these relations except the ansatz for $\hat{A}'_{(3)}$ were given in the paper [35] in 2000.

As a first step towards finding $\hat{F}_{(4)}$, one may calculate $A'_{(3)}$. The ansatz for this field is given by eqn (5.5) of [33] as follows

$$\hat{A}'_{(3)} = \tfrac{1}{2} A_{\alpha\hat{\beta}\hat{\gamma}}\, d\mu_\alpha \wedge (d\phi_\beta - g\, A^\beta_{(1)}) \wedge (d\phi_\gamma - g\, A^\gamma_{(1)})\,, \tag{3.23}$$

where $A_{\alpha\hat{\beta}\hat{\gamma}}\, d\mu_\alpha$ can be derived from eqn (4.19) of [33]. Here we have introduced the hat notation in $\hat{\alpha}$ where $(\hat{1}, \hat{2}, \hat{3}, \hat{4}) = (5, 6, 7, 8)$. We do not repeat somewhat lengthy relations for $A_{\alpha\hat{\beta}\hat{\gamma}}\, d\mu_\alpha$ here, however, to obtain the result, one needs to find out W_i, introduced in eqn (4.20) of [33] first. They are

$$W_1 = (1 + b^2)^2\, e^{-2\varphi}\, s^2\,, \qquad W_a = e^{2\varphi}\, c^2 + (1 + b^2)\, s^2\, (1 - \nu_a^2)\,, \qquad a = 2, 3, 4\,. \tag{3.24}$$

Hence after calculation, one can write down

$$\begin{aligned}
A_{\alpha 56}\, d\mu_\alpha &= \frac{b\beta}{2\Xi g^3}\big[c^2\, d(s^2\,\nu_2^2) - s^2\,\nu_2^2\, e^{-2\varphi}(1 + b^2)\, d(c^2)\big],\\
A_{\alpha 78}\, d\mu_\alpha &= \frac{b\beta\, s^2}{2\Xi g^3}\big[\nu_4^2\, d(s^2\,\nu_3^2) - \nu_3^2\, d(s^2\,\nu_4^2)\big],\\
A_{\alpha 57}\, d\mu_\alpha &= \frac{b\beta}{2\Xi g^3}\big[c^2\, d(s^2\,\nu_3^2) - s^2\,\nu_3^2\, e^{-2\varphi}\,(1 + b^2)\, d(c^2)\big],\\
A_{\alpha 68}\, d\mu_\alpha &= \frac{b\beta\, s^2}{2\Xi g^3}\big[\nu_4^2\, d(s^2\,\nu_2^2) - \nu_2^2\, d(s^2\,\nu_4^2)\big],\\
A_{\alpha 58}\, d\mu_\alpha &= \frac{b\beta}{2\Xi g^3}\big[c^2\, d(s^2\,\nu_4^2) - s^2\,\nu_4^2\, e^{-2\varphi}\,(1 + b^2) d(c^2)\big],\\
A_{\alpha 67}\delta\mu_\alpha &= \frac{b\beta\, s^2}{2\Xi g^3}\big[\nu_3^2\, d(s^2\,\nu_2^2) - \nu_2^2\, d(s^2\,\nu_3^2)\big].
\end{aligned} \tag{3.25}$$

Now, upon using (3.23), the ansatz for $A'_{(3)}$ becomes as follows

$$\begin{aligned}
A'_3 = \frac{b\beta}{2\Xi g^3}\Bigg\{ \sum_a \Bigg[& c^2\, d(s^2\,\nu_a^2) \wedge d\phi_1 \wedge d\phi_a - e^{-2\varphi}\,(1 + b^2)\, s^2\,\nu_a^2 d(c^2) \wedge d\phi_1 \wedge d\phi_a \Bigg]\\
& + \tfrac{1}{2}\, s^2 \sum_{a,b} \big[\nu_b^2\, d\left(s^2\,\nu_a^2\right) - \nu_a^2\, d(s^2\,\nu_b^2)\big] \wedge d\phi_a \wedge d\phi_b \Bigg\},
\end{aligned} \tag{3.26}$$

where we have set $A_i = 0$ for simplicity and it will be recovered in the last step.

Now, considering the $\mathbb{CP}^2$ relations introduced in (3.7), one can write

$$\begin{aligned}
A'_3 = \frac{b\beta}{2\Xi g^3}\Big[& 2cs\left(c^2 + e^{-2\varphi}\,(1 + b^2)\, s^2\right) d\xi \wedge d\phi_1 \wedge (d\psi + B)\\
& - 2c^2 s^2\, J \wedge d\phi_1 + 2s^4\, J \wedge (d\psi + B)\Big]\,,
\end{aligned} \tag{3.27}$$

where we have used

$$d\psi + B = \sum_a \nu_a^2\, d\phi_a \quad \Rightarrow \quad dB = \sum_a d(\nu_a^2) \wedge d\phi_a = 2J\,, \tag{3.28}$$

from relations in (3.7).

Finally, one may recover the gauge potentials by using (3.20) and write

$$\hat{A}'_{(3)} = \frac{sc\,\chi}{g^3}\, d\xi \wedge (d\phi_1 - g\widetilde{A}_{(1)}) \wedge (d\psi + B - gA_{(1)}) - \frac{s^2c^2}{\beta g^3}\,\chi\, e^{2\varphi}\,(d\phi_1 - g\widetilde{A}_{(1)}) \wedge J\,. \tag{3.29}$$

Other terms of the uplifting ansatz for $\hat{F}_{(4)}$ can readily be found from equations 41, 42 and 43 of [35], hence the final result for $\hat{F}_{(4)}$ is

$$\hat{F}_{(4)} = -2gU\,\epsilon_{(4)} + \hat{G}_{(4)} + d\hat{A}'_{(3)} + \hat{F}''_{(4)}\,, \tag{3.30}$$

where

$$\begin{aligned}
U &= 2(Y^2\,c^2 + \widetilde{Y}^2\,s^2) + Y^2\,,\\
\hat{G}_{(4)} &= -\frac{2sc}{g}\,(*d\varphi - \chi\, e^{2\varphi} *d\chi) \wedge d\xi\,,\\
&\quad + \frac{s^4}{\beta g^3}\,\chi\, e^{2\varphi}\,(d\psi + B - gA_{(1)}) \wedge J\,,\\
\hat{F}''_{(4)} &= \frac{sc}{g^2\,|W|^2}\, d\xi \wedge \widetilde{R} \wedge (d\phi_1 - g\widetilde{A}_{(1)}) - \frac{sc}{g^2\,|W|^2}\, d\xi \wedge R \wedge (d\psi + B - gA_{(1)})\\
&\quad - \frac{s^2}{g^2\,|W|^2}\, R \wedge J\,,
\end{aligned} \tag{3.31}$$

and $A'_{(3)}$ is given by (3.29). One can find W from eqn (37) of [35] as

$$|W|^2 = (1 + 4b^2)(1 + b^2)^2 \tag{3.32}$$

and also from eq (40) of [35] one has

$$\widetilde{R} = R_1\,, \qquad R = R_2 = R_3 = R_4\,, \tag{3.33}$$

with

$$\begin{aligned}\widetilde{R} &= \widetilde{Y}^6\,[(1+3b^2)\,{*\widetilde{F}_{(2)}} + 2b^3\,\widetilde{F}_{(2)}] + 3b\,(1+b^2)\,\widetilde{Y}^2\,[b\,{*F_{(2)}} + (1+2b^2)\,F_{(2)}]\,,\\ R &= Y^2\,(1+b^2)^2\,[{*F_{(2)}} - 2b\,F_{(2)}] + b\,(1+b^2)\,\widetilde{Y}^2\,[b\,{*\widetilde{F}_{(2)}} + (1+2b^2)\,\widetilde{F}_{(2)}]\,.\end{aligned} \tag{3.34}$$

3.2.2.1 Finding the Lagrangian

The bosonic Lagrangian can be obtained from eqn (34) of [35], and in the $3+1$ specialization, it shall be

$$\mathcal{L}_4 = R\,{*\mathbb{1}} - \tfrac{3}{2}\left({*d\varphi}\wedge d\varphi + e^{2\varphi}\,*\,d\chi\wedge d\chi\right) - V\,{*\mathbb{1}} + \mathcal{L}_{KinA} + \mathcal{L}_{CS}, \tag{3.35}$$

where V is the potential for scalars (φ,χ) and $\mathcal{L}_{KinA}$ and $\mathcal{L}_{CS}$ are the kinetic and Chern-Simons terms for one-form potentials $A_{(1)}$ and $\tilde{A}_{(1)}$. Following the equations 35, 36 and 38 of [35] and using $3+1$ specialization, V , $\mathcal{L}_{Kin}$ and $\mathcal{L}_{CS}$ can be written as follows

$$\begin{aligned}V &= -12g^2\left(e^{\varphi} + e^{-\varphi}(1+\chi^2\,e^{2\varphi})\right),\\ \mathcal{L}_{KinA} &= -\tfrac{1}{2(1+4\chi^2\,e^{2\varphi})}\left[6\chi^2\,e^{\varphi}F\wedge{*\widetilde{F}} + e^{-3\varphi}(1+3\chi^2e^{2\varphi})(1+\chi^2e^{2\varphi})\widetilde{F}\wedge{*\widetilde{F}} + 3e^{\varphi}\,{*\,F}\,\wedge F\right],\\ \mathcal{L}_{CS} &= -\tfrac{\chi}{(1+4\chi^2\,e^{2\varphi})}\left[-3e^{2\varphi}F\,\wedge\,F + 3(1+2\chi^2\,e^{2\varphi})\,\widetilde{F}\wedge\,F + \chi^2(1+\chi^2\,e^{2\varphi})\,\widetilde{F}\,\wedge\,\widetilde{F}\right].\end{aligned} \tag{3.36}$$

One may compare these uplifting ansätze for the metric, four-form $\hat{F}_{(4)}$ and the Lagrangian with the previous results of [49] where the dilatons are not identical, but axions were set to zero. If one set all of the dilatons equal to each other as well as three gauge potentials in that paper, it turns out the result matches with that of our result in this section, after truncating the axion to zero. In other words, by setting the axion to zero, one can set $b=\chi e^{\varphi}=0$, therefore, as one can observe from (3.29), $A_3' = 0$ and the involved calculation for obtaining A_3' can be avoided.

3.3 $2+2$ Truncation of gauged STU supergravity

3.3.1 The embedding of the metric

For our purpose of $2+2$ truncation, defined in (3.2), it is convenient to define μ_i as follows

$$\mu_1 = c\,\cos\tfrac{1}{2}\theta\,, \qquad \mu_2 = c\,\sin\tfrac{1}{2}\theta\,, \qquad \mu_3 = s\,\cos\tfrac{1}{2}\tilde{\theta}\,, \qquad \mu_4 = s\,\sin\tfrac{1}{2}\tilde{\theta}\,, \tag{3.37}$$

where $c = \cos\xi$ and $s = \sin\xi$ as before.

The next step is finding b_i, Y_i and $\widetilde{Y}_i$ in this specialization as follows

$$\begin{aligned} &b_1 = b = \chi\, e^{\varphi}, \qquad b_2 = b_3 = 0, \qquad Y_1 = Y = e^{\frac{\varphi}{2}}, \qquad Y_2 = Y_3 = 1 \\ &\widetilde{Y}_1 = \widetilde{Y} = (1+\chi^2\, e^{2\varphi})^{\frac{1}{2}}\, e^{-\frac{\varphi}{2}}, \qquad \widetilde{Y}_2 = \widetilde{Y}_3 = 1\,. \end{aligned} \tag{3.38}$$

Therefore, one can find out Z_i from eqn (20) of [35] as follows

$$\begin{aligned} Z_1 &= Z_2 = \widetilde{Y}^2\, s^2 + c^2\,, \quad Z_3 = Z_4 = Y^2\, c^2 + s^2\,, \\ \Xi &= Y^2\, c^4 + \widetilde{Y}^2\, s^4 + (1 + Y^2\, \widetilde{Y}^2)\, c^2 s^2 = Z_1\, Z_3\,. \end{aligned} \tag{3.39}$$

With these preliminaries, one can calculate the metric from eqn (28) of [35] as follows

$$\begin{aligned} d\hat{s}_{11}^2 = \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\, \Xi^{-\frac{2}{3}} \Big\{ & Z_1 \left(d\mu_1^2 + \mu_1^2\, d\phi_1^2 + d\mu_2^2 + \mu_2^2\, d\phi_2^2\right) \\ & + Z_3 \left(d\mu_3^2 + \mu_3^2\, d\phi_3^2 + d\mu_4^2 + \mu_4^2\, d\phi_4^2\right) \\ & + \tfrac{1}{2}\, b^2 \left[(\mu_1\, d\mu_1 + \mu_2\, d\mu_2)^2 + (\mu_3\, d\mu_3 + \mu_4\, d\mu_4)^2\right] \Big\}\,. \end{aligned} \tag{3.40}$$

Again as we emphasized in the case of $3+1$ truncation, the above metric is ungauged and one needs to follow (3.15) to gauge it. Using the definitions for μ_i in (3.37), one can write

$$d\hat{s}_{11}^2 = \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\,\Xi^{-\frac{2}{3}} \Big\{ Z_1 \Big[s^2\, d\xi^2 + \tfrac{1}{4} c^2 d\theta^2 + c^2 \left(\cos^2 \tfrac{1}{2}\theta\, d\phi_1^2 + \sin^2 \tfrac{1}{2}\theta\, d\phi_2^2 \right) \Big] + Z_3 \Big[c^2\, d\xi^2 + \tfrac{1}{4} s^2 d\tilde{\theta}^2 + s^2 \left(\cos^2 \tfrac{1}{2}\tilde{\theta}\, d\phi_3^2 + \sin^2 \tfrac{1}{2}\tilde{\theta}\, d\phi_4^2 \right) \Big] + \chi^2\, e^{2\varphi}\, c^2 s^2 d\xi^2 \Big\} \,. \quad (3.41)$$

It is convenient to introduce the following relation for the four azimuthal angles

$$\phi_1 = \tfrac{1}{2}(\psi + \phi)\,, \qquad \phi_2 = \tfrac{1}{2}(\psi - \phi)\,, \qquad \phi_3 = \tfrac{1}{2}(\widetilde{\psi} + \tilde{\phi})\,, \qquad \phi_4 = \tfrac{1}{2}(\widetilde{\psi} - \tilde{\phi})\,. \qquad (3.42)$$

Therefore, after some algebra, one can write

$$d\hat{s}_{11}^2 = \Xi^{\frac{1}{3}}\, ds_4^2 + g^{-2}\,\Xi^{-\frac{2}{3}} \Big\{ \Xi\, d\xi^2 + \tfrac{1}{4} c^2 \,(\widetilde{Y}^2\, s^2 + c^2) \left(d\theta^2 + d\psi^2 + d\phi^2 + 2 d\psi\, d\phi\, \cos\theta \right) + \tfrac{1}{4} s^2\, (Y^2\, c^2 + s^2) \big(d\tilde{\theta}^2 + d\tilde{\psi}^2 + d\tilde{\phi}^2 + 2 d\tilde{\psi}\, d\tilde{\phi}\, \cos\tilde{\theta} \big) \Big\} \,. \qquad (3.43)$$

One needs to find out how the the gauge potentials incorporate in the Euler angles . Following (3.15) and above definitions for azimuthal angles , one can readily find out

$$\begin{aligned} d\phi &\longrightarrow d\phi\,, \qquad d\psi \longrightarrow d\psi - 2g\, A_{(1)}\,, \\ d\tilde{\phi} &\longrightarrow d\tilde{\phi}\,, \qquad d\tilde{\psi} \longrightarrow d\tilde{\psi} - 2g\, \widetilde{A}_{(1)}\,. \end{aligned} \qquad (3.44)$$

Having obtained these, one can easily complete the square in the metric ansatz (3.43), and after considering the gauge potentials as it is stated in (3.44), one can write

$$d\hat{s}_{11}^2 = \Xi^{\frac{1}{3}}\, ds_4^2 + \frac{\Xi^{\frac{1}{3}}}{g^2} \Big\{ d\xi^2 + \frac{\cos^2\xi}{4Z_3} \Big[d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi + \cos\theta\, d\phi - 2gA_{(1)})^2 \Big] + \frac{\sin^2\xi}{4Z_1} \Big[d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\phi}^2 + (d\widetilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi} - 2g\widetilde{A}_{(1)})^2 \Big] \Big\} \,, \quad (3.45)$$

as it was presented in eqn (6.10) of [33].

One can compare the metric ansazt we have found, with the metric presented in eqn (1) of [34].

The latter result, studying the embedding of $D=4$, $N=4$ with $SO(4)$ gauging supergravity in eleven dimensions, is more general than what it was obtained here. However, one should find the same result after considering an Abelian truncation of $SO(4)$ gauging, i.e. $U(1)^2$ gauging, with $A_{(1)}$ and $\widetilde{A}_{(1)}$ gauge potentials defined by (3.2). To make a connection with the eqn (1) of [34], one needs to write down h_i and $\tilde{h}_i$ [2], appeared in their metric, as follows

$$h_i = \sigma_i - g\, A_i\,, \qquad \tilde{h}_i = \tilde{\sigma}_i - g\, \widetilde{A}_i\,, \tag{3.46}$$

where σ_i are three left-invariant one-forms in $S^3 = SU(2)$. Note, since $SO(4) = SU(2)\times SU(2)$, we have two copies of S^3 here, where their corresponding gauge potentials are denoted by A_i and $\widetilde{A}_i$. One may explicitly write σ_i in terms of the Euler angles as follows

$$\sigma_1 = \cos\psi\, d\theta + \sin\psi\, \sin\theta\, d\phi\,, \quad \sigma_2 = -\sin\psi\, d\theta + \cos\psi\, \sin\theta\, d\phi\,, \quad \sigma_3 = d\psi + \cos\theta\, d\phi\,. \tag{3.47}$$

With these preliminaries, one can find out $\sum_i h_i^2$ and $\sum_i \tilde{h}_i^2$ in the metric ansatz in eqn (1) of [34] as follows

$$\begin{aligned}\sum_i h_i^2 &= d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi + \cos\theta\, d\phi - 2\, gA_{(1)})^2 \\ \sum_i \tilde{h}_i^2 &= d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\phi}^2 + (d\widetilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi} - 2\, g\widetilde{A}_{(1)})^2\,,\end{aligned} \tag{3.48}$$

where as we have emphasized, just two gauge potentials $A_3 \equiv A_{(1)}$ and $\widetilde{A}_3 \equiv \widetilde{A}_{(1)}$ out of 6 gauge potentials of $SO(4)$ are kept.

Finally, Setting $\Xi = \Delta^2$, and rescaling $g = \frac{1}{\sqrt{2}} g^{\text{Ref. [34]}}$, one can retrieve eqn (1) of [34] from the metric presented in (3.45).

One may set $\chi = 0$ to find a further truncation which was studied in [49] for the first time.

[2] In chapter 4, the combination of the left-invariant one-forms and the gauge potentials will be denoted by ν_i.

Hence, with this assumption, one has

$$Y = e^{\frac{1}{2}\varphi}\,, \qquad \widetilde{Y} = e^{-\frac{1}{2}\varphi}\,, \qquad \Xi \equiv \Delta^2 = e^{-\varphi}\left(e^{\varphi}\,c^2 + s^2\right)^2\,, \tag{3.49}$$

and it can be observed the resulting ansatz is the same as that of eqn (3.1) of [49] and also eqn (44) of [34].

3.3.2 The embedding of the four-form

Full ansatz of the embedding of the four-form of gauged STU supergravity in eleven dimensions is given in section 4 and 5 of [33]. In this part, we find out the uplift of the four-form in $2+2$ truncation of gauged STU supergravity and we will show it is in full agreement with the result derived in [34].

One can follow the same route as it was taken in the $3+1$ truncation, and use the ansatz in (3.22). Let us calculate $A'_{(3)}$ contribution in four-form field strength first. To do so, one needs to find out W_i in eqn (4.20) of [33] as follows

$$\begin{aligned} W_1 &= c^2\sin^2\tfrac{1}{2}\theta + s^2\widetilde{Y}^2\,, \qquad W_2 = c^2\cos^2\tfrac{1}{2}\theta + s^2\widetilde{Y}^2\,, \\ W_3 &= s^2\sin^2\tfrac{1}{2}\tilde{\theta} + c^2 Y^2\,, \qquad W_4 = s^2\cos^2\tfrac{1}{2}\tilde{\theta} + c^2 Y^2\,. \end{aligned} \tag{3.50}$$

The next step is finding $A_{\alpha\hat{\beta}\hat{\gamma}}\,d\mu_\alpha$ from eqn (4.19) of [33]. Here we have introduced the hat notation in $\hat{\alpha}$ where $(\hat{1},\hat{2},\hat{3},\hat{4}) = (5,6,7,8)$. Since $b_2 = b_3 = 0$, it leads to a considerable simplification. The results become

$$\begin{aligned} A_{\alpha 56}\,d\mu_\alpha &= \frac{b}{2\,\Xi\,g^3}\Big[\mu_1^2\,W_2\,d(\mu_2^2) - \mu_2^2\,W_1\,d(\mu_1^2) - \mu_1^2\,\mu_2^2\,d(\alpha_2+\alpha_3)\Big]\,, \\ A_{\alpha 78}\,d\mu_\alpha &= \frac{b}{2\,\Xi\,g^3}\Big[\mu_4^2\,W_3\,d(\mu_3^2) - \mu_3^2\,W_4\,d(\mu_4^2) + \mu_3^2\,\mu_4^2\,d(\alpha_2-\alpha_3)\Big]\,, \\ A_{\alpha 57}\,d\mu_\alpha &= A_{\alpha 68}\,d\mu_\alpha = A_{\alpha 58}\,d\mu_\alpha = A_{\alpha 67}\,d\mu_\alpha = 0\,, \end{aligned} \tag{3.51}$$

where

$$\alpha_1 = \mu_1^2 + \mu_2^2 \qquad \alpha_2 = \mu_1^2 + \mu_3^2\,, \qquad \alpha_3 = \mu_1^2 + \mu_4^2\,. \tag{3.52}$$

Now, one needs to make use of the reparametrizations introduced in (3.37) and the result is

$$A_{\alpha 56}\, d\mu_\alpha = \tfrac{b\, c^4}{4\,\Xi\, g^3}\, (s^2\, \widetilde{Y}^2 + c^2)\, \sin\theta\, d\theta\,, \qquad A_{\alpha 78}\, d\mu_\alpha = \tfrac{-b\, s^4}{4\,\Xi\, g^3}\, (s^2 + Y^2\, c^2)\, \sin\tilde{\theta}\, d\tilde{\theta}\,. \tag{3.53}$$

The next step is using the expression (5.5) in [33] for finding $A'_{(3)}$, which we already presented in (3.23). According to eqn (3.51), just two terms contribute in (3.23) relation, hence one can write down

$$\begin{aligned} \hat{A}'_{(3)} = \frac{\chi\, e^{\varphi}}{4\,\Xi\, g^3} \Big(& c^4 \sin\theta\, (c^2 + s^2\, \widetilde{Y}^2)\, d\theta \wedge (d\phi_1 - g\, A_{(1)}) \wedge (d\phi_2 - g\, A_{(1)}) \\ & - s^4 \sin\tilde{\theta}\, (s^2 + c^2\, Y^2)\, d\tilde{\theta} \wedge (d\phi_3 - g\, \tilde{A}_{(1)}) \wedge (d\phi_4 - g\, \tilde{A}_{(1)}) \Big)\,. \end{aligned} \tag{3.54}$$

Now, making use of (3.42), one can obtain

$$\begin{aligned} \hat{A}'_{(3)} = \frac{\chi\, e^{\varphi}}{8\, g^3} \Big(& \frac{c^4}{s^2 + c^2\, Y^2}\, \sin\theta\, d\theta \wedge d\phi \wedge\, (d\psi - 2g\, A_{(1)}) \\ & - \frac{s^4}{c^2 + s^2\, \widetilde{Y}^2}\, \sin\tilde{\theta}\, d\tilde{\theta} \wedge d\tilde{\phi} \wedge\, (d\tilde{\psi} - 2g\, \widetilde{A}_{(1)}) \Big)\,. \end{aligned} \tag{3.55}$$

To relate the above result to that of [34] in eqn (7), one has to find out

$$\begin{aligned} \epsilon_{(3)} & \equiv \tfrac{1}{6} \varepsilon_{ijk}\, h^i \wedge h^j \wedge h^k = h^1 \wedge h^2 \wedge h^3 = \sigma^1 \wedge \sigma^2 \wedge (\sigma^3 - 2g\, A_{(1)}) \\ & = \sin\theta\, d\theta \wedge d\phi \wedge\, (d\psi - 2g\, A_{(1)})\,, \end{aligned} \tag{3.56}$$

where we have used (3.47). One can obtain the similar result for $\tilde{\epsilon}_{(3)}$ and hence, can verify

$$\hat{A}'_{(3)} = f\, \epsilon_{(3)} + \tilde{f}\, \tilde{\epsilon}_{(3)}\,, \tag{3.57}$$

where

$$f=\frac{\chi\, e^{\varphi}\, c^4}{8g^3\,(s^2+c^2\,Y^2)}\,,\qquad \tilde{f}=-\frac{\chi\, e^{\varphi}\, s^4}{8g^3\,(c^2+s^2\,\widetilde{Y}^2)}\,. \tag{3.58}$$

It turns out this is the same result as eqn (8) of [34] with the above mentioned rescaling for the coupling constant g, i.e. $g=\frac{1}{\sqrt{2}}g^{\text{Ref. [34]}}$.

To obtain the full ansatz for the four-form field strength, one needs to find out other parts of it. To do so, one may calculate $\hat{F}''_{(4)}$, presented in eq (43) of [35] as follows

$$\hat{F}''_{(4)}=-\frac{1}{2g^2}\,|W|^{-2}\sum_i d\mu_i^2\wedge(d\phi_i-g\,A^i_{(1)})\wedge R_i\,. \tag{3.59}$$

Here R_i are two-forms introduced in eqn (40) of [35] and according to $2+2$ specialization, they read

$$R_1=R_2=\widetilde{Y}^2\,(1+b^2)\,(*F_{(2)}+bF_{(2)})\,,\qquad R_3=R_4=Y^2\,(1+b^2)\,(*\widetilde{F}_{(2)}-b\widetilde{F}_{(2)})\,, \tag{3.60}$$

where $F_{(2)}=dA_{(1)}$ and $\widetilde{F}_{(2)}=d\widetilde{A}_{(1)}$. Also, as it can be obtained easily from eqn (37) of [35], $W=1+b^2$. Now, making use of the following relations

$$\begin{aligned} d\phi_1-g\,A &= \tfrac{1}{2}(d\psi+d\phi-2g\,A)=\tfrac{1}{2}h_3+\sin^2\tfrac{1}{2}\theta\,d\phi\,,\\ d\phi_2-g\,A &= \tfrac{1}{2}(d\psi-d\phi-2g\,A)=\tfrac{1}{2}h_3-\cos^2\tfrac{1}{2}\theta\,d\phi\,, \end{aligned} \tag{3.61}$$

one can write down the following result for $\hat{F}''_{(4)}$

$$\begin{aligned} \hat{F}''_{(4)} =& \frac{1}{2g^2\,(1+b^2)}\Big[e^{-\varphi}\,(cs\,d\xi\wedge h_3+\tfrac{1}{2}c^2\,h_1\wedge h_2)\wedge\,(*F_{(2)}+bF_{(2)})\\ &+e^{\varphi}\,(-cs\,d\xi\wedge\tilde{h}_3+\tfrac{1}{2}s^2\,\tilde{h}_1\wedge\tilde{h}_2)\wedge\,(*\widetilde{F}_{(2)}-b\widetilde{F}_{(2)})\Big]\,. \end{aligned} \tag{3.62}$$

Again with the rescling of the coupling constant , one can obtain the same result as eqn (10) of [34], after applying an appropriate truncations.

The only remaining term for finding the complete uplift ansazt for four-form is $\hat{G}_{(4)}$ which can be readily obtained by using eqn (5.8) of [33]. Hence the full ansatz reads

$$\hat{F}_{(4)} = -2g\, U\, \epsilon_{(4)} + d\hat{A}'_{(3)} + \hat{F}''_{(4)} + \frac{cs}{g}\,(-*d\varphi + e^{2\varphi}\,\chi * d\chi) \wedge d\xi\,, \tag{3.63}$$

where $U = c^2Y^2 + s^2\widetilde{Y}^2 + 2$ and $A'_{(3)}$ and $\hat{F}''_{(4)}$ are given by (3.57) and (3.62) respectively. Again, one may check the above ansatz is consistent with the truncated result in [34] after applying the rescaling in the coupling constant.

3.3.3 Finding the bosonic Lagrangian

The bosonic Lagrangian, as we mentioned in $3+1$ case, can be derived from eqn (34) of [35]. It reads

$$\mathcal{L}_4 = R*\mathbb{1} - \tfrac{1}{2}\sum_{i=1}^{3}\Big(*d\varphi_i \wedge d\varphi_i + e^{2\varphi_i}*d\chi_i \wedge d\chi_i\Big) - V*\mathbb{1} + \mathcal{L}_{Kin} + \mathcal{L}_{CS}\,. \tag{3.64}$$

Note that since $b_2 = b_3 = 0$, this leads to a great deal of simplicity in calculation. One may readily calculate V, $\mathcal{L}_{Kin}$ and $\mathcal{L}_{CS}$ from eqn (35) , eqn (36) and eqn (38) of [35] respectively and the result is

$$\begin{aligned}\mathcal{L} =\; & R*\mathbb{1} - \tfrac{1}{2}*d\varphi \wedge d\varphi - \tfrac{1}{2}e^{2\varphi}*d\chi \wedge d\chi - V*\mathbb{1}\\ & - Y^{-2}*F_{(2)} \wedge F_{(2)} - \widetilde{Y}^{-2}*\widetilde{F}_{(2)} \wedge \widetilde{F}_{(2)}\\ & - \chi\, F_{(2)} \wedge F_{(2)} + \chi Y^2\, \widetilde{Y}^{-2}\, \widetilde{F}_{(2)} \wedge \widetilde{F}_2\,,\end{aligned} \tag{3.65}$$

where

$$V = -4g^2\,(Y^2 + \widetilde{Y}^2 + 4)\,. \tag{3.66}$$

Therefore, we could be able to recover all results which previously presented in [34] for case of $2+2$ truncation of gauged STU supergravity.

4. ON THE PAULI REDUCTION OF MINIMAL SUPERGRAVITY IN FIVE DIMENSIONS

4.1 Introduction

In this chapter, we address the possibility of the consistent Pauli S^2 Reduction of minimal supergravity in five dimensions. The main motivation for this investigation, is the resemblance between the bosonic Lagrangian of this theory and that of eleven-dimensional supergravity, which can be observed from the following expressions

$$\begin{aligned}\mathcal{L}_5 &= R{*}\mathbb{1} - \tfrac{1}{2}{*}F_{(2)} \wedge F_{(2)} - \tfrac{1}{3\sqrt{3}}\, F_{(2)} \wedge F_{(2)} \wedge A_{(1)}\,, \\ \mathcal{L}_{11} &= R{*}\mathbb{1} - \tfrac{1}{2}{*}F_{(4)} \wedge F_{(4)} - \tfrac{1}{6}\, F_{(4)} \wedge F_{(4)} \wedge A_{(3)}\,, \end{aligned} \tag{4.1}$$

where $F_{(2)} = dA_{(1)}$ and $F_{(4)} = dA_{(3)}$. Since there are S^7, S^4 and S^5 consistent Pauli reductions of the latter theory, one wonders about the existence of S^2 or S^3 consistent Pauli reductions of the former one.

One may study this problem by writing a trial ansatz, and then considering the five-dimensional equations of motion, if the internal manifold coordinates (S^2 components in this case) remarkably conspire and cancel out in these equations, one can claim the consistent ansatz has been found. The important point is, this ansatz should include the gauge bosons with gauge group of the isometry of the internal manifold (in this case, three gauge bosons with $SU(2)$ gauging). Using this method, the construction of the ansatz was not successful. Therefore, we investigated another method, which is a more systematic one, and can be used to study the other cases as well.

This method, originally presented in [7], was named the "Hopf fibration technique" in [45]. The idea is starting from a higher dimensional theory and performing a (necessary consistent) DeWitt reduction on a group manifold G, then again from the initial theory, one can perform another DeWitt reduction on a group manifold H, where the latter group is a sub-group of the former one. Now, viewing the group manifold G as an H Hopf fibration over G/H, it is guaranteed by a group-theoretical argument that the coset reduction G/H is indeed consistent. In our case, the higher dimensional theory is minimal six-dimensional supergravity. We implement a consistent DeWitt $SU(2)$ reduction on this theory and find a three-dimensional space-time. In addition to this, we perform a consistent Kaluza-Klein S^1 reduction to obtain minimal five-dimensional supergravity coupled to a vector multiplet. Now, writing the group manifold $SU(2)$ as a $U(1)$ Hopf fibration over $SU(2)/U(1) = S^2$ coset space, one can obtain a consistent Pauli S^2 reduction from the five-dimensional theory to the three-dimensional one. However, to accomplish the task of finding the S^2 reduction of minimal five-dimensional supergravity, one needs to perform a consistent truncation in five dimensions. But, as we will clarify later, this truncation gives rise to the vanishing of the field strengths, and hence it implies the impossibility of the S^2 reduction of this theory.

The rest of this chapter organizes as follows. In section 4.2 , we will obtain an S^1 reduction of minimal six-dimensional supergravity, and find a relation, due to the self-duality of three-form field strength in six dimensions, between two- and three- form field strengths in five dimensions. In section 4.3, we will present an $S^3 = SU(2)$ DeWitt reduction from six-dimensional minimal supergravity down to a three-dimensional space-time. Also, we consider this S^3 as a Hopf fibration of $U(1)$ over an S^2, and by this means, write down the $SU(2)$ DeWitt ansätze in a Hopf fibration fashion. Hence by comparing these ansätze with those of the circle reduction in section 4.2.2, one can find finally the ansätze for S^2 reduction of minimal supergravity coupled to a vector multiplet. However, as we will show in section 4.4, the truncation we have found for obtaining pure minimal supergravity in five dimensions is not compatible with our ansätze, meaning that, one cannot find a consistent Pauli S^2 reduction of minimal $D = 5$ supergravity using the Hopf fibration technique.

4.2 S^1 reduction of minimal $D = 6$ supergravity

The field content of the bosonic sector of minimal supergravity in six dimensions consists of a metric $\hat{g}_{MN}$ and a two-form potential $\hat{B}_{(2)}$ whose field strength is a self-dual field, i.e. $\hat{H}_{(3)} = d\hat{B}_{(2)} = \hat{*}\hat{H}_{(3)}$. Our convention is to insert a hat on the six-dimensional and a bar on five-dimensional fields to avoid any ambiguity. In any $4n + 2$ dimensions, self duality of the field strength, $*H_{(2n+1)} = H_{(2n+1)}$ yields to

$$* H_{(2n+1)} \wedge H_{(2n+1)} = H_{(2n+1)} \wedge H_{(2n+1)} = -H_{(2n+1)} \wedge H_{(2n+1)} = 0\,. \tag{4.2}$$

Hence, it is not possible to write down a Lagrangian for this theory and one should state equations of motion instead. However, minimal supergravity in six dimensions can be derived from the following six-dimensional bosonic string Lagrangian with an appropriate truncation

$$\hat{\mathcal{L}} = \hat{R}\,\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}d\hat{\phi} \wedge d\hat{\phi} - \tfrac{1}{4}e^{a\,\hat{\phi}}\,\hat{*}\hat{H}_{(3)} \wedge \hat{H}_{(3)}\,, \tag{4.3}$$

where $\hat{H}_{(3)} = d\hat{B}_{(2)}$, and $a^2 = \frac{8}{D-2} = 2$. Here, $\hat{*}$, $\bar{*}$ and $*$ denote the Hodge dual of forms in 6, 5 and 3 dimensions respectively. Therefore, the equations of motion read

$$\begin{aligned} \hat{R}_{MN} &= -\frac{e^{a\hat{\phi}}}{12(D-2)}\,\hat{H}^2\,\hat{g}_{MN} + \tfrac{1}{2}\partial_M\hat{\phi}\,\partial_N\hat{\phi} + \tfrac{e^{a\hat{\phi}}}{8}\hat{H}^2_{MN}, \\ \hat{\Box}\hat{\phi} &= \tfrac{a}{24}\,e^{a\hat{\phi}}\,\hat{H}^2, \\ d\big(e^{a\hat{\phi}}\hat{*}\hat{H}_{(3)}\big) &= 0, \end{aligned} \tag{4.4}$$

where $\hat{H}^2 = \hat{H}_{MNP}\,\hat{H}^{MNP}$, and $\hat{H}^2_{MN} = \hat{H}_{MPQ}\,\hat{H}_N{}^{PQ}$. Now, upon imposing the self duality condition on three-form field strength, one obtains $\hat{H}^2 = 0$. Then to have a consistent truncation, due to the second equation of (4.4), the scalar field should be set to zero. Therefore, the truncated theory, which is the bosonic sector of minimal supergravity in six dimensions, has the following equations of motion

$$\hat{R}_{MN} = \tfrac{1}{8}\hat{H}^2_{MN}, \quad d\hat{*}\hat{H}_{(3)} = 0, \quad \hat{*}\hat{H}_{(3)} = \hat{H}_{(3)}. \tag{4.5}$$

4.2.1 A Kaluza-Klein circle reduction to $5D$

According to the standard Kaluza-Klein S^1 reduction presented in chapter 2, one can write

$$d\hat{s}_6^2 = e^{2\bar{\alpha}\phi}\, d\bar{s}_5^2 + e^{2\bar{\beta}\phi}\,(g^{-1}\,d\tau + \mathcal{A}_{(1)})^2\,, \tag{4.6}$$
$$\hat{H}_{(3)} = d\hat{B}_{(2)}\,, \qquad \hat{B}_{(2)} = B_{(2)} + B_{(1)}\,\wedge g^{-1}\,d\tau\,, \qquad \hat{H}_{(3)} = H_{(3)} +\, H_{(2)} \wedge (g^{-1}\,d\tau + \mathcal{A}_{(1)})\,,$$

where we choose $\bar{\alpha}^2 = \frac{1}{24}$ and $\bar{\beta} = -3\bar{\alpha}$ to find a canonically normalized kinetic term for the "breathing mode" ϕ in five-dimensional bosonic Lagrangian. Here $z = g^{-1}\,\tau$ has the dimensions of length, while τ is a dimensionless coordinate. As usual, the higher dimensional relations above yield the following relations in five dimensions

$$H_{(3)} = dB_{(2)} - dB_{(1)} \wedge \mathcal{A}_{(1)}\,, \qquad H_{(2)} = dB_{(1)}\,. \tag{4.7}$$

One can find the six-dimensional dual of the field strength as follows

$$\begin{aligned}\hat{*}\hat{H}_{(3)} &= \hat{*}H_{(3)} + \hat{*}\big(H_{(2)} \wedge (dz + \mathcal{A}_{(1)})\big) = (-1)^{3\times 1}\, e^{-\bar{\alpha}\phi}\, e^{\bar{\beta}\phi}\, \bar{*}H_{(3)} \wedge\, (dz + \mathcal{A}_{(1)})\\ &\quad +(-1)^{2\times 0}\, e^{\bar{\alpha}\phi}\, e^{-\bar{\beta}\phi}\, \bar{*}H_{(2)} = -e^{-4\bar{\alpha}\phi}\, \bar{*}H_{(3)}\, \wedge (dz + \mathcal{A}_{(1)}) + e^{4\bar{\alpha}\phi}\, \bar{*}H_{(2)}\ .\end{aligned} \tag{4.8}$$

Now, the self duality condition implies

$$H_{(3)} = e^{4\bar{\alpha}\phi}\, \bar{*}H_{(2)}\,, \tag{4.9}$$

in five dimensions. Therefore, the six-dimensional three-form field strength shall be

$$\hat{H}_{(3)} = e^{4\bar{\alpha}\phi}\, \bar{*}H_{(2)} + H_{(2)}\, \wedge (dz + \mathcal{A}_{(1)})\,. \tag{4.10}$$

4.2.2 Finding the bosonic Lagrangian in $5D$

The six-dimensional Einstein equation, writing in the flat indices, has a simple form

$$\hat{R}_{AB} = \tfrac{1}{8}\hat{H}_A{}^{CD}\hat{H}_{BCD}\,. \tag{4.11}$$

Now one needs to find the components of $\hat{H}^2_{AB} = \hat{H}_A{}^{CD}\hat{H}_{BCD}$ in five dimensions. To do so, from (4.7) one can obtain the following relations for the flat components of the five- and six-dimensional field strengths as usual

$$\hat{H}_{abc} = e^{-3\bar{\alpha}\phi}\,H_{abc}\,, \qquad \hat{H}_{ab6} = e^{\bar{\alpha}\phi}\,H_{ab}\,, \tag{4.12}$$

where the above five-dimensional field strengths are clearly a three-form and a two-form fields respectively, and for simplicity we refrain to insert the subscripts when there is no ambiguity. Recalling the self duality relation (4.9), one has the following result for the components of $H_{(3)}$ and $H_{(2)}$

$$H_{abc} = \tfrac{1}{2}e^{4\bar{\alpha}\phi}\,\epsilon_{abc}{}^{de}\,H_{de}\,. \tag{4.13}$$

Therefore, using the above relation, one can obtain the following expressions for the components of $\hat{H}^2_{AB}$

$$\begin{aligned}
\hat{H}^2_{ab} &= \hat{H}_{acd}\,\hat{H}_b{}^{cd} + 2\hat{H}_{ac6}\,\hat{H}_b{}^{c6} = e^{-6\bar{\alpha}\phi}\,H_{acd}\,H_b{}^{cd} + 2e^{\bar{\alpha}\phi}\,H_{ac}\,H_b{}^c \\
&= \tfrac{1}{4}e^{2\bar{\alpha}\phi}\,\epsilon_{acdef}\,H^{df}\,\epsilon_b{}^{cdgh}\,H_{gh} + 2e^{\bar{\alpha}\phi}\,H_{ac}\,H_b{}^c = e^{2\bar{\alpha}\phi}\,(-\eta_{ab}\,H_{cd}\,H^{cd} + 4H_{ac}\,H_b{}^c)\,, \\
\hat{H}^2_{a6} &= \hat{H}_{abc}\,\hat{H}_6{}^{bc} = e^{-2\bar{\alpha}\phi}\,H_{abc}\,H^{bc} = \tfrac{1}{2}e^{2\bar{\alpha}\phi}\,\epsilon_{abcde}\,H^{bc}\,H^{de}\,, \\
\hat{H}^2_{66} &= \hat{H}_{6ab}\,\hat{H}_6{}^{ab} = e^{2\bar{\alpha}\phi}\,H_{ab}\,H^{ab}\,.
\end{aligned} \tag{4.14}$$

Using the expressions for higher dimensional Ricci components in (2.13) of chapter 2, one has the following five-dimensional relations for the higher dimensional Einsteins equation in (4.11)

$$\begin{aligned}
\hat{R}_{ab} &= e^{-2\bar{\alpha}\phi}\left(R_{ab} - \tfrac{1}{2}\partial_a\phi\,\partial_b\phi - \tfrac{1}{2}e^{-8\bar{\alpha}\phi}\,\mathcal{F}_{ab}^2 - \bar{\alpha}\,\eta_{ab}\,\Box\phi\right)\\
&= \tfrac{1}{8}e^{2\bar{\alpha}\phi}\left(-\eta_{ab}\,H_{cd}\,H^{cd} + 4H_{ac}\,H_b{}^c\right),\\
\hat{R}_{az} &= \hat{R}_{zb} = \tfrac{1}{2}e^{2\,\bar{\alpha}\phi}\,\nabla^b\left(e^{-8\bar{\alpha}\phi}\,\mathcal{F}_{ab}\right) = \tfrac{1}{16}e^{2\bar{\alpha}\phi}\,\epsilon_{abcde}\,H^{bc}\,H^{de}\,,\\
\hat{R}_{zz} &= e^{-2\bar{\alpha}\phi}\left(3\bar{\alpha}\,\Box\phi + \tfrac{1}{4}e^{-8\bar{\alpha}\phi}\,\mathcal{F}^2\right) = \tfrac{1}{8}e^{2\bar{\alpha}\phi}\,H_{ab}\,H^{ab}\,.
\end{aligned} \tag{4.15}$$

In differential geometry, there is an operator which is the adjoint of the exterior derivative (sometimes called the interior derivative) which can be written as follows

$$(\delta\,\omega_{(p)})_{\mu_1\cdots\mu_{p-1}} \equiv (-1)^{np+t}\,(*d*\omega_{(p)})_{\mu_1\cdots\mu_{p-1}} = -\nabla^\mu\,\omega_{\mu\mu_1\cdots\mu_{p-1}}\,, \tag{4.16}$$

where $\omega_{(p)}$ is a p-form and n and t are the number of space-time and time-like dimensions respectively. Having used this relation, one can write the second equation of (4.15) in the form language as follows

$$d(e^{-8\bar{\alpha}\phi}\,\bar{*}\mathcal{F}_{(2)}) = -\tfrac{1}{2}H_{(2)}\wedge H_{(2)}\,. \tag{4.17}$$

Having obtained the above equation, now it is not hard to find a five-dimensional Lagrangian which yields the three equations in (4.15) as an Einstein, a one-form gauge potential $\mathcal{A}_{(1)}$ and a scalar equations of motion. From the right hand side of (4.17), one needs to include the term $-\frac{1}{2}H_{(2)}\wedge H_{(2)}\wedge\mathcal{A}_{(1)}$ in the Lagrangian, while the pre-factors of the kinetic terms may be found from the third equation in (4.15), i.e. the scalar equation of motion. The five-dimensional bosonic Lagrangian shall be written as

$$\mathcal{L}_5 = \bar{R}\,\bar{*}\mathbb{1} - \tfrac{1}{2}\bar{*}d\phi\wedge d\phi - \tfrac{1}{2}e^{-8\bar{\alpha}\phi}\,\bar{*}\mathcal{F}_{(2)}\wedge\mathcal{F}_{(2)} - \tfrac{1}{2}e^{4\bar{\alpha}\phi}\,\bar{*}H_{(2)}\wedge H_{(2)} - \tfrac{1}{2}H_{(2)}\wedge H_{(2)}\wedge\mathcal{A}_{(1)}\,. \tag{4.18}$$

The first equation of (4.15), upon using the scalar equation of motion, gives rise to the Einstein equation derived from the above Lagrangian. However, the equation for $B_{(1)}$, the gauge potential whose field strength is $H_{(2)} = dB_{(1)}$, did not appear in (4.15), but it can be derived from the Bianchi identity, i.e. $d\hat{H}_{(3)} = 0$. Namely, using (4.10), one has

$$d\hat{H}_{(3)} = d\big(e^{4\bar{\alpha}\phi}\,\bar{*}H_{(2)}\big) + H_{(2)}\wedge\mathcal{F}_{(2)} = 0. \tag{4.19}$$

This is exactly the equation for the gauge potential $B_{(1)}$ derived from the above Lagrangian.

4.2.3 Truncations to minimal supergravity in $5D$

The Lagrangian we found in (4.18) is five dimensional minimal supergravity coupled to one vector multiplet. One needs to perform the following truncation to obtain minimal supergravity in five dimensions

$$\phi = 0\,, \qquad H_{(2)} = \sqrt{2}\,\mathcal{F}_{(2)}\,. \tag{4.20}$$

The factor of $\sqrt{2}$ is crucial here, since, to satisfy the third equation of (4.15), upon truncation of the scalar field to zero, the presence of this factor guarantees the source term for the truncated scalar field is also vanishing.

To have a canonical normalization for the kinetic term of the field strength, one may use the following field redefinition

$$\widetilde{A}_{(1)} = \sqrt{3}\,\mathcal{A}_{(1)}\,. \tag{4.21}$$

Hence, the bosonic Lagrangian shall be that of pure minimal supergravity in five dimensions

$$\mathcal{L}_5 = \bar{R}\,\bar{*}\mathbb{1} - \tfrac{1}{2}\bar{*}\widetilde{F}_{(2)}\wedge\widetilde{F}_{(2)} - \tfrac{1}{3\sqrt{3}}\,\widetilde{F}_{(2)}\wedge\widetilde{F}_{(2)}\wedge\widetilde{A}_{(1)}\,. \tag{4.22}$$

4.3 $SU(2)$ DeWitt reduction from $D = 6$ to $D = 3$

In this section, we consider a consistent $SU(2)$ DeWitt reduction of minimal supergravity in six dimensions down to a three-dimensional theory. This construction was already presented in [44] and we shall review it here.

It is convenient to present the general DeWitt reduction ansatz of the Einstein-Hilbert action in $D = n + q$ dimensions reduced on a q-dimensional group manifold. The metric ansatz, stated in [7], and it is

$$d\hat{s}^2_{(n+q)} = e^{2\alpha\varphi}\, ds^2_{(n)} + g^{-2} e^{2\beta\varphi}\, \widetilde{T}_{ij}\, \nu^i\, \nu^j, \tag{4.23}$$

where one-forms ν^i

$$\nu^i = \sigma^i - gA^i\,, \tag{4.24}$$

are written in terms of the left-invariant one-forms σ^i in the group manifold. The unimodular matrix $\widetilde{T}_{ij}$ parameterizes the remaining scalar fields of the n-dimensional theory. The scalar field φ is a “breathing mode" and constants (α, β) shall be determined by demanding the lower dimensional Lagrangian has no scalar pre-factor in the Einstein-Hilbert term and the scalar has a canonically normalized kinetic term. Hence the constants are

$$\alpha = -\sqrt{\frac{q}{2(n-2)(n+q-2)}}\,, \qquad \beta = -\frac{\alpha\,(n-2)}{q}\,. \tag{4.25}$$

We investigate the case of $G = SU(2)$ in the following.

4.3.1 The ansätze for an $SU(2)$ group manifold reduction

The reduction ansätze for the metric and the self-dual three-form are the following

$$d\hat{s}^2_6 = e^{2\alpha\varphi} ds^2_3 + g^{-2}\, e^{2\beta\varphi}\, \widetilde{T}_{ij}\;\, \nu^i\, \nu^j\,, \tag{4.26}$$

$$\hat{H}_{(3)} = mg^{-3}\Omega_{(3)} + me^{4\alpha\varphi}\,\epsilon_{(3)} + \tfrac{1}{2}g^{-2}\,\varepsilon_{ijk}\, B^i \wedge \nu^j \wedge \nu^k - g^{-1} e^{\frac{4\alpha\varphi}{3}}\, \widetilde{T}_{ij} *B^i \wedge\, \nu^j\,, \tag{4.27}$$

where the constants α and β can be found from (4.25) by setting $(n, q) = (3, 3)$ and they are given by $\alpha^2 = \frac{3}{8}$ and $\beta = -\frac{\alpha}{3}$. One-forms ν^i are given by (4.24) as $\nu^i = \sigma^i - gA^i$ and three left invariant one-form σ^i are expressed as the following expression in terms of the Euler angles (ψ, θ, τ)

$$\sigma_1 = \cos\psi\, d\theta{+}\sin\psi\, \sin\theta\, d\tau\,, \quad \sigma_2 = -\sin\psi\, d\theta{+}\cos\psi\, \sin\theta\, d\tau\,, \quad \sigma_3 = d\psi{+}\cos\theta\, d\tau\,, \tag{4.28}$$

and they satisfy the following relation

$$d\sigma_i = -\tfrac{1}{2}\varepsilon_{ijk}\,\sigma_j \wedge \sigma_k\,. \tag{4.29}$$

The three-form $\Omega_{(3)}$ is defined by

$$\Omega_{(3)} \equiv \nu^1 \wedge \nu^2 \wedge \nu^3\,, \tag{4.30}$$

where $\epsilon_{(3)}$ is the volume form of the three-dimensional space-time, and B^i denotes an $SU(2)$ triplet of one-form fields.

One can define the covariant derivative and the field strength of the $SU(2)$ gauge potentials A^i as usual

$$D\nu^i \equiv d\nu^i + g\,\varepsilon_{ijk}\,A^j \wedge \nu^k \qquad F^i = dA^i + \tfrac{1}{2}g\,\varepsilon_{ijk}\,A^j \wedge A^k\,. \tag{4.31}$$

Then, one can find an expression for $D\nu^i$ as follows

$$\begin{aligned} D\nu^i &\equiv d\nu^i + g\varepsilon_{ijk}\,A^j \wedge \nu^k = d\sigma^i - g\,dA^i + g\varepsilon_{ijk}\,A^j \wedge \nu^k = -\tfrac{1}{2}\varepsilon_{ijk}\,\sigma_j \wedge \sigma_k \\ &\quad -g\,dA^i + g\varepsilon_{ijk}\,A^j \wedge \nu^k = -\tfrac{1}{2}\varepsilon_{ijk}\,(\nu_j + gA_j)\,\wedge(\nu_k + gA_k) - g\,dA^i + g\varepsilon_{ijk}\,A^j \wedge \nu^k \\ &= -\tfrac{1}{2}\varepsilon_{ijk}\,\nu_j \wedge \nu_k - \tfrac{1}{2}g\varepsilon_{ijk}\,\nu_j \wedge A_k - \tfrac{1}{2}g\varepsilon_{ijk}\,A_j \wedge \nu_k - \tfrac{1}{2}g^2\,\varepsilon_{ijk}\,A_j \wedge A_k - g\,dA^i \\ &\quad +g\,\varepsilon_{ijk}\,A^j \wedge \nu^k = -g\,dA^i - \tfrac{1}{2}g^2\,\varepsilon_{ijk}\,A_j \wedge A_k - \tfrac{1}{2}\varepsilon_{ijk}\,\nu_j \wedge \nu_k = -\tfrac{1}{2}\varepsilon_{ijk}\,\nu_j \wedge \nu_k - g\,F^i\,. \end{aligned} \tag{4.32}$$

Using (4.27) and the Bianchi identity $d\hat{H}_{(3)} = 0$ one has

$$\begin{aligned} d\hat{H}_{(3)} &= D\hat{H}_{(3)} = mg^{-3}\,d\Omega_{(3)} + \tfrac{1}{2}g^{-2}\varepsilon_{ijk}\,DB^i \wedge \nu^j \wedge \nu^k - g^{-2}\varepsilon_{ijk}\,B^i \wedge D\nu^j \wedge \nu^k \\ &\quad -g^{-1}\tfrac{4}{3}\alpha\,e^{\frac{4\alpha\varphi}{3}}\,\widetilde{T}_{ij}\,d\varphi \wedge *B^i \wedge \nu^j - g^{-1}\,e^{\frac{4\alpha\varphi}{3}}\,D\widetilde{T}_{ij} \wedge *B^i \wedge \nu^j \\ &\quad -g^{-1}\,e^{\frac{4\alpha\varphi}{3}}\,\widetilde{T}_{ij} \wedge *B^i \wedge D\nu^j - g^{-1}\,e^{\frac{4\alpha\varphi}{3}}\,\widetilde{T}_{ij} \wedge D * B^i \wedge \nu^j = 0\,. \end{aligned} \tag{4.33}$$

We are interested in terms involving $\nu^j \wedge \nu^k$, and since they are independent of those terms which involving ν^i, then the former terms should be vanishing according to the above equation. Also using (4.30), the following expression for $d\Omega_{(3)}$ can be written

$$d\Omega_{(3)} = \tfrac{3}{6}\varepsilon_{ijk}\, D\nu^i \wedge \nu^j \wedge \nu^k = -\tfrac{1}{2} g\, \varepsilon_{ijk}\, F^i \wedge \nu^j \wedge \nu^k \,. \tag{4.34}$$

Then, according to the above argument about the vanishing of terms involving $\nu^j \wedge \nu^k$ in (4.33), one can conclude

$$\varepsilon_{ijk}\, \nu^j \wedge \nu^k \wedge \big(-\tfrac{1}{2} m g^{-2}\, F^i + \tfrac{1}{2} g^{-2} DB^i + \tfrac{1}{2} g^{-1} e^{\frac{4\alpha\varphi}{3}}\, \widetilde{T}_{i\ell} * B^\ell \big) = 0 \,. \tag{4.35}$$

In other words

$$DB^i - mF^i + g e^{\frac{4}{3}\alpha\varphi}\, \widetilde{T}_{ij} *B^j = 0 \,, \tag{4.36}$$

where $DB^i = dB^i + g\, \varepsilon_{ijk}\, A^j \wedge B^k$.

Upon plugging in the ansätze (4.26) and (4.27) into the six dimensional equations of motion (4.5), one can find the three-dimensional equations of motion and it can be derived from a bosonic Lagrangian presented in [44].

4.3.2 $SU(2)$ as a Hopf fibration

So far, we have obtained a circle reduction of minimal supergravity in six dimensions and also an $SU(2)$ reduction of that theory. Now, one can re-interpret the latter reduction as a $U(1)$ Hopf fibration of the circle reduction over S^2. Hence, one can find a consistent S^2 reduction of five-dimensional theory. The crucial point here is, one needs to re-write elements of the $SU(2)$ reduction in the Hopf fibration manner.

To describe the S^2 reduction, one may introduce three Cartesian coordinates μ^i, where $\mu^i\, \mu^i = 1$, to describe the unit two-sphere in $\mathbb{R}^3$. One may use the following re-parametrization in terms of the Euler angles (θ, ψ)

$$\mu_1 = \sin\psi\, \sin\theta \,, \quad \mu_2 = \cos\psi\, \sin\theta \,, \quad \mu_3 = \cos\theta \,. \tag{4.37}$$

Now, one needs to find relations between elements of the $SU(2)$ reduction and these three parameters. The starting point is the following expression between the three parametrization coordinates and the three left-invariant one-forms

$$d\mu^i = \varepsilon_{ijk}\,\mu^j\,\sigma^k\,, \tag{4.38}$$

where σ^i are defined in (4.28). One can easily verify the above relation by a direct calculation. Now, considering the gauge fields, the above relation yields to the following

$$D\mu^i = \varepsilon_{ijk}\,\mu^j\,\nu^k\,, \tag{4.39}$$

where the covariant derivative is defined as $D\mu^i \equiv d\mu^i + g\,\varepsilon_{ijk}\,A^j\,\mu^k$. Now, having obtained that, let us find the following combination

$$\varepsilon_{ijk}\,\mu^j\,D\nu^k = \varepsilon_{ijk}\,\mu^j\,\varepsilon_{kmn}\,\mu^m\,\nu^n = \mu^j\,(\mu^i\,\nu^j - \mu^j\,\nu^i) = \mu^i\,(\mu^j\,\nu^j) - \nu^i\,. \tag{4.40}$$

Therefore one can write

$$\begin{aligned} \nu^i &= \sigma^i - g\,A^i = -\varepsilon_{ijk}\,\mu^j\,D\mu^k + \mu^i\,\sigma\,, \\ \sigma &\equiv \mu^i\,\nu^i = d\tau + \cos\theta\,d\psi - g\,\mu^i\,A^i\,. \end{aligned} \tag{4.41}$$

At this stage, it is useful to introduce some relations which will be needed to perform the calculations. First, we present two well-known relations in linear algebra about the determinant and the inverse of a general $n \times n$ matrix

$$\begin{aligned} \det A &= \varepsilon_{i_1 i_2 \cdots i_n}\,A_{1i_1}\,A_{2i_2}\cdots A_{ni_n} = \tfrac{1}{n!}\varepsilon_{i_1 i_2 \cdots i_n}\,\varepsilon j_1 j_2 \cdots j_n\,A_{i_1 j_1}\,A_{i_2 j_2}\cdots A_{i_n j_n}\,, \\ A^{-1}_{ij} &= \tfrac{1}{(n-1)!}\,\tfrac{1}{\det A}\,\varepsilon_{i\,i_2\cdots i_n}\,\varepsilon_{j\,j_2\cdots j_n}\,A_{j_2 i_2}\cdots A_{j_n i_n}\,, \end{aligned} \tag{4.42}$$

where the second relation above can be derived from the first one, by multiplying A^{-1}_{ij} in both sides of the first relation and using a useful identity named Schouten's trick. The latter is as follows

$$\varepsilon_{[i_1 i_2 \cdots i_n}\, M_{i]} = 0 \Rightarrow \varepsilon_{i_1 i_2 \cdots i_n}\, M_i = \varepsilon_{i\, i_2 \cdots i_n}\, M_{i_1} + \varepsilon_{i_1\, i \cdots i_n}\, M_{i_2} + \cdots \varepsilon_{i_1 i_2 \cdots i}\, M_{i_n}\,. \tag{4.43}$$

Since the anti-symmetrization of $n+1$ components where each of them runs over n value, is always vanishing, hence the result is obvious, but, in spite of its simplicity, it is a very powerful tool and we have used it frequently in our calculations.

Let us employ (4.42) to find a useful lemma

$$\varepsilon_{ijk}\, A_{jm}\, A_{kn} = \det A\ \varepsilon_{\ell mn}\, A^{-1}_{\ell i}\,, \tag{4.44}$$

where it can be proved as follows

$$\begin{aligned}\varepsilon_{\ell mn}\, A^{-1}_{\ell i} &= \tfrac{1}{2}\tfrac{1}{\det A}\, \varepsilon_{\ell mn}\, \varepsilon_{\ell pq}\, \varepsilon_{ijk}\, A_{jp}\, A_{kq} = \tfrac{1}{2}\tfrac{1}{\det A}\, \varepsilon_{ijk}\, (A_{jm}\, A_{kn} - A_{jn}\, A_{km}) \\ &= \tfrac{1}{\det A}\, \varepsilon_{ijk}\, A_{jm}\, A_{kn}\,. \end{aligned} \tag{4.45}$$

Therefore, for the case of $A_{ij} = \widetilde{T}_{ij}$, one has

$$\varepsilon_{ijk}\, \widetilde{T}_{jm}\, \widetilde{T}_{kn} = \widetilde{T}^{-1}_{i\ell}\, \varepsilon_{\ell mn}\,, \tag{4.46}$$

where we have used the fact that $\widetilde{T}_{ij}$ is uni-modular and symmetric.

To obtain the Hopf fibration result for $SU(2)$ reduction metric ansatz in (4.26), one needs to find a relation for $\widetilde{T}_{ij}\, \nu^i\, \nu^j$. Using (4.41) one can calculate

$$\begin{aligned}\widetilde{T}_{ij}\, \nu^i\, \nu^j &= \widetilde{T}_{ij}\, (-\varepsilon_{imn}\, \mu^m\, D\mu^n + \mu^i\, \sigma)\, (-\varepsilon_{jpq}\, \mu^p\, D\mu^q + \mu^j\, \sigma) = \widetilde{T}_{ij}\, \varepsilon_{imn}\, \varepsilon_{jpq}\, \mu^m\, \mu^p\, D\mu^n\, D\mu^q \\ &\quad -2\widetilde{T}_{ij}\, \varepsilon_{ik\ell}\, \mu^k\, \mu^j\, D\mu^\ell\, \sigma + \widetilde{\Delta}\, \sigma^2 = \widetilde{\Delta}\, \big(\sigma - \widetilde{\Delta}^{-1}\, \widetilde{T}_{ij}\, \varepsilon_{ik\ell}\, \mu^k\, \mu^j\, D\mu^\ell\big)^2 + M\,, \end{aligned} \tag{4.47}$$

where $\widetilde{\Delta} = \widetilde{T}_{ij}\, \mu^i \mu^j$. We try to complete the square and M denotes the remaining terms to be found later. Our motivation to complete the square term is to find a $(d\tau + g\mathcal{A})^2$ term which appears in the S^1 reduction. Taking into account ν^i is a dimensionless quantity, one may have the following relation

$$\sigma - \widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\;\varepsilon_{ik\ell}\,\mu^k\,\mu^j\,D\mu^\ell = d\tau + g\mathcal{A}\,, \tag{4.48}$$

and considering σ from (4.41), one can write the following relation for the Kaluza-Klein vector gauge potential

$$\mathcal{A} = g^{-1}\,\cos\theta\,d\psi - \mu^i A^i - g^{-1}\,\widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\,\varepsilon_{ik\ell}\,\mu^j\mu^k\,D\mu^\ell\,. \tag{4.49}$$

Now, one needs to find M. From (4.47), one can write the following expression for M

$$M = \widetilde{\Delta}^{-1}\left(\widetilde{\Delta}\,\widetilde{T}_{ij}\,\varepsilon_{imn}\,\varepsilon_{jpq}\,\mu^m\,\mu^p\,D\mu^n\,D\mu^q - \left(\widetilde{T}_{ij}\,\varepsilon_{ik\ell}\,\mu^k\,\mu^j\,D\mu^\ell\right)^2\right). \tag{4.50}$$

Writing $\widetilde{\Delta} = \widetilde{T}_{rs}\,\mu^r\mu^s$, the first term of the right hand side of the above relation shall be written as

$$\begin{aligned}
\widetilde{\Delta}\,\widetilde{T}_{ij}\,\varepsilon_{imn}\,\varepsilon_{jpq}\,\mu^m\,\mu^p\,D\mu^n\,D\mu^q &= \widetilde{T}_{rs}\,\mu^r\mu^s\,\widetilde{T}_{ij}\,\varepsilon_{imn}\,\varepsilon_{jpq}\,\mu^m\,\mu^p\,D\mu^n\,D\mu^q \\
&= \widetilde{T}_{rs}\,\widetilde{T}_{ij}\,(\varepsilon_{rpq}\,\mu^j + \varepsilon_{jrq}\,\mu^p + \varepsilon_{jpr}\,\mu^q)\,\varepsilon_{imn}\,\mu^s\mu^m\mu^p\,D\mu^n\,D\mu^q \\
&= (\widetilde{T}_{rs}\,\varepsilon_{rpq}\,\mu^s\mu^p\,D\mu^q)(\widetilde{T}_{ij}\,\varepsilon_{imn}\,\mu^j\mu^m\,D\mu^n) \\
&\quad + \varepsilon_{isp}\,\widetilde{T}^{-1}_{pq}\,\varepsilon_{imn}\,\mu^s\mu^m\,D\mu^n\,D\mu^q \\
&= \left(\widetilde{T}_{ij}\,\varepsilon_{ik\ell}\,\mu^k\,\mu^j\,D\mu^\ell\right)^2 + \widetilde{T}^{-1}_{ij}\,D\mu^i\,D\mu^j\,,
\end{aligned} \tag{4.51}$$

where we have use the following relations

$$\begin{aligned}
&\mu^r\,\varepsilon_{jpq} = \varepsilon_{rpq}\,\mu^j + \varepsilon_{jrq}\,\mu^p + \varepsilon_{jpr}\,\mu^q\,, \\
&\varepsilon_{jrq}\,\widetilde{T}_{rs}\,\widetilde{T}_{ij} = \varepsilon_{isp}\,\widetilde{T}^{-1}_{pq}\,, \qquad \mu^i\mu^i = 1\,, \qquad \mu^i\,D\mu^i = \tfrac{1}{2}D(\mu^i\mu^i) = 0\,.
\end{aligned} \tag{4.52}$$

Now, upon using (4.50) and (4.51) then M has a simple form

$$M = \widetilde{\Delta}^{-1}\,\widetilde{T}^{-1}_{ij}\,D\mu^i\,D\mu^j\,. \tag{4.53}$$

Finally, the metric ansatz (4.26) can be written in the Hopf fibration form

$$d\hat{s}_6^2 = e^{2\alpha\varphi}\, ds_3^2 + g^{-2}\, e^{2\beta\varphi}\, \widetilde{\Delta}^{-1}\, \widetilde{T}_{ij}^{-1}\, D\mu^i\, D\mu^j + g^{-2}\, e^{2\beta\varphi}\, \widetilde{\Delta}\, (d\tau + g\mathcal{A})^2\,, \tag{4.54}$$

where the Kaluza-Klein vector potential $\mathcal{A}$ has defined in (4.49).

The next step is finding a Hopf fibration expression for the three-form field strength $\hat{H}_{(3)}$. Before calculating it, let us introduce some useful relations

$$\nu^i = \mu^i\,(d\tau + g\mathcal{A}) - \widetilde{\Delta}^{-1}\,\widetilde{T}_{jk}\,\varepsilon_{ij\ell}\,\mu^k\,D\mu^\ell\,, \tag{4.55}$$

$$\tfrac{1}{2}\varepsilon_{ijk}\,\nu^j \wedge \nu^k = (d\tau + g\mathcal{A}) \wedge D\mu^i + \widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\,\mu^j\,\omega_{(2)}\,, \tag{4.56}$$

$$\Omega_{(3)} = \tfrac{1}{6}\varepsilon_{ijk}\nu^i \wedge \nu^j \wedge \nu^k = (d\tau + g\mathcal{A}) \wedge \omega_{(2)}\,, \tag{4.57}$$

where

$$\omega_{(2)} = \tfrac{1}{2}\varepsilon_{ijk}\,\mu^i\,D\mu^j\,\wedge\,D\mu^k\,. \tag{4.58}$$

To prove (4.55), one can start from (4.41) to write

$$\begin{aligned}
\nu^i &= -\widetilde{\Delta}^{-1}\,\widetilde{T}_{mn}\,\mu^m\mu^n\,\varepsilon_{ijk}\,\mu^j\,D\mu^k + \mu^i\left(d\tau + g\mathcal{A} + \widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\,\varepsilon_{ik\ell}\,\mu^j\mu^k\,D\mu^\ell\right)\\
&= -\widetilde{\Delta}^{-1}\left(\widetilde{T}_{mn}\,(\varepsilon_{mjk}\,\mu^i + \varepsilon_{imk}\,\mu^j + \varepsilon_{ijm}\,\mu^k)\,\mu^n\mu^j\,D\mu^k - \widetilde{T}_{ij}\,\varepsilon_{ik\ell}\,\mu^i\mu^j\mu^k\,D\mu^\ell\right)\\
&\quad + \mu^i\,(d\tau + g\mathcal{A}) = \mu^i\,(d\tau + g\mathcal{A}) - \widetilde{\Delta}^{-1}\,\widetilde{T}_{jk}\,\varepsilon_{ij\ell}\,\mu^k\,D\mu^\ell\,,
\end{aligned} \tag{4.59}$$

where we have used the Schouten's identity in the second line above and using the following expression for σ

$$\sigma = d\tau + \cos\theta\,d\psi - g\,\mu^i\,A^i = d\tau + g\mathcal{A} + \widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\,\varepsilon_{ik\ell}\,\mu^j\mu^k\,D\mu^\ell\,. \tag{4.60}$$

To prove (4.56) , one can use (4.55)

$$
\begin{aligned}
\tfrac{1}{2}\varepsilon_{ijk}\,\nu^j\wedge\nu^k &= \tfrac{1}{2}\varepsilon_{ijk}\left(\mu^j\,(d\tau+g\mathcal{A})-\widetilde{\Delta}^{-1}\,\varepsilon_{jm\ell}\,\widetilde{T}_{mn}\,\mu^n\,D\mu^\ell\right)\\
&\quad\times\left(\mu^k\,(d\tau+g\mathcal{A})-\widetilde{\Delta}^{-1}\,\varepsilon_{kpq}\,\widetilde{T}_{pr}\,\mu^r\,D\mu^q\right) = -\widetilde{\Delta}^{-1}\,\varepsilon_{ijk}\,\varepsilon_{jm\ell}\,\widetilde{T}_{mn}\,\mu^n\mu^k\,D\mu^\ell\\
&\quad\wedge(d\tau+g\mathcal{A})+\tfrac{1}{2}\widetilde{\Delta}^{-2}\,\varepsilon_{ijk}\,\varepsilon_{jm\ell}\,\widetilde{T}_{mn}\,\mu^n\,\varepsilon_{kpq}\,\widetilde{T}_{pr}\,\mu^r\,\varepsilon_{\ell qs}\,\mu^s\,\omega_{(2)} \qquad (4.61)\\
&= -\widetilde{\Delta}^{-1}\,(\widetilde{T}_{kn}\,D\mu^i-\widetilde{T}_{in}\,D\mu^k)\mu^n\mu^k\,\wedge\,(d\tau+g\mathcal{A})\\
&\quad+\tfrac{1}{2}\widetilde{\Delta}^{-2}\,(\widetilde{T}_{ir}\,\varepsilon_{\ell js}-\widetilde{T}_{jr}\,\varepsilon_{\ell is})\,\varepsilon_{jm\ell}\,\widetilde{T}_{mn}\,\mu^n\mu^r\mu^s\,\omega_{(2)} = (d\tau+g\mathcal{A})\wedge\,D\mu^i\\
&\quad+\widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\,\mu^j\,\omega_{(2)}-\tfrac{1}{2}\widetilde{\Delta}^{-2}\,(\widetilde{T}_{ir}\,\widetilde{T}_{sn}-\widetilde{T}_{sr}\,\widetilde{T}_{in})\,\mu^n\mu^r\mu^s\,\omega_{(2)}\\
&= (d\tau+g\mathcal{A})\wedge D\mu^i+\widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\,\mu^j\,\omega_{(2)}\,,
\end{aligned}
$$

where we have used

$$D\mu^i\wedge D\mu^j=\varepsilon_{ijk}\,\mu^k\,\omega_{(2)}\,. \qquad (4.62)$$

Its proof is the following

$$
\begin{aligned}
D\mu^i\,\wedge D\mu^j &= \tfrac{1}{2}\varepsilon_{ijk}\,\varepsilon_{kpq}\,D\mu^p\,\wedge D\,\mu^q = \tfrac{1}{2}\varepsilon_{ijk}\,\varepsilon_{kpq}\,D\mu^p\,\wedge D\mu^q\,\mu^m\,\mu^m\\
&= \tfrac{1}{2}\varepsilon_{ijk}\,(\varepsilon_{mpq}\,\mu^k+\varepsilon_{kmq}\,\mu^p+\varepsilon_{kpm}\,\mu^q)\,D\mu^p\,\wedge D\mu^q\,\mu^m\\
&= \tfrac{1}{2}\varepsilon_{ijk}\,\varepsilon_{mpq}\,\mu^m D\mu^p\,\wedge D\mu^q\,\mu^k=\varepsilon_{ijk}\,\mu^k\,\wedge\,\omega_{(2)}\,. \qquad (4.63)
\end{aligned}
$$

Note that from (4.56) by multiplying μ^i in both sides, one obtains

$$\tfrac{1}{2}\varepsilon_{ijk}\,\mu^i\,\nu^j\wedge\nu^k=\omega_{(2)}\,. \qquad (4.64)$$

There is also another expression for $\omega_{(2)}$. Multiplying both sides of the first relation of (4.41) by $D\mu^i$, one can easily find

$$\omega_{(2)}=\tfrac{1}{2}D\mu^i\wedge\nu^i\,. \qquad (4.65)$$

Making use of (4.65) , (4.32) and (4.41), one may calculate

$$\omega_{(2)} = -\tfrac{1}{2}\varepsilon_{ijk}\,\mu^i\,\nu^j\wedge\nu^k + 2\omega_{(2)} = \mu^i\,(D\nu^i + gF^i) + D\mu^i\wedge\nu^i = D(\mu^i\nu^i) + g\mu^i F^i$$
$$= d\sigma + g\mu^i F^i = \sin\theta\,d\psi\wedge\,d\theta - g\,d(\mu^i A^i) + g\,\mu^i F^i\,. \qquad (4.66)$$

Finally (4.57) can be proved as follows

$$(d\tau + g\mathcal{A})\wedge\omega_{(2)} = \mu^m\nu^m\wedge\omega_{(2)} = \tfrac{1}{2}\varepsilon_{ijk}\,\mu^i\,\mu^m\nu^m\;\wedge\nu^j\wedge\nu^k = \tfrac{1}{2}\varepsilon_{ijk}\,\mu^i\,\mu^m\,\varepsilon_{mjk}\,\Omega_{(3)} = \Omega_{(3)}\,,$$

where we multiply μ^i in both sides of (4.55) and also employ the relation

$$D\mu^i\wedge D\mu^j\wedge D\mu^k = \varepsilon_{ijk}\,\omega_{(3)} \Rightarrow \omega_{(3)} = \tfrac{1}{6}\varepsilon_{ijk}\,D\mu^i\wedge D\mu^j\wedge D\mu^k = \tfrac{1}{6}\mu^m\mu^m\,\varepsilon_{ijk}\,D\mu^i\wedge D\mu^j\wedge D\mu^k$$
$$= \tfrac{1}{6}\mu^m(\varepsilon_{mjk}\,\mu^i + \varepsilon_{imk}\,\mu^j + \varepsilon_{ijm}\,\mu^k)\,D\mu^i\wedge D\mu^j\wedge D\mu^k = 0\,. \qquad (4.67)$$

After these preliminaries, now we may find the Hopf fibration form of the self-dual field strength $\hat{H}_{(3)}$ given by (4.27)

$$\hat{H}_{(3)} = mg^{-3}\,d\tau + g\mathcal{A})\wedge\omega_{(2)}\; + me^{4\alpha\varphi}\,\epsilon_{(3)} + g^{-2}\,B^i\wedge\Big((d\tau + g\mathcal{A})\wedge D\mu^i + \widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\,\mu^j\,\omega_{(2)}\Big)$$
$$-g^{-1}e^{\frac{4\alpha\varphi}{3}}\,\widetilde{T}_{ij}*B^j\wedge\;\Big(\mu^i\,(d\tau + g\mathcal{A}) - \widetilde{\Delta}^{-1}\,\varepsilon_{im\ell}\,\widetilde{T}_{mk}\,\mu^k\,D\mu^\ell\Big) \qquad (4.68)$$
$$= d(\tau + gA)\wedge\Big[m\,g^{-3}\,\omega_{(2)} - g^{-2}\,B^i\wedge D\mu^i - g^{-1}\,e^{\frac{4}{3}\alpha\varphi}\,\widetilde{T}_{ij}\,\mu^i*B^j\Big]$$
$$+me^{4\alpha\varphi}\,\epsilon_{(3)} + g^{-2}\,\widetilde{\Delta}^{-1}\,\widetilde{T}_{ij}\,\mu^i B^j\wedge\omega_{(2)} + g^{-1}e^{\frac{4}{3}\alpha\varphi}\,\widetilde{\Delta}^{-1}\,\varepsilon_{jkm}\,\mu^\ell\,\widetilde{T}_{ij}\,\widetilde{T}_{k\ell}*B^i\wedge D\mu^m\,.$$

Now we have all ingredients of re-interpreting the DeWitt $SU(2)$ reduction of initial minimal six-dimensional supergravity as a Pauli S^2 reduction of five dimensional theory. To obtain this, one may compare the metric ansätze of the $SU(2)$ expressed in the Hopf fibration form in (4.54) with that of the S^1 reduction in (4.6)

$$d\bar{s}_5^2 = e^{2\alpha\varphi - 2\bar{\alpha}\phi}\, ds_3^2 + g^{-2}\, e^{-\frac{2}{3}\alpha\varphi - 2\bar{\alpha}\phi}\, \widetilde{\Delta}^{-1}\, \widetilde{T}_{ij}^{-1}\, D\mu^i D\mu^j \,, \tag{4.69}$$

$$e^{-6\bar{\alpha}\phi} = e^{-\frac{2}{3}\alpha\varphi}\, \widetilde{\Delta}\,. \tag{4.70}$$

Moreover, by comparison of the three-form self-dual field strength reduction ansätze of the $SU(2)$ in (4.68) with that of the circle reduction in (4.10), one finds

$$\begin{aligned} H_{(2)} &= mg^{-2}\, \omega_{(2)} - g^{-1}\, B^i \wedge D\mu^i - e^{\frac{4}{3}\alpha\varphi}\, \widetilde{T}_{ij}\, \mu^i *B^j \,, \\ e^{4\bar{\alpha}\phi}\, \bar{*}H_{(2)} &= me^{4\alpha\varphi}\, \epsilon_{(3)} + g^{-2}\, \widetilde{\Delta}^{-1}\, \widetilde{T}_{ij}\, \mu^i B^j \wedge \omega_{(2)} + g^{-1} e^{\frac{4}{3}\alpha\varphi}\, \widetilde{\Delta}^{-1}\, \varepsilon_{jkm}\, \mu^\ell\, \widetilde{T}_{ij}\, \widetilde{T}_{k\ell} *B^i \wedge D\mu^m \,. \end{aligned} \tag{4.71}$$

One can follow [7] to define the three-dimensional scalar fields

$$T_{ij} = Y^{\frac{1}{3}}\, \widetilde{T}_{ij} \,, \qquad Y = e^{4\alpha\varphi}\,. \tag{4.72}$$

Hence, the Pauli S^2 reduction ansätze of five-dimensional minimal supergravity coupled with a vector multiplet whose Lagrangian is given by (4.18) has the following form

$$\begin{aligned} d\bar{s}_5^2 &= Y^{\frac{1}{3}}\, \Delta^{\frac{1}{3}}\, ds_3^2 + g^{-2}\, Y^{\frac{1}{3}}\, \Delta^{-\frac{2}{3}}\, T_{ij}^{-1}\, D\mu^i D\mu^j \,, \\ e^{6\bar{\alpha}\phi} &= Y^{\frac{1}{2}}\, \Delta^{-1} \,, \\ \mathcal{A}_{(1)} &= g^{-1}\, \cos\theta d\psi - \mu^i\, A^i - g^{-1}\, \Delta^{-1}\, T_{ij}\, \varepsilon_{ik\ell}\, \mu^j \mu^k\, D\mu^\ell \,, \\ B_{(1)} &= mg^{-2}\, \cos\theta\, d\psi - mg^{-1}\, \mu^i A^i + g^{-1}\, \mu^i B^i + d\omega_{(0)} \\ \mathcal{F}_{(2)} &= -\tfrac{1}{2}\varepsilon_{ijk}\, \Big(g^{-1}\, U\, \Delta^{-2}\, \mu^i\, D\mu^j \wedge D\mu^k - 2g^{-1}\, \Delta^{-2}\, D\mu^i \wedge DT_{j\ell}\, T_{km}\, \mu^\ell\, \mu^m\Big) - \Delta^{-1}\, T_{ij}\, \mu^i\, F^j \\ H_{(2)} &= mg^{-2}\, \omega_{(2)} - g^{-1}\, B^i \wedge D\mu^i - T_{ij}\, \mu^i *B^j \,, \end{aligned} \tag{4.73}$$

where $\Delta = T_{ij}\, \mu^i \mu^j = Y^{\frac{1}{3}}\, \widetilde{\Delta}$ and $U = 2\mu^i\, T_{ij}\, T_{jk}\, \mu^k - \Delta\, T_{ii}$.

Using equations (4.36) and (4.66), one can write $H_{(2)}$ given by (4.71) as

$$H_{(2)} = mg^{-2}\,\omega_{(2)} - g^{-1}B^i \wedge D\mu^i + g^{-1}\,\mu^i\,DB^i - mg^{-1}\,\mu^i F^i = mg^{-2}\,d\sigma + g^{-1}D(\mu^i\,B^i), \quad (4.74)$$

therefore, considering $H_{(2)} = dB_{(1)}$, hence $B_{(1)}$ can be written up to a total derivative denoted by $d\omega_{(0)}$, as the expression given by (4.73).

4.4 Impossibility of the Pauli reduction of $5D$ minimal supergravity

We constructed a consistent Pauli S^2 reduction of five-dimensional minimal supergravity coupled with a vector multiplet in the last section. However, to obtain a Pauli S^2 reduction of pure minimal supergravity in five dimensions, one needs to perform a consistent truncation given by (4.20). In this section, we examine this condition in five dimensions and study its consequences in three-dimensional space-time fields. Thus, we find out whether obtaining the consistent Pauli reduction of pure minimal supergravity is possible by this method.

From the Pauli reduction ansatz for ϕ given in (4.73), one can observe $\phi = 0$ yields to

$$Y^{\frac{1}{2}} = \Delta = T_{ij}\,\mu^i\mu^j\,. \qquad (4.75)$$

The important point is Y and T_{ij} depend just upon the three-dimensional space-time, and are independent of the S^2 coordinates μ^i, thus

$$T_{ij} = f\delta_{ij}\,, \qquad Y^{\frac{1}{2}} = \Delta = f\,, \qquad (4.76)$$

where f is a general function of the three-dimensional space-time. Considering the determinant of T_{ij} and according to (4.72), one has

$$\det T_{ij} = f^3 = Y \det \widetilde{T}_{ij} = Y = f^2\,, \qquad (4.77)$$

where we have used the fact that $\widetilde{T}_{ij}$ is uni-modular. Hence $f = 1$, so $T_{ij} = \delta_{ij}$. Now, considering the truncation $B_{(1)} = \sqrt{2}A$, one concludes from (4.73)

$$m = \sqrt{2}\, g\,, \qquad \mu^i B^i = -g\, d\omega_{(0)}\,, \tag{4.78}$$

and the two-sphere coordinates independence of B^i yields

$$B^i = 0\,. \tag{4.79}$$

Therefore, from (4.36) we find out $B^i = 0$ implies $F^i = 0$, and thus A^i is pure gauge.

Consequently, although there is a consistent Pauli S^2 reduction of minimal supergravity coupled to a vector multiplet, the investigation of finding such a consistent reduction in case of pure minimal supergravity fails due to the impossibility of performing a consistent truncation in five dimensions. However, this failure does not provide a rigorous mathematical proof of the impossibility of the consistent Pauli S^2 reduction of pure minimal supergravity in the case where all $SU(2)$ gauge bosons are retained.

5. AN ALTERNATIVE M-THEORY ORIGIN OF THE SALAM-SEZGIN THEORY

5.1 Introduction

Salam and Sezgin [36], based on a work of Nishino and Sezgin [37], found a chiral $N = (1, 0)$ Einstein-Maxwell supergravity in six dimensions. Among many interesting features of this theory, its spontaneous compactification on $Minkowski_4 \times S^2$ is more striking and a natural question of whether there is a full non-linear Pauli consistent reduction of it on S^2 had been raised. Gibbons and Pope in [38] studied this question and found out a remarkable consistent ansatz for this reduction. Although they presented the ansatz, but the underlying principle of why this ansatz works remains unclear. One may use the “Hopf fibration technique” introduced in [7] to investigate this problem. To do so, one needs to find a seven-dimensional theory which upon reducing on a circle gives rise to the Salam-Sezgin theory, or in other words, find out an embedding of the latter theory in a higher dimensional theory. Fortunately, such an embedding was constructed in [39] and one can use it to perform the above mentioned technique. However, unfortunately, the Kaluza-Klein vector potential obtained by a circle reduction from seven-dimensional theory has been truncated to zero in that work. It means the Dirac monopole on two-sphere is vanishing and thus the Hopf fibration in that case, is just a trivial fibration of $S^2 \times S^1$ instead of the expected S^3 one.

By further investigation, we have been able to find an alternative embedding of the Salam-Sezgin theory in M-theory where the Kaluza-Klein vector potential is present. Hence, by using the Hopf fibration technique, which we will elaborate in chapter 6, we have a group-theoretical understanding of Gibbons-Pope remarkable S^2 reduction of the Salam-Sezgin theory.

The first step towards finding the higher-dimensional origin of the Salam-Sezgin theory is starting from an $SO(4)$, half maximal, i.e. $N = 2$ seven-dimensional supergravity. The latter can be derived from a consistent truncation of the maximal theory, i.e. $N = 4$, $SO(5)$ in seven dimensions found by a Noether method in [40]. Then since the maximal theory itself can be understood as a consistent Pauli S^4 reduction of M-theory [13–15], thus the half maximal theory can be derived from type I or Heterotic string theory as it was shown in [15].

Having said that the seven-dimensional $SO(4)$, half maximal supergravity, is a higher-dimensional origin of the Salam-Sezgin theory, now at this stage, there are two possibilities to proceed:

- The first one, considered in the original work of [39] and our work in [45], is passing from a compact $SO(4)$ group to a non-compact $SO(2,2)$ one in seven dimensions. The justification of the possibility of this process was discussed in [41] and [39]. We will consider this method, to find the **bosonic sector** of the Salam-Sezgin theory in the next chapter.

- The second possibility, which will be addressed in this chapter, is based on the Wick rotation $\delta_{\bar{A}\bar{B}} \to -\mathrm{i}\ \delta_{\bar{A}\bar{B}}$. Again, based on the work in [41], we shall discuss in subsection 5.2.4 how this possibility is obtained. This method changes the sign of the Yang-Mills kinetic term and also the potential term in the bosonic Lagrangian of $SO(4)$ theory. However, although the supersymmetry transformations maintain and also the Lagrangian is invariant under them, the reality conditions on fermionic fields are ambiguous. We **expect** a *Wick rotated* seven-dimensional supergravity whose bosonic Lagrangian is given by a "wrong" sign for the Yang-Mills kinetic and the potential terms, exists and we try to find the fermionic Lagrangian and also supersymmetry transformations. Moreover, we will show the **bosonic** sector of the Salam-Sezgin theory can be derived from that theory.

As we have mentioned above, in this chapter we follow the second option . Then, one needs to perform a Kaluza-Klein S^1 reduction down to six dimensions, and apply some bosonic truncations. the Yang-Mills field strengths are truncated to zero in six dimensions. In that sense the "wrong" sign of the kinetic terms for Yang-Mills fields does not appear in the Salam-Sezgin theory.

The rest of this chapter organizes as follows. Since the initial stages of the embedding of the Salam-Sezgin theory in higher dimensions, constructed in [39], are the same as our alternative embedding, section 5.2 is a brief review of how one can obtain $N = 2$ gauge $SO(4)$ supergravity in seven dimensions from M-theory. Also, the above mentioned Wick rotation will be addressed and the reason why this scheme has been followed will be discussed. Section 5.3 devotes to the Kaluza-

Klein circle reduction from seven-dimensions down to six, and finding the bosonic truncations of the embedding. Finally in section 5.4, we will discuss about the consequence of our Wick rotation and will show, the Wick rotated seven-dimensional theory can be obtained by a time-like reduction from ten dimensions.

5.2 Obtaining $N = 2$ gauged $SO(4)$ from $N = 4$ gauged $SO(5)$ supergravity in seven dimensions

We review the different steps needed to obtain $N = 2$ gauged $SO(4)$ supergravity from $N = 4$ gauged $SO(5)$ theory in seven dimensions. The starting point is seven-dimensional $N = 4$, $SO(5)$ gauged maximal supergravity. It was first constructed in 1984 by Noether method in [40], and about fifteen years later, it was shown [13, 14], this theory can be understood as a remarkable Pauli S^4 reduction of eleven-dimensional supergravity, the low energy limit of the M-theory. This is one of the most significant examples of the Pauli reduction. We shall review this theory in the following.

5.2.1 Review of $N = 4$ gauged $SO(5)$ supergravity in $7D$

The bosonic part of the theory comprises a graviton, 10 Yang-Mills vector potentials $A_{(1)A}{}^B$ with $SO(5)_g$ gauging, 14 scalars $\Pi_A{}^i$ parametrize the coset manifold $SL(5,\mathbb{R})/SO(5)_c$ (however the rigid $SL(5,\mathbb{R})$ symmetry is changed to $SO(5)_c$ during the gauging procedure), and 5 three-form potentials $S_{(3)A}$ which are vectors in $SO(5)_g$ gauging. The fermionic part consists of 4 gravitini $\psi_M{}^I$, which is a vector-spinor of space-time and a spinor of $SO(5)_c$ composite group (the spinor index of the latter group denotes by I while that of the space-time suppressed), and also 16 gaugini λ_i^I, which is a spinor of space-time with suppressed indices, and a vector-spinor of the composite group $SO(5)_c$ where the indices are denoted by indices i and I respectively.

The bosonic sector $(e_M^A, A_{(1)A}{}^B, S_{(3)A}, \Pi_A{}^i)$ has $14 + 10 \times 5 + 5 \times 10 + 14 = 128$ degrees of freedom, the same as the fermionic part $(\psi_M{}^I, \lambda_i^I)$ with $4 \times 4 \times 4 + 16 \times 4 = 128$ degrees of freedom.

In this theory space-time curved (flat) indices are denoted by $M, N, P, \ldots$ $(A, B, C, \ldots)$ respectively and run over $0, 1, ..., 6$. The $SO(5)_g$ indices are denoted by $A, B, C, ...$ (not to be confused with the flat indices) and run over $0, 1, 2, 3, 4$. The $SO(5)_c$ composite group vector indices are presented by $i, j, k, ...$, and also run over $0, 1, 2, 3, 4$. The $SO(5)_c$ composite group spinor indices are expressed by $I, J, K, ...$ and run over $1, 2, 3, 4$. Space-time spinor indices are suppressed here.

Since the original work [40] and the paper [39] which we follow its notation and convention, do not use the same conventions, one needs to find field re-definitions which is necessary for relating results of the former to those of the latter. We consider this issue in appendix A. All results of this section already presented in [40], and we review them here. The bosoinc Lagrangian of this theory reads

$$\begin{aligned}\mathcal{L}_7 =\; & R *\mathbb{1} - *P_{ij} \wedge P^{ij} - \tfrac{1}{4}\Pi_A{}^i\, \Pi_B{}^j\, \Pi_C{}^i\, \Pi_D{}^j *F^{AB}_{(2)} \wedge F^{CD}_{(2)} - \tfrac{1}{2}\Pi^{-1}{}_i{}^A\, \Pi^{-1}{}_i{}^B *S_{(3)\,A} \wedge S_{(3)\,B} \\ & + \frac{1}{2g}\,\eta^{AB}\, S_{(3)\,A} \wedge DS_{(3)\,B} - \frac{1}{8g}\,\epsilon_{AC_1\cdots C_4}\,\eta^{AB}\, S_{(3)\,B} \wedge F^{C_1C_2}_{(2)} \wedge F^{C_3C_4}_{(2)} - \frac{1}{g}\,\Omega - V *\mathbb{1}\,, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} & F_{(2)A}{}^B = dA_{(1)A}{}^B + g\, A_{(1)A}{}^C \wedge A_{(1)C}{}^B\,, \\ & V = \tfrac{1}{2}g^2\left(2T_{ij}\,T_{ij} - (T_{ii})^2\right), \qquad T_{ij} = \Pi^{-1}{}_i{}^A\, \Pi^{-1}{}_j{}^B\, \eta_{AB}\,, \\ & \Pi^{-1}{}_i{}^A\left(\delta_A{}^B\, d + g\, A_{(1)A}{}^B\right)\Pi_B{}^k\,\delta_{kj} = P_{ij} + Q_{ij}\,; \quad P_{ij} = P_{(ij)}\,, \quad Q_{ij} = Q_{[ij]}\,, \end{aligned} \tag{5.2}$$

and by definition we have

$$A^{AB}_{(1)} \equiv \eta^{AC}\, A_{(1)C}{}^B\,, \qquad \text{with } A^{AB}_{(1)} = -A^{BA}_{(1)}\,. \tag{5.3}$$

The above relations, besides the factor of $\frac{1}{4}$ we introduced in the kinetic term of $F^{AB}_{(2)}$ instead of that of $\frac{1}{2}$ in [39], are exactly the same as equations (1), (2) and (3) of [39].

Next, we review the supersymmetry transformations given by eqn (9) of [39]. Those of the bosonic fields are [1]

$$
\begin{aligned}
\delta e_M^A &= \tfrac{1}{2}\bar{\epsilon}\,\Gamma^A\,\psi_M\,,\\
\Pi_A{}^i\,\Pi_B{}^j\,\delta A_M^{AB} &= \tfrac{1}{2}\,\bar{\epsilon}\,\gamma^{ij}\,\psi_M + \tfrac{1}{4}\,\bar{\epsilon}\,\Gamma_M\,\gamma^k\,\gamma^{ij}\,\lambda_k\,,\\
\Pi^{-1}{}_i{}^A\,\delta\Pi_A{}^j &= \tfrac{1}{4}(\bar{\epsilon}\,\gamma_i\,\lambda^j + \bar{\epsilon}\,\gamma^j\,\lambda_i)\,, \qquad (5.4)\\
\delta S_{MNP,A} &= -\tfrac{3}{8}\,\Pi_A{}^i\,(2\bar{\epsilon}\,\gamma_{ijk}\,\psi_{[M} + \bar{\epsilon}\,\Gamma_{[M}\,\gamma^\ell\,\gamma_{ijk}\,\lambda_\ell)\,\Pi_B{}^j\,\Pi_C{}^k\,F_{NP]}^{BC}\\
&\quad -\tfrac{3}{2}\,\delta_{ij}\,\Pi_A{}^j\,\mathcal{D}_{[M}\,(2\bar{\epsilon}\,\Gamma_N\,\gamma^i\,\psi_{P]} + \bar{\epsilon}\,\Gamma_{NP]}\,\lambda^i)\\
&\quad +\tfrac{1}{2}\,g\,\delta_{AB}\,\Pi^{-1}{}_i{}^B\,(3\bar{\epsilon}\,\Gamma_{[MN}\,\gamma^i\,\psi_{P]} - \bar{\epsilon}\,\Gamma_{MNP}\,\lambda^i)\,, \qquad (5.5)
\end{aligned}
$$

where we add a factor of g, which was missing in [39], in the last line of the above relations. Note the covariant derivative appeared in the supersymmetry transformation of the three-form potential above, is denoted by $\mathcal{D}$, which is the most general covariant derivative introduced in this work. In the original paper [40], there is a typographical error and instead of $\mathcal{D}$ the covariant derivative D appeared in that supersymmetry transformation, however in [13, 14], this has been corrected to the more general covariant derivative $\mathcal{D}$ [2]. The relation between these two covariant derivatives shall be clarified in the following. First, consider the covariant derivative D acting on a spinor as follows

$$
D_M\,\epsilon = \partial_M\,\epsilon + \tfrac{1}{4}\omega_{MAB}\,\Gamma^{ab}\,\epsilon + \tfrac{1}{4}Q_{Mij}\,\gamma^{ij}\,\epsilon\,. \qquad (5.6)
$$

The covariant derivative acts on a field with the Yang-Mills and the composite vector indices in a standard way

$$
D\,T_{(1)\,iA} = d\,T_{(1)\,iA} + g\,A_{(1)\,A}{}^B\,\wedge T_{(1)\,iB} + Q_{(1)\,i}{}^j\,\wedge T_{(1)\,jA}\,. \qquad (5.7)
$$

[1] One should not confuse the flat indices in seven dimensions $A, B, C, \cdots$ with the $SO(5)_g$ Yang-Mills gauge group indices

[2] The notion was used in [13, 14] for the covariant derivative and the general covariant derivative are ∇ and D respectively, while we have used D and $\mathcal{D}$ for them respectively.

Now, the general covariant derivative involves $P_{(1)\,ij}$ as well as $Q_{(1)\,ij}$ as follows

$$\mathcal{D}\,T_{(1)\,iA} = D\,T_{(1)\,iA} + P_{(1)\,i}{}^{j} \;\wedge T_{(1)jA}\,. \tag{5.8}$$

Note the only place this general covariant derivative appears is the supersymmetry transformation of the three-form potential, and in all other equations, covariant derivative D appears.

The supersymmetry transformations rules of fermions, originally presented in eqn (9) of [40], and then in eqn (6) of [39] are as follows

$$\begin{aligned}
\delta\psi_M &= D_M\,\epsilon + \tfrac{1}{20}\,g\,T_{ii}\,\Gamma_M\,\epsilon - \tfrac{1}{80}\,(\Gamma_M{}^{NP} - 8\delta_M^N\,\Gamma^P)\,\gamma_{ij}\,\epsilon\,\Pi_A{}^i\,\Pi_B{}^j\,F_{NP}{}^{AB}\\
&\quad -\tfrac{1}{60}\,(\Gamma_M{}^{NPQ} - \tfrac{9}{2}\delta_M^N\,\Gamma^{PQ})\,\gamma^i\,\epsilon\,\Pi^{-1}{}_i{}^A\,S_{NPQ\,,A}\,,\\
\delta\lambda_i &= \tfrac{1}{32}\,\Gamma^{MN}\,(\gamma_{k\ell}\,\gamma_i - \tfrac{1}{5}\gamma_i\,\gamma_{k\ell})\,\epsilon\,\Pi_A{}^k\,\Pi_B{}^\ell\,F_{MN}{}^{AB} - \tfrac{1}{120}\,\Gamma^{MNP}\,(\gamma_i{}^j - 4\delta_i^j)\,\epsilon\,\Pi^{-1}{}_j{}^A\,S_{MNP\,,A}\\
&\quad +\tfrac{1}{2}g\,(T_{ij} - \tfrac{1}{5}T_{kk}\,\delta_{ij})\,\gamma^j\,\epsilon + \tfrac{1}{2}\Gamma^M\,\gamma^j\,\epsilon\,P_{M\,ij}\,.
\end{aligned} \tag{5.9}$$

The original sign which appeared in the second term of the gravitino supersymmetry transformation in in eqn (9) of [40] written by Pernici, Pilch, and van Nieuwenhuizen in 1984 , was an equal sign, which is definitely a typographical error. However, in two works published in 1999 by Nastase, Vaman and van Nieuwenhuizen [13, 14], it was stated to be a minus sign. During the construction of the alternative embedding of the Salam-Sezgin theory, we found out if this sign would change to a plus sign, then the alternative reduction would work. Upon direct calculation regarding to verify the correct sign, we found, surprisingly, this sign should be a plus sign. After reviewing the literature, it turns out this sign was correctly reported as a plus sign in [41] (couple of months later after the original work in 1984). Some later works [42,43] followed this paper, and hence presenting the correct sign.

According to equations (5) and (6) of [15] one can write the following relation for the three-form potential

$$DS^A_{(3)} \equiv dS^A_{(3)} + g\,A^A{}_B \wedge S^B_{(3)} = g\,T_{AB} * S^B_{(3)} + \tfrac{1}{8}\varepsilon_{ABCDE}\,F^{BC}_{(2)} \;\wedge F^{DE}_{(2)}\,, \tag{5.10}$$

where

$$T^{AB} \equiv \Pi^{-1}{}_i{}^A\,\Pi^{-1}{}_i{}^B\,. \tag{5.11}$$

5.2.2 The Inönü-Wigner group contraction limit of the $SO(5)$

The next step shall be using the Inönü-Wigner group contraction limit of $SO(5)$ gauged supergravity mentioned above, to find maximal supergravity with $SO(4)$ gauge group. The result, $N = 4$, $SO(4)$ gauged maximal supergravity, could be interpreted as a consistent Pauli S^3 reduction of type IIA supergravity [15].

One may divide both $SO(5)_g$ and $SO(5)_c$ vector indices, A and i, as follows

$$i = (0, \alpha)\,, \qquad A = (0, \bar{A}), \tag{5.12}$$

where $\bar{A}$ and α are now vector indices of $SO(4)_g$ and $SO(4)_c$ gauging respectively. If one applies different rescalings on bosonic fields and the gauge coupling constant introduced in equations (11) and (14) of [39] , then by taking a singular limit, one can achieve the $SO(4)$ gauging. The procedure described in that paper and was originally presented in [15].

For our final purpose to obtain the Salam-Sezgin theory, one does not need the detail of maximal $SO(4)$ gauge supergravity, however, one has to find a further truncation to $N = 2$, $SO(4)$ gauge supergravity. In the following, we will address this truncation.

5.2.3 Review of $N = 2$ gauged $SO(4)$ supergravity in seven dimensions

One needs to perform consistent truncations of the bosonic and fermionic fields of the maximal theory, to obtain an $N = 2$ theory with $SO(4)$ gauge group in seven dimensions, as described in [39]. This theory can be considered as a consistent Pauli S^3 reduction of the type I, or Heterotic supergravity. Let us review the field content, the Lagrangian, and supersymmetry transformations of this theory. All of these results presented in [39] and we repeat it here for convenience.

The bosonic truncations we apply are the following

$$\chi^\alpha = 0, \qquad A^{0\bar{A}}_{(1)} = 0\,, \qquad S^\alpha_{(3)} = 0\,, \tag{5.13}$$

where four scalars χ^α were introduced in T_{ij} in eqn (11) of [39], as a part of the Inönü-Wigner procedure.

To retain the supersymmetry, one needs to truncate some fermionic fields as well. For that purpose, one may write $\epsilon = \epsilon^+ + \epsilon^-$, where the superscripts $\pm$ show the positive and negative chiralities under the $SO(5)_c$ chirality operator (i.e. γ_0). In other words, $\gamma_0\,\epsilon^\pm = \pm\epsilon^\pm$. According to eqn (12) of [39], one can make the following truncations

$$\epsilon^- = 0\,, \qquad \psi_\mu^- = 0\,, \qquad \lambda_0^- = 0\,, \qquad \lambda_\alpha^+ = 0\,. \tag{5.14}$$

Hence from $\gamma^i\,\lambda_i = 0$, we can find

$$\gamma^i\,\lambda_i = \lambda_0^+ - \lambda_0^- + \gamma^\alpha\,\lambda_\alpha^+ + \gamma^\alpha\,\lambda_\alpha^- = \lambda_0^+ + \gamma^\alpha\,\lambda_\alpha^- = 0\,, \tag{5.15}$$

then

$$\lambda_0^+ = -\gamma^\alpha\,\lambda_\alpha^-\,. \tag{5.16}$$

Therefore, after the above truncations, the remaining independent fermionic fields are ψ_μ^+ and λ_α^-. One has to verify the supersymmetry transformations (5.5) and (5.9) shall be consistent under the bosonic and fermionic truncations mentioned above (i.e. the supersymmetry transformation of a truncated field should be vanished) and ffter some calculation, it is clear that they are consistent.

The bosonic field content of half maximal $SO(4)$ supergravity comprises a graviton e_M^A, 9 scalar fields described by a unimodular matrix $\pi_{\bar{A}}{}^\alpha$ which parametrizes a coset of $SL(4,\mathbb{R})/SO(4)_c$, a scalar field denoted by Φ which results from the Inönü-Wigner group contraction procedure as explained in eqn (14) of [39], 6 Yang-Mills vector potentials $A_{(1)}^{\bar{A}\bar{B}}$ in the vector representation of $SO(4)_g$, and a three-form field strength $H_{(3)}$. The fermionic fields include two gravitini ψ_M and 8 gaugini λ_α. The bosonic degrees of freedom for the multiplet of $(e_M^A, \pi_{\bar{A}}{}^\alpha, \Phi, A_{(1)}^{\bar{A}\bar{B}}, H_{(3)})$ are $14+9+1+6\times5+10 = 64$ and the fermionic degrees of freedom for the multiplet of (ψ_M, λ_α) are $2\times4\times4+8\times4 = 64$ as we expect for the half maximal theory in seven dimensions.

Here $\bar{A}, \bar{B}, ...$ $(\alpha, \beta, ...)$, run over values $1,2,3,4$, are the vector indices of $SO(4)_g$ (respectively composite group $SO(4)_c$). Seven dimensional curved (flat) indices denote by $M, N, P, \cdots$ (respectively $A, B, C, \cdots$) run over $0, 1, \cdots, 6$. Seven-dimensional Dirac matrices denote by Γ_M, while the composite Dirac matrices denote by γ_α.

The bosonic Lagrangian of this theory reads

$$\begin{aligned}\mathcal{L}_7 = & \, R*\mathbb{1} - \tfrac{5}{16}\,\Phi^{-2}*d\Phi\wedge d\Phi - *p_{\alpha\beta}\wedge p^{\alpha\beta} - \tfrac{1}{2}\Phi^{-1}*H_{(3)}\wedge H_{(3)} \\ & -\tfrac{1}{4}\Phi^{-1/2}\,\pi_{\bar{A}}{}^{\alpha}\,\pi_{\bar{B}}{}^{\beta}\,\pi_{\bar{C}}{}^{\alpha}\,\pi_{\bar{D}}{}^{\beta}*F^{\bar{A}\bar{B}}_{(2)}\wedge F^{\bar{C}\bar{D}}_{(2)} - \frac{1}{g}\,\Omega - V*\mathbb{1}\,,\end{aligned} \tag{5.17}$$

where

$$\pi^{-1}{}_{\alpha}{}^{\bar{A}}\,[\delta_{\bar{A}}{}^{\bar{B}}\,d + g\,A_{(1)\bar{A}}{}^{\bar{B}}]\,\pi_{\bar{B}}{}^{\gamma}\,\delta_{\beta\gamma} = p_{\alpha\beta} + Q_{\alpha\beta}\,, \qquad p_{\alpha\beta} = p_{(\alpha\beta)}\,, \qquad Q_{\alpha\beta} = Q_{[\alpha\beta]}\,, \tag{5.18}$$

and the scalar potential in the Lagrangian is

$$V = \tfrac{1}{2}g^2\,\Phi^{1/2}\,(2M_{\alpha\beta}\,M_{\alpha\beta} - (M_{\alpha\alpha})^2)\,. \tag{5.19}$$

where $M_{\alpha\beta} = \pi^{-1}{}_{\alpha}{}^{\bar{A}}\,\pi^{-1}{}_{\beta}{}^{\bar{B}}\,\eta_{\bar{A}\bar{B}}$. Also the gauge potentials are raised and lower by $SO(4)_g$ invariant tensor $\eta_{\bar{A}\bar{B}}$. Finally, Ω denotes the Chern-Simons term, which its detail form can be found in [40]. Covariant derivatives are the same as the case of the maximal theory presented in (5.6), (5.7) and (5.8).

One can obtain from (5.10), and considering the bosonic truncation we have made, the following relation

$$dH_{(3)} = \tfrac{1}{8}\,\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}\,F^{\bar{A}\bar{B}}_{(2)}\wedge F^{\bar{C}\bar{D}}_{(2)}\,. \tag{5.20}$$

Starting from the fermionic Lagrangian in $SO(5)$ gauged maximal seven-dimensional supergravity, given in eqn (8) of [40], one can write the fermionic Lagrangian of the $SO(4)$ limit of this theory. It is understood all of the fields and Dirac matrices and the covariant derivative are in seven dimensions. The fermionic Lagrangian reads

$$
\begin{aligned}
e^{-1}\mathcal{L}_{Fermi} = & -\bar{\psi}_M\Gamma^{MNP}D_N\psi_P - \bar{\lambda}^i\Gamma^M D_M\lambda_i + \tfrac{1}{8}g\,\Phi^{\frac{1}{4}}\,(M\bar{\lambda}^i\lambda_i - 8M_{\alpha\beta}\bar{\lambda}_\alpha\lambda_\beta) \\
& +\tfrac{1}{2}g\,\Phi^{\frac{1}{4}}M_{\alpha\beta}\bar{\lambda}_\alpha\gamma_\beta\Gamma^M\psi_M + \tfrac{1}{2}\bar{\psi}_M\,\Gamma^N\Gamma^M\,\lambda^0\,\Phi^{-1}\partial_N\Phi + \bar{\psi}_M\,\Gamma^N\Gamma^M\gamma^\alpha\lambda^\beta p_{N\alpha\beta} \\
& -\tfrac{1}{8}\bar{\psi}_M\,\Gamma^N\Gamma^M\gamma^\alpha\lambda_\alpha\,\Phi^{-1}\partial_N\Phi + \tfrac{1}{8}g\,\Phi^{\frac{1}{4}}\,M\,\bar{\psi}_M\,\Gamma^{MN}\psi_N \\
& +\tfrac{1}{8\sqrt{2}}\,\Phi^{-\frac{1}{4}}\,\Big(\bar{\psi}_M(\Gamma^{MNPQ} - 2\delta^{MN}\,\delta^{PQ})\,\gamma_{\alpha\beta}\psi_Q\,\pi_{\bar{A}}{}^\alpha\,\pi_{\bar{B}}{}^\beta F_{NQ}{}^{\bar{A}\bar{B}} \\
& +4\bar{\psi}_M\Gamma^{NP}\,\Gamma^M\,\gamma_\alpha\lambda_\beta\pi_{\bar{A}}{}^\alpha\,\pi_{\bar{B}}{}^\beta\,F_{NP}{}^{\bar{A}\bar{B}} + \tfrac{1}{2}\bar{\lambda}_i\gamma^j\,\gamma_{\alpha\beta}\,\gamma^i\,\Gamma^{MN}\,\lambda_j\,\pi_{\bar{A}}{}^\alpha\,\pi_{\bar{B}}{}^\beta\,F_{MN}{}^{\bar{A}\bar{B}}\Big) \\
& -\tfrac{1}{24}\,\Phi^{-\frac{1}{2}}\,\Big(\bar{\psi}_M\,(\Gamma^{MNPQR} + 6\delta^{MN}\,\Gamma^P\,\delta^{QR})\,\psi_R \\
& -2\bar{\psi}_M(\Gamma^{MNPQ} - 3\delta^{MN}\,\delta^{PQ})\,\lambda^0 - \bar{\lambda}^0\,\Gamma^{NPQ}\,\lambda_0 + \bar{\lambda}^\alpha\,\Gamma^{NPQ}\,\lambda_\alpha\Big)H_{NPQ}\,. \qquad (5.21)
\end{aligned}
$$

Supersymmetry transformations for the fermionic fields shall be given by [3]

$$
\begin{aligned}
\delta\psi_M = & \; D_M\,\epsilon + \tfrac{1}{20}\,g\,M_{\alpha\alpha}\,\Phi^{1/4}\,\Gamma_M\,\epsilon - \tfrac{1}{80}\,(\Gamma_M{}^{NP} - 8\delta_M^N\,\Gamma^P)\,\gamma_{\alpha\beta}\,\epsilon\,\Phi^{-1/4}\,\pi_{\bar{A}}{}^\alpha\,\pi_{\bar{B}}{}^\beta\,F_{NP}^{\bar{A}\bar{B}} \\
& -\tfrac{1}{60}\,(\Gamma_M{}^{NPQ} - \tfrac{9}{2}\delta_M^N\,\Gamma^{PQ})\,\epsilon\,\Phi^{-1/2}\,H_{NPQ}\,, \\
\delta\lambda_\alpha = & \; \tfrac{1}{2}\Gamma^M\,\gamma^\beta\,\epsilon\,P_{M,\alpha\beta} + \tfrac{1}{32}\,\Gamma^{MN}\,(\gamma_{\beta\gamma}\,\gamma_\alpha - \tfrac{1}{5}\gamma_\alpha\,\gamma_{\beta\gamma})\,\epsilon\,\Phi^{-1/4}\,\pi_{\bar{A}}{}^\beta\,\pi_{\bar{B}}{}^\gamma\,F_{MN}^{\bar{A}\bar{B}} \\
& -\tfrac{1}{120}\,\Gamma^{MNP}\,\gamma_\alpha\,\epsilon\,\Phi^{-1/2}\,H_{MNP} + \tfrac{1}{2}g\,(M_{\alpha\beta} - \tfrac{1}{5}M_{\gamma\gamma}\,\delta_{\alpha\beta})\,\Phi^{1/4}\,\gamma^\beta\,\epsilon\,. \qquad (5.22)
\end{aligned}
$$

Also, correcting a typographical error in the three-form supersymmetry transformation in eqn

[3] The typographical error in the second term of the right hand side of eqn (19) in the gravitino supersymmetry transformation is corrected here.

(20) of [39], the bosonic supersymmetric transformations read [4]

$$\delta e^A_M = \tfrac{1}{2}\bar\epsilon\,\Gamma^A\,\psi_M\,,$$
$$\Phi^{-1/4}\,\pi_{\bar A}{}^{\alpha}\,\pi_{\bar B}{}^{\beta}\,\delta A_M^{\bar A\bar B} = \tfrac{1}{2}\,\bar\epsilon\,\gamma^{\alpha\beta}\,\psi_M + \tfrac{1}{2}\,\bar\epsilon\,\Gamma_M\,(\gamma^\beta\,\lambda^\alpha - \gamma^\alpha\,\lambda^\beta)\,,$$
$$\pi^{-1}{}_{\alpha}{}^{\bar A}\,\delta\pi_{\bar A}{}^{\beta} = \tfrac{1}{4}(\bar\epsilon\,\gamma_\alpha\,\lambda^\beta + \bar\epsilon\,\gamma^\beta\,\lambda_\alpha) - \tfrac{1}{8}\bar\epsilon\,\gamma^\gamma\,\lambda_\gamma\,\delta^\beta_\alpha\,, \qquad \Phi^{-1}\,\delta\Phi = -\bar\epsilon\,\gamma^\alpha\,\lambda_\alpha\,,$$
$$\delta H_{MNP} = -\tfrac{3}{4}\,\Phi^{1/4}\,(\bar\epsilon\,\gamma_{\alpha\beta}\,\psi_{[M} - \bar\epsilon\,\Gamma_{[M}\,\gamma_{\alpha\beta\gamma}\,\lambda^\gamma)\,\pi_{\bar B}{}^{\alpha}\,\pi_{\bar C}{}^{\beta}\,F_{NP]}{}^{\bar B\bar C}$$
$$-\tfrac{3}{2}\,\Phi^{1/2}\,\mathcal{D}_{[M}\,(2\bar\epsilon\,\Gamma_N\,\psi_{P]} - \bar\epsilon\,\Gamma_{NP]}\,\gamma^\alpha\,\lambda_\alpha). \tag{5.23}$$

5.2.4 Towards a Wick rotated supergravity in seven dimensions

The way non-compact $SO(2,2)$ gauging was achieved in [39], is based on the argument first presented in [41]. Let us briefly state the argument here. It is in the context of $N = 4$, $SO(5)$ seven-dimensional supergravity which had been discovered in [40] couple of months earlier than that work. However, one can use it to a less general case of our interest, i.e. $N = 2$, $SO(4)$ seven-dimensional supergravity.

They perform a "Wick rotation" represented by a matrix $E_A{}^{A'} \in SL(5,\mathbb{C})$, and satisfies

$$E_A{}^{A'}\,E_B{}^{B'}\delta^{AB} = \eta^{A'B'}\,. \tag{5.24}$$

All fields of the theory rotate according to this Wick rotation, i.e.

$$A_{M\,A}{}^{B} = E_A{}^{A'}\,A'_{M\,A'}{}^{B'}\,E^{-1}{}_{B'}{}^{B}\,, \qquad \Pi_A{}^{i} = E_A{}^{A'}\,\Pi'_{A'}{}^{i}\,, \qquad S_{MNP,A} = E_A{}^{A'}\,S'_{MNP,A'}\,. \tag{5.25}$$

Then, if one assumes $\det E_A{}^{A'} = 1$, the Lagrangian and supersymmetry transformation shall be replaced by the 'prime' fields, moreover, δ_{AB} shall be replaced by η_{AB}. According to [41], since the supersymmetry transformations maintain and the Lagrangian is invariant under them, then **one**

[4] The typographical sign error of the supersymmetry transformation of the three-form in eqn (20) of [39] is corrected (the sign of the second term after the covariant derivative should be minus rather than plus). Also, because of the rescaling introduced in equations (11) and (14) of that paper, the last term in this transformation (term with coupling constant g in (5.5) should be vanished. The missing term in the scalar transformation, $\pi_{\bar A}{}^{\alpha}$ has been added.

can declare all of the primed fields are real.

Now, having said about their argument, we introduce $E_A{}^{A'}$ as follows

$$E_A{}^{A'} = \text{diag}(-1, e^{\frac{\mathrm{i}\,\pi}{4}}, e^{\frac{\mathrm{i}\,\pi}{4}}, e^{\frac{\mathrm{i}\,\pi}{4}}, e^{\frac{\mathrm{i}\,\pi}{4}})\,, \tag{5.26}$$

where clearly $\det E_A{}^{A'} = 1$. The invariant tensor has become

$$\eta^{A'B'} = E_A{}^{A'}\, E_B{}^{B'} \delta^{AB} = \text{diag}(1, \mathrm{i}\,, \mathrm{i}\,, \mathrm{i}\,, \mathrm{i}\,)\,. \tag{5.27}$$

Now, if we consider the Inönü-Wigner group contraction limit of the $SO(5)$, the invariant tensor becomes

$$\eta^{\bar{A}\bar{B}} = \mathrm{i}\ \delta^{\bar{A}\bar{B}}\,, \qquad \text{and} \qquad \eta_{\bar{A}\bar{B}} = -\mathrm{i}\ \delta_{\bar{A}\bar{B}}\,. \tag{5.28}$$

Note that the above result follows exactly the same line of reasoning as passing from $SO(4)$ compact group to $SO(2,2)$ non-compact one. In the latter work one should replace $\delta_{\bar{A}\bar{B}} \to \eta_{\bar{A}\bar{B}}$ and in our work the replacement is $\delta_{\bar{A}\bar{B}} \to -\mathrm{i}\ \delta_{\bar{A}\bar{B}}$.

Having obtained this Wick rotated theory, one may consider its consequence in the supersymmetry transformations and the Lagrangian. First of all, recall the fundamental Yang-Mill gauge potentials have the form of $A_{(1)\bar{A}}{}^{\bar{B}}$, and hence to raise the index, one may write

$$A_{(1)}^{\bar{A}\bar{B}} = \eta^{\bar{A}\bar{C}}\, A_{(1)\bar{C}}{}^{\bar{B}} = \mathrm{i}\ \delta^{\bar{A}\bar{C}}\, A_{(1)\bar{C}}{}^{\bar{B}}\,. \tag{5.29}$$

In other words, the kinetic term of $A_{(1)\bar{A}}{}^{\bar{B}}$ in the bosonic Lagrangian has a positive sign instead of the usual negative one.

Also, the scalar potential in the Lagrangian undergoes a sign change and is given by

$$V = \tfrac{1}{2} g^2\, \Phi^{1/2}\, (2M_{\alpha\beta}\, M_{\alpha\beta} - (M_{\alpha\alpha})^2)\,. \tag{5.30}$$

where

$$M_{\alpha\beta} = {\pi^{-1}}_{\alpha}{}^{\bar{A}}\, {\pi^{-1}}_{\beta}{}^{\bar{B}}\, \eta_{\bar{A}\bar{B}} = -\mathrm{i}\; \delta_{\bar{A}\bar{B}}\, {\pi^{-1}}_{\alpha}{}^{\bar{A}}\, {\pi^{-1}}_{\beta}{}^{\bar{B}}\,. \tag{5.31}$$

We will not present all of the supersymmetry transformations and the entire Lagrangian here. One can easily obtained them by the replacing $\delta_{\bar{A}\bar{B}} \to -\mathrm{i}\; \delta_{\bar{A}\bar{B}}$ in (5.17), (5.21), (5.22) and (5.23).

One can provide another Wick rotation to obtain the same result. Assuming the standard $SO(4)$ gauging, if the scalar field Phi is analytically continued to a pure imaginary field, i.e. $\Phi^{1/4} \to -\mathrm{i}\; \Phi^{1/4}$, together with $H_{MNP} \to -H_{MNP}$, one has the same result as the previous case of $\delta_{\bar{A}\bar{B}} \to -\mathrm{i}\; \delta_{\bar{A}\bar{B}}$. Since it is easier to work with this Wick rotation, we will consider this case and hence we have $\eta_{\bar{A}\bar{B}} = \delta_{\bar{A}\bar{B}}$.

5.2.5 Why the Wick rotated theory is necessary to study the fermionic sector?

The previous work [39] about finding a higher-dimensional origin for the Salam-Sezgin theory, as we have emphasized, is based on non-compact $SO(2,2)$ gauging of the half maximal supergravity in seven dimensions. Also, there is a possibility, as we shall show in the next chapter, to construct the bosonic sector of the Salam-Sezgin theory by this non-compact gauging and at the same time, retain the kaluza-Klein vector potential $\mathcal{A}_{(1)}$. Hence, one may speculate the possibility of obtaining the fermionic sector with the same gauging. However, one can show it is inconsistent to assume the same ferminic truncations as [39] and also keep $\mathcal{A}_{(1)}$. It can be observed from the vielbein transformation in (5.23), since both ψ_μ^- and ψ_7^+ have been truncated to zero, as it was assumed in [39], then the supersymmetry transformation of $\mathcal{A}_{(1)}$ shall be vanishing. Hence, $\mathcal{A}_{(1)}$ itself should be truncated.

One may argue, other fermionic truncation may be employed to keep $\mathcal{A}_{(1)}$ and using the non-compact gauging. The form of gaugino transformation in (5.22) forces us to assume the following truncation for the positive chirality [5] part of the gaugino: $\lambda_\alpha^+ = \eta_{\alpha\beta}\, \gamma_\beta\, \lambda^+$. Also, other consequence of this reduction is $H_{(2)} = 0$. Then considering the Yang-Mills supersymmetry transformation in (5.23) leads to a fermionic truncation of $\psi_\mu^- = 0$. The latter result is based on the assumption

[5] Note here the positive and negative chiralities are with respect to the space-time chirality operator Γ^7, and it should not be confused with that of the composite one which we discussed previously.

from the bosonic sector where the Chern-Simons contribution vanishes if $A^{12}_{(1)} = \pm A^{34}_{(1)}$. If one considers the supersymmetry transformation of ψ_7^+ in (5.22), then since $H_{(2)} = 0$, then one has $\delta\psi_7^+ \sim g(\chi^{12} + \chi^{34})\,\gamma_{12}\epsilon$.[6] In addition to this, since the fermionic degrees of freedom are 20, then one needs to truncate the latter field (since it is not possible to make this field proportional to other fermionic fields), meaning that $\delta\mathcal{A}_{(1)} = 0$, or $\mathcal{A}_{(1)}$ should be truncated. This argument shows the bosonic and fermionic ansatz, presented in [39], is the only consistent possibility if one considers $SO(2,2)$ non-compact gauging in seven dimensions.

It is worth to mention that the truncation discussed in [39] does not consistent with the compact $SO(4)$ gauging. To see this, one may find the supersymmetry transformation of the field ψ_μ^-, which is truncated to zero in that work, is non-vanishing. According to (5.22), there is a $M_{\alpha\alpha} = \eta_{\alpha\alpha}$ contribution in that transformation, which actually becomes zero for the case of $SO(2,2)$, but it shall be 4 in the case of $SO(4)$ gauging. It clearly indicates that this scheme of truncations does not consistent with the compact gauging, and one needs to use the non-compact one, as it was employed in [39].

In the next section, we perform a Kaluza-Klein circle down to six dimensions and consider the bosonic truncations and equations of motion.

5.3 The Kaluza-Klein circle reduction to six dimensions: bosonic sector

The Kaluza-Klein circle reduction is a standard calculation which we presented in the chapter 2. Using those results, one can write the metric ansatz as follows

$$d\hat{s}_7^2 = e^{2\alpha\varphi}\, ds_6^2 + e^{-8\alpha\varphi}\,(dz + \mathcal{A}_{(1)})^2\,, \tag{5.32}$$

where φ is a "breathing mode", and $\alpha^2 = \frac{1}{40}$. Also in this section, we insert hat on seven-dimensional fields to distinguish them from the six-dimensional ones. The bosonic ansätze for a circle reduction are standard and shall be addressed in the next part.

[6]The scalar has been ignored, this is why "$\sim$" has been used

5.3.1 The ansätze for gauge potentials

For obtaining the Salam-Sezgin theory, one does not need six gauge potentials of the $SO(4)$ half maximal theory, thus one may truncate them. We consider a consistent truncation where only the maximal Abelian subgroup, i.e. $U(1)^2$, of $SO(4)_g$ gauged Yang-Mills fields are retained. Namely, $\hat{A}^{12}_{(1)} = \hat{A}_{(1)}$ and $\hat{A}^{34}_{(1)} = \hat{A}'_{(1)}$ are kept. Also, one may truncate all scalars in the coset $SL(4,\mathbb{R})/SO(4)$, i.e. $\pi_{\bar{A}}{}^{\alpha} = \delta_{\bar{A}}{}^{\alpha}$. Consequently, with the assumption of keeping the compact gauge group $SO(4)$, then $M_{\alpha\beta} = \delta_{\alpha\beta}$. Obviously, this truncation is consistent with all equations of motion. Having assumed this truncation, and using (5.20), then one can write

$$d\hat{H}_{(3)} = \hat{F}_{(2)} \wedge \hat{F}'_{(2)}\,, \qquad \hat{F}_{(2)} = d\hat{A}_{(1)}, \qquad \hat{F}'_{(2)} = d\hat{A}'_{(1)}\,. \tag{5.33}$$

Based on the above relations, one can write

$$d\hat{H}_{(3)} = \tfrac{1}{2}\, d(\hat{F}_{(2)} \wedge \hat{A}'_{(1)} + \hat{A}_{(1)} \wedge \hat{F}'_{(2)})\,, \tag{5.34}$$

and hence

$$\hat{H}_{(3)} = d\hat{B}_{(2)} + \tfrac{1}{2}\, \hat{F}_{(2)} \wedge \hat{A}'_{(1)} + \tfrac{1}{2}\, \hat{F}'_{(2)} \wedge \hat{A}_{(1)}\,. \tag{5.35}$$

The ansätze for the Kaluza-Klein circle reduction for the bosonic fields are standard and can be written as follows

$$\begin{aligned}
\hat{H}_{(3)} &= H_{(3)} + H_{(2)} \wedge (dz + \mathcal{A}_{(1)}),\\
\hat{A}_{(1)} &= A_{(1)} + \chi\, dz, \quad \hat{A}'_{(1)} = A'_{(1)} + \chi'\, dz,\\
\hat{B}_{(2)} &= B_{(2)} + B_{(1)} \wedge dz,\\
\hat{F}_{(2)} &= F_{(2)} + d\chi \wedge (dz + \mathcal{A}_{(1)}), \quad \hat{F}'_{(2)} = F'_{(2)} + d\chi' \wedge (dz + \mathcal{A}_{(1)})\,.
\end{aligned} \tag{5.36}$$

Therefore, one may find the following relations for six-dimensional fields

$$\begin{aligned}
H_{(3)} &= dB_{(2)} - dB_{(1)} \wedge \mathcal{A}_{(1)} + \tfrac{1}{2}\, dA_{(1)} \wedge A'_{(1)} + \tfrac{1}{2}\, dA'_{(1)} \wedge A_{(1)} \\
&\quad -\tfrac{1}{2}\,(\chi' dA_{(1)} + \chi dA'_{(1)}) \wedge \mathcal{A}_{(1)} - \tfrac{1}{2}\,(A_{(1)} \wedge d\chi' + A'_{(1)} \wedge d\chi) \wedge \mathcal{A}_{(1)}, \\
H_{(2)} &= dB_{(1)} + \tfrac{1}{2}\,(\chi' dA_{(1)} + \chi dA'_{(1)} + A'_{(1)} \wedge d\chi + A_{(1)} \wedge d\chi'), \\
F_{(2)} &= dA_{(1)} - d\chi \wedge \mathcal{A}_{(1)}, \quad F'_{(2)} = dA'_{(1)} - d\chi' \wedge \mathcal{A}_{(1)}, \quad \mathcal{F}_{(2)} = d\mathcal{A}_{(1)}\,.
\end{aligned} \tag{5.37}$$

Writing in the flat indices, one can find the following relations between components of seven- and six- dimensional fields

$$\begin{aligned}
\hat{H}_{abc} &= e^{-3\alpha\varphi}\, H_{abc}, \quad \hat{H}_{ab7} = e^{2\alpha\varphi}\, H_{ab}, \\
\hat{A}_a &= e^{-\alpha\varphi}\,(A_a - \chi\, \mathcal{A}_a), \quad \hat{A}'_a = e^{-\alpha\varphi}\,(A'_a - \chi'\, \mathcal{A}_a), \quad \hat{A}_7 = e^{4\alpha\varphi}\,\chi, \quad \hat{A}'_7 = e^{4\alpha\varphi}\,\chi' \\
\hat{e} &= e^{2\alpha\varphi}\, e\,.
\end{aligned} \tag{5.38}$$

In addition to this, one can easily obtain relations between the components of seven- and six-dimensional fields in the curved indices

$$\begin{aligned}
\hat{H}_{(3)\,\mu\nu\rho} &= H_{(3)\,\mu\nu\rho} + 3H_{(2)\,[\mu\nu}\, \mathcal{A}_{\rho]}, \quad \hat{H}_{(3)\,\mu\nu z} = H_{(2)\,\mu\nu}, \\
\hat{A}_\mu &= A_\mu, \quad \hat{A}'_\mu = A'_\mu, \quad \hat{A}_z = \chi, \quad \hat{A}'_z = \chi', \\
\hat{B}_{(2)\,\mu\nu} &= B_{(2)\,\mu\nu}, \quad \hat{B}_{(2)\,\mu z} = B_{(1)\,\mu}\,.
\end{aligned} \tag{5.39}$$

At this stage, following [45], a re-parametrization of two scalar fields Φ and φ is introduced as follows

$$\Phi = e^{\frac{2}{5}\psi - \frac{4}{5}\phi}, \qquad 20\alpha\varphi = -2\psi - \phi\,. \tag{5.40}$$

With these preliminaries, the six-dimensional bosonic Lagrangian becomes

$$\begin{aligned}\mathcal{L}_6 &= R{*}\mathbb{1} - \tfrac{1}{4}{*}d\phi\wedge d\phi - \tfrac{1}{4}{*}d\psi\wedge d\psi - \tfrac{1}{2}e^{\frac{1}{2}\phi+\psi}\,{*}\mathcal{F}_{(2)}\wedge\mathcal{F}_{(2)} - \tfrac{1}{2}e^{\phi}\,{*}H_{(3)}\wedge H_{(3)}\\ &\quad -\tfrac{1}{2}e^{\frac{1}{2}\phi-\psi}\,{*}H_{(2)}\wedge H_{(2)} + \tfrac{1}{2}\,e^{\frac{1}{2}\phi}\,({*}F_{(2)}\wedge F_{(2)} + {*}F'_{(2)}\wedge F'_{(2)})\\ &\quad +\tfrac{1}{4}e^{-\psi}\,({*}d\chi\wedge d\chi + {*}d\chi'\wedge d\chi') - \tfrac{\Omega_{(6)}}{g} - V{*}\mathbb{1}\,,\end{aligned} \tag{5.41}$$

where

$$V = 4g^2 e^{-\frac{1}{2}\phi}\,, \tag{5.42}$$

and $\Omega_{(6)}$ is the Chern-Simons term in six dimensions which shall be studied in the next subsection. Note that the Yang-Mills kinetic terms have a wrong sign.

5.3.2 The Chern-Simons term in seven dimensions

Let us calculate the Chern-Simons variation for the Abelian truncation, i.e. $\hat{A}_{(2)} = \hat{A}^{12}_{(2)}$, $\hat{A}'_{(2)} = \hat{A}^{34}_{(2)}$. Since all of the fields in this part is seven-dimensional, then we omit the hat for simplicity. The expression we have used here to calculate the Chern-Simons term is eqn (10) of [40]. Since the normalization convention for the trace operator does not specify clearly in the latter, we generally assume

$$\mathrm{Tr}(F\wedge F) = c\,F^{ij}\wedge F^{ji}\,, \tag{5.43}$$

where this c is either 1 or $\frac{1}{2}$. We have the following relations for $\delta\Omega_{(3)}$

$$\begin{aligned}\delta\Omega_{(3)} &= 4c^2(F\wedge F + F'\wedge F')\wedge(F\wedge\delta A + F'\wedge\delta A')\\ &= 4c^2(F\wedge F\wedge F\wedge\delta A + F\wedge F\wedge F'\wedge\delta A'\\ &\quad +F'\wedge F'\wedge F\wedge\delta A + F'\wedge F'\wedge F'\wedge\delta A')\,.\end{aligned} \tag{5.44}$$

Also, $\delta\Omega_{(5)}$ shall be written as follows

$$\delta\Omega_{(5)} = 2c\,(F\wedge F\wedge F\wedge\delta A + F'\wedge F'\wedge F'\wedge\delta A')\,. \tag{5.45}$$

Now, combining the above relations, one may find the following relation for the Chern-Simons variation

$$\delta\Omega_{(7)} = 2\delta\Omega_{(5)} - \delta\Omega_{(3)} = 4c\,\big[(1-c)F\wedge F\wedge F\wedge\delta A + (1-c)F'\wedge F'\wedge F'\wedge\delta A' \\ -cF\wedge F\wedge F'\wedge\delta A' - cF'\wedge F'\wedge F\wedge\delta A\big]\,. \tag{5.46}$$

Therefore, for the specific value of $c = \frac{1}{2}$, we have the following

$$\delta\Omega_{(7)} = F\wedge F\wedge F\wedge\delta A + F'\wedge F'\wedge F'\wedge\delta A' \\ -F\wedge F\wedge F'\wedge\delta A' - F'\wedge F'\wedge F\wedge\delta A\,. \tag{5.47}$$

After some algebra, one finds the $7D$ Chern-Simons term, up to a total derivative is

$$\Omega_{(7)} = \tfrac{1}{4}F\wedge F\wedge F\wedge A + \tfrac{1}{4}F'\wedge F'\wedge F'\wedge A' - \tfrac{1}{2}F\wedge F\wedge F'\wedge A'\,. \tag{5.48}$$

The above relation for seven-dimensional Chern-Simons term can be vanished in specific cases, e.g. $\hat{F} = \pm\hat{F}'$. Therefore, we apply a further truncation and assume $\hat{A} = \hat{A}'$, or in six dimensions $A = A'$ and $\chi = \chi'$.

5.3.3 The bosonic equations of motion

Now, we shall find equations of motion of the six dimensional theory derived from the Lagrangian above (5.41). To ensure the truncation we have imposed on the Yang-Mills fields, i.e. $A = A'$ and $\chi = \chi'$ is consistent, we present equations of motion for both gauge potentials and the axions, and then we will deduce the above mentioned truncation is consistent with the equations we will present. However, the Chern-Simons term is vanishing according to the argument presented in the previous part. First we state scalar equations of motion

$$\Box\,\phi = \tfrac{1}{4}e^{\frac{1}{2}\phi+\psi}\,\mathcal{F}_{(2)}^2 + \tfrac{1}{6}e^{\phi}\,H_{(3)}^2 + \tfrac{1}{4}e^{\frac{1}{2}\phi-\psi}\,H_{(2)}^2 - \tfrac{1}{4}e^{\frac{1}{2}\phi}\,(F_{(2)}^2 + F_{(2)}'^2) - 4g^2e^{-\frac{1}{2}\phi}, \\ \Box\,\psi = \tfrac{1}{2}e^{\frac{1}{2}\phi+\psi}\,\mathcal{F}_{(2)}^2 - \tfrac{1}{2}e^{\frac{1}{2}\phi-\psi}\,H_{(2)}^2 + \tfrac{1}{2}\,e^{-\psi}\,\big((\partial\chi)^2 + (\partial\chi')^2\big). \tag{5.49}$$

Next, we consider equations of motion for one-form potentials $A_{(1)}$ and $A'_{(1)}$ as follows:

$$
\begin{aligned}
2d(e^{\frac{1}{2}\phi} * F_{(2)}) &= -d(e^{\frac{1}{2}\phi-\psi} * H_{(2)}\chi') - e^{\frac{1}{2}\phi-\psi} * H_{(2)} \wedge d\chi' \\
&\quad -d\big(e^{\phi} * H_{(3)} \wedge (A'_{(1)} - \chi' \mathcal{A}_{(1)})\big) + e^{\phi} * H_{(3)} \wedge F'_{(2)}\,, \\
2d(e^{\frac{1}{2}\phi} * F'_{(2)}) &= -d(e^{\frac{1}{2}\phi-\psi} * H_{(2)}\chi) - e^{\frac{1}{2}\phi-\psi} * H_{(2)} \wedge d\chi \\
&\quad -d\big(e^{\phi} * H_{(3)} \wedge (A_{(1)} - \chi \mathcal{A}_{(1)})\big) + e^{\phi} * H_{(3)} \wedge F_{(2)}\,. \qquad (5.50)
\end{aligned}
$$

The equation of motion for the Kaluza-Klein vector potential $\mathcal{A}_{(1)}$ reads

$$
\begin{aligned}
d\big(e^{\frac{1}{2}\phi+\psi} * \mathcal{F}_{(2)}\big) &= -e^{\phi} * H_{(3)} \wedge \big[dB_{(1)} - \tfrac{1}{2}\left(\chi' dA_{(1)} + \chi dA'_{(1)} + A_{(1)} \wedge d\chi' + A'_{(1)} \wedge d\chi\right)\big] \\
&\quad + \tfrac{1}{2}\, e^{-\frac{1}{2}\phi} \left(*F_{(2)} \wedge d\chi + *F'_{(2)} \wedge d\chi'\right). \qquad (5.51)
\end{aligned}
$$

Next, we consider equations of motion for potentials $B_{(1)}$ and $B_{(2)}$ in the following

$$
\begin{aligned}
d(e^{\frac{1}{2}\phi-\psi} * H_{(2)}) &= d(e^{\phi} * H_{(3)} \wedge \mathcal{A}_{(1)}), \\
d(e^{\phi} * H_{(3)}) &= 0\,. \qquad (5.52)
\end{aligned}
$$

Finally, equations of motion for axionic scalar fields χ and χ' are

$$
\begin{aligned}
d(e^{-\psi} * d\chi) &= e^{\phi} * H_{(3)} \wedge dA'_{(1)} \wedge \mathcal{A}_{(1)} - d(e^{\phi} * H_{(3)} \wedge A'_{(1)} \wedge \mathcal{A}_{(1)}) - e^{\frac{1}{2}\phi-\psi} * H_{(2)} \wedge dA'_{(1)} \\
&\quad -d(e^{\frac{1}{2}\phi-\psi} * H_{(2)} \wedge A'_{(1)}) - d(e^{\frac{1}{2}\phi} * F_{(2)} \wedge \mathcal{A}_{(1)})\,, \\
d(e^{-\psi'} * d\chi') &= e^{\phi} * H_{(3)} \wedge dA_{(1)} \wedge \mathcal{A}_{(1)} - d(e^{\phi} * H_{(3)} \wedge A_{(1)} \wedge \mathcal{A}_{(1)}) - e^{\frac{1}{2}\phi-\psi} * H_{(2)} \wedge dA_{(1)} \\
&\quad -d(e^{\frac{1}{2}\phi-\psi} * H_{(2)} \wedge A_{(1)}) - d(e^{\frac{1}{2}\phi} * F'_{(2)} \wedge \mathcal{A}_{(1)})\,. \qquad (5.53)
\end{aligned}
$$

Now it is a straightforward task to check the ansazt $A = A'$ and $\chi = \chi'$ is consistent with the above mention equations. therefore, it is possible to find more convenient relations. Plugging

(5.52) in (5.50), one can obtain the following relations for the six-dimensional field strengths

$$d(e^{\frac{1}{2}\phi} * F_{(2)}) = -e^{\frac{1}{2}\phi-\psi} * H_{(2)} \wedge d\chi + e^{\phi} * H_{(3)} \wedge F_{(2)}\,. \tag{5.54}$$

Using the equations (5.50) and (5.52), one can find the following equations of motion for the axionic scalar field

$$\begin{aligned} d(e^{-\psi} * d\chi) = {} & e^{\phi} * H_{(3)} \wedge dA_{(1)} \wedge \mathcal{A}_{(1)} - e^{\frac{1}{2}\phi-\psi} * H_{(2)} \wedge (2dA_{(1)} - d\chi \wedge d\mathcal{A}_{(1)}) \\ & -e^{\frac{1}{2}\phi} * F_{(2)} \wedge d\mathcal{A}_{(1)}\,. \end{aligned} \tag{5.55}$$

The equation of motion for Kaluza-Klein vector field strengths, can be easily found by using $H_{(2)}$ relation from (5.37)

$$d\big(e^{\frac{1}{2}\phi+\psi} * \mathcal{F}_{(2)}\big) = -e^{\phi} * H_{(3)} \wedge H_{(2)} + e^{-\frac{1}{2}\phi} * F_{(2)} \wedge d\chi\,. \tag{5.56}$$

5.3.4 The bosonic truncations

There is only one scalar field appears in the Salam-Sezgin theory, while the theory presented in the last part has three scalars ϕ, ψ and χ . Hence, one needs to perform a consistent truncation of these scalars. The axions were truncated to zero in [39], but, if these fields would be just constant numbers, then, obviously, they did not contribute in the kinetic part of the Lagrangian. Upon investigation in the supersymmetry transformations, one has to truncate both A and A' to zero. However, since the covariant derivative depends on the gauge field, **one needs to keep the axion, but one may assume it is not a dynamical field, but a constant number to be determined by supersymmetry transformations**.

In that sense, one can retain the standard gauging in the covariant derivative, however, instead of the usual Yang-Mills vector potential, one has the Kaluza-Klein vector potential $\mathcal{A}_{(1)}$ as a gauge field.

Now, we may check whether our proposed truncation is consistent with the bosonic equations of motion. It can be easily verified that equation of motion for the axion (5.55) , and also the field strength $F_{(2)}$ (5.50) are trivially satisfied. The remaining equations after this truncation shall be

$$
\begin{aligned}
\Box\,\phi &= \tfrac{1}{4}e^{\frac{1}{2}\phi+\psi}\,\mathcal{F}_{(2)}^2 + \tfrac{1}{6}e^{\phi}\,H_{(3)}^2 + \tfrac{1}{4}e^{\frac{1}{2}\phi-\psi}\,H_{(2)}^2 - 4g^2 e^{-\frac{1}{2}\phi},\\
\Box\,\psi &= \tfrac{1}{2}e^{\frac{1}{2}\phi+\psi}\,\mathcal{F}_{(2)}^2 - \tfrac{1}{2}e^{\frac{1}{2}\phi-\psi}\,H_{(2)}^2,\\
d(e^{\frac{1}{2}\phi-\psi} * H_{(2)}) &= -e^{\phi} * H_{(3)} \wedge \mathcal{F}_{(2)},\\
d(e^{\phi} * H_{(3)}) &= 0,\\
d\big(e^{\frac{1}{2}\phi+\psi} * \mathcal{F}_{(2)}\big) &= -e^{\phi} * H_{(3)} \wedge H_{(2)}\,.
\end{aligned}
\tag{5.57}
$$

Also, the six-dimensional bosonic Lagrangian has the following form

$$
\begin{aligned}
\mathcal{L}_6 = {} & R{*}\mathbb{1} - \tfrac{1}{4}{*}d\phi\wedge d\phi - \tfrac{1}{4}{*}d\psi\wedge d\psi - \tfrac{1}{2}e^{\frac{1}{2}\phi+\psi}\,{*}\mathcal{F}_{(2)}\wedge\mathcal{F}_{(2)} - \tfrac{1}{2}e^{\phi}\,{*}H_{(3)}\wedge H_{(3)}\\
& -\tfrac{1}{2}e^{\frac{1}{2}\phi-\psi}\,{*}H_{(2)}\wedge H_{(2)} - 4g^2\,e^{-\frac{1}{2}\phi}\,{*}\mathbb{1}\,.
\end{aligned}
\tag{5.58}
$$

This is still not the bosonic Salam-Sezgin Lagrangian, and one more truncation is needed to this theory be achieved. Assume the scalar field ψ is vanishing, and at the same time, $\mathcal{F}_{(2)}^2 = H_{(2)}^2$. They are consistent with equations of motion (5.57). Therefore, due to the re-parametrization given in (5.40) one has

$$
\Phi = e^{-\frac{4}{5}\phi}\,, \quad 20\alpha\varphi = -\phi\,, \quad \Phi = e^{16\alpha\varphi}\,. \tag{5.59}
$$

Finally, the bosonic Lagrangian shall be that of the Salam-Sezgin

$$
\mathcal{L}_6 = R{*}\mathbb{1} - \tfrac{1}{4}{*}d\phi\wedge d\phi - e^{\frac{1}{2}\phi}\,{*}\mathcal{F}_{(2)}\wedge\mathcal{F}_{(2)} - \tfrac{1}{2}e^{\phi}\,{*}H_{(3)}\wedge H_{(3)} - 4g^2\,e^{-\frac{1}{2}\phi}\,{*}\mathbb{1}\,, \tag{5.60}
$$

with the rescaling of $\mathcal{F}_{(2)}^2 = H_{(2)}^2 = 2\,\tilde{F}_{(2)}^2$, where $\tilde{F}_{(2)}$ is the field strength in the Salam-Sezgin theory. Let us summarize all the bosonic truncations we have found so far

$$\begin{aligned}
&\psi = 0, \qquad \Phi = e^{16\alpha\varphi}, \qquad 20\alpha\varphi = -\phi, \qquad \pi_{\bar{A}}{}^{\alpha} = \delta_{\bar{A}}{}^{\alpha}, \qquad M_{\alpha\beta} = \delta_{\alpha\beta}, \\
&\hat{F}^{\bar{A}\bar{B}} = 0, \quad \hat{A}_a = \hat{A}'_a = -\chi\mathcal{A}_a, \quad \hat{A}_7 = \hat{A}'_7 = \chi, \quad \mathcal{F}^2_{(2)} = H^2_{(2)} = 2\,\tilde{F}^2_{(2)}, \quad \chi = \chi' = \text{const.}\,, \\
&H_{(3)} = dB_{(2)} - dB_{(1)} \wedge \mathcal{A}_{(1)}, \quad H_{(2)} = dB_{(1)}, \quad \mathcal{F}_{(2)} = d\mathcal{A}_{(1)}.
\end{aligned} \tag{5.61}$$

Since $\mathcal{F}^2_{(2)} = H^2_{(2)}$, one has options of $\mathcal{F}_{(2)} = \pm H_{(2)}$. The important question of which sign should be chosen, is not answered by the bosonic sector, and it will be determined by the supersymmetry considerations. Also, the constant value of the axion shall be determined by the supersymmetry transformations as well.

5.3.5 The supersymmetry transformations?

We stated in 5.2.4 that the starting point theory in seven dimensions is a Wick rotated theory which we may find it by $\delta_{\bar{A}\bar{B}} \to -\mathrm{i}\,\delta_{\bar{A}\bar{B}}$ Wick rotation. However, the reality condition for the fermionic fields becomes vague here. In spite of that, we found a fermionic truncation for the circle reduction of the Wick rotated theory and we could recover the fermionic Lagrangian of the Salam-Sezgin theory. This is a very encouraging hint about the presence of such an exotic theory in seven dimensions.

We do not present our calculations in the fermionic sector, since the seven dimensional theory is not yet rigorously proved to be existed.

5.4 Final remarks

It is instructive to find out the embedding of the seven-dimensional metric in ten dimensions. Recall our Wick rotation $\delta_{\bar{A}\bar{B}} \to -\mathrm{i}\,\delta_{\bar{A}\bar{B}}$, or the other option $\Phi^{1/4} \to -\mathrm{i}\,\Phi^{1/4}$. One needs to investigate the consequences of this odd choice.

The embedding of seven-dimensional metric ansatz to ten dimensions, as in eqn (46) of [15], is as follows

$$ds^2_{10} = g^{-\frac{1}{4}}\,\Phi^{\frac{3}{16}}\,\Delta^{\frac{1}{4}}\left(ds^2_7 + g^{-2}\,\Phi^{-\frac{1}{2}}\,\Delta^{-1}\,M^{-1}_{\alpha\beta}\,D\mu^{\alpha}\,D\mu^{\beta}\right). \tag{5.62}$$

Hence, after applying either of the above Wick rotations, one can write

$$ds_{10}^2 = \Omega \left(ds_7^2 - g^{-2} \, \Phi^{-\frac{1}{2}} \, D\mu^\alpha \, D\mu^\alpha \right) . \tag{5.63}$$

It means the dimensional reduction has been performed on three **time-like** coordinates instead of the usual space-like reduction. One may conclude this reduction is a case of Hull's [50, 51] program about time-like reduction. Actually he found there is an exotic type IIA in (6,4) signature, i.e. 6 space-like and 4 time-like coordinates [50]. Then one may speculate the Wick rotated supergravity we are looking for, is actually the three time-like reduction of type IIA (6,4).

This is an interesting possibility and further work is needed to find a more elaborated connection between the Wick rotated seven dimensional supergravity and the time-like reduction.

6. PAULI S^2 REDUCTION OF THE SALAM-SEZGIN THEORY

6.1 Introduction

In the previous chapter we saw an alternative higher dimensional origin for the Salam-Sezgin theory where the Kaluza-Klein vector potential $\mathcal{A}_{(1)}$ is non-zero. As we have emphasized there, the reason for keeping this field is to use the "Hopf fibration technique" and to find a group-theoretical explanation for the consistent Pauli S^2 reduction of the Salam-Sezgin theory, constructed by Gibbons and Pope in [38].

The first step towards using the Hopf fibration technique, is starting from a seven-dimensional theory. As we have discussed in the previous chapter, **if one considers just the bosonic sector**, this theory can be $SO(2,2)$ gauged half maximal, i.e. $N=2$ seven-dimensional supergravity. This theory has been investigated in [39], and it may be derived from type I or Heterotic string theory as it was shown in [15].

The next step is performing the Kaluza-Klein S^1 reduction down to six dimensions to obtain a non-chiral $N=(1,1)$ supergravity. Then, one may apply consistent truncations to obtain the Salam-Sezgin theory. However, the difference between the six-dimensional bosonic truncation imposed in [39] and ours is the Kaluza-Klein field strength $\mathcal{F}_{(2)}$ and two-form field strength $H_{(2)}$ in one hand and the Yang-Mills field strengths $F^{12}_{(2)}$ and $F^{34}_{(2)}$ in the other hand. Let us compare two truncations in the following

$$\begin{aligned}
&\text{Truncation given by [39] :} \qquad && \mathcal{F}_{(2)} = H_{(2)} = 0\,, \qquad && F^{12}_{(2)} = -F^{34}_{(2)}\,,\\
&\text{Our truncation :} && \mathcal{F}_{(2)} = -H_{(2)}\,, && F^{12}_{(2)} = F^{34}_{(2)} = 0\,.
\end{aligned} \tag{6.1}$$

Note both of these truncations are consistent and give rise to the Salam-Sezgin theory in the bosonic sector.

Then one needs to perform an $SU(2)$ DeWitt reduction of the seven-dimensional theory to obtain a four-dimensional one [44]. Now, having obtained a consistent reduction of the Salam-

Sezgin theory with its Dirac monopole $\mathcal{A}_{(1)}$, it is a straightforward calculation to write the the $SU(2)$ as a $U(1)$ Hopf fibration over S^2 . The crucial point here is, the ansatz we may write in the Hopf fibration manner, should be consistent with the truncation we have applied in six dimensions to obtain the Salam-Sezgin theory. As we will observe, this truncation is consistent with the Hopf ansatz. Thus, one can read off the consistent S^2 reduction of the Salam-Sezgin theory, and find the exact same results as [38] after some field rescalings. Therefore, by this method, we have a group-theoretic understanding of why the Pauli S^2 reduction of the Salam-Sezgin theory works.

In this chapter, we deal with fields in seven, six and four dimensions. To avoid ambiguity, the convention is as follow. We insert a hat on seven-dimensional fields and four-dimensional ones are appearing without any symbol on them everywhere. Especially in section 6.5, and where there is a chance of an ambiguity, we shall insert a bar on six-dimensional quantities.

The rest of this chapter is organized as follows. In section 6.2, we shall consider the non-compact $SO(2,2)$ half maximal seven-dimensional supergravity and perform some truncatuions. In section 6.3, we perform a Kaluza-Klein S^1 reduction followed by our bosonic truncations to obtain the Salam-Sezgin theory. We perform an $SU(2)$ DeWitt reduction in section 6.4, and we consider the Hopf fibration technique in section 6.5 to find the S^2 reduction of the Salam-Sezgin theory. Also, we will verify the consistency of our ansatz by inserting it in six-dimensional equations of motion.

6.2 Truncation of $N = 2$ gauged $SO(2,2)$ supergravity in seven dimensions

The gauged $SO(4)$ half maximal supergravity in seven dimensions, as we described in length in chapter 5, can be obtained from maximal $SO(5)$ seven-dimensional supergravity found in [40] in two steps:

First, one may use the Inönü-Wigner group contraction limit of $SO(5)$ gauged supergravity to find a maximal supergravity (i.e. $N = 4$) with $SO(4)$ gauge group.

Second, one may perform a consistent truncations on bosonic and fermionic fields of the maximal $SO(4)$ theory to obtain a half maximal theory with $N = 2$, but with the same $SO(4)$ gauging in seven dimensions.

The entire Lagrangian and the supersymmetry transformations of seven-dimensional gauged $SO(4)$ half maximal supergravity were given in 5.2.3.

6.2.1 Pass to a non-compact $SO(2,2)$ gauging

As it was illustrated in [41], it is possible to pass from a compact Yang-Mills gauging to a non-compact one. Following that idea, it was assumed in [39] the gauging can be the non-compact $SO(2,2)$ rather than the compact $SO(4)$ one. One needs to simply assume $\eta_{\bar{A}\bar{B}} = \mathrm{diag}\,(1,1,-1,-1)$.

The main motivation of choosing the non-compact gauging is obtaining the positive definite potential of the Salam-Sezgin theory, which is one of the main obstacle to find a higher dimensional embedding of this theory. According to a no-go theorem by Maldacena and Nunez [43], if the lower-dimensional space-time has a positive definite potential, then the non-singular internal space should be non-compact. This is why the non-compact group was chosen in [39].

Now, having described the bosonic part of the half maximal theory with non-compact $SO(2,2)$ gauging in seven dimensions, one may perform a consistent bosonic truncation to find a simpler theory in seven dimensions.

6.2.2 Truncation of $SO(2,2)$ half maximal theory in seven dimensions

If one interests in just the **bosonic section** of the Salam-Sezgin theory and its Pauli S^2 reduction down to four-dimensional space-time, then it is more convenient to impose truncations in seven-dimensional $SO(2,2)$ half maximal theory, before performing the S^1 Kaluza-Klein reduction. The truncation is as follows

$$\pi_{\bar{A}}{}^{\alpha} = \delta_{\bar{A}}{}^{\alpha}\,, \qquad\qquad A_{(1)}^{\bar{A}\bar{B}} = 0\,. \tag{6.2}$$

The above truncation is obviously consistent, i.e. the equations of motion of the bosonic Lagrangian (5.17) is consistent with them. The final bosonic Lagrangian after applying the above truncations has become

$$\hat{\mathcal{L}}_7 = \hat{R}\,\hat{*}\mathbb{1} - \tfrac{5}{16}\,\hat{\Phi}^{-2}\,\hat{*}d\hat{\Phi}\wedge d\hat{\Phi} - \tfrac{1}{2}\hat{\Phi}^{-1}\,\hat{*}\hat{H}_{(3)}\wedge\hat{H}_{(3)} - 4g^2\,\hat{\Phi}^{\frac{1}{2}}\,\hat{*}\mathbb{1}\,, \tag{6.3}$$

where we insert hat on fields to emphasize they are seven-dimensional ones. Note we have used the non-compact gauging to obtain the potential term, i.e. $\eta_{\bar{A}\bar{B}} = \mathrm{diag}\,(1,1,-1,-1)$, and also $M_{\alpha\beta} = \eta_{\alpha\beta}$, according to truncation of the scalar fields.

The Bianchi identity (5.20), after the above truncations reads

$$d\hat{H}_{(3)} = 0\,. \tag{6.4}$$

Now, one may perform the S^1 Kaluza-Klein reduction to find the bosonic sector of the Salam-Sezgin theory.

6.3 The Kaluza-Klein circle reduction down to six dimensions

The Kaluza-Klein circle reduction is a standard calculation which we presented in the chapter 2. Using these results, one can write the metric ansatz as follows

$$d\hat{s}_7^2 = e^{2\alpha\varphi}\, ds_6^2 + e^{-8\alpha\varphi}\,(dz + \mathcal{A}_{(1)})^2\,, \tag{6.5}$$

where φ is a "breathing mode", and $\alpha^2 = \frac{1}{40}$.

Making use of the seven-dimensional Bianchi identity $d\hat{H}_{(3)} = 0$, one can write

$$\hat{H}_{(3)} = d\hat{B}_{(2)}\,. \tag{6.6}$$

The ansätze for the Kaluza-Klein circle reduction for the bosonic fields are standard and can be written as follows

$$\hat{B}_{(2)} = B_{(2)} + B_{(1)} \wedge dz\,, \qquad \hat{H}_{(3)} = H_{(3)} + H_{(2)} \wedge (dz + \mathcal{A}_{(1)})\,, \tag{6.7}$$

where

$$H_{(3)} = dB_{(2)} - dB_{(1)} \wedge \mathcal{A}_{(1)}\,, \qquad H_{(2)} = dB_{(1)}\,, \qquad \mathcal{F}_{(2)} = d\mathcal{A}_{(1)}\,. \tag{6.8}$$

The relations between the flat and curved components of seven- and six- dimensional fields are as follows

$$
\begin{aligned}
&\hat{H}_{abc} = e^{-3\alpha\varphi}\, H_{abc}, \qquad \hat{H}_{ab7} = e^{2\alpha\varphi}\, H_{ab}\,, \qquad \hat{e} = e^{2\alpha\varphi}\, e \\
&\hat{H}_{(3)\,\mu\nu\rho} = H_{(3)\,\mu\nu\rho} + 3H_{(2)\,[\mu\nu}\,\mathcal{A}_{\rho]}, \qquad \hat{H}_{(3)\,\mu\nu z} = H_{(2)\,\mu\nu}\,, \\
&\hat{B}_{(2)\,\mu\nu} = B_{(2)\,\mu\nu}, \quad \hat{B}_{(2)\,\mu z} = B_{(1)\,\mu}\,.
\end{aligned} \tag{6.9}
$$

The convenient re-parametrization of scalar fields is introduced again as follows

$$
\Phi = e^{\frac{2}{5}\psi - \frac{4}{5}\phi}\,, \qquad 20\alpha\varphi = -2\psi - \phi\,. \tag{6.10}
$$

Having obtained the circle reduction, the six-dimensional bosonic Lagrangian becomes

$$
\begin{aligned}
\mathcal{L}_6 = {}& R{*}\mathbb{1} - \tfrac{1}{4}{*}d\phi \wedge d\phi - \tfrac{1}{4}{*}d\psi \wedge d\psi - \tfrac{1}{2}e^{\frac{1}{2}\phi + \psi}\,{*}\mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} - \tfrac{1}{2}e^{\phi}\,{*}H_{(3)} \wedge H_{(3)} \\
&-\tfrac{1}{2}e^{\frac{1}{2}\phi - \psi}\,{*}H_{(2)} \wedge H_{(2)} - 4g^2 e^{-\frac{1}{2}\phi}{*}\mathbb{1}\,,
\end{aligned} \tag{6.11}
$$

Now, we shall find equations of motion of the six dimensional theory derived from the Lagrangian in (6.11). They are exactly as eqn (5.57).

At this stage, we impose truncations in six-dimensional fields to obtain the Salam-Sezgin theory.

6.3.1 The bosonic truncations in six dimensions

The bosonic sector of the Salam-Sezgin theory can be achieved by the following truncations in six dimensions

$$
\psi = 0\,, \qquad \mathcal{A}_{(1)} = -\bar{B}_{(1)} \equiv \tfrac{1}{\sqrt{2}}\,A_{(1)} \tag{6.12}
$$

where $A_{(1)}$ is the gauge potential appears in the Salam-Sezgin theory. It is easy to check the above ansatz is consistent with the six-dimensional equations we have found in (5.57).

Now, the six-dimensional bosonic Lagrangian in (6.11) becomes that of the Salam-Sezgin theory

$$\mathcal{L}_{SS} = \bar{R}\,\bar{*}\mathbb{1} - \tfrac{1}{4}\bar{*}d\phi \wedge d\phi - \tfrac{1}{2}e^{\frac{1}{2}\phi}\,\bar{*}F_{(2)} \wedge F_{(2)} - \tfrac{1}{2}e^{\phi}\,\bar{*}\bar{H}_{(3)} \wedge \bar{H}_{(3)} - 4g^2\,e^{-\frac{1}{2}\phi}\,\bar{*}\mathbb{1}\,, \qquad (6.13)$$

where we have used bar to emphasize the fields are six-dimensional ones. (No bar on $F_{(2)}$ since there will be no such a field in four dimensions below and therefore it will be surplus.)

The next step is performing a DeWitt $SU(2)$ reduction of seven dimensions down to four, which will be the subject of the next section.

6.4 DeWitt $SU(2)$ reduction from $7D$ theory

In this section, we perform a group-theoretic DeWitt reduction of the seven-dimensional theory, the half maximal $SO(2,2)$ supergravity, on a group manifold $S^3 = SU(2)$. The consistency of this reduction is guaranteed by a group-theoretic argument as we have emphasized in chapter 1. The method we have used here is analogous to that of chapter 4.

The ansätze for the metric and three-form $\hat{H}_{(3)}$ in terms of four-dimensional fields read

$$\begin{aligned} d\hat{s}_7^2 &= e^{2\alpha'\varphi'}\,ds_4^2 + \tfrac{1}{4}g^{-2}\,e^{-\frac{4\alpha'}{3}\varphi'}\,T_{ij}\,\nu^i\,\nu^j\,, \\ \hat{H}_{(3)} &= mg^{-3}\,\Omega_{(3)} + g^{-2}\,\tfrac{1}{2}\varepsilon_{ijk}\,B^i \wedge \nu^j \wedge \nu^k + g^{-1}\,C^i \wedge \nu^i + H_{(3)}, \end{aligned} \qquad (6.14)$$

where $\nu^i = \sigma^i - gA^i$, and σ^i are the left-invariant one-forms of $SU(2)$, as described in eqn (4.28) of chapter 4, and $\alpha'^2 = \frac{3}{20}$. The matrix T_{ij} describing the scalar fields is unimodular. The three-form $\Omega_{(3)}$ is $\Omega_{(3)} \equiv \nu^1 \wedge \nu^2 \wedge \nu^3$, as it was mentioned in chapter 4. Also four-dimensional fields B^i and C^i are one-form and two-form triplets of $SU(2)$ respectively. Finally $H_{(3)}$ is a four-dimensional three-form.

Having obtained the complete reduction ansäztze for an $SU(2)$ reduction of seven-dimensional theory, the next step is writing $S^2 = S^3/S^1$ and using the Hopf fibration method.

6.5 $SU(2)$ as a $U(1)$ Hopf fibration over S^2

Now, following the same line of reasoning as in section 4.3.2, one can write $SU(2)$ as a $U(1)$ Hopf fibration over S^2. We may use extensively the calculations we presented in chapter 4. One can write

$$\begin{aligned}
\nu^i &= 2g\mu^i\,(dz + \mathcal{A}_{(1)}) - \Delta^{-1}\,T_{jk}\,\varepsilon_{ij\ell}\,\mu^k\,D\mu^\ell\,, \\
\tfrac{1}{2}\varepsilon_{ijk}\,\nu^j \wedge \nu^k &= 2g(dz + \mathcal{A}_{(1)}) \wedge D\mu^i + \Delta^{-1}\,T_{ij}\,\mu^j\,\omega_{(2)}\,, \\
\Omega_{(3)} &= \tfrac{1}{6}\varepsilon_{ijk}\nu^i \wedge \nu^j \wedge \nu^k = 2g(dz + \mathcal{A}_{(1)}) \wedge \omega_{(2)}\,,
\end{aligned} \tag{6.15}$$

where we already provided the proof for all of above relations in section 4.3.2. However, we rescale g and use z instead of τ.

Now, with these preliminaries, one may write down DeWitt $SU(2)$ ansätze in (6.14) in a Hopf fibration manner. To do so, one may start from the metric. It reads

$$d\hat{s}_7^2 = e^{2\alpha'\varphi'}\,ds_4^2 + \tfrac{1}{4}g^{-2}\,e^{-\frac{4\alpha'}{3}\varphi'}\,\Delta^{-1}\,T^{-1}_{ij}\,D\mu^i\,D\mu^j + e^{-\frac{4\alpha'}{3}\varphi'}\,\Delta\,(dz + \mathcal{A}_{(1)})^2\,, \tag{6.16}$$

where, as in section 4.3.2, the gauge potential is

$$\mathcal{A}_{(1)} = \tfrac{1}{2}g^{-1}\,\cos\theta\,d\psi - \tfrac{1}{2}\mu^i A^i - \tfrac{1}{2}g^{-1}\,\Delta^{-1}\,T_{ij}\,\varepsilon_{ik\ell}\,\mu^j\mu^k\,D\mu^\ell\,, \qquad \Delta = T_{ij}\,\mu^i\mu^j\,. \tag{6.17}$$

After some calculations, one can find the field strength of the above Kaluza-Klein gauge potential as follows

$$\mathcal{F}_{(2)} = -\tfrac{1}{2}g^{-1}\,U\Delta^{-2}\,\omega_{(2)} + \tfrac{1}{2}g^{-1}\Delta^{-2}\varepsilon_{ijk}\,D\mu^i \wedge DT_{j\ell}\,T_{km}\,\mu^\ell\mu^m - \tfrac{1}{2}\Delta^{-1}T_{ij}\mu^i\,F^j\,, \tag{6.18}$$

where $U = 2T_{ik}\,T_{kj}\,\mu^i\mu^j - \Delta\,T_{ii}$.

Now, one needs to compare the above result with the metric we found from a circle reduction in (6.5). It turns out

$$e^{-8\alpha\varphi} = e^{-\frac{4\alpha'}{3}\varphi'}\,\Delta\,, \tag{6.19}$$

and also

$$d\bar{s}_6^2 = e^{-2\alpha\varphi+2\alpha'\varphi'} ds_4^2 + \tfrac{1}{4} g^{-2}\, e^{-2\alpha\varphi-\frac{4\alpha'}{3}\varphi'}\,\Delta^{-1}\, T_{ij}^{-1}\, D\mu^i\, D\mu^j\,. \tag{6.20}$$

The next step is finding the Hopf fibration form of the seven-dimensional three-form $\hat{H}_{(3)}$, using (6.14) and the relations (6.15). It reads

$$\begin{aligned}\hat{H}_{(3)} &= 2m\, g^{-2}\,(dz+\mathcal{A}_{(1)}) \wedge \omega_{(2)} + g^{-2}\, B^i \wedge \big(2g\,(dz+\mathcal{A}_{(1)}) \wedge D\mu^i + \Delta^{-1}\, T_{ij}\,\mu^j\,\omega_{(2)}\big)\\ &\quad + g^{-1}\, C^i \wedge \big(2g\,\mu^i\,(dz+\mathcal{A}_{(1)}) - \Delta^{-1}\, T_{jk}\,\varepsilon_{ij\ell}\,\mu^k\, D\mu^\ell\big) + H_{(3)}\,. \end{aligned} \tag{6.21}$$

Recall $\hat{H}_{(3)}$ can be written as

$$\begin{aligned}\hat{H}_{(3)} &= d\hat{B}_{(2)} = d\bar{B}_{(2)} + d\bar{B}_{(1)} \wedge dz = d\bar{B}_{(2)} - d\bar{B}_{(1)} \wedge \mathcal{A}_{(1)} + d\bar{B}_{(1)} \wedge (dz+\mathcal{A}_{(1)})\\ &= \bar{H}_{(3)} + \bar{H}_{(2)} \wedge (dz+\mathcal{A}_{(1)})\,. \end{aligned} \tag{6.22}$$

Hence, comparing the above relations one can read off the six-dimensional two-form and three-form field strengths $\bar{H}_{(2)}$ and $\bar{H}_{(3)}$ as follows

$$\begin{aligned}\bar{H}_{(2)} &= 2mg^{-2}\,\omega_{(2)} - 2g^{-1} B^i \wedge D\mu^i + 2\mu^i C^i,\\ \bar{H}_{(3)} &= g^{-2}\,\Delta^{-1}\, T_{ij}\,\mu^j\, B^i \wedge \omega_{(2)} - g^{-1}\,\Delta^{-1}\, T_{jk}\,\varepsilon_{ij\ell}\,\mu^k\, C^i \wedge D\mu^\ell + H_{(3)}. \end{aligned} \tag{6.23}$$

6.5.1 Imposing the six-dimensional truncations

Now, all the bosonic fields of six-dimensional theory have been written in terms of the four-dimensional ones. The next step is considering the truncations in six dimensions necessary to obtain the Salam-Sezgin theory, and applying those constraints on the Hopf ansatz we have found. Recall to obtain the Salam-Sezgin theory one needs to perform the following truncations in six dimensions

$$\mathcal{F}_{(2)} = -\bar{B}_2 \ , \qquad\qquad \psi = 0\,. \tag{6.24}$$

Now, let us explore the consequences of the above constraints. Comparing results of (6.18) and (6.23), one can observe

$$U = 4m\, g^{-1}\,\Delta^2\,, \qquad B^i = -\tfrac{1}{4}\Delta^{-2}\varepsilon_{ijk}\, DT_{j\ell}\, T_{km}\,\mu^\ell\mu^m\,, \qquad C^i = \tfrac{1}{4}\Delta^{-1}T_{ij}\,F^j\,. \tag{6.25}$$

Now, pursuing the same argument as presented in section 4.4, one can claim since B^i is a four-dimensional field, it shall not depend on the internal components μ^i, therefore $T_{ij} = \delta_{ij}$, and hence $\Delta = T_{ij}\,\mu^i\mu^j = 1$. Also from the expression for U, one finds $U = 2T_{ik}\,T_{kj}\,\mu^i\mu^j - \Delta\, T_{ii} = -1$. In summary, one should have

$$T_{ij} = \delta_{ij}\,, \qquad m = -\tfrac{1}{4}g\,, \qquad B^i = 0\,, \qquad C^i = \tfrac{1}{4}F^i\,. \tag{6.26}$$

The other constraint is $\psi = 0$, and from (6.10), it can be seen that $\phi = -20\alpha\varphi$, and $\Phi = e^{-\frac{4}{5}\phi}$. Then, from relation (6.19), one has $6\alpha\varphi = \alpha'\varphi'$.

Note here in contrast with chapter 4, it is possible to satisfy the constraints resulted from the needed truncation in six dimensions. Now, having found an interpretation of the $SU(2)$ as a $U(1)$ Hopf fibration over S^2, one can write down the ansäzte for the Pauli S^2 reduction of the Salam-Sezgin theory down to four dimensions.

Using the above relations between different scalar fields, from the metric ansatz given by (6.20), one may read off the ansatz for Pauli S^2 reduction of the Salam-Sezgin theory as follows

$$d\bar{s}_6^2 = e^{-\frac{1}{2}\phi}\, ds_4^2 + \tfrac{1}{4}g^{-2}\, e^{\frac{1}{2}\phi}\, D\mu^i D\mu^i\,. \tag{6.27}$$

The next step is using the relations given by (6.26) in results (6.23) to write down the ansäzte for the Salam-Sezgin two-form and three-form field strengths

$$
\begin{aligned}
F_{(2)} &= \sqrt{2}\mathcal{F}_{(2)} = -\sqrt{2}\bar{H}_{(2)} = \tfrac{1}{\sqrt{2}}g^{-1}\omega_{(2)} - \tfrac{1}{\sqrt{2}}\mu^i\, F^i\,, \\
\bar{H}_{(3)} &= H_{(3)} - \tfrac{1}{4}g^{-1}\varepsilon_{ijk}\, F^i\mu^j\, \wedge D\mu^k\,.
\end{aligned}
\tag{6.28}
$$

The six-dimensional scalar field ϕ is the same as four-dimensional one, meaning that in six dimensions it does not depend upon the internal manifold. Now the reduction ansäzte are complete.

6.5.2 Consistency of the ansatz

The way we have found our reduction ansatz, i.e. the Hopf fibration technique, guarantees its consistency by group-theoretical argument. However, it is useful to directly verify this important issue. One may check the consistency of the ansatz, by insert it in six-dimensional equations of motion. If all of the internal manifold dependence would cancel, then the remaining equations, all would depend upon four-dimensional fields, could be considered as the space-time equations of motion which may be derived from a four-dimensional Lagrangian.

First, we consider the Bianchi identity. From (6.8), one can write

$$
d\bar{H}_{(3)} = -\bar{H}_{(2)} \wedge \mathcal{F}_{(2)} = \tfrac{1}{2}F_{(2)} \wedge F_{(2)}\,. \tag{6.29}
$$

To find out the consequence of this six-dimensional equations, one may employ the following relations which we already proved in section 4.3.2.

$$
DF^i = 0\,, \quad \tfrac{1}{2}\varepsilon_{ijk}\, D\mu^j\, \wedge\, D\mu^k = \mu^i\, \omega_{(2)}\,, \quad D(D\mu^i) = g\,\varepsilon_{ijk}\, F^j\, \mu^k\,, \quad D\omega_{(2)} = gD\mu^i \wedge F^i\,. \tag{6.30}
$$

Hence, $d\bar{H}_{(3)}$ becomes

$$
\begin{aligned}
d\bar{H}_{(3)} &= D\bar{H}_{(3)} = dH_{(3)} - \tfrac{1}{4}g^{-1}\,\varepsilon_{ijk}\, F^i\, \wedge D\mu^j\, \wedge D\mu^k\, - \tfrac{1}{4}g^{-1}\,\varepsilon_{ijk}\, F^i\, \mu^j\, \wedge D(D\mu^k) \\
&= dH_{(3)} - \tfrac{1}{2}g^{-1}\,\mu^i\, F^i\, \wedge \omega_{(2)} - \tfrac{1}{4}F^i\, \wedge F^i + \tfrac{1}{4}\mu^i\mu^j\, F^i \wedge F^j\,.
\end{aligned}
\tag{6.31}
$$

To find out the right hand side of (6.29), one can write

$$F_{(2)} \wedge F_{(2)} = -g^{-1}\,\mu^i\,F^i\;\wedge\omega_{(2)} + \tfrac{1}{2}\mu^i\mu^j\,F^i \wedge F^j\,. \tag{6.32}$$

Therefore, the Bianchi identity in six dimensions becomes

$$dH_{(3)} = \tfrac{1}{4}F^i\;\wedge F^i\,. \tag{6.33}$$

As we expected, the internal manifold dependence, terms involving μ^i, $D\mu^i$ and $\omega_{(2)}$ cancels out and the final result just depends upon four-dimensional space-time.

The next step is considering the six-dimensional equations of motion. Since they involve finding the Hodge-dual calculation, let us first present it. We have employed the result in appendix B. The $\bar{*}$ means the Hodge-dual with respect to six-dimensional metric ds_6^2. Then $\bar{H}_{(3)}$ is

$$\begin{aligned}\bar{*}\bar{H}_{(3)} &= \bar{*}H_{(3)} - \tfrac{1}{4}g^{-1}\varepsilon_{ijk}\,\bar{*}(\,F^i\mu^j\;\wedge D\mu^k) = (-1)^{2\times 3}\,e^{\frac{1}{2}\phi}\,\tfrac{1}{4g^2}\,e^{\frac{1}{2}\phi}\;*H_{(3)}\wedge\omega_{(2)} \\ &\quad -\tfrac{1}{4}g^{-1}\varepsilon_{ijk}\,\mu^j\,(-1)^{2\times 1}\;*F^i\wedge\varepsilon_{kmn}\,\mu^m\,D\mu^n = \tfrac{1}{4g^2}\,e^{\phi}\;*H_{(3)}\wedge\omega_{(2)} \\ &\quad +\tfrac{1}{4g}\,*F^i\wedge D\mu^i\,. \end{aligned} \tag{6.34}$$

Also, to find the six-dimensional Hodge-dual of $F_{(2)}$, one can write

$$\begin{aligned}\bar{*}F_{(2)} &= \tfrac{1}{\sqrt{2}}g^{-1}\bar{*}\omega_{(2)} - \tfrac{1}{\sqrt{2}}\bar{*}(\mu^i\,F^i) = \tfrac{1}{\sqrt{2}}g^{-1}\,e^{-\phi}\,(4g^2)\,e^{-\frac{1}{2}\phi}\,*\mathbb{1} \\ &\quad -\tfrac{1}{\sqrt{2}}\mu^i\,(-1)^{2\times 2}\,\tfrac{1}{4g^2}\,e^{\frac{1}{2}\phi}\;*F^i\wedge\omega_{(2)} = 2\sqrt{2}\,g\,e^{-\frac{3}{2}\phi}\;*\mathbb{1} \\ &\quad -\tfrac{1}{4\sqrt{2}g^2}\,e^{\frac{1}{2}\phi}\,\mu^i *F^i\wedge\omega_{(2)}\,. \end{aligned} \tag{6.35}$$

One may find $\bar{*}\mathbb{1}$ as follows

$$\bar{*}\mathbb{1} = e^{-\phi}\,\tfrac{1}{4g^2}\,e^{\frac{1}{2}\phi}\;*\mathbb{1}\wedge\omega_{(2)} = \tfrac{1}{4g^2}\,e^{-\frac{1}{2}\phi}\;*\mathbb{1}\wedge\omega_{(2)}\,. \tag{6.36}$$

Finally, $\bar{*}d\phi$ can be written as

$$\bar{*}d\phi = (-1)^{1\times 2}\, e^{-\frac{1}{2}\phi}\, \tfrac{1}{4g^2}\, e^{\frac{1}{2}\phi} * d\phi \wedge \omega_{(2)} = \tfrac{1}{4} g^{-2} *d\phi \wedge \omega_{(2)}\,. \tag{6.37}$$

Having obtained the above relations, now we can consider the equations of motion in six dimensions. From the bosonic Salam-Sezgin Lagrangian in (6.13) , we can derive the following equation for the scalar field

$$d\bar{*}d\phi + \tfrac{1}{2} e^{\frac{1}{2}\phi}\, \bar{*}F_{(2)} \wedge F_{(2)} + e^{\phi}\, \bar{*}\bar{H}_{(3)} \wedge \bar{H}_{(3)} - 4g^2\, e^{-\frac{1}{2}\phi}\, \bar{*}\mathbb{1} = 0\,. \tag{6.38}$$

Now, if one substitutes the six-dimensional ansätze given in (6.28), and using the above result for $\bar{*}d\phi$, the six-dimensional scalar equations yields

$$\begin{aligned}
&d(\tfrac{1}{4} g^{-2} *d\phi \wedge \omega_{(2)}) + \tfrac{1}{2} e^{\frac{1}{2}\phi} \big(2\sqrt{2}\, g\, e^{-\frac{3}{2}\phi} * \mathbb{1} - \tfrac{1}{4\sqrt{2}g^2}\, e^{\frac{1}{2}\phi}\, \mu^i * F^i \wedge \omega_{(2)}\big) \wedge \\
&(\tfrac{1}{\sqrt{2}} g^{-1} \omega_{(2)} - \tfrac{1}{\sqrt{2}} \mu^i\, F^i) + e^{\phi} \big(\tfrac{1}{4g^2}\, e^{\phi} * H_{(3)} \wedge \omega_{(2)} + \tfrac{1}{4g} * F^i \wedge D\mu^i\big) \wedge \\
&\big(H_{(3)} - \tfrac{1}{4} g^{-1} \varepsilon_{ijk}\, F^i \mu^j \wedge D\mu^k\big) - 4g^2\, e^{-\frac{1}{2}\phi} \big(\tfrac{1}{4g^2}\, e^{-\frac{1}{2}\phi} * \mathbb{1} \wedge \omega_{(2)}\big) = 0\,.
\end{aligned} \tag{6.39}$$

Now, the surviving terms above have a form of $G_{(4)} \wedge \omega_{(2)}$, where $G_{(4)}$ is a four-form in space-time. Hence $G_{(4)} \wedge \omega_{(2)} = 0$, implies $G_{(4)} = 0$. It means

$$\begin{aligned}
&\tfrac{1}{4g^2}\, d{*}d\phi + e^{-\phi} * \mathbb{1} + \tfrac{1}{16g^2}\, e^{\phi}\, \mu^i \mu^j * F^i \wedge F^j + \tfrac{1}{4g^2}\, e^{2\phi} * H_{(3)} \wedge H_{(3)} \\
&- \tfrac{1}{16g^2}\, e^{\phi} \left(\mu^i \mu^j * F^j \wedge F^i - *F^i \wedge F^i\right) - e^{-\phi} * \mathbb{1} = 0\,.
\end{aligned} \tag{6.40}$$

As it is clear from the above relation, all S^2 dependence of fields cancels out and the remaining equation is that of a four-dimensional scalar field, which reads

$$d{*}d\phi + e^{2\phi} *H_{(3)} \wedge H_{(3)} + \tfrac{1}{4} e^{\phi} *F^i \wedge F^i = 0\,. \tag{6.41}$$

Now, let us consider the one-form gauge potential six-dimensional equation of motion. From the Lagrangian (6.13), it reads

$$d(e^{\frac{1}{2}\phi}\bar{*}F_{(2)}) - e^{\phi}\,\bar{*}\bar{H}_{(3)} \wedge F_{(2)} = 0\,. \tag{6.42}$$

Using relations (6.28), (6.34) and (6.35), one can write

$$\begin{aligned}
&2\sqrt{2}\,g\,d(e^{-\phi}\,*\,\mathbb{1}) - \tfrac{1}{4\sqrt{2}g^2}\,d(e^{\phi}\,\mu^i * F^i \wedge \omega_{(2)}) - e^{\phi}\left(\tfrac{1}{4g^2}\,e^{\phi}\,*\,H_{(3)} \wedge \omega_{(2)} + \tfrac{1}{4g} * F^i \wedge D\mu^i\right) \wedge \\
&\left(\tfrac{1}{\sqrt{2}}g^{-1}\omega_{(2)} - \tfrac{1}{\sqrt{2}}\mu^i\,F^i\right) = -\tfrac{1}{4\sqrt{2}g^2}\,e^{\phi}\,\Big(d\phi \wedge \mu^i * F^i \wedge \omega_{(2)} + \mu^i D * F^i \wedge \omega_{(2)} \\
&+g\mu^i * F^i \wedge D\mu^j \wedge F^j - e^{\phi}\,*\,H_{(3)} \wedge \mu^i F^i \wedge \omega_{(2)} - g * F^i \wedge F^j \wedge D\mu^i\,\mu^j\Big) = 0\,,
\end{aligned} \tag{6.43}$$

where we have used (6.30). Now, it is apparent that the S^2 dependence vanishes and the four-dimensional equation becomes

$$d\phi \wedge *F^i + D * F^i - e^{\phi}\,*\,H_{(3)} \wedge F^i = 0\,. \tag{6.44}$$

Finally the six-dimensional equation of motion for $\bar{B}_{(2)}$ is

$$d(e^{\phi}\,\bar{*}\bar{H}_{(3)}) = 0\,. \tag{6.45}$$

Using (6.34), and the relations in (6.30), the above equation implies

$$\begin{aligned}
&\tfrac{1}{4g^2}\,d(e^{2\phi}\,*\,H_{(3)} \wedge \omega_{(2)}) + \tfrac{1}{4g}d(e^{\phi}\,*\,F^i \wedge D\mu^i) = \tfrac{1}{4g^2}\,\Big[d(e^{2\phi}\,*\,H_{(3)}) \wedge \omega_{(2)} \\
&-e^{2\phi}\,*\,H_{(3)} \wedge gD\mu^i\,F^i + gD(e^{\phi}\,*\,F^i) \wedge D\mu^i + g^2\,e^{\phi}\,*\,F^i \wedge \varepsilon_{ijk}\,F^j\mu^k\Big] = 0\,.
\end{aligned} \tag{6.46}$$

Since $*F^i \wedge F^j = \frac{1}{2}F^i.F^j * \mathbb{1}$, then it is symmetric under the exchange of i and j indices, hence the last term in the above relation vanishes. Then, again all of the internal manifold dependence vanishes and one concludes the following equations

$$d(e^{2\phi} *H_{(3)}) = 0\,, \qquad \text{and} \quad D(e^{\phi} *F^i) - e^{2\phi} *H_{(3)} \wedge F^i = 0\,. \tag{6.47}$$

Having obtained all of the field equations for four-dimensional space-time, one may observe that those fields can be derived from the following bosonic Lagrangian in four dimensions.

$$\mathcal{L}_4 = R *\mathbb{1} - \tfrac{1}{2} {*d\phi} \wedge d\phi - \tfrac{1}{2} e^{2\phi} *H_{(3)} \wedge H_{(3)} - \tfrac{1}{4} e^{\phi} *F^i \wedge F^i\,. \tag{6.48}$$

Finally one may compare the ansäzte we have found here with those of [38]. With the following rescalings, one can recover the Pauli S^2 reduction of the Salam-Sezgin theory presented in [38]

$$A^i \to \sqrt{2}\, A^i\,, \qquad g \to \sqrt{2}\, g\,, \qquad \phi \to -\phi\,. \tag{6.49}$$

7. CONCLUSION AND OUTLOOK

In this dissertation, we have investigated different consistent Kaluza-Klein-Pauli reductions. All the dimensional reductions we have considered are Pauli (coset) reductions, which besides few exceptions, they are generally *inconsistent*.

In chapter 2, we considered a toy example of the Klein-Gordon scalar field, and explaining why one has an infinite massive tower of fields in the lower dimensional theory, after a circle reduction of a massless scalar in the higher dimensions. Also, we presented the reduction ansätze for the metric and gauge potential for the circle reduction.

In chapter 3, we found the embedding of two specific truncations, i.e. $3 + 1$ and $2 + 2$ of STU gauge supergravity, in eleven dimensions. STU gauged supergravity is a maximal Abelian subgroup of the eminent $N = 8$, gauged $SO(8)$ four-dimensional supergravity. We have found the metric and four-form field strength ansätze for $3 + 1$ case for the first time, and we could recover the results of [34] in $2 + 2$ truncation.

We have used a systematic method to find Pauli reductions. This method was introduced in [7] and named "Hopf fibration technique" in [45]. One performs a consistent DeWitt reduction of a higher dimensional initial theory, say T_1, on a group manifold G down to the lower-dimensional space-time theory, named T_3. Furthermore, one may construct another consistent DeWitt reduction of T_1 on a group manifold H, where $G \supset H$, and obtains a theory T_2 whose dimension is higher than that of the space-time. The claim is, there is a consistent Pauli reduction of T_2 on a coset space G/H to give the space-time theory, T_3. It can be justified, if one considers a Hopf fibration of G as a group manifold H bundle over a coset space G/H, then the consistency of the Pauli reduction on a coset space G/H is guaranteed based on the consistencies of both DeWitt reductions of T_1 on group manifolds G and H. Thanks to this Hopf fibration technique, one may find out a deeper understanding of why a "miraculous" Pauli reduction works.

We have employed the Hopf fibration technique to investigate the possibility of the Pauli reductions in two different scenarios. First, in chapter 4, we consider an S^2 reduction of the minimal supergravity in five dimensions. The main motivation for this investigation is the resemblance of the bosonic Lagrangian of the latter theory and that of eleven-dimensional supergravity. As a result of this analogy, and since there are S^7, S^4 and S^5 Pauli reductions of eleven-dimensional supergravity, one may conjecture there are S^2 or S^3 reductions of minimal supergravity in five dimensions. Making use of this technique, one may start from minimal supergravity in six dimensions as the initial theory, and perform an S^1 reduction to find minimal supergravity coupled to a vector multiplet in five dimensions. However, to obtain pure five-dimensional minimal supergravity, one needs to perform consistent truncations in five dimensions. The next step, shall be an $SU(2)$ DeWitt reduction of the initial theory to give a three-dimensional space-time. Now, viewing the $S^3 = SU(2)$ as a Hopf fibration of an S^1 bundle over coset space $S^2 = S^3/S^1$, one can find the consistent Pauli S^2 reduction of the five-dimensional minimal supergravity coupled to a vector multiplet. While it is an interesting result in its own right, this construction has not been our main goal of the investigation, and we are looking to find an S^2 reduction of pure five-dimensional minimal supergravity. As we have mentioned above, one needs to perform truncations in five dimensions to obtain the latter theory from the former one. However, a simple consideration shows these truncations are not compatible with the ansatz we have found for minimal supergravity coupled to a vector multiplet, and the search for constructing the Pauli reduction of minimal supergravity fails. Although this analysis does not provide a rigorous mathematical proof of the impossibility of the reduction, it is very promising. Especially, we have tried to find a trial ansatz and by inspecting the higher dimensional equations of motion, we hoped to find the correct ansatz. Again, this method failed to give a correct ansatz. Therefore, by considering these examinations, it seems very persuading to believe there is no consistent Pauli S^2 of the minimal supergravity in five dimensions.

Our second inquiry, in chapter 6 is revisiting an interesting question of the possibility of a consistent S^2 Pauli reduction of six-dimensional $N = (1, 0)$ Einstein-Maxwell supergravity. The latter theory found by Salam and Sezgin in 1984, and hence it has been named Salam-Sezgin theory. Since this theory admits a $Minkowski_4 \times S^2$ vacuum with $N = 1$ supersymmetry, it is an interesting inquiry to find out whether this theory admits the full non-linear Pauli S^2 reduction. Gibbons and Pope found out a "remarkable" consistent reduction ansatz in 2003, however, the underlying reason of why this ansatz is achievable remains obscure. Furthermore, Salam and Sezgin observed in their original paper [36], since the integer S^2 Dirac monopole charge n has been fixed by the equations of motion to be $n = \pm 1$, then regarding the fact that an S^2 with a singly charge Dirac monopole on it is an S^3, they conjectured the possibility of the presence of an S^3 dimensional reduction of a seven-dimensional theory to give the four-dimensional Minkowski vacuum they found from the spontaneous compactification of their theory. This is exactly what we have found with the Hopf fibration technique.

To use the Hopf fibration technique, one needs to find an embedding of the Salam-Sezgin theory in a higher-dimensional one. This has been done in [39] and an M-theory origin of this theory has been found. However, the Kaluza vector potential $\mathcal{A}_{(1)}$, arises from a circle reduction of the metric of a seven-dimensional theory, was set to zero in their work. Since this vector field is corresponding to a Dirac monopole on S^2, its vanishing means the monopole charge has to be set to zero, and hence the resulting Hopf fibration is just a trivial $S^2 \times S^1$ instead of the expected S^3 one. To remedy this problem, one has to find another embedding of the Salam-Sezgin theory where the vector potential $\mathcal{A}_{(1)}$ is non-zero. We have found such an embedding.

The alternative embedding of the Salam-Sezgin theory in M-theory has been found based on the assumption that the Kaluza vector potential $\mathcal{A}_{(1)}$ is not zero. The striking feature of this embedding is the role of a constant axion. According to the standard Kaluza-Klein circle reduction, one may obtain a scalar, named axion, from a reduction of a vector one-form gauge potential. While this axion has been always assumed either to be truncated to zero or to be a scalar field depends on the space-time coordinate in the literature, we presume this field is a constant to be determined by the supersymmetry transformations. By this premise, one can retain the usual covariant derivative. Since the six-dimensional Yang-Mills gauge potentials are truncated to zero in our scheme, one needs a vector potential in the covariant derivative to be gauged. Thanks to the circle reduction, the Kaluza vector potential $\mathcal{A}_{(1)}$ appears along with an axion in the covariant derivative of six-dimensional theory, and this vector potential can play a role of the usual gauge vector in the covariant derivative as a result of the constant axion.

In the first embedding of the Salam-Sezgin theory in M-theory found in [39], it was assumed the compact $SO(4)$ gauge group of the seven-dimensional theory is changed to a non-compact group of $SO(2,2)$. They could recover the positive definite potential of the Salam-Sezgin theory by this premise. We constructed the alternative embedding, in the **bosonic sector**, with the non-compact gauging in chapter 6, where the vector potential $\mathcal{A}_{(1)}$ is non-zero. However, to retain the vector potential $\mathcal{A}_{(1)}$, if one would consider the fermionic sector as well, one has to start from an $N = 2$ with $SO(4)$ gauging, but a 'Wick rotated' supergravity in seven dimensions, as it was shown in chapter 5. The Wick rotation can be achieved by either $\delta_{\bar{A}\bar{B}} \to -\mathrm{i}\, \delta_{\bar{A}\bar{B}}$ or $\Phi^{1/4} \to -\mathrm{i}\, \Phi^{1/4}$, together with $H_{MNP} \to -H_{MNP}$. However, as we have emphasized in chapter 5, the reality condition of this theory is not very clear. This is why we did not present our fermionic calculation in this dissertation. Nevertheless, both supersymmetry transformations maintain the Lagrangian is invariant under them. More promising, we could find exactly the same Lagrangian, both bosonic and fermionic sectors, as that of the Salam-Sezgin theory. Also, according to our fermionic truncations, the supersymmetry transformations become the same as those of the Salam-Sezgin theory.

Upon writing down the embedding of seven-dimensional metric ansatz in ten dimensions, one can observe that the Wick rotated theory in seven dimensions, results from a time-like reduction of ten-dimensional type I theory. In other words, the metric of the internal manifold, has an overall negative sign as opposed to the usual positive sign. Time-like reduction is an interesting possibility in the dimensional reduction program, which explored by Hull in 1990s [50, 51]. Since reduction is time-like, then the higher dimensional theory should have a different space-time signature than that of Minkowski. In this case, to obtain a $(t, s) = (1, 6)$ theory with three time-like compact coordinates, the ten dimensional theory should have a signature of $(t, s) = (4, 6)$. This is a possibility in Hull's work in [50] as a type IIA (4,6). It is intriguing that the Salam-Sezgin theory has an embedding with the usual space-like reduction and also an exotic time-like one.

Having found an alternative embedding of the Salam-Sezgin theory with non-zero vector potential $\mathcal{A}_{(1)}$, one can follow the Hopf fibration technique to find an S^2 reduction of the latter theory. In chapter 6, we showed in detail how this works. After performing a circle reduction of seven-dimensional theory, one needs to impose a truncation to obtain the Salam-Sezgin theory in six dimensions. The crucial point is to verify whether this truncation is consistent with the Hopf ansatz we find. Unlike the case of chapter 4, we showed this truncation is actually consistent with the Hopf ansatz in chapter 6, meaning that, there is a consistent Pauli S^2 reduction of the Salam-Sezgin theory down to four dimensions. The two contrasting examples of chapters 4 and 6 show how a truncation can be critical for the possibility of finding a consistent Pauli reduction.

One may employ the Hopf reduction technique to find more Pauli reductions of different supergravities. Especially the case of Pauli S^2 reduction is the easiest one to implement. However, one may consider more involved geometries and may find more interesting Pauli reductions. In addition to this, one may use the assumption of the constant axion to find different truncations, especially when there is a Maxwell gauge potential in the higher-dimensional theory. Since one may truncate the lower-dimensional Maxwell gauge potential to zero and by keeping the axion as a constant number, one can still retain the usual gauging in the lower-dimensional theory by making use of the Kaluza-Klein vector potential.

The other exciting possibility is the time-like reduction which may give rise to more consistent dimensional reductions or even constructing of unknown supergravities.

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APPENDIX A

FINDING THE FIELD RE-DEFINITIONS

It is important to find the field re-definitions between [36], the original paper of Salam and Sezgin, [40] where $SO(5)$ gauged maximal supergravity in seven dimensions was initially constructed, and [39] where the first embedding of the Salam-Sezgin theory was introduced. Especially since the fermionic Lagrangian was not presented (beside the two kinetic terms) in the last work, one needs to find the field re-definitions to obtain the the full fermionic Lagrangian of the Salam-Sezgin theory, from reduction and truncations of the original theory of [40] and compare it with that of the Salam and Sezgin work in [36].

Let us label fields and coupling constant of [36] and [40] with tilde and prime respectively, while those of [39] are without any labeling. To begin with, one may compare the bosonic Lagrangian of $SO(5)$ gauged maximal supergravity in seven dimensions, i.e. eqn (8) of [40] and eqn (1) of [39]. Hence one can find the following

$$\mathcal{L}_7 = 2\mathcal{L}'_7, \quad g = \tfrac{1}{2}g', \quad F_{(2)} = \sqrt{2}F'_{(2)}, \quad S_{(3)} = -2\sqrt{3}\,\mathrm{i}\; mS'_{(3)}, \tag{A.1}$$

where $m = \frac{1}{2}g'$ in the former paper. One can observe that the supersymmetry transformations, eqn (9) of [40] and equations (6) and (9) of [39], agree with each other upon considering the above field redefinitions. However, there is a typographical error in eqn (2) of [39], where the field strength was defined. According to [40] (below eqn (9)), the field strength was defined as usual

$$F'_{(2)A}{}^{B} = dA'_{(1)A}{}^{B} + g'\, A'_{(1)A}{}^{C} \wedge A'_{(1)C}{}^{B}\,, \tag{A.2}$$

where we rewrite it in the form-language, and also rename the vector potential from $B_{(1)}$ to $A'_{(1)}$.

Upon using the above field redefinitions, one has to find the following relation in [39]

$$F_{(2)A}{}^{B} = dA_{(1)A}{}^{B} + \sqrt{2}\, g\, A_{(1)A}{}^{C} \wedge A_{(1)C}{}^{B}\,, \tag{A.3}$$

where $\sqrt{2}$ is missing in eqn (2) of [39]. However, since we consider the abelian truncation of the original theory to obtain the Salam-Sezgin theory, this issue shall be irrelevant. However, this issue reflects in the definition of Q_{ij}. If one accepts the definition of this quantity as it is presented in eqn (6) of [40], then Q_{ij} defined in eqn (2) of [39] should be modifies as follows

$$\Pi^{-1}{}_{i}{}^{A}\,(\delta_{A}{}^{B}\, d + \sqrt{2}\, g\, A_{(1)A}{}^{B})\,\Pi_{B}{}^{k}\,\delta_{kj} = P_{ij} + Q_{ij}\,; \qquad P_{ij} = P_{(ij)}\,, \quad Q_{ij} = Q_{[ij]}, \tag{A.4}$$

where factor of $\sqrt{2}$ has been added. Since it is somewhat inconvenient to have the factor of $\sqrt{2}$, we redefine the gauge potential in the way that this factor becomes absorbed in that field. This is the convention which used in [15] and we follow it.

APPENDIX B

FINDING THE HODGE-DUALITY RELATION WITH A SPHERICAL CONSTRAINT

B.1 The notation

We follow the convention and notion of [26]. For convenience, we present the main definitions here. First of all, to specify our notation, consider a general p-form in a general D dimensions

$$\omega_{(p)} = \tfrac{1}{p!}\,\omega_{\mu_1\mu_2\cdots\mu_p}\,dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}\,. \tag{B.1}$$

Then, the Hodge-dual of this p-form field is defined as follows

$$*\,(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}) = \tfrac{1}{q!}\,\epsilon_{\nu_1\cdots\nu_q}{}^{\mu_1\cdots\mu_p}\,dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q}\,, \tag{B.2}$$

where $q = D - p$. Here

$$\epsilon_{\mu_1\cdots\mu_D} = \sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_D} \tag{B.3}$$

is a totally anti-symmetric Levi-Civita tensor while the Levi-Civita tensor density $\varepsilon_{\mu_1\cdots\mu_D} = (1,-1,0)$ for an even, odd or no permutation of $(0,1,\cdots,D-1)$ respectively. All upstairs indices Levi-Civita density tensor is related to the above one as follows

$$\varepsilon^{\mu_1\cdots\mu_D} = (-1)^t\,\varepsilon_{\mu_1\cdots\mu_D}\,, \tag{B.4}$$

where t is number of time-like component in the metric. Here, since the Levi-Civita density tensor is not a tensor, hence its indices do not raise and lower by the metric. However those of the Levi-Civita tensor do raise and lower by the metric, then one can find out

$$\epsilon^{\mu_1\cdots\mu_D} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{\mu_1\cdots\mu_D}\,. \tag{B.5}$$

One can use (B.2) to find the following relation for the components of a Hodge-dual form field

$$
\begin{aligned}
(*\omega)_{(q)} &\equiv \tfrac{1}{q!}\,(*\omega)_{\nu_1\cdots\nu_q}\,dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q}\,,\\
(*\omega)_{\nu_1\cdots\nu_q} &= \tfrac{1}{p!}\,\epsilon_{\nu_1\cdots\nu_q}{}^{\mu_1\cdots\mu_p}\,\omega_{\mu_1\cdots\mu_p}\,.
\end{aligned}
\tag{B.6}
$$

Also, again using (B.2) for the case of $p=0$, one can write

$$
*\,\mathbb{1} = \tfrac{1}{D!}\,\epsilon_{\mu_1\cdots\mu_D}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D}\,. \tag{B.7}
$$

Other important relation which we have been used it extensively is the applying the Hodge-dual twice on a general p-form. The result is as follows

$$
\begin{aligned}
\,\omega_{(p)} &= \tfrac{1}{p!}\,\omega_{\mu_1\cdots\mu_p}\;(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}) = \tfrac{1}{p!q!}\,\omega_{\mu_1\cdots\mu_p}\,\epsilon_{\nu_1\cdots\nu_q}{}^{\mu_1\cdots\mu_p}\;*\,(dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q})\\
&= \tfrac{1}{p!q!}\,\omega_{\mu_1\cdots\mu_p}\,\epsilon_{\nu_1\cdots\nu_q}{}^{\mu_1\cdots\mu_p}\,\epsilon_{\rho_1\cdots\rho_p}{}^{\nu_1\cdots\nu_q}\,dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}\\
&= (-1)^{pq+t}\,\omega_{\mu_1\cdots\mu_p}\,\delta^{\mu_1\cdots\mu_p}_{\rho_1\cdots\rho_p}\,dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p} = (-1)^{pq+t}\,\omega_{\mu_1\cdots\mu_p}\,dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}\\
&= (-1)^{pq+t}\,\omega_{(p)},
\end{aligned}
\tag{B.8}
$$

where $\delta^{\mu_1\cdots\mu_p}_{\rho_1\cdots\rho_p} = \delta^{[\mu_1}_{[\rho_1}\cdots\delta^{\mu_p]}_{\rho_p]}$. We have employed the following relation

$$
\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\,\epsilon^{\mu_1\cdots\mu_p\rho_1\cdots\rho_q} = p!q!\,(-1)^t\,\delta^{\rho_1\cdots\rho_q}_{\nu_1\cdots\nu_q}\,. \tag{B.9}
$$

Note according to the above definition one has $\delta^{\mu_1\cdots\mu_D}_{\mu_1\cdots\mu_D} = 1$.

Another useful relation is $*A_{(p)}\wedge B_{(p)} = \frac{1}{p!}\,A.B\;*\,\mathbb{1}$. The proof is as follows

$$
\begin{aligned}
*A_{(p)}\wedge B_{(p)} &= \tfrac{1}{p!^2q!}\,A_{\mu_1\cdots\mu_p}\,\epsilon_{\nu_1\cdots\nu_q}{}^{\mu_1\cdots\mu_p}\,B_{\rho_1\cdots\rho_p}\,dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q}\wedge dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}\\
&= \tfrac{1}{p!^2q!}\,A^{\mu_1\cdots\mu_p}\,B_{\rho_1\cdots\rho_p}\;\epsilon_{\nu_1\cdots\nu_q\mu_1\cdots\mu_p}(-1)^t\,\epsilon^{\nu_1\cdots\nu_q\rho_1\cdots\rho_p}\;*\,\mathbb{1}\\
&= \tfrac{1}{p!}\,\delta^{\rho_1\cdots\rho_p}_{\mu_1\cdots\mu_p}\,A^{\mu_1\cdots\mu_p}\,B_{\rho_1\cdots\rho_p}\;*\,\mathbb{1} = \tfrac{1}{p!}\,A^{\mu_1\cdots\mu_p}\,B_{\mu_1\cdots\mu_p}\;*\,\mathbb{1} = \tfrac{1}{p!}\,A.B\;*\,\mathbb{1}\,,
\end{aligned}
\tag{B.10}
$$

where we have used (B.9) and also

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} = (-1)^t \, \epsilon^{\mu_1 \cdots \mu_D} \, * \mathbb{1} \, , \tag{B.11}$$

where one can readily find out this relation from (B.7).

B.2 The Hodge-dual on a sphere

In this section, we consider an important problem of finding the Hodge-dual of a general p-form on a sphere. For simplicity one can assume the sphere has a unit radius. Then, one may parametrize the $(D-1)$-sphere in D dimensions by x^μ where $\delta_{\mu\nu} x^\mu x^\nu = 1$. Note that the index in x^μ can raise and lower by the delta Dirac metric and hence one may simply write the constraint as $x^\mu x^\mu = 1$. Then considering the following metric

$$ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu \, , \tag{B.12}$$

one needs to find the following Hodge-dual $*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p})$. Because of the constraint, $D-1$ components are independent, and hence the Hodge-dual of a p-form shall be a $D - p - 1 = q - 1$ form. Considering this point, the most general form of the Hodge-dual of a p-form is as follows

$$* \left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \right) = \tfrac{1}{(q-1)!} \, \epsilon_{\nu \nu_1 \cdots \nu_{q-1}}{}^{\mu_1 \cdots \mu_p} \, A^\nu \, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{q-1}} \, , \tag{B.13}$$

when A^ν is a general function to be determined. To find this function, one may apply the Hodge-dual operation twice on a p-form as follows

$$\begin{aligned}
**\,(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}) &= \tfrac{1}{(q-1)!}\,\epsilon_{\nu\nu_1\cdots\nu_{q-1}}{}^{\mu_1\cdots\mu_p}\,A^\nu\;*\,(dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{q-1}})\\
&= \tfrac{1}{p!(q-1)!}\,\epsilon_{\nu\nu_1\cdots\nu_{q-1}}{}^{\mu_1\cdots\mu_p}\,A^\nu\,\epsilon_{\rho\rho_1\cdots\rho_p}{}^{\nu_1\cdots\nu_{q-1}}\,A^\rho\,dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}\\
&= \tfrac{(-1)^{p(q-1)}}{p!(q-1)!}\,\epsilon^{\nu\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{q-1}}\,A_\nu\,\epsilon_{\rho\rho_1\cdots\rho_p}{}^{\nu_1\cdots\nu_{q-1}}\,A^\rho\,dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}\\
&= (p+1)\,(-1)^{p(q-1)+t}\,\delta^{\nu\mu_1\cdots\mu_p}_{\rho\rho_1\cdots\rho_p}\,A_\nu\,A^\rho\,dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}\\
&= (-1)^{p(q-1)+t}\,(\delta^\nu_\rho\,\delta^{\mu_1\cdots\mu_p}_{\rho_1\cdots\rho_p} - p\,\delta^\nu_{\rho_1}\,\delta^{\mu_1\cdots\mu_p}_{\rho\cdots\rho_p})\,A_\nu\,A^\rho\,dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}\\
&= (-1)^{p(q-1)+t}\,\big(A_\nu\,A^\nu\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} - p\,A_\nu\,dx^\nu\,A^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\big)\\
&= (-1)^{p(q-1)+t}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,,
\end{aligned}\tag{B.14}$$

where we have used $(p+1)\,\delta^{\nu\mu_1\cdots\mu_p}_{\rho\rho_1\cdots\rho_p} = \delta^\nu_\rho\,\delta^{\mu_1\cdots\mu_p}_{\rho_1\cdots\rho_p} - p\,\delta^\nu_{[\rho_1}\,\delta^{\mu_1\cdots\mu_p}_{|\rho|\cdots\rho_p]}$, and the anti-symmetrization has been dropped since this term is coupled to $dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}$. The last line was written according to the general result of (B.8). To achieve this result, the following two conditions should be satisfied

$$A_\nu\,A^\nu = 1\,,\qquad A_\nu\,dx^\nu = 0\,.\tag{B.15}$$

We claim the following expression for A^μ will fulfill both of the above conditions

$$A^\mu = \tfrac{1}{\sqrt{\Delta}}\,g^{\mu\nu}\,x^\nu\,,\tag{B.16}$$

where $\Delta = g^{\mu\nu}\,x^\mu x^\nu$. Now, one may readily check the above expression satisfies both conditions in (B.15) as follows

$$\begin{aligned}
A_\mu\,dx^\mu &= g_{\mu\nu}\,A^\mu dx^\nu = \tfrac{1}{\sqrt{\Delta}}\,g_{\mu\nu}\,g^{\mu\rho}\,x^\rho\,dx^\nu = \tfrac{1}{\sqrt{\Delta}}\,x^\nu\,dx^\nu = \tfrac{1}{2\sqrt{\Delta}}\,d(x^\nu\,x^\nu) = 0\,,\\
A_\mu\,A^\mu &= g_{\mu\nu}\,A^\mu A^\nu = \tfrac{1}{\Delta}\,g_{\mu\nu}\,g^{\mu\rho}x^\rho\,g^{\nu\sigma}x^\sigma = \tfrac{1}{\Delta}\,g^{\nu\sigma}\,x^\nu x^\sigma = 1\,.
\end{aligned}\tag{B.17}$$

This completes the proof. Thus the final result for a Hodge-dual of a general p-form with the constraint of $x^\mu x^\mu = 1$ shall be

$$* \left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\right) = \tfrac{1}{(q-1)!}\, \tfrac{1}{\sqrt{\Delta}}\, \epsilon_{\nu\nu_1\cdots\nu_{q-1}}{}^{\mu_1\cdots\mu_p}\, g^{\nu\rho}\, x^\rho\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{q-1}}\,. \tag{B.18}$$

One can may consider two important special cases of the above general result.

Corollary 1. Consider the case $p = 0$. Then we have

$$\begin{aligned} *\mathbb{1} &= \tfrac{1}{(D-1)!}\, \tfrac{1}{\sqrt{\Delta}}\, \epsilon_{\nu\nu_1\cdots\nu_{D-1}}\, g^{\nu\rho}\, x^\rho\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-1}} \\ &= \tfrac{1}{(D-1)!}\, \tfrac{1}{\sqrt{\Delta}}\, \epsilon_{\nu\nu_1\cdots\nu_{D-1}}\, g^{\nu\rho}\, x^\rho\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-1}}\, x^\mu x^\mu \\ &= \tfrac{1}{(D-1)!}\, \tfrac{1}{\sqrt{\Delta}}\, (\epsilon_{\mu\nu_1\cdots\nu_{D-1}}\, x^\nu + (D-1)\, \epsilon_{\nu\mu\cdots\nu_{D-1}}\, x^{\nu_1}) g^{\nu\rho}\, x^\mu\, x^\rho\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-1}} \\ &= \tfrac{1}{(D-1)!}\, \sqrt{\Delta}\, \epsilon_{\mu_1\mu_2\cdots\mu_D}\, x^{\mu_1}\, dx^{\mu_2} \wedge \cdots \wedge dx^{\nu_D}\,, \end{aligned} \tag{B.19}$$

where we have used the following Schouten's identity

$$\epsilon_{[\mu_1\mu_2\cdots\mu_D}\, V_{\mu]} = 0 \Rightarrow \epsilon_{\mu_1\mu_2\cdots\mu_D}\, V_\mu = \epsilon_{\mu\mu_2\cdots\mu_D}\, V_{\mu_1} + \cdots + \epsilon_{\mu_1\mu_2\cdots\mu}\, V_{\mu_D}\,, \tag{B.20}$$

and also the relation $x^\mu dx^\mu = 0$.

Corollary 2. Consider the case $p = 1$. Hence, according to (B.18), one can write

$$\begin{aligned} *dx^\mu &= \tfrac{1}{(D-2)!}\, \tfrac{1}{\sqrt{\Delta}}\, \epsilon_{\nu\nu_1\cdots\nu_{D-2}}{}^{\mu}\, g^{\nu\rho}\, x^\rho\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-2}} \\ &= \tfrac{1}{(D-2)!}\, \tfrac{1}{\sqrt{\Delta}}\, g^{\mu\sigma}\, g^{\nu\rho}\, \epsilon_{\nu\nu_1\cdots\nu_{D-2}\sigma}\, x^\rho\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-2}}\,. \end{aligned} \tag{B.21}$$

Considering an S^2 reduction, which has been studied frequently in the literature, one needs to examine the case of $D = 3$. From the above formula, one can write

$$* \, dx^i = \tfrac{1}{\sqrt{\Delta}}\, g^{-1}_{im}\, g^{-1}_{jn}\, \sqrt{|g|}\, \varepsilon_{jkm}\, x^n\, dx^k = \tfrac{1}{\sqrt{\Delta}}\, \sqrt{|g|}\, |g|^{-1}\, \varepsilon_{in\ell}\, g_{k\ell}\, x^n\, dx^k = \tfrac{1}{\sqrt{\Delta}}\, \tfrac{1}{\sqrt{|g|}}\, \varepsilon_{ijk}\, x^j\, g_{k\ell}\, dx^\ell\,, \tag{B.22}$$

where we have used the identity (4.44) in chapter... $\varepsilon_{ijk}\, g^{-1}_{jm}\, g^{-1}_{kn} = |g|^{-1}\, \varepsilon_{\ell mn}\, g_{\ell i}$. Note $i, j, k, \cdots =$ $1, 2, 3$ and they raise and lower by δ_{ij}, thus one may not worry about up and down indices in this situation.