PRESSURE AND RATE-TRANSIENT PERFORMANCE BEHAVIORS OF A HORIZONTAL WELL INTERCEPTING MULTIPLE HYDRAULIC FRACTURES WITHIN A SHALE RESERVOIR

A Dissertation

by

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Submitted to the Office of Graduate and Professional Studies of Texas A&M University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2018

Major Subject: Petroleum Engineering

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The primary goal of this work is to develop semi-analytical models to predict the pressure and rate performance behaviors of unconventional reservoirs — specifically, shale reservoirs. In these types of reservoirs, there are multi-scale heterogeneities that can hinder the modeling and the diagnostic analyses. Additionally, the use of large stimulation treatments can further complicate the modeling of these heterogeneous reservoir systems. Our approach is to extend the existing models for a multi-fractured horizontal well (MFHW) in a homogenous reservoir to consider the fractal reservoir scenario.

This work presents the detailed derivation of the model of a horizontal well intercepting a single finite-conductivity fracture within a fractal reservoir. The solution of this model is semi-analytical. This is developed by discretizing the hydraulic fracture, which defines a system of equations, the solution of which provides the pressure at any position inside the fracture. The shape of the imposed hydraulic fracture can be either circular or rectangular. By modifying the solution in the Laplace domain of the diffusivity equation for the reservoir, we have investigated different reservoir conditions, such as single porosity fractal reservoirs with typical or anomalous diffusions and double porosity reservoirs with typical diffusion.

We have extended the semi-analytical solution for a horizontal well intercepting a single finite-conductivity fracture to the MFHW case. For this purpose, we have used the principle of superposition in space. We show that the pressure transient response of a MFHW within a fractal reservoir can exhibit a maximum of four (4) distinct periods of flow — (1) fracture (dominated)
flow, (2) early fracture-reservoir interaction, (3) late fracture-reservoir interaction, and (4) reservoir-dominated flow.

To provide an alternative explanation to the anomalous diffusion phenomenon in petroleum reservoirs, we have also developed a double porosity model considering matrix blocks with fractal geometry and a fractal fracture network. We assumed transient interporosity transfer conditions and we modeled it using the classical convolution scheme given in the literature. Under particular conditions, the resulting model acquires a similar mathematical shape to the so-called anomalous diffusion equation.
DEDICATION

To my parents, Marta and Ismael.
ACKNOWLEDGEMENTS

I would like to express my gratitude to:

Prof. Tom Blasingame who has distinguished me with his mentorship and friendship. A friendship based on hard work and high standards of ethics and values...always with a great sense of humor.

Prof. Maria Barrufet, Prof. Peter Valkó and Prof. Jean-Luc Guermond, for serving as my committee members and for their excellent teachings in each one of the courses I took with them.

Prof. Leif Larsen, for his recommendation to this program, his contributions for this work, and his willingness to answer questions.

Dr. Dilhan Ilk, for giving me the opportunity to collaborate in projects related to unconventional reservoirs.

Dr. Andres Moctezuma for his support in the matters related to my scholarship.

Prof. Heber Cinco, Prof. Fernando Samaniego and Mr. Frode Hveding, for their recommendation to this program and their support earlier in my career.

My siblings (Ivan and Karina) and uncles (Raul and Javier), for their support in each one of the periods of my life.

My colleagues (who turned out to be fantastic friends), for their support, the knowledge I acquired from them and their company in this journey. Nefeli Moridis, Basayir al-Lawati, Jorge Garcia-P, Pedro Sanchez-L, Erick Martinez-A, Nick Wiewiorowski, Adaiyibo Kio, Matias Brolli, Chulhwan Song, Hefdhi Abdennadher and Rui Kou.
The friends I met in College Station, Mariana Valdez, Carla Isnaldi, Husniye Kantarci, and Miguel Ortiz. Thank you for your friendship…

My friends living around the Globe, who were close to me during these years, regardless of the distance. Thank you…

Texas A&M University for the education I received in this institution.

Mr. Hector Pulido, for his mentoring and teachings at the beginning of my career. Without his influence, I would not have achieved this goal.
CONTRIBUTORS AND FUNDING SOURCES

This work was supported by a dissertation committee consisting of Professors Thomas A. Blasingame (chair), Maria Barrufet, Peter Valko of the Department of Petroleum Engineering, and Professor Jean-Luc Guermond of the Department of Mathematics.

The modeling of finite conductivity hydraulic fractures (Chapters III and IV) was additionally supervised by Professor Leif Larsen from the University of Stavanger (retired). The algorithm for numerical inversion used in Chapter V was provided by Professor Peter Valko.

I would like to thank to the High Performance Research Computing (HPRC) group at Texas A&M University for access to its computational resources to generate some of the results presented in this work.

All other work conducted for the dissertation was completed by the student independently.

This graduate study was supported by a scholarship from Instituto Mexicano del Petróleo (IMP).
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CHAPTER I
INTRODUCTION

In this chapter, we present the general overview of this dissertation. We divided this chapter into three sections. In the first section, we state the motivation of this research problem and define the strategy that we followed to find a solution. Finally, we define the basic, and recurrent, concepts used in this work in the third section.

1.1. Research Problem

To produce the hydrocarbons contained in highly heterogeneous formations with low/ultralow permeability, such as shale oil and shale gas reservoirs, two main strategies have been implemented: (1) drill horizontal wells to maximize the flowing area and to minimize the number of wells, and (2) stimulate the horizontal well by large hydraulic fracturing treatments that creates multiple hydraulic fractures along the wellbore.

The system created by a horizontal well intercepting multiple hydraulic fractures within a shale gas/oil reservoir becomes a challenging system to model for flow diagnosis purposes given three factors: (1) the heterogeneous and low/ultralow permeability nature of the reservoir, (2) the geometry of the well and fractures, and (3) the properties of the petroleum fluids. In this work, we will address the first two factors to develop diffusivity models and semi-analytical solutions to be used for flow diagnosis.

Some authors have suggested the use of fractal models as the best practice to depict the transient performance behavior of highly heterogeneous reservoirs (e.g., Naturally Fractured Reservoir and
shale reservoirs). Therefore, we modeled the flow of petroleum fluids within a shale reservoir as a fractal object (Chang et al., 1990). In Chapter II, we introduce the classic models related to this topic.

To model the flow towards the horizontal well intercepting multiple hydraulic fractures, we will use the approach proposed by Larsen et al. (1994), which applies the principle of superposition in space to extend the models developed for a horizontal well intercepting a single finite-conductivity hydraulic fracture to the multifractured horizontal well case (MFHW).

In Chapter III, we present the pressure and rate transient analyses for a horizontal well intercepting a single finite-conductivity fracture. We investigated two conditions for the geometry of the hydraulic fractures and three conditions of the reservoir. For the geometry of the hydraulic fractures, we assumed them to be either (1) circular transverse or (2) rectangular longitudinal. For the reservoir, we consider the three cases: (1) single porosity fractal reservoir with typical diffusion, (2) double porosity reservoir with typical diffusion, and (3) single porosity fractal reservoir with anomalous diffusion.

In Chapter IV, we apply the principle of superposition in space to extend the finite-conductivity fracture models presented in Chapter III to horizontal wells intercepting multiple hydraulic fractures. We restricted our pressure and rate transient analyses to single porosity fractal reservoirs with typical diffusion.

As an attempt to provide a physical explanation to the anomalous diffusion phenomenon in shale reservoirs, we introduce the double fractal model in Chapter V. This model considers a double
porosity reservoir with transient interporosity transfer, where both the fracture network and the
matrix blocks are fractal objects.

We summarize our conclusion and recommendations in Chapter VI. Additionally, we provide an
outline for the future work related to the modeling of wells intercepting hydraulic fractures in
fractal reservoirs.

1.2. Research Objectives

The main objectives of this dissertation are:

- To develop semi-analytical reservoir models to predict the pressure and rate-transient
  performance behaviors of a horizontal well intercepting multiple hydraulic fractures within
  an unconventional reservoir—specifically, shale reservoirs.
- To investigate the combined effect of the parameters of an unconventional reservoir (i.e.,
  fractal dimension, conductivity index and/or anomalous diffusivity exponent) and the
  characteristics of the hydraulic fracture (fracture geometry and conductivity) on the flow
  periods of a horizontal well intercepting multiple hydraulic fractures within a fractal
  reservoir.
- To derive diagnostic interpretation relations to estimate the parameters of the hydraulic
  fractures for the evaluation of hydraulic fracturing treatments.

1.3. Basic Concepts

In this section, we define the fundamental concepts used in this work. It is not our objective to
discuss these concepts in depth (which can be very abstract), but to provide a review and references
that are helpful to understand the fundamental theory behind this dissertation and some of the methods used in it.

**Unconventional reservoirs**

These are hydrocarbon reservoirs that must be stimulated to be able to produce at commercially viable flowrates. This classification is comprehensive and includes resources such as shale oil/gas reservoirs, heavy oil reservoirs, tight sands reservoirs, etc. In this work, we delimitated our research to shale reservoirs.

Shale reservoirs are highly heterogeneous media that need massive hydraulic fracturing treatments, due to their low porosity and permeability, to produce at economic flowrates. To model the transient performance behavior in these types of reservoirs, some authors have suggested that the use of fractal models is the most appropriate practice given their highly heterogeneous nature.

**Fractals**

Mandelbrot (1977) defined a fractal as a family of shapes with irregular and fragmented patterns. These irregularities are statistical and identical at all scales. The number of shapes \( n \) approaches to infinity as their size \( l \) approaches to zero. This relation is defined as:

\[
 n \sim l^{-D},
\]

where \( D \) is the fractal dimension.

**Fractals in Petroleum Reservoir Engineering**

The application of the fractal theory in petroleum reservoir engineering became popular in the 1990s, but the mathematical models with similar definition to Eq. 1.1 have been used for over fifty
years. Sahimi et al. (1990) presented a review of the applications of the fractal geometry theory in reservoir engineering.

Based on the definition of fractals proposed by Mandelbrot, we can extend such a concept to petroleum reservoirs as follows: a fractal reservoir can be defined as the family of permeable sites (e.g. lithofacies with hydrocarbon content) with irregular shapes and fragmented patterns which are identical at all scales (sizes). Therefore, the number of the permeable sites is related to a characteristic length (e.g. pore radius), \( R \), using a power-law function:

\[
N(R) = \sigma R^{D_f-1},
\]

where \( \sigma \) is the density of the permeable sites and \( D_f \) is the fractal dimension of the reservoir. A log-log plot of Eq. 1.2 will yield a straight line with a \( D_f-1 \) slope. Based on the expression given by Eq. 1.2, the porosity and the permeability for a fractal reservoir can also be modeled as power-law functions (see Appendix B for details). The porosity of a fractal reservoir has been defined as space-dependent by:

\[
\phi(R) = \phi_0 R^{D_f-d},
\]

where \( \phi_0 \) is a reference porosity and \( d \) is the Euclidean dimension of the reservoir. Given that there is relation between the permeability and the porosity of a reservoir, the fundamental Darcy’s law should also be modified to a power-law expression. This is given by:

\[
q_R = \frac{\sigma V_s}{\phi_0} \frac{R^H}{k_0} \frac{\partial p}{\partial R},
\]
where $V_s$ is the volume of the permeable sites and $k_0$ is a reference permeability. The parameter $\beta$ in Eq. 1.4 deserves special attention. It is defined as:

$$\beta = D_f - \theta - 1,$$

(1.5)

where $\theta$ is the conductivity index, which depicts the connectivity between the permeable sites. It can acquire values equal or greater than zero. A value of zero represents that the permeable sites are perfectly connected, whereas a high value represents poorly connected permeable sites.

Doe (1991) defined the spatial dimension of the reservoir as a manner to describe irregular patterns of flow (e.g., sublinear or hyperspherical flow). Although the author pointed out that this parameter is not necessarily related to the possible fractal nature of the reservoir, we believe that the spatial dimension can be related to $\beta$, given that these parameters have similar influence in the pressure and pressure derivative functions of their corresponding models (fractal and fractional models).

**Anomalous Diffusion**

This concept is related to the random walk theory. It is defined as a process where the mean-square displacement of a random walker is given by a power law function. This is mathematically defined as (Metzler et al., 1994):

$$\langle r^2(t) \rangle - t^{2/d_w}, \quad \text{..............................................................} (1.6)$$

where $d_w$ is the anomalous diffusion exponent.
Fracture Conductivity

This parameter is used for the evaluation of hydraulic fracturing treatments. It is defined by the product of the permeability, $k_f$, and the width, $w$, of the hydraulic fracture. In this sense, the fracture conductivity is a concept analogous to the flow capacity of a homogenous reservoir. The dimensionless form of the fracture conductivity can be defined by:

$$F_{cD} = \frac{k_f w}{k l_r},$$ (1.7)

where $l_r$ is a reference length and $k$ is the permeability of the reservoir. In general, a hydraulic fracture with a large $F_{cD}$-value (e.g., 300 for a fracture in a cylindrical reservoir) is classified as an infinite-conductivity fracture, which is the desired outcome.

Laplace Transform

The Laplace transform is the standard method in petroleum engineering to obtain the analytical solutions of reservoir models. For a function $f(t)$ defined in the real domain, the Laplace transform is defined as:

$$\tilde{f}(u) = \int_0^{\infty} f(t)e^{-ut}dt,$$ (1.8)

Duhamel's Principle

Using this principle, the dimensionless wellbore pressure, $p_{wD,cr}(t_D)$, can be related with the dimensionless flowrate, $q_{wD}(t_D)$, in the Laplace domain as follows:
This principle permits to obtain the constant-pressure solution of a reservoir model by using the constant-rate solution and vice-versa.

**Wellbore Storage**

This effect depicts the capability of a wellbore to store or unload a volume of fluids, when it is subjected to a change in the pressure. It can be included using the relation in the Laplace domain:

\[
\bar{p}_{WD}(u,s,C_D) = \frac{\bar{p}_{WD}(u)+s}{1+C_Du^2[\bar{p}_{WD}(u)+s]}
\]  

(1.10)

where \(C_D\) is the dimensionless wellbore storage coefficient, \(s\) is the skin (additional drop of pressure) around the wellbore and \(p_{WD}(u)\) is the constant-rate solution of a reservoir model.

**Numerical Inversion of the Laplace Transform**

Most of the reservoir models are solved analytically in the Laplace domain due to the fact that it allows the use of Duhamel's principle (Eq. 1.6) and/or the inclusion of the wellbore storage effects (Eq. 1.7). However, the resulting equations become problematic when inverting from the Laplace domain to the real domain. Therefore, a numerical method to express the solution in the real domain should be applied. The standard methods in petroleum industry are the ones based on Gaver's algorithm (Gaver, 1965).

Based on the probability theory, Gaver (1965) developed an algorithm that estimate the inverse Laplace transform. Later, Stehfest (1970) modified Gaver's work by utilizing the Salzer
summation as an accelerator of convergence. This resulted in the algorithm for numerical inversion of the Laplace transform defined by:

\[ f(t) = \frac{\ln[2]}{t} \sum_{i=1}^{N} V_i \left[ \frac{\ln[2]}{t} \right] \]

where \( N \) is an even number of terms to be used and the coefficients \( V_i \) are calculated using the expression:

\[ V_i = [-1]^{N/2+i} \min[i,N/2] \sum_{k=[i+1]/2}^{k N/2} k^{(2k)!} (2k)! \]

The use of a large number of terms, \( N \), can cause numerical instability in the algorithm due to the limitations of the programming software. In Fig. 1.1, we present an example of this numerical instability by comparing the exponential integral function in the real domain:

\[ f(t) = -\frac{1}{2} Ei\left[-\frac{a^2}{4t}\right] \] (where \( a \) is a positive constant),

and the numerical inversion of its Laplace transform is given by:

\[ \bar{f}(s) = \frac{K_0\left[u^{1/2}\right]}{u} \]

for different values of \( N \).

In the example presented in Fig. 1.1, we can see that the Stehfest algorithm can show divergence when using a high number of approximation terms, \( N \).
In general, the numerical instability, or divergence, of the Stehfest algorithm is the result of truncation errors. This can be exacerbated by the type of the function to be inverted, e.g., equations that involve very high or very low arguments of the exponential function, power-law functions, special functions, etc.

As an alternative to the Stehfest algorithm, Valko et al. (2004) investigated non-linear methods to accelerate the convergence of the Gaver algorithm. They implemented the Wynn-Rho algorithm as a convergence accelerator in Mathematica and concluded that the use of this approach provides accurate results in the numerical inversion of a function in the Laplace domain by reducing the propagation of truncation errors.

![Figure 1.1](image_url) — Example of the Numerical Stability of the Stehfest algorithm.
CHAPTER II

LITERATURE REVIEW

The analysis of pressure-transient data as a method to evaluate fracturing jobs has been a topic of research for over forty years. During the 1970s decade, the attention was focused on the problem of vertical wells intercepting vertical fractures and the techniques to analyze the pressure-transient data from these type of wells were presented. In the 1990s decade, these techniques were extended to horizontal wells intercepting multiple vertical fractures within homogeneous reservoirs. Given the development and increase in production of shale oil and shale gas reservoirs in the last decade, some authors have extended the methods and techniques used in conventional reservoirs to unconventional reservoirs.

In this chapter, we have summarized the classic works related to the analysis of pressure-transient data of wells intercepting hydraulic fractures and the models used to depict the performance behavior of unconventional reservoirs — specifically, fractal reservoirs and reservoirs with anomalous diffusion.

2.1. Vertical Wells Intercepting Hydraulic Fractures

Gringarten et al. (1974) applied Green and Source Functions to develop solutions for the problem of a well intercepting a vertical fracture within a homogeneous reservoir. The authors considered two cases: (1) infinite conductivity and (2) uniform flux fractures. The solution for the first case was obtained semi-analytically by discretizing the fracture into segments and establishing a system of equations whose solution provides the pressure-transient behavior anywhere in the fracture. For
the second case, the solution was derived by superimposing the line-source solution along the fracture length.

The uniform flux solution provides an exact solution at early times and it implicitly assumes that pressure will vary along the fracture. Gringarten et al. (1974) found that the infinite conductivity pressure-transient behavior at the wellbore can be reproduced by evaluating the uniform flux solution at a dimensionless position in the fracture, \( x_D \), equal to 0.732. Houze et al (1988) extended these ideas to double porosity reservoirs considering both, transient and pseudosteady-state interporosity transfers.

Cinco-Ley et al. (1978) presented a general semi-analytical solution for the pressure-transient response of a vertical well intersecting a vertical finite conductivity hydraulic fracture within a homogenous reservoir. This model can reproduce several flow periods observed in wells intercepting a hydraulic fracture by varying the dimensionless fracture conductivity. The authors concluded that the assumption of infinite conductivity applies only when the dimensionless fracture conductivity is equal to or greater than 300. This contribution improved the diagnosis of fracturing jobs, allowing the identification long and/or low permeability fractures.

Cinco-Ley et al. (1981a) identified the flow periods generated by the finite conductivity fracture model and introduced the concepts of fracture linear flow and bilinear flow, which were used as a diagnostic technique of pressure-transient data. The flow periods that can be observed in a vertical hydraulic fracture are:
1. **Fracture Linear flow**: At "very early transient" times the pressure derivative signature shows a half-slope straight line that corresponds to a linear flow occurring in the hydraulic fracture. This period is theoretical and is not observed in practice.

2. **Bilinear flow**: For low values of the dimensionless conductivity of the fracture, at "early transient" times the pressure derivative signature shows a quarter-slope straight line that corresponds to two linear flows acting simultaneously: a linear flow in the hydraulic fracture and a linear flow from the formation to the fracture.

3. **Pseudo-Linear flow**: For intermediate and high values of the dimensionless conductivity of the fracture, at "intermediate transient" times the linear flow from the formation to the fracture dominates and the pressure derivative signature shows a half-slope straight line.

4. **Pseudo-Radial flow**: At "late transient" times the pressure derivative signature exhibits a zero-slope that corresponds to a flow dominated by the reservoir.

Cinco-Ley et al. (1981b) included the effects of damage for a vertical well intercepting a hydraulic fracture. The authors considered two types of damage conditions: (1) damaged zone around fracture caused by loss of fluid in the formation (fracture skin), and (2) damaged zone around wellbore caused by crushing, embedding or loss of proppant within the fracture (chocked fracture skin).

Cinco-Ley et al. (1981b) pointed out the differences between the pressure-transient response at early times of a finite conductivity fracture and a hydraulic fracture affected by any of the two types of damage mentioned before. They found that for small values of the fracture skin, the pressure-transient response approximates to the infinite conductivity fracture case, whereas for
large values the response converges to the uniform flux case. Additionally, Wong et al. (1986) and Valdes-Perez et al. (2011) studied the impact of the fracture skin in the pressure-transient and pressure derivative behaviors of the bilinear model and presented techniques to analyze pressure-transient data.

Lee et al. (1986) developed a trilinear flow model to provide an analytical approach to the finite conductivity fracture model. In their paper, Lee et al. (1986) found the solutions for both constant-pressure and constant-rate cases, including the effects of fracture skin, wellbore storage and fracture storage. This approximate model can match the "early time" period of the finite conductivity fracture model for two logarithmic cycles.

Cinco-Ley et al. (1988) extended the finite conductivity fracture model to double porosity reservoirs. In their development, the authors considered both transient and pseudosteady-state interporosity transfers. Based on a fully analytical model for "early times," Cinco-Ley et al. (1988) found that under certain circumstances this type of system can yield a one-eight-slope (trilinear flow) in the log-log pressure-transient and pressure derivative plot and not one-quarter (bilinear flow) as in the homogeneous case. Similarly for "intermediate times," a quarter-slope (bilinear flow) could be shown instead a one-half-slope (pseudo-linear flow).

To provide an expression simpler to compute than the semi-analytical solution to the finite conductivity fracture model, Blasingame et al. (1993) combined the trilinear solution developed by Lee et al. (1986) and the alternative uniform flux solution obtained by Ozkan et al. (1991) to develop an equivalent solution in the Laplace domain valid for values of the dimensionless fracture conductivity greater than 0.5.
Cossio et al. (2013) presented an application of the fractal theory to provide an improved semi-analytical solution for the flow of a single fluid within a single vertical fracture that fully penetrates a homogeneous infinite-acting reservoir. The authors modified the trilinear flow model (Lee et al., 1986) to introduce fractal parameters in each one of the three regions. The essence of this work is the modification of the diffusivity equation for linear flow to include both permeability and porosity as distance-dependent properties using power-law models. This diffusivity equation was analytically solved for a closed system and validated with a 1D finite-volume black-oil reservoir simulator. The authors verified their fractal-based trilinear solution with the solution developed by Cinco-Ley et al. (1988).

2.2. Horizontal Wells Intercepting Hydraulic Fractures

Larsen et al. (1991) developed a semi-analytical model of a horizontal well intersecting a finite conductivity vertical fracture within a 3D homogenous reservoir (see the detailed derivation of these models in Appendix A). The authors considered two geometries for the fracture: (1) circular fracture perpendicular to the axis of the wellbore (Fig. 2.1) and (2) rectangular fracture parallel to the axis of the wellbore (Fig. 2.2).

Fig. 2.3 and Fig 2.4 show the pressure-transient behavior of the radial and rectangular fractures, respectively. Based on these models, the authors investigated the flow periods that can exist in the pressure-transient response of horizontal wells with multiple vertical fractures (Larsen et al. 1994). They found that at early flow periods (before pressure interference between fractures occur), the pressure-transient behavior of a multi-fractured horizontal well can be appropriately analyzed using the single fracture models.
Figure 2.1  — Vertical circular hydraulic fracture transverse to the axis of a horizontal well within an infinite 3D reservoir.

Figure 2.2  — Vertical rectangular hydraulic fracture longitudinal to the axis of a horizontal well within an infinite 3D reservoir.
Figure 2.3 — Schematic example of the pressure and pressure derivative of a horizontal well intercepting a single circular transverse finite conductivity fracture within a 3D reservoir.

Figure 2.4 — Schematic example of the pressure and pressure derivative of a horizontal well intercepting a single rectangular longitudinal finite conductivity fracture within a 3D reservoir.
Raghavan *et al.* (1997) applied the commingled-reservoir concepts to single fracture solutions and developed a model to analyze the pressure-transient response of a multi-fractured horizontal well in a homogeneous reservoir. Using this approach, the authors concluded (similar to Larsen *et al.* 1994) that a multi-fractured well behaves as an equivalent well intercepting a single fracture with an equivalent conductivity and fracture length equal to the distance between the hydraulic fractures at the edges, where the lowest rate towards the wellbore comes from the hydraulic fractures at the center of the well, whereas the highest production is observed in the outermost fractures.

Based on the point source solution developed by Ozkan *et al.* (1991), Chen *et al.* (1997) derived expressions for pressure distribution caused by a multi-fractured horizontal wells in reservoirs with rectangular shape. A discussion of the flow geometries and regimes that may appear (depending on the properties/characteristics of the hydraulic fractures) in these sort of well completion-reservoir configurations was presented. According to the authors, the main objective of this work was the development of an algorithm capable of incorporating boundary effects without relying on approximations or the use of image wells.

Similar to the developments made by Lee *et al.* (1986), Brown *et al.* (2011) derived a trilinear flow solution for the pressure-transient analysis of multi-fractured horizontal wells in unconventional shale reservoirs. The authors divided the reservoir into three subsystems (hydraulic fracture, inner and outer reservoir zones) with the following characteristics:

- Each hydraulic fracture is considered to be a finite conductivity porous medium
- Double porosity behavior in the inner reservoir zone (zone between hydraulic fractures) with transient interporosity transfer
• Linear flow in a homogenous reservoir in the outer reservoir zone.

This approach was verified with more rigorous analytical (Chen et al. 1997 and Raghavan et al. 1997) and numerical (Medeiros et al. 2008) solutions. One application of this model was presented by Ozkan et al. (2011), who studied the impact of reservoir properties such as permeability in the transient performance of a fractured horizontal-well drilled in an unconventional reservoir.

Ozcan et al. (2014) modified the inner reservoir zone of the trilinear model proposed by Brown et al. (2011) to include the effects of a nano-porous media by using an anomalous diffusivity model, based on the fractional derivative in time.

2.3. Vertical Wells within Fractal Reservoirs

Barker (1988) presented a generalized diffusivity model for hydraulic tests. The author presented the constant flow rate and constant head (constant pressure) solutions for this model. The solutions are given in terms of Modified Bessel Functions. This model and its solutions are able to represent the transient performance (pressure and flow rate) of the flow of a Newtonian fluid in linear, radial or spherical systems. This work is helpful as background to understand the solutions of the diffusivity models for fractal reservoirs.

Based on the work of O'Shaughnessy et al. (1985), Chang et al. (1990) developed a diffusivity equation to represent the flow of a single phase fluid within a fractal reservoir (Fig. 2.5). The system was idealized as a Naturally Fractured Reservoir (NFR) and two cases were studied: (1) without matrix participation, and (2) with pseudosteady-state interporosity transfer. The authors presented the analytical solution for the first case, whereas for the second case the solution was numerical. Appendix D shows the development of this model (case 1) and its procedure of solution
using the Laplace transform. **Fig. 2.6** shows the pressure-transient and pressure derivative for a vertical well within a fractal reservoir for selected values of the fractal dimension, $D_f$, and the conductivity index, $\theta$.

**Figure 2.5** — Schematic of a vertical well within a reservoir with fractal fracture network.
Figure 2.6  — Schematic example of the pressure and pressure derivative of a vertical well within a fractal reservoir for selected values of the fractal dimension ($D_f$) and fixed conductivity index ($\theta=0$).

Doe (1991) analyzed the impact of the spatial dimension on constant-pressure tests. The spatial dimension describes the variation in the exposed to flow area with distance from the well and the reservoir properties (heterogeneities). Defining $r$ as the distance from the well, the area exposed to flow for linear flow varies proportionally to $r^0$ (i.e., the area remains unchanged). For radial flow, the area exposed to flow changes linearly (i.e., proportionally to $r^1$). For spherical flow, the area exposed to flow changes proportionally to $r^2$. The author pointed out that fractional dimension may or may not indicate fractal reservoir geometries and can be classified as sublinear flow when the exponent of the distance from the well is less than zero and as hyperspherical if it is greater than two.

Acuna et al. (1991) proposed a numerical method to represent natural fracturing processes with fractal geometry. The authors used the "Iteration Function System" technique (Barnsley, 1988)
and concepts from the theory of fragmentation to create "numerical fractal fracture networks." The authors created several cases of 2D synthetic fractal fracture networks. For each case, they simulated the pressure-transient response of a well intercepting one of these networks assuming single phase flow of a slightly compressible fluid.

Abbaszadeh (1995) developed a three region composite model divided into three regions. The inner and the outer regions were considered to be radial and the middle region was considered to be fractal. This model attempted to represent phenomena such as precipitation of solids, chemical dissolution processes, matrix acidizing, etc., in the pressure-transient response of a vertical well. The author presented an example of the application of the solution to this model using real data from the Prudhoe Bay reservoir, where laboratory studies of the reservoir indicate precipitation of scales and organic material in the vicinity of producer wells.

Acuna et al. (1995) presented analyses of real well tests from NFRs located in western Venezuela, Monterey formation and Geyser geothermal field, applying the model developed by Chang et al. (1990). Based on the power-law response of the real pressure-transient data, the authors determined the range of fractal parameters (mass fractal dimension and conductivity index). Subsequently, they used the fractal parameters determined from the analyses to generate fracture networks of the reservoirs, using techniques previously presented by Acuna et al. (1991).

Olarewaju (1996) proposed a method to build a heterogeneous reservoir permeability field with stochastic fractal functions. The author used the Weiestrass Mandelbrot fractal unconditional simulation method to generate the permeability field. Such a method is based on the fractional Brownian motion (fBm), given that the processes that follow fBm are selfisimilar. The input parameters to generate such field are obtained from pressure-transient data (effective permeability,
fractal dimension and conductivity index). To estimate the input parameters, Olarewaju (1996) developed a double porosity model, considering transient interporosity transfer. He obtained an analytical solution in Laplace domain that is able to reproduce the three flow periods of an infinite double porosity reservoir (flow in fracture network, interaction between porous media, and single system behavior).

Based on the works of Chang et al. (1990) and Olarewaju (1996), Flamenco-Lopez et al. (2001) deduced approximate analytical solutions for late transient times for a double porosity system (i.e., when the double porosity reservoir behaves as a single system) and for pseudosteady-state flow period (boundary dominated). One of the conclusions of this work was the necessity of analyzing transient and pseudosteady state flow periods to properly characterize a fractal reservoir. In a revised version of this paper (Flamenco-Lopez et al. 2003), the authors included a trial and error methodology to estimate the fractal parameters during transient regime.

To investigate the production decline behavior in fractal reservoirs, Camacho-Velazquez et al. (2008) obtained the constant pressure analytical solutions for the models proposed by Chang et al. (1990). The authors developed approximate analytical solutions for the single and double porosity models for both, transient and pseudosteady-state flow periods.

### 2.4. Wells Intercepting Hydraulic Fractures within Fractal Reservoirs

The first work related to wells intercepting a hydraulic fracture in fractal reservoirs was developed by Beier (1994). Motivated by the fractal behavior observed in the pressure-transient response in wells from the San Andres formation, Beier (1994) derived the infinite conductivity and the
uniform flux solutions for a vertical well using a similar approach as the one used by Gringarten et al. (1974) and the fractal model presented by Chang et al. (1990).

Beier (1994) found that this well-reservoir configuration exhibits two power-law flowing periods. The first one corresponds to a distortion in the formation linear flow due to the superimposed flow of the reservoir towards the plane of the fracture, whereas the second one corresponds to the flow dominated by the fractal reservoir. Fig. 2.7 shows an example of the two power-law flowing periods generated by a vertical well intercepted by a hydraulic fracture within a fractal reservoir. At early times (formation linear flow), the pressure and pressure derivative exhibit a three-quarters-slope, whereas at late times (infinite-fractal reservoir) a half-slope is observed.

![Figure 2.7](image)

**Figure 2.7** — Schematic example of the pressure and pressure derivative of a vertical well intercepting a uniform flux hydraulic fracture within a fractal reservoir for selected values of the fractal dimension ($D$) and fixed conductivity index ($\theta=0$).
2.5. Reservoir Models considering Anomalous Diffusion

In 1994, Metzler et al. presented a generalization of the diffusion model of a fractal structure developed by O'Shaugnessy et al. (1985). Such a generalization consists in the inclusion of the "anomalous diffusion" concept in the diffusion equation for fractal media. The anomalous diffusivity phenomenon is modeled by a fractional derivative of a probability density function. The authors showed that the solution to their anomalous diffusion model considering an infinite fractal object is given by Fox's H-functions, of which asymptotic case when the time tends to infinite is given by an stretched exponential.

Camacho-Velazquez et al. (2008) introduced the concept of anomalous diffusion in the petroleum reservoir engineering, and derived constant-pressure and constant-rate solutions for the analysis of pressure-rate performance behavior in fractal reservoirs. The authors presented complete solutions in Laplace domain in terms of Modified Bessel Functions and developed asymptotic solutions in the real domain, defined by power-law functions. Fig. 2.8 and Fig. 2.9 show the pressure and rate-transient behaviors, respectively, of a vertical well within a fractal reservoir with anomalous diffusion.
Figure 2.8 — Schematic example of the pressure and pressure derivative of a vertical well within a fractal reservoir with anomalous diffusion for selected values of the conductivity index (θ) and fixed fractal dimension (D_f=2.5).

Figure 2.9 — Schematic example of the rate and rate derivative of a vertical well within a fractal reservoir with anomalous diffusion for selected values of the conductivity index (θ) and fixed fractal dimension (D_f=2.5).
Razminia et al. (2014) extended the model presented by Camacho-Velazquez et al. (2008) to double porosity reservoirs with pseudosteady interporosity transfer, to composite radial systems (Razminia et al. 2015a), and to vertical wells intercepting a uniform flux fracture (Razminia et al. 2015b).

Raghavan (2012a) modeled a 2D diffusivity equation considering anomalous diffusion phenomenon assuming a time-dependent version of Darcy's Law. Such a variation of the Darcy's Law is modeled by a convolution integral of the gradient of the pressure weighted by a power-law function of the time. Raghavan (2012a) presented the "constant-rate" solution for the proposed diffusivity model, considering an infinite reservoir. The "constant-rate" solution for this type of models implies a time-dependent inner boundary condition that creates a power-law behavior of the pressure and pressure derivative functions (see Fig. 2.10). The development of this model and its solution is shown in Appendix F. Raghavan (2012b) and Raghavan et al. (2013) applied this approach to the fractured well and multi-fractured –horizontal well cases, respectively.
Figure 2.10 — Schematic example of the pressure and pressure derivative of a vertical well within a 2D reservoir with anomalous diffusion for selected values of the anomalous diffusion index ($\alpha$).
CHAPTER III
PRESSURE AND RATE-TRANSIENT BEHAVIOR OF A HORIZONTAL WELL
INTERCEPTING A SINGLE HYDRAULIC FRACTURE WITHIN A FRACTAL
RESERVOIR

In this chapter, we present a summary of the semi-analytical solution for the pressure and rate transient behaviors of a horizontal well intercepting a single finite-conductivity hydraulic fracture (either circular or rectangular fracture) within a fractal reservoir considering either single or naturally-fractured/dual porosity reservoir conditions. Naturally-fractured/dual porosity and anomalous diffusion effects are included by modifying the solution of the diffusivity equation for the reservoir in the Laplace domain. The detailed derivations of these models are presented in Appendix D.

3.1. Model Assumptions

Similar to Larsen et al. (1991), we have considered two geometries for the hydraulic fractures. Fig. 3.1a shows the schematics of a horizontal well intercepting a circular transverse hydraulic fracture within a fractal reservoir and Fig. 3.1b shows the rectangular longitudinal hydraulic fracture case. The assumptions of the systems are summarized in Table 3.1.

---

Figure 3.1 — a. Vertical circular hydraulic fracture transverse to the axis of a horizontal well within an infinite fractal reservoir, and b. Vertical rectangular hydraulic fracture longitudinal to the axis of a horizontal well within an infinite fractal reservoir.

Table 3.1 — Assumptions used to develop the proposed reservoir models

<table>
<thead>
<tr>
<th>Medium</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>● Flow to the wellbore occurs only through the hydraulic fracture.</td>
</tr>
<tr>
<td></td>
<td>● Pressure-squared gradients are negligible.</td>
</tr>
<tr>
<td></td>
<td>● The well produces at constant flowrate, $q_w$.</td>
</tr>
<tr>
<td></td>
<td>● Uniform initial pressure, $p_i$.</td>
</tr>
<tr>
<td></td>
<td>● Single slightly-compressible fluid flow with constant compressibility, $c_o$, and constant viscosity, $\mu$.</td>
</tr>
<tr>
<td>Hydraulic Fracture</td>
<td>● Closed fracture of constant half-length ($r_f$ for the circular fracture or $x_f$ for the rectangular fracture).</td>
</tr>
<tr>
<td></td>
<td>● Flow obeys Darcy's Law.</td>
</tr>
<tr>
<td></td>
<td>● The fracture has constant properties: compressibility, $c_f$, permeability, $k_f$, porosity, $\phi_f$, width, $w_f$.</td>
</tr>
<tr>
<td>Reservoir</td>
<td>● Unbounded fractal reservoir of $D_f$-dimension.</td>
</tr>
<tr>
<td></td>
<td>● Flow obeys modified Darcy's Law for fractal systems of $\theta$-conductivity index.</td>
</tr>
<tr>
<td></td>
<td>● Both porosity and permeability vary according to power law functions.</td>
</tr>
<tr>
<td></td>
<td>● The reservoir has constant formation compressibility, $c$.</td>
</tr>
<tr>
<td>Matrix blocks (naturally-fractured/dual porosity reservoir case)</td>
<td>● Finite and single size Euclidean matrix block (either slabs or spheres).</td>
</tr>
<tr>
<td></td>
<td>● Flow obeys Darcy's Law.</td>
</tr>
<tr>
<td></td>
<td>● The matrix blocks have constant properties: compressibility, $c_{ma}$, permeability, $k_{ma}$, porosity, $\phi_{ma}$.</td>
</tr>
</tbody>
</table>
The flow of fluids in the hydraulic fracture and in the fractal reservoir is governed by its corresponding diffusivity equation. For a circular fracture, the flow is modeled by the diffusivity equation in radial coordinates with a source term. This equation in its dimensionless form is:

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial p_{D,cr}}{\partial r_D} \right] + \frac{2}{F_{cD}} \frac{\partial p_{D,cr}}{\partial z_D} \bigg|_{z_D=0} = \frac{1}{\eta_{fD}} \frac{\partial p_{fD,cr}}{\partial t_D} \tag{3.1}
\]

For the rectangular fracture case, the flow is modeled by the diffusivity equation for a linear system with a source term, written in dimensionless form, our starting point is given as:

\[
\frac{\partial^2 p_{fD,cr}}{\partial x_D^2} + \frac{2}{F_{cD}} \frac{\partial p_{D,cr}}{\partial z_D} \bigg|_{z_D=0} = \frac{1}{\eta_{fD}} \frac{\partial p_{fD,cr}}{\partial t_D} \tag{3.2}
\]

The flow within the fractal reservoir considering typical diffusion (Chang et al., 1990) is modeled by:

\[
\frac{1}{R_D^{D_f-1}} \frac{\partial}{\partial R_D} \left[ R_D^\beta \frac{\partial p_{D,cr}}{\partial R_D} \right] \frac{\partial p_{D,cr}}{\partial t_D} \bigg|_{D_f-1} = \frac{\partial p_{D,cr}}{\partial t_D} \tag{3.3}
\]

For the case of a fractal reservoir considering anomalous diffusion (Camacho-Velazquez et al., 2008), the flow model is:

\[
\frac{1}{R_D^{D_f-1}} \frac{\partial}{\partial R_D} \left[ R_D^\beta \frac{\partial p_{D,cr}}{\partial R_D} \right] \frac{\partial p_{D,cr}}{\partial t_D} \bigg|_{D_f-1} = \frac{\partial^\gamma p_{D,cr}}{\partial t_D^\gamma}, \tag{3.4}
\]

where \( \gamma = 2/[\theta+2] \). The definitions of the dimensionless variables are summarized in Table 3.2.
3.2. Development of the Models and Solution Procedure

Given that the source term in the left hand side of Eq. 3.1 and Eq. 3.2 is variable, the solution for this "coupled" model has to be semi-analytical. The procedure we have used is the same applied by Larsen et al. (1991). The workflow of this procedure is:

1. **Discretize the fracture into N-segments.** The discretization permits us to approximate the source term to a constant value for each segment. For a \( j \)-segment of a circular fracture, the diffusivity equation is approximated as

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ \frac{\partial p_{fD,cr}}{\partial r_D} \right] - \frac{2 \pi h_D}{F_c D} \partial q_{Dj} = \frac{1}{\eta_{fD}} \frac{\partial p_{fD,cr}}{\partial t_D}, .................................................. (3.5)
\]

whereas for a \( j \)-segment of a rectangular fracture, the diffusivity equation is approximated as

\[
\frac{\partial^2 p_{fD,cr}}{\partial x_D^2} - \frac{2 \pi h_D}{F_c D} \partial q_{Dj} = \frac{1}{\eta_{fD}} \frac{\partial p_{fD,cr}}{\partial t_D}, .................................................. (3.6)
\]

2. **Obtain the analytical general solutions in the Laplace domain for each one of the segments of the hydraulic fracture.** Given that the source term is approximated to a constant value, Eqs. 3.5 and 3.6 can be solved using the Laplace transform. The general solution for a \( j \)-segment of the circular fracture is:

\[
\bar{p}_{fD,cr,j}(r_D, u) = A_j K_0 \left[ \frac{r_D}{\eta_{fD}} \right] + B_j I_0 \left[ \frac{r_D}{\eta_{fD}} \right] - \frac{2 \pi h_D \eta_{fD}}{u F_c D} \partial q_{Dj}, .................................................. (3.7)
\]
Table 3.2 — Dimensionless variables for the model of a horizontal well intercepting a single finite conductivity fracture within a fractal reservoir.

<table>
<thead>
<tr>
<th>Dimensionless Variable</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure in the fractal reservoir</td>
<td>( P_{D,D}(R_D,t_D) = \frac{2\pi k_0h[p_i - p(R,t)]}{q_wB\mu L_r^\theta} )</td>
</tr>
<tr>
<td>Pressure in the hydraulic fracture</td>
<td>( P_{f,D}(\psi_D,t_D) = \frac{2\pi k_0h[p_i - p(\psi,t)]}{q_wB\mu L_r^\theta} )</td>
</tr>
<tr>
<td>Fractal reservoir rate</td>
<td>( \delta q_D = \frac{4\pi}{\alpha_{D,f}} \left[ \frac{b}{L_r} \right]^{D_f - 3} \frac{\delta q_r}{q_w} )</td>
</tr>
<tr>
<td>Time</td>
<td>( t_D = \frac{k_0}{\phi_0 \mu c_t L_r^{2+\theta}} t )</td>
</tr>
<tr>
<td>Position in the circular fracture</td>
<td>( r_D = r / r_f )</td>
</tr>
<tr>
<td>Position in the z-direction</td>
<td>( z_D = z / L_r )</td>
</tr>
<tr>
<td>Position in the x-direction</td>
<td>( x_D = x / L_r )</td>
</tr>
<tr>
<td>Position in the fractal reservoir</td>
<td>( R_D = R / L_r )</td>
</tr>
<tr>
<td>Fracture length along the wellbore</td>
<td>( y_D = y / x_f )</td>
</tr>
<tr>
<td>Height</td>
<td>( h_D = h / L_r )</td>
</tr>
<tr>
<td>Fracture conductivity</td>
<td>( F_{c,D} = \frac{k_f w}{kL_r} )</td>
</tr>
<tr>
<td>Fracture fractal conductivity</td>
<td>( F_{hc,D} = \frac{k_f w L_r^{\theta}}{kh} )</td>
</tr>
<tr>
<td>Fractal hydraulic diffusivity of the hydraulic fracture</td>
<td>( \eta_{f,D} = \frac{\phi_0 c_t k_f L_r^\theta}{\phi_f c_{tf} k_0} )</td>
</tr>
<tr>
<td>Hydraulic diffusivity of the matrix blocks</td>
<td>( \eta_{ma,D} = \frac{k_{ma} [\phi_i] t_r^\theta}{k_0 f_i c_{tfb} ma c_{ima}} )</td>
</tr>
<tr>
<td>Storativity ratio</td>
<td>( \omega = \frac{\phi_0 f_i c_{tfb}}{\phi_0 f_i c_{tfb} + \phi_{ma} c_{ima}} )</td>
</tr>
<tr>
<td>Fracture area</td>
<td>( A_{f,D} = A_{f,ma} h_{ma} )</td>
</tr>
<tr>
<td>Matrix block size</td>
<td>( h_{ma,D} = h_{ma} / L_r )</td>
</tr>
<tr>
<td>Interporosity skin</td>
<td>( S_{int} = \frac{k_{ma} d}{k_d h_{ma}} )</td>
</tr>
</tbody>
</table>

Where \( L_r \) is the reference length. For the circular fracture it is equal to \( r_f \), whereas for a rectangular fracture it is \( x_f \).
For a $j$-segment of a rectangular fracture, the general solution is:

$$
\bar{p}_{jD,cr,j}(x_D,u) = A_j \exp\left[-x_D \sqrt{\frac{u}{\eta_jD}}\right] + B_j \exp\left[x_D \sqrt{\frac{u}{\eta_jD}}\right] - \frac{2\pi bD\eta_jD}{uF_{cD}} \xi_jD_{j} \quad \cdots (3.8)
$$

3. Define a system of equations for $3N$-unknowns. Since constants $A_j$, $B_j$ and $\xi_jD_{j}$ are unknowns for each $j$-segment, a system of $3N$ equations must be defined and solved to determine such constants. The system of equations is defined by:

- Applying the inner boundary condition (constant-rate) to the general solution governing segment 1. For a circular fracture:

$$
\left[ r_D \frac{\partial p_{jD,cr,1}}{\partial r_D} \right]_{r_D=r_{wD}} = -\frac{1}{F_{hcD}} \quad \cdots (3.9)
$$

For a rectangular fracture:

$$
\left[ \frac{\partial p_{jD,cr,1}}{\partial x_D} \right]_{x_D=0} = -\frac{\pi}{2F_{hcD}y_D} \quad \cdots (3.10)
$$

- Applying the outer boundary condition (closed fracture) to the general solution governing segment $N$:

$$
\left[ \frac{\partial p_{jD,cr,N}}{\partial \xi_D} \right]_{\xi_D=1} = 0 \quad \cdots (3.11)
$$

Where $\xi_D=r_D$ for a circular fracture and $\xi_D=x_D$ for a rectangular fracture:

- Establishing pressure continuity at each of the interfaces:

$$
\bar{p}_{jD,cr,j-1}(\xi_{D_j-1},u) = \bar{p}_{jD,cr,j}(\xi_{D_j-1},u) \quad \cdots (3.12)
$$
• Establishing the flowrate continuity at each of the interfaces:

\[
\begin{bmatrix}
\frac{\partial f_{D,cr,j}}{\partial \xi_D} \\
\frac{\partial f_{D,cr,j-1}}{\partial \xi_D}
\end{bmatrix}_{\xi_D=\xi_{Dj-1}} = \begin{bmatrix}
\frac{\partial f_{D,cr,j}}{\partial \xi_D} \\
\frac{\partial f_{D,cr,j-1}}{\partial \xi_D}
\end{bmatrix}_{\xi_D=\xi_{Dj-1}} \quad \text{................................. (3.13)}
\]

• Establishing pressure continuity between each segment of the fracture and the fractal reservoir.

\[
\bar{p}_{D,cr,j}(\xi_{Dj}^*,u) = \bar{p}_{D,cr,j}(\xi_{Dj}^*,u) \quad \text{.................................................. (3.14)}
\]

where \(\xi_{Dj}^*\) is the middle point in the segment. The pressure in the fractal reservoir is the result of superimposing the constant rate solution of the diffusivity equation for fractal reservoirs over the area, \(A\), of the segment (see Appendix D for more details).

For a fractal reservoir considering typical diffusivity, the pressure behavior is given by:

\[
\bar{p}_{D,cr}(R_D,u) = \frac{\delta \bar{p}_{D} h_D}{[\theta + 2][1\nu]} \left[ \frac{\theta + 2}{\nu} \right]^\nu \int_A \left[ R_D^{[\theta+2]/2} K_{\nu} \left[ R_D^{[\theta+2]/2} \frac{2\nu^2}{\theta + 2} \right] dA \quad (\text{where } \nu = \frac{1-\beta}{\theta + 2}) \ldots \text{ (3.15)}
\]

Similarly, for a fractal reservoir considering anomalous diffusion:

\[
\bar{p}_{D,cr}(R_D,u) = \frac{\delta \bar{p}_{D} h_D}{[\theta + 2][1\nu]} \left[ \frac{\theta + 2}{\nu^2} \right]^\nu \int_A \left[ R_D^{[\theta+2]/2} K_{\nu} \left[ R_D^{[\theta+2]/2} \frac{2\nu^2}{\theta + 2} \right] dA \quad \text{......................... (3.16)}
\]

These models can be extended to the naturally-fractured/dual porosity reservoir case as:

\[
\bar{p}_{D,cr}(R_D,u) = \frac{\delta \bar{p}_{D} h_D}{[\theta + 2][1\nu]} \left[ \frac{\theta + 2}{\sqrt{u}} \right]^\nu \int_A \left[ R_D^{[\theta+2]/2} K_{\nu} \left[ R_D^{[\theta+2]/2} \frac{2\sqrt{u}}{\theta + 2} \right] dA, \quad \text{................................. (3.17)}
\]

where the interporosity transfer function is defined by:
For slab matrix blocks:

$$f(u) = \omega u^{\gamma - 1} + \frac{[1 - \omega] A_f D \bar{g}(\eta_{maD}, h_{maD}, u)}{1 + \frac{h_{maD}^2 s_{int}}{\eta_{maD}}} \bar{g}(\eta_{maD}, h_{maD}, u)$$ ................................. (3.18)

and for spherical matrix blocks:

$$\bar{g}(\eta_{maD}, h_{maD}, u) = \frac{1}{h_{maD}} \sqrt{\frac{\eta_{maD}}{u}} \tanh \left( \frac{h_{maD}}{2} \sqrt{\frac{u}{\eta_{maD}}} \right)$$ ................................. (3.19)

Eq. 3.18 considers the anomalous diffusion phenomenon. To restrict it to the typical diffusion case, \(\gamma\) should be equal to unity and considered to be independent of the conductivity index, \(\theta\).

4. **Solve the system of equations and evaluate the solution at the wellbore.** For the results presented in this dissertation, we have used the "mldivide" function of MATLAB. Once the constants \(A_j\), \(B_j\) and \(\delta q_{Dj}\) \((j=1,\ldots,N)\) are determined, the solution is evaluated at the wellbore. For a circular fracture, the pressure at the wellbore is determined by:

$$p_{fD, cr, 1}(r_{wD}, u) = A_1 K_0 \left( r_{wD} \sqrt{\frac{u}{\eta/J}} \right) + B_1 I_0 \left( r_{wD} \sqrt{\frac{u}{\eta/J}} \right) - \frac{2 \pi h_D \eta_{jD}}{u F_{Dj}} \delta q_{Dj}$$ ................................. (3.21)

Similarly, for a rectangular fracture:

$$p_{fD, cr, 1}(0, u) = A_1 + B_1 \frac{2 \pi h_D}{u F_{Dj}} \delta q_{Dj}$$ ................................. (3.22)
The wellbore storage effects can be included using the classic (constant) wellbore storage relation in the Laplace domain presented in Chapter I. To include the skin effect (i.e., using a choked fracture) for the circular fracture case, the effective wellbore radius should be used. For the rectangular fracture case, the skin factor should be added to the dimensionless drop of pressure without wellbore storage:

\[
\bar{p}_{wd}(u, s, C_{FD} = 0) = \bar{p}_{wd}(u, C_{FD} = 0) + s \]

\[ (3.23) \]

5. **Apply the Stehfest algorithm to numerically invert the Laplace domain solution.**

The Stehfest algorithm is implemented in MATLAB and has been tested for consistency and accuracy. In addition to the dimensionless pressure and rate solutions, we can also use the Stehfest algorithm to provide both the cumulative and the derivative of any given formulation in the Laplace domain.

*Constant-Pressure Solution*

As defined in Chapter I, Duhamel's principle is used to obtain the models for rate-transient analysis (constant pressure solution) from the models for pressure-transient analysis (constant rate solution). The dimensionless cumulative production at a given time is computed by integrating the dimensionless rate profile as presented in Chapter I.


3.3. Results and Discussion

Single Porosity Fractal Reservoirs

In this section we present the results of our sensitivity analyses which consider the influence of key parameters for the fractal reservoir and the hydraulic fracture in terms of the pressure and rate transient performance behaviors. Based on the configuration of the system, the expected diagnostic signatures for our proposed semi-analytical solution are:

- Period 1 (Fracture flow): Radial or linear flow (depending on the geometry of the fracture) at very early times. As in classic studies for the case of a single finite-conductivity fracture, this period will never be observed in practice.

- Period 2 (Fracture-reservoir interaction): "Radial-Fractal" or "Linear-Fractal" at intermediate-transient times. This period can be subdivided into two sub-periods: (1) early-intermediate and (2) late-intermediate. The early-intermediate period is analogous to the bilinear flow regime for a finite-conductivity vertical fracture in an infinite-acting homogeneous reservoir, whereas the late-intermediate is analogous to the formation-linear flow regime observed at most times for a case with a very high conductivity vertical fracture and at later times for cases with a medium to high conductivity vertical fracture.

- Period 3 (Reservoir dominated flow): "Pseudo-Fractal" flow. This flow period is dominated by the reservoir and yields power-law behavior (i.e., a straight line in the pressure drop and pressure derivative functions versus time on a log-log plot).
The schematics for the expected diagnostic signatures (i.e., the dimensionless pressure and dimensionless pressure derivative functions for the constant rate case) for the circular and rectangular fracture cases are shown in Fig 3.2 and Fig 3.3, respectively.

Figure 3.2 — Schematic example of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity within a single porosity fractal reservoir (constant rate case).
In Fig. 3.4, we show the impact of the fractal dimension of the reservoir ($D_f$) on the behavior of the dimensionless pressure and dimensionless pressure derivative functions (constant rate case) for the circular fracture case, considering a "low" dimensionless conductivity. Based on the pressure derivative response, we observe that at late-transient times (dimensionless times greater than 1), pseudo-fractal flow is exhibited for all cases. However; at early-transient times (dimensionless times < $10^{-2}$), for a fractal dimension of 1.5, a power-law response is observed, instead of the expected radial fracture flow regime. For each of the cases shown in Fig. 3.4 only one sub-period of flow is observed at intermediate-transient times. These "sub-periods" exhibit a power-law behavior that is the result of the interaction between the fractal reservoir and the circular fracture.
Figure 3.4 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the fractal dimension ($D_f$) (constant rate case).

Fig. 3.5 shows the (constant pressure) rate-transient case for the example presented in Fig. 3.4. At late-transient times (i.e., dimensionless times $> 7$), this case exhibits higher flowrates at higher values of the fractal dimension. However, at early and intermediate-transient times (dimensionless times $< 7$), the well exhibits higher flowrates at lower values of the fractal dimension. This behavior implies that the cumulative production of the well is higher for low values of the fractal dimension until a certain point in time, after which the cumulative production is higher for high values of the fractal dimension (see Fig. 3.6 for the cumulative production behavior).
Figure 3.5 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the fractal dimension ($D_f$) (constant pressure case).

Figure 3.6 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the fractal dimension ($D_f$). (constant pressure case)
Similar to our previous comparisons for the circular fracture case, we now consider the rectangular fracture case — and we begin with the constant rate solution for the rectangular fracture case (as shown in Fig. 3.7) where we plot the dimensionless pressure and dimensionless pressure derivative functions. We do note a "spectra" of solutions based on the given fractal dimension ($D_f$) — in particular, the fractal dimension case of 2.5 shows the most "aggressive" behavior, where the $D_f = 1.5$ case is the most "conservative." We present the constant pressure cases (i.e., the dimensionless rate and dimensionless rate derivative profiles) in Fig. 3.8 and the dimensionless cumulative production profile is shown in Fig. 3.9 — and we also note the "aggressive" and "conservative" behaviors as caused by the fractal dimension parameter ($D_f$) for these cases.

**Figure 3.7** — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the fractal dimension ($D_f$). (constant rate case).
Figure 3.8 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the fractal dimension ($D_f$). (constant pressure case).

Figure 3.9 — Log-log plot of the dimensionless cumulative production for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the fractal dimension ($D_f$). (constant pressure case).
We next consider the impact of the (fractal) conductivity index ($\theta$) on the proposed constant rate and constant pressure solutions — and we begin with the constant rate, circular fracture case where the dimensionless pressure and dimensionless pressure derivative functions are shown in Fig. 3.10. The influence of the conductivity index ($\theta$) is similar to that for the fractal dimension case ($D_f$) — however; in the case of the conductivity index the features are much more "subtle" compared to the fractal dimension cases. The constant pressure cases are shown in Fig. 3.11 (i.e., the dimensionless rate and dimensionless rate derivative functions) and Fig. 3.12 (the dimensionless cumulative production functions) — and much like the constant rate cases, the influence of the conductivity index ($\theta$) tends to be very precise, as in the case of the dimensionless rate derivative functions (Fig. 3.11), which are essentially parallel over the entire time scale. Lastly, the dimensionless cumulative production profiles shown in Fig. 3.12 have very little independent character (i.e., the trends are almost indistinguishable over time).

Figure 3.10 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the conductivity index ($\theta$). (constant rate case)
Figure 3.11 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the conductivity index ($\theta$). (constant pressure case).

Figure 3.12 — Log-log plot of the dimensionless cumulative production for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the conductivity index ($\theta$). (constant pressure case).
Finally, we consider the influence of the conductivity index ($\theta$) for the constant rate and constant pressure cases for the case of a rectangular fracture of finite conductivity (see Figs. 3.13 and 3.14, respectively). In Fig. 3.13 we note that both the dimensionless pressure and the dimensionless pressure derivative functions are dominated by the "linear flow" feature (1:2 slope) at early times ($i.e.$, for $t_D < 10^{-2}$), then by the fracture-reservoir transient regime, followed by the fractal-influenced reservoir behavior.

We next present the constant pressure solution in Fig. 3.14 ($i.e.$, the dimensionless rate and dimensionless rate derivative functions) and we immediately note that these cases are essentially "mirror images" of their corresponding constant rate cases. This behavior is probably due to the uniqueness of this selection of parameters for this particular model. To finish this suite of comparisons, we present the dimensionless cumulative production profile for our selected cases (Fig. 3.15) and we find that all of the proposed flow regimes are observed — in particular, the early-time "linear flow" feature (1:2 slope) for $t_D < 10^{-2}$, then the fracture-reservoir transient regime, then the fractal-influenced reservoir behavior.
Figure 3.13 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the conductivity index ($\theta$). (constant rate case)

Figure 3.14 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the conductivity index ($\theta$). (constant pressure case).
Figure 3.15 — Log-log plot of the dimensionless cumulative production for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the conductivity index ($\theta$). (constant pressure case).

In Fig. 3.16, we illustrate the effect of the dimensionless fracture conductivity on our proposed solutions, in particular for the dimensionless pressure and dimensionless pressure derivative functions for the circular fracture case (constant rate case). Analogous to the non-fractal models, we have observed that the flow within the fractal reservoir dominates during all periods of flow at very low values of the dimensionless fracture conductivity ($F_{CD}=1$ in Fig. 3.16). At high values of conductivity (e.g., $F_{CD}=150$), the three expected periods of flow are well defined: (1) fracture flow for dimensionless time $< 10^{-2}$, (2) fracture-reservoir interaction period for dimensionless times between $10^{-2}$ and 10, and (3) pseudo-fractal flow for dimensionless times $> 10$. For intermediate values of dimensionless fracture conductivity (e.g., $F_{CD}=10$), the fracture flow and the interaction periods yield the expected power-law signatures, where this is a result of a narrower (or perhaps, shorter) fracture.
Figure 3.16 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless fracture conductivity ($F_{c,D}$). (constant rate case)

Fig. 3.17 shows the influence of the dimensionless fracture conductivity on the rate-transient performance behavior for the circular fracture case — specifically the dimensionless rate and dimensionless rate derivative functions (constant pressure case). As expected, the highest dimensionless flowrate and dimensionless cumulative production profiles (see Fig. 3.18 for the dimensionless cumulative production profiles) are given by the higher values of the dimensionless fracture conductivity. We believe that the dispersions observed in the dimensionless rate derivative functions for dimensionless times > 30 as shown in Fig. 3.17 for the lowest dimensionless fracture conductivity case (i.e., $F_{c,D}=1$) are the result of numerical instabilities for small arguments of the modified Bessel functions in the solution for this particular case. The Stehfest algorithm (for numerical inversion) will amplify small scale instabilities, particularly for any derivative functions.
Figure 3.17 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless fracture conductivity ($F_cD$). (constant pressure case).

Figure 3.18 — Log-log plot of the dimensionless cumulative production for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless fracture conductivity ($F_cD$). (constant pressure case)
The sensitivity analyses considering variations in the dimensionless fracture conductivity on our proposed solutions for the rectangular fracture case are shown in Fig. 3.19 and Fig. 3.20, respectively. Again we use the dimensionless pressure (constant rate case) and dimensionless rate (constant pressure) solutions as appropriate. Using these plots we observe that the rectangular fracture cases have a very unusual "late-time" effect in that, for the constant rate case, the dimensionless pressure function appears to flatten, but the dimensionless pressure derivative function exhibits an approximate 1/4 slope that is decreasing (see Fig. 3.19). A similar feature is observed for the constant pressure case in Fig. 3.20.

**Figure 3.19** — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless fracture conductivity ($F_{cD}$). (constant rate case)
Figure 3.20 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless fracture conductivity ($F_{cD}$). (constant pressure case).

The characteristic signature (i.e., unit-slope in the dimensionless pressure derivative function) of the "transition to reservoir" effect is observed over the dimensionless time period from $10^{-2}$ to $10^{-1}$ in Fig. 3.19. We also note abrupt changes in the dimensionless rate derivative function over the dimensionless time period from $10^{-2}$ to $10^{1}$ (see Fig. 3.20) we believe that this is also a consequence of the "transition to reservoir" effect. The dimensionless cumulative production profile is shown in Fig. 3.21, where we note that the early transient flow is dominated by the $F_{cD}$-value (i.e., the higher the $F_{cD}$ value the higher the production during this period), we also not that the unit slope in the dimensionless cumulative production profile for dimensionless times $> 10$ reflect an essentially constant flowrate at late times.
Figure 3.21 — Log-log plot of the dimensionless cumulative production for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless fracture conductivity ($F_{cD}$). (constant pressure case)

*Naturally Fractured Porosity/Dual Porosity Fractal Reservoir with Typical Diffusion*

To extend these models to naturally-fractured/dual porosity reservoirs, we have used the reservoir model presented by Valdes-Perez (2013), considering the "slab" model for the fracture-matrix systems of all of the cases considered in this work. This model considers transient interporosity transfer and can reproduce pseudosteady-state interporosity transfer conditions at high values of interporosity skin. Regardless of the interporosity transfer conditions, the model solution (as represented by the dimensionless pressure and dimensionless pressure derivative signatures) yields three distinct flow periods, as governed by the model parameters. A schematic log-log plot of the dimensionless pressure and dimensionless pressure derivative functions is shown in Fig. 3.22.
The following flow periods are evident for these case:

- Period 1 — (fractal fracture network flow): Early-fractal flow — this period of flow is dominated by the properties of the fractal network and exhibits a power-law signature in both the dimensionless pressure and dimensionless pressure derivative functions (see Fig. 3.22).

- Period 2 — (interaction between the fractal fracture network and the matrix blocks): The matrix blocks contribute fluid to the fractal fracture network. The dimensionless pressure and dimensionless pressure derivative functions exhibit a "combined" power-law behavior, different from the behavior observed in Period 1.

- Period 3 — (single porosity fractal reservoir flow): Both porous media (i.e., the fractal fracture network and the matrix blocks) behave as a (total) single porosity fractal system. The dimensionless pressure and dimensionless pressure derivative functions should exhibit the same power-law behavior as that observed in Period 1.

As presented by Cinco-Ley et al. (1988), a finite conductivity hydraulic fracture within a naturally-fractured/dual porosity reservoir can exhibit a maximum of twelve (12) sub-periods of flow. The number of sub-periods of flow that may be observed depends on the properties of the hydraulic fracture and the reservoir.
In this section, we present selected cases that show the influence of the naturally-fractured/dual porosity effects on the transient performance of a horizontal well intercepting a single finite conductivity fracture.

Sensitivity to the $\eta_{maD}$-Parameter: Circular Fracture Case

The plot shown in Fig. 3.23 presents three (constant rate) cases of the dimensionless hydraulic diffusivity of the matrix ($\eta_{maD}$): (1) high ($\eta_{maD} = 10^{-4}$), (2) intermediate ($10^{-10}$), and (3) low ($10^{-16}$). These three cases exhibit radial flow (in the fracture) at early times (dimensionless times $< 10^{-5}$). At intermediate-transient times (dimensionless times within the range $10^{-5}$ to $10^{-3}$), the dimensionless pressure derivative signatures exhibit a power-law behavior, and we note that the slope is less steep for low $\eta_{maD}$-values. At late-transient times (dimensionless times $> 10^{-3}$), the naturally-fractured/dual porosity reservoir component dominates the flow behavior.
Figure 3.23 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless hydraulic diffusivity of the matrix ($\eta_{maD}$). (constant rate case)

The constant pressure cases are shown in Figs. 3.24 and 3.25. In Fig. 3.24 we present the dimensionless rate and dimensionless rate derivative functions for the constant rate example cases presented in Fig. 3.23. As expected, the trends in Fig. 3.24 confirm that the influence of the dimensionless hydraulic diffusivity of the matrix ($\eta_{maD}$) in the sense that the lower the value of the $\eta_{maD}$-parameter, the lower the rate performance of these cases. The dimensionless cumulative production cases are presented in Fig. 3.25 and although there is little character in these curves, it is clear that the highest production corresponds to the highest value of the $\eta_{maD}$-parameter.
Figure 3.24 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless hydraulic diffusivity of the matrix ($\eta_{m3D}$). (constant pressure case)

Figure 3.25 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless hydraulic diffusivity of the matrix ($\eta_{m3D}$). (constant pressure case).
Sensitivity to the $\eta_{maD}$-Parameter: Rectangular Fracture Case

In Fig. 3.26 we present three (constant rate) cases for the rectangular fracture case where we have varied the dimensionless hydraulic diffusivity of the matrix ($\eta_{maD}$) as follows: (1) high ($\eta_{maD} = 10^{-4}$), (2) intermediate ($10^{-12}$), and (3) low ($10^{-20}$). These three cases exhibit linear flow (in the fracture) at very early times (dimensionless times $< 10^{-5}$). At intermediate-transient times (dimensionless times within the range $10^{-3}$ to $10^{2}$), the dimensionless pressure derivative signatures exhibit an apparent "transient radial flow" behavior, but we understand that this could also be interpreted as a very shallow power-law behavior. Lastly, we observe the influence of the naturally-fractured/dual porosity reservoir component, which manifests a late-time radial flow regime for dimensionless times $> 10^4$. We acknowledge that these behaviors are uniquely dependent on the selection of the input parameters, and we can clearly see the evidence of the dimensionless hydraulic diffusivity of the matrix parameter ($\eta_{maD}$) during the intermediate and late times.
Figure 3.26 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless hydraulic diffusivity of the matrix ($\eta_{maD}$). (constant rate case)

The constant pressure cases are shown in Figs. 3.27 and 3.28. The dimensionless rate and dimensionless rate derivative functions reflect very closely the behaviors exhibited for the constant rate example cases presented in Fig. 3.26 (with the anomalies in the dimensionless rate derivative functions for the $\eta_{maD}$-parameter values of $10^{-12}$ and $10^{-20}$ duly noted — these are artifacts of the character of the dimensionless rate for these cases). In Fig. 3.28 we present the dimensionless cumulative production and we clearly note that the $\eta_{maD} = 10^{-12}$ and $10^{-20}$ cases are quite similar across the entire time-scale, but that these cases differ significantly from the $\eta_{maD} = 10^{-4}$ case.
Figure 3.27 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless hydraulic diffusivity of the matrix ($\eta_{mD}$). (constant pressure case)

Figure 3.28 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the dimensionless hydraulic diffusivity of the matrix ($\eta_{mD}$). (constant pressure case)
Sensitivity to the $\omega$-Parameter: Circular Fracture Case

In this section we consider the influence of the storativity ratio ($\omega$) for the naturally-fractured/dual porosity reservoir component of the solution. We first consider the constant rate cases for a single circular fracture of finite conductivity where we have varied the storativity ratio ($\omega$) as follows: $\omega = 1$ (high/homogeneous), $\omega = 10^{-2}$ (intermediate), and $\omega = 10^{-16}$ (very low) as shown in Fig. 3.29. As expected, each case exhibits radial flow (in the fracture) at early times (dimensionless times $< 10^{-4}$). At intermediate-transient times (dimensionless times within the range $10^{-4}$ to $10^0$), the dimensionless pressure derivative signatures exhibit a near-unity power-law behavior which reflects the transition to the fractal solution component. At late-transient times (dimensionless times $> 10^1$) we observe an approximate slope of 1:4 in the dimensionless pressure derivative profiles, which suggests that the response is now dominated by the fractal reservoir solution component.

We next present the constant pressure cases in Figs. 3.30 and 3.31 (i.e., the dimensionless rate and dimensionless rate derivative functions are shown in Fig. 3.30 and the dimensionless cumulative production is shown in Fig. 3.31). In relative terms, the dimensionless rate profiles in Fig. 3.30 roughly reflect the same features we observed for the dimensionless pressure solutions in Fig. 3.29. However; the dimensionless rate derivative profiles shown in Fig. 3.30 are significantly affected by the "transition" features observed in the dimensionless rate profiles — in short, the dimensionless rate derivative profiles exhibit numerous artifacts that could make interpretation of these trends non-unique.
Figure 3.29 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the storativity ratio ($\omega$). (constant rate case)

Figure 3.30 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the storativity ratio ($\omega$). (constant pressure case)
In Fig. 3.31, we present the dimensionless cumulative production for these cases and we note that each case is strongly distinctive and that the behavior of these functions is uniquely related to the storativity ratio ($\omega$) for the naturally-fractured/dual porosity reservoir component of the solution.

**Sensitivity to the $\omega$-Parameter: Rectangular Fracture Case**

We continue to consider the influence of the storativity ratio ($\omega$) on the naturally-fractured/dual porosity reservoir component of the solution, but now we move to the constant rate cases for a single rectangular fracture of finite conductivity. We use the following cases for the storativity ratio ($\omega$): $\omega = 1$ (high/homogeneous), $\omega = 10^{-4}$ (intermediate), and $\omega = 10^{-10}$ (very low), where the dimensionless pressure and dimensionless pressure derivative profiles for these cases are shown.
in Fig. 3.32. At very early times (dimensionless times \(< 10^{-5}\) to \(10^{-2}\), depending on the \(\omega\)-parameter), we observe linear flow (in the fracture). Similar to the circular fracture case, for intermediate-transient times (dimensionless times within the range \(10^{-6}\) to \(10^{-3}\) for the \(\omega = 10^{-4}\) and \(10^{-10}\) cases, and dimensionless times within the range \(10^{-1}\) to \(10^{2}\) for the \(\omega = 10^{0}\) case) we observe a near-unity power-law behavior in the dimensionless pressure derivative signatures, which reflects the transition to the fractal solution component. At late-transient times (dimensionless times depending on the \(\omega\)-parameter) we observe a very low power-law slope of 1:10 in the dimensionless pressure derivative profiles for the \(\omega = 10^{0}\) and \(10^{-4}\) cases.

The constant pressure cases are presented in Figs. 3.33 and 3.34 (i.e., the dimensionless rate and dimensionless rate derivative functions are shown in Fig. 3.33 and the dimensionless cumulative production is shown in Fig. 3.34). As with the circular fracture case, the dimensionless rate profiles in Fig. 3.33 roughly reflect the same features we observed for the dimensionless pressure solutions in Fig. 3.32 — and again (as in the circular fracture case), the dimensionless rate derivative profiles shown in Fig. 3.33 are significantly affected by the "transition" features observed in the dimensionless rate profiles. The dimensionless cumulative production profiles for these cases are shown in Fig. 3.34, and as with the circular fracture cases, we note that each trend is quite unique and we believe that the behavior of these functions is exceptionally related to the storativity ratio (\(\omega\)) for the naturally-fractured/dual porosity reservoir component of the solution.
Figure 3.32 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the storativity ratio ($\omega$). (constant rate case)

Figure 3.33 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the storativity ratio ($\omega$). (constant pressure case)
Sensitivity to the $S_{int}$-Parameter: Circular Fracture Case

In this section we consider the influence of the interporosity skin ($S_{int}$) on the naturally-fractured/dual porosity reservoir component of the solution. This is a fairly straightforward exercise, but it is important to recognize that the interporosity skin ($S_{int}$) is not an actual reservoir parameter, but is instead, an "additional/imposed" dimensionless pressure drop. For the constant rate case of a single circular fracture of finite conductivity, we will test the solution behavior for $S_{int} = 0$ (no "additional/ imposed" dimensionless pressure drop) and $S_{int} = 0.1$, which for this solution is a fairly large "additional/imposed" dimension-less pressure drop.

We present the (constant rate) dimensionless pressure and dimensions less pressure derivative profiles for the $S_{int} = 0$ and $S_{int} = 0.1$ cases in Fig. 3.35. As with all of our other sensitivity cases,
we observe radial flow (in the fracture) at early times (dimensionless times $< 10^{-4}$). At intermediate-transient times (dimensionless times within the range $10^{-4}$ to $10^3$), the dimension-less pressure and dimensionless pressure derivative profiles exhibit a sharp response that reflects the transition to the fractal solution component. For the case of $S_{int} = 0$, it appears that the solution simply "shifts" to an approximate power-law flow regime. For the case of $S_{int} = 0.1$, we note a very strong response, particularly in the dimensionless pressure derivative profile where a "hump" feature appears (where such features are typically associated with wellbore storage distortion). We believe this behavior is a characteristic of the (relatively) large $S_{int}$-parameter. At late-transient times (dimensionless times $> 10^3$) we observe an approximate slope of 1:4 in the dimensionless pressure derivative profiles, which suggests that the response is now dominated by the fractal reservoir solution component.

Figure 3.35 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the interporosity skin ($S_{int}$). (constant rate case)
The constant pressure cases are presented in Figs. 3.36 and 3.37 (i.e., the dimensionless rate and dimensionless rate derivative functions are shown in Fig. 3.36 and the dimensionless cumulative production is shown in Fig. 3.37). As has become a common observation, the dimensionless rate profiles in Fig. 3.36 roughly reflect the same features we observed for the dimensionless pressure solutions in Fig. 3.35 — and as has also been observed, the dimensionless rate derivative profiles shown in Fig. 3.36 are dramatically affected by the sharp features observed in the dimensionless rate profiles, which yield artifacts in the dimensionless rate derivative profiles (particularly so for the $S_{int} = 0.1$ case). We present the dimensionless cumulative production for these interporosity skin ($S_{int}$) cases in Fig. 3.37, we note (somewhat surprisingly) that the $S_{int}$-parameter only affects the intermediate time behavior of the dimensionless cumulative production profiles.

![Figure 3.36](image_url) — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the interporosity skin ($S_{int}$). (constant pressure case).
Figure 3.37 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting a single circular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the interporosity skin ($S_{int}$). (constant pressure case).

Sensitivity to the $S_{int}$-Parameter: Rectangular Fracture Case

We now consider the effect of the interporosity skin ($S_{int}$) on the naturally-fractured/dual porosity reservoir component of the solution for the constant rate case of a single rectangular fracture of finite conductivity. Similar to our efforts for the circular fracture cases, we will test the solution behavior for $S_{int} = 0$ (no "additional/ imposed" dimensionless pressure drop), but because of the nature of the rectangular fracture, we will use $S_{int} = 0.01$ for these cases (where we noted that $S_{int} = 0.01$ is still a relatively large "additional/ imposed" dimensionless pressure drop, particularly for the rectangular fracture cases).
We present the (constant rate) dimensionless pressure and dimensions less pressure derivative profiles for the $S_{int} = 0$ and $S_{int} = 0.1$ cases in Fig. 3.38 and we immediately note the existence of a very strong linear flow (in the fracture) signature at very early times (dimensionless times $< 10^{-5}$). At intermediate-transient times (dimensionless times within the range $10^{-5}$ to $10^{3}$), the dimensionless pressure and dimensionless pressure derivative profiles exhibit a sharp response that reflects the transition to the fractal solution component. As with the circular fracture example, for the case of $S_{int} = 0$, it appears that the solution simply "shifts" to an approximate power-law flow regime and for the case of $S_{int} = 0.01$, we note a very strong feature in the dimensionless pressure derivative profile where a "hump" appears. We believe this behavior is a characteristic of the (relatively) large $S_{int}$ -parameter. At late-transient times (dimensionless times $> 10^{3}$) we observe what appears to be slope of approximately 1:4 in the dimensionless pressure derivative profiles, which suggests that the response is now dominated by the fractal reservoir solution component.
Figure 3.38 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the interporosity skin ($S_{int}$). (constant rate case).

The dimensionless rate and dimensionless rate derivative functions for the constant pressure rate cases are presented in Fig. 3.39 and it appears that the dimensionless rate profiles in Fig. 3.39 roughly reflect the same features we observed for the dimensionless pressure solutions in Fig. 3.38 (in fact, these profiles are almost "mirror" images). As has been the observation throughout this work, the dimensionless rate derivative profiles shown in Fig. 3.39 are dramatically affected by the sharp features observed in the dimensionless rate profiles, which yields significant artifacts in the dimensionless rate derivative profiles for the $S_{int} = 0.01$ case. In Fig. 3.40 we present the dimensionless cumulative production for these interporosity skin ($S_{int}$) cases and we note that, for the rectangular fracture case, the $S_{int}$ -parameter significantly affects the character of the dimensionless cumulative production profiles.

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Figure 3.39 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the interporosity skin ($S_{int}$). (constant pressure case).

Figure 3.40 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting a single rectangular hydraulic fracture of finite conductivity in a dual porosity fractal reservoir considering typical (constant) diffusion, for selected values of the interporosity skin ($S_{int}$). (constant pressure case)
In Figs. 3.41 and 3.42 we present the comparisons of the constant rate solutions (dimensionless pressure and dimensionless pressure drop derivative) for the typical (constant) and anomalous diffusion models, for both the circular and rectangular fracture cases, respectively. We observed that the anomalous diffusion phenomena creates an additional pressure drop at early and intermediate-transient times, but the anomalous diffusion cases also exhibit a lower pressure drop at late times.

Figure 3.41 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative profiles for a horizontal well intercepting a circular fracture of finite conductivity in a fractal reservoir considering typical (constant) and anomalous diffusions, for selected values of the (fractal) conductivity index ($\theta$). (constant rate case)
Figure 3.42 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative profiles for a horizontal well intercepting a rectangular fracture of finite conductivity in a fractal reservoir considering typical (constant) and anomalous diffusions, for selected values of the (fractal) conductivity index ($\theta$). (constant rate case)

The constant rate behaviors shown in Figs. 3.41 and 3.42 imply by induction that at early and intermediate-transient times the typical (constant) diffusion cases yield higher flowrates and have lower flowrates at late-intermediate times (see Figs. 3.43 and 3.44). This behavior is confirmed as shown in Figs. 3.43 and 3.44, respectively — and it is important to note that the reservoir-transition features (occurring from $10^{-2} < t_D < 10^1$) are less salient in all of the anomalous diffusion cases.
Figure 3.43 — Log-log plot of the dimensionless rate and dimensionless rate derivative profiles for a horizontal well intercepting a circular fracture of finite conductivity in a fractal reservoir considering typical (constant) and anomalous diffusions, for selected values of the (fractal) conductivity index ($\theta$). (constant pressure case)

Figure 3.44 — Log-log plot of the dimensionless rate and dimensionless rate derivative profiles for a horizontal well intercepting a rectangular fracture of finite conductivity in a fractal reservoir considering typical (constant) and anomalous diffusions, for selected values of the (fractal) conductivity index ($\theta$). (constant pressure case)
In our final comparison of dimensionless cumulative production profiles (see Figs. 3.45 and 3.46), we note that in terms of the ultimate cumulative production, the anomalous diffusion phenomenon does "improve" the hydrocarbon "recovery" in a fractal reservoir — however; this observation may not be general and should only be considered as "guidance." Obviously, the selection of a model (any model) must be validated based on diagnostic comparison with the actual performance data and all geological, completion, and reservoir data should also be incorporated into the model selection process.

**Figure 3.45** — Log-log plot of the dimensionless cumulative production profiles for a horizontal well intercepting a circular fracture of finite conductivity in a fractal reservoir considering typical (constant) and anomalous diffusions, for selected values of the (fractal) conductivity index ($\theta$). (constant pressure case)
Figure 3.46 — Log-log plot of the dimensionless cumulative production profiles for a horizontal well intercepting a rectangular fracture of finite conductivity in a fractal reservoir considering typical (constant) and anomalous diffusions, for selected values of the (fractal) conductivity index ($\theta$). (constant pressure case).

3.4. Summary

Ultimately, our goal is to demonstrate the viability of the fractal reservoir concept for the transient pressure and rate behavior of unconventional reservoirs. In this chapter we presented a simplified case considers a horizontal well intersecting a single hydraulic fracture, and while this concept can be generalized to consider an arbitrary number of fractures, the purpose of this simplified case is to establish the basis and feasibility of the concept from the standpoint of comparative behavior and diagnostic analysis functions. The extension to horizontal wells intercepting multiple hydraulic fractures is presented in Chapter IV.
CHAPTER IV
PRESSURE AND RATE-TRANSIENT ANALYSIS OF HORIZONTAL WELLS INTERCEPTING MULTIPLE HYDRAULIC FRACTURES WITHIN A FRACTAL RESERVOIR²

In this chapter, we present the procedure to apply the principle of superposition in space to extend the models presented in Chapter III to the case of a "multi-fractured horizontal well" (or MFHW) within a fractal reservoir. Additionally, we use the technique of image wells to create vertical boundaries that represent a fractal reservoir with finite thickness.

4.1. Development of the Model

Analogous to Larsen et al. (1994), we have used the principle of superposition in space to model the transient performance behavior of a MFHW in a fractal reservoir. We have considered two scenarios: (1) an unbounded fractal reservoir (infinite thickness) and (2) a fractal reservoir vertically bounded by two parallel boundaries (i.e., a finite thickness reservoir).

The principle of superposition in space takes into account the pressure effects of the individual elements (i.e., the hydraulic fractures and boundaries) which define a system — specifically a MFHW within a fractal reservoir with infinite or finite thickness. In general, the pressure at the

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wellbore for a system defined by a horizontal well intercepting multiple vertical fractures, along a length $L_D$, in a reservoir with boundaries can be defined by:

$$P_{wD,cr}(t_D) = PHF + P_{BOU} \quad \text{............................................................ (4.1)}$$

where $PHF$ is the sum of the dimensionless drop of pressure of the hydraulic fractures and $P_{BOU}$ is the sum of the dimensionless drop of pressure of the $N_B$ boundaries. The sum of the dimensionless drop of pressure of the hydraulic fractures ($PHF$) is mathematically defined as:

$$PHF = \sum_{i=1}^{N_f} P_{fD,cr}(d_i, t_D) \quad \text{............................................................ (4.2)}$$

where $N_f$ is the number of hydraulic fractures and $d_i$ is the distance from a defined observation point to the $i$-hydraulic fracture. In this work, we have assumed evenly spaced hydraulic fractures by a distance $F_s$ between fractures (see Fig. 4.1a). The shape and orientation of the hydraulic fractures can be either circular and transverse or rectangular and longitudinal along the wellbore.

The dimensionless drop of pressure of each hydraulic fracture is computed using the models presented in Chapter III (Eq. 3.21 for circular fractures and Eq. 3.22 for rectangular fractures). Therefore, all the assumptions and dimensionless variables established in Chapter III are applicable in this chapter. To use such models in the case of a MFHW, we consider that the well produces only through the hydraulic fractures at a rate equal to:

$$q_i = \frac{q_t}{N_f} \quad \text{............................................................ (4.3)}$$

where $q_t$ is the total flowrate (production) of the well. Eq. 4.3 implies that the inner boundary condition (i.e., the "rate per fracture") is defined as the total flowrate ($q_t$) divided by the number
of fractures \( (N_f) \). For the case of a reservoir with infinite thickness, the dimensionless drop of pressure of the boundaries \( (p_{BOU}) \) is zero. For the finite thickness case, we have assumed that the horizontal well is drilled along the center of the formation, which is vertically delimited by two parallel and impermeable boundaries (see Fig. 4.1b). Using the method of image wells, the dimensionless pressure drop at the boundaries \( (p_{BOU}) \) is:

\[
p_{BOU} = 2 \sum_{i=1}^{\infty} p_{D,cr}(l_{Di}, t_D), \quad (4.4)
\]

where \( l_{Di} \) is the distance from the wellbore to the vertical axis of the \( i \)-image well. Given that the well is placed in a fractal reservoir, the dimensionless pressure of the image wells is:

\[
p_{D,cr}(l_{Di}, t_D) = \frac{r_D^{(1-\beta)/2}}{2(\theta + 2)[1 - \nu]} \left[ -v, \frac{f_D^{(\theta+2)}}{[\theta + 2]^2 t_D} \right] \quad (where \nu = \frac{1-\beta}{[\theta+2]}), \quad (4.5)
\]

For these models, the wellbore storage effects can be included by applying the Laplace transform to Eq. 4.1 and subsequently, using the scheme defined in Chapter I. The inverse Laplace transform for the cases presented in this chapter is numerically made using the Stehfest algorithm.
4.2. Results and Discussion

Larsen et al. (1994) and Raghavan (1997) showed that a MFHW behaves as an equivalent system of a well intercepting a single hydraulic fracture. We have constructed cases similar to those presented by Larsen et al. (1994) and generated type curves for the resulting pressure and rate-transient performance behaviors. We developed these type curves by varying the number of fractures ($N_f$) for fixed values of the conductivity of the fractures ($F_cD$) and the fractal parameters ($D_f$ and $\theta$). We present these type curves in Appendix G.

We analyzed the influence of the number of fractures ($N_f$) in the pressure and rate transient performance behaviors of a MFHW in a fractal reservoir and we concluded that: (1) the pressure and rate-transient signatures of a horizontal well intercepting multiple hydraulic fractures in a fractal reservoir can behave as an equivalent system created by a horizontal well intercepting a single hydraulic fracture, and (2) the spacing of the fractures can create a flow period that corresponds to the interference of flow between hydraulic fractures. In short, the influence of the number of fractures ($N_f$) and the spacing between the fractures ($F_s$) in the pressure transient performance behavior of a horizontal well intercepting multiple hydraulic fractures in a fractal reservoir is analogous to the one studied by Larsen et al. (1994) for a 3D (spherical) reservoir.

*Periods of Flow of Horizontal Wells Intercepting Multiple Hydraulic Fractures in a Fractal Reservoir*

For the analyses presented in this section, we selected cases with a short spacing between the hydraulic fractures ($F_s = 5/8$) and considered that all the fractures in the system have the same characteristics, *i.e.*, they have the same dimensionless fracture conductivity ($F_cD$) and geometry.
We studied three scenarios of the dimensionless conductivity and hydraulic diffusivity of the fractures: (1) high \( F_{cD} = 1000 \) and \( \eta_{D} = 10^6 \), (2) intermediate \( F_{cD} = 10 \) and \( \eta_{D} = 10^5 \), and (3) low \( F_{cD} = 1 \) and \( \eta_{D} = 10^4 \).

In Fig. 4.3 we present the pressure transient behavior of a horizontal well intercepting multiple circular transverse vertical fractures in a fractal reservoir for selected values of the conductivity of the fractures \( F_{cD} \), considering infinite or finite thickness. We present the analogous case for rectangular longitudinal fractures in Fig. 4.4. In these plots, we have identified the characteristic three period of flows for a well intercepting hydraulic fractures: (1) fracture flow at early times, (2) fracture-reservoir interaction flow at intermediate times, and (3) pseudo-fractal flow at late-times. The fracture-reservoir interaction flow is divided into two sub-periods: Early Radial-Fractal \( (ERF) \) and Late Radial-Fractal \( (LRF) \) occur for circular fractures and Early Linear-Fractal \( (ELF) \) and Late Linear-Fractal \( (LLF) \) occur for rectangular fractures. The \( ERF \) and \( ELF \) sub-periods are analogous "bilinear flow" which exists for the case of finite-conductivity fractures in a homogenous, infinite-acting reservoir.

We observed that the effect of interference between fractures is more significant for wells intercepting multiple low conductivity fractures. We conclude from the cases with hydraulic fractures having low \( F_{cD} \)-values (presented in Figs. 4.3 and 4.4) that these cases exhibit additional sub-periods of flow during intermediate times compared to cases with high \( F_{cD} \)-values. This behavior is clear for the case with circular fractures in a reservoir with infinite thickness \( (i.e., \text{Fig. 4.3}) \), where the pressure derivative signature of the \( ERF \) sub-period flow is followed by a power-law (straight-line with slope approximately of 1:3) in the range \( 10^{-2} < t_{D} < 10^{0} \). We also observed this phenomenon for the case with rectangular fractures case \( (i.e., \text{Fig. 4.4}) \) — however, we noted
that the interference between fractures for this case is more subtle, exhibiting a smooth transition from the \( ELF \) sub-period into the pseudo-fractal flow period.

\[ \text{Figure 4.2} \quad \text{— Schematics of a horizontal well intercepting multiple hydraulic fractures in a fractal reservoir with infinite thickness: a. circular transverse, and b. rectangular longitudinal hydraulic fractures.} \]

\[ \text{Figure 4.3} \quad \text{— Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting multiple circular transverse hydraulic fractures in a fractal reservoir with} \ D_f = 2.5 \ \text{and} \ \theta = 0 \ \text{for selected values of the Fracture Conductivity} \ ( F_{cD} ). \]
Figure 4.4 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting multiple rectangular longitudinal hydraulic fractures in a fractal reservoir with $D_f=2.5$ and $\theta=0$ for selected values of the Fracture Conductivity ($F_cD$).

The pressure derivative function for the $F_cD=1$ case given in Fig. 4.3 shows numerical instability at $t_D>10^3$, which is caused by the computation of extremely small arguments of the Modified Bessel Functions in the analytical solution for the circular fracture, and the subsequent numerical solution of the system of equations. This computational instability is systematic, and we have also observed it in the constant-pressure solutions — there may be a mechanism to eliminate/mitigate this instability, but for the purpose of the present work we will note this as an anomaly.

At late-times (pseudo-fractal flow), we noted that the inclusion of parallel impermeable boundaries in a fractal reservoir creates power-law signatures in the pressure and pressure derivative functions (Figs. 4.3 and 4.4). As might be expected, the slopes of these power-law signatures are related to the fractal parameters of the reservoir ($\nu$ and $D_f$) used in Eq. 4.5. Although different combinations of the $D_f$ and $\theta$-values can yield the same value of $\nu$, the power-law behavior (specifically, the
slopes of the straight-line in the log-log plot) for the pseudo-fractal flow period depends on the combination of these parameters. We ran several combinations of these parameters and found linear correlations between the fractal parameters and the slope of the pressure derivative function for a fractal reservoir bounded by two parallel impermeable boundaries and present these correlations in Fig. 4.5. Except for the upper end of the trend for the $D_f=3$ case shown in Fig. 4.5, the slope of the pressure derivative function for a fractal reservoir vertically bounded by parallel impermeable boundaries can be generated by the equation:

$$v' = v + \frac{1 - v}{D_f}$$

(Eq. 4.6)

**Figure 4.5** — Correlation between the grouping parameter ($v$) and the slope of the pressure derivative of a fractal reservoir delimitated by two impermeable boundaries.
Influence of the Fractal Parameters in the Pressure and Rate Transient Performance Behaviors

In Chapter III, we showed that a horizontal well intercepting a single hydraulic fracture in a fractal reservoir with low fractal dimension \(D_f\) and/or higher conductivity index \(\theta\) yields lower values of the dimensionless pressure drop during early and intermediate times, compared to a well with the same conditions but placed in a reservoir with higher \(D_f\) and/or low \(\theta\)-values. This trend is reversed at late times, i.e., the drop of pressure is lower in a well intercepting a hydraulic fracture in a fractal reservoir with high \(D_f\) and/or low \(\theta\)-values. Consequently, the higher flow rates at early and intermediate times are observed in wells intercepting a single hydraulic fracture in a fractal reservoir with a low fractal dimension \(D_f\) and/or a high conductivity index \(\theta\) the trend is also reversed at late times. As a summary statement, we expect to observe similar behavior for the multi-fracture horizontal well (MHFW) cases considered in this Chapter.

Sensitivity to the Fractal Dimension \((D_f)\) — Circular Hydraulic Fractures

*Figure 4.6* shows the impact of the fractal dimension \((D_f)\) on the pressure and pressure transient derivative functions for a horizontal well intercepting multiple (9) hydraulic circular transverse fractures with high-conductivity. In this plot, we observe that the higher \(D_f\)-values yield a higher dimensionless pressure drop during early and intermediate times, whereas at late times the pressure drop is lower for high \(D_f\)-values. Consequently, lower \(D_f\)-values yield higher flowrates (and cumulative production) at early and intermediate times and lower flowrates (and cumulative production) at late times (see *Figs. 4.7 and 4.8*). The change in these trends is defined by an intersection/inflection point of all the dimensionless pressure curves located at \(t_D \approx 2\) in *Fig. 4.6*. We have also observed intersection points in the pressure derivative \((t_D \approx 0.5)\), in the dimensionless
rate ($t_D \approx 0.07$ of Fig. 4.7), and in the dimensionless cumulative production curves ($t_D \approx 1.5$ of Fig. 4.8).

Figure 4.6 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting multiple circular transverse hydraulic fractures with high conductivity in a fractal reservoir for selected values of the fractal dimension ($D_f$), assuming infinite thickness. (constant-rate case)

Figure 4.7 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting multiple transverse hydraulic fractures with high conductivity in a fractal reservoir for selected values of the fractal dimension ($D_f$), assuming infinite thickness. (constant-pressure case)
Figure 4.8 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting multiple transverse hydraulic fractures with high conductivity in a fractal reservoir for selected values of the fractal dimension ($D_f$), assuming infinite thickness. (constant-pressure case)

Based on the signatures of the pressure derivatives of the cases presented in Fig. 4.6, we summarized the start and the end of their periods of flow in Table 1. In these cases, the pressure derivative functions provide constant values during "Period 1," which is characteristic of the radial (fracture) flow. The constant value of the pressure derivative is also lower as the $D_f$-value increases. The three cases presented in Fig. 4.6 for "Period 3" yield straight-lines in the pressure derivative function with slope equal to the $v$-parameter, which is characteristic of the pseudo-fractal flow regime. During "Period 2," the pressure derivative curves show power-law behaviors that correspond to the $LRF$ sub-period of flow and are followed by a prolonged transition period (approximately two log-cycles), where this transition period is the result of the interference between hydraulic fractures. For $D_f=2.5$, the $LRF$ sub-period yields a straight-line with a slope of
3:4, whereas the slope for $D_f=2$ is slightly below 1 and above 1 (5:4) for $D_f=1.5$. We consider that the fractional values of the slopes that approach unity (exhibited by the $LRF$ sub-periods for $D_f=2$ and $D_f=1.5$) are the result of the combined effect of the fractal nature of the reservoir and the finite extent of the hydraulic fractures.

We conclude that horizontal wells intercepting multiple fractures in fractal reservoir with low fractal dimension, $D_f$ (consequently low $v$-values) (1) show shorter fracture flow periods and $LRF$ sub-periods and (2) yield steeper straight-lines in the pressure derivative during "Period 2."

The dimensionless flowrate function for $D_f=1.5$ presented in Fig. 4.7 shows a disruptive trend within the period of dimensionless time defined by $10^{-1}$ and $10^1$. We associate this erratic behavior with the computation of the uniform flux solution using the "Fractal Point Source" function for $v$-values less than 2 (see Appendix H).

<table>
<thead>
<tr>
<th>$D_f$</th>
<th>$v$</th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>-0.25</td>
<td>$t_D&lt;2\times10^{-5}$</td>
<td>$2\times10^{-4}&lt;t_D&lt;4\times10^{-2}$ (LRF)</td>
<td>$2\times10^{-2}&lt;t_D$</td>
</tr>
<tr>
<td>2.0</td>
<td>0</td>
<td>$t_D&lt;6\times10^{-4}$</td>
<td>$2\times10^{-3}&lt;t_D&lt;3\times10^{-2}$ (LRF)</td>
<td>$2\times10^{-2}&lt;t_D$</td>
</tr>
<tr>
<td>1.5</td>
<td>0.25</td>
<td>$t_D&lt;2\times10^{-3}$</td>
<td>$5\times10^{-3}&lt;t_D&lt;2\times10^{-2}$ (LRF)</td>
<td>$2\times10^{-2}&lt;t_D$</td>
</tr>
</tbody>
</table>

**Sensitivity to the Conductivity Index (θ) — Circular Fractures**

**Figure 4.9** presents the influence of the conductivity index (θ) on the pressure and pressure transient derivative functions for a horizontal well intercepting multiple circular transverse vertical fractures. Analogous to the sensitivity analysis for the fractal dimension ($D_f$), we observe an intersection/inflection point of the dimensionless pressure and pressure derivative curves located
at $t_D \approx 3$ and $t_D \approx 0.7$, respectively. Similarly, the constant-pressure solutions also show such intersection points. Considering the rate-transient performance behavior shown in Fig. 4.10 this feature is located at $t_D \approx 0.5$, whereas for the dimensionless cumulative production shown in Fig. 4.11 this feature is observed at $t_D \approx 2$, where it is logical that such features would be "delayed" for the cumulative production case considering the nature of a cumulative function.

For dimensionless times ($t_D$) greater than 3, the systems with higher $\theta$-values (consequently, high $\nu$-values) yield larger dimensionless pressure drops. This implies that for dimensionless times greater than 0.5, the systems with higher $\theta$-values yield lower flowrates. These tendencies are opposite for $t_D < 3$ and $t_D < 0.5$ in regard to the pressure and the flowrate, respectively.

Figure 4.9 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting multiple transverse hydraulic fractures with high conductivity in a fractal reservoir for selected values of the conductivity index ($\theta$), assuming infinite thickness. (constant-rate case)
Figure 4.10 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting multiple transverse hydraulic fractures with high conductivity in a fractal reservoir for selected values of the conductivity index ($\theta$), assuming infinite thickness. (constant-pressure case)

Figure 4.11 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting multiple transverse hydraulic fractures with high conductivity in a fractal reservoir for selected values of the conductivity index ($\theta$), assuming infinite thickness. (constant-pressure case)
As in the previous analyses, we have identified the flow periods in Fig. 4.9, using the pressure derivative functions and summed them up in Table 4.2. In this case, the pressure derivative for all cases collapse into a single trend at early times (depicting radial (fracture) flow). The duration of the fracture flow is longer for higher $\theta$-values (higher $v$-values). After the fracture flow period, we observe the appearance of the LRF sub-period, which yields straight-lines with slopes of 3:4, 4:5, and 6:7 for $\theta$-values of 0, 0.5 and 1.3, respectively. At late times (Period 3), the signatures of the pressure derivative functions yield straight-lines on a log-log scale with slopes equal to the $v$-parameter. Similar to the sensitivity analyses presented for the fractal dimension ($D_f$), we identified a transition period between the LRF and the pseudo-fractal flow for all of the cases which illustrate pressure interference between the hydraulic fractures.

In general, we note that the conductivity index ($\theta$) has a similar influence on the pressure transient performance behavior in horizontal wells intercepting multiple fractures in a fractal reservoir as the fractal dimension ($D_f$) — that is, the systems with higher $\theta$-values (consequently lower $v$-values) (1) show longer fracture flow periods and shorter LRF sub-periods and (2) yield steeper straight-lines in the pressure derivative during "Period 2." However, we also observe that the behavioral features caused by the conductivity index ($\theta$) are less dramatic than those caused by the fractal dimension ($D_f$).
Table 4.2 — Definition of flow periods, based on the pressure derivative, for the sensitivity analysis of the conductivity index ($\theta$) on the pressure transient performance behavior for a horizontal well intercepting multiple circular transverse hydraulic fractures.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$v$</th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.25</td>
<td>$t_D &lt; 2 \times 10^{-5}$</td>
<td>$2 \times 10^{-4} &lt; t_D &lt; 4 \times 10^{-2}$ (LRF)</td>
<td>$2 \times 10^{-2} &lt; t_D$</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>$t_D &lt; 1 \times 10^{-4}$</td>
<td>$3 \times 10^{-4} &lt; t_D &lt; 1 \times 10^{-2}$ (LRF)</td>
<td>$2 \times 10^{-2} &lt; t_D$</td>
</tr>
<tr>
<td>1.3</td>
<td>0.25</td>
<td>$t_D &lt; 2 \times 10^{-4}$</td>
<td>$5 \times 10^{-4} &lt; t_D &lt; 8 \times 10^{-3}$ (LRF)</td>
<td>$2 \times 10^{-2} &lt; t_D$</td>
</tr>
</tbody>
</table>

Sensitivity to the Fractal Dimension ($D_f$) — Rectangular Hydraulic Fractures

In Fig. 4.12, we present the sensitivity analysis of the fractal dimension ($D_f$) in the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting multiple (9) rectangular longitudinal hydraulic fractures of high fracture conductivity. Similar to the cases for a horizontal well intercepting multiple circular transverse fractures, we observe an intersection/inflection point of the dimensionless pressure curves located at $t_D \approx 2$ and at $t_D \approx 0.5$ in the dimensionless pressure derivative curves. For $t_D < 2$, the high $D_f$-values (i.e., low $v$-values) yield a higher dimensionless pressure drop, which results in lower flowrates (see Fig. 4.13). These trends are reversed at $t_D$-values above the intersection point, i.e., lower pressure drop and higher flowrates for higher $D_f$-values — consequently, lower $v$-values — for $t_D > 2$. In the case of the cumulative production (Fig. 4.14), we observe three intersection points instead of one: (1) at $t_D \approx 1$ between the $D_f$-curves of 2 and 2.5, (2) at $t_D \approx 7$ between the $D_f$-curves of 2.5 and 1.5, and (3) at $t_D \approx 15$ between the $D_f$-curves of 1.5 and 2.
Figure 4.12 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting multiple longitudinal hydraulic fractures with high conductivity in a fractal reservoir for selected values of the fractal dimension ($D_f$), assuming infinite thickness. (constant-rate case)

Figure 4.13 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting multiple longitudinal hydraulic fractures with high conductivity in a fractal reservoir for selected values of the fractal dimension ($D_f$), assuming infinite thickness. (constant-pressure case)
Figure 4.14 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting multiple longitudinal hydraulic fractures with high conductivity in a fractal reservoir for selected values of the fractal dimension \((D_f)\), assuming infinite thickness. (constant-pressure case)

In Table 4.3 we present a summary of the flow periods that we identified in the signatures of the dimensionless pressure derivative functions for the cases presented in Fig. 4.12. At late times, the pseudo-fractal flow period behaves in the same manner as in the circular fracture cases — i.e., the dimensionless pressure derivative functions yield straight lines with slopes equal to their corresponding \(v\)-value in the log-log plot. For the case of "Period 2," for \(D_f=2.5\) we observe two power-law trends with slopes of approximately of 2:5 and 3:5 which correspond to the ELF and LLF sub-periods. Similarly, during "Period 2," for \(D_f=2\), we observe a half-slope behavior during the ELF sub-period and a power-law behavior with slope slightly below 1 during the LLF sub-period.
We also observed that, for the case defined by $D_f = 1.5$, a power-law trend with a slope approximately of 2:3 that corresponds to the ELF sub-period is observed. We could not (specifically) identify the LLF sub-period for the $D_f = 1.5$ case — instead, we observed that the dimensionless pressure derivative function is higher than the dimensionless pressure function in the $t_D$ range defined by $0.03 < t_D < 1$ (which is a somewhat anomalous behavior). Similar to the circular transverse fractures case, this case has an anomalous flowrate function (Fig. 4.13) in the $t_D$ range defined by $10^{-2} < t_D < 10^2$ - specifically, zero flowrates are observed in this region. The behavior of the flowrate functions in Fig. 4.13 leads to observation of the "wavy" and "flat" portions observed in the dimensionless cumulative production (Fig. 4.14).

<table>
<thead>
<tr>
<th>$D_f$</th>
<th>$v$</th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>-0.25</td>
<td>—</td>
<td>$1 \times 10^{-6} &lt; t_D &lt; 2 \times 10^{-5}$ (ELF)</td>
<td>$2 \times 10^{-2} &lt; t_D$</td>
</tr>
<tr>
<td>2.0</td>
<td>0</td>
<td>—</td>
<td>$1 \times 10^{-6} &lt; t_D &lt; 5 \times 10^{-4}$ (ELF)</td>
<td>$2 \times 10^{-2} &lt; t_D$</td>
</tr>
<tr>
<td>1.5</td>
<td>0.25</td>
<td>—</td>
<td>$1 \times 10^{-6} &lt; t_D &lt; 3 \times 10^{-3}$ (ELF)</td>
<td>$2 \times 10^{-2} &lt; t_D$</td>
</tr>
</tbody>
</table>

Table 4.3 — Definition of flow periods, based on the pressure derivative, for the sensitivity analysis of the fractal dimension ($D_f$) on the pressure transient performance behavior for a horizontal well intercepting multiple rectangular longitudinal hydraulic fractures.

Sensitivity to the Conductivity Index ($\theta$) — Rectangular Fractures

The sensitivity analyses considering variations in the conductivity index ($\theta$) on the pressure and rate-transient performance behaviors are shown in Figs. 4.15 and 4.16, respectively. Again, we observed an intersection point for the dimensionless pressure curves at $t_D \approx 1$ and at $t_D \approx 0.2$ in the dimensionless pressure derivative curves (Fig. 4.15). However, the dimensionless flowrate (Fig. 4.16) and the dimensionless cumulative production (Fig. 4.17) do have exhibit "crossovers," but
not a unique intersection point. Although we do not observe a unique intersection point in the rate-transient performance behavior (Fig. 4.16) and the dimensionless cumulative production (Fig. 4.17), the influence of the conductivity index ($\theta$) is consistent with the circular transverse vertical fractures case. This means that the higher the $\theta$-value, the higher flow dimensionless pressure drop and the lower the dimensionless flowrate at early and inter-mediate times and the trend reverses at late times.

Figure 4.15 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting multiple longitudinal hydraulic fractures with high conductivity in a fractal reservoir for selected values of the conductivity index ($\theta$), assuming infinite thickness. (constant-rate case)
Figure 4.16 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting multiple longitudinal hydraulic fractures with high conductivity in a fractal reservoir for selected values of the conductivity index ($\theta$), assuming infinite thickness. (constant-pressure case)

Figure 4.17 — Log-log plot of the dimensionless cumulative production function for a horizontal well intercepting multiple longitudinal hydraulic fractures with high conductivity in a fractal reservoir for selected values of the conductivity index ($\theta$), assuming infinite thickness. (constant-pressure case)
As in the previous analyses, we identified the periods of flow using the pressure derivative function and summarized these results in Table 4.4. The ELF and the LLF sub-periods are evident and exhibit power-law behaviors (i.e., straight-lines) on the log-log plot of the dimensionless pressure derivative functions for all cases (Fig. 4.15). The ELF sub-periods show straight-lines with slopes of 3:7, 1:3, and 3:5 for $\theta$-values of 0.4, 1 and 4, respectively. For the LLF sub-periods, the straight-lines have slopes of 4:7, 3:5 and 7:8, for the values of the conductivity index ($\theta$) of 0.4, 1 and 4.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\nu$</th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>-0.25</td>
<td>—</td>
<td>$1\times10^6 &lt; t_D &lt; 4\times10^6$ (ELF)</td>
<td>$2\times10^1 &lt; t_D$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$5\times10^4 &lt; t_D &lt; 7\times10^1$ (LLF)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>—</td>
<td>$1\times10^6 &lt; t_D &lt; 1\times10^5$ (ELF)</td>
<td>$4\times10^1 &lt; t_D$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1\times10^4 &lt; t_D &lt; 1\times10^1$ (LLF)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>$3\times10^6 &lt; t_D$</td>
<td>$3\times10^6 &lt; t_D &lt; 2\times10^4$ (ELF)</td>
<td>$10^2 &lt; t_D$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1\times10^3 &lt; t_D &lt; 10^0$ (LLF)</td>
<td></td>
</tr>
</tbody>
</table>

4.3. Summary

In this chapter, we used the principle of superposition in space to extend the models developed in Chapter III for a horizontal well intercepting a single finite conductivity fracture to the so-called "multi-fractured well" case. We have observed that our solutions based on the "fractal point source" approach yield unexpected behaviors in the pressure and rate-transient performance behaviors at intermediate times when $\nu \geq 0$. In Appendix H we present analyses of the application of the traditional line/point source approaches for fractal reservoirs and introduce the concept of fractional integrals as an alternative to model the pressure and rate-transient behavior in fractured
wells in fractal reservoirs. We applied the image well method to model impermeable parallel vertical boundaries in a fractal reservoir.
CHAPTER V

PRESSURE AND RATE-TRANSIENT BEHAVIOR OF DOUBLE POROSITY RESERVOIR WITH TRANSIENT INTERPOROSITY TRANSFER WITH FRACTAL MATRIX BLOCKS

In this chapter we present double porosity models considering matrix blocks with fractal geometry and fracture networks with either radial or fractal fracture networks. The recent development of anomalous diffusion models has opened the possibility of adapting double porosity models to estimate reservoir (and related) parameters for unconventional reservoirs. The primary objective of the development of these models is to provide physical explanations for the anomalous diffusion phenomenon.

Traditionally, fractal diffusivity models have been used to model highly heterogeneous reservoirs (e.g., NFRs and shale reservoirs). In double porosity systems (fractures and matrix blocks), such models are used to describe the flow of fluids within the network of natural fractures (e.g., Chang et al., 1990, Olarewaju, 1996 and Valdes-Perez, 2013). To the best of our knowledge, no attempt to apply the fractal theory to model the flow from the matrix blocks to the fractures has been made. Given that the porosity of the matrix blocks in shale oil/gas reservoirs is a combination of multiple organic and inorganic porosities, we believe that it is appropriate (in terms of geological evidence) to model the matrix blocks as fractal objects.

5.1. Theory of the Double Porosity Models with Transient Interporosity Transfer

The use of double porosity models to depict the pressure-transient behavior of NFRs has been studied since the 1960s. These models idealize reservoirs with multiple porous media (e.g., micro/nano fractures, micro/nano vugs, matrix blocks, etc.) as double porosity reservoirs (fracture network and matrix blocks). The double porosity models can be classified as transient or pseudosteady-state interporosity transfer models. In this chapter, we will focus only on the transient interporosity transfer condition.

In 1976, de Swaan presented a double porosity analytical model with transient interporosity transfer. For this model, de Swaan established a radial fracture network and considered either slabs or spherical matrix blocks. The transient interporosity transfer was modeled with a convolution integral of the flux from the matrix blocks weighted by the derivative of the pressure in the radial network of fractures with respect to time. The flux of the matrix blocks was determined by solving the diffusivity equation for either linear or spherical systems considering constant-pressure at the matrix-fracture interface (see Appendix E for detailed derivation).

The periods of flow exhibited by the double porosity model with transient interporosity transfer conditions are (see Fig. 5.1):

- Period 1 — (fractal fracture network flow): This period exhibits a flat slope (radial flow) in the pressure derivative function that represents that the flow is dominated by the properties of the radial fracture network.

- Period 2 — (interaction between the fractal fracture network and the matrix blocks): This period can be subdivided into three sub-periods: (1) a transition sub-period
dominated by the early interaction of the matrix blocks, (2) a pseudoradial flow, and (3) a transition sub-period, dominated by the closed boundaries of the matrix blocks.

- **Period 3** — (single porosity fractal reservoir flow): In this period of flow, the double porosity reservoir has achieved equilibrium and behaves as a single porosity reservoir. However, this period is dominated by the geometry of the fracture

![Figure 5.1](image)

**Figure 5.1** — Schematics of the pressure transient performance behavior of a double porosity reservoir considering slab matrix blocks, and spherical matrix blocks.

In 1982, three independent research groups modified de Swaan's model and obtained asymptotic solutions in the real domain to be used for well-test analysis (Cinco-Ley et al., Serra et al., and Streltsova). In this work, we will use the model presented by Cinco-Ley et al. (1982) to include the effects of a fractal matrix. A summary of the development of this model is presented in Appendix E.

### 5.2. Models Assumptions

In this chapter, we present models that consider two porous media: (1) matrix blocks and (2) a fracture network. All the models consider the matrix blocks to be fractal, whereas the fracture
network is considered to be either radial or fractal. The assumptions of the models are summarized in Table 5.1.

<table>
<thead>
<tr>
<th>Medium</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>• Flow to the wellbore occurs only through the fracture network (radial or fractal).</td>
</tr>
<tr>
<td></td>
<td>• Pressure-squared gradients are negligible.</td>
</tr>
<tr>
<td></td>
<td>• Uniform initial pressure, ( p_i )</td>
</tr>
<tr>
<td></td>
<td>•Single slightly-compressible fluid flow with constant compressibility, ( c_o ), and constant viscosity, ( \mu )</td>
</tr>
<tr>
<td>Fractal matrix blocks</td>
<td>• Single size of the matrix blocks of ( D_{ma} )-dimension</td>
</tr>
<tr>
<td></td>
<td>• Flow obeys modified Darcy's Law for fractal systems of ( \theta_{ma} )-conductivity index. This implies that both porosity and permeability varies according to power law functions. Reference matrix porosity, ( \phi_{ma} ), and matrix permeability, ( k_{ma} ), are considered.</td>
</tr>
<tr>
<td></td>
<td>• The matrix blocks have constant compressibility, ( c_{ma} ).</td>
</tr>
<tr>
<td>Radial fracture network</td>
<td>• Unbounded radial network</td>
</tr>
<tr>
<td></td>
<td>• Flow obeys Darcy's Law.</td>
</tr>
<tr>
<td></td>
<td>• The radial fracture network has constant properties: compressibility, ( c_{fb} ), permeability, ( k_{fb} ), and porosity, ( \phi_{fb} ).</td>
</tr>
<tr>
<td>Fractal fracture network</td>
<td>• Unbounded fractal fracture network of ( D_f )-dimension</td>
</tr>
<tr>
<td></td>
<td>• Flow obeys modified Darcy's Law for fractal systems of ( \theta )-conductivity index with reference fractured bulk porosity, ( \phi_{fb} ), and matrix of the fractured bulk, ( k_{fb} ), are considered.</td>
</tr>
<tr>
<td></td>
<td>• The fractal fracture network has constant compressibility, ( c_{fb} ).</td>
</tr>
</tbody>
</table>

5.3. Development of the Models

**Fluid Transfer Function considering Fractal Matrix Blocks**

The fluid transfer function from fractal matrix blocks to the fracture network, \( F(\eta_{maD}, h_{maD}, I_D) \), is defined by the geometry and properties of the matrix blocks. To derive this function, we have used an analogous procedure to the one presented by de Swaan (1976) for repetitive elements of matrix blocks (see Appendix E for details). We have investigated two scenarios: (1) closed matrix blocks and (2) infinite-acting matrix blocks. For the closed matrix blocks case, this function is defined by:

\[
F(\eta_{maD}, h_{maD}, I_D) = \frac{1}{\sqrt{\eta_{maD}}} \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma}/2} L^{-1} \left[ \frac{1}{\sqrt{u}} \left[ \frac{\Gamma[\nu_{ma}]\Gamma[1-\nu_{ma}]\Gamma[\nu_{ma}-1] - 2K_{ma} - 1(\xi)}{2K_{ma} \xi + \Gamma[\nu_{ma}]\Gamma[1-\nu_{ma}]\Gamma[\nu_{ma}-1] \xi} \right] \right],
\]

(5.1)
where \( v_{ma} = \frac{1 - \beta_{ma}}{\theta_{ma} + 2} \), and:

\[
\xi = \frac{2}{\theta_{ma} + 2} \left[ \frac{h_{ma} D}{2} \right]^{\theta_{ma} + 2}/2 \sqrt{\frac{u}{\eta_{ma} D}}. \tag{5.2}
\]

\( \beta_{ma} \) is the spatial dimension of the matrix. It is defined as \( \beta_{ma} = D_{ma} - \theta_{ma} - 1 \). The expression given by Eq. 5.1 applies for \( v_{ma} \neq 0 \). If \( v_{ma} = 0 \), the following expression should be used:

\[
F(\eta_{ma} D, h_{ma} D, t_D) = \frac{1}{\sqrt{\eta_{ma} D}} \left[ \frac{h_{ma} D}{2} \right]^{\theta_{ma} / 2} L^{-1} \left[ \frac{1}{\sqrt{u}} I \sqrt{v_{ma} - 1} [\xi] \right]. \tag{5.3}
\]

Due to the extremely low permeability of the matrix blocks in shale oil/gas reservoirs, we modeled the behavior of the matrix blocks as infinite-acting (fully transient) media. The fluid transfer function for infinite-acting matrix blocks is:

\[
F(\eta_{ma} D, h_{ma} D, t_D) = \frac{1}{\sqrt{\eta_{ma} D}} \left[ \frac{h_{ma} D}{2} \right]^{\theta_{ma} / 2} L^{-1} \left[ \frac{1}{\sqrt{u}} K \sqrt{v_{ma} - 1} [\xi] \right]. \tag{5.4}
\]

The dimensionless variables for the models presented in this chapter are summarized in Table 5.2.

**Asymptotic behaviors**

As pointed out by Cinco-Ley et al. (1982), the flow from the matrix blocks to the fracture network is linear at early and intermediate times (Periods 1 and 2) regardless of the geometry of the matrix block. During these periods of flow, the behavior of the fluid transfer function for closed and infinite-acting matrix blocks are the same. Therefore, the fluid transfer function for all cases at early and intermediate times can be simplified to:
At late times, the fluid transfer function for the infinite-acting matrix blocks case can be simplified to:

\[
F(\eta_{maD}, h_{maD}, t_D) = \frac{1}{\sqrt{\pi \eta_{maD}}} \left[ \frac{h_{maD}}{2} \right]^{\eta_{maD}/2} t_D^{-1/2} \tag{5.5}
\]

\[
F(\eta_{maD}, h_{maD}, t_D) = \left[ \frac{\theta_{ma} + 2}{\eta_{maD}} \right]^{\eta_{maD}} t_D^{-\eta_{maD}} \tag{5.6}
\]

**Table 5.2** — Dimensionless variables for the proposed double porosity reservoir models

<table>
<thead>
<tr>
<th>Dimensionless Variable</th>
<th>Definition for the radial fracture network model</th>
<th>Definition for the double fractal model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure in the fracture network</td>
<td>[ p_{rfnD,cr}(r_D \cdot t_D) = \frac{k_{fb} h_i</td>
<td>p_i - p_{rfn}(r, t)</td>
</tr>
<tr>
<td>Pressure in the fractal matrix blocks</td>
<td>[ p_{maD,cr}(r_D \cdot t_D) = \frac{k_{fb} h_i</td>
<td>p_i - p_{ma}(r, t)</td>
</tr>
<tr>
<td>Time</td>
<td>[ t_Dr = \frac{k_{fb}}{[\phi_r]_l w^2} \frac{t}{\mu^2} ]</td>
<td>[ t_Df = \frac{k_0 fb}{[\phi_r]_l w^{\theta + 2}} \frac{t}{\mu^2} ]</td>
</tr>
<tr>
<td>Radius</td>
<td>[ r_D = \frac{r}{r_w} ]</td>
<td>[ r_D = \frac{r}{r_w} ]</td>
</tr>
<tr>
<td>Storativity ratio</td>
<td>[ \omega = \frac{\phi_{fb} c_{fb}}{[\phi_r]_l} ]</td>
<td>[ \omega = \frac{\phi_{fb} c_{fb}}{[\phi_r]_l} ]</td>
</tr>
<tr>
<td>Fracture area</td>
<td>[ A_{fD} = A_{maD} h_{ma} ]</td>
<td>[ A_{fD} = A_{maD} h_{ma} t_w^\theta ]</td>
</tr>
<tr>
<td>Hydraulic diffusivity of the fractal matrix blocks</td>
<td>[ \eta_{maD} = \frac{k_{0ma} [c_{fb}]<em>l}{c</em>{ma} \phi_{0ma} k_{fb}^2} ]</td>
<td>[ \eta_{maD} = \frac{k_{0ma} [c_{fb}]<em>l}{c</em>{ma} \phi_{0ma} k_{fb}^2} ]</td>
</tr>
<tr>
<td>Matrix block size</td>
<td>[ h_{maD} = \frac{h_{ma}}{r_w} ]</td>
<td>[ h_{maD} = \frac{h_{ma}}{r_w} ]</td>
</tr>
</tbody>
</table>

**Double Porosity Model considering Radial Fracture Network and Fractal Matrix Blocks**

To model the flow within the radial fracture network, we have considered the diffusivity equation presented by Cinco-Ley et al. (1982). This expression in its dimensionless form is given by:
The solution of Eq. 5.1 in the Laplace domain considering constant flow rate at the wellbore and infinite fracture network is given by:

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial \hat{p}_{rfn,cr}}{\partial r_D} \right] = \omega \frac{\partial \hat{p}_{rfn,cr}}{\partial t_D} + \left[ 1 - \omega \right] \frac{A \gamma_{maD}}{h_{maD}} \int_0^t \frac{\partial \hat{p}_{rfn,cr}}{\partial t_D} F(\eta_{maD}, h_{maD}, t_D - \tau) d\tau \quad (5.7)
\]

To obtain the performance of Eq. 5.8 in the real domain, a numerical inversion should be used. Additionally, we have derived asymptotic solutions, evaluated at the wellbore, for the main periods of flow depicted in this model. These solutions are summarized in Table 5.3.

**Analogy to the Anomalous Diffusion Model**

The 2D diffusivity model presented by Raghavan (2012) included the anomalous diffusion effects by using a version of Darcy's Law expressed in terms of a fractional derivative. The definition of the fractional derivative involves a convolution term where the derivative of the pressure with respect to time is weighted by a power-law function of the time. A similar diffusivity model can be obtained from the double porosity model considering a radial fracture network and fractal matrix blocks. Such a model is obtained by neglecting the storativity ratio (\(\omega=0\)) and considering
the asymptotic behavior for late times of the fluid transfer function for infinite-acting fractal matrix blocks with positive \( v_{ma} \)-values (Eq.5.6). The resulting diffusivity equation is:

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial \tilde{p}_{rfinD,cr}}{\partial r_D} \right] = \frac{\alpha_1}{\Gamma[1-v_{ma}]} \int_0^{t_{Dr}} \frac{\partial \tilde{p}_{rfinD,cr}}{\partial \tau} [t_{Dr} - \tau]^{-v_{ma}} d\tau \nonumber \tag{5.10}
\]

where:

\[
\alpha_1 = \frac{AfDi_{maD}^{1-v_{ma}}}{2} \left[ \theta_{ma} + 2 \right]^{1-2v_{ma}} \Gamma[1-v_{ma}] \left[ h_{maD} \right]^{\beta_{ma}-1} \nonumber \tag{5.11}
\]

To provide a model that is comparable to the anomalous diffusion model presented by Raghavan (2012a), we have considered a time-dependent inner boundary condition modeled by a power-law function mathematically defined by:

\[
\left[ r_D \frac{\partial \tilde{p}_{jD}}{\partial r_D} \right]_{r_D=1} = -t_{Dr}^{v_0-v_{ma}} \nonumber \tag{5.12}
\]

where \( v_0 \) is an arbitrary reference exponent. We have assumed a unit value of this parameter.

The line source approximation in the Laplace domain of Eq. 5.10 considering variable flow rate (Eq. 5.12) and infinite radial fracture network is given by:

\[
\tilde{p}_{rfinD,cr}(r_D,u) = \frac{\Gamma[2-v_{ma}]K_0 [r_D \sqrt{ug(u)}]}{u^{2-v_{ma}}} \nonumber \tag{5.13}
\]

where:

\[
g(u) = \alpha_1 u^{v_{ma}-1} \nonumber \tag{5.14}
\]
Table 5.3  — Asymptotic constant-rate solutions in the real domain for the double porosity model considering radial fracture network

<table>
<thead>
<tr>
<th>Asymptotic solution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Early times</strong></td>
</tr>
<tr>
<td>$P_{wD,cr}(t_{Dr}) = \frac{1}{2} \ln \left[ \frac{\exp[\gamma]}{\omega} t_{Dr} \right]$</td>
</tr>
<tr>
<td><strong>Intermediate times</strong></td>
</tr>
<tr>
<td>$P_{wD,cr}(t_{Dr}) = \frac{1}{4} \ln[t_{Dr}] + \frac{1}{2} \ln \left[ \exp[\gamma/2] \frac{h_{maD}}{[1-\omega]} \frac{h_{maD}}{A_{fD}} \left[ \frac{\theta_{ma}}{2} \frac{\eta_{maD}}{2} \right]^{-1/2} \right]$</td>
</tr>
<tr>
<td><strong>Late times</strong></td>
</tr>
</tbody>
</table>
| $P_{wD,cr}(t_{Dr}) = \frac{1}{2} \ln[\exp[\gamma]]_{Dr}$  
(for closed matrix blocks) |
| $P_{wD,cr}(t_{Dr}) = \frac{\nu_{ma}}{2} \ln[\exp[\gamma]]_{Dr} \left[ \frac{A_{fD} \left[ \theta_{ma} + 2 \right] \frac{1-2\nu_{ma}}{\Gamma[1-\nu_{ma}]} \frac{2}{\Gamma[\nu_{ma}]} \left[ \frac{\beta_{ma} + 1}{2} \right] \right]_{maD}$  
(for infinite-acting matrix blocks) |

**Double Porosity Model considering Fractal Fracture Network and Fractal Matrix Blocks (Double Fractal Model)**

The motivation of the double fractal model is to provide a reservoir model to depict the transient performance behavior of highly heterogeneous multi-porosity systems, such as shale oil and shale gas reservoirs. The objective of this model is to take into account the randomness and heterogeneity of the natural fractures and the matrix blocks.

The presence of natural fractures in shale reservoirs is significantly lower than in carbonate reservoirs. Therefore, the porosity of the natural fractures in shale reservoirs is very small, and the storativity ratio would be negligible. However, it is important to consider the presence of the natural fractures because it has a significant impact in the flow of fluids towards hydraulic fractures and/or the wellbore. To model the flow within the fractal fracture network, we have considered the diffusivity equation presented by Valdes-Perez (2013). Such an equation in its dimensionless form is defined by:
The solution of Eq. 5.15 in the Laplace domain considering constant-rate at the wellbore and infinite fracture network is given by:

\[
\frac{1}{R_D^{D_f} \partial R_D} \left[ R_D^\beta \hat{\partial} p \hat{f}nD_{cr} \right] = \omega \frac{\hat{\partial} p \hat{f}nD_{cr}}{\partial t_D} + [1 - \omega] \frac{A_f D \eta_{maD}}{h_{maD}} \int_0^{D_f} \frac{\hat{\partial} p \hat{f}nD_{cr}}{\partial \tau} F(\eta_{maD} \cdot h_{maD} \cdot t_{D_f} - \tau) d\tau.
\]

\[
(5.15)
\]

where \( v = \frac{1 - \beta}{\theta + 2} \). The transfer function \( f(u) \) has the same shape as the one presented in Eq. 5.9. For this model, we have considered only the behavior of the fluid transfer function of the matrix blocks as infinite-acting (Eq. 5.4) due to their extremely low permeability in shale oil/gas reservoirs.

For systems with negligible storativity ratio and considering the fluid transfer function at intermediate times (Eq. 5.5) the model given by Eq. 15 can be simplified to:

\[
\frac{1}{R_D^{D_f} \partial R_D} \left[ R_D^\beta \hat{\partial} p \hat{f}nD_{cr} \right] = \frac{\alpha_0}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} \frac{\hat{\partial} p \hat{f}nD_{cr}}{\partial \tau} [t_{D_f} - \tau]^{-1/2} d\tau \quad \ldots \quad (5.17)
\]

where:

\[
\alpha_0 = \frac{A_f D \sqrt{\eta_{maD}}}{2} \left[ \frac{h_{maD}}{2} \right]^{-\theta_{maD}/2 - 1}
\]

\[
(5.18)
\]
A similar model for radial fracture networks and Euclidean matrix blocks was developed by Cinco-Ley et al. (1982).

For late times, substituting Eq. 5.6 in Eq. 5.14 and neglecting storativity ratio, the following model is derived:

\[
\frac{1}{R_D^{D_f^{-1}}} \frac{\partial}{\partial R_D} \left[ R_D^\theta \frac{\partial p_{ffnD,cr}}{\partial R_D} \right] = \alpha_1 t_D^D \frac{\partial p_{ffnD,cr}}{\partial \tau} \left[ t_Df - \tau \right]^{-\nu_m} d\tau. \tag{5.19}
\]

Eq. 5.19 has the same shape as the models assuming the anomalous diffusion phenomena (Camacho-Velazquez et al., 2008 and Raghavan, 2012a). This model relates its parameters to the physical properties related to the geology of the matrix blocks. The asymptotic constant-rate solutions in the real domain of Eq. 5.17 and Eq. 5.19 are summarized in Table 5.4.

| Asymptotic solution | \( P_{wD,cr}(t|D_f) = \frac{1}{\Gamma([5/4] \sqrt{\alpha_0} t_Df)} \) |
|---------------------|-------------------------------------------------------------|
| Early-Intermediate times | \( p_{wD,cr}(t|D_f) = \frac{[\theta + 2v^{-1}] \Gamma(v)}{\alpha_0^v [1 - v] \Gamma(v/2 + 1)} t_Df^{v/2} \) (\( v > 0 \)) |
| Late-Intermediate times | \( p_{wD,cr}(t|D_f) = \frac{[\theta + 2v^{-1}] \Gamma(v)}{\alpha_1^v [1 - v] \Gamma(v/ma + 1)} t_Df^{vma} \) (\( v > 0 \)) |
| Late times | \( p_{wD,cr}(t|D_f) = \frac{[\theta + 2v^{-1}] \Gamma(v)}{\alpha_1^v [1 - v] \Gamma(v/ma + 1)} t_Df^{vma} \) (\( v > 0 \)) |

**Constant Pressure Solutions**

The rate-transient performance behavior of the models presented in this chapter is obtained using Dunhamel's principle, which relates the constant-rate and constant-pressure solutions in the Laplace domain, as presented in Chapter I.
5.4. Results and Discussion

In this section, we present the sensitivity analyses of the characterizing parameters involved in the models presented in Section 5.3. To perform such analyses, we numerically inverted the solutions in the Laplace domain using the Gaver-Wynn-Rho algorithm implemented in Mathematica.

Double Porosity Model considering Radial Fracture Network and Fractal Matrix Blocks

Sensitivity to the $D_{fma}$-Parameter: Closed Matrix Blocks

**Fig. 5.2** exhibits the effect of the fractal dimension of the matrix ($D_{fma}$) in the pressure transient performance behavior of a double porosity reservoir considering a radial fracture network (Eq. 5.8) and closed matrix blocks (Eq. 1 for $D_{fma} \neq 2$ and Eq. 3 for $D_{fma} = 2$). This plot shows no evident differences among the pressure-transient signatures when the fractal dimension of the matrix ($D_{fma}$) is varied. However, the influence of the $D_{fma}$-parameter is clear in the pressure derivative function. For all cases, the pressure derivative function exhibits two radial flows. The first one (fracture network flow) occurs at early times and represents the geometry and the expansion within the radial fracture network. The second radial flow (single porosity reservoir flow) occurs at late times and represents that the radial fracture network and the matrix blocks behave as a single porosity system where the geometry of the fracture network prevails. At intermediate times (interaction between the fracture network and the matrix blocks), we have observed that the interaction between the fracture network and the matrix blocks and the boundary effects of the matrix blocks occur faster when the matrix blocks have higher values of the fractal dimension (keeping unchanged the other parameters). This model can reproduce the behavior of the transfer functions.
developed by de Swaan (1976) when the conductivity index of the matrix is zero (i.e., \( \theta_{ma}=0 \)) and \( D_{fma}=1 \) for slabs and \( D_{fma}=3 \) for spheres.

Figure 5.2 — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir considering radial fracture network and closed fractal matrix blocks for selected values of the fractal dimension of the matrix (\( D_{fma} \)).

Fig. 5.3 shows the rate-transient case for the example presented in Fig. 5.2. Based on the rate derivative function in this plot, we observed that is not possible to distinguish the first period of flow (i.e., the radial fracture network flow). However, the acceleration effect of the fractal dimension of the matrix (\( D_{fma} \)) is also observed in this plot, i.e., the higher the \( D_{fma} \)-value, the sooner the boundary effects of the matrix will appear.
Sensitivity to the $\theta_{ma}$-Parameter: Closed Matrix Blocks

Fig. 5.4 presents the sensitivity analysis of the conductivity index of the matrix ($\theta_{ma}$) in the pressure transient performance behavior for the radial fracture network and closed matrix blocks case (Eq. 5.1 for $\theta_{ma} \neq 1$ and Eq. 3 for $\theta_{ma} = 1$). For the $\theta_{ma}$-parameter instance, we have observed two phenomena: (1) there is a delay in the interaction between the fracture network when the matrix blocks have higher $\theta_{ma}$-values and (2) the boundary effects of the matrix blocks occur at earlier times for high values of the conductivity index of the matrix. High $\theta_{ma}$-values represent that the permeable sites inside the matrix blocks are poorly connected.

Based on Fig. 5.5, we concluded that better connected matrix blocks (lower $\theta_{ma}$-values) yield higher flow rates. Similar to the fractal dimension of the matrix case, the boundary effects of the matrix appear at earlier times for high $\theta_{ma}$-values.
Figure 5.4 — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir considering radial fracture network and closed fractal matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$).

Figure 5.5 — Log-log plot of the rate-transient and rate derivative performance behaviors of a double porosity reservoir considering radial fracture network and closed fractal matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$).
Sensitivity to the $D_{fma}$-Parameter: Infinite-Acting Matrix Blocks

In Fig. 5.6, we present the sensitivity analysis of the fractal dimension of the matrix ($D_{fma}$) in the pressure transient performance behavior for the radial fracture network considering infinite-acting matrix blocks (Eq. 5.4). This plot shows that fractal matrix blocks with lower $D_{fma}$-values yield a higher drop of pressure of the double porosity system. The pressure derivative function in this plot exhibits the same behavior as the closed fractal matrix case at early times (fracture network flow) and intermediate times (interaction between the fracture network and the matrix blocks). However at late times, the flow is dominated by the fluid transfer from the matrix to the fractures. For $D_{fma}$-values higher than two, the pressure derivative function yields a negative slope equal to the $v_{ma}$-parameter. If $D_{fma}$ is lower than two (and higher than one), the pressure derivative function stabilizes to a constant value equal to $v_{ma}/2$.

The rate-transient performance behavior in Fig. 5.7 shows that fractal matrix blocks with higher $D_{fma}$-values yield higher flow rates. In addition, the rate derivative function has a response at early and intermediate times similar to the one observed in the closed matrix blocks case. At late times, the infinite-acting nature of the matrix blocks dominates.
Figure 5.6 — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir considering radial fracture network and infinite-acting matrix blocks for selected values of the fractal dimension of the matrix ($D_{fma}$).

Figure 5.7 — Log-log plot of the rate-transient and rate derivative performance behaviors of a double porosity reservoir considering radial fracture network and infinite-acting fractal matrix blocks for selected values of the fractal dimension of the matrix ($D_{fma}$).
Sensitivity to the $\theta_{ma}$-Parameter: Infinite-Acting Matrix Blocks

The plot in Fig. 5.8 presents the sensitivity analysis of the conductivity index of the matrix ($\theta_{ma}$) in the pressure transient performance behavior of a double porosity reservoir considering radial fracture network and infinite-acting matrix blocks. The example shown in this plot considers a small storativity ratio ($\alpha=10^{-10}$). Therefore, the flat slope of the pressure derivative function for dimensionless times from $10^0$ to $10^7$ corresponds to the interaction between the fracture network and the matrix blocks. Fig. 5.8 shows that better connected permeable sites inside matrix blocks (low $\theta_{ma}$ -values) yield a lower drop of pressure. In the rate-transient case, this implies that low $\theta_{ma}$ -values yield higher flow-rates (see Fig. 5.9).

In Fig. 5.8, we observe that poorly connected permeable sites inside matrix blocks (high $\theta_{ma}$ -values) accelerate the appearance of the single system behavior period. Similar to the fractal dimension case, the pressure derivative function yields a negative slope equal to the $v_{ma}$-parameter for $\theta_{ma}$ -values lower than one. If $\theta_{ma}$ is greater than one, the pressure derivative function stabilizes to a constant value equal to $v_{ma}/2$. Therefore, by increasing the $v_{ma}$ -value to very high values, the pressure derivative function approaches to a constant value of 0.5.
Figure 5.8 — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir considering radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$).

Figure 5.9 — Log-log plot of the rate-transient and rate derivative performance behaviors of a double porosity reservoir considering radial fracture network and infinite-acting fractal matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$).
Wellbore Storage Effects

The wellbore storage effects for all the models presented in this section are incorporated using the standard scheme presented in Chapter I. **Fig. 5.10** and **Fig. 5.11** exhibit the impact of the wellbore storage in the pressure transient behavior presented in **Fig. 5.6** and **Fig. 5.8**, respectively. These plots show that wellbore storage effects can hinder the radial fracture flow when the storativity ratio is relatively high (**Fig. 5.10**) and shorten the transition period between early and intermediate times when the storativity ratio is relatively low (**Fig. 5.11**).

**Analogy to the Anomalous Diffusion Model: Sensitivity to the $D_{fma}$-Parameter**

**Fig. 5.12** shows the influence of the fractal dimension of the matrix ($D_{fma}$) in the pressure and pressure derivative functions of the double porosity model considering radial fracture network and infinite-acting fractal matrix blocks, assuming the time-dependent inner boundary condition.

**Figure 5.10** — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir considering radial fracture network and infinite-acting fractal matrix blocks for selected values of the fractal dimension of the matrix ($D_{fma}$).
Figure 5.11 — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir considering radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$) considering wellbore storage.

Figure 5.12 — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the fractal dimension of the matrix ($D_{fra}$).
We have observed that none of the cases in Fig. 5.12 show the characteristic signature of radial flow (0-slope of the pressure derivative function), even though the geometry of the fracture network was assumed to be radial. Instead, the responses of the pressure and pressure derivative functions of this model yield power-law behaviors equal to the exponent of the inner boundary condition as shown by the $\beta$-pressure derivative function in Fig. 5.13. We conclude that the higher the $D_{fma}$-parameter (lower $v_{ma}$) the steeper the log-log straight-line.

Fig. 5.14 presents the constant-pressure version for the example presented in Fig. 5.12. The rate and rate derivative functions yield power-law behaviors whose slope in the log-log plot is equal to the negative value of slope in the constant-rate case (i.e., $v_{ma} - 1$). This is confirmed by the $\beta$-rate derivative (Fig. 5.15). We have observed that at dimensionless times greater than $10^{-1}$, the fractal matrix blocks with higher $D_{fma}$-values yield higher flow rates. However, an unrealistic scenario is observed at earlier times ($t_{D} < 10^{-1}$), where the highest flow-rates are yielded by fractal matrix blocks with lower $D_{fma}$-values. The reason why this scenario is unrealistic is because high $D_{fma}$-values represent, basically, more permeable sites within the matrix blocks.

Figure 5.13 — Log-log plot of the $\beta$-pressure derivative function of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting fractal matrix blocks for selected values of the fractal dimension of the matrix ($D_{fma}$).
**Figure 5.14** — Log-log plot of the rate-transient and rate derivative performance behaviors of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the fractal dimension of the matrix ($D_{fma}$).

**Figure 5.15** — Log-log plot of the $\beta$-rate derivative function of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the fractal dimension of the matrix ($D_{fma}$).
Analogy to the Anomalous Diffusion Model: Sensitivity to the $\theta_{ma}$-Parameter

In Fig. 5.16, we present the sensitivity analysis of the conductivity index of the matrix ($\theta_{ma}$). At late times ($t_D > 10^9$), this plot shows a behavior similar to the one observed for the $D_{fma}$-parameter, i.e., the signatures of the pressure and pressure derivative functions yield power-law behaviors equal to the exponent of the inner boundary condition (i.e., $1 - \nu_{ma}$). This is confirmed by the $\beta$-pressure derivative function presented in Fig. 5.17. At early and intermediate times ($t_D < 10^9$), the $\beta$-pressure derivative function exhibits a variable behavior for all the cases presented. We concluded that the better connected permeable sites inside the matrix (i.e., lower $\theta_{ma}$- and consequently $\nu_{ma}$-values), the steeper the log-log straight-line of the pressure and pressure derivative functions.

Figure 5.16 — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$).
Figure 5.17  — Log-log plot of the $\beta$-pressure derivative function of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$).

Based on Fig. 5.18, we conclude that better connected permeable sites inside the matrix blocks (low $\theta_{ma}$-values) yield higher flow rate. Similar to the fractal dimension of the matrix ($D_{fma}$), the $\beta$-rate derivatives presented in Fig. 5.19 show that rate and rate derivative functions exhibited in Fig. 5.18 yield power-law behaviors with a slope equal to $\nu_{ma} - 1$, for $\theta_{ma}$-values equal to 5 (only after $t_D > 10^7$) and 10. The non-constant behavior of the $\beta$-rate derivative for $\theta_{ma}$ equal to 100 indicates that the rate-transient for this case does not correspond to a power-law behavior.
Figure 5.18 — Log-log plot of the rate-transient and rate derivative performance behaviors of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$).

Figure 5.19 — Log-log plot of the $\beta$-rate derivative function of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{ma}$).
Given that the scheme to include the wellbore storage effects was developed assuming a constant inner boundary condition, it is not appropriate to use it to models with a variable inner boundary condition.

**Double Porosity Model considering Fractal Fracture Network and Infinite-Acting Fractal Matrix Blocks (Double Fractal Model)**

In general, the most significant parameters in a diffusivity model for a fractal object are the fractal dimension and the conductivity index. The fractal dimension can get values from one to three, whereas the conductivity index can get values higher than zero. Both parameters have a similar influence in the slope of the pressure derivative function. Therefore, we will restrict our sensitivity analyses only to the conductivity indexes (both, fractures and matrix) and consider that the fractal dimension is equal to three for both media.

**Sensitivity to the \( \theta \)-Parameter**

**Fig. 5.20** presents the sensitivity analysis of the conductivity index of the fracture network (\( \theta \)) in the pressure transient performance behavior of the double fractal model, considering both cases, with and without wellbore storage effects. **Fig. 5.21** shows the \( \beta \)-pressure derivative of the cases presented in **Fig. 5.20**. In the no-wellbore storage case, we observe that there are two periods of flow separated by a smooth transition period. The first period corresponds to the interaction between the fracture network and the matrix blocks, and the second period corresponds to single system behavior. In both figures, the first period is not observed when the wellbore storage effects are taken into account. This leads to the hypothesis that for practical applications in pressure transient analysis, the only period of flow that could be observed is the single system behavior.
period, and the fluid transfer function of the matrix could be reduced to the expression given by Eq. 5.5.

**Figure 5.20** — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta$).

**Figure 5.21** — Log-log plot of the $\beta$-pressure derivative function of a double porosity reservoir with time-dependent inner boundary, considering radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta$).
In Fig. 5.22, we observe that a better connected fractal fracture network yield higher flow rates (i.e., the lower the $q$-value, the higher flow rate). Given that the standard (constant) wellbore storage effects do not affect the rate transient behavior, it is possible to observe the two periods of flow of the transient behavior of the double fractal reservoir. As in the pressure-transient case, the two periods of flow can be detected in the $\beta$-rate derivative presented in Fig. 5.23.

![Log-log plot of the rate-transient and rate derivative performance behaviors of a double porosity reservoir considering fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta$).](image)

**Figure 5.22** — Log-log plot of the rate-transient and rate derivative performance behaviors of a double porosity reservoir considering fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta$).
Figure 5.23  — Log-log plot of the $\beta$-rate derivative function of a double porosity reservoir considering fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta$).

Sensitivity to the $\theta_{ma}$-Parameter

Fig. 5.24 shows the influence of the conductivity index of the matrix ($\theta_{ma}$) in the pressure transient performance behavior of the double fractal model, considering both cases, with and without wellbore storage effects. Fig. 5.25 shows the $\beta$-pressure derivative of the cases presented in Fig. 24. Similar to the case of the conductivity index of the fracture network ($\theta$), we observe that in the no-wellbore storage case two periods of flow are separated by a smooth transition period. In this case, the first period of flow is also hindered by wellbore storage effects. The main difference with the $\theta$-case (Fig. 5.21) is that the first period of flow in Fig. 5.24 is the same for all the $\theta_{ma}$-values. Such a period of flow is shorter, and the appearance of the transition period occurs faster for higher $\nu_{ma}$-values (higher $\theta_{ma}$-values). The second period of flow in Fig. 5.24 yields a straight-line with slope equal to the product $\nu_{ma}\nu$. This is confirmed by their $\beta$-pressure derivative plot (Fig. 5.25).
Figure 5.24 — Log-log plot of the pressure-transient and pressure derivative performance behaviors of a double porosity reservoir considering fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{\text{mat}}$).

Figure 5.25 — Log-log plot of the $\beta$-pressure derivative function of a double porosity reservoir considering fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{\text{mat}}$).
**Fig. 5.26** shows that better connected permeable sites inside the fractal matrix blocks yield higher flow rates (*i.e.*, the lower the $\theta_\text{mu}$-value, the higher flow rate). **Fig. 5.27** shows two periods of flow for each one of the cases presented in **Fig. 5.27**. We observed that the first period of flow corresponds to the interaction between the fracture network and the matrix blocks and it is the same for all the cases. The behavior of the second period is dominated by the $\theta_\text{mu}$-value, and we have observed that the higher the conductivity index of the matrix, the sooner the second period appears.

![Sensitivity Analysis of the Conductivity Index of the Matrix on the Rate Transient Performance Behavior of a Double Porosity Reservoir with Transient Interporosity Transfer considering Fractal Fracture Network and Infinite-Acting Fractal Matrix Blocks](image)

**Figure 5.26** — Log-log plot of the rate-transient and rate derivative performance behaviors of a double porosity reservoir considering fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_\text{mu}$).
Figure 5.27 — Log-log plot of the $\beta$-rate derivative function of a double porosity reservoir considering fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix ($\theta_{\text{mat}}$).

5.5. Summary

The purpose of the development of the models presented in this chapter is to provide analytical models for the estimation of parameters of unconventional reservoirs, using pressure and rate transient data.
CHAPTER VI
CONCLUSIONS AND RECOMMENDATIONS

The following conclusions have been derived from this work:

1. We have utilized the fractal geometry theory to develop reservoir models for shale reservoirs. We have extended the existing models for multi-fractured wells to consider the fractal reservoir scenario.

2. Using our new solutions for a horizontal well intercepting a single finite-conductivity, we have found that at early-transient (hydraulic fracture flow) and intermediate-transient times (interaction between the hydraulic fracture and the fractal reservoir), are very sensitive to the fractal parameters of the reservoir. In addition, the influence of the fractal parameters at early and intermediate-transient times is more evident at low values of the dimensionless fracture conductivity.

3. When considering a double porosity reservoir with a fractal fracture network and Euclidean matrix blocks, the presence of a single finite conductivity hydraulic fracture can yield as many as twelve (12) power-law sub-periods of flow as observed in the dimensionless pressure and dimensionless pressure derivative functions as exhibited by our new solutions.

4. We applied the principle of superposition in space to the single finite-conductivity solution to generate the multi-fractured well case, which is a more practical/realistic scenario. Analogous to the case where the horizontal wells are drilled in conventional/homogenous reservoirs, we
found that the transient performance of these types of wells in fractal reservoirs behaves as an equivalent system of a horizontal well intercepting a single fracture.

5. We have studied the impact of the anomalous diffusion phenomenon in the pressure and rate transient performance behaviors and compared these responses to the typical (constant) diffusion cases. We have found that the anomalous diffusion phenomenon shows an additional influence on the pressure and rate responses during early and intermediate-transient times. However, at late times, the anomalous diffusion phenomenon exhibits an overall lower pressure drop for the constant rate solution and consequently, the inclusion of the anomalous diffusion phenomenon results in higher values of flowrate and cumulative production at late-transient times.

6. Our partial results and the mathematical background of the models considering anomalous diffusion made us believe that the fractal reservoir model with typical diffusion is more likely in theory and in practice than the anomalous diffusion case. To provide an explanation to the anomalous diffusion based on the concepts of petroleum reservoir engineering, we developed the double fractal model.

7. We considered that the highly heterogeneous and the low/ultralow nature of the shale reservoirs make appropriate (1) the use of transient interporosity transfer functions and (2) the modeling of the fractal matrix blocks as infinite-acting media. This approach results in a diffusivity equation with the same shape as the so-called anomalous diffusion equation. Therefore, we concluded that the anomalous diffusivity phenomenon in unconventional reservoirs can be related to the fractal geometry and the heterogeneities of the fractal matrix blocks.
8. To model the matrix blocks as fractal objects, we investigated the closed and the infinite-acting matrix blocks scenarios. We found that the fractal dimension of the matrix blocks does not have a significant impact in the signature of the pressure and pressure-transient derivative functions, when the blocks are considered closed (small blocks and/or high hydraulic diffusivity). However, when the matrix blocks behave as "infinite-acting," the pressure- and rate-transient performance behaviors are sensitive to a combined effect of the properties of the fracture network and the matrix blocks at late times.

Based on the results presented in this work, we have defined the following recommendations for future work:

1. In this dissertation, we followed the traditional approaches to superimpose the flow of a reservoir (fractal reservoir) on the plane of the hydraulic fracture. Nonetheless, we have questioned ourselves on the validity of these methods when coupling a fractal and a Euclidean object. The reason is the lack of a physical explanation to observe higher flowrates at early and intermediate times in fractal reservoirs with poorly connected permeable sites (high $\theta$-values) and/or lower fractal dimension ($D_f$). Therefore, our first recommendation is to explore other alternatives to define the continuity between a hydraulic fracture and a fractal reservoir.

2. Our second recommendation is related to the development of analytical solutions for RTA. In shale reservoirs, the Power-Law/Stretched exponential empirical model is widely used because of its accuracy and flexibility. The shape of this model corresponds to the asymptotic behavior of Fox's functions for small arguments. The solution to the double fractal model can be expressed in terms of this type of function. We believe that the parameters of the double fractal
model can be related to the empirical parameters of the Power-Law/Stretched Exponential model. Consequently, we recommend the development analytical relations between this empirical model and the asymptotic solutions of the double fractal model.
NOMENCLATURE

Field Variables

\[ A_{fma} = \text{Fracture area per unit of matrix volume, } L^{-1} \text{ [m}^{-1}] \text{ or [ft}^{-1}] \]

\[ B = \text{Oil formation volume factor, [bbl/STB] or [Rm}^{3}/Sm^{3}] \]

\[ c_f = \text{Hydraulic fracture compressibility, } (M/Lt^2)^{-1} \text{ [Pa}^{-1}] \text{ or [psi}^{-1}] \]

\[ c_{fb} = \text{Radial fracture network compressibility, } (M/Lt^2)^{-1} \text{ [Pa}^{-1}] \text{ or [psi}^{-1}] \]

\[ c_o = \text{Fluid compressibility (oil), } (M/Lt^2)^{-1} \text{ [Pa}^{-1}] \text{ or [psi}^{-1}] \]

\[ c_{ma} = \text{Matrix blocks compressibility, } (M/Lt^2)^{-1} \text{ [Pa}^{-1}] \text{ or [psi}^{-1}] \]

\[ c_t = \text{Total compressibility, } (M/Lt^2)^{-1} \text{ [Pa}^{-1}] \text{ or [psi}^{-1}] \]

\[ c_{tfb} = \text{Total compressibility (natural fractures, radial or fractal), } (M/Lt^2)^{-1} \text{ [Pa}^{-1}] \text{ or [psi}^{-1}] \]

\[ c_{tma} = \text{Matrix blocks total compressibility, } (M/Lt^2)^{-1} \text{ [Pa}^{-1}] \text{ or [psi}^{-1}] \]

\[ c_{tf} = \text{Hydraulic fracture total compressibility, } (M/Lt^2)^{-1} \text{ [Pa}^{-1}] \text{ or [psi}^{-1}] \]

\[ h = \text{Formation thickness, } L \text{ [m] or [ft]} \]

\[ h_{ma} = \text{Matrix block size, } L \text{ [m] or [ft]} \]

\[ k = \text{Permeability, } L^2 \text{ [mD] or [m}^2] \]

\[ k_0 = \text{Reference permeability for a fractal reservoir, } L^2 \text{ [mD] or [m}^2] \]

\[ k_{0fb} = \text{Reference permeability for a fractal fracture network, } L^2 \text{ [mD] or [m}^2] \]

\[ k_{0ma} = \text{Reference permeability for the fractal matrix blocks, } L^2 \text{ [mD] or [m}^2] \]

\[ k_f = \text{Hydraulic fracture permeability, } L^2 \text{ [mD] or [m}^2] \]

\[ k_{fb} = \text{Radial fracture network permeability, } L^2 \text{ [mD] or [m}^2] \]

\[ k_{ma} = \text{Matrix permeability, } L^2 \text{ [mD] or [m}^2] \]
$L_r$ = Reference Length, $L$ [m] or [ft]

$p$ = Pressure, $M/Lt^2$ [Pa] or [psi]

$p_i$ = Initial reservoir pressure, $M/Lt^2$ [Pa] or [psi]

$p_f$ = Fracture pressure, $M/Lt^2$ [Pa] or [psi]

$p_{wf}$ = Wellbore flowing pressure, $M/Lt^2$ [Pa] or [psi]

$p_{wf}$ = Wellbore phase redistribution pressure, $M/Lt^2$ [Pa] or [psi]

$p_{tf}$ = Tubing flowing pressure at surface, $M/Lt^2$ [Pa] or [psi]

$r$ = Radial distance, $L$ [m] or [ft]

$R$ = Radial distance (fractal or spherical systems), $L$ [m] or [ft]

$r_f$ = Radius of a circular hydraulic fracture, $L$ [m] or [ft]

$r_w$ = Wellbore radius, $L$ [m] or [ft]

$s$ = Skin factor, dimensionless

$q_w$ = Well Flowrate, $L^3/t$ [$m^3/sec$] or [ft$^3$/s]

$t$ = Time, $t$ [sec]

$u$ = Laplace transform variable

$w$ = Fracture width, $L$ [m] or [ft]

$x_f$ = Fracture half length, $L$ [m] or [ft]

**Dimensionless Variables**

$A_{fD}$ = Dimensionless natural fracture area

$A_{ffD}$ = Dimensionless fractal fracture area

$D_f$ = Fractal dimension of a fractal reservoir

$D_{fb}$ = Fractal dimension of a fractal fracture network
\(D_{fma} =\) Fractal dimension of the matrix blocks  \\
\(C_{fD} =\) Dimensionless wellbore storage constant  \\
\(F_{cD} =\) Dimensionless fracture conductivity  \\
\(h_D =\) Dimensionless formation thickness  \\
\(h_{maD} =\) Dimensionless matrix block size  \\
\(N_{pD} =\) Dimensionless cumulative production  \\
\(p_{D,cr} =\) Dimensionless pressure in the reservoir defined for the constant-rate solution  \\
\(p_{fD,cr} =\) Dimensionless pressure in the hydraulic fracture defined for the constant-rate solution  \\
\(p_{fhnD,cr} =\) Dimensionless pressure in the fracture network defined for the constant-rate solution  \\
\(p_{rfnD,cr} =\) Dimensionless pressure in the fractal-fracture network for the constant-rate solution  \\
\(p_{maD,cr} =\) Dimensionless pressure in the fractal matrix blocks for the constant-rate solution  \\
\(p_{wD} =\) Dimensionless wellbore flowing pressure  \\
\(r_D =\) Dimensionless radius for a circular system  \\
\(r_{wD} =\) Dimensionless radius wellbore  \\
\(R_D =\) Dimensionless radius for a spherical or fractal system  \\
\(q_{wD} =\) Dimensionless flowrate  \\
\(S_{int} =\) Interporosity skin  \\
\(t_D =\) Dimensionless time  \\
\(t_{Dr} =\) Dimensionless time (dimensionless variables for a radial fracture network)  \\
\(t_{Df} =\) Dimensionless time (dimensionless variables for a fractal fracture network)  \\
\(x_D =\) Dimensionless linear position in the \(x\)-direction  \\
\(z_D =\) Dimensionless linear position in the \(z\)-direction
\( \nu \) = Grouping parameter of the fractal variables of a fractal fracture network/reservoir.

\( \nu_{ma} \) = Grouping parameter of the fractal variables of the fractal matrix blocks

**Greek Symbols**

\( \alpha_0 \) = Grouping parameter of the properties the fractal matrix block, dimensionless

\( \alpha_1 \) = Grouping parameter of the properties the fractal matrix block, dimensionless

\( \alpha_{Df} \) = Area of a unit sphere in \( D_f \) dimensions

\( \beta \) = Spatial dimension, dimensionless

\( \beta_{ma} \) = Spatial dimension of the fractal fracture network, dimensionless

\( \gamma \) = Anomalous diffusivity exponent as function of fractal parameters, dimensionless

\( \delta q_D \) = Point density fluid withdrawal, dimensionless

\( \theta \) = Conductivity index, dimensionless

\( \theta_{ma} \) = Conductivity index of the fractal matrix blocks, dimensionless

\( \eta_{Df} \) = Dimensionless hydraulic diffusivity of the hydraulic fracture, dimensionless

\( \eta_{maD} \) = Dimensionless hydraulic diffusivity of the matrix, dimensionless

\( \mu \) = Newtonian Viscosity, \( M/Lt \) [cp] or [lb\text{m}/ft\text{s}]

\( \xi \) = Grouping parameter, dimensionless

\( \xi_{RD} \) = Function of \( R_D \), dimensionless

\( \phi \) = Porosity, fraction

\( \phi_0 \) = Reference porosity for a fractal reservoir, fraction

\( \phi_{0fb} \) = Reference porosity for fractal fracture network, fraction

\( \phi_f \) = Hydraulic fracture porosity, fraction
\( \phi_{fb} \) = Natural fracture network porosity, fraction

\( \phi_{0ma} \) = Reference porosity for the fractal matrix blocks, fraction

\( \phi_{ma} \) = Matrix porosity, fraction

\( \omega \) = Storativity ratio, dimensionless

**Mathematical Functions**

\( E_i(x) \) = Exponential Integral Function

\( I_v(x) \) = Modified Bessel Functions of the first kind, \( v \)-order

\( K_v(x) \) = Modified Bessel Functions of the second kind, \( v \)-order

\( \Gamma(x) \) = Gamma function

\( \Gamma(a,x) \) = Incomplete Gamma function
REFERENCES


APPENDIX A

PRESSURE-TRANSIENT BEHAVIOR OF HORIZONTAL WELLS WITH FINITE-CONDUCTIVITY VERTICAL FRACTURES

In this Appendix, we present the detailed derivation and the procedure to obtain the solution of the model presented by Larsen et al. (1991).

A.1. Development of the Radial Diffusivity Equation for a Circular Transverse Finite Conductivity Fracture

To develop the radial diffusivity equation that describes the flow inside the circular hydraulic fracture, consider the control volume shown in Fig. A.1. The control volume is constructed by two concentric horizontal cylinders. The inner and outer cylinders have radii $r$ and $r+\Delta r$, respectively. For this model, the flow occurs in the $r$- and $z$-directions (angular flow is neglected).

![Figure A.1](image_url)  

**Figure A.1** — Control volume of a circular vertical hydraulic fracture.
The volume of the geometric shape exhibit in Fig. A.1 is given by:

\[ V_c = r \Delta r \Delta \theta \Delta z \] \hspace{1cm} (A.1)

The incoming mass of a fluid, \( m_{in,r} \), throughout the outer cylinder is:

\[ m_{in,r} = -\rho v_r[r + \Delta r] \Delta \theta \Delta z \Delta t \] \hspace{1cm} (A.2)

and the outgoing mass, \( m_{out,r} \), throughout the outer cylinder is given by:

\[ m_{out,r} = [\rho v_r + \Delta (\rho v_r)]r \Delta \theta \Delta z \Delta t \] \hspace{1cm} (A.3)

Analogously, the incoming mass of fluid in the \( z \)-direction is:

\[ m_{in,z} = -\rho v_z r \Delta r \Delta \theta \Delta z \Delta t \] \hspace{1cm} (A.4)

and the outgoing mass in the same direction is defined by:

\[ m_{out,z} = [\rho v_z + \Delta (\rho v_z)]r \Delta r \Delta \theta \Delta z \Delta t \] \hspace{1cm} (A.5)

The cumulative mass of the fluid is determined by:

\[ m_c = m_{out,r} + m_{out,z} - [m_{in,r} + m_{in,z}] \] \hspace{1cm} (A.6)

The following expression results from combining Eqs. A.2 through A.6 and arraying:

\[ m_c = \Delta (\rho v_r) r \Delta \theta \Delta z \Delta t + \rho v_r \Delta r \Delta \theta \Delta z \Delta t + \Delta (\rho v_z) r \Delta r \Delta \theta \Delta z \Delta t \] \hspace{1cm} (A.7)

The mass of fluid at an initial time can be expressed as:

\[ m_{in} = \phi f S_f \rho v_r \Delta r \Delta \theta \Delta z \] \hspace{1cm} (A.8)
At a final time, the mass of fluid is given by:

\[ m_{t_2} = \phi f S_f \rho v_r \Delta r \Delta \theta \Delta z + \Delta (\phi f S_f \rho) v_r \Delta r \Delta \theta \Delta z \cdot \] \hspace{1cm} \text{(A.9)}

The cumulative mass of fluid for the time-dependent case is given by:

\[ m_c = m_{t_2} - m_t \text{ } . \hspace{1cm} \text{(A.10)}\]

Substituting Eq. A.8 and Eq. A.9 in Eq. A.10:

\[ m_c = \Delta (\phi f S_f \rho) v_r \Delta r \Delta \theta \Delta z \text{ } . \hspace{1cm} \text{(A.11)}\]

Equating Eq. A.7 and Eq. A.11:

\[ \Delta (\rho v_r) v_r \Delta r \Delta \theta \Delta t + \rho v_r \Delta r \Delta \theta \Delta z \Delta t + \Delta (\rho v_z) v_r \Delta r \Delta \theta \Delta t = \Delta (\phi f S_f \rho) v_r \Delta r \Delta \theta \Delta z \cdot \] \hspace{1cm} \text{(A.12)}

Dividing Eq. A.12 by \( V_c \Delta t \), such an equation reduces to:

\[ \frac{\Delta (\rho v_r)}{\Delta r} + \frac{\rho v_r}{r} + \frac{\Delta (\rho v_z)}{\Delta z} = \frac{\Delta (\phi f S_f \rho)}{\Delta t} \] \hspace{1cm} \text{(A.13)}

Taking the limits of \( \Delta r, \Delta t \) and \( \Delta z \) to zero, Eq. A.13 can be written in its differential form. Such an equation becomes:

\[ \frac{\partial (\rho v_r)}{\partial r} + \frac{\rho v_r}{r} + \frac{\partial (\rho v_z)}{\partial z} = \frac{\partial (\phi f S_f \rho)}{\partial t} \] \hspace{1cm} \text{(A.14)}

To obtain the diffusivity equation for radial flow, the continuity equation defined by Eq. A.14 must be combined with an equation of motion and an equation of state. For this model, Darcy's Law flow will be considered in both \( r \)- and \( z \)-directions. For the \( r \)-direction:
\[ v_r = \frac{k_r}{\mu} \frac{\partial p_f}{\partial r}, \]  

(A.15)

and in the \( z \)-direction:

\[ v_r = \frac{k_z}{\mu} \frac{\partial p_f}{\partial z}. \]  

(A.16)

Substituting Eq. A.15 and Eq. A.16 in Eq. A.14 and assuming single fluid flow (i.e., \( S_f=1 \)):

\[ \frac{\partial}{\partial r} \left[ \frac{k_r}{\mu} \frac{\partial p_f}{\partial r} \right] + \frac{\rho}{r} \left[ \frac{k_r}{\mu} \frac{\partial p_f}{\partial r} \right] + \frac{\partial}{\partial z} \left[ \frac{k_z}{\mu} \frac{\partial p_f}{\partial z} \right] = \frac{\partial (\phi_f \rho)}{\partial t} \]  

(A.17)

Expanding Eq. A.15 and assuming constant permeability and viscosity, such an equation is rewritten as:

\[ \frac{\partial}{\partial r} \left[ \frac{k_r}{\mu} \frac{\partial p_f}{\partial r} \right] + \frac{\rho}{r} \left[ \frac{k_r}{\mu} \frac{\partial p_f}{\partial r} \right] + \frac{\partial}{\partial z} \left[ \frac{k_z}{\mu} \frac{\partial p_f}{\partial z} \right] = \frac{\partial (\phi_f \rho)}{\partial t} \]  

(A.18)

Expanding Eq. A.15 and assuming constant permeability and viscosity, such an equation is rewritten as:

\[ \frac{\partial^2 p_f}{\partial r^2} + \frac{1}{r} \frac{\partial p_f}{\partial r} + \frac{k_z}{\mu} \frac{\partial^2 p_f}{\partial z^2} + \frac{1}{\rho} \frac{\partial p_f}{\partial \rho} + 1 \frac{\partial k_z}{\partial z} \frac{\partial p_f}{\partial z} - \frac{\partial p_f}{\partial \rho} \frac{\partial \rho}{\partial t} + \frac{\partial \phi_f}{\partial \rho} \frac{\partial \rho}{\partial t} \]  

(A.19)

To include the effects of a slightly compressible fluid, it is convenient to express some of the derivatives in Eq. A.19 in terms of the change in density of porosity of the hydraulic fracture with respect to pressure. Therefore, using the chain rule:

\[ \frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial \rho_f} \frac{\partial \rho_f}{\partial r}, \]  

(A.20)
\[
\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial p_f} \frac{\partial p_f}{\partial z}, \quad \text{.................................................(A.21)}
\]

\[
\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p_f} \frac{\partial p_f}{\partial t}, \quad \text{.................................................(A.22)}
\]

\[
\frac{\partial \phi_f}{\partial t} = \frac{\partial \phi_f}{\partial p_f} \frac{\partial p_f}{\partial t}, \quad \text{.................................................(A.23)}
\]

Substituting Eq. A.20 through Eq. A.23 in Eq. A.19 and arraying:

\[
\frac{\partial^2 p_f}{\partial r^2} + \frac{1}{r} \frac{\partial p_f}{\partial r} + \frac{k_z}{k_r} \frac{\partial^2 p_f}{\partial z^2} + \frac{1}{\rho} \frac{\partial \rho}{\partial p_f} \left[ \frac{\partial p_f}{\partial r} \right]^2 + \frac{k_z}{k_r} \left[ \frac{\partial p_f}{\partial z} \right]^2 = \frac{\phi_f \mu}{k_r} \left[ \frac{1}{\rho} \frac{\partial \rho}{\partial p_f} + \frac{1}{\phi_f} \frac{\partial \phi_f}{\partial p_f} \right] \frac{\partial p_f}{\partial t}, \quad \text{.................................................(A.24)}
\]

The compressibility of the fluid is defined by:

\[
ce_o = \frac{1}{\rho} \frac{\partial \rho}{\partial p_f}, \quad \text{.................................................(A.25)}
\]

and the compressibility of the hydraulic fracture is:

\[
ce_f = \frac{1}{\phi_f} \frac{\partial \phi_f}{\partial p_f}, \quad \text{.................................................(A.26)}
\]

Substituting Eqs. A.25 and A.26 in Eq. A.24 and assuming negligible squared pressure gradients, Eq. A.24 becomes:

\[
\frac{\partial^2 p_f}{\partial r^2} + \frac{1}{r} \frac{\partial p_f}{\partial r} + \frac{k_z}{k_r} \frac{\partial^2 p_f}{\partial z^2} + \frac{\phi_f \mu c_{\text{ef}}}{k_r} \frac{\partial p_f}{\partial t}, \quad \text{.................................................(A.27)}
\]

where the total compressibility of the fracture is defined by:
\[ c_{if} = c_{a} + c_{f} \] .................................................................(A.28)

Rewriting Eq. A.27 in the compact form and considering the fracture as an isotropic media (i.e., \( k_f = k_r = k_z \)):

\[
\frac{1}{r} \left[ \frac{\hat{r}}{\partial r} \right] + \frac{\partial^2 p_f}{\partial z^2} = \frac{\phi_f \mu c_{if}}{k_f} \frac{\hat{z}}{\partial t} \] .................................................................(A.29)

To couple the two porous media (the circular hydraulic fracture and the 3D reservoir), the pressure and the flowrate in both media must be the same. Hence, for the pressure:

\[ p_f(z = \pm w/2, t) = p(z = \pm w/2, t) \approx p(z = 0, t), \] .................................................................(A.30)

and for the flowrate:

\[
\left[ k_f \frac{\hat{p}_f}{\partial z} \right]_{z = \pm w/2} \approx \left[ k \frac{\hat{p}}{\partial z} \right]_{z = 0} \] .................................................................(A.31)

The second derivative of the pressure in the circular fracture with respect to \( z \) can be reduced by taking an averaged pressure in the fracture:

\[
\frac{1}{r} \left[ \frac{\hat{r}}{\partial r} \left[ \frac{1}{w - w/2} \int_{z = w/2}^{z = -w/2} p_f \, dz \right] \right] + \frac{1}{w - w/2} \frac{\partial^2 p_f}{\partial z^2} \, dz = \frac{\phi_f \mu c_{if}}{k_f} \frac{\hat{z}}{\partial t} \left[ \frac{1}{w - w/2} \int_{z = w/2}^{z = -w/2} p_f \, dz \right] \] .................................................................(A.32)

Applying the integral and evaluating the integration limits Eq. A.32 becomes:

\[
\frac{1}{r} \left[ \frac{\hat{r}}{\partial r} \left[ \frac{1}{w - w/2} \int_{z = w/2}^{z = -w/2} p_f \, dz \right] \right] + \frac{1}{w} \left[ \frac{\hat{p}_f}{\partial z} \right]_{z = w/2} - \left[ \frac{\hat{p}_f}{\partial z} \right]_{z = -w/2} = \frac{\phi_f \mu c_{if}}{k_f} \frac{\hat{z}}{\partial t} \left[ \frac{1}{w - w/2} \int_{z = w/2}^{z = -w/2} p_f \, dz \right] \] ...........(A.33)
Assuming no significant difference between the averaged pressure in the circular fracture and the pressure and any point of it, i.e.:

\[ p_f = \frac{1}{w} \int_{-w/2}^{w/2} p_f \, dz \] \hspace{1cm} (A.34)

and according to Eq. A.31, Eq. A.33 reduces to:

\[ \frac{1}{r} \left[ \frac{\partial p_f}{\partial r} \right] + \frac{2k}{wk_f} \left[ \frac{\partial p}{\partial z} \right] \bigg|_{z=0} = \frac{\phi_f \mu \kappa f}{k_f} \frac{\partial p_f}{\partial t} \] \hspace{1cm} (A.35)

Larsen et al. (1991) presented solutions considering constant flowrate at the wellbore. Therefore, to generalize the problem, the following dimensionless variables were defined. The dimensionless pressure for the circular fracture is:

\[ p_{fD,cr}(r_D, t_D) = \frac{2\pi kh}{qB\mu} [p_i - p_f(r, t)] \] \hspace{1cm} (A.36)

and for the 3D reservoir case:

\[ p_{D,cr}(R_D, t_D) = \frac{2\pi kh}{qB\mu} [p_i - p(R, t)] \] \hspace{1cm} (A.37)

The dimensionless time for this model is defined by:

\[ t_D = \frac{k}{\phi \mu \kappa r_f^2} \] \hspace{1cm} (A.38)

The dimensionless (radial) position within the fracture is:

\[ r_D = \frac{r}{r_f} \] \hspace{1cm} (A.39)
The horizontal position is:

\[ z_D = \frac{z}{r_f} \]  \hspace{1cm} \text{(A.40)}

Using the chain rule, the derivative of the pressure in the circular fracture with respect to the time can be expressed as:

\[ \frac{\partial p_f}{\partial t} = \frac{\partial p_f}{\partial t_D} \frac{\partial t_D}{\partial D} \]  \hspace{1cm} \text{(A.41)}

Then, based on Eq. A.37 and Eq. A.38:

\[ \frac{\partial p_f}{\partial D} = \frac{qB \mu}{2\pi k h f} \frac{\partial p_f}{\partial D} \]  \hspace{1cm} \text{(A.42)}

Similarly, for the first derivative of the pressure in the circular fracture with respect to \( r \):

\[ \frac{\partial p_f}{\partial r} = \frac{qB \mu}{2\pi k h f} \frac{\partial p_f}{\partial r} \]  \hspace{1cm} \text{(A.43)}

The derivative of pressure in the 3D reservoir with respect to \( z \) has a similar shape:

\[ \frac{\partial p}{\partial z} = \frac{qB \mu}{2\pi k h f} \frac{\partial p}{\partial z} \]  \hspace{1cm} \text{(A.44)}

Considering the definition of the dimensionless radius (Eq. A.39) and substituting Eqs. A.41 through A.44 in Eq. A.35 results in:

\[ \frac{1}{r_D} \left[ 1 - \frac{qB \mu}{2\pi k h f} \frac{\partial p_D}{\partial D} \right] + \frac{2k}{w k f} \left[ -\frac{qB \mu}{2\pi k h f} \frac{\partial p_D}{\partial t} \right] \]  \hspace{1cm} \text{(A.45)}
Eliminating terms, Eq. A.45 reduces to the radial diffusivity equation with a source term in its dimensionless form:

\[
\frac{1}{t_{DD}} \left[ F_{DD} \frac{\partial p_{D,cr}}{\partial r_{D}} \right] + \frac{2}{F_{cD}} \left[ \frac{\partial p_{D,cr}}{\partial z_{D}} \right]_{z_{D}=0} = \frac{1}{\eta_{fD}} \frac{\partial p_{fD,cr}}{\partial t_{D}},
\]

(A.46)

where the dimensionless hydraulic diffusivity of the radial fracture is defined by:

\[
\eta_{fD} = \frac{k_{f} \phi_{f}}{\phi_{f} c_{if} k},
\]

(A.47)

and the dimensionless fracture conductivity is:

\[
F_{cD} = \frac{w k_{f}}{k r_{f}},
\]

(A.48)

To obtain the general solution of Eq. A.46 it is convenient to eliminate one of the independent variables. This reduction of variables is made by discretizing the circular fracture into segments such that the gradient in the \(z\)-direction can be approximated by:

\[
\left[ \frac{\partial p}{\partial z} \right]_{z=0} = \frac{\delta q_{r,j} B \mu}{2 k r_{f}^{2}},
\]

(A.49)

where \(\delta q_{r,j}\) is the flowrate from the reservoir flowing towards the \(j\)-segment of the discretized circular fracture. Transforming Eq. A.49 to dimensionless variables:

\[
\left[ \frac{\partial p_{D,cr}}{\partial z_{D}} \right]_{z_{D}=0} \approx \pi \delta q_{D,j} h_{D},
\]

(A.50)

where the dimensionless reservoir rate in the \(j\)-segment is defined by:
\[ \delta q_{D,j} = \frac{\delta q_{r,j}}{q}, \quad \text{.................................................................(A.51)} \]

and the dimensionless thickness is:

\[ h_D = \frac{h}{r_f}, \quad \text{..........................................................................................(A.52)} \]

Substituting Eq. A.50 in Eq. A.46:

\[ \frac{1}{r_D} \left[ \frac{\partial p_{fD,cr}^*}{\partial r_D} \right] - \frac{2\pi}{F_{hcD}} \delta q_{D,j} = \frac{1}{\eta_{fD}} \frac{\partial p_{fD,cr}^*}{\partial t_D}, \quad \text{.................................................................(A.53)} \]

where \( F_{hcD} \) is an alternate parameter to describe the fracture conductivity. It is defined as:

\[ F_{hcD} = \frac{F_{cD}}{h_D}, \quad \text{..........................................................................................(A.54)} \]

Applying the Laplace transform to Eq. A.53 and treating the dimensionless reservoir rate in the j-interval as a function of time:

\[ \frac{1}{r_D} \left[ \frac{\partial p_{fD,cr}^*}{\partial r_D} \right] - \frac{2\pi}{F_{hcD}} \delta q_{D,j} = \frac{1}{\eta_{fD}} \left[ u \frac{\partial p_{fD,cr}^*}{\partial r_D} - p_{fD,cr}^* (r_D, t_D = 0) \right], \quad \text{.................................................................(A.55)} \]

For this model, it is assumed that the pressure is initially distributed. In dimensionless variables it is expressed as:

\[ p_{fD,cr}^* (r_D, t_D = 0) = 0, \quad \text{.................................................................(A.56)} \]

Hence, Eq. A.55 can be rewritten as follows:
Eq. A.57 is a non-homogenous partial differential equation. The particular solution to the non-homogeneous part is obtained by removing the derivatives of the dimensionless pressure in the circular fracture. Such a particular solution is given by:

\[
\overline{p}_{jD,cr} = \frac{2\pi \eta_{jD}}{u F_{hcD}} \overline{q}_{D,j},
\]

To solve the homogenous part of Eq. A.57 consider the transformation function:

\[
\overline{G}(\xi) = \overline{p}_{jD,cr},
\]

and the transformation variable:

\[
\xi = r_D \sqrt{\frac{u}{\eta_{jD}}},
\]

Using the chain rule, the derivative of the dimensionless pressure in the fracture in the Laplace domain with respect to the dimensionless radius is:

\[
\frac{d\overline{p}_{jD,cr}}{dr_D} = \frac{d\overline{G}(\xi)}{d\xi} \frac{d\xi}{dr_D} = \sqrt{\frac{u}{\eta_{jD}}} \frac{dG(\xi)}{d\xi} \frac{u}{\eta_{jD}} \frac{dG(\xi)}{d\xi}.
\]

Applying the second derivative to Eq. A.61 with respect to \(r_D\):

\[
\frac{d^2 \overline{p}_{jD,cr}}{dr_D^2} = \frac{u}{\eta_{jD}} \frac{d^2 G(\xi)}{d\xi^2}.
\]
Taking the homogenous part of Eq. A.57 and substituting Eqs. A. 59 through A. 61:

\[ \xi^2 \frac{d^2 \bar{G}(\xi)}{d\xi^2} + \xi \frac{d\bar{G}(\xi)}{d\xi} - \xi^2 \bar{G}(\xi) = 0. \] \hspace{2cm} (A.63)

By inspection, the general solution of Eq. A.63 is given by:

\[ \bar{G}(\xi) = A_j K_0(\xi) + B_j I_0(\xi). \] \hspace{2cm} (A.64)

The general solution to Eq. A.57 is given by adding Eq. A.58 and Eq. A.64 and using the definitions given by Eqs. A.59 and A.60:

\[ \bar{p}_{fD,cr}(r_D, u) = A_j K_0 \left[ r_D \sqrt{\frac{u}{\eta_D}} \right] + B_j I_0 \left[ r_D \sqrt{\frac{u}{\eta_D}} \right] - \frac{2\pi \eta_D}{uF_{hcD}} \bar{q}_{D,cr} \] \hspace{2cm} (A.65)

**A.2. Development of the Linear Diffusivity Equation for a Rectangular Longitudinal Finite Conductivity Fracture**

To develop the diffusivity equation for a linear system with a source to represent the flow within a rectangular fracture, consider the control volume shown in Fig. A.2. The control volume is constructed by a rectangular prism whose volume is defined by:

\[ V_c = \Delta x \Delta y \Delta z. \] \hspace{2cm} (A.66)
The incoming mass of fluid, $m_{in,x}$, throughout the rectangular prism through in the $x$-direction is:

$$m_{in,x} = -\rho v_x \Delta y \Delta z \Delta t$$ \hspace{1cm} (A.67)

and the outgoing mass, $m_{out,x}$, in the same direction is:

$$m_{out,x} = [ -\rho v_x + \Delta (\rho v_x) ] \Delta y \Delta z \Delta t$$ \hspace{1cm} (A.68)

Analogously, the incoming mass of fluid in the $z$-direction is:

$$m_{in,z} = -\rho v_z \Delta x \Delta y \Delta t$$ \hspace{1cm} (A.69)

and the outgoing mass is:

$$m_{out,z} = [ -\rho v_z + \Delta (\rho v_z) ] \Delta x \Delta y \Delta t$$ \hspace{1cm} (A.70)

The cumulative mass of fluid is given by:
The mass of fluid at an initial time can be expressed as (considering a single fluid, i.e., $S_f=1$):

$$m_i = \phi_f \rho \Delta x \Delta y \Delta z,$$

(A.72)

and at a final time, the mass of the fluid is given by:

$$m_f = \phi_f \rho \Delta x \Delta y \Delta z + \Delta (\rho \nu_f) \Delta x \Delta y \Delta z,$$

(A.73)

Combining Eqs. A.71 through A.73:

$$m_c = \Delta (\phi_f \rho) \Delta x \Delta y \Delta z,$$

(A.74)

Equating Eqs. A.71 and A.74:

$$\Delta (\rho \nu_f) \Delta y \Delta z \Delta t + \Delta (\rho \nu_x) \Delta x \Delta y \Delta t = \Delta (\phi_f \rho) \Delta x \Delta y \Delta z,$$

(A.75)

Dividing Eq. A.75 by $\Delta x \Delta y \Delta z \Delta t$:

$$\frac{\Delta (\rho \nu_f)}{\Delta x} + \frac{\Delta (\rho \nu_x)}{\Delta z} = \frac{\Delta (\phi_f \rho)}{\Delta t}.$$

(A.76)

Taking the limits of $\Delta x$, $\Delta z$, and $\Delta t$ to zero, Eq. A.76 results the 2D continuity equation:

$$\frac{\partial (\rho v_f)}{\partial x} + \frac{\partial (\rho v_x)}{\partial z} = \frac{\partial (\phi_f \rho)}{\partial t}.$$

(A.77)

Darcy's law for a linear system in $i$-direction is defined by

$$v_i = k_i \frac{\partial p_f}{\partial i}.$$

(A.78)
Substituting Eq. A.78 in Eq. A.77 for $x$ and $z$ directions and assuming (1) isotropic media with constant permeability, $k_f$, and (2) a fluid with constant viscosity, the following expression is obtained:

$$\frac{k_f}{\mu} \frac{\partial}{\partial x} \left[ \rho \frac{\partial p_f}{\partial x} \right] + \frac{k_f}{\mu} \frac{\partial}{\partial z} \left[ \rho \frac{\partial p_f}{\partial z} \right] = \frac{\partial (\phi_f \rho)}{\partial t} \quad \text{(A.79)}$$

Applying the derivatives in Eq. A.79:

$$\frac{\partial^2 p_f}{\partial x^2} + \frac{\partial^2 p_f}{\partial z^2} + \frac{1}{\rho} \frac{\partial p_f}{\partial x} \frac{\partial p_f}{\partial x} + \frac{1}{\rho} \frac{\partial p_f}{\partial z} \frac{\partial p_f}{\partial z} = \frac{\mu}{k_f} \left[ \frac{1}{\rho} \frac{\partial p_f}{\partial x} + 1 \frac{\partial \phi_f}{\partial t} \right] \quad \text{(A.80)}$$

Using the chain rule as in the previous section:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial p_f} \frac{\partial p_f}{\partial x} \quad \text{(A.81)}$$

$$\frac{\partial p}{\partial z} = \frac{\partial p}{\partial p_f} \frac{\partial p_f}{\partial z} \quad \text{(A.82)}$$

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial p_f} \frac{\partial p_f}{\partial t} \quad \text{(A.83)}$$

$$\frac{\partial \phi_f}{\partial t} = \frac{\partial \phi_f}{\partial p_f} \frac{\partial p_f}{\partial t} \quad \text{(A.84)}$$

Substituting Eqs. A.81 through A.84 in Eq. A.80:

$$\frac{\partial^2 p_f}{\partial x^2} + \frac{\partial^2 p_f}{\partial z^2} + \frac{1}{\rho} \frac{\partial p_f}{\partial x} \frac{\partial p_f}{\partial x} + \frac{1}{\rho} \frac{\partial p_f}{\partial z} \frac{\partial p_f}{\partial z} = \frac{\mu}{k_f} \left[ \frac{1}{\rho} \frac{\partial p_f}{\partial x} + 1 \frac{\partial \phi_f}{\partial t} \right] \quad \text{(A.85)}$$
According to the definition of the compressibility of the fluid (Eq. A.25) and the compressibility of the hydraulic fracture (Eq. A.26), Eq. A.85 can be rewritten as:

\[
\frac{\partial^2 p_f}{\partial x^2} + \frac{\partial^2 p_f}{\partial z^2} + c_o \left( \frac{\partial p_f}{\partial x} \right)^2 + \left( \frac{\partial p_f}{\partial z} \right)^2 = \frac{\phi_f \mu c_{tf}}{k_f} \frac{\partial p_f}{\partial t} . \tag{A.86}
\]

where \(c_{tf}\) is the total compressibility and is defined by Eq. A.28. Neglecting the squared-pressure gradient results in the 2D diffusivity equation:

\[
\frac{\partial^2 p_f}{\partial x^2} + \frac{\partial^2 p_f}{\partial z^2} = \frac{\phi_f \mu c_{tf}}{k_f} \frac{\partial p_f}{\partial t} . \tag{A.87}
\]

The second derivative of the pressure in the fracture with respect to \(z\) in Eq. A.86 should be treated as in section A.1, i.e.: the average pressure of the fracture should be taken and the flowrate at the interface between the rectangular fracture and the 3D reservoir should be considered equal (i.e. Eq. A.31). After this mathematical treatment, Eq. A.86 becomes:

\[
\frac{\partial^2 p_f}{\partial x^2} + \frac{2k}{k_f w} \left[ \frac{\partial p}{\partial z} \right]_{z=0} = \frac{\phi_f \mu c_{tf}}{k_f} \frac{\partial p_f}{\partial t} . \tag{A.88}
\]

For this model, the dimensionless variables are analogously defined by Eqs. A. 36 through A.40, but the reference length for the rectangular fracture is the half-length \(x_f\) instead of the radius of the fracture \(r_f\). Therefore, Eq. A.88 expressed in dimensionless variables is:

\[
\frac{\partial^2 p_{fD,cr}}{\partial x_D^2} + \frac{2g}{F_{cD}} \left[ \frac{\partial p_{D,cr}}{\partial z_D} \right]_{z_D=0} = \frac{1}{\eta fD} \frac{\partial p_{fD,cr}}{\partial t_D} . \tag{A.89}
\]

where the dimensionless hydraulic diffusivity of the radial fracture is defined by Eq. A.47 and the
dimensionless fracture conductivity for this case is:

\[ F_{cD} = \frac{wkJ_D}{kx_f} \] .................................(A.90)

To reduce the number of variables in Eq. A.89 and make the process of solving easier, the same procedure depicted in Section A.1 should be applied. For this case, the approximation of the gradient of the pressure with respect of \( z \) is given by:

\[
\left[ \frac{\partial p}{\partial z} \right]_{z=0} \approx \frac{\delta t_{r,j} B \mu}{2kx_f^2} \] .................................(A.91)

Transforming Eq. A.91 to dimensionless variables yields the same shape as Eq. A.50. Substituting such an expression in Eq. A.89:

\[
\frac{\partial^2 \tilde{p}_{fD,cr}}{\partial x_D^2} - \frac{2\pi}{F_{hcD}} \frac{1}{\eta_{fD}} \frac{\partial \tilde{p}_{fD,cr}}{\partial t_D} = \frac{1}{\eta_{fD}} \frac{\partial q_{D,j}}{\partial t_D} \] .................................(A.92)

where \( F_{hcD} \) is defined exactly as for the circular fracture case (Eq. A.54).

Applying the Laplace transform to Eq. A.92:

\[
\frac{d^2 \tilde{p}_{fD,cr}}{dx_D^2} - \frac{2\pi}{F_{hcD}} \frac{1}{\eta_{fD}} [u\tilde{p}_{fD,cr} - \tilde{p}_{fD,cr}(x_D, t_D = 0)] = \frac{1}{\eta_{fD}} \frac{\partial q_{D,j}}{\partial t_D} \] .................................(A.93)

Considering that the initial pressure is uniformly distributed, i.e.:

\[ p_{fD,cr}(x_D, t_D = 0) = 0 \] .....................................................(A.94)

Eq. A.93 is rewritten as:
\[
\frac{d^2 \overline{p}_{fD,cr}}{dx_D^2} - \frac{2\pi}{F_{hcD}} \bar{Q}_{D,j} = \frac{u}{\eta_{fD}} \overline{p}_{fD,cr}, \quad \text{.................................................(A.95)}
\]

Similar to Eq. A.57, Eq. A.95 is also a non-homogenous partial differential equation. Therefore, Eq. A.95 can be solved following the same procedure used for Eq. A.57. The solution of the non-homogenous part is:

\[
\overline{p}_{fD,cr,nh} = -\frac{2\pi \eta_{fD}}{F_{hcD}u} \bar{Q}_{D,j}, \quad \text{.................................................(A.96)}
\]

The homogenous part of Eq. A.95 is:

\[
\frac{d^2 \overline{p}_{fD,cr}}{dx_D^2} - \frac{u}{\eta_{fD}} \overline{p}_{fD,cr} = 0, \quad \text{.................................................(A.97)}
\]

Eq. A.97 is an ordinary differential equation of second order with constant coefficients. Hence, its solution can be written as:

\[
\overline{p}_{fD,cr,h} = A_j \exp \left[ -x_D \frac{u}{\eta_{fD}} \right] + B_j \exp \left[ x_D \frac{u}{\eta_{fD}} \right], \quad \text{.................................................(A.98)}
\]

The solution of Eq. A.95 is given by adding the solution of the non-homogenous part (Eq. A.96) and the homogenous part (Eq. A.97):

\[
\overline{p}_{fD,cr,j}(x_D, u) = A_j \exp \left[ -x_D \frac{u}{\eta_{fD}} \right] + B_j \exp \left[ x_D \frac{u}{\eta_{fD}} \right] - \frac{2\pi \eta_{fD}}{F_{hcD}u} \bar{Q}_{D,j}, \quad \text{.................................................(A.99)}
\]

The value of the constants \(A_j\) and \(B_j\) depends on the segment of the discretized fracture where the solution is evaluated. These values are determined by solving a system of equations. The
construction of such a system and the procedure to obtain its solution will be addressed in Section A.5

**A.3. Development of the Diffusivity Equation for a 3D reservoir (Spherical Flow) and its constant-rate solution**

Based on Fig. A.3, the control volume for a spherical system is constructed by two concentric spheres, and it is determined by:

\[ V_c = R^2 \Delta \theta \Delta \omega \Delta R \sin \theta \]  

\[ (A.100) \]

![Control volume of a 3D reservoir.](image)

The incoming mass of a fluid, \( m_{in,R} \), throughout the outer sphere is:

\[ m_{in,R} = -\rho v_R [R + \Delta R]^2 \sin \theta \Delta \theta \Delta \omega \Delta \tau , \]  

\[ (A.101) \]

and the outgoing mass, \( m_{out,r} \), throughout the outer cylinder is given by:
The cumulative mass of the fluid is determined by:

\[ m_{out,R} = [-\rho v_R + \Delta(\rho v_R)]R^2 \sin \theta \Delta \theta \Delta \omega \Delta t \] (A.102)

and it is reduced to:

\[ m_c = m_{out,R} - m_{in,R} = [-\rho v_R + \Delta(\rho v_R)]R^2 \sin \theta \Delta \theta \Delta \omega \Delta t + \rho v_R[R^2 + 2R\Delta R + \Delta R^2] \sin \theta \Delta \theta \Delta \omega \Delta t \] (A.103)

The mass of a single fluid \((S_f = 1)\) at an initial time can be expressed as:

\[ m_n = \phi \rho R^2 \Delta \theta \Delta \omega \Delta R \sin \theta \] (A.105)

At a final time, the mass of fluid is given by:

\[ m_{f2} = \phi \rho R^2 \Delta \theta \Delta \omega \Delta R \sin \theta + \Delta(\phi \rho)R^2 \Delta \theta \Delta \omega \Delta R \sin \theta \] (A.106)

The cumulative mass of fluid for the time-dependent case is given by:

\[ m_c = m_{f2} - m_n \] (A.107)

Substituting Eq. A.105 and Eq. A.106 in Eq. A.107:

\[ m_c = \Delta(\phi \rho)R^2 \Delta \theta \Delta \omega \Delta R \sin \theta \] (A.108)

Equating Eq. A.104 and Eq. A.108:

\[ \Delta(\rho v_R)R^2 \sin \theta \Delta \theta \Delta \omega \Delta t + [2R + \Delta R]\rho v_R \Delta R \sin \theta \Delta \theta \Delta \omega \Delta t = \Delta(\phi \rho)R^2 \Delta \theta \Delta \omega \Delta R \sin \theta \] (A.109)

Dividing Eq. A.109 by \(V_c \Delta t\), such an equation reduces to:
\[ \frac{\Delta (\rho v_R)}{\Delta R} + \frac{[2R + \Delta R]v_R}{R^2} = \frac{\Delta (\phi \rho)}{\Delta t} \] ............................................................ (A.110)

Taking the limits of \( \Delta R \) and \( \Delta t \) to zero, Eq. A.110 can be written in its differential form:

\[ \frac{\partial (\rho v_R)}{\partial R} + \frac{2\rho v_R}{R} = \frac{\partial (\phi \rho)}{\partial t} \] ............................................................ (A.111)

Recall Darcy's law:

\[ v_R = \frac{k}{\mu} \frac{\partial p}{\partial R} \] ............................................................ (A.112)

Substituting Eq. A.112 in Eq. A.111 and applying the derivatives assuming constant permeability and viscosity results in:

\[ \frac{\partial^2 p}{\partial R^2} + \frac{1}{\rho} \frac{\partial \rho}{\partial R} \frac{\partial p}{\partial R} + \frac{2}{R} \frac{\partial p}{\partial R} = \frac{\mu}{k \rho} \left[ \rho \frac{\partial \phi}{\partial t} + \phi \frac{\partial \rho}{\partial t} \right] \] ............................................................ (A.113)

Use the chain rule as in the previous sections to apply the definitions of the compressibility of a fluid (Eq. A.25) and the compressibility of the reservoir which is defined:

\[ c = \frac{1}{\phi} \frac{\partial \phi}{\partial p} \] ............................................................ (A.114)

As a result of this, Eq. A.113 is rewritten as:

\[ \frac{\partial^2 p}{\partial R^2} + \frac{2}{R} \frac{\partial p}{\partial R} = \frac{\phi \mu c_1}{k} \frac{\partial p}{\partial t} \] ............................................................ (A.115)

Consider the dimensionless variables defined in section A.1, i.e., the pressure of the 3D reservoir (Eq. A.37), the dimensionless time (Eq. A.38) and the dimensionless radius of the 3D reservoir:
\[ R_D = \frac{R}{L_r} \] .................................................................(A.116)

where \( L_r \) is the reference length (\( r_f \) for the circular fracture case and \( x_f \) for the rectangular fracture).

Transforming Eq. A.115 to dimensionless variables results in the following equation results (in the compact form):

\[
\frac{1}{R_D^2} \frac{\partial}{\partial R_D} \left[ R_D^2 \frac{\partial P_D}{\partial R_D} \right] = \frac{\partial P_D}{\partial t_D} \] .................................................................(A.117)

To solve Eq. A.115, consider the following initial and boundary conditions:

\[ p_D(R_D,t_D = 0) = 0 \] .................................................................(initial condition), (A.118)

\[ \left[ \frac{\partial p_D}{\partial R_D} \right]_{R_D = R_{wD}} = - \frac{\Delta q_D h_D}{2 R_{wD}^2} \] .................................................................(inner boundary condition), (A.119)

\[ \lim_{R_D \to \infty} p_D(R_D,t_D) = 0 \] .................................................................(outer boundary condition), (A.120)

where the dimensionless reservoir rate is defined by:

\[ \Delta q_D = \frac{\Delta q}{q} \] .................................................................(A.121)

The dimensionless thickness is:

\[ h_D = \frac{h}{L_r} \] .................................................................(A.122)

and dimensionless radius of the source is:
\[ R_{wD} = \frac{R_w}{L_r}, \]  
(A.123)

Consider the transformation:

\[ p_D = \frac{b_D}{R_D}, \]  
(A.124)

then use the chain rule:

\[ \frac{\partial p_D}{\partial t_D} = \frac{1}{R_D} \frac{\partial b_D}{\partial t_D}, \]  
(A.125)

and:

\[ \frac{\partial p_D}{\partial R_D} = -\frac{b_D}{R_D^2} + \frac{1}{R_D} \frac{\partial b_D}{\partial R_D}, \]  
(A.126)

Substituting Eq. A.125 and Eq. A.126 in Eq. A.117:

\[ \frac{1}{R_D} \frac{\partial}{\partial R_D} \left[ -b_D + R_D \frac{\partial b_D}{\partial R_D} \right] = \frac{\partial b_D}{\partial t_D}, \]  
(A.127)

Applying the derivative on the left hand side of Eq. A.127 reduces to:

\[ \frac{\partial^2 b_D}{\partial R_D^2} = \frac{\partial b_D}{\partial t_D}, \]  
(A.128)

Applying the transformation defined in Eq. A.114 in the initial condition (Eq. A.118):

\[ b_D(R_D, t_D) = 0, \]  
(A.129)
Applying the Laplace transform to Eq. A.128 and considering the initial condition (Eq. A.129), the following equation results in:

\[ \frac{d^2 b_D}{dR_D^2} - ub_D = 0. \] (A.130)

Mathematically, Eq. A.130 has the same shape of Eq. A.97. Hence, the general solution of Eq. A.130 is given by:

\[ b_D(R_D,u) = C_1 \exp[-R_D \sqrt{u}] + C_2 \exp[R_D \sqrt{u}], \] (A.131)

or, in terms of the dimensionless pressure, it is given by:

\[ \bar{p}_D(R_D,u) = \frac{C_1}{R_D} \exp[-R_D \sqrt{u}] + \frac{C_2}{R_D} \exp[R_D \sqrt{u}], \] (A.132)

Applying the outer boundary condition (Eq. A.120) to Eq. A.132:

\[ \lim_{R_D \to \infty} \bar{p}_D(R_D,u) = C_1 \lim_{R_D \to \infty} \left[ \frac{1}{R_D} \exp[-R_D \sqrt{u}] \right] + C_2 \lim_{R_D \to \infty} \left[ \frac{1}{R_D} \exp[R_D \sqrt{u}] \right] = 0, \] (A.133)

It is concluded that \( C_2 \) must be zero. Therefore, the bounded solution is:

\[ \bar{p}_D(R_D,u) = \frac{C_1}{R_D} \exp[-R_D \sqrt{u}], \] (A.134)

To apply the inner boundary condition, the derivative of the dimensionless pressure with respect to \( R_D \) of Eq. A.134 should be considered:

\[ \frac{\partial \bar{p}_D}{\partial R_D} = -C_1 \left[ \frac{1}{R_D^2} + \frac{\sqrt{u}}{R_D} \right] \exp[-R_D \sqrt{u}], \] (A.135)
Evaluating Eq. A.134 in $R_{wd}$ and comparing it with the inner boundary condition (Eq. A.119) expressed in the Laplace domain, it is concluded that:

$$C_1 = \frac{\overline{Q}_{D} h_{D}}{2[1 + R_{wd} \sqrt{u}]} \exp[R_{wd} \sqrt{u}]$$

(A.136)

Substituting Eq. A.136 in Eq. A.134, the solution for this model is obtained:

$$\tilde{p}_D(R_{D,u}) = \frac{\overline{Q}_{D} h_{D}}{2 R_{D}[1 + R_{wd} \sqrt{u}]} \exp\left[R_{wd} - R_{D} \sqrt{u}\right]$$

(A.137)

Assuming a point source (i.e., $R_{wd}$=0), Eq. A.138 reduces to:

$$\tilde{p}_D(R_{D,u}) = \frac{\overline{Q}_{D} h_{D}}{2 R_{D}} \exp\left[-R_{D} \sqrt{u}\right]$$

(A.138)

A.4. Constant-Rate Solution for the Model of a Horizontal Well Intercepting a Circular Transverse Finite Conductivity Fracture within a 3D Reservoir

Recall the general solution of the radial flow model of a circular fracture (Eq.A.65):

$$\tilde{p}_{jD,cr,j}(r_{D,u}) = A_j K_0 \left[ r_{D} \frac{u}{\eta_{jD}} \right] + B_j I_0 \left[ r_{D} \frac{u}{\eta_{jD}} \right] - \frac{2 \pi \eta_{jD}}{uF_{hc,D}} \overline{Q}_{D,j}.$$  

The last term on the right hand side of the equation represents the "contribution of pressure" due to the flow from the 3D reservoir to the circular fracture. Such a term results from the approximation of the gradient in the $z$-direction.

Fig A.4 shows a schematic example of a circular fracture logarithmically discretized in five segments. Larsen (2016) suggested that a discretization in eleven segments provides accurate
results. However, for the sake of giving an example, five segments will be used. The responses of the dimensionless pressure for each one of the segments in Fig.A.4 are:

\[
\bar{p}_{D, cr, 1}(r_D, u) = A_1 K_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] + B_1 I_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] - \frac{2\pi \eta_{fD}}{u F_{hcD}} \bar{q}_{D, 1} (r_{D0} \leq r_D \leq r_{D1}), \quad \ldots \ldots \ldots (A.139)
\]

\[
\bar{p}_{D, cr, 2}(r_D, u) = A_2 K_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] + B_2 I_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] - \frac{2\pi \eta_{fD}}{u F_{hcD}} \bar{q}_{D, 2} (r_{D1} \leq r_D \leq r_{D2}), \quad \ldots \ldots \ldots (A.140)
\]

\[
\bar{p}_{D, cr, 3}(r_D, u) = A_3 K_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] + B_3 I_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] - \frac{2\pi \eta_{fD}}{u F_{hcD}} \bar{q}_{D, 3} (r_{D2} \leq r_D \leq r_{D3}), \quad \ldots \ldots \ldots (A.141)
\]

\[
\bar{p}_{D, cr, 4}(r_D, u) = A_4 K_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] + B_4 I_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] - \frac{2\pi \eta_{fD}}{u F_{hcD}} \bar{q}_{D, 4} (r_{D3} \leq r_D \leq r_{D4}), \quad \ldots \ldots \ldots (A.142)
\]

\[
\bar{p}_{D, cr, 5}(r_D, u) = A_5 K_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] + B_5 I_0 \left[ r_D \sqrt{\frac{u}{\eta_{fD}}} \right] - \frac{2\pi \eta_{fD}}{u F_{hcD}} \bar{q}_{D, 5} (r_{D4} \leq r_D \leq r_{D5}). \quad \ldots \ldots \ldots (A.143)
\]

This approach leads to fifteen unknowns (in general \(3N\) unknowns for \(N\)-segments of the circular fracture). To set a system of fifteen equations (\(3N\) in general) four conditions must be considered: (1) boundary conditions of the circular fracture, (2) continuity of pressure at the interfaces of the discretized fracture, (3) continuity of flowrate at the interfaces of the discretized fracture, and (4) pressure continuity between the fracture and the 3D reservoir.
1. Boundary conditions of the circular fracture.

It is assumed that the well at the center of the circular fracture is produced at a constant rate and such a fracture is closed. Therefore, the inner boundary condition is defined by:

\[
\left[\frac{\partial p_f}{\partial r}\right]_{r=r_w} = \frac{qB\mu}{2\pi k f_w}, \qquad \text{..........................(A.144)}
\]

and the outer boundary condition is defined by

\[
\left[\frac{\partial p_f}{\partial r}\right]_{r=r_f} = 0, \qquad \text{..........................(A.145)}
\]

Transforming Eqs. A.144 and A.145 to dimensionless variables, they become:

\[
\left[\frac{\partial p_f}{\partial r}\right]_{r=r_{wD}} = \frac{1}{F_{hcD}}, \qquad \text{..........................(A.146)}
\]
and:

\[
\left. \frac{\partial p_{fD,cr}}{\partial r_D} \right|_{r_D=1} = 0, \tag{A.147}
\]

respectively. Given that the inner boundary condition corresponds to the segment defined by \(r_{D0}\) and \(r_{D1}\), it must be applied to Eq. A.139. Applying the Laplace transform to Eq. A.146 and then the boundary condition to Eq. A.139, the following expression is derived:

\[
\left. \frac{\partial p_{fD,cr,1}}{\partial r_D} \right|_{r_D=r_{wD}} = r_{wD} \sqrt{\frac{u}{\eta_{fD}}} A_1 K_1 \left[ r_{wD} \sqrt{\frac{u}{\eta_{fD}}} \right] - B_1 I_1 \left[ r_{wD} \sqrt{\frac{u}{\eta_{fD}}} \right] = -\frac{1}{F_{hcD} u} \tag{A.148}
\]

Rewriting Eq. A.148:

\[
A_1 K_1 \left[ r_{wD} \sqrt{\frac{u}{\eta_{fD}}} \right] - B_1 I_1 \left[ r_{wD} \sqrt{\frac{u}{\eta_{fD}}} \right] = \frac{\sqrt{\eta_{fD}}}{F_{hcD} r_{wD} u^{3/2}} \tag{A.149}
\]

Similarly, applying the outer boundary condition to Eq. A.143:

\[
A_5 K_1 \left[ \sqrt{\frac{u}{\eta_{fD}}} \right] - B_5 I_1 \left[ \sqrt{\frac{u}{\eta_{fD}}} \right] = 0, \tag{A.150}
\]

2. Continuity of pressure at the interfaces of the discretized fracture.

To establish continuity of the pressure along the circular fracture, the pressure must be equal at every interface, i.e.,

\[
\bar{p}_{fD,cr,j-1}(r_{Dj-1}, u) = \bar{p}_{fD,cr,j}(r_{Dj-1}, u), \tag{A.151}
\]

for \(2 \leq j \leq N\). For the segments defined in Fig. A.4:
3. Continuity of flowrate at the interfaces of the discretized fracture.

In addition to the continuity of the pressure, continuity of the flowrate at every interface along the circular fracture must also be considered, \(i.e.,\)

\[
\left[ \frac{\partial \bar{p}_{jD,cr,j-1}}{\partial r_D} \right]_{r_D = r_{Dj-1}} = \frac{\partial \bar{p}_{jD,cr,j}}{\partial r_D} \bigg|_{r_D = r_{Dj-1}}
\]  

\(.................(A.156)\)

for \(2 \leq j \leq N\). For the segments defined in \textbf{Fig. A.4}:
4. Continuity of pressure between the fracture and the 3D reservoir.

So far, ten out of the fifteen equations needed to obtain the fifteen unknowns in Eqs.A.139 – A.143 have been set up. The remaining five are defined by superposing the pressure of the 3D reservoir on the plane of the circular fracture (see Fig. A.5). To show this procedure, consider the "point well" solution obtained in section A.3 (Eq. A.138):

$$\bar{p}_D(R_D,u) = \frac{\bar{Q}_D h D}{2R_D} \exp\left[-R_D \sqrt{u}\right].$$

The superposition of Eq. A.138 over the area of the fracture is given by the double integral over the domain of the angle, \(\theta\), and the radius \(r_D\):

$$\bar{p}_{Dcr, j} = \frac{\bar{Q}_D j h D}{2} \frac{r_D \max}{r_D \min} \int_0^\theta D \exp\left[-R_D \sqrt{u}\right] d\theta dr_D.$$

\((A.161)\)
Integrating Eq. A.161 over the domain of $\theta$:

$$
\overline{p}_{Dcr,j} = \frac{\delta h_{Dj} \max_r}{2} \int_{r_{Dmin}}^{r_{Dmax}} \frac{r_D}{R_D} \theta(r_D) \exp[-R_D \sqrt{\theta}] dr_D \cdots (A.162)
$$

The value of the angle $\theta$ depends on the position along the $r_D$-axis. To obtain a function that relates these two variables, consider an observation point $(r^*_D, 0, 0)$ in cartesian coordinates. On the other hand, the equation of a circumference with center $(h, k)$ and radius $r_D$ in the $XY$-plane is:

$$
(x-h)^2 + (y-k)^2 = r^2_D \cdots (A.163)
$$

Taking $r^*_D$ as the center of the circumference, then Eq. A.163 becomes:

$$
x^2 - 2xr^*_D + (r^*_D)^2 + y^2 = r^2_D \cdots (A.164)
$$

Based on the definition of the radius of a circle in cartesian coordinates:
\[ r_D^2 = x^2 + y^2, \quad \text{.................................................................}(A.165) \]

and the projection of the radius on the x-axis:

\[ x = r_D \cos \theta. \quad \text{.................................................................}(A.166) \]

Substituting Eqs. A.165 and A.166 in Eq. A.164:

\[-2r_{Dj}r_D \cos \theta + (r_{Dj}^*)^2 + r_D^2 = r_{Dj}^2 \quad \text{.................................................................}(A.167)\]

Solving Eq. A.167 for \( \theta \):

\[ \theta = \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - r_{Dj}^2 + r_D^2}{2r_{Dj}r_D} \right]. \quad \text{.................................................................}(A.168) \]

Substituting Eq. A.168 in Eq. A.162:

\[ \overline{p}_{Dcr,j}(R_D, z_{0D}, u) = \frac{\delta h_D}{2} \frac{r_D \max}{R_D \cos^{-1}} \left[ \frac{(r_{Dj}^*)^2 - r_{Dj}^2 + r_D^2}{2r_{Dj}r_D} \right] \exp \left[ -R_D \sqrt{u} \right] r_D \quad \text{.................................................................}(A.169) \]

The radius of the spherical reservoir \( (R_D) \) is related to the radius of the circular fracture \( (r_D) \) as:

\[ R_D^2 = r_D^2 + z_{0D}^2. \quad \text{.................................................................}(A.170) \]

where \( z_{0D} \) is the observation point along the wellbore.

To provide the remaining expressions to complete the system of equations, the solution of the pressure at the circular fracture should be evaluated at certain \( r_{Dj}^* \)-value and equated to an
expression that represents the pressure of the reservoir at the same point. Total pressure acting on a \( r_{Dj}^* \) point located at \( j \)-segment defined in Fig. A.5 is determined by:

\[
\bar{p}_{Dcr, j}(r_{Dj}^*, u) = \frac{h_D}{2} \sum_{i=1}^{5} \Delta q_{D, j}[G_c(r_{Dj}^*, r_{Di}, u) - G_c(r_{Dj}^*, r_{Di-1}, u)]
\]  ..................................................(A.171)

where:

\[
G_c(r_{Dj}^*, r_{Di}, u) = \frac{r_{Dj}^{max}}{r_{Dj}^{min}} \frac{r_D}{R_D} \cos^{-1}\left[\frac{(r_{Dj}^*)^2 - r_{Di}^2 + r_D^2}{2r_{Dj}^*r_D}ight] \exp\left[-R_D\sqrt{u}\right]dR_D
\]  ..................................................(A.172)

Larsen et al. (1991) used Simpson's rule to perform the integral in Eq. A.172 in their calculations. To perform the integral in Eq. A.172, such an equation must be arranged to consider the relative position of the observation point and the integration limits. This arrangement is shown at the end of this subsection.

It is convenient to define the observation points, \( r_{Dj}^* \), at the midpoint of every segment of the discretized circular fracture (see Fig. A.6). At the observation points, the dimensionless pressure of the circular fracture and the dimensionless pressure of the 3D reservoir are equal. Evaluating Eq. A.65 at the generic observation point, \( r_{Dj}^* \), and equating this expression to Eq. A.171 results in:

\[
A_j K_0 \left[ r_{Dj}^* \sqrt{\frac{u}{\eta_{jD}}} \right] + B_j I_0 \left[ r_{Dj}^* \sqrt{\frac{u}{\eta_{jD}}} \right] - \frac{2\pi \eta_{jD}}{uF_{hcD}} \Delta q_{D, j} = \frac{h_D}{2} \sum_{i=1}^{5} \Delta q_{D, j}[G_c(r_{Dj}^*, r_{Di}, u) - G_c(r_{Dj}^*, r_{Di-1}, u)]
\]  ..................................................(A.173)
Figure A.6 — Definition of the observation points to define the continuity of pressure between the circular fracture and the 3D reservoir.

Then, the following set of equations should be used for the case presented in Fig. A.6:

\[
A_1 K_0 \left[ r_{D1}^* \frac{u}{\eta_f D} \right] + B_1 I_0 \left[ r_{D1}^* \frac{u}{\eta_f D} \right] - \frac{2\pi \eta_f D}{uF_{hcD}} \bar{\alpha}_{D1} = \\
\frac{h_D}{2} \sum_{i=1}^{5} \alpha_{D1,i} \left[ G_c(r_{D1}^*,r_{Di},u) - G_c(r_{Di},r_{Di-1},u) \right]
\]

\[
A_2 K_0 \left[ r_{D2}^* \frac{u}{\eta_f D} \right] + B_2 I_0 \left[ r_{D2}^* \frac{u}{\eta_f D} \right] - \frac{2\pi \eta_f D}{uF_{hcD}} \bar{\alpha}_{D2} = \\
\frac{h_D}{2} \sum_{i=1}^{5} \alpha_{D2,i} \left[ G_c(r_{D2}^*,r_{Di},u) - G_c(r_{D2},r_{Di-1},u) \right]
\]

\[
A_3 K_0 \left[ r_{D3}^* \frac{u}{\eta_f D} \right] + B_3 I_0 \left[ r_{D3}^* \frac{u}{\eta_f D} \right] - \frac{2\pi \eta_f D}{uF_{hcD}} \bar{\alpha}_{D3} = \\
\frac{h_D}{2} \sum_{i=1}^{5} \alpha_{D3,i} \left[ G_c(r_{D3}^*,r_{Di},u) - G_c(r_{D3},r_{Di-1},u) \right]
\]
Considerations for the integration of the $G_c$-function

Recall Eq. A.172:

$$G_c(r_j^*, r_{D_k}, u) = \int_{r_{D_j}^*}^{r_{D_k}} \frac{r_d}{R_D} \cos^{-1} \left[ \frac{(r_d)^2 - r_{D_k}^2 + r_d^2}{2r_d r_D} \right] \exp \left[ -R_D \sqrt{u} \right] dr_d .$$

If the observation point, $r_j^*$, in Eq. A.172 is greater than the upper integration limit ($r_{D_k} < r_j^*$), the integration limits should be adjusted to consider the circumference created by radius $r_{D_k}$ (see Fig. A.7). The lower integration limit is redefined as:

$$r_{D_{min}} = r_j^* - r_{D_k} \hspace{1cm} \text{.................................(A.179)}$$

and the upper integration limit is:

$$r_{D_{max}} = r_j^* + r_{D_k} \hspace{1cm} \text{.................................(A.180)}$$
Figure A.7 — Definition of the integration limits for $r_D$ when the observation point, $r^*_Dj$, is greater than the limit $r_Dk$.

If the observation point in Eq. A.172 is lower than the upper integration limit ($r_Dk > r^*_Dj$), then the lower limit, $r_{Dmin}$, will be zero and the upper integration limit, $r_{Dmax}$, is equal to $r_Dk$ (see Fig. A.8).

For this case, there is a "natural lower bound" defined by:

$$r_\alpha = r_Dk - r^*_Dj$$

\[\text{(A.181)}\]

Figure A.8 — Definition of the integration limits for $r_D$ when the observation point, $r^*_Dj$, is lower than the limit $r_Dk$.

The numerical integration when $r_Dk > r^*_Dj$ causes some difficulties given that the lower integration limit is zero. Therefore, it is convenient to use the "natural lower bound" to split the integral to avoid such difficulties. To do so, it is more convenient to write Eq. A.172 in terms of $R_D$:

$$G_c(r^*_Dj, r_Dk, \mu) = \frac{R_{Dmax}}{R_{Dmin}} \left[ \frac{(r^*_Dj)^2 - r_Dk^2 + R_D^2 - z_0^2}{2r_D^2 \sqrt{R_D^2 - z_0^2}} \right] \exp \left[ - R_D \sqrt{\mu} \right] R_D \text{ } \left[ \text{.....(A.182)} \right]$$

where:

$$R_{Dmin} = \sqrt{r_D^2 + z_0^2} \text{ } \left[ \text{.....(A.183)} \right]$$
and:

\[
R_{D\text{max}} = \sqrt{\frac{r_{Dk}^2 + z_{0D}^2}{R_D^2}} \] ..............................(A.184)

Splitting the integral in Eq. A.182 at the "natural lower bound" it becomes:

\[
G_c(r_{Dj}^*, r_{Dk}, u) = \int_{R_D\min}^{R_D\max} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - r_{Dk}^2 + R_D^2 - z_{0D}^2}{2r_{Dj}^* \sqrt{R_D^2 - z_{0D}^2}} \right] \exp \left[ - R_D \sqrt{u} \right] dR_D + \\
\int_{R_D\alpha}^{R_D\max} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - r_{Dk}^2 + R_D^2 - z_{0D}^2}{2r_{Dj}^* \sqrt{R_D^2 - z_{0D}^2}} \right] \exp \left[ - R_D \sqrt{u} \right] dR_D ..........................(A.185)
\]

where:

\[
R_D\alpha = \sqrt{(r_{Dk} - r_{Dj}^*)^2 + z_{0D}^2} \] ...............................(A.186)

Considering the trigonometric identity:

\[
\cos^{-1}(-x) = \pi - \cos^{-1}(x), \] .................................(A.187)

Eq. A. 185 is rewritten as:

\[
G_c(r_{Dj}^*, r_{Dk}, u) = \pi \int_{R_D\min}^{R_D\alpha} \exp \left[ - R_D \sqrt{u} \right] dR_D - \\
\int_{R_D\alpha}^{R_D\min} \cos^{-1} \left[ \frac{r_{Dk}^2 + z_{0D}^2 - (r_{Dj}^*)^2 + R_D^2}{2r_{Dj}^* \sqrt{R_D^2 - z_{0D}^2}} \right] \exp \left[ - R_D \sqrt{u} \right] dR_D + \\
\int_{R_D\alpha}^{R_D\max} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - r_{Dk}^2 + R_D^2 - z_{0D}^2}{2r_{Dj}^* \sqrt{R_D^2 - z_{0D}^2}} \right] \exp \left[ - R_D \sqrt{u} \right] dR_D ..........................(A.188)
\]

Integrating the first term on the right hand side:
Given that the lower limit, $r_{D\text{min}}$, is zero and the observation point $z_{0D}$ is also zero, Eq. A.183 reduces to:

$$R_{D\text{min}} = 0$$ .................................................................(A.190)

and Eq. A.186 becomes:

$$R_{D\alpha} = R_{Dk} - r_{Dj}^*$$ .................................................................(A.191)

Consequently, the integration interval of the integral of the second term in the right hand side of Eq. A.189 is defined over the $[1, \infty)$-interval, which is out of the real domain of the $\cos^{-1}$-function. Therefore, such an integral is equal to zero and Eq. A.189 reduces to:

$$G_c(r_{Dj}^*, r_{Dk}, u) = \pi[\exp[-R_{D\alpha}\sqrt{u}] - \exp[-R_{D\text{min}}\sqrt{u}]] +$$

$$R_{D\text{max}} \left[ \frac{(r_{Dj}^* - r_{Dk}^2 + R_{D}^2 - z_{0D}^2)}{2r_{Dj}^* \sqrt{R_{D}^2 - z_{0D}^2}} \right] \exp[-R_{D\sqrt{u}}]dR_{D}$$ ...........................................(A.192)
A.5. Constant-Rate Solution for the Model of a Horizontal Well Intercepting a Rectangular Longitudinal Finite Conductivity Fracture within a 3D Reservoir

The procedure to develop the constant-rate solution for the rectangular longitudinal fracture case within a 3D reservoir is essentially the same as in the previous section. Therefore, recall the solution developed in section A.2 for a discretized rectangular fracture (Eq. A.99):

\[
\bar{p}_{jD,cr,j}(x_D,u) = A_j \exp \left( -x_D \frac{u}{\eta_D} \right) + B_j \exp \left( x_D \frac{u}{\eta_D} \right) - \frac{2\pi \eta_D}{F_{hcDu}} \bar{Q}_{D,j}.
\]

**Fig A.9** shows an example of a rectangular fracture logarithmically discretized into three segments. A discretization in thirty segments is recommended. However, for the sake of providing an example, three segments will be used. The dimensionless pressure response of the discretized rectangular fracture shown in **Fig. A.9** is given by the set of equations:

\[
\bar{p}_{jD,cr,1}(x_D,u) = A_1 \exp \left( -x_D \frac{u}{\eta_D} \right) + B_1 \exp \left( x_D \frac{u}{\eta_D} \right) - \frac{2\pi \eta_D}{F_{hcDu}} \bar{Q}_{D,1} (x_{D0} \leq x_D \leq x_{D1}), \quad \text{......(A.193)}
\]

\[
\bar{p}_{jD,cr,2}(x_D,u) = A_2 \exp \left( -x_D \frac{u}{\eta_D} \right) + B_2 \exp \left( x_D \frac{u}{\eta_D} \right) - \frac{2\pi \eta_D}{F_{hcDu}} \bar{Q}_{D,2} (x_{D1} \leq x_D \leq x_{D2}), \quad \text{......(A.194)}
\]

\[
\bar{p}_{jD,cr,3}(x_D,u) = A_3 \exp \left( -x_D \frac{u}{\eta_D} \right) + B_3 \exp \left( x_D \frac{u}{\eta_D} \right) - \frac{2\pi \eta_D}{F_{hcDu}} \bar{Q}_{D,3} (x_{D2} \leq x_D \leq x_{D3}), \quad \text{......(A.195)}
\]

This approach results in a set of nine unknowns (3N-unknowns for N-segments of the rectangular fracture). To determine the values of the unknowns the procedure shown in section A.4 is also applied for this case.
1. Boundary conditions of the rectangular fracture.

The inner boundary condition for the well intercepting the rectangular fracture and producing at constant rate is:

\[
\left[ \frac{\partial p}{\partial x} \right]_{x=0} = \frac{qB\mu}{4kf_w^w} \]

\( \text{(A.196)} \)

Eq. A.196 expressed in dimensionless variables is:

\[
\left[ \frac{\partial p f_{D,\text{cr}}}{\partial x_D} \right]_{x_D=0} = -\frac{\pi}{2F_{hcD}y_wf} \]

\( \text{(A.197)} \)

where dimensionless horizontal length of the fracture is:

\[
y_{fD} = \frac{y}{x_f} \]

\( \text{(A.198)} \)
For this model, it is considered a closed hydraulic fracture. Therefore the outer boundary condition is defined by:

$$\left[ \frac{\partial p_f}{\partial x} \right]_{x=x_f} = 0,$$

(A.199)

or, expressed in dimensionless variables:

$$\left[ \frac{\partial p_{fcr}}{\partial x_D} \right]_{x_D=1} = 0,$$

(A.200)

Transforming the inner boundary condition (A.197) to the Laplace domain and then applying it to Eq. A.193 (segment defined by \( x_{D0} \) and \( x_{Di} \)), this results in the expression:

$$A_1 - B_1 = \frac{\pi \sqrt{\eta_{/D}}}{2F_{hcD} y_{/D} u^{3/2}}.$$

(A.201)

Similarly, applying the outer boundary condition (Eq. A.200) to Eq. A.195:

$$A_3 \exp\left[ -\frac{u}{\eta_{/D}} \right] - B_3 \exp\left[ \frac{u}{\eta_{/D}} \right] = 0.$$

(A.202)

2. Continuity of pressure at the interfaces of the discretized fracture.

The continuity of the pressure for this case is defined by:

$$\bar{p}_{fcr,j} (x_{Dj-1}, u) = \bar{p}_{fcr,j} (x_{Dj-1}, u),$$

(A.203)

for \( 2 \leq j \leq N \). For the segments defined in Fig. A.9:
3. Continuity of flowrate at the interfaces of the discretized fracture.

The continuity of flowrate is defined by:

\[
\begin{align*}
\frac{\bar{\rho}_{fD,cr,j-1}}{\bar{\rho}_{fD,cr,j}} &= \frac{\bar{\rho}_{fD,cr,j}}{\bar{\rho}_{fD,cr,j-1}}_{x_D = x_{Dj-1}} \quad \text{for } 2 \leq j \leq N.
\end{align*}
\]

For the example given in Fig. A.9:

\[
\begin{align*}
-A_1 \exp\left[-x_{D1} \sqrt{\frac{u}{\eta / fD}}\right] + B_1 \exp\left[x_{D1} \sqrt{\frac{u}{\eta / fD}}\right] &= \\
-A_2 \exp\left[-x_{D1} \sqrt{\frac{u}{\eta / fD}}\right] + B_2 \exp\left[x_{D1} \sqrt{\frac{u}{\eta / fD}}\right] \quad (x_{D1}\text{-interface}), \ldots (A.207)
\end{align*}
\]

\[
\begin{align*}
-A_2 \exp\left[-x_{D2} \sqrt{\frac{u}{\eta / fD}}\right] + B_2 \exp\left[x_{D2} \sqrt{\frac{u}{\eta / fD}}\right] &= \\
-A_3 \exp\left[-x_{D2} \sqrt{\frac{u}{\eta / fD}}\right] + B_3 \exp\left[x_{D2} \sqrt{\frac{u}{\eta / fD}}\right] \quad (x_{D2}\text{-interface}), \ldots (A.208)
\end{align*}
\]

\[
\begin{align*}
\frac{2\pi \eta fD}{F_{hcDu}} \Delta \bar{q}_{D,1} &= \\
\frac{2\pi \eta fD}{F_{hcDu}} \Delta \bar{q}_{D,2} &= \\
\frac{2\pi \eta fD}{F_{hcDu}} \Delta \bar{q}_{D,3} &= \\
\end{align*}
\]
4. Continuity of pressure between the fracture and the 3D reservoir.

To provide the remaining expressions to complete the system of equations for this model, consider the superposition of the dimensionless pressure of the 3D reservoir on the plane of the rectangular fracture (analogous to the circular fracture case):

$$
\bar{p}_{Dcr,j}(r_{Dj}^*, u) = \frac{h_D}{2} \sum_{i=1}^{N} \delta_j \delta_{Dj} G_R(r_{Di}^*, \theta, u). \tag{A.209}
$$

where:

$$
G_R(r_{Di}^*, \theta, u) = \int_{r_D}^{R_D} \int_{\theta}^{\theta_D} \frac{1}{r_D \theta R_D} \exp[-R_D \sqrt{u}] r_D d\theta dr_D. \tag{A.210}
$$

The integral in Eq. A.210 is defined in polar coordinates. To integrate a rectangular surface using polar coordinates consider Fig. A.10. Note that a rectangular area can be subdivided into three semicircular areas, each one defined by minimum and maximum radii and an angle.

Given that $r_D$ and $\theta$ are defined depending on the observation point $(x_{Dk}^*, y_{Dk}^*)$ and any coordinate $(x_{iD}, y_{jD})$, it is convenient to use the following notation in terms of function $Z_R$:

$$
G_R(r_{Di}^*, \theta, u) = Z_R(x_{Dk}^*, y_{Dk}^*, x_{iD}, y_{jD}, x_{2D}, y_{2D}, u). \tag{A.211}
$$

Based on Fig A.10, observe that the minimum and maximum radii and angle depend on the segment (semicircular area) of the rectangle. Hence, the integrals $n Z_R$-function can be written as:
Figure A.10 — Subdivision of a rectangular area into semicircular areas.

Integrating Eq. A.212 with respect to $\theta$

$$Z_R(x_{Dk}^*,y_{Dk}^*,x_{1D},y_{1D},x_{2D},y_{2D},u) = \frac{r_{D2}^2 \theta_{\text{max},I}}{r_{D1} \theta_{\text{min},I}} \frac{1}{R_D} \exp[-R_D \sqrt{u}] \int_{\theta_{\text{min},I}}^{\theta_{\text{max},I}} \frac{r_D}{R_D} d\theta dr_D + \frac{r_{D3}^2 \theta_{\text{max},II}}{r_{D2} \theta_{\text{min},II}} \frac{1}{R_D} \exp[-R_D \sqrt{u}] \int_{\theta_{\text{min},II}}^{\theta_{\text{max},II}} \frac{r_D}{R_D} d\theta dr_D + \frac{r_{D4}^2 \theta_{\text{max},III}}{r_{D3} \theta_{\text{min},III}} \frac{1}{R_D} \exp[-R_D \sqrt{u}] \int_{\theta_{\text{min},III}}^{\theta_{\text{max},III}} \frac{r_D}{R_D} d\theta dr_D$$

(A.212)

where:

$$\theta_{\text{min},I} = \sin^{-1} \left[ \frac{x_{1D}}{r_D} \right]$$

(A.214)
\[
\theta_{\text{max}, I} = \cos^{-1} \left[ \frac{y_1 D}{r_D} \right] \hspace{1cm} \text{(A.215)}
\]

\[
\theta_{\text{min}, II} = \sin^{-1} \left[ \frac{x_1 D}{r_D} \right] \hspace{1cm} \text{(A.216)}
\]

\[
\theta_{\text{max}, II} = \sin^{-1} \left[ \frac{x_2 D}{r_D} \right] \hspace{1cm} \text{(A.217)}
\]

\[
\theta_{\text{min}, III} = \cos^{-1} \left[ \frac{y_2 D}{r_D} \right] \hspace{1cm} \text{(A.218)}
\]

\[
\theta_{\text{max}, III} = \sin^{-1} \left[ \frac{x_2 D}{r_D} \right] \hspace{1cm} \text{(A.219)}
\]

Because of the reference system, angles \( \theta_{\text{min}, II} \) and \( \theta_{\text{max}, II} \) are different from the ones reported by Larsen et al. (1991). However the expressions yield the same results.

If the observation point is located at the origin, then \( r_{D1} \) and \( \theta_{\text{min}, I} \) (Eq. A.214) become zero, and \( \theta_{\text{max}, I} = \pi/2 \). Consequently, the first integral on the right hand side of Eq. A.213 can be performed analytically. The resulting equation is:

\[
Z_R(x_i^*, y_i^*, x_1D, y_1D, x_2D, y_2D, u) = \frac{\pi}{2\sqrt{u}} \left[ \exp[-z_0D\sqrt{u}] - \exp\left[-\sqrt{u}(r_{D2}^2 - z_0^2D)\right] \right] + \\
\int_{r_{D2}}^{r_{D3}} \left[ \theta_{\text{max}, II} - \theta_{\text{min}, II} \right] \frac{r_D}{R_D} \exp[-R_D\sqrt{u}] dr_D + \ldots \ldots \hspace{1cm} \text{(A.220)}
\]

The observation point \((x_i^*D, y_i^*D)\) should be selected exactly where the pressure of the 3D reservoir will be evaluated. Such points should correspond to the center point of each segment of the
discretized rectangular fracture (see Fig. A.8). To perform the integrals given in Eq. A.220, it is
convenient to always translate the observation point to the origin of the coordinate system by
resetting the observation point \((x^*_D, y^*_D)\) as the origin \((0,0)\). The translation of the rest of the points
are made by rescaling the \(x\)-axis as

\[
x'_{jD} = x_{jD} - x^*_D, \quad \text{................................................................. (A.221)}
\]

and \(y\)-axis as:

\[
y'_{jD} = y_{jD} - y^*_D, \quad \text{................................................................. (A.222)}
\]

This translation of axes allows some of the variables of the \(Z_R\)-function to drop. It is redefined as:

\[
Z_R(x'_{1D}, y'_{1D}, x'_{2D}, y'_{2D}, u) = Z_R(x^*_{Dk}, y^*_{Dk}, x_{1D}, y_{1D}, x_{2D}, y_{2D}, u) \quad \text{................................................................. (A.223)}
\]

After the axes translation, some of the coordinates can become negative. If that is the case, the
area (or areas) of the rectangle that relies on the negative section of the translated reference system
can be translated again to be considered in the positive section. These cases are summarized and
graphically shown in Table A.1. For such cases, the \(Z_R\)-function should be mathematically treated
as follows:

\[
Z_R(x'_{1D}, y'_{1D}, x'_{2D}, y'_{2D}, u) = Z_R(0, x'_{1D}, |x'_{1D}|, y'_{2D}, u) + Z_R(0, y'_{1D}, x'_{2D}, y'_{2D}, u) \quad \text{................................................................. (A.224)}
\]

\[
Z_R(x'_{1D}, y'_{1D}, x'_{2D}, y'_{2D}, u) = Z_R(x'_{1D}, 0, x'_{2D}, |y'_{1D}|, u) + Z_R(x'_{1D}, 0, y'_{2D}, y'_{2D}, u) \quad \text{................................................................. (A.225)}
\]
\[ Z_R(x'_{1D}, y'_{1D}, x'_{2D}, y'_{2D}, u) = Z_R(0,0, x'_{2D}, y'_{2D}, u) + \\
Z_R(0,0, x'_{2D}, y'_{1D}, u) + \\
Z_R(0,0, x'_{1D}, y'_{1D}, u) + \\
Z_R(0,0, x'_{1D}, y'_{2D}, u) \]

\((x'_{1D} < 0 < x'_{2D} \text{ and } y'_{1D} < 0 < y'_{2D}), \ldots (A.226)\)

As shown in Fig. A.9, the continuity of pressure between the rectangular fracture and the 3D reservoir generates three equations by comparing the dimensionless pressure of the fracture (Eq. A.99) and the superposed dimensionless pressure of the 3D reservoir over each segment of the fracture (Eq. A.209). Both equations should be evaluated at the observation points defined in Fig. A.9. These equations are:

\[ A_1 \exp \left[-x_{D1}^* \frac{u}{\eta fD} \right] + B_1 \exp \left[x_{D1}^* \frac{u}{\eta fD} \right] - \frac{2\pi \eta fD}{F_{hcD} u} \sum_{i=1}^{3} \widetilde{\alpha} \left(x_{D1,i}^* \right) = 0 \]

\[ A_2 \exp \left[-x_{D2}^* \frac{u}{\eta fD} \right] + B_2 \exp \left[x_{D2}^* \frac{u}{\eta fD} \right] - \frac{2\pi \eta fD}{F_{hcD} u} \sum_{i=1}^{3} \widetilde{\alpha} \left(x_{D2,i}^* \right) = 0 \]

\[ A_3 \exp \left[-x_{D3}^* \frac{u}{\eta fD} \right] + B_3 \exp \left[x_{D3}^* \frac{u}{\eta fD} \right] - \frac{2\pi \eta fD}{F_{hcD} u} \sum_{i=1}^{3} \widetilde{\alpha} \left(x_{D3,i}^* \right) = 0 \]

In general, for a discretization into \(N\)-segments:
$$A_j \exp \left[ -x_{Dj}^* \sqrt{\frac{u}{\eta fD}} \right] + B_j \exp \left[ x_{Dj}^* \sqrt{\frac{u}{\eta fD}} \right] - \frac{2\pi \eta fD}{F_hC\nu} \bar{\eta}D_{ij} =$$

$$= \frac{hD}{2} \sum_{i=1}^{N} \bar{\eta}D_{ij} Z_R (x_{Dj}^*, y_{Dj}^*, x_{Dj-1}, y_{Dj-1})$$

---

**Table A.1** — Translation of the observation points and rectangular areas.

<table>
<thead>
<tr>
<th>Original set</th>
<th>After Translation</th>
<th>Equivalent geometry due to symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Case I**
- Observation point

**Case II**
- Observation point

**Case III**
- Observation point
APPENDIX B
PRESSURE-TRANSIENT ANALYSIS OF FRACTAL RESERVOIRS

This Appendix is divided into two sections. In the first section, we show the detailed derivation of the diffusivity equation presented by Chang et al. (1990) to depict the pressure-transient behavior of a fractal reservoir. In the second section, we present the constant-rate solution of such a diffusivity equation, assuming an infinite fractal reservoir.

B.1. Development of the Diffusivity Equation for a Fractal Reservoir

Consider that the number of permeable sites contained in a volume with characteristic length \( R \) is determined by a power-law:

\[
N(R) = \sigma R^{D_f - 1}, \text{ where } D_f \text{ is the fractal dimension of the permeable sites.}
\]

where \( D_f \) is the fractal dimension of the permeable sites. The sign of the exponent \( D_f \) depicts the proportionality between the permeable sites and the volume.

Assuming that all permeable sites in the reservoir have the same volume, \( V_s \), the pore volume of the permeable sites is determined by:

\[
PV(R) = V_s N(R), \text{ where } PV(R) \text{ is the pore volume of the permeable sites.}
\]

To obtain the porosity of the fractal reservoir, the pore volume must be divided by the volume of the reservoir. Chang et al., (1990) defined the size of the reservoir in terms of a geometry factor, \( G \). However, it is more appropriate to use the generalized definition (Barker, 1988):

\[
V_R(R) = \alpha_d R^{d-1} b^{3-d} \Delta R, \text{ where } \alpha_d \text{ is a constant.}
\]
where $d$ is the Euclidean dimension and $\alpha_d$ is the area of a unit sphere in $d$ dimensions:

$$\alpha_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \ldots$$

Then, if $d=1$, the reservoir would have the shape of a rectangular prism with two sides of size $b$ and one of size $2R$. If $d=2$, the reservoir would have the shape of a cylinder of radius $R$ and height $b$. If $d=3$, the reservoir would have the shape of a sphere of radius $R$.

Combining Eq. B.2 and Eq. B.3:

$$\phi(R) = \frac{PV}{V_r} = \phi_0 R^{D_f-d}, \ldots$$

where $\phi_0$ is a reference porosity defined by:

$$\phi_0 = \frac{\sigma V_s}{\alpha_d b^{3-d}}, \ldots$$

To develop the Darcy's law equation for a fractal system, consider the Hagen-Poiseuille equation to determine the flowrate of a fluid of viscosity $\mu$ through a capillary tube of radius $r_c$ and length $R$:

$$q_u = \frac{\pi r_c^4}{8\mu} \frac{\partial p}{\partial R}, \ldots$$

For a system of $n_p$ capillary tubes with the same shape, the total flow rate of the system is given by:

$$q_u = n_p \frac{\pi r_c^4}{8\mu} \frac{\partial p}{\partial R}, \ldots$$
An equivalent number of capillary tubes of volume $V_c$ contained in a reservoir volume $V_r$ with pore volume $PV$ is determined by:

$$n_p = \frac{PV}{V_c}$$  \hspace{1cm} (B.9)

Combining Eqs. B.1, B.2, B.8 and B.9 gives us:

$$q_u = \frac{\sigma R D f^{-1} V_s}{V_c} \frac{\pi c^4}{8\mu} \frac{\partial p}{\partial R}$$  \hspace{1cm} (B.10)

Assuming that the volume of the permeable site and the volume of the capillary tube are equal and that the radius of the capillary tube is a power-law function of the characteristic length of the reservoir, $R$:

$$r_c = R^{-\theta/4}$$  \hspace{1cm} (B.11)

Substituting Eq. B.11 in Eq. B.10 results in the following expression:

$$q_u = \frac{c^* R D f^{-1} \theta - 1}{\mu} \frac{\partial p}{\partial R}$$  \hspace{1cm} (B.12)

where:

$$c^* = \frac{\pi c^4}{8}$$  \hspace{1cm} (B.13)

On the other hand, consider Darcy's law with variable permeability:

$$q_R = \frac{k(R) A \frac{\partial p}{\partial R}}{\mu}$$  \hspace{1cm} (B.14)
where $A$ is the area exposed to flow. Based on Eq. B.3, the area exposed to flow can be defined as:

$$A = \alpha_d b^{3-d}.$$ .................................................................(B.15)

Assuming that the variation of permeability can be represented by a power-law function:

$$k(R) = k_0 R^\beta,$$ .................................................................(B.16)

where $k_0$ is a reference permeability. Then, substituting Eq. B.15 and Eq. B.16 in Eq. B.14 results in:

$$q_R = R^\beta \alpha_d b^{3-d} \frac{k_0}{\mu} \frac{\partial p}{\partial R}.$$ .................................................................(B.17)

Alternatively, using Eq. B.6:

$$q_R = \frac{\sigma V_s}{\phi_0} R^\beta k_0 \frac{\partial p}{\partial R}.$$ .................................................................(B.18)

Comparing Eq. B.12 and Eq. B.17, it is concluded that

$$\xi^* = \frac{\sigma V_s}{\phi_0}.$$ .................................................................(B.19)

and that index $\beta$ is related to the distribution of the sites, $D_f$ and the connectivity of them:

$$\beta = D_f - \theta - 1.$$ .................................................................(B.20)

To develop the diffusivity equation for a fractal reservoir, consider the incoming mass of fluid, $m_{in,R}$ through an infinitesimal section of the reservoir can be expressed as:
Similarly, the outgoing mass of fluid:

\[ m_{out, R} = [-\rho q R + \Delta(\rho q R)] \Delta t \] \hspace{1cm} (B.22)

The cumulative mass of fluid is determined by:

\[ m_c = m_{out, R} - m_{in, R} = \Delta(\rho q R) \Delta t \] \hspace{1cm} (B.23)

The mass of a single saturating fluid at an initial time can be expressed as:

\[ m_{t1} = \phi q V_r \] \hspace{1cm} (B.24)

Analogously, at a final time the mass of fluid is:

\[ m_{t2} = \phi q V_r + \Delta(\phi q) V_r \] \hspace{1cm} (B.25)

The cumulative mass of fluid can be also given by:

\[ m_c = m_{t2} - m_{t1} = \Delta(\phi q) V_r \] \hspace{1cm} (B.26)

Equating Eq. B.23 and Eq. B.26:

\[ \Delta(\rho q R) \Delta t = \Delta(\phi q) V_r \] \hspace{1cm} (B.27)

Based on Eq. B.3, Eq. B.26 can be rewritten as:

\[ \frac{1}{\alpha_d R^{d-1} b^{d-1} \Delta R} = \frac{\Delta(\phi q)}{\Delta t} \] \hspace{1cm} (B.28)

Taking the limits of \( \Delta R \) and \( \Delta t \) to zero, Eq. B.28 takes the differential form:
\[
\frac{1}{\alpha_d R^{d-1} b^{3-d}} \frac{\partial (\rho q_R)}{\partial R} = \frac{\partial (\phi)}{\partial t} \hspace{1cm} \text{(B.29)}
\]

Applying the derivative on the right hand side of Eq. B.29 and arraying, it results in:

\[
\frac{1}{\alpha_d R^{d-1} b^{3-d}} \frac{\partial (\rho q_R)}{\partial \phi} = \frac{\partial (\rho)}{\partial t} \hspace{1cm} \text{(B.30)}
\]

Substituting the definition of the porosity (Eq. B.5) and Darcy's equation in its fractal form (Eq. B.18) in Eq. B.30, such an expression reduces to:

\[
\frac{1}{R^{D_f-1}} \frac{\partial}{\partial R} \left[ \rho \frac{R^\beta k_0 \partial \phi}{\mu \partial R} \right] = \rho \left[ \frac{1}{\phi} \frac{\partial \phi}{\partial t} + \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right] \hspace{1cm} \text{(B.31)}
\]

Applying the derivative on the left hand side of Eq. B.29 gives the expression:

\[
\frac{1}{R^{D_f-1}} \left[ \frac{\partial}{\partial R} \left( R^\beta \frac{\partial p}{\partial R} + \frac{R^\beta}{\rho} \frac{\partial \rho}{\partial R} \right) \right] = \frac{\phi_0 k}{k_0} \left[ \frac{1}{\phi} \frac{\partial \phi}{\partial t} + \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right] \hspace{1cm} \text{(B.32)}
\]

Using the chain rule similar to the cases shown in Appendix A:

\[
\frac{\partial p}{\partial R} = \frac{\partial \phi}{\partial R} \frac{\partial p}{\partial \phi} \hspace{1cm} \text{(B.33)}
\]

\[
\frac{\partial p}{\partial t} = \frac{\partial \phi}{\partial t} \frac{\partial p}{\partial \phi} \hspace{1cm} \text{(B.34)}
\]

Substituting Eqs. B.33-B.35 in Eq. B.32 results in:

\[
\frac{\partial p}{\partial R} = \frac{\partial \phi}{\partial R} \frac{\partial p}{\partial \phi} \hspace{1cm} \text{(B.35)}
\]
The compressibility of the fluid can be defined by:

\[
c_o = \frac{1}{\rho} \frac{\partial \rho}{\partial p}
\] (B.37)

and the compressibility of the fractal reservoir as:

\[
c = \frac{1}{\phi} \frac{\partial \phi}{\partial p}
\] (B.38)

Substituting Eqs. B.37 and B.38 in Eq. B.36 and neglecting the squared gradient on the left hand side of the equation, this yields the expression:

\[
\frac{1}{R^{D_f-1}} \frac{\partial}{\partial R} \left[ R^\beta \frac{\partial p}{\partial R} \right] = \frac{\phi_0 \mu c_t}{k_0} \frac{\partial p}{\partial t}
\] (B.39)

where the total compressibility of the fractal reservoir is defined by:

\[
c_t = c_o + c
\] (B.40)

For this model, consider the following dimensionless variables: Dimensionless pressure:

\[
p_{D,cr}(R_D,t_D) = \frac{\sigma V_s k_0}{q B_l \mu_w^{1-\beta} \phi_0}[p_i - p(R,t)]
\] (B.41)

dimensionless time:

\[
t_D = \frac{k_0}{\phi_0 \mu c_t r_w^{\theta + 2}}
\] (B.42)

and dimensionless position in the fractal reservoir:

\[206\]
Using the chain rule, the derivative of the pressure with respect to time is expressed as:

\[
\frac{\partial p}{\partial t} = \frac{\partial p}{\partial p_{D,cr}} \frac{\partial p_{D,cr}}{\partial t_D} - \frac{q B \mu_w^{1-\beta} \phi_0}{\sigma v_s k_0} \phi_0 \mu_e r_w^{\theta+2} \frac{\partial p_{D,cr}}{\partial t_D} .
\]  \hspace{1cm} (B.44)

Similarly, for the first derivative of the pressure with respect to \( R \):

\[
\frac{\partial p}{\partial R} = \frac{\partial p}{\partial p_{D,cr}} \frac{\partial p_{D,cr}}{\partial R_D} - \frac{q B \mu_w^{1-\beta} \phi_0}{\sigma v_s k_0} \frac{1}{r_w} \frac{\partial p_{f,cr}}{\partial r_D} .
\]  \hspace{1cm} (B.45)

Substituting Eqs. B.43-B.45 in Eq. B.39 gives the diffusivity equation for a fractal reservoir in its dimensionless form:

\[
\frac{1}{R^f} \frac{1}{R_D} \frac{\partial}{\partial R_D} \left[ R_D^{\beta} \frac{\partial p_{D,cr}}{\partial R_D} \right] = \frac{\partial p_{D,cr}}{\partial t_D} .
\]  \hspace{1cm} (B.46)

**B.2. Constant-Rate Solution for the Fractal Reservoir Model**

To solve Eq. B.46, consider the following initial and boundary conditions:

\[
p_D(R_D, t_D = 0) = 0 \hspace{1cm} \text{(initial condition),} \hspace{1cm} \text{..................................................(B.47)}
\]

\[
\left[ \frac{R^\beta}{R_D} \frac{\partial p_{D,cr}}{\partial R_D} \right]_{R_D=1} = -1 \hspace{1cm} \text{(inner boundary condition),} \hspace{1cm} \text{..................................................(B.48)}
\]

\[
\lim_{R_D \to \infty} p_D(R_D, t_D) = 0 \hspace{1cm} \text{(outer boundary condition),} \hspace{1cm} \text{..................................................(B.49)}
\]
Applying the derivative on the left hand side of Eq. B.46 and arraying, it yields the following equation:

\[ R_D^2 \frac{\partial^2 P_{D,cr}}{\partial R_D^2} + \beta R_D \frac{\partial P_{D,cr}}{\partial R_D} - R_D^{\theta+2} \frac{\partial P_{D,cr}}{\partial t_D} = 0 \]  \hspace{1cm} (B.50)

Applying the Laplace transform to Eq. B.50, it becomes:

\[ \frac{d^2 \bar{P}_{D,cr}}{d R_D^2} + \beta \frac{d \bar{P}_{D,cr}}{d R_D} - u R_D^\theta \bar{P}_{D,cr} = 0 \]  \hspace{1cm} (B.51)

Consider the transformation function:

\[ G(z) = \bar{P}_{D,cr}(R_D,u) \]  \hspace{1cm} (B.52)

and the transformation variable:

\[ z = \frac{2\sqrt{u}}{\theta + 2} R_D^{(\theta+2)/2} \]  \hspace{1cm} (B.53)

Using the chain rule, the derivative of the dimensionless pressure in the Laplace domain with respect to the dimensionless radius is:

\[ \frac{d \bar{P}_{D,cr}}{d R_D} = \frac{d \bar{P}_{D,cr}}{d G(z)} \frac{d G(z)}{d z} \frac{d z}{d R_D} = R_D^{\theta/2} \sqrt{u} \frac{d G(z)}{d z} \]  \hspace{1cm} (B.54)

Applying the second derivative to Eq. B.52 with respect to \( R_D \):

\[ \frac{d^2 \bar{P}_{D,cr}}{d R_D^2} = R_D^\theta u \frac{d^2 G(z)}{d z^2} + \theta R_D^{(\theta-2)/2} \sqrt{u} \frac{d G(z)}{d z} \]  \hspace{1cm} (B.55)

Substituting Eqs. B.52 and B.53 in Eq. B.49 and arraying, it reduces to the following expression:
\[ z^2 \frac{d^2 G(z)}{dz^2} + [1 - 2v] \frac{dG(z)}{dz} - z^2 G(z) = 0 \] (where \( v = [1 - \beta] / [\theta + 2] \)) ...........................................(B.56)

Defining a new transformation function:

\[ B(z) = \left[ \frac{2\sqrt{u}}{\theta + 2} \right]^{-v} z^v G(z) \] .................................................................(B.57)

the derivative of the function \( G(z) \) with respect to \( z \) can be expressed as:

\[ \frac{dG(z)}{dz} = \left[ \frac{2\sqrt{u}}{\theta + 2} \right]^{-v} \left[ z^v \frac{dB(z)}{dz} + z^{v-1} \frac{d}{dz} B(z) \right] \] .................................................................(B.58)

Taking the second derivative to Eq. B.56 yields:

\[ \frac{d^2 G(z)}{dz^2} = \left[ \frac{2\sqrt{u}}{\theta + 2} \right]^{-v} \left[ z^v \frac{d^2 B(z)}{dz^2} + z^{v-1} \frac{d}{dz} B(z) \right] + \left[ \frac{2\sqrt{u}}{\theta + 2} \right]^{-v} \left[ z^{v-1} \frac{d}{dz} B(z) + [v - 1] z^{v-2} B(z) \right] \] ...........(B.59)

Substituting Eqs. B.57-B.59 in Eq. B.56 and arraying, it reduces to the following differential equation:

\[ z^2 \frac{d^2 B(z)}{dz^2} + z \frac{dB(z)}{dz} - [v^2 + z^2] B(z) = 0 \] .................................................................(B.60)
By inspection, the general solution to Eq. B.60 is given by:

\[ B(z) = C_1 K_v(z) + C_2 I_v(z) \] ...................................................... (B.61)

Expressing Eq. E.61 in terms of the pressure:

\[ \bar{\rho}_{D,cr}(R_D, u) = R_D^{[1-\beta]/2} \left[ C_1 K_v \left( \frac{2\sqrt{u}}{\theta + 2} R_D^{[\theta + 2]/2} \right) + C_2 I_v \left( \frac{2\sqrt{u}}{\theta + 2} R_D^{[\theta + 2]/2} \right) \right] \] ...........................................(B.62)

Applying the outer boundary condition (Eq.B.49) to Eq. B.62, we can conclude that \( C_2 \) must be zero. Then, the bounded solution for this problem is:

\[ \bar{\rho}_{D,cr}(R_D, u) = R_D^{[1-\beta]/2} C_1 K_v \left[ \frac{2\sqrt{u}}{\theta + 2} R_D^{[\theta + 2]/2} \right] \] ...................................................... (B.63)

The value of the constant \( C_1 \) results from applying the inner boundary condition (Eq.B.48 in the Laplace domain) to Eq. B.63. Such a value is:

\[ C_1 = \frac{1}{u^{3/2}} \frac{1}{K_{\nu -1} \left[ \frac{2\sqrt{u}}{\theta + 2} \right]} \] ...................................................... (B.64)

The particular solution is obtained by substituting Eq. B.64 in Eq. B.63:

\[ \bar{\rho}_{D,cr}(R_D, u) = \frac{R_D^{[1-\beta]/2}}{u^{3/2}} \frac{K_v \left[ \frac{2\sqrt{u}}{\theta + 2} R_D^{[\theta + 2]/2} \right]}{K_{\nu -1} \left[ \frac{2\sqrt{u}}{\theta + 2} \right]} \] ...................................................... (B.65)

To provide the particular solution in the real domain, consider the "point source" approximation, \textit{i.e.}:

\[ \text{To be continued...} \]
\[ K_{v-1} \left[ \frac{2\sqrt{u}}{\theta+2} \right] = K_{1-v} \left[ \frac{2\sqrt{u}}{\theta+2} \right] \approx \Gamma[1-v] \left[ \frac{\sqrt{u}}{\theta+2} \right]^{-[1-v]} \] 

(for \( v>0 \)) .................................................................(B.66)

Substituting Eq. B.66 in Eq. B.65:

\[ \bar{p}_{D,cr}(R_D,u) = \frac{2^{[1-\beta]/2}}{[\theta+2]^{-v}[\Gamma[1-v]]^{2+2/v} K_v} \left[ \frac{2\sqrt{u}}{\theta+2} R_D^{[\theta+2]/2} \right] \] .................................................................(B.67)

Applying the inverse Laplace transform to Eq. B.67 (Eq. A32 from Barker, 1988):

\[ p_{D,cr}(R_D,t_D) = \frac{R_D^{[1-\beta]}}{[\theta+2][\Gamma[1-v]]^{-v}} \left[ \frac{R_D^{[\theta+2]}}{[\theta+2]^2 t_D} \right] \] .................................................................(B.68)
APPENDIX C
PRESSURE-TRANSIENT BEHAVIOR OF A VERTICALLY FRACTURED WELL IN A FRAC TAL RESERVOIR

In this Appendix, we show the detailed derivation of the uniform-flux solution of a vertical well intercepted by a hydraulic fracture in a fractal reservoir (Beier, 1994).

C.1. Uniform-Flux Solution of a Vertically Fractured Well within a Fractal Reservoir

Recall the constant-rate solution for an infinite fractal reservoir (Eq. B.68):

\[ p_{D,cr}(R_D,t_D) = \frac{R_{D}^{[1-\beta]}}{[\theta + 2][1-v]} \left[ -v, \frac{R_{D}^{\theta + 2}}{[\theta + 2]^{2} t_D} \right] \text{ (where } v = \frac{[1-\beta]}{[\theta + 2]}) \]

Consider the redefinition of the dimensionless time:

\[ t_{D,Be} = \frac{[\theta + 2]^2}{4} \left[ \frac{r_w}{x_f} \right]^{\theta + 2} \left[ t_D, \text{ .................................................................(C.1) } \right] \]

and:

\[ p_{DBe}(R_D,t_{D,Be}) = \frac{[\theta + 2]}{4} p_{D,cr}(R_D,t_D) \text{ .................................................................(C.2)} \]

Then, Eq.B.66 can be written as:

\[ p_{DBe}(R_D,t_{D,Be}) = \frac{R_{D}^{[1-\beta]}}{4\Gamma[1-v]} \left[ -v, \frac{R_{D}^{\theta + 2}}{4t_{D,Be}} \right] \text{ .................................................................(C.3)} \]

Based on the definition of the Incomplete Gamma Function, Eq. C.3 can be written as:
\[ p_{DBe}(R_{D}, t_{D}, Be) = \frac{R_{D}^{1-\beta}}{4\Gamma[(1-\nu)]} \int_{-\infty}^{\infty} \frac{[z^{-\nu-1} \exp[-z]]}{\psi} \, dz \] \quad (C.4) 

where:

\[ \psi = \frac{R_{D}^{\theta+2}}{4t_{D}, Be} \] \quad (C.5)

According to the grouping variable \( z \):

\[ z = \frac{R_{D}^{\theta+2}}{4t_{D}} \] \quad (C.6)

Eq. C.3 can be rewritten as:

\[ p_{DBe}(R_{D}, t_{D}, Be) = \frac{1}{\Gamma[1-\nu]} \int_{0}^{t_{D}, Be} \frac{1}{[4\tau]^{1-\nu}} \exp\left[-\frac{R_{D}^{\theta+2}}{4\tau} \right] \, d\tau \quad \text{………………………………………(C.7)} \]

To be consistent with the solution presented by Beier (1994), consider the change of variables:

\[ d_{s} = \frac{2D_{f}}{\theta + 2} \] \quad (C.8)

consequently:

\[ 1 - \nu = \frac{d_{s}}{2} \] \quad (C.9)

Using Eq. C.8 and C.9, Eq. C.7 can be written as:

\[ p_{DBe}(R_{D}, t_{D}, Be) = \frac{1}{\Gamma[d_{s}/2]} \int_{0}^{t_{D}, Be} \frac{1}{[4\tau]^{d_{s}/2}} \exp\left[-\frac{R_{D}^{2D_{f}/d_{s}}}{4\tau} \right] \, d\tau \] \quad \text{………………………………………………(C.10)}
Assuming that the "fractal radius" is given by:

\[
R_D^{2f/d_s} = [x_D - x_{wD}]^{2f/d_s} + [y_D - y_{wD}]^{2f/d_s},
\]

then, Eq. C.10 is rewritten as:

\[
P_{DBe}(R_D, t_D, B_e) = \frac{1}{\Gamma(d_s/2)} \int_0^{t_D, B_e} \exp \left[ -\frac{[x_D - x_{wD}]^{2f/d_s}}{4\tau} - \frac{[y_D - y_{wD}]^{2f/d_s}}{4\tau} \right] \frac{d\tau}{[4\tau]^{d_s/2}}.
\]

Assuming that \( y_{wD} \) is zero and integrating over the domain of the \( x_D \)-axis results in:

\[
P_{DBe}(x_D, y_D, t_D, B_e) = \frac{1}{\Gamma(d_s/2)} \int_0^{t_D, B_e} \exp \left[ -\frac{[y_D]^{2f/d_s}}{4\tau} \right] \frac{d\tau}{[4\tau]^{d_s/2}}\cdot d_{wD} \cdot \frac{d\tau}{[4\tau]^{d_s/2}}.
\]

To integrate Eq. C.13 over the \( x_{wD} \) interval, consider the change of variables:

\[
w = \frac{x_D - x_{wD}}{[4\tau]^{d_s/2d_f}},
\]

and consequently:

\[
dw = \frac{-dx_{wD}}{[4\tau]^{d_s/2d_f}}.
\]

Evaluating at \( y_D = 0 \) and substituting Eq. C.14 and Eq. C.15 into Eq. C.13, it becomes:
\[ p_{DBe}(x_D \cdot t_D, Be) = \frac{1}{\Gamma[d_s / 2]} \]

\[
\int_0^{t_D} \frac{1}{\Gamma[d_s / 2]} \left[ \exp\left[-w \frac{2D_f / d_s}{[4\tau]} d_s\right] - \left[\frac{2D_f}{d_s}\right]^{d_s / 2} \int_0^{w} d\tau \frac{1+x_D}{[4\tau] d_s / 2D_f} \right] d\tau 
\]

where:

\[
w_l = \frac{1+x_D}{[4\tau] d_s / 2D_f} \]

and:

\[
w_u = \frac{-1+x_D}{[4\tau] d_s / 2D_f} \]

Splitting and arraying the inner integral in Eq. C.16 results in:

\[ p_{DBe}(x_D \cdot t_D, Be) = \frac{1}{\Gamma[d_s / 2]} \]

\[
\int_0^{t_D} \left[ \frac{d_s}{2} \left( 1 - \frac{1}{D_f} \right) \right] \left[ \int_0^{w_l} \exp\left[-w \frac{2D_f / d_s}{[4\tau]} d_s\right] d\tau + \int_0^{w_u} \exp\left[-w \frac{2D_f / d_s}{[4\tau]} d_s\right] d\tau \right] d\tau 
\]

Using the change of variables:

\[ z = w \frac{2D_f / d_s}{[4\tau]} \]

The first integral inside the brackets in Eq. C.19 becomes:

\[
\int_0^{w_l} \exp\left[-w \frac{2D_f / d_s}{[4\tau]} d_s\right] d\tau = \frac{d_s}{2D_f} \int_0^{z_u} \frac{d_s}{2D_f} \exp[-z] dz \]

where:

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According to the definition of the lower incomplete gamma function, Eq. C.21 can be expressed as:

\[
\int_0^{w_{ul}} \exp[-w\frac{2D_f}{d_s}] \, dw = \frac{d_s}{2D_f} \gamma\left[\frac{d_s}{2D_f}, \frac{[1-x_D]^{2D_f/d_s}}{4\tau}\right]
\]  \hspace{1cm} \text{(C.23)}

Similarly, for the second integral inside the brackets into Eq. C.19:

\[
\int_0^{w_{l}} \exp[-w\frac{2D_f}{d_s}] \, dw = \frac{d_s}{2D_f} \gamma\left[\frac{d_s}{2D_f}, \frac{[1+x_D]^{2D_f/d_s}}{4\tau}\right]
\]  \hspace{1cm} \text{(C.24)}

Substituting Eqs. C.23 and C.24 in Eq. C.19:

\[
p_{DBe}(x_{D, lD, Be}) = \frac{4^\nu}{\Gamma[d_s/2]} \frac{d_s}{2D_f} \int_0^{\tau_D, Be} \left[\gamma\left[\frac{d_s}{2D_f}, \frac{[1-x_D]^{2D_f/d_s}}{4\tau}\right] + \gamma\left[\frac{d_s}{2D_f}, \frac{[1+x_D]^{2D_f/d_s}}{4\tau}\right]\right] \, d\tau
\]  \hspace{1cm} \text{(C.25)}

where:

\[
\nu = \left[\frac{d_s}{2}\right] \left[1 - \frac{1}{D_f}\right]
\]  \hspace{1cm} \text{(C.26)}

Applying the distribution property to the integral on the right hand side of Eq. C.26 results in two definite integrals of the product of a power-law function and a lower incomplete gamma function.

The integral for these types of functions is:
Using Eq. C.30 in the first integral on the right hand side of Eq. C.26 results in:

\[
\left[ \frac{\tau^a \gamma(b, c \tau^{-1})}{0} \right] \frac{d \tau}{(a + 1) \gamma(b, c \tau^{-1})} - \left[ \frac{\tau^a \gamma(b, c \tau^{-1})}{0} \right] \frac{\partial}{\partial \tau} = - \frac{\tau^{a+1}}{(a+1)} \frac{\partial \gamma[b, c \tau^{-1}]}{\partial \tau} d \tau.
\] ..........................(C.27)

The derivative of the lower incomplete gamma function is obtained by using the Leibniz rule:

\[
\frac{\partial \gamma[b, c \tau^{-1}]}{\partial \tau} = \frac{\partial}{\partial \tau} \int_0^\tau z^{b-1} \exp[-z] \, dz = - c \tau \exp[-c \tau] - \tau^{a-1} \exp[-c \tau] d \tau.
\] ..........................(C.28)

Substituting Eq. C.28 into Eq. C.27 and arraying results in:

\[
\frac{\tau^a \gamma(b, c \tau^{-1})}{0} \frac{d \tau}{(a + 1) \gamma(b, c \tau^{-1})} + \frac{\tau^{a+1}}{(a+1)} \frac{\partial \gamma[b, c \tau^{-1}]}{\partial \tau} d \tau = - \frac{\tau^{a+1}}{(a+1)} \frac{\partial \gamma[b, c \tau^{-1}]}{\partial \tau} d \tau.
\] ..........................(C.29)

Using the change of variable:

\[
u = c \tau^{-1},
\] ..........................(C.30)

Eq. C.29 becomes:

\[
\frac{\tau^a \gamma(b, c \tau^{-1})}{0} \frac{d \tau}{(a + 1) \gamma(b, c \tau^{-1})} = - \frac{\tau^{a+1}}{(a+1)} \frac{\partial \gamma[b, c \tau^{-1}]}{\partial \tau} d \tau.
\] ..........................(C.31)

Eq. C.32 can be written in terms of the upper incomplete gamma function as:

\[
\frac{\tau^a \gamma(b, c \tau^{-1})}{0} \frac{d \tau}{(a + 1) \gamma(b, c \tau^{-1})} = - \frac{\tau^{a+1}}{(a+1)} \Gamma(b-a-1, c \tau^{-1})
\] ..........................(C.33)

Using Eq. C.30 in the first integral on the right hand side of Eq. C.26 results in:

\[
\frac{tDBe}{0} \frac{d \tau}{\tau^{\gamma} \gamma[\frac{d_s}{2D_f}, \frac{1-x_D}{d_s}, \frac{2D_f/d_s}{4}, \frac{1}{\tau}]} + \frac{tDBe}{v+1} \gamma[\frac{d_s}{2D_f}, \frac{1-x_D}{d_s}, \frac{2D_f/d_s}{4}, \frac{1}{\tau}]
\] ..........................(C.34)

\[
\frac{1}{v+1} \left[ \frac{1-x_D}{d_s}, \frac{2D_f/d_s}{4}, \frac{1}{\tau} \right] + \frac{1}{v+1} \Gamma[\frac{d_s}{2D_f}, -v-1, \frac{1-x_D}{d_s}, \frac{2D_f/d_s}{4}, \frac{1}{\tau}]
\]
Substituting Eqs. C.32 and C.33 into Eq. C.25:

\[
\frac{t_{DBe}}{2D_f} \gamma \left[ \frac{d_s}{2D_f}, \frac{[1+x_D]^{2D_f/d_s}}{4} \right]^{\nu+1} _\tau \frac{d_s}{2D_f} \gamma \left[ \frac{d_s}{2D_f}, \frac{[1-x_D]^{2D_f/d_s}}{4t_{DBe}} \right] + \frac{1}{\nu+1} \left[ \frac{[1+x_D]^{2D_f/d_s}}{4} \right]^{\nu+1} \Gamma \left[ \frac{d_s}{2D_f}, \frac{[1-x_D]^{2D_f/d_s}}{4t_{DBe}} \right] + \cdots \tag{C.35}
\]

Substituting Eqs. C.32 and C.33 into Eq. C.25:

\[
P_{DBe}(x_D, t_{DBe}) = \frac{4^\nu}{\Gamma(d_s/2)} \frac{d_s}{2D_f} \gamma \left[ \frac{d_s}{2D_f}, \frac{[1-x_D]^{2D_f/d_s}}{4t_{DBe}} \right] + \frac{1}{\nu+1} \left[ \frac{[1-x_D]^{2D_f/d_s}}{4} \right]^{\nu+1} \Gamma \left[ \frac{d_s}{2D_f}, \frac{[1-x_D]^{2D_f/d_s}}{4t_{DBe}} \right] + \cdots \tag{C.36}
\]

Substituting Eq. C.26 in Eq. C.36 and arraying results in the expression provided by Beier (1994):

\[
P_{DBe}(x_D, t_{DBe}) = C_1 \Gamma \left[ \frac{d_s}{2D_f}, \frac{[1-x_D]^{2D_f/d_s}}{4t_{DBe}} \right] + \frac{1}{\nu+1} \left[ \frac{[1-x_D]^{2D_f/d_s}}{4} \right]^{\nu+1} \Gamma \left[ \frac{d_s}{2D_f}, \frac{[1-x_D]^{2D_f/d_s}}{4t_{DBe}} \right] + C_2 \left[ 1 + x_D \right]^{C_3} \Gamma \left[ \frac{d_s}{2}, 1 - \frac{[1-x_D]^{2D_f/d_s}}{4t_{DBe}} \right] + \left[ 1 - x_D \right]^{C_3} \Gamma \left[ \frac{d_s}{2}, 1 - \frac{[1-x_D]^{2D_f/d_s}}{4t_{DBe}} \right] \tag{C.37}
\]

where:

\[
C_1 = \frac{2^{-d_s[1-1/D_f]}}{\Gamma(d_s/2)} \frac{d_s}{2D_f} \frac{t_{DBe}^{1-d_s[1-1/D_f]/2}}{1-d_s[1-1/D_f]}/2 \tag{C.38}
\]

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\[ C_2 = \frac{d_s / D_f}{8[1 - d_s(1 - 1/D_f)/2]} \] ...(C.39)

and:

\[ C_3 = 2D_f[1 - d_s(1 - 1/D_f)/2]/d_s \] ...(C.40)
APPENDIX D

DERIVATION OF THE MODEL OF A HORIZONTAL WELL INTERCEPTING A SINGLE FINITE CONDUCTIVITY FRACTURE WITHIN A FRACTAL RESERVOIR

In this Appendix, we show the mathematical development of the models of a horizontal well intercepting a single finite conductivity fracture within a fractal reservoir considering either typical or anomalous diffusion.

D.1. Point Source Constant-Rate Solution of the Diffusivity Equation for a Fractal Reservoir with Typical Diffusion

Flow Model and Initial and Boundary Conditions

In this section, we show the procedure to obtain the particular solution of the diffusivity equation for an infinite fractal reservoir producing at a constant rate through a "point well." Consider the diffusivity equation for a fractal object presented by Chang et al. (1990):

\[
\frac{1}{R^{D_f-1}} \frac{\partial}{\partial R} \left[ R^\beta \frac{\partial p}{\partial R} \right] = \frac{\phi_0 k_c}{k_0} \frac{\partial p}{\partial t}, \tag{D.1}
\]

where \( \beta \) is the spatial dimension and it is defined as \( \beta = D_f - \theta - 1 \). Assuming that the initial pressure is uniformly distributed along the reservoir, the initial condition is:

\[
p(R,t=0) = p_i, \tag{D.2}
\]

Considering constant-rate conditions, the inner boundary condition is:
For an infinite fractal reservoir, the outer boundary condition is defined as:

$$\lim_{R \to \infty} p(R, t) = p_i \quad \text{..........................................................(D.4)}$$

**Transformation to Dimensionless Variables**

For the constant-rate case, the dimensionless pressure in the fractal reservoir is defined as:

$$p_{D,cr}(R_D, t_D) = \frac{2\pi k_0 h}{q_w B_H l_D} [p_i - p(R, t)], \quad \text{..........................................................(D.5)}$$

where $L_r$ is the reference length. It must be changed for the radius of the fracture, $r_f$, in the case of the circular fracture or the half-length of the fracture, $x_f$, for a rectangular fracture. Dimensionless time is defined as:

$$t_D = \frac{k_0}{\phi_0 \mu_r L_r^2} \quad \text{..........................................................(D.6)}$$

The dimensionless position in the fractal reservoir is defined by:

$$R_D = \frac{R}{L_r} \quad \text{..........................................................(D.7)}$$

Using the chain rule, Eq. D.1 is transformed to dimensionless variables as:

$$\frac{1}{R_D^{D_f-1}} \frac{\partial}{\partial R_D} \left[ R_D^\beta \frac{\partial p_{D,cr}}{\partial R_D} \right] = \frac{\partial p_{D,cr}}{\partial t_D} \quad \text{..........................................................(D.8)}$$

Similarly, initial and boundary conditions become:
The dimensionless parameters involved in Eq. D.10 are defined as follows. Dimensionless fractal reservoir rate:

\[
\delta t_D = \frac{4\pi}{a_{Df}} \left( b \frac{D_f}{L_r} \right)^{D_f-3} \frac{\delta \theta_D}{q_w} . \]

Dimensionless height:

\[
h_D = \frac{h}{L_r} .
\]

Dimensionless fractal source radius:

\[
R_{wD} = \frac{R_w}{L_r} .
\]

**Constant-Rate Solution to the Diffusivity Equation for an Infinite Fractal Reservoir**

As shown in Appendix B, the general solution in the Laplace domain of the diffusivity equation for a fractal reservoir is given by:

\[
\bar{p}_{D,cr}(R_D, u) = R_D^{[1-\beta]/2} \left[ C_1 K_v \left[ R_D^{(\theta+2)/2} \frac{2\sqrt{u}}{\theta+2} \right] + C_2 I_v \left[ R_D^{(\theta+2)/2} \frac{2\sqrt{u}}{\theta+2} \right] \right] .
\]
where \( v \) is a grouping parameter and it is defined as:

\[
v = \frac{1 - \beta}{\theta + 2} \]..........................(D.16)

Applying the outer boundary condition to Eq. D.15, a bounded solution is obtained:

\[
\bar{p}_{D,cr}(R_D, u) = C_1 R_D^{[1 - \beta]/2} K_v \left[ R_D^{[\theta + 2]/2} \frac{2 \sqrt{u}}{\theta + 2} \right] \]..........................(D.17)

Applying the inner boundary condition to Eq. D.17, the value of the constant \( C_1 \) is obtained:

\[
C_1 = \frac{\partial \bar{p}_{D,cr}(R_D, u)}{2 R_D^{D_f/2} \sqrt{u} K_{1-v} \left[ R_D^{[\theta + 2]/2} \frac{2 \sqrt{u}}{\theta + 2} \right]} \]..........................(D.18)

The particular solution for this model is obtained by substituting Eq. D.18 in Eq. D.17. It is given by:

\[
\bar{p}_{D,cr}(R_D, u) = \frac{\partial \bar{p}_{D,cr}(R_D, u)}{2 R_D^{D_f/2} \sqrt{u} K_{1-v} \left[ R_D^{[\theta + 2]/2} \frac{2 \sqrt{u}}{\theta + 2} \right]} \]..........................(D.19)

The point source solution is obtained by taking the limit to Eq. D.19 when \( R_D \) tends to zero. This implies that the modified Bessel function in the denominator in Eq. D.19 can be approximated to a power-law function:

\[
K_{1-v} \left[ R_D^{[\theta + 2]/2} \frac{2 \sqrt{u}}{\theta + 2} \right] \approx \frac{\Gamma[1 - v]}{2} \left[ \frac{\theta + 2}{R_D^{[\theta + 2]/2} \sqrt{u}} \right]^{1-v} \]..........................(D.20)

The following expression results after Substituting Eq. D.20 in Eq. D.19:
Extension to Double Porosity Reservoirs (Valdes-Perez, 2013)

It is well known that a solution with the shape of Eq. D.21 can be extended to models for double porosity reservoirs if the Laplace parameter $u$ is replaced by $uf(u)$ where $f(u)$ is the interporosity transfer function. Hence, the following expression is obtained:

$$
\overline{p}_{D,cr}(R_D,u) = \frac{\bar{\alpha}_D h_D R_D^{1-\beta/2}}{[\theta+2]^{1/[1-v]}} \left[ \frac{\theta+2}{\sqrt[\theta+2]{uf(u)}} \right]^\nu K_v \left[ R_D^{\theta/2} \frac{2\sqrt[\theta+2]{uf(u)}}{\theta+2} \right] \quad \text{(D.22)}
$$

In this model, the interporosity transfer function to be used is:

$$
f(u) = \omega + \frac{[1-\omega] A_{D} \bar{F}(\eta_{maD}, h_{maD}, u)}{1 + \frac{h_{maD}^2 S_{int}}{\eta_{maD}} \bar{F}(\eta_{maD}, h_{maD}, u)} \quad \text{(D.23)}
$$

where the function $F(\eta_{maD}, h_{maD}, u)$ depends on the shape and properties of the matrix blocks (see Appendix E). The dimensionless hydraulic diffusivity of the matrix is defined as:

$$
\eta_{maD} = \frac{k_{ma}[\phi_{t}] \theta}{k_{fb} \phi_{ma c_{tma}}} \quad \text{(D.24)}
$$

The dimensionless size of the matrix blocks is:

$$
h_{maD} = \frac{h_{ma}}{L_r} \quad \text{(D.25)}
$$

the storativity ratio is:

$$
\eta_{maD} \left[ \frac{\bar{\alpha}_D h_D R_D^{1-\beta/2}}{[\theta+2]^{1/[1-v]}} \left[ \frac{\theta+2}{\sqrt[\theta+2]{uf(u)}} \right]^\nu K_v \left[ R_D^{\theta/2} \frac{2\sqrt[\theta+2]{uf(u)}}{\theta+2} \right] \right] \quad \text{(D.21)}
$$
\[ \omega = \frac{\phi_0 \phi_b c \phi_b}{\phi_0 \phi_b c \phi_b + \phi_{ma} c \phi_{ma}} \] .................................................................(D.26)

and the interporosity skin is:

\[ S_{int} = \frac{k_{ma} d}{k_d h_{ma}} \] .................................................................(D.27)

For matrix blocks with the shape of slabs, the transfer function is:

\[ F(\eta_{maD}, h_{maD}, u) = \frac{1}{h_{maD}} \sqrt{\eta_{maD}} \tanh \left[ \frac{h_{maD}}{2} \sqrt{\frac{u}{\eta_{maD}}} \right] \] .................................................................(D.28)

whereas for matrix blocks with the shape of spheres, the transfer function is:

\[ F(\eta_{maD}, h_{maD}, u) = \frac{1}{h_{maD}} \sqrt{\eta_{maD}} \coth \left[ \frac{h_{maD}}{2} \sqrt{\frac{u}{\eta_{maD}}} \right] - \frac{2}{h_{maD}} \sqrt{\frac{u}{\eta_{maD}}} \] .................................................................(D.29)

**D.2. Point Source Constant-Rate Solution of the Diffusivity Equation for a Fractal Reservoir with Anomalous Diffusion**

The diffusivity equation for fractal reservoir with anomalous diffusion expressed in dimensionless variables was presented by Camacho-Velazquez et al. (2008). The model is given by:

\[ \frac{1}{R_D^{D_f-1}} \frac{\partial}{\partial R_D} \left[ R_D^p \frac{\partial p_{D,cr}}{\partial R_D} \right] = \frac{\partial^\gamma p_{D,cr}}{\partial t_D^\gamma} \] .................................................................(D.30)

Where \( \gamma \) is the fractional derivative order and it is a function of the conductivity index. Such a function is defined as:
\[ \gamma = \frac{2}{2 + \theta} \]  

The time fraction derivative of order \( \gamma \) in Eq. D.30 is defined as:

\[ \begin{align*}
\frac{D^\gamma p_{D,cr}}{dt_D} &= \frac{1}{\Gamma[1-\gamma]} \frac{d}{dt_D} \int_0^{t_D} (t_D - \tau)^{-\gamma} p_{D,cr}(R_D, \tau) d\tau. 
\end{align*} \]  

Analogous to the case presented in the previous section, the initial and boundary condition for Eq. D.22 are defined by Eq. D.9 though D.11. The procedure to solve Eq. D.30 is similar to the one shown in Appendix B and in section D.1. Then, applying the Laplace transform to Eq. D.30:

\[ \frac{1}{R_D^{\beta - 1}} \frac{\partial}{\partial R_D} \left[ R_D^{\beta} \frac{\partial p_{D,cr}}{\partial R_D} \right] = u^\gamma \overline{p}_{D,cr}. \]  

To find the general solution for this case, the transformation variable to be used is:

\[ z = \frac{2}{\theta + 2} \left[ u^\gamma R_D^{(\theta + 2)} \right]^{1/2} \]  

Using the transformation variable defined in Eq. D.34 and the procedure shown in Appendix B, the general solution for this model is given by:

\[ \overline{p}_{D,cr}(R_D, u) = R_D^{(1 - \beta)/2} \left[ C_1 K_v \left( R_D^{(\theta + 2)/2} \frac{2u^\gamma}{\theta + 2} \right) + C_2 I_v \left( R_D^{(\theta + 2)/2} \frac{2u^\gamma}{\theta + 2} \right) \right], \]

Applying the outer boundary condition, the resulting bounded solution is:

\[ \overline{p}_{D,cr}(R_D, u) = C_1 R_D^{(1 - \beta)/2} K_v \left( R_D^{(\theta + 2)/2} \frac{2u^\gamma}{\theta + 2} \right) \]
The value of the constant $C_1$ is obtained by applying the inner boundary condition to Eq. D.36:

$$
C_1 = \frac{\bar{\alpha}_D h_D}{2R_w^{D/2} u^{\gamma/2} K_{1-v}} \left[ R_w^{(\theta+2)/2} \frac{2u^{\gamma/2}}{\theta+2} \right] \tag{D.37}
$$

Then, the particular solution for this model is:

$$
\bar{\rho}_{D,cr}(R_D, u) = \frac{\bar{\alpha}_D h_D R_D^{(1-\beta)/2}}{2R_w^{D/2} u^{\gamma/2} K_{1-v}} \left[ R_w^{(\theta+2)/2} \frac{2u^{\gamma/2}}{\theta+2} \right]^{\theta/2} \tag{D.38}
$$

For this case, the point source solution is given by:

$$
\bar{\rho}_{D,cr}(R_D, u) = \frac{\bar{\alpha}_D h_D R_D^{(1-\beta)/2}}{(\theta+2) \Gamma(1-v)} \left[ \frac{u^{\gamma/2}}{\theta+2} \right]^{\theta/2} \left[ R_w^{(\theta+2)/2} \frac{2u^{\gamma/2}}{\theta+2} \right] \tag{D.39}
$$

To consider double porosity conditions, Eq. D.22 should be used considering the following transfer function, $f(u)$:

$$
f(u) = \omega u^{\gamma-1} + \frac{[1-\omega] A_f D \bar{F}(\eta_{maD}, h_{maD}, u)}{1 + \frac{h_{maD}^2 S_{int}}{\eta_{maD}} \bar{F}(\eta_{maD}, h_{maD}, u)} \tag{D.40}
$$

**D.3. Constant-Rate Solution for the Model of a Horizontal Well Intercepting a Circular Transverse Finite Conductivity Fracture within a Fractal Reservoir**

This model considers radial flow within a hydraulic fracture as previously presented by Larsen et al. (1991). The flow within the circular fracture obeys the diffusivity equation developed in Appendix C and given by:
Considering that the initial pressure is uniformly distributed along the hydraulic fracture, the initial condition is:

\[
p_f(r,t=0) = p_i.\].................................................................(D.42)

For a well intercepting a hydraulic fracture producing at a constant rate, the inner boundary condition is:

\[
\left[ \frac{\partial p_f}{\partial r} \right]_{r=r_w} = \frac{q_w B \mu}{2 \pi k_f w} .................................................................(D.43)
\]

For an closed hydraulic fracture, the outer boundary condition is defined as:

\[
\left[ \frac{\partial p_f}{\partial r} \right]_{r=r_f} = 0 .................................................................(D.44)
\]

Similar to the reservoir model shown in section D.1, the dimensionless pressure in the circular hydraulic fracture is defined as:

\[
p_{fD,cr}(r_D,t_D) = \frac{2 \pi k_0 h}{q_w B \mu \theta_f} [p_i - p_f(r,t)], .................................................................(D.45)
\]

Dimensionless time is defined by Eq. D.6 and the dimensionless radius of the circular hydraulic fracture is:

\[
r_D = \frac{r}{r_f}, .................................................................(D.46)
\]
and the dimensionless position in the z-direction is:

\[ z_D = \frac{z}{r_f} \] .................................(D.47)

Transforming Eq. D.41 to dimensionless variables, it becomes:

\[ \frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial p_{fD,cr}}{\partial r_D} \right] + \frac{2}{F_{cD}} \frac{\partial p_{D,cr}}{\partial z_D} \bigg|_{z_D=0} = \frac{1}{\eta_{fD}} \frac{\partial p_{fD,cr}}{\partial t_D} , \] .................................(D.48)

where dimensionless fracture conductivity is:

\[ F_{cD} = \frac{k_f w}{k r_f} \] .................................(D.49)

and the dimensionless fractal hydraulic diffusivity of the circular fracture is:

\[ \eta_{fD} = \frac{\phi_0 c_i k_f r_f}{\phi_f c_f k_0} \] .................................(D.50)

Similarly, initial and boundary conditions become:

\[ p_{fD,cr}(r_D, t_D = 0) = 0 \] .................................(initial condition), (D.51)

\[ \left[ \frac{\partial p_{fD,cr}}{\partial r_D} \right]_{r_D=r_wD} = -\frac{1}{F_{fD,crD}} \] .................................(inner boundary condition), (D.52)

\[ \left[ \frac{\partial p_{D,cr}}{\partial r_D} \right]_{r_D=1} = 0 \] .................................(outer boundary condition), (D.53)

where dimensionless fracture fractal conductivity is:
Analogous to the approach made by Larsen et al. (1991), the circular fracture should be discretized into sufficiently small segments so the pressure gradient in Eq. D.48 can be approximated as:

\[
\left[ \frac{\partial p_{D,cr}}{\partial z_D} \right]_{z_D=0} \approx -\pi \bar{q} D_j h_D \tag{D.55}
\]

Then, Eq. D.48 becomes:

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \begin{bmatrix} r_D \frac{\partial p_{D,cr}}{\partial r_D} \end{bmatrix} - \frac{2 \pi h_D}{F_{cD}} \bar{q} D_j = \frac{1}{\eta_j D} \frac{\partial p_{D,cr}}{\partial t_D} \tag{D.56}
\]

Following the procedure presented in Appendix A, the general solution to Eq. D.56 in the Laplace domain of a \( j \)-segment of the discretized circular hydraulic fracture is:

\[
\bar{p}_{D,cr} = A_j K_0 \left[ r_D \sqrt{\frac{u}{\eta_j D}} \right] + B_j I_0 \left[ r_D \sqrt{\frac{u}{\eta_j D}} \right] - \frac{2 \pi h_D \eta_j D}{u F_{cD}} \bar{q} D_j \tag{D.57}
\]

This discretization generates a system of equations with \( 3N \) unknowns. Therefore, it is necessary to define \( 3N \) equations to solve such a system.

Two out of the \( 3N \) equations of the system are obtained by applying the boundary conditions to the corresponding segment. The first equation is defined by applying the inner boundary condition to segment one. The resulting expression is:

\[
A_1 K_1 \left[ r_{wD} \sqrt{\frac{u}{\eta_j D}} \right] - B_1 I_1 \left[ r_{wD} \sqrt{\frac{u}{\eta_j D}} \right] = \frac{\sqrt{\eta_j D}}{F_{hncD} r_{wD}^{3/2}} \tag{D.58}
\]
The second equation is obtained by applying the outer boundary condition to segment $N$. The resulting equation is:

$$A_N K_1 \left[ r_D \sqrt{\frac{u}{\eta \beta D}} \right] - B_N I_1 \left[ r_D \sqrt{\frac{u}{\eta \beta D}} \right] = 0.$$  \hspace{1cm} (D.59)

$N-1$ equations are defined by establishing pressure continuity between each one of the segments. This is obtained by equating the pressure of a segment $j$ with the pressure of the adjacent segment $j-1$, both evaluated at the interface $j-1$:

$$A_{j-1} K_0 \left[ r_{Dj-1} \sqrt{\frac{u}{\eta \beta D}} \right] + B_{j-1} I_0 \left[ r_{Dj-1} \sqrt{\frac{u}{\eta \beta D}} \right] - \frac{2\pi h_D \eta \beta D}{u F_c D} \delta I_{Dj-1} = 0.$$  \hspace{1cm} (D.60)

Similarly, $N-1$ more equations are defined by establishing rate continuity between each one of the segments. Then, for this case:

$$A_{j-1} K_1 \left[ r_{Dj-1} \sqrt{\frac{u}{\eta \beta D}} \right] - B_{j-1} I_1 \left[ r_{Dj-1} \sqrt{\frac{u}{\eta \beta D}} \right] = A_{j-1} K_1 \left[ r_{Dj-1} \sqrt{\frac{u}{\eta \beta D}} \right] - B_{j-1} I_1 \left[ r_{Dj-1} \sqrt{\frac{u}{\eta \beta D}} \right] \hspace{1cm} (D.61)$$

To complete the system of equations, it is necessary to establish pressure continuity between the segments of the circular fracture and the fractal reservoir. First, the flow from the fractal reservoir to the circular fracture must be integrated over the area of the circular segment:

$$\bar{p}_{D,cr}(R_D, u) = \frac{\delta I_D h_D}{[\theta + 2] \Gamma[1 - v]} G_F(r_{Dj}, r_{Dj}, u).$$  \hspace{1cm} (D.62)
Then, defining the observations points at the midpoint of each one of the $N$-segments of the discretized fracture, the dimensionless pressure of the circular fracture and dimensionless pressure of the fractal reservoir should be equated. The following expression is obtained:

$$
A_j K_0 \left[ r_{Dj} \sqrt{\frac{u}{\eta_f I_D}} \right] + B_j l_0 \left[ r_{Dj} \sqrt{\frac{u}{\eta_f I_D}} \right] - 2\pi h_D \eta_f I_D \frac{u}{u f_{cD}} \delta q_{Dj} = \frac{h_D}{[\theta + 2]\Gamma[1-\nu]} \sum_{i=1}^{N} \delta q_{Di} [G_F(r_{Dj}, r_{Di-1}, u) - G_F(r_{Dj}^*, r_{Di-1}, u)]
$$

...(D.63)

The function $G_F$ for a fractal reservoir with typical diffusion is:

$$
G_F(r_{Dj}, r_{Di}, u) = \left[ \frac{\theta + 2}{\sqrt{u}} \right]^\nu \int_{r_{Di}}^{r_{Dj}} \cos^{-1} \left[ \frac{\left( r_{Dj}^* \right)^2 - r_{Di}^2 + r_{Dj}^2}{2 r_{Dj}^* r_{Dj}} \right] K_v \left[ R_{Dj}^{[\theta + 2]/2} \frac{2\sqrt{u}}{\theta + 2} \right] dr_D \quad ............(D.64)
$$

For a fractal reservoir with anomalous diffusion is:

$$
G_F(r_{Dj}^*, r_{Di}, u) = \left[ \frac{\theta + 2}{u^{\gamma/2}} \right]^\nu \int_{r_{Di}}^{r_{Dj}} \cos^{-1} \left[ \frac{\left( r_{Dj}^* \right)^2 - r_{Di}^2 + r_{Dj}^2}{2 r_{Dj}^* r_{Dj}} \right] K_v \left[ R_{Dj}^{[\theta + 2]/2} \frac{2u^{\gamma/2}}{\theta + 2} \right] dr_D \quad .......(D.65)
$$

For double porosity reservoirs:

$$
G_F(r_{Dj}^*, r_{Di}, u) = \left[ \frac{\theta + 2}{\sqrt{u f(u)}} \right]^\nu \int_{r_{Di}}^{r_{Dj}} \cos^{-1} \left[ \frac{\left( r_{Dj}^* \right)^2 - r_{Di}^2 + r_{Dj}^2}{2 r_{Dj}^* r_{Dj}} \right] K_v \left[ R_{Dj}^{[\theta + 2]/2} \frac{2u f(u)}{\theta + 2} \right] dr_D \quad .......(D.66)
$$

For all cases:

$$
R_D = \sqrt{\frac{2}{D} + \frac{2}{0D}} \quad ...........................................................(D.67)
$$

Similar to the model developed by Larsen et al. (1991), the integrals involved in Eq. D.64 through D.66 must be performed numerically. For this case, we have used the adaptative quadrature
method (*MATLAB's integral*-function). Besides, the cases when the observation point is greater or less than the upper integration limit should also be taken into account. Therefore, if the observation point is greater than the upper integration limit, the following expression for a fractal reservoir with typical diffusion should be used:

\[
G_F(r_{Dj}^*, r_{Di}^*, u) = \left[ \frac{\theta + 2}{\sqrt{u}} \right]^\nu \int_{r_{Dj}^*-r_{Dk}^*}^{r_{Dj}^*+r_{Dk}^*} \frac{r_D^*}{R_D^{[\beta-1]/2}} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - (r_{Dj}^*)^2 + r_D^*}{2r_{Dj}^*r_D^*} \right] K_v \left[ R_D^{[\theta+2]/2} \frac{2\sqrt{u}}{\theta+2} \right] dR_D \ldots \text{(D.68)}
\]

For a fractal reservoir with anomalous diffusion:

\[
G_F(r_{Dj}^*, r_{Di}^*, u) = \left[ \frac{\theta + 2}{\sqrt{u}} \right]^\nu \int_{r_{Dj}^*-r_{Dk}^*}^{r_{Dj}^*+r_{Dk}^*} \frac{r_D^*}{R_D^{[\beta-1]/2}} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - (r_{Dj}^*)^2 + r_D^*}{2r_{Dj}^*r_D^*} \right] K_v \left[ R_D^{[\theta+2]/2} \frac{2u^{\gamma/2}}{\theta+2} \right] dR_D \ldots \text{(D.69)}
\]

For double porosity reservoirs:

\[
G_F(r_{Dj}^*, r_{Di}^*, u) = \left[ \frac{\theta + 2}{\sqrt{u}} \right]^\nu \int_{r_{Dj}^*-r_{Dk}^*}^{r_{Dj}^*+r_{Dk}^*} \frac{r_D^*}{R_D^{[\beta-1]/2}} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - (r_{Dj}^*)^2 + r_D^*}{2r_{Dj}^*r_D^*} \right] K_v \left[ R_D^{[\theta+2]/2} \frac{2\sqrt{u}}{\theta+2} \right] dR_D \ldots \text{(D.70)}
\]

If the observation point is smaller than the upper integration limit, the following expression for a fractal reservoir with typical diffusion should be used:

\[
G_F(r_{Dj}^*, r_{Di}^*, u) = \left[ \frac{\theta + 2}{\sqrt{u}} \right]^\nu \int_{R_{D_{mid}}}^{R_{D_{max}}} \frac{r_D^{[\beta-1]/2}}{R_D^{[\beta-1]/2}} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - (r_{Dj}^*)^2 + r_D^*}{2r_{Dj}^*r_D^*} \right] K_v \left[ R_D^{[\theta+2]/2} \frac{2\sqrt{u}}{\theta+2} \right] dR_D + \\
\left[ \frac{\theta + 2}{\sqrt{u}} \right]^\nu \int_{R_{D_{mid}}}^{R_{D_{min}}} \frac{r_D^{[\beta-1]/2}}{R_D^{[\beta-1]/2}} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - (r_{Dj}^*)^2 + r_D^*}{2r_{Dj}^*r_D^*} \right] K_v \left[ R_D^{[\theta+2]/2} \frac{2\sqrt{u}}{\theta+2} \right] dR_D \ldots \text{(D.71)}
\]

For a fractal reservoir with anomalous diffusion:
For double porosity reservoirs:

\[
G_F(r_{Dj}^*,r_{Di},u) = \left[ \frac{\theta + 2}{u^{\gamma/2}} \right] \pi \frac{R_{Dmid}}{R_{Dmin}} \frac{R_D^{[3-\beta]/2}}{2\theta + 1} K_v \int R_D^{[\theta+2]/2} \frac{2u^{\gamma/2}}{2\theta + 1} dR_D + \]

\[
\left[ \frac{\theta + 2}{u^{\gamma/2}} \right] \pi \frac{R_{Dmax}}{R_{Dmid}} \frac{R_D^{[3-\beta]/2}}{2\theta + 1} \cos^{-1} \left[ \frac{(r_{Dj}^*)^2 - r_{Dj}^2 + R_D^2 - z_{0D}^2}{2r_{Dj} \sqrt{R_D^2 - z_{0D}^2}} \right] K_v \int R_D^{[\theta+2]/2} \frac{2u^{\gamma/2}}{2\theta + 1} dR_D . \tag{D.72}
\]

Where:

\[
R_{Dmin} = \sqrt{\frac{2}{R_{Dmin}^2} + z_{0D}^2} , \tag{D.74}
\]

\[
R_{Dmid} = \sqrt{[r_{Dk} - r_{Dj}^*]^2 + z_{0D}^2} , \tag{D.75}
\]

and:

\[
R_{Dmax} = \sqrt{\frac{2}{R_{Dk}^2} + z_{0D}^2} , \tag{D.76}
\]

**D.4. Constant-Rate Solution for the Model of a Horizontal Well Intercepting a Circular Transverse Finite Conductivity Fracture within a Fractal Reservoir**

The flow within the rectangular fracture is governed by the diffusivity equation for linear flow with a source term. It is given by:
Analogous to the circular fracture case, the initial and boundary conditions for the present case are:

\[
p_f(x, t = 0) = p_i \quad \text{(initial condition), } \text{.........................(D.78)}
\]

\[
\left[ \frac{\partial p_f}{\partial x} \right]_{x=0} = \frac{q_w B \mu}{4 k_f w f} \quad \text{(inner boundary condition), } \text{.........................(D.79)}
\]

\[
\left[ \frac{\partial p_f}{\partial x} \right]_{x=x_f} = 0 \quad \text{(outer boundary condition). } \text{.........................(D.80)}
\]

For the rectangular hydraulic fracture case, the dimensionless pressure in the fracture is:

\[
p_{fD,cr}(x_D, t_D) = \frac{2 \pi k_0 h}{q_w B \mu f} [p_i - p_f(x, t)] \text{..........................(D.81)}
\]

The dimensionless time is defined by Eq. D.6 and the dimensionless position in the \(x\)-direction is:

\[
x_D = \frac{x}{x_f} \text{.................................................................(D.82)}
\]

The dimensionless position in the \(z\)-direction is:

\[
z_D = \frac{z}{x_f} \text{.................................................................(D.83)}
\]

Transforming Eq. D.77 to dimensionless variables, it becomes:
where dimensionless fracture conductivity is:

\[ F_{cD} = \frac{k_f w}{k_x f}, \]  \hspace{1cm} \text{(D.85)}

and the dimensionless fractal hydraulic diffusivity of the rectangular fracture is:

\[ \eta_{fD} = \frac{\phi_0 c_f k_f x_f^\theta}{\phi_f c_f k_0}, \]  \hspace{1cm} \text{(D.86)}

Similarly, initial and boundary conditions become:

\[ p_{fD,cr}(x_D, t_D = 0) = 0 \]  \hspace{1cm} \text{(initial condition), \hspace{1cm} (D.87)}

\[ \left[ \frac{\partial p_{fD,cr}}{\partial x_D} \right]_{r_D = r_w D} = -\frac{\pi}{2F_{fhcD}y_f J_D} \]  \hspace{1cm} \text{(inner boundary condition), \hspace{1cm} (D.88)}

\[ \left[ \frac{\partial p_{fD,cr}}{\partial x_D} \right]_{x_D = 1} = 0 \]  \hspace{1cm} \text{(outer boundary condition), \hspace{1cm} (D.89)}

Where the dimensionless fracture fractal conductivity is:

\[ F_{fhcD} = \frac{k_f w x_f^\theta}{k h}, \]  \hspace{1cm} \text{(D.90)}

and the dimensionless fracture length along the wellbore is:
\[ y_{jD} = \frac{y_f}{x_f} \] (D.91)

For this case, the discretization of the rectangular fracture into sufficiently small segments permits the approximation of the pressure gradient in Eq. D.84 by using Eq. D.55. Consequently, Eq. D.84 becomes:

\[ \frac{\partial^2 p_{jD,cr}}{\partial x_D^2} - \frac{2 \pi h_D}{F_{cD}} \frac{\partial q_{Dj}}{\partial t_D} = \frac{1}{\eta_{jD}} \frac{\partial p_{jD,cr}}{\partial t_D}. \] (D.92)

Following the procedure presented in Appendix A, the general solution to Eq. D.92 in the Laplace domain of a \( j \)-segment of the discretized rectangular hydraulic fracture is:

\[ \bar{p}_{jD,cr} = A_j \exp \left[ -x_D \sqrt{\frac{u}{\eta_{jD}}} \right] + B_j \exp \left[ x_D \sqrt{\frac{u}{\eta_{jD}}} \right] - \frac{2 \pi h_D \eta_{jD}}{u F_{cD}} \partial \bar{q}_{Dj}. \] (D.93)

Analogous to the circular fracture case, the first equation to establish the system of \( 3N \) equations results from applying the inner boundary condition to segment one. The resulting expression is:

\[ A_1 - B_1 = \frac{\pi \sqrt{\eta_{jD}}}{2 F_{hD}^{1/2} y_{jD} u^{3/2}} \] (D.94)

The second equation is obtained by applying the outer boundary condition to segment \( N \). The resulting equation is:

\[ A_N \exp \left[ -\sqrt{\frac{u}{\eta_{jD}}} \right] - B_N \exp \left[ \sqrt{\frac{u}{\eta_{jD}}} \right] = 0. \] (D.95)
The pressure continuity between each one of the segments is obtained by equating the pressure of a segment \( j \) with the pressure of the adjacent segment \( j-1 \), both evaluated at the interface \( j-1 \):

\[
A_{j-1} \exp \left[ -x_{Dj-1} \sqrt{\frac{u}{\eta_f D}} \right] + B_{j-1} \exp \left[ x_{Dj-1} \sqrt{\frac{u}{\eta_f D}} \right] - \frac{2\pi D \eta_f D}{uF_c D} \phi_j = \]

\[
A_j \exp \left[ -x_{Dj-1} \sqrt{\frac{u}{\eta_f D}} \right] + B_j \exp \left[ x_{Dj-1} \sqrt{\frac{u}{\eta_f D}} \right] - \frac{2\pi D \eta_f D}{uF_c D} \phi_j
\]

.........................(D.96)

Similarly, the rate continuity between each one of the segments is defined by:

\[
-A_{j-1} \exp \left[ -x_{Dj-1} \sqrt{\frac{u}{\eta_f D}} \right] + B_{j-1} \sqrt{I} \exp \left[ x_{Dj-1} \sqrt{\frac{u}{\eta_f D}} \right] =
\]

\[
-A_j \exp \left[ x_{Dj-1} \sqrt{\frac{u}{\eta_f D}} \right] - B_j \exp \left[ r_{Dj-1} \sqrt{\frac{u}{\eta_f D}} \right]
\]

.........................(D.97)

The pressure continuity between each one of the segments of the hydraulic fracture and the fractal reservoir is obtained by equating the pressure of a segment \( j \) with the pressure of the fractal reservoir, superimposed over the area of the segment \( j \):

\[
A_j \exp \left[ -x^*_{Dj} \sqrt{\frac{u}{\eta_f D}} \right] + B_j \exp \left[ x^*_{Dj} \sqrt{\frac{u}{\eta_f D}} \right] - \frac{2\pi D \eta_f D}{uF_c D} \phi_j = \frac{hD}{[\theta + 2\Gamma[1-\nu]}
\]

\[
\sum_{i=1}^{N} \phi_i \left[ Z_F (x^*_{Dj}, y^*_{Dj}, x_{Dj-1}, y_{Dj-1}, x_{Dj}, y_{Dj} \mid u) + Z_F (x_{Dj}, y_{Dj}, -x_{Dj}, y_{Dj-1}, -x_{Dj-1}, y_{Dj} \mid u) \right]
\]

......(D.98)

The function \( Z_F \) for a fractal reservoir with typical diffusion is:
For double porosity reservoirs:

\[
Z_F(x_D^*, y_D^*, x_{1D}, y_{1D}, x_{2D}, y_{2D}, u) = \\
\left[\frac{\theta + 2}{\sqrt{u}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2\sqrt{u}}{\theta + 2} \right] dr_D + \\
\left[\frac{\theta + 2}{\sqrt{u}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2\sqrt{u}}{\theta + 2} \right] dr_D + \\
\left[\frac{\theta + 2}{\sqrt{u}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2\sqrt{u}}{\theta + 2} \right] dr_D
\]

For a fractal reservoir with anomalous diffusion, the function is:

\[
Z_F(x_D^*, y_D^*, x_{1D}, y_{1D}, x_{2D}, y_{2D}, u) = \\
\left[\frac{\theta + 2}{u^{\gamma/2}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2u^{\gamma/2}}{\theta + 2} \right] dr_D + \\
\left[\frac{\theta + 2}{u^{\gamma/2}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2u^{\gamma/2}}{\theta + 2} \right] dr_D + \\
\left[\frac{\theta + 2}{u^{\gamma/2}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2u^{\gamma/2}}{\theta + 2} \right] dr_D
\]

For double porosity reservoirs:

\[
Z_F(x_D^*, y_D^*, x_{1D}, y_{1D}, x_{2D}, y_{2D}, u) = \\
\left[\frac{\theta + 2}{\sqrt{u}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2\sqrt{u(\theta)} (\theta + 2)}{\theta + 2} \right] dr_D + \\
\left[\frac{\theta + 2}{\sqrt{u}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2\sqrt{u(\theta)} (\theta + 2)}{\theta + 2} \right] dr_D + \\
\left[\frac{\theta + 2}{\sqrt{u}}\right] r_D^{\beta} \frac{R_D^{[\theta + 2]/2} (\theta + 2)}{R_D^{[\beta - 1]/2}} K_v \left[ R_D^{[\theta + 2]/2} \frac{2\sqrt{u(\theta)} (\theta + 2)}{\theta + 2} \right] dr_D
\]

For all cases:
\[ \vartheta_{\text{min},I} = \sin^{-1}\left[ \frac{y_{1D}}{r_D} \right] \] \hspace{1cm} \text{(D.102)}

\[ \vartheta_{\text{max},I} = \cos^{-1}\left[ \frac{x_{1D}}{r_D} \right] \] \hspace{1cm} \text{(D.103)}

\[ \vartheta_{\text{min},II} = \cos^{-1}\left[ \frac{x_{2D}}{r_D} \right] \] \hspace{1cm} \text{(D.104)}

\[ \vartheta_{\text{max},II} = \cos^{-1}\left[ \frac{x_{1D}}{r_D} \right] \] \hspace{1cm} \text{(D.105)}

\[ \vartheta_{\text{min},III} = \cos^{-1}\left[ \frac{x_{2D}}{r_D} \right] \] \hspace{1cm} \text{(D.106)}

\[ \vartheta_{\text{max},III} = \sin^{-1}\left[ \frac{y_{2D}}{r_D} \right] \] \hspace{1cm} \text{(D.107)}

and:

\[ r_{D1} = \sqrt{x_{1D}^2 + y_{1D}^2} \] \hspace{1cm} \text{(D.108)}

\[ r_{D2} = \sqrt{x_{2D}^2 + y_{1D}^2} \] \hspace{1cm} \text{(D.109)}

\[ r_{D3} = \sqrt{x_{1D}^2 + y_{2D}^2} \] \hspace{1cm} \text{(D.110)}

\[ r_{D4} = \sqrt{x_{2D}^2 + y_{2D}^2} \] \hspace{1cm} \text{(D.111)}
The same considerations of symmetry and change of variables shown in Appendix A for the calculation of $Z_R$ must be made for $Z_F$. 
In this Appendix, we present the derivation of the classic transient interporosity transfer functions developed by de Swaan (1976) and their application in the double porosity flow model presented by Cinco-Ley et al. (1982). In addition, we show the development of the transient interporosity transfer function considering fractal matrix blocks and the derivation of the asymptotic solution of the double fractal model.

E.1. Transient Interporosity Transfer Functions (de Swaan, 1976)

Development of the transfer interporosity function considering slab matrix blocks

The diffusivity equation for the linear flow occurring in slab matrix blocks as shown in Fig. E.1 is given by:

\[ \frac{\partial^2 p_{ma}}{\partial z^2} = \frac{1}{\eta_{ma}} \frac{\partial p_{ma}}{\partial t} \]  \hspace{1cm} (E.1)

Where the hydraulic diffusivity of the matrix is defined by:

\[ \eta_{ma} = \frac{k_{ma}}{\phi_{ma} k_{ima}} \]  \hspace{1cm} (E.2)
The matrix blocks are assumed to have uniformly distributed initial pressure, \( p_i \). Hence the initial condition for Eq E.1 is given by:

\[
p_{ma}(z, t = 0) = p_i \quad \text{..........................................................(E.3)}
\]

The interface between the matrix blocks and the natural fractures is assumed to be unrestricted (i.e., without interporosity skin). Therefore, the inner boundary condition is:

\[
p_{ma}(z = 0, t) = p_f(r, t) \quad \text{..........................................................(E.4)}
\]

Since the matrix blocks are assumed to be closed systems, the outer boundary condition for Eq. 1 is:

\[
\left[ \frac{\partial p_{ma}}{\partial z} \right]_{z = h_{ma}/2} = 0 \quad \text{..........................................................(E.5)}
\]

To solve Eq. E.1 in the Laplace domain, it is more convenient to express it in terms of a normalized drop of pressure. For this case, the normalized drop of pressure is defined as:
\[ \Delta p_{ma}(z,t) = \frac{p_i - p_{ma}(z,t)}{p_i - p_{f}(r,t)} \]  \hspace{1cm} \text{(E.6)}

Using the chain rule, Eq. E.1 is rewritten as:

\[ \frac{\partial^2 \Delta p_{ma}}{\partial z^2} = \frac{1}{\eta_{ma}} \frac{\partial \Delta p_{ma}}{\partial t} \]  \hspace{1cm} \text{(E.7)}

and the initial and boundary conditions become:

\[ \Delta p_{ma}(z,t = 0) = 0 \]  \hspace{1cm} \text{(initial condition), \hspace{1cm} (E.8)}

\[ \Delta p_{ma}(z = 0, t) = 1 \]  \hspace{1cm} \text{(inner boundary condition), \hspace{1cm} (E.9)}

\[ \left[ \frac{\partial \Delta p_{ma}}{\partial z} \right]_{z = h_{ma}/2} = 0 \]  \hspace{1cm} \text{(outer boundary condition), \hspace{1cm} (E.10)}

Applying the Laplace transform to Eq. E.7 and according to the initial condition, the following differential equation is obtained:

\[ \frac{d^2 \Delta \bar{p}_{ma}}{dz^2} - \frac{u}{\eta_{ma}} \Delta \bar{p}_{ma} = 0 \]  \hspace{1cm} \text{(E.11)}

By inspection, the general solution of Eq. E.11 can be given by a linear combination of hyperbolic sines and hyperbolic cosines. Therefore, for this problem the general solution is:

\[ \Delta \bar{p}_{ma}(z,u) = C_1 \sinh \left[ z \sqrt{\frac{u}{\eta_{ma}}} \right] + C_2 \cosh \left[ z \sqrt{\frac{u}{\eta_{ma}}} \right] \] \hspace{1cm} \text{(E.12)}

Applying the outer boundary condition to the general solution, the following expression results in:

\[ \left[ \frac{d \Delta \bar{p}_{ma}(z,u)}{dz} \right]_{z = h_{ma}/2} = C_1 \cosh \left[ \frac{h_{ma}}{2} \sqrt{\frac{u}{\eta_{ma}}} \right] + C_2 \sinh \left[ \frac{h_{ma}}{2} \sqrt{\frac{u}{\eta_{ma}}} \right] = 0 \] \hspace{1cm} \text{(E.13)}
From Eq. E.13, it is concluded that the constant $C_1$ is:

$$C_1 = -C_2 \tanh \left[ \frac{h_{ma}}{2} \sqrt{\frac{u}{\eta_{ma}}} \right]$$  \hspace{1cm} (E.14)

Substituting Eq. E.14 in Eq. E.12, a bounded solution for this problem is:

$$\Delta \bar{p}_{ma}(z,u) = -C_2 \tanh \left[ \frac{h_{ma}}{2} \sqrt{\frac{u}{\eta_{ma}}} \right] \sinh \left[ z \sqrt{\frac{u}{\eta_{ma}}} \right] + C_2 \cosh \left[ z \sqrt{\frac{u}{\eta_{ma}}} \right]$$  \hspace{1cm} (E.15)

Applying the inner boundary condition to Eq. 15, it is concluded that the constant $C_2$ is:

$$\Delta \bar{p}_{ma}(z = 0, u) = C_2 = \frac{1}{u}$$  \hspace{1cm} (E.16)

The particular solution for this problem is obtained by substituting $C_2$ in Eq. E.15:

$$\Delta \bar{p}_{ma}(z,u) = -\frac{1}{u} \tanh \left[ \frac{h_{ma}}{2} \sqrt{\frac{u}{\eta_{ma}}} \right] \sinh \left[ z \sqrt{\frac{u}{\eta_{ma}}} \right] + \frac{1}{u} \cosh \left[ z \sqrt{\frac{u}{\eta_{ma}}} \right]$$  \hspace{1cm} (E.17)

The derivative of the pressure with respect to $z$ evaluated in zero is required to estimate the flow rate transferred from the matrix to the fractures. Based on Eq. E.17, such a derivative is:

$$\left[ \frac{d\Delta \bar{p}_{ma}(z,u)}{dz} \right]_{z=0} = -\frac{1}{u \eta_{ma}} \tanh \left[ \frac{h_{ma}}{2} \sqrt{\frac{u}{\eta_{ma}}} \right].$$  \hspace{1cm} (E.18)

Applying the inverse Laplace transform to Eq. E.18 (Eq.8.51 from Oberhettinger et al. (1973), p.294) the following expression is obtained:

$$\left[ \frac{d\Delta p_{ma}(z,t)}{dz} \right]_{z=0} = -\frac{2}{h_{ma}} \theta_2 \left[ \frac{4\eta_{ma}}{h_{ma}^2} \right] = -\frac{4}{h_{ma}^2} \sum_{n=0}^{\infty} \exp \left[ -4\pi^2 (2n+1)^2 \frac{\eta_{ma}}{h_{ma}^2} \right]$$  \hspace{1cm} (E.19)
Under the transient regime, the flow rate per unit of fracture volume from the matrix blocks to the fractures surrounding the matrix blocks can be expressed using the convolution integral. The convolution is defined as:

\begin{equation}
q = -\frac{2}{A_{ma} h_f} \int_0^t \frac{\partial \Delta p_f}{\partial \tau} q_{uma}(t - \tau) d\tau \nonumber \end{equation}

(E.20)

where \( q_{uma} \) is the fluid transfer rate from the matrix blocks to the fracture. Based on Darcy's Law, the fluid transfer rate from the matrix blocks to the fracture is:

\begin{equation}
q_{uma} = -\frac{k_{ma} A_{ma}}{\mu} \left[ \frac{\partial \Delta p_{ma}}{\partial z} \right]_{z=0} \nonumber \end{equation}

(E.21)

Substituting Eq. E.19 and Eq. E. 21 in Eq. E.20:

\begin{equation}
q = \frac{8 k_{ma}}{h_f h_{ma} \mu} \int_0^t \frac{\partial \Delta p_f}{\partial \tau} \sum_{n=0}^{\infty} \exp \left[ -\frac{4 \pi^2 (2n+1)^2 \eta_{ma}}{h_{ma}^2} \right] \Delta t \nonumber \end{equation}

(E.22)

Development of the transfer interporosity function considering spherical matrix blocks

For this case, consider an array as the one shown in Fig. E.2. The diffusivity equation for a system with the shape of a sphere is given by:

\begin{equation}
\frac{\partial^2 p_{ma}}{\partial R^2} + \frac{2 \Delta p_{ma}}{R} \frac{\partial}{\partial R} = \frac{1}{\eta_{ma}} \frac{\partial p_{ma}}{\partial t} \nonumber \end{equation}

(E.23)

where hydraulic diffusivity of the matrix, \( \eta_{ma} \) is defined exactly as for the case of slab matrix blocks (Eq. E.2). The initial and the boundary conditions for this case are the same as in the previous section, \textit{i.e.}:
\[ p_{ma}(R, t = 0) = p_i \] (initial condition), .................................................. (E.24)

\[ \left[ \frac{\partial p_{ma}}{\partial R} \right]_{R=0} = 0 \] (inner boundary condition),............................................. (E.25)

\[ p_{ma}(R = R_{ma}, t) = p_f(r, t) \] (outer boundary condition)................................. (E.26)

Figure E.2 — Schematics of a Naturally Fractured Reservoir with spherical matrix blocks and horizontal fractures.

Analogous to the previous subsection, the normalized drop of pressure for this case is:

\[ \Delta p_{ma}(R, t) = \frac{p_i - p_{ma}(R, t)}{p_i - p_f(r, t)} \] ................................................................. (E.27)

Consequently, Eq. E.23 is rewritten as:

\[ \frac{\partial^2 \Delta p_{ma}}{\partial R^2} + \frac{2}{R} \frac{\partial \Delta p_{ma}}{\partial R} = \frac{1}{\eta_{ma}} \frac{\partial \Delta p_{ma}}{\partial t} \] ................................................................. (E.28)

and, the initial and boundary conditions become:

\[ \Delta p_{ma}(R, t = 0) = 0 \] (initial condition), ................................................................. (E.29)
\[ \Delta p_{ma}(R = R_{ma}, t) = 1 \]  
(inner boundary condition)................................. (E.30)

\[ \left[ \frac{\partial \Delta p_{ma}}{\partial R} \right]_{R=0} = 0 \]  
(outer boundary condition)................................. (E.31)

As shown in Appendix A, the transformation defined by:

\[ \Delta p_{ma}(R, t) = \frac{b(R, t)}{R} \] ................................................................. (E.32)

facilitates the procedure to find a solution to an equation with the shape as of Eq. E.28. Applying such a transformation, Eq. E.28 becomes:

\[ \frac{\partial^2 b}{\partial R^2} = \frac{1}{\eta_{ma}} \frac{\partial b}{\partial t} \] ................................................................. (E.33)

Since Eq. E.33 has exactly the same shape as Eq. E.7, its general solution in the Laplace domain is:

\[ \bar{b}(R, u) = C_1 \sinh \left[ R \frac{u}{\eta_{ma}} \right] + C_2 \cosh \left[ R \frac{u}{\eta_{ma}} \right] \] ................................................................. (E.34)

In terms of the drop of pressure in the matrix (Eq. E.32):

\[ \Delta p_{ma}(R, u) = \frac{C_1}{R} \sinh \left[ R \frac{u}{\eta_{ma}} \right] + \frac{C_2}{R} \cosh \left[ R \frac{u}{\eta_{ma}} \right] \] ................................................................. (E.35)

Applying the outer boundary condition, it is concluded that the constant \( C_2 \) is zero. Therefore, the bounded solution for this problem is:

\[ \Delta p_{ma}(R, u) = \frac{C_1}{R} \sinh \left[ R \frac{u}{\eta_{ma}} \right] \] ................................................................. (E.36)
Applying the inner boundary condition, the value of the constant $C_1$ is obtained. It is given by:

$$C_1 = \frac{R_{ma}}{u} \frac{1}{\sinh \left[ R_{ma} \frac{u}{\eta_{ma}} \right]}$$ ................................................................. (E.37)

Substituting Eq. E.37 in Eq. E.36, the particular solution for this problem is:

$$\Delta \bar{p}_{ma}(R,u) = \frac{R_{ma}}{\sinh \left[ R_{ma} \frac{u}{\eta_{ma}} \right]} \frac{1}{u} \sinh \left[ R \frac{u}{\eta_{ma}} \right]$$ ................................................................. (E.38)

The derivative of Eq. E.38 with respect of the radius $R$ and evaluated in $R_{ma}$ is:

$$\left[ \frac{d \Delta \bar{p}_{ma}(R,u)}{dR} \right]_{R=R_{ma}} = -\frac{1}{R_{ma}u} + \frac{1}{\sqrt{\eta_{ma}u}} \coth \left[ R_{ma} \frac{u}{\eta_{ma}} \right].$$ ................................................................. (E.39)

The Inverse Laplace Transform of Eq. E.39 is (Eq. 8.52, Oberhettinger et al., 1973, p. 294):

$$\left[ \frac{d \Delta \bar{p}_{ma}(R,t)}{dR} \right]_{R=R_{ma}} = \frac{1}{R_{ma}} - 1 + \theta_3 \left[ \frac{\eta_{ma}}{R_{ma}^2} \right] = \frac{2}{R_{ma}} \sum_{n=1}^{\infty} \exp \left[ -\frac{\pi^2 n^2 \eta_{ma}}{R_{ma}^2} t \right].$$ ................................................................. (E.40)

E.2. Double Porosity Model with Transient Interporosity Transfer

Constant-rate solution (Cinco-Ley et al., 1982)

The flow in a radial fracture network considered by Cinco-Ley et al. (1982) is modeled by:

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial p_f}{\partial r} \right] = \frac{\phi_f \mu_k c_f b}{k_f b} \frac{\partial p_f}{\partial t} + \frac{\mu}{k_f b} q^*,$$ ................................................................. (E.41)
where $q^*$ is the fluid transfer from the matrix to the fracture network per unit of bulk volume. As shown by de Swaan (1976), it can be modeled by the convolution integral:

$$q^* = t \frac{\partial p_f}{\partial \tau} q^*_{uma}(t-\tau) d\tau,$$

(E.42)

where:

$$q^*_{uma} = \frac{A_{fb} k_{ma}}{\mu} [\nabla p_{ma}]_{sur},$$

(E.43)

Substituting Eq. E.42 and Eq. E.43 in Eq. E.41:

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial p_f}{\partial r} \right] = \frac{\phi_{fb} k_{fb}}{k_{fb}} \frac{\partial p_f}{\partial \tau} + k_{ma} A_{fb} t \frac{\partial p_f}{\partial \tau} [\nabla p_{ma}]_{sur} (t-\tau) d\tau.$$  

(E.44)

The initial and boundary conditions for this model are:

$$p_f(r, t=0) = p_i$$ (initial condition), .............................................................. (E.45)

$$\left[ \frac{\partial p_f}{\partial r} \right]_{r=r_w} = \frac{2\pi k_{fb} h}{B\mu}$$ (inner boundary condition), .............................................................. (E.46)

$$\lim_{r \to \infty} p_f(r, t) = p_i$$ (outer boundary condition).............................................. (E.47)

The dimensionless variables defined for this model are summarized in Table E.1.
Table E.1 — Dimensionless variables for the Cinco-Ley et al. (1982) model.

<table>
<thead>
<tr>
<th>Dimensionless Variable</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure in the fracture network</td>
<td>( p_{fD,cr}(r_D,t_D) = \frac{k_f h}{q B \mu} [p_i - p_f(r,t)] )</td>
</tr>
<tr>
<td>Pressure in the matrix blocks</td>
<td>( p_{maD,cr}(r_D,t_D) = \frac{k_f h}{q B \mu} [p_i - p_{ma}(r,t)] )</td>
</tr>
<tr>
<td>Time</td>
<td>( t_D = \frac{k_f}{[\phi \epsilon]_t \mu r_w} t )</td>
</tr>
<tr>
<td>Radius</td>
<td>( r_D = \frac{r}{r_w} )</td>
</tr>
<tr>
<td>Storativity ratio</td>
<td>( \omega = \frac{\phi_f c_f \mu_h}{[\phi \epsilon]_t} )</td>
</tr>
<tr>
<td>Fracture area</td>
<td>( A/f_D = A_{fmaD}/h_{ma} )</td>
</tr>
</tbody>
</table>

Eq. E.44 expressed in dimensionless form is:

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial p_{fD,cr}}{\partial r_D} \right] = \omega \frac{\partial p_{fD,cr}}{\partial t_D} + [1 - \omega] \frac{A_{fD} h_{maD}}{h_{maD}} \frac{\partial p_{fD,cr}}{\partial t_D} - \frac{\partial}{\partial \tau} F(\eta_{maD}, h_{maD}, t_D = \tau) d\tau, \quad \text{...........(E.48)}
\]

where:

\[
F(\eta_{maD}, h_{maD}, t_D = \tau) = [\nabla p_{maD,cr}]_{z_D=\text{surface}} \quad \text{...........(E.49)}
\]

The initial and boundary conditions in dimensionless variables become:

\[
p_{fD}(r_D,t_D = 0) = 0 \quad \text{..................(initial condition), \quad \text{...........(E.50)}}
\]

\[
\left. \begin{bmatrix} \frac{\partial p_{fD}}{\partial r_D} \\ \frac{\partial p_{fD}}{\partial r_D} \end{bmatrix} \right|_{r_D = 1} = -1 \quad \text{..................(inner boundary condition), \quad \text{...........(E.51)}}
\]

\[
\lim_{r_D \to \infty} p_{fD}(r_D,t_D) = 0 \quad \text{..................(outer boundary condition), \quad \text{...........(E.52)}}
\]
Applying the Laplace transform to Eq. E.48 and arraying it can be written as:

\[
\frac{1}{r_D} \frac{d}{dr_D} \left[ r_D \frac{d \bar{P}_{J_D,cr}}{dr_D} \right] - f(u) \mu \bar{P}_{J_D,cr} = 0 \tag{E.53}
\]

Considering the boundary conditions for this model, the particular solution of Eq. E.53 is:

\[
\bar{P}_{J_D,cr}(r_D,u) = \frac{K_0 r_D \sqrt{uf(u)}}{u \sqrt{uf(u)} K_1 \sqrt{uf(u)}} \tag{E.54}
\]

where \( f(u) \) is the interporosity transfer function. Such a function depends on the properties of the matrix blocks (geometry and petrophysical properties) and the interporosity transfer regime. For unrestricted flow (without interporosity skin), it is defined as:

\[
f(u) = \omega + [1 - \omega] A_{J_D} \frac{h_{maD}}{h_{maD}} \sqrt{\frac{\eta_{maD}}{u}} \tanh \left[ \frac{h_{maD}}{2} \frac{u}{\eta_{maD}} \right] \tag{E.55}
\]

For the classic cases (slab and spheres matrix blocks) presented by Cinco-Ley et al., (1982) the functions are:

for slabs, and:

\[
f(u) = \omega + [1 - \omega] A_{J_D} \frac{h_{maD}}{h_{maD}} \sqrt{\frac{\eta_{maD}}{u}} \coth \left[ \frac{h_{maD}}{2} \frac{u}{\eta_{maD}} \right] - \frac{2}{h_{maD}} \sqrt{\frac{\eta_{maD}}{u}} \tag{E.57}
\]

for spheres.
Solution considering time-depend inner boundary condition

For this model, we have considered the diffusivity equation given defined by Eq. E.48 and the same initial and outer boundary conditions as in the previous section (*i.e.*, Eq. 50 and Eq. E.52). For the inner boundary condition, we have assumed a power-law time dependent pressure gradient around the wellbore, *i.e.*,:

\[
\left. r_D \frac{\partial \tilde{p}}{\partial r_D} \right|_{r_D=1} = -r_0^{\nu_0 - \nu} \quad \text{(inner boundary condition),}\ ...
\]  

(E.58)

Where \(\nu_0\) is an arbitrary reference exponent. We have considered a unit value of this parameter.

Applying the outer boundary condition, the bounded solution of Eq. E.48 in the Laplace domain is:

\[
\tilde{p}_{fD,cr}(r_D, u) = C_1 K_0 \left[ r_D \sqrt{u_0(u)} \right] \quad \text{................................. (E.59)}
\]

Applying the inner boundary condition, the value of the constant \(C_1\) is obtained:

\[
C_1 = \frac{\Gamma[2-\nu]}{u^{2-\nu} \sqrt{u_0(u)} K_1 \sqrt{u_0(u)}} \quad \text{.................................................... (E.60)}
\]

Hence, the particular solution for this problem is:

\[
\tilde{p}_{fD,cr}(r_D, u) = \frac{\Gamma[2-\nu]\sqrt{u_0(u)} K_0 \left[ r_D \sqrt{u_0(u)} \right]}{u^{2-\nu} \sqrt{u_0(u)} K_1 \sqrt{u_0(u)}} \quad \text{.................................................... (E.61)}
\]

Taking the line source approximation, Eq. E.61 reduces to:
\[
\mathbf{\bar{p}}_{D,cr}(r_D,u) = \frac{\Gamma(2-v)K_0[r_D \sqrt{uf(u)}]}{u^{2-v}}. \tag{E.62}
\]

**E.3. Transient Interporosity Transfer Function Considering Fractal Matrix Blocks**

*Development of the transfer interporosity function considering closed fractal matrix blocks*

If matrix blocks with fractal geometry are considered (see Fig. E.3), the model developed by Chang et al., (1990) for fractal fracture networks can be extended to fractal matrix blocks. For this case, consider the diffusivity equation for a fractal matrix block:

\[
\frac{1}{R^{D_{fma}-1}} \frac{\partial}{\partial R} \left[ R^{D_{fma}} \frac{\partial p_{ma}}{\partial R} \right] = \frac{1}{\eta_{ma}} \frac{\partial p_{ma}}{\partial t}. \tag{E.63}
\]

where \( \eta_{ma} \) is the geometry hydraulic diffusivity of the matrix, exactly defined as for the cases of slab and sphere matrix blocks. \( D_{fma} \) is the fractal dimension of the matrix block and \( \beta_{ma} \) is the spatial dimension of the matrix, defined as \( \beta_{ma} = D_{fma} - \theta_{ma} - 1 \).

Analogous to the cases shown in Section E.1 of this Appendix, the initial and the boundary conditions for this problem are:

\[
p_{ma}(R, t = 0) = p_i \quad \text{(initial condition)}, \tag{E.64}
\]

\[
p_{ma}(R = R_0, t) = p_f(r, t) \quad \text{(inner boundary condition)}, \tag{E.65}
\]

\[
\left[ \frac{\partial p_{ma}}{\partial R} \right]_{R = h_{ma}/2} = 0 \quad \text{(outer boundary condition)}. \tag{E.66}
\]
Figure E.3 — Schematics of a Naturally Fractured Reservoir with fractal matrix blocks.

For this model, the dimensionless pressure in the matrix is:

\[
p_{ma, cp}(R_D, t_D) = \frac{p_i - p_{ma}(R, t)}{p_i - p_f(r, t)}
\] (E.67)

The dimensionless time is defined as:

\[
t_D = \frac{k_{fb}}{[\phi_c]_t \mu_w^2} = \frac{\eta_f}{r_w^2} t
\] (E.68)

and the dimensionless position in the fractal matrix block is:

\[
R_D = \frac{R}{r_w}
\] (E.69)

Applying the definitions of the dimensionless variables, Eq. E.63 becomes:

\[
\frac{1}{R_{D}^{Dma-1}} \frac{\partial}{\partial R_D} \left[ R_{D}^{Dma} \frac{\partial p_{ma, cp}}{\partial R_D} \right] = \frac{1}{\eta_{ma}} \frac{\partial p_{ma, cp}}{\partial t_D}
\] (E.70)
where the dimensionless hydraulic anomalous diffusivity is defined as:

\[ \eta_{maD} = \frac{\eta_{ma}}{\eta_{fr \theta ma}} \] ................................................................. (E.71)

The initial and boundary conditions in dimensionless variables become:

\[ p_{maD,cp}(R_D, t_D = 0) = 0 \] (initial condition), ................................................. (E.72)

\[ \left[ \frac{\partial p_{maD,cp}}{\partial R_D} \right]_{R_D = R_0} = 0 \] (inner boundary condition), ................................................. (E.73)

\[ p_{maD,cp}(R_D = h_{maD}/2, t_D = 1) = 1 \] (outer boundary condition) ................................................. (E.74)

As shown in Appendix B, the general solution to Eq. E.70 in the Laplace domain is:

\[ \tilde{p}_{maD,cp}(R_D, \tilde{u}) = R_D^{[1-\beta_{ma}]/2} \left[ C_1 \left[ v_{ma} \left[ \tilde{\xi}_{RD} \right] \right] + C_2 I_{v_{ma}} \left[ \tilde{\xi}_{RD} \right] \right] \] ................................................................. (E.75)

where:

\[ v_{ma} = \frac{1-\beta_{ma}}{\theta_{ma} + 2}, \] ................................................................................. (E.76)

and:

\[ \tilde{\xi}_{RD} = R_D^{[\theta_{ma} + 2]/2} \left[ \frac{2}{\theta_{ma} + 2} \sqrt{\frac{u}{\eta_{maD}}} \right]. \] ................................................................. (E.77)

Applying inner boundary condition, the resulting bounded solution is:

\[ \tilde{p}_{maD,cp}(R_D, \tilde{u}) = C_2 R_D^{[1-\beta_{ma}]/2} \left[ \frac{I_{v_{ma}-1}[\tilde{\xi}_0]}{K_{v_{ma}-1}[\tilde{\xi}_0]} \right] \left[ v_{ma} \left[ \tilde{\xi}_{RD} \right] \right] + I_{v_{ma}} \left[ \tilde{\xi}_{RD} \right], \] ................................................................. (E.78)
where:

\[ \xi_0 = \lim_{R_D \to R_{0D}} \xi_{R_D} \] ......................................................................................... (E.79)

Given that \( R_{0D} \) is located at zero, the following approximation can be used:

\[ \frac{I_{\nu_{ma}}}{K_{\nu_{ma}} \xi_0} = \frac{2}{\Gamma[\nu_{ma}] \Gamma[1 - \nu_{ma}]} \] ......................................................................................... (E.80)

Then, Eq. E.78 becomes:

\[ \bar{p}_{ma,D,cp}(R_D,u) = R_D^{[1 - \beta_{ma}]/2} C_2 \left[ \frac{2}{\Gamma[\nu_{ma}] \Gamma[1 - \nu_{ma}]} K_{\nu_{ma}} \bar{\xi}_{R_D} + I_{\nu_{ma}} \xi_{R_D} \right] \] ........................................ (E.81)

Applying outer boundary condition, the particular solution for this problem is obtained:

\[ \bar{p}_{ma,D,cp}(R_D,u) = \frac{1}{u} \left[ \frac{2 R_D}{h_{ma,D}} \right]^{[1 - \beta_{ma}]/2} \left[ \frac{2}{\Gamma[\nu_{ma}] \Gamma[1 - \nu_{ma}]} K_{\nu_{ma}} \bar{\xi}_{R_D} + I_{\nu_{ma}} \xi_{R_D} \right] \] ........................................ (E.82)

where:

\[ \bar{\xi} = \left[ \frac{h_{ma,D}}{2} \right]^{[\nu_{ma} + 2]/2} \frac{2}{\nu_{ma} + 2} \sqrt{\frac{u}{\eta_{ma,D}}} \] ......................................................................................... (E.83)

Taking the derivative to Eq. E.83 with respect of \( R_D \):

\[ \frac{\partial \bar{p}_{ma,D,cp}}{\partial R_D} = \frac{R_D^{[1 - \beta_{ma}]/2}}{\sqrt{u \eta_{ma,D}}} \left[ \frac{2 R_D}{h_{ma,D}} \right]^{[1 - \beta_{ma}]/2} \left[ \frac{2}{\Gamma[\nu_{ma}] \Gamma[1 - \nu_{ma}]} K_{\nu_{ma}} \bar{\xi}_{R_D} + I_{\nu_{ma}} \xi_{R_D} \right] \] ........................................ (E.84)

Evaluating at the interface, Eq. E.84 can be expressed in terms of the function \( F(\eta_{ma,D}, h_{ma,D}, t_D) \):
Development of the transfer interporosity function considering infinite-acting fractal matrix blocks

To model infinite-acting matrix blocks, we considered matrix blocks of infinite length. Therefore, we solved Eq. E.70 using the following initial and boundary conditions:

\[ p_{maD,cp}(R_D, t_D = 0) = 0 \] (initial condition), .........................................................(E.86)

\[ p_{maD,cp}(R_D = h_{maD}/2, t_D) = 1 \] (inner boundary condition), .........................................................(E.87)

\[ \lim_{R_D \to \infty} p_{maD,cp}(R_D, t_D) = 0 \] (outer boundary condition).........................................................(E.88)

Applying the boundary conditions to the general solution (Eq. E.75) the particular solution is:

\[ \bar{p}_{maD,cp}(R_D, u) = \left[ \frac{2R_D}{h_{maD}} \right]^{[1-\beta_{ma}]/2} \frac{K_{v_{ma}} \left[ \xi_{RD} \right]}{uK_{v_{ma}} \left[ \xi \right]} \] .........................................................(E.89)

where \( \xi \) is given by Eq. E.83. Taking the derivative to Eq. 89 the following equation is derived:

\[ \frac{\partial p_{maD,cp}}{\partial R_D} = -R_D^{\beta_{ma}/2} \frac{1}{\sqrt{u\eta_{maD}}} \left[ \frac{2R_D}{h_{maD}} \right]^{[1-\beta_{ma}]/2} \frac{K_{v_{ma}}^{-1} \left[ \xi_{RD} \right]}{K_{v_{ma}} \left[ \xi \right]} \] .........................................................(E.90)

Evaluating at the interface:

\[ \left[ \frac{\partial p_{maD,cp}}{\partial R_D} \right]_{R_D = h_{maD}/2} = \left[ \frac{h_{maD}}{2} \right]^{\beta_{ma}/2} \frac{1}{\sqrt{u\eta_{maD}}} \frac{K_{v_{ma}}^{-1} \left[ \xi \right]}{K_{v_{ma}} \left[ \xi \right]} \] .........................................................(E.91)
At early and intermediate times (large values of the Laplace parameter $u$), the Modified Bessel function of second order ($K_v(x)$) can be approximated as:

$$K_v(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \quad \text{.................................................................} \quad \text{(E.92)}$$

Hence, at early and intermediate times Eq. E.91 can be written in terms of Eq. E.49 as:

$$\bar{F}(\eta_{maD}, h_{maD}, u) = \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma}/2} \frac{1}{\sqrt{u \eta_{maD}}} \quad \text{.................................................................} \quad \text{(E.93)}$$

Applying the inverse Laplace transform to Eq. E.93:

$$F(\eta_{maD}, h_{maD}, t_D) = \frac{1}{\sqrt{\pi \eta_{maD}}} \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma}/2} t_D^{-1/2} \quad \text{.................................................................} \quad \text{(E.94)}$$

Eq. E.94 applies for both closed and infinite-acting matrix blocks.

At late times (small values of the Laplace parameter $u$), the Modified Bessel function of second order ($K_v(x)$) can be approximated as:

$$K_v(x) \approx \frac{\Gamma[v]}{2} \left[ \frac{\pi}{2} \right]^{-v} \quad \text{.................................................................} \quad \text{(E.95)}$$

Taking the approximation given by Eq. E.95, Eq. E.91 is reduced to (in terms of Eq. E.49):  

$$\bar{F}(\eta_{maD}, h_{maD}, u) = \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma}/2} \frac{1}{\sqrt{u \eta_{maD}}} \frac{\Gamma[1 - \nu_{ma}]}{\Gamma[\nu_{ma}]} \left[ \frac{\varepsilon}{2} \right]^{2\nu_{ma} - 1} \quad \text{.................................................................} \quad \text{(E.96)}$$

Applying the inverse Laplace transform to Eq. E.96:
E.4. Asymptotic Solutions of the Double Porosity Model with Transient Interporosity Transfer considering Fractal Matrix Blocks

To develop the asymptotic solutions of the constant-rate case of this model, consider the line-source approximation of Eq. E.54 and evaluate such an expression at wellbore (i.e., \( r_D=1 \)). The resulting equation is:

\[
F(\eta_{maD}, h_{maD}, t_D) = \frac{[\theta_{ma} + 2]^{1-2\nu_{ma}}}{\Gamma[\nu_{ma}]h_{maD}^\nu_{ma}} \left( \frac{h_{maD}}{2} \right)^{-\beta_{ma}} t_D^{-\nu_{ma}} \]

(E.97)

Additionally, the Modified Bessel function in Eq. 98 for small arguments can be approximated using Eq. 9.6.8 from Abramowitz et al. (1970). Therefore, the resulting expression is:

\[
\bar{p}_{D,cr}(u) = \frac{K_0\sqrt{uf(u)}}{u} \]

(E.98)

To obtain the asymptotic constant-pressure solutions for this model, consider the approximation (Earlougher, 1977):

\[
p_{D,cr}(t_D) = \frac{1}{q_{D}(t_D)} \]

(E.100)

Closed Matrix Blocks

For this case, the interporosity transfer function defined by Eq. E.85 should be considered. At early times, such a function approximates to the value of the storativity ratio (i.e., \( f(u) = \omega \)). Therefore, the asymptotic constant-rate solution in the Laplace domain at early times is:
\( \bar{p}_{wD,cr}(u) = -\frac{1}{2u}[\ln[u] + \ln[\omega]]. \) ................................................................. (E.101)

Applying the inverse Laplace transform to Eq. E.101 results in:

\( p_{wD,cr}(t_D) = \frac{1}{2}\ln[t_D] + \frac{1}{2}\ln\left[\frac{\exp[\gamma]}{\omega}\right]. \) ................................................................. (E.102)

Hence, using the relation given by Eq. E.100 we obtain:

\( q_{wD}(t_D) = \frac{2}{\ln[t_D] + \ln\left[\frac{\exp[\gamma]}{\omega}\right]} \) ................................................................. (E.103)

where \( \gamma \) is Euler's constant (0.57721…). As pointed out by Cinco-Ley et al. (1982), the flow from the matrix blocks to the fracture network is linear at early and intermediate times regardless the geometry of the matrix block. For these periods of flow, the function \( F(\eta_{maD}, h_{maD}, t_D) \) takes asymptotic behavior given by Eq. E.94 and the interporosity transfer function is approximated as:

\( f(u) \approx [1 - \omega] \frac{A_{fD}}{h_{maD}} \sqrt{\frac{h_{maD}}{2}} \frac{\theta_{ma}}{u} \) ................................................................. (E.104)

Substituting Eq. 104 in Eq. E.99 and arraying it results in:

\( \bar{p}_{wD,cr}(u) = -\frac{1}{4u}\ln[u] + 2\ln\left[1 - \omega\right]\frac{A_{fD}}{h_{maD}}\sqrt{\frac{h_{maD}}{2}}\frac{\theta_{ma}}{\eta_{maD}} \] ................................................................. (E.105)

Applying inverse Laplace transform to Eq. E.105, it results:

\( p_{wD,cr}(t_D) = \frac{1}{4}\ln[t_D] + \frac{1}{2}\ln\left[\frac{\exp[\gamma/2]}{1 - \omega}\right] \frac{h_{maD}}{A_{fD}}\sqrt{\frac{h_{maD}}{2}}\frac{\theta_{ma}}{\eta_{maD}}^{-1/2} \] ................................................................. (E.106)
Consequently, the constant-pressure asymptotic solution for this case is:

\[
q_{wD}(t_D) = \frac{2}{\ln[t_D] + \ln \left[ \frac{\exp[\gamma/2]}{1-\omega} \frac{h_{maD}}{A_{fD}} \left[ \frac{h_{maD}}{2} \gamma_{ma} \eta_{maD} \right]^{-1/2} \right]^{-1/2}} \]

(E.107)

At late times, the behavior of the interporosity transfer function approaches one (i.e., \( f(u) = 1 \)). Substituting such a value in Eq. E.94 and applying inverse Laplace transform, the asymptotic constant-rate solution is:

\[
p_{wD,cr}(t_D) = \frac{1}{2} \ln[t_D] + \gamma \]

(E.108)

where \( \gamma \) is Euler's constant (0.57721…). Similar to the previous cases, the constant-pressure solution is:

\[
q_{wD}(t_D) = \frac{2}{\ln[t_D] + \gamma} \]

(E.109)

Infinite-Acting Matrix Blocks

The flow behavior of the matrix blocks assuming either closed or infinite-acting conditions is the same at early and intermediate times (i.e., before the outer boundary condition of the matrix blocks are reached). Therefore, the asymptotic solutions developed for closed matrix blocks, for early and intermediate times (constant-rate and constant-pressure cases), also apply for the infinite-acting matrix blocks. The asymptotic constant-rate solution in the Laplace domain for \( \nu_{ma} > 0 \) is obtained by plugging Eq. E.97 in the interporosity transfer function (Eq. E.55). Substituting the resulting equation in Eq. E.99, gives us:
\[ \bar{p}_{wD,cr}(u) = -\frac{1}{2u} v ma \ln[u] + \ln \left[ \frac{A_fD[\theta_{ma} + 2]^{1-2v_{ma}}}{h_{maD}n_{maD}^{v_{ma} - 1}} \frac{\Gamma[1-v_{ma}]}{\Gamma[v_{ma}]} \left[ \frac{2}{h_{maD}} \right]^{[\beta_{ma} + 1]/2} \right] \] .................................. (E.110)

(\gamma\text{ is Euler’s constant}). The expression in Eq. E.110 in the real domain is given by:

\[ p_{wD,cr}(t_D) = \frac{v_{ma}}{2} \ln[t_D] + \frac{v_{ma}}{2} \gamma - \frac{1}{2} \ln \left[ \frac{A_fD[\theta_{ma} + 2]^{1-2v_{ma}}}{h_{maD}n_{maD}^{v_{ma} - 1}} \frac{\Gamma[1-v_{ma}]}{\Gamma[v_{ma}]} \left[ \frac{2}{h_{maD}} \right]^{[\beta_{ma} + 1]/2} \right] \] .................................. (E.111)

Applying the relation defined by Eq. E.100, the constant-pressure solution is:

\[ q_{wD}(t_D) = \frac{1}{v_{ma} \ln[t_D] + \frac{v_{ma}}{2} \gamma - \frac{1}{2} \ln \left[ \frac{A_fD[\theta_{ma} + 2]^{1-2v_{ma}}}{h_{maD}n_{maD}^{v_{ma} - 1}} \frac{\Gamma[1-v_{ma}]}{\Gamma[v_{ma}]} \left[ \frac{2}{h_{maD}} \right]^{[\beta_{ma} + 1]/2} \right] \} \] .................................. (E.112)

E.5. Asymptotic Solutions of the Double Fractal Model

Pressure –Transient Solutions

Consider the constant-rate solution of the diffusivity equation to model the flow within the infinite fractal fracture network of a double porosity reservoir:

\[ \bar{p}_{wD,cr}(u) = \frac{K_v}{u \sqrt[\theta + 2]{\frac{2\sqrt{u}(u)}}} \] .................................. (E.113)

Using the approximation given in Eq. E.92, at early-intermediate times, Eq. E.113 reduces to:

\[ \bar{p}_{wD,cr}(u) = \frac{1}{u \sqrt{u}(u)} \] .................................. (E.114)

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The asymptotic behavior of the function \( F(\eta_{maD},h_{maD},t_D) \) at early-intermediate times is given by Eq. E. 93. Substituting such an expression in the interporosity transfer function (Eq. E.55) and neglecting the storativity ratio \( (\omega = 0) \) derives the following expression:

\[
f(u) = \alpha_0 u^{1/2}, \quad \text{..................................................} \quad (E.115)
\]

where:

\[
\alpha_0 = \frac{A_{fD}}{2} \left[ \frac{\eta_{maD}}{2} \right]^{1/2 - 1} \quad \text{..................................................} \quad (E.116)
\]

Substituting Eq. E.115 in Eq. E.114:

\[
\tilde{p}_{wD,cr}(u) = \frac{u^{-5/4}}{\sqrt{\alpha_0}} \quad \text{..................................................} \quad (E.117)
\]

Applying the inverse Laplace transform to Eq. E.116 the asymptotic solution in the real domain at early-intermediate times is:

\[
p_{wD,cr}(t_{Df}) = \frac{1/4}{\Gamma[5/4]}[\alpha_0] \quad \text{..................................................} \quad (E.118)
\]

At late-intermediate and late times, the Modified Bessel function of second kind can be approximated using Eq. E. 95. Applying such an approximation in Eq. E. 113 results in:

\[
\tilde{p}_{wD,cr}(u) = \frac{\Gamma[v]}{\Gamma[\theta + 2 - 2v]} \left[ \frac{u^{f}(u)}{u} \right]^{-\theta - 2v} \quad \text{..................................................} \quad (E.119)
\]

The following expression results from substituting Eq. E.115 in Eq. E.119:
Applying the inverse Laplace transform to Eq. E.120 the asymptotic solution in the real domain at late-intermediate times is:

$$\bar{p}_{wD,cr}(u) = \frac{[\theta + 2]^{\nu_0 - 1} \Gamma[\nu]_u^{-[(\nu + 1)/2]}\alpha_0^{\nu_0(\nu - 1)/2}}{\Gamma[\nu_0(\nu - 1)/2]}$$  \hfill (E.120)$$

The asymptotic behavior of the function $F(\eta_{maD}, h_{maD}, t_D)$ at late times is given by Eq. E.97. Substituting such an expression in the interporosity transfer function (Eq. E.55) and neglecting the storativity ratio ($\omega = 0$) derives in the following expression:

$$f(u) = \alpha_1 u^{\nu_{ma}/2}, \hfill \text{..................................................................................................................... (E.122)}$$

where:

$$\alpha_1 = \frac{A_f D \eta_{maD}^{1-\nu_{ma}}}{2} \frac{[\theta_{ma} + 2]^{1 - 2\nu_{ma}} \Gamma[\nu_{ma}]^{1 - \nu_{ma}}}{\Gamma[\nu_{ma}]} \left[ \frac{h_{maD}}{2} \right]^{\beta_{ma} - 1}$$  \hfill (E.123)$$

Substituting Eq. E.122 in Eq. E.119:

$$\bar{p}_{wD,cr}(u) = \frac{[\theta + 2]^{\nu_0 - 1} \Gamma[\nu]_u^{-[(\nu + 1)/2]}\alpha_0^{\nu_0(\nu - 1)/2}}{\Gamma[\nu_0(\nu - 1)/2]} \hfill (E.124)$$

Applying the inverse Laplace transform to Eq. E.124 the asymptotic solution in the real domain at late times is:

$$\bar{p}_{wD,cr}(u) = \frac{[\theta + 2]^{\nu_0 - 1} \Gamma[\nu]_u^{-[(\nu + 1)/2]}\alpha_0^{\nu_0(\nu - 1)/2}}{\Gamma[\nu_0(\nu - 1)/2]} \hfill (E.124)$$
\[ p_{\text{WD,cr}}(t_{\text{Df}}) = \frac{[\theta + 2]^{\gamma - 1} \Gamma[\gamma]}{\alpha_1^\gamma \Gamma[1 - \gamma] \Gamma[\gamma \nu_{\text{ma}} + 1]} t_{\text{Df}}^{\nu_{\text{ma}}} \] (E.125)
APPENDIX F

ANOMALOUS DIFFUSIVITY MODEL CONSIDERING TIME-DEPENDENT DARCY'S LAW

In this Appendix, we present the derivation of the model presented by Raghavan (2012a) for a cylindrical reservoir. Consider the conservation equation in cartesian coordinates:

\[
\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = \phi c_t \frac{\partial p}{\partial t} \tag{F.1}
\]

This model considers a time-dependent version of Darcy's Law, which applies that the flux is not local in time and space for a Continuous Time Random Walk. For this case, Darcy's Law is defined as (Eq. 2.10 from Raghavan, 2012a):

\[
q = -\frac{k_l A}{\mu} \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}[\nabla p] = -\frac{k_l A}{\mu} \frac{1}{\Gamma[\alpha]} \int_0^t (t-\tau)^{\alpha-1} \frac{d\nabla p}{dt} d\tau. \tag{F.2}
\]

Applying the Laplace transform to Eq. F.2 results in:

\[
\tilde{q}_i = -\frac{k_l A}{\mu} u^{1-\alpha} \varphi p, \tag{F.3}
\]

where \(i\) defines the directions \(x, y\) or \(z\). Applying the Laplace transform to Eq. F.1 results in

\[
\frac{\partial \tilde{q}_x}{\partial x} + \frac{\partial \tilde{q}_y}{\partial y} + \frac{\partial \tilde{q}_z}{\partial z} = \phi c_t[u \tilde{p} - p(t = 0)]. \tag{F.4}
\]

Substituting and arraying Eq. F.3 in Eq. F.4 gives us:

\[
\frac{\partial}{\partial x} \left[ -\frac{k_l A}{\mu} \nabla \tilde{p} \right] + \frac{\partial}{\partial y} \left[ -\frac{k_l A}{\mu} \nabla \tilde{p} \right] + \frac{\partial}{\partial z} \left[ -\frac{k_l A}{\mu} \nabla \tilde{p} \right] = \phi c_t u^{1-\alpha}[u \tilde{p} - p(t = 0)] \tag{F.5}
\]
Applying the inverse Laplace transform to Eq. F.5 yields:

\[
\frac{\partial}{\partial x} \left[ -\frac{k_l A}{\mu} \nabla p \right] + \frac{\partial}{\partial y} \left[ -\frac{k_l A}{\mu} \nabla p \right] + \frac{\partial}{\partial z} \left[ -\frac{k_l A}{\mu} \nabla p \right] = \phi \frac{\partial}{\partial t} \nabla p \]

(F.6)

The expression inside the brackets is the classic version of Darcy's Law:

\[
q_l = -\frac{k_l A}{\mu} \nabla p \]

(F.7)

Transforming Eq. F.6 to cylindrical coordinates results in:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial p}{\partial r} \right] = \frac{\phi \mu}{k_r} \frac{\partial}{\partial t} \nabla p \]

(F.8)

For a cylindrical system (considering only the radial coordinate), Eq. F.2 is expressed as:

\[
q = -\frac{2\pi k_r h}{\mu} \frac{\partial}{\partial t} \nabla p \]

(F.9)

Applying the Laplace transform to Eq. F.9 and solving for the product of the radius and the gradient of the pressure results in:

\[
\left[ r \frac{\partial p}{\partial r} \right] = -\frac{\mu}{2\pi k_r h} u^{1-\alpha} \]

(F.10)

The initial condition to solve Eq. F.8 is:

\[
p(r, t = 0) = p_i \]

(F.11)

Assuming constant flow rate at the wellbore, \( q \), the inner boundary condition is (based on Eq. F.9):

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\[
\left[ r \frac{\hat{\Delta p}}{\partial r} \right]_{r=r_w} = -\frac{q\mu}{2\pi k_r h\Gamma[2-\alpha]}t^{1-\alpha}. \quad \text{................................................................. (F.12)}
\]

The outer boundary condition assumes an infinite cylindrical reservoir, given by:

\[ \lim_{r \to \infty} p(r,t) = p_i. \quad \text{................................................................. (F.13)} \]

Defining the drop of pressure as:

\[ \Delta p(r,t) = p_i - p(r,t), \quad \text{................................................................. (F.14)} \]

Eq. F.8 becomes:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\hat{\Delta p}}{\partial r} \right] = \frac{\phi \mu e_i}{k_r} \frac{\hat{\alpha} \Delta p}{\hat{\alpha} t^\alpha}. \quad \text{................................................................. (F.15)}
\]

Similarly, the initial and boundary conditions in terms of the drop of pressure are:

\[ \Delta p(r, t = 0) = 0 \quad \text{(initial condition), ................................................................. (F.16)} \]

\[
\left[ r \frac{\hat{\Delta p}}{\partial r} \right]_{r=r_w} = -\frac{q\mu}{2\pi k_r h\Gamma[2-\alpha]}t^{1-\alpha} \quad \text{(inner boundary condition), ................................................................. (F.17)}
\]

\[ \lim_{r \to \infty} \Delta p(r,t) = 0 \quad \text{(outer boundary condition). ................................................................. (F.18)} \]

Applying the Laplace transform to Eq. F.15 and substituting the initial condition gives us:

\[
\frac{d^2 \hat{\Delta p}}{dr^2} + \frac{1}{r} \frac{d\hat{\Delta p}}{dr} - \frac{\phi \mu e_i}{k_r} t^\alpha \Delta p = 0, \quad \text{................................................................. (F.19)}
\]

Multiplying Eq. F.19 by \( r^2 \):

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\[
\frac{r^2}{dr^2} \frac{d^2 \Delta \bar{p}}{dr^2} + r \frac{d \Delta \bar{p}}{dr} - r^2 u \frac{\alpha \mu u}{k_r} \Delta \bar{p} = 0, \quad \text{.................................................. (F.20)}
\]

and defining the transformation variable \(z\) as:

\[
z = ru^{\alpha/2} \sqrt{\frac{\phi \mu u}{k_r}} \quad \text{.................................................. (F.21)}
\]

Then, using the chain rule the first derivative of the drop of pressure with respect to \(r\) is:

\[
\frac{d \Delta \bar{p}}{dr} = \frac{d \Delta \bar{p}}{dz} \frac{dz}{dr} = u^{\alpha/2} \sqrt{\frac{\phi \mu u}{k_r}} \frac{d \Delta \bar{p}}{dz} \quad \text{.................................................. (F.22)}
\]

Similarly, the second derivative of Eq. F.22 is:

\[
\frac{d^2 \Delta \bar{p}}{dr^2} = u^{\alpha/2} \sqrt{\frac{\phi \mu u}{k_r}} \frac{d^2 \Delta \bar{p}}{dz^2} \quad \text{.................................................. (F.23)}
\]

Substituting Eq. F.22 and Eq. F.23 in Eq. F.20 results in:

\[
r^2 u \frac{\alpha \mu u}{k_r} \frac{d^2 \Delta \bar{p}}{dz^2} + ru^{\alpha/2} \sqrt{\frac{\phi \mu u}{k_r}} \frac{d \Delta \bar{p}}{dz} - r^2 u^{\alpha/2} \sqrt{\frac{\phi \mu u}{k_r}} \Delta \bar{p} = 0 \quad \text{.................................................. (F.24)}
\]

Then, using the definition given by Eq. F.21, Eq. F.24 is rewritten as:

\[
z^2 \frac{d^2 \Delta \bar{p}}{dz^2} + z \frac{d \Delta \bar{p}}{dz} - z^2 \Delta \bar{p} = 0 \quad \text{.................................................. (F.25)}
\]

By inspection, the general solution to Eq. F.25 is:

\[
\Delta \bar{p} = AK_0 \left[ ru^{\alpha/2} \sqrt{\frac{\phi \mu u}{k_r}} \right] + Bf_0 \left[ ru^{\alpha/2} \sqrt{\frac{\phi \mu u}{k_r}} \right] \quad \text{.................................................. (F.26)}
\]

Applying outer boundary condition (infinite reservoir) to Eq. F.26, the bounded solution is:
The derivative of the drop of pressure with respect to \( r \) is:

\[
\frac{d\Delta \bar{p}}{dr} = -Ap_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} K_1 \left[ ru_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} \right].
\] (F.28)

Then, applying the inner boundary condition to Eq. F.26 gives us:

\[
\left[ r \frac{d\Delta \bar{p}}{dr} \right]_{r=r_w} = -Ap_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} K_1 \left[ ru_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} \right] = -\frac{aq}{2\pi k_r h u^{2-\alpha}}.
\] (F.29)

Solving for \( A \):

\[
A = \frac{q\mu}{2\pi k_r h u^{2-\alpha} r_w u_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} K_1 \left[ ru_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} \right]}.
\] (F.30)

Hence, the particular solution for this problem is:

\[
\Delta \bar{p} = \frac{q\mu K_0 \left[ ru_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} \right]}{2\pi k_r h u^{2-\alpha} r_w u_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} K_1 \left[ ru_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} \right]}.
\] (F.31)

For small arguments, the Modified Bessel Function \( K_1(x) \) is approximated as:

\[
K_1 \left[ ru_{r/2} \sqrt{\frac{\phi \mu c}{k_r}} \right] \approx \frac{1}{r_w u_{r/2} \sqrt{\frac{\phi \mu c}{k_r}}}. \] (F.32)
Then, Eq. F.31 reduces to:

\[ \Delta \bar{p} = \frac{q \mu}{2\pi \mu_r h} \frac{K_0}{u^{2-\alpha}} \left[ \frac{\mu/2}{\sqrt{\mu/k_r}} \right] \] ................................. (F.33)

Eq. F.33 is equivalent to the line-source solution (Eq. 4.10 in Raghavan, 2012a).
APPENDIX G

SENSITIVITY ANALYSES OF PRESSURE AND RATE TRANSIENT BEHAVIORS OF HORIZONTAL WELLS INTERCEPTING MULTIPLE FRACTURES

G.1. Horizontal well intercepting $N_f$ fractures in a Fractal Reservoir with Infinite Thickness

*Circular Transverse Fractures — High Conductivity*

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**Figure G.1** — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.2 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=1.5) \) and conductivity index \( (\theta=0) \) (constant-pressure case).

Figure G.3 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2) \) and conductivity index \( (\theta=0) \) (constant-rate case).
Figure G.4 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.5 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-rate case).
**Figure G.6** — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

**Figure G.7** — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-rate case).
Figure G.8 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-pressure case).

Figure G.9 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1.3$) (constant-rate case).
Figure G.10 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1.3$) (constant-pressure case).

Circular Transverse Fractures — Intermediate Conductivity

Figure G.11 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.12 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \((D)=1.5\) and conductivity index \((\theta)=0\) (constant-pressure case).

Figure G.13 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \((D)=2\) and conductivity index \((\theta)=0\) (constant-rate case).
Figure G.14 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.15 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.16 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N \) circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \((D)=2.5\) and conductivity index \((\theta)=0\) (constant-pressure case).

Figure G.17 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N \) circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \((D)=2.5\) and conductivity index \((\theta)=0.5\) (constant-rate case).
Figure G.18 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-pressure case).

Figure G.19 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1.3$) (constant-rate case).
**Figure G.20** — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( D_f=2.5 \) and conductivity index \( (\theta=1.3) \) (constant-pressure case).

**Circular Transverse Fractures — Low Conductivity**

**Figure G.21** — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( D_f=1.5 \) and conductivity index \( (\theta=0) \) (constant-rate case).
Figure G.22 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=1.5) \) and conductivity index \( (\theta=0) \) (constant-pressure case).

Figure G.23 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2) \) and conductivity index \( (\theta=0) \) (constant-rate case).
Figure G.24 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.25 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.26 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.27 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-rate case).
Figure G.28 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \( (\theta=0.5) \) (constant-pressure case).

Figure G.29 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \( (\theta=1.3) \) (constant-rate case).
Figure G.30 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) circular transverse hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \( (\theta=1.3) \) (constant-pressure case).

Rectangular Transverse Fractures — High Conductivity

Figure G.31 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=1.5) \) and conductivity index \( (\theta=0) \) (constant-rate case).
Figure G.32 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($θ=0$) (constant-pressure case).

Figure G.33 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($θ=0$) (constant-rate case).
Figure G.34 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_r$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.35 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_r$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.36 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_r$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.37 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_r$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=0.4$) (constant-rate case).
Figure G.38 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=0.4$) (constant-pressure case).

Figure G.39 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=1$) (constant-rate case).
Figure G.40 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D=3) \) and conductivity index \( (\theta=1) \) (constant-pressure case).

Figure G.41 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D=3) \) and conductivity index \( (\theta=4) \) (constant-rate case).
Figure G.42 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=4$) (constant-pressure case).

**Rectangular Transverse Fractures — Intermediate Conductivity**

Figure G.43 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-rate case).

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Figure G.44  — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D=1.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.45  — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D=2$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.46 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.47 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.48 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.49 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.4$) (constant-rate case).
Figure G.50 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D=2.5$) and conductivity index ($\theta=0.4$) (constant-pressure case).

Figure G.51 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D=2.5$) and conductivity index ($\theta=1$) (constant-rate case).
Figure G.52 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \( (\theta=1) \) (constant-pressure case).

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Figure G.53 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=1.5) \) and conductivity index \( (\theta=0) \) (constant-rate case).
Figure G.54 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.55 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.56 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2) \) and conductivity index \( (\theta=0) \) (constant-pressure case).

Figure G.57 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \( (\theta=0) \) (constant-rate case).
Figure G.58 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \((\theta=0)\) (constant-pressure case).

Figure G.59 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \((\theta=0.4)\) (constant-rate case).
Figure G.60 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.4$) (constant-pressure case).

Figure G.61 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=4$) (constant-rate case).
Figure G.62 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of infinite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=4$) (constant-pressure case).

G.2. Horizontal well intercepting $N_f$ fractures in a Fractal Reservoir with Finite Thickness

*Circular Transverse Fractures — High Conductivity*

Figure G.63 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_c$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.64 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.65 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.66 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.67 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.68 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.69 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-rate case).
Figure G.70 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-pressure case).

Figure G.71 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1.3$) (constant-rate case).
Figure G.72  — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1.3$) (constant-pressure case).

Circular Transverse Fractures — Intermediate Conductivity

Figure G.73  — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-rate case).

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Figure G.74 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.75 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.76 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \(N_f\) circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension \((D_f=2)\) and conductivity index \((\theta=0)\) (constant-pressure case).

Figure G.77 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \(N_f\) circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension \((D_f=2.5)\) and conductivity index \((\theta=0)\) (constant-rate case).
Figure G.78 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.79 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-rate case).
Figure G.80 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-pressure case).

Figure G.81 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1.3$) (constant-rate case).
Figure G.82 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1.3$) (constant-pressure case).

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Figure G.83 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.84  — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f = 1.5$) and conductivity index ($\theta = 0$) (constant-pressure case).

Figure G.85  — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f = 2$) and conductivity index ($\theta = 0$) (constant-rate case).
Figure G.86 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.87 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.88 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.89 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-rate case).
Figure G.90 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.5$) (constant-pressure case).

Figure G.91 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1.3$) (constant-rate case).
Figure G.92 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ circular transverse hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\phi=1.3$) (constant-pressure case).

**Rectangular Longitudinal Fractures — High Conductivity**

Figure G.93 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\phi=0$) (constant-rate case).
Figure G.94 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.95 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.96  — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension \( (D_f=2) \) and conductivity index \( (\theta=0) \) (constant-rate case).

Figure G.97  — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \( (\theta=0) \) (constant-rate case).
Figure G.98 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.99 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=0.4$) (constant-rate case).
Figure G.100 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=0.4$) (constant-pressure case).

Figure G.101 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=1$) (constant-rate case).
Figure G.102 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=1$) (constant-pressure case).

Figure G.103 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=4$) (constant-rate case).
Figure G.104 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with high conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=3$) and conductivity index ($\theta=4$) (constant-pressure case).

Rectangular Longitudinal Fractures — Intermediate Conductivity

Figure G.105 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.106— Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f = 1.5$) and conductivity index ($\theta = 0$) (constant-pressure case).

Figure G.107— Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f = 2$) and conductivity index ($\theta = 0$) (constant-rate case).
Figure G.108 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\Theta=0$) (constant-pressure case).

Figure G.109 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\Theta=0$) (constant-rate case).
Figure G.110 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.111 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.4$) (constant-rate case).
Figure G.112— Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.4$) (constant-pressure case).

Figure G.113— Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1$) (constant-rate case).
Figure G.114—Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with intermediate conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=1$) (constant-pressure case).

Rectangular Longitudinal Fractures — Low Conductivity

Figure G.115—Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\theta=0$) (constant-rate case).
Figure G.116— Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=1.5$) and conductivity index ($\Theta=0$) (constant-pressure case).

Figure G.117— Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2$) and conductivity index ($\Theta=0$) (constant-rate case).
Figure G.118— Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension \( (D_f=2) \) and conductivity index \( (\theta=0) \) (constant-pressure case).

Figure G.119— Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension \( (D_f=2.5) \) and conductivity index \( (\theta=0) \) (constant-rate case).
Figure G.120— Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0$) (constant-pressure case).

Figure G.121— Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=0.4$) (constant-rate case).
Figure G.122 — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension (\( D_f = 2.5 \)) and conductivity index (\( \theta = 0.4 \)) (constant-pressure case).

Figure G.123 — Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a horizontal well intercepting \( N_f \) rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension (\( D_f = 2.5 \)) and conductivity index (\( \theta = 4 \)) (constant-rate case).
**Figure G.124** — Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a horizontal well intercepting $N_f$ rectangular longitudinal hydraulic fractures with low conductivity in a fractal reservoir of finite thickness with fixed fractal dimension ($D_f=2.5$) and conductivity index ($\theta=4$) (constant-pressure case).
APPENDIX H

THE USE OF FRACTIONAL INTEGRALS TO MODEL THE TRANSIENT PERFORMANCE BEHAVIOR OF WELLS INTERCEPTING HYDRAULIC FRACTURES

H.1. Wells Intercepting Hydraulic Fractures in Euclidean Reservoirs: Classic Models

The finite conductivity fracture models are based on the discretization of the hydraulic fracture, which defines a system of equations. The "coupling" between the hydraulic fracture and the reservoir is made by the use of the uniform flux solution of a hydraulic fracture, which implies the superposition of the reservoir (line/point source) solution. Gringarten et al. (1974) presented the uniform flux solution for a vertical well intercepting a rectangular fracture within a 2D (cylindrical) reservoir (schematics in Fig. H.1). The authors developed the solution by assuming that \( y_D = y_{wD} = 0 \) and superimposing the line source solution over the lateral extent of the hydraulic fracture (\(-x_{fD}, x_{fD}\)), \textit{i.e.}:

\[
p_{D,cr}(x_D, x_{wD}, y_D, y_{wD}, l_D) = \frac{1}{4} \int_{-1}^{1} E_1 \left[ \frac{(x_D - x_{wD})^2}{4l_D} \right] dx_{wD}, \text{.........................................................}(H.1)
\]

which results in the following expression:

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4 Parts of this Appendix are reprinted with permission from "Pressure and Rate Transient Behavior of a Horizontal Well Intercepting Multiple Hydraulic Fractures within a Fractal Reservoir" by Valdes-Perez, A. R., Larsen, L., and Blasingame, T.A., 2018. Unconventional Resources Technology Conference (URTeC) Proceedings, URTeC-2902854. Copyright [2018] by Society of Petroleum Engineers, Inc.
Evaluating Eq. H.2 at the wellbore (i.e., $x_D=0$) and taking the logarithmic derivative gives us:

$$\frac{dp_{wD,cr}(t_D)}{d\ln t_D} = \frac{\sqrt{\pi t_D}}{2} \text{erf} \left[ \frac{1}{2\sqrt{t_D}} \right] \left( \text{erf} \left[ \frac{1+x_D}{2\sqrt{t_D}} \right] + \frac{1+x_D}{4} E_1 \left[ \frac{(1+x_D)^2}{4t_D} \right] + \frac{1-x_D}{4} E_1 \left[ \frac{(1-x_D)^2}{4t_D} \right] \right). \quad (H.3)$$

**Fig. H.2** shows the pressure data for a uniform flux fracture published by Gringarten *et al.* (1974) and the pressure derivative function calculated with Eq. H.3. At early times, it exhibits the characteristic formation-linear flow (half-slope in the pressure derivative) followed by the pseudo-radial flow (flat slope in the pressure derivative).

![Schematic of the uniform flux fracture solution using the line source function.](image)

**Figure H.1** — Schematic of the uniform flux fracture solution using the line source function.
Figure H.2 — Log-log plot of the pressure and pressure derivative functions of the uniform flux fracture solution using the line source.

In 1991, Larsen et al. used the point source solution in the Laplace domain to develop the model of a horizontal well intercepting a finite conductivity rectangular fracture within a 3D (spherical) reservoir. Fig. H.3 shows a schematic of this system defined for a vertical (instead of a horizontal) well. The authors applied a double integral in polar coordinates to superimpose the point source solution in the Laplace domain over the surface of the segments of the rectangular fracture. In cartesian coordinates, this expression is given by:

\[
\bar{P}_{D,cr}(x_D, x_{wD}, y_D, y_{wD}, t) = \frac{1}{2u} \int_{-1}^{1} \int_{0}^{1} \frac{1}{\sqrt{(x_D-x_{wD})^2 + (y_D-y_{wD})^2}} \exp \left[ -\sqrt{u} \sqrt{(x_D-x_{wD})^2 + (y_D-y_{wD})^2} \right] d y_{wD} d x_{wD}
\]

\[..............(H.4)\]
In their Appendix B, Larsen et al. (1991) showed that the uniform flux solution of a hydraulic fracture within a $3D$ (spherical) reservoir (Eq. H.4) converges to the $2D$ (cylindrical) reservoir case if the longitudinal extension of the fracture ($y_{FD}$) goes to infinity. Utilizing MATLAB's integral2-function to numerically integrate Eq. H.4 and subsequently applying the Stehfest algorithm, we generated the pressure and pressure derivative functions of this model for increasing $y_{FD}$-values. Then, we compared these results to the data presented by Gringarten et al. (1974). We present such a comparison in **Fig. H.4**. We observed that the use of the double integral approach in the point source reproduces the formation linear flow at early times and the characteristic negative half slope for spherical systems at late times. The pressure transient behavior of this type of system is similar to the one observed in partially penetrated wells. This solution can also reproduce the pseudoradial flow if the $y_{FD}$-value is sufficiently high.
Based on the line source solution, Cinco-Ley et al. (1978) developed the finite conductivity fracture model considering a rectangular fracture. Depending on the characteristics (parameters) of the fracture, this model can exhibit the linear fracture flow at early times, the bilinear and formation linear flows at intermediate times, and the pseudoradial flow at late times. Fig. H.5 shows the pressure data published by the authors. We estimated the pressure derivative data using MATLAB's `spline`-function. Given that the point source function can reproduce the results of the uniform flux fracture using the line source function if one of the axes goes to infinity, the finite conductivity model developed by Larsen et al. (1991) should converge to the model presented by Cinco-Ley et al. (1978) at long longitudinal extensions of the hydraulic fracture ($y/f_D$).
To test our hypothesis, we have used the input data (dimensionless fracture conductivity and dimensionless hydraulic diffusivity) provided by Cinco-Ley et al. (1978) in the model developed by Larsen et al. (1991) to compare the results of both models. We noted that to compare the results of these models, the pressure \( p_{wD,L} \) and pressure derivative signatures generated with the Larsen et al. model should be rescaled as follows:

\[
p_{wD,cr} = \left[ \frac{2y_{fD}}{h_D} \right] p_{wD,L}, \quad ...............................................................(H.5)
\]

where \( h_D \) is the dimensionless formation thickness. We tested both scenarios for a short longitudinal fracture \( (y_{fD}=2) \) and for a long longitudinal fracture \( (y_{fD}=80) \). Subsequently, we compared these results to the data provided by Cinco-Ley et al. (1978). The results are shown in
Based on these results, we came to the following conclusions: (1) for high $y_{FD}$-values, the model proposed by Larsen et al. (1991) converges to the model presented by Cinco-Ley et al. (1978), and (2) the pressure and pressure derivative functions at early and intermediate times ($t_D<10^{-1}$) are sensitive only to the $F_{cD}$- and $\eta_{FD}$-parameters, regardless of the longitudinal length of the hydraulic fracture ($y_{FD}$) and the source solution related to the geometry of the reservoir (line source for a cylindrical reservoir or point source for a spherical reservoir).

**Fig. H.6** — Convergence of the finite conductivity fracture model using the point source solution (Larsen et al., 1991) to the finite conductivity fracture model using the line source solution (Cinco-Ley et al., 1978)

**H.2. Extension of the Line/Point Source approaches to Fractal Reservoirs**

We have observed that our solutions based on the "fractal point" source approach yield erratic behaviors in the pressure and rate transient performance behaviors at intermediate times when $\nu \geq 0$. Beier (1994) redefined the parameters of the fractal reservoir model presented by Chang et al.
(1990) and presented the infinite conductivity and uniform flux rectangular fracture solutions using a "fractal line" approach. We have observed that the use of this approach can produce unstable pressure and rate behaviors when \( r < 0 \). These erratic behaviors are related to the treatment of the fractal source function. Beier (1994) used a "line-source" approach (single integral), which involves the superposition of the fractal source along the extent of the rectangular fracture; whereas, we have applied a "point source" approximation (double integral), which implies the superposition of the fractal source over the entire surface of the fracture. We recognize that none of these approaches are appropriate for a fractal reservoir.

The main convergence requirement of the fractal reservoir model establishes that this model should converge to the Euclidean reservoir models (Chang et al., 1990) when (1) \( D_f = 3 \) and \( \theta = 0 \) (\( r = -0.5 \)) for the spherical reservoir model (Chatas, 1966), (2) \( D_f = 2 \) and \( \theta = 0 \) (\( r = 0 \)) for the cylindrical reservoir model (van Everdingen et al., 1949), and (3) \( D_f = 1 \) and \( \theta = 0 \) (\( r = 0.5 \)) for the linear reservoir model (Miller, 1962). Therefore, the models of wells intercepting hydraulic fractures in fractal reservoirs should be able to reproduce all the Euclidean scenarios. For a well intercepting a finite conductivity rectangular fracture within a fractal reservoir, the model should be able to reproduce the results presented by Cinco-Ley et al. (1978) when \( D_f = 2 \) and \( \theta = 0 \) and the ones presented by Larsen et al. (1991) when \( D_f = 3 \) and \( \theta = 0 \). Although the model for a rectangular fracture presented in Chapter III fulfills the requirement for the Larsen et al. model, it fails to reproduce the results presented by Cinco-Ley et al. (1978). Similarly, the infinite conductivity fracture model presented by Beier (1994) fails to converge to the model presented by Larsen et al. (1991) (considering a high conductivity fracture).
In Section H.1, we showed that the double integral used for the point source solution (spherical homogeneous reservoir, \( i.e., D_f=3 \) and \( \theta=0 \)) and the single integral used for the line source solution (cylindrical homogeneous reservoir, \( i.e., D_f=2 \) and \( \theta=0 \)) provide exactly the same pressure and pressure derivative performance behavior at early and intermediate times of a well intercepting a hydraulic fracture. In short, the transient behavior for a fractured well is unaffected by the geometry of the reservoir (fractal dimension, \( D_f \)) at early and intermediate times. These results are contradictory to the analyses presented in Chapters III and IV and the ones presented by Beier (1994). Hence, we have concluded that the use of the traditional line/point source approaches for fractal reservoirs is inappropriate. The use of the traditional schemes to model hydraulic fractures in fractal reservoirs can cause the "over-interpretation" of transient data by the introduction of phenomena unlikely to be distinguished in the macro-scale of a reservoir (e.g. anomalous diffusion parameters).

H.3. Fractional Integration to Model the Transient Performance Behavior of a Well Intercepting a Rectangular Hydraulic Fracture within a Fractal Reservoir

The traditional schemes to model the transient behavior of a well intercepting a hydraulic fracture in a homogeneous reservoir consist in the "coupling" of two Euclidean objects: the reservoir (cylinder or sphere) and the fracture (rectangular or circular). This process involves the integration of a function over a domain in a "linear" axis or axes. Because a fractal reservoir is an irregular object, it is inappropriate to use the same schemes because the axes are not necessarily linear for all cases. To overcome this problem, we propose the use of fractional integrals. Ortigueira et al. (2017) defined the superposition of a function over a rectangular region defined by \((a_1, b_1) \times (a_2, b_2)\), utilizing the fractional integrals, as follows:
The use of Eq. H.6 implies a rescaling of the axes of a given system using power-law functions.

For a fractal reservoir, we consider that this is the most appropriate procedure to couple the irregular (fractal) geometry of the reservoir to the Euclidean shape of the hydraulic fracture. For this work, we have considered the case where \(a_1=a_2\).

To show the application of the concept defined in Eq. H.6 in the transient performance behavior of a fractured well in a fractal reservoir, consider the fractal source solution in the Laplace domain developed in Appendix D:

\[
\bar{p}_{fD,cr}(R_D,u) = \frac{h_D^{\theta-2} u^{-1/2} R_D^{-1/2}}{[\theta+2][1-\nu]} K_\nu \left[ R_D^{\theta/2} 2\frac{\sqrt{u}}{\theta+2} \right].
\]  

(H.7)

Similar to Beier (1994), we defined the fractal radius as:

\[
R_D^{\theta+2} = [x_D' - x'_{wD}]^{\theta+2} + [y_D' - y'_{wD}]^{\theta+2}.
\]  

(H.8)

In this development, it is important to point out that the variables \(x_D', x'_{wD}, y_D', y'_{wD}\) do not correspond to the classic cartesian (linear) axes. This is the main difference to Beier's conceptualization. To provide an equivalent expression in cartesian (linear) axes \((x_{wD}, y_{wD})\), we considered the following transformations:

\[
[x_D' - x'_{wD}] = [x_D - x_{wD}]^{2/\beta},
\]  

(H.9)

and,
After combining Eqs. H.7 to H.10, the fractal source solution becomes:

\[
\begin{align*}
\bar{p}_{fD,cr}(x_D, x_{wD}, y_D, y_{wD}, u) & = \frac{h_D^{Df-2}}{[\theta + 2]^{v+2} \Gamma[1-v]} u^{v+2/2} \left[ (x_D - x_{wD})^{2\xi} + (y_D - y_{wD})^{2\xi} \right]^{v/2} \\
K_v & \left[ (x_D - x_{wD})^{2\xi} + (y_D - y_{wD})^{2\xi} \right]^{1/2} 2\sqrt{v} \theta + 2 \\
\end{align*}
\]  

(H.11)

where \( \xi = [\theta + 2]/\beta \). In this work, we use a "fractal point" approach (i.e., a double integral) to superimpose the solution of the fractal reservoir (Eq. H.12) over the surface of the hydraulic fracture. The fracture is defined in the domain (-1,1) in the \( x_{wD} \)-axis and (0,\( y_{fD} \)) in the \( y_{wD} \)-axis. Therefore, the uniform-flux fracture solution, is defined by:

\[
\begin{align*}
\bar{p}_{fD,cr}(x_D, x_{wD}, y_D, y_{wD}, u) & = \frac{h_D^{Df-2}}{[\theta + 2]^{v+2} \Gamma[1-v]} u^{v+2/2} \int_{-1}^{1} \int_{0}^{y_{fD}} K_v \left[ \psi 2\sqrt{v} \theta + 2 \right] dy_{wD} \ dy_{fD} \\
\end{align*}
\]  

(H.12)

where:

\[
\psi = \sqrt{[x_D - x_{wD}]^{2\xi} + [y_D - y_{wD}]^{2\xi}} \]  

(H.13)

We could not obtain an analytical solution to the general case of the integral defined in Eq. H.12. The results presented in this Appendix were developed using numerical integration (MATLAB’s integral2-function).

**Convergence to the Classic Uniform-Flux Fracture Models**

According to the convergence requirements defined in Section H.2 (taken from Chang et al., 1990), the solution given by Eq. H.12 must converge to the uniform-flux fracture solution proposed
by Gringarten et al. (1974) for $D_f=2$ and $\theta=0$, and to the point source solution used by Larsen et al. (1991) for $D_f=3$ and $\theta=0$. In Fig. H.7, we present the convergence of the model for the first case (Gringarten et al., 1974), and the convergence to the second case (Larsen et al., 1991) is presented in Fig. H.8. We considered a unit thickness ($h_D=1$) for both scenarios, whereas the longitudinal length of the fracture ($y_D$) is 1 for the first scenario and 2 for the second. We observed that the results generated with Eq. H.12 provide excellent matches to the pressure, pressure derivative, and $\beta$-pressure derivative data from Gringarten et al. (1974) and a perfect match to the results generated using the point source solution proposed by Larsen et al. (1991). The convergence to the Larsen et al. model can be analytically derived by substituting the fractal parameters $D_f=3$ and $\theta=0$ in Eq. H.12 and it will collapse to the point source approach (Eq. H.4).

**Figure H.7** — Convergence of the uniform flux fracture model within a fractal reservoir using the fractional integral solution to the uniform flux fracture model in a cylindrical reservoir (Gringarten et al., 1974)
Influence of the Fractal Parameters — Fractal Dimension ($D_f$)

As presented in Fig. H.9, the fractal dimension of the reservoir ($D_f$) does not have influence in the slope of the pressure and pressure derivative functions at early times ($t_D < 4 \times 10^{-2}$). All the cases presented in Fig. H.9 show the characteristic formation linear flow at early times, followed by the pseudo-fractal flow at late times (the slope of the pressure derivative is equal to $\nu$). This is confirmed by the $\beta$-pressure derivatives presented in Fig. H.10. We observed that our proposed solution (Eq. H.12) produces a slight offset towards the right of the plot as the fractal dimension ($D_f$) decreases. Such an offset is more evident for $D_f < 2$ ($\nu>0$). This offset is also presented in the rate and rate derivative functions (Fig. H.11) and causes slightly higher flowrates at early times, for lower $D_f$-values.
Figure H.9 — Influence of the fractal dimension \((D_f)\) in the pressure and pressure derivative functions of a uniform flux fracture within a fractal reservoir (fractional integral approach).

Figure H.10 — Influence of the fractal dimension \((D_f)\) in the \(\beta\)-pressure derivative functions of a uniform flux fracture within a fractal reservoir (fractional integral approach).
Figure H.11 — Influence of the fractal dimension ($D_f$) in the rate and rate derivative functions of a uniform flux fracture within a fractal reservoir (fractional integral approach).

*Influence of the Fractal Parameters — Conductivity Index ($\theta$)*

We present the impact of the conductivity index ($\theta$) in the pressure and pressure derivative signatures in Fig. H.12. This parameter describes the connectivity between the permeable sites. Low $\theta$-values represent better connected permeable sites within a fractal system (reservoir). We observed that this parameter does have an impact on the performance behavior of the pressure and pressure derivative (slope) at early and late times. Based on analytical approximations and confirmation with the $\beta$-pressure derivative (Fig. H.13), we concluded that at early times (early fractal formation flow), the slope of the pressure and pressure derivative functions is equal to $[\theta+2]^{-1}$, whereas at late times (pseudo-fractal flow) the slope of the pressure derivative is equal to $\nu$. The relevance of this model is its potential use to directly determine the fractal reservoir
parameters without the use of trial and error methods \( (e.g. \text{ Flamenco-Lopez et al., 2003}) \). The conductivity index \( \theta \) can be estimated from the slope at early times. Once the \( \theta \)-value is obtained, it would also allow the straight calculation of the fractal dimension \( (D_f) \) from the slope of the pressure derivative function \( (v) \) if the test is long enough to observe the pseudo-fractal flow.

**Figure H.12** — Influence of the conductivity index \( (\theta) \) in the pressure and pressure derivative functions of a uniform flux fracture within a fractal reservoir (fractional integral approach).
In Fig. H.14, we present the constant-pressure solution ("mirror" images) of the cases presented in Fig. H.12. This plot shows that the variation of the $\theta$-parameter is consistent to the physics, i.e., better connected permeable sites in the fractal reservoir (low $\theta$-values), yielding higher flowrates during the entire test. These results are contradictory to the ones generated by using the traditional point source/double integral approach and presented in Chapters III and IV, where the flowrate signatures showed that the better connected systems yielded higher flowrates only at late times.
Figure H.14 — Influence of the conductivity index ($\theta$) in the rate and rate derivative functions of a uniform flux fracture within a fractal reservoir (fractional integral approach).

Relation to the Anomalous Diffusion Model

We noted that the shape of the signatures presented in Fig. H.12 are very similar to ones generated using the anomalous diffusion model presented by Camacho-Velazquez et al. (2008) (Fig. 2.8). Although their model is for unfractured wells (the model of a uniform flux fracture within a fractal reservoir with anomalous diffusion was presented by Razminia et al. 2015b, who used a traditional single/integer integral), we consider that these models are comparable given the use of fractional calculus, i.e., the fractional derivative to model the anomalous diffusion in fractal reservoirs and the fractional integral used to model the fractured well in fractal reservoirs with typical diffusion. We present a comparison of these models in Fig. H.15 (the input parameters for the uniform-flux fracture model are the same from Fig. H.12). We confirmed that the shape of the signatures of
these models are very similar, particularly at early times where the signatures of the $\beta$-pressure derivative function from both models overlap (see Fig. H.16).

Given that the anomalous diffusion models are entirely based on mechanical statistics and lacks any geological basis, we consider that the fractional integral approach is a more appropriate treatment to reproduce signatures as the one shown in Fig. H.15.

![Comparison Between the Unfractured Well Model within a Fractal Reservoir with Anomalous Diffusion and the Uniform-Flux Fracture Model within a Fractal Reservoir with Typical Diffusion using the Fractional Integral approach](image)

**Figure H.15** — Comparison of the pressure and pressure derivative functions between the uniform-flux fracture model using the fractional integral and the anomalous diffusion model for an unfractured well within a fractal reservoir (Camacho-Velazquez et al., 2008).
Application of the Fractional Integral Uniform Flux Fractures in 2D (Cylindrical) Reservoirs

In Section H.1, we presented the influence of the longitudinal length of the fracture in the point source solution (Fig. H.4). We observed that long longitudinal fractures can create two subperiods of flow at late-times: (1) pseudoradial and (2) spherical flow. As an exercise, we performed the same analysis using the double integral approach in the line source solution (see Fig. H.17). Although the solution at late times shows the expected pseudoradial flow for all cases, the results show also that at high \( y_{fD} \)-values, the characteristic half-slopes at early times (formation linear flow) of the pressure and pressure derivative functions collapse in an unitary slope with no physical significance.
We investigated the possibility of applying the fractional integration in the line source solution \((D_f=2 \text{ and } \theta=0 \text{ in Eq. H.12})\) to perform a similar analysis to the ones presented in Figs. H.4 and H.17. We present such an analysis in Fig. H.18. For \(y_{Df}\)-values different than one, we observed that the fractional integration approach can reproduce the following sequence of three periods of flow: (1) linear, (2) bilinear, and (3) pseudoradial flows. This sequence is the same as the one observed in low conductivity fractures. Given that the fractional integral approach does not consider the flow inside the fracture, the linear and bilinear flows reproduced with the fractional integral approach correspond to the formation linear flow and a "formation bilinear" flow. In a sense, the "formation bilinear flow" can be conceptualized as the reservoir portion of the classic trilinear flow model proposed by Lee et al. (1986).
Figure H.18 — Transient performance behavior of the pressure and pressure derivative functions of the uniform flux fracture model using the line source solution and applying the fractional integral approach.

Convergence to the Classic Finite Conductivity Models for Rectangular Fractures

The finite conductivity model for a rectangular fracture in a fractal reservoir presented in Chapter III was developed using a "point source" approach (i.e., double integral). This model can reproduce the results presented by Larsen et al. (1991), when the fractal parameters converge to the Euclidean limit of a sphere (i.e., $D_f=3$ and $q=0$). Consequently, it can also reproduce the results presented by Cinco-Ley et al. (1978) for high $y/p$-values, similar to the results presented in Section H.1 (Fig. H.6). However, this convergence criterion is weak for a fractal model, which should achieve the convergence based on the fractal properties ($D_f=2$ and $q=0$, for the Cinco-Ley et al. model). Therefore, we implemented the fractional derivative approach in scheme of the finite conductivity model presented in Chapter III.
In Fig. H.19, we present the convergence of the finite conductivity fracture in a fractal model using the fractional integral approach to the model proposed by Larsen et al. (1991). We present three cases: (1) low \( (F_{cD} = 0.2 \pi \text{ and } \eta_{\text{fD}} = 10^3) \), (2) intermediate \( (F_{cD} = \pi \text{ and } \eta_{\text{fD}} = 10^4) \), and high \( (F_{cD} = 100 \pi \text{ and } \eta_{\text{fD}} = 10^6) \) conductivity fractures. We observed that the match of these models is perfect for all these cases.

Similarly in Fig. H.20, we show the convergence of the fractal model using the fractional integral approach to the Cinco-Ley et al. model, using the data provided in the original publication. We observed that the pressure \( (p_{wD,F}) \) and pressure derivative signatures of the generated by the fractal model using the fractional integral approach model are similar (in terms of the shapes of the curves) to the ones presented by Cinco-Ley et al. (1978). We found that these models match by rescaling the fractal model as follows:
\[ p_{WD,cr} = c_1 \left[ \frac{2y_f D}{h_D} \right] p_{WD,F}, \]  \hspace{1cm} (H.14)

and

\[ t_D = c_2 t_{D,F}, \]  \hspace{1cm} (H.15)

where \( t_{D,F} \) is the dimensionless time of the fractal model and \( c_1 \) and \( c_2 \) are adjusting factors. We determined these factors by trial and error. For the low conductivity fracture the factors are \( c_1 = 1.325 \) and \( c_2 = 0.08 \), for the intermediate conductivity fracture the factors are \( c_1 = 1.085 \) and \( c_2 = 0.16 \), and for the high conductivity fracture the factors are \( c_1 = 1 \) and \( c_2 = 5 \). We present the normalized models in Fig. H.21, where we observed excellent matches for the three cases at early and intermediate times. At late times (pseudofractal/pseudoradial flow), the proposed solution shows numerical instability for the high conductivity case.

![Diagram](image-url)

**Figure H.20** — Convergence of the finite conductivity fracture model fractal reservoir using the fractional integral solution to the finite conductivity fracture model in a cylindrical reservoir (Cinco-Ley et al., 1978)
The discussion should be focused now on the development of analytical functions to determine the adjusting parameters $c_1$ and $c_2$. In their literature review, Ortigueira et al. (2017) pointed out that there is not a generalized definition for the concept of the fractional integral, e.g., some fractional integrals which involve a convolution scheme have the gamma function of the order of the integral as an integration coefficient. Although we obtained good results using the definition given by Eq. H.6, we believe that the adjusting parameters $c_1$ and $c_2$ could be related to an integration coefficient of the fractional integral.
H.4. Mathematical Development of the Fractional Integral Approach

Consider the solution for a fractal reservoir in terms of Green's functions (Chang et al., 1993):

\[
p_{D,cr}(R_D, R_wD, t_D) = \alpha \left[ \frac{D_f}{\tau} \right]^{\frac{\theta + 2}{2}} \exp \left[ -\frac{R_D^{\theta + 2}}{[\theta + 2]^2 \tau} \right] d\tau, \quad \text{(H.16)}
\]

where \( \alpha \) is a constant. The solution given by Eq. H.16 is general and applicable for \( D_f \leq 3 \), which includes the three Euclidean dimensions. Intuitively, Eq. H.16 can be restricted to consider only one direction as follows:

\[
p_{D,cr}(x'_{D}, x'_wD, t_D) = \alpha \left[ \frac{D_{fx}}{\tau} \right]^{\frac{\theta\_x + 2}{2}} \exp \left[ -\frac{(x'_D - x'_{wD})^{\theta\_x + 2}}{[\theta\_x + 2]^2 \tau} \right] d\tau, \quad \text{(H.17)}
\]

where \( 0 < D_{fx} \leq 1 \) and \( \theta\_x \geq 0 \). It is important to point out that the \( x'_{D} \)-axis corresponds to the traditional (1D-linear) axis only if \( D_{fx} = 1 \) and \( \theta\_x = 0 \). Similarly for the \( y'_{D} \), and the \( z'_{D} \) axes, the solutions are:

\[
p_{D,cr}(y'_{D}, y'_{wD}, t_D) = \alpha \left[ \frac{D_{fy}}{\tau} \right]^{\frac{\theta\_y + 2}{2}} \exp \left[ -\frac{(y'_D - y'_{wD})^{\theta\_y + 2}}{[\theta\_y + 2]^2 \tau} \right] d\tau, \quad \text{(H.18)}
\]

and:

\[
p_{D,cr}(z'_{D}, z'_{wD}, t_D) = \alpha \left[ \frac{D_{fz}}{\tau} \right]^{\frac{\theta\_z + 2}{2}} \exp \left[ -\frac{(z'_D - z'_{wD})^{\theta\_z + 2}}{[\theta\_z + 2]^2 \tau} \right] d\tau, \quad \text{(H.19)}
\]

Analogous to Gringarten et al. (1973), we used the Newman's method to combine Eqs. H.17-H.19 to provide a "3D" solution:
Assuming that the connectivity between permeable sites is the same in all directions (i.e., \( q = q_x = q_y = q_z \)), Eq. H.20 reduces to

\[
p_{D,cr}(x_D', x_w', y_D', y_w', z_D', z_w') = -\alpha \left[ \frac{D_{fx'}}{\theta_x + 2} + \frac{D_{fy'}}{\theta_y + 2} + \frac{D_{fz'}}{\theta_z + 2} \right] \exp \left[ -\frac{[x_D' - x_w']^2}{(\theta_x + 2) \tau} \right] \exp \left[ -\frac{[y_D' - y_w']^2}{(\theta_y + 2) \tau} \right] \exp \left[ -\frac{[z_D' - z_w']^2}{(\theta_z + 2) \tau} \right] d\tau
\]

where the fractal dimension \( D_f \) is given by:

\[
D_f = D_{fx} + D_{fy} + D_{fz} \quad \text{.................................(H.22)}
\]

Comparing Eq. H.16 and H.21, we can concluded that:

\[
R_D^{\theta+2} = [x_D' - x_w']^{\theta+2} + [y_D' - y_w']^{\theta+2} + [z_D' - z_w']^{\theta+2} \quad \text{.................................(H.23)}
\]

Given that the modeling of a hydraulic fracture is a planar (2D) structure defined in cartesian coordinates, it is fair to neglect the last term in Eq. H.23 (i.e., \( z'_{D} = z'_{wD} \)). Additionally, the axes \( x_D' \) and \( y_D' \) can be rescaled using a power law function. Assuming that the superposition of the fractal reservoir on the 2D hydraulic fracture occurs equally in both directions, we related the 2D cartesian coordinates to the coordinates of the fractal reservoir as (here, only for the \( x \)-axis):

\[
[x_D - x_w] = [x_D' - x_w']^{[D_f - 1]/2} \quad \text{.................................(H.24)}
\]
The exponent \([D_f-1]/2\) depicts the exposure of the fractal reservoir to the 2D hydraulic fracture. The expression given by Eq. H.24 is applicable for perfectly connected systems (\(\theta=0\)). We found that the general case of Eq. H.24 is given by:

\[
[ x_D - x_{WD} ] = [ x'_D - x'_{WD} ]^{\beta/2} 
\]

(Eq. H.24)

For \(\beta>0\). Eq. H.25 introduces the use of the spatial dimension \(\beta\), which (in a sense) provides an "effective geometry" of the reservoir.