ITERATED MONODROMY GROUPS OF RATIONAL MAPPINGS

A Thesis

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

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August 2018

Major Subject: Mathematics

ABSTRACT

We give the wreath recursion presentations of iterated monodromy groups of post-critically finite quadratic rational mappings f_c whose ramification portrait are of the form

$$0 \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto \infty \rightleftarrows 1$$

To find a pattern of these wreath recursions, we compute the wreath recursions of iterated monodromy groups of capture maps composed with the Basilica polynomial. This computation gives rise to the notion of addresses, which is used to represent wreath recursions. Then we conjecture that each capture map composed with the Basilica polynomial is topologically equivalent to a post-critically finite quadratic rational mapping, and thus we conclude that the iterated monodromy groups of f_c can be represented by addresses.

ACKNOWLEDGMENTS

I would like to express my very great appreciation to Prof. Nekrashevych, who comes up with this project and gives crucial ideas during the process of research. I would also like to thank my committee members for reviewing this thesis.

A special thanks to one of my fellow students, Qinfeng Zhu, for offering technical support on coding.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a thesis committee consisting of Professors Volodymyr Nekrashevych and Zhizhang Xie of the Department of Mathematics and Professor Michael Longnecker of the Department of Statistics.

The ideas of Captures, Choosing Generating Sets and Addresses was provided by Professor Volodymyr Nekrashevych. The Matlab code in the appendices was provided by Qinfeng Zhu. All other work conducted for the thesis was completed by the student independently.

Funding Sources

There is no funding source.

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1. INTRODUCTION

1.1 Background and sketch of the method

Iterated monodromy groups of post-critically finite branched coverings can be presented by wreath recursions. Let $\widehat{\mathbb{C}} \cup \{\infty\}$ be the Riemann sphere. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a degree d ($1 < d < \infty$) rational mapping of $\widehat{\mathbb{C}}$ to itself. Denote by C_f the set of all critical points of f and by $P_f := \{f^{\circ n}(c_0) : c_0 \in C_f, n \geq 1\}$ the set of all post-critical points of f. The mapping f is called post-critically finite if P_f is finite. The iterated monodromy group of f (denoted IMG(f)) is defined to be the quotient of the fundamental group $\pi_1(\widehat{\mathbb{C}} \backslash P_f, t)$ by the kernel of its monodromy action of the tree of preimages of the base point t (see [1] for a detailed discussion of the definition of iterated monodromy groups). The wreath recursion of IMG(f) can be computed using Proposition 2.1 in [2]. The computation requires a choice of generating sets for $\pi_1(\widehat{\mathbb{C}} \backslash P_f, t)$.

There is a rich amount of examples of iterated monodromy groups presented by wreath recursions. One of the most important classes is the iterated monodromy groups of quadratic polynomials (degree 2 branched coverings). These groups are studied in the paper [2]. The wreath recursions for these groups are associated with the *kneading sequences* of the corresponding polynomials, and thus the pattern of the wreath recursions is completely determined by kneading sequences. However, in other degree 2 cases, we do not know a pattern for the wreath recursions, which makes it hard to study the corresponding iterated monodromy groups.

The goal of this paper is to give a method to compute the wreath recursions of iterated monodromy groups of all post-critically finite quadratic rational mappings that are of the form $f_c = \dfrac{z^2 + c}{z^2 - 1}$, $c \in \mathbb{C}$, with ramification portrait

$$0 \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto \infty \rightleftarrows 1$$

Note that f_c's are also degree 2 mappings. Especially, we define a new type of "sequences" called *addresses* to determine the wreath recursions. To do so, we pick a point a_2 in a Fatou component on the Basilica Julia set (which is the dynamics of the Basilica polynomial $p_{-1} = z^2 - 1$) and iterate p_{-1} on this point. The orbit will be quite similar:

$$a_2 \mapsto \cdots \mapsto a_m \longmapsto 0 \rightleftarrows -1$$

This motivates the usage of *captures* (see [3] for a discussion of captures). The critical points of p_{-1} are 0 and ∞. Let σ_β be a capture map (define in section 2.5 of [3]) such that $\sigma_\beta(\infty) = a_2$. Then the ramification portrait of $\sigma_\beta \circ p_{-1}$ is

$$\infty \mapsto a_2 \mapsto \cdots \mapsto a_m \longmapsto 0 \rightleftarrows -1$$

Then we can make the following conjecture:

Conjecture. The function $\sigma_\beta \circ p_{-1}$ is topologically equivalent to some post-critically finite rational functions f_c.

By assuming the conjecture, we can conclude that the wreath recursion of IMG(f_c) is the same as that of IMG($\sigma_\beta \circ p_{-1}$) (Theorem 4.3). Hence IMG(f_c) will be completely determined by addresses. It remains compute IMG($\sigma_\beta \circ p_{-1}$) and give a complete description of addresses.

1.2 Outline of this thesis

Chapter 2 introduces a method of choosing a generating set for $\pi_1(\widehat{\mathbb{C}} \backslash P_{\sigma_\beta \circ p_{-1}}, t)$, gives the first observation of wreath recursions, and defines the notion of addresses. Chapter 3 describes the properties of addresses and use them to represent wreath recursion. Chapter 4 is the conclusion.

1.3 Notice about the conjecture

It is reasonable to make the above conjecture since $\sigma_\beta \circ p_{-1}$ is also a topological branched covering [3], and thus it makes sense to compute its iterated monodromy group; also, if we know the above ramification portrait is the post-critical orbit of a degree 2 rational mapping, then the only form of this mapping is $\dfrac{z^2 + c}{z^2 - 1}$ (Lemma 4.1).

The proof of this conjecture is beyond the scope of this thesis and will be given in later research. Another similar problem was studied in the paper [4].

1.4 Notations

(1) We use letters without brackets to denote homotopy classses of curves. For example, we use g_∞ instead of $[g_\infty]$ to denote a homotopy class of loops going around the point ∞. The homotopy product is denoted $\cdot$.

(2) Denote by $\mathcal{B}$ the Basilica Julia set, and $\mathcal{RB}$ the reversed Basilica Julia set. Let F be a Fatou component on $\mathcal{B}$ or $\mathcal{RB}$; the boundary of F is denoted ∂F.

(3) For a Fatou component F on the $\mathcal{B}$ or $\mathcal{RB}$, those points on ∂F parameterized by the internal angles $j/2^i$, $i, j \in \mathbb{N}$ are called *joint points*. The point parameterized by 0 is call the *root point* of this Fatou component. A joint point on ∂F is the root point for the next Fatou component attached to F.

2. COMPUTING WREATH RECURSIONS

2.1 The problem of choosing generating sets

Choose a point c on the parameter plane such that $f_c : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a post-critically finite branched covering with ramification portrait of the form

$$0 \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto \infty \rightleftarrows 1$$

Critical points of f_c are $0, \infty$. Let $C_{f_c} := \{0, \infty\}$; let P_{f_c} be the set of post-critical points of f_c. Note, in particular, that $f_c(\infty) = 1$ and $f_c(1) = \infty$. Hence $1, \infty \in P_{f_c}$, and ∞ is in the cycle of period 2, and thus the family of **all** quadratic rational mappings of the form $f_c = \dfrac{z^2 + c}{z^2 - 1}, c \in \mathbb{C}$ is denoted V_2 [5]. Let $\mathcal{C}_c = \widehat{\mathbb{C}} \backslash P_{f_c}$. Fixing a base point $t \in \mathcal{C}_c$, the fundamental group $\pi_1(\mathcal{C}_c, t)$ is a free group generated by $|P_{f_c}| - 1$ elements. The generators of $\pi_1(\mathcal{C}_c, t)$ can be taken to be loops going around each point in P_{f_c} in the positive direction. The choice of generating loops of $\pi_1(\mathcal{C}_c, t)$, however, depends not only on the coordinates of each post-critical point, but also on the relative positions of loops and the coordinate of the base point t. For instance, let $a, b \in P_{f_c}$ such that $b = f_c(a)$ and suppose a is to the left of b on $\widehat{\mathbb{C}}$, and let a_1, a_2 be the first and second inverses of a under f_c, respectively. Let γ_a and γ_b be the loops going around a and b, respectively. Here γ_a might be chosen to the left of γ_b or the other way around (See Figure 2.1 below), yet they might result in different wreath recursions: for example, in the former case, the first inverse image of γ_a might be a loop going around a_1, while in the latter case, the first inverse image of γ_a might be a loop going around a_2.

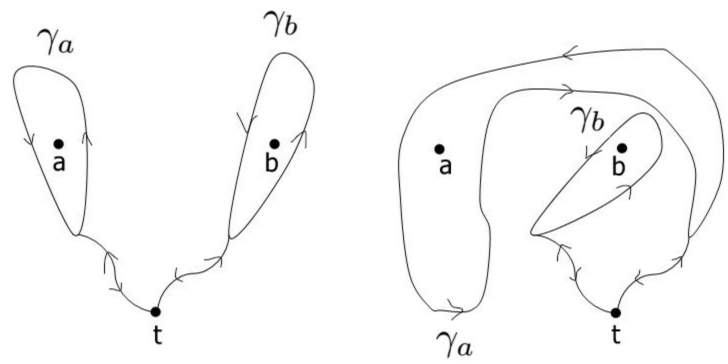

Figure 2.1: Different ways to choose a generating set.

The issue is that we do not know which of these choices is the best, *a priori*, in the sense of dynamical consistency of all IMG(f_c). Also, if the two inverse images of γ_a are simple paths (i.e. not loops), then there is an issue of choosing the connecting paths of the inverse image of t with t to form a new loop. Hence we need a consistent way of choosing generating sets. Ideally, we would like the inverse images of loops to be consistent with the inverse images of points, i.e. the first inverse image of γ_a being a loop going around a_1 and the second inverse image of γ_a being a loop going around a_2.

2.2 First glance at IMG(f_c)

Let f_c be a post-critically finite quadratic rational mapping, and consider the forward orbit of 0 under f_c. It eventually goes to ∞ which forms a cycle with 1. Hence $\{f^{\circ n}(0)|n \in \mathbb{N}\} = P_{f_c}$. Let m be the smallest integer such that $f^{\circ m}(0) = \infty$. Fix a base point t. It follows that the size of the generating set of $\pi_1(\mathcal{C}_c, t)$ is m. Let the generators g_2, g_3,...,g_m, g_1 be loops going around $f(0)$, $f^{\circ 2}(0)$,...,$f^{\circ(m-1)}(0)$, 1, respectively, in counterclockwise (positive) orientation. Also denote by g_∞ the loop going around ∞ in positive orientation, which is clockwise orientation around all finite post-critical points. Note that g_∞ is a product of all generators in a certain order and thus it is not in the generating set we choose. Nevertheless, we will view g_∞ as a generator to maintain consistency with the extended wreath recursions on Basilica groups to be described below. The (first observation of) wreath recursion of IMG(f_c) is as follows (using notations from [2]):

$$\begin{aligned} g_2 &= \langle\langle *, * \rangle\rangle \sigma \\ g_{k+1} &= \langle\langle g_k, 1 \rangle\rangle \text{ or } \langle\langle 1, g_k \rangle\rangle, k = 2, ..., m-1 \\ g_1 &= \langle\langle *, * \rangle\rangle \sigma \\ g_\infty &= \langle\langle g_1, g_m \rangle\rangle \end{aligned} \tag{2.1}$$

Where $*$ stands for undetermined loops, and the presentation of g_{k+1} is undetermined.

Proposition 2.1. The iterated monodromy group of each post-critically finite f_c has wreath recursion (2.1).

Proof. Let P be a point in $\widehat{\mathbb{C}}$. By a standard result of complex analysis, we have that there exist open neighborhoods U and U' of P and $f_c(P)$ respectively, and open neighborhoods V and V' of 0 in $\mathbb{C}$, and biholomorphisms $g : U \to V$ and $g' : U' \to V'$ sending 0 and $f_c(0)$ respectively to 0, such that the map $g' \circ f_c \circ g^{-1} : V \to V'$ is equal to $z \mapsto z^i$, $i = 1$ if P is not a critical point, and $i = 2$ if P is a critical point. Since 0 and ∞ are critical points of f_c, and thus $f_c(0)$ and -1 are critical values of f_c, it follows that the lift of g_2 by f_c is mapped biholomorphically to the lift of a loop γ going around $0 \in \mathbb{C}$ by $z \mapsto z^2$. Since each point other that 0 in $\mathbb{C}$ has two preimages under $z \mapsto z^2$, it follows that the lift of γ is a simple path. Hence the lift of g_2 is also a simple path. Same argument holds for g_1. Moreover, all other post-critical points are not critical values, and thus the lifts of each g_k, $k = 3, ..., m-1, \infty$ are all loops. By Proposition 2.1 in [2], the wreath recursion can be written as (2.1). ☐

2.3 Extended wreath recursion on the Basilica group

The problem stated in section 2.1 can be solved by computing the extended wreath recursion on the Basilica group and then applying captures on the Basilica Julia set $\mathcal{B}$, which is the Julia set of the quadratic polynomial $p_{-1} = z^2 - 1$. Later we will use the Basilica Julia set and the reversed

Basilica Julia set $\mathcal{RB}$ (which is the Julia set of $f_0 = \dfrac{z^2}{z^2-1}$) interchangeably, since they have the same dynamics.

We start by picking an arbitrary finite Fatou component, denoted F_2, on $\mathcal{B}$ and apply p_{-1} on F_2. Then the image will be another Fatou component. Iterating this process, we obtain an orbit of Fatou components and the image will eventually be F_0, which is the Fatou component containing 0, and form a cycle with F_{-1}, the Fatou component containing -1. In fact, p_{-1} is a homeomorphism between the closure of each Fatou component, except for F_0, on this orbit. In practice, we only need to pick a point a_2 inside F_2 and iterate p_{-1} on a_2. The orbit of this point is in one-to-one correspondence of that of F_2, i.e. $p_{-1}^{\circ k}(a_2) \in p_{-1}^{\circ k}(F_2)$. Also let m be the smallest integer such that $p_{-1}^{\circ m}(a_2) \in p_{-1}^{\circ m}(F_2) = F_0$. Note that $m < \infty$ since it takes only finitely many steps for F_2 to go to F_0 under iteration of p_{-1}.

The Basilica group is described in Section 3.3 of [6]. To compute the extended wreath recursion on the Basilica group at F_2, we first impose one puncture on each $p_{-1}^{\circ k}(F_2)$ by simply removing the point $p_{-1}^{\circ k}(a_2)$, for $k = 0, 1, ..., m+1$. Define $P_{F_2} := \{p_{-1}^{\circ k}(a_2) | k = 0, ..., m+1\}$. Then fix a base point $t = \frac{1-\sqrt{5}}{2}$. In this way we obtained a partial self-covering $p_{-1} : \widehat{\mathbb{C}} \setminus p_{-1}^{-1}(P_{F_2}) \to \widehat{\mathbb{C}} \setminus P_{F_2}$.

Definition 1. The *extended wreath recursion* on the Basilica group at F_2 is the wreath recursion of the partial self-covering $p_{-1} : \widehat{\mathbb{C}} \setminus p_{-1}^{-1}(P_{F_2}) \to \widehat{\mathbb{C}} \setminus P_{F_2}$.

Remark. The extended wreath recursions on the Basilica group do not define new iterated monodromy groups. Indeed, if we define IMG(F_2, t):$= \pi_1(\widehat{\mathbb{C}} \setminus P_{F_2}, t)/Ker(\Phi)$, where $Ker(\Phi) = \{\gamma \in \pi_1(\widehat{\mathbb{C}} \setminus P_{F_2}, t) | \forall x \in \bigsqcup_n p_{-1}^{\circ(-n)}(t), \gamma x = x\}$ [6], then all loops going around the imposed puncture points will be in $Ker(\Phi)$, and thus IMG(F_2, t) is isomorphic to the Basilica group. Nevertheless, for brevity, we still say that the extended wreath recursion on the Basilica group at F_2 is the wreath recursion for IMG(F_2, t), or IMG(F_2) when t is specified.

We can also use Proposition 2.1 in [2] to compute the extended wreath recursions. The fundamental group $\pi_1(\widehat{\mathbb{C}} \setminus P_{F_2}, t)$ is generated by $m+1$ elements. Let the generators g_2, g_3,...,g_0, g_{-1} be loops going around a_2, $a_3 = p_{-1}(a_2)$,...,$0 = p_{-1}^{\circ m}(a_2)$, $-1 = p_{-1}^{\circ(m+1)}(a_2)$, respectively. It is worth noting that technically we require a_2 to be a strictly preperiod point of p_{-1} that eventually gets mapped to 0, so that we are in fact extending the Basilica group by imposing more post-critical points. Yet, in practice, it does not matter which point is removed inside a Fatou component because of the one-one correspondence of $p_{-1}^{\circ k}(a_2)$ and $p_{-1}^{\circ k}(F_2)$. We only need to draw loops close to the boundary of each Fatou components to make sure those loops will go around the (imposed) post-critical points.

Remark. We need to show that there is a consistent way of choosing generating sets for all IMG(F_2). This requires the notion of *addresses*. Therefore, we will give the definition of addresses in this chapter and leave the detailed description to the next chapter.

It is known (see [7]) that the boundary of each Fatou component on $\mathcal{B}$ is homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$ and that two adjacent Fatou components share only one joint point. We now give a consistent parametrization of the boundary of each Fatou component. First, let the homeomorphism $\theta_0 : \mathbb{R}/\mathbb{Z} \to \partial F_0$ be the unique conjugacy with z^2 on the unit circle such that $\theta_0(0) = \theta_0(1) = t$. Especially, all joint points are parameterized by angles $j/2^i$, $i, j \in \mathbb{N}$. Then pick an arbitrary joint point b on ∂F_0 and consider the attached Fatou component F_b. We define another homeomorphism $\theta_b : \mathbb{R}/\mathbb{Z} \to \partial F_b$ such that $\theta_b(0) = \theta_b(1) = b$ and all other points are parameterized in the exact same way as that of F_0. Continuing this process, we get a consistent way of parameterizations, and we will identify the boundary of each Fatou component with $\mathbb{R}/\mathbb{Z}$ in this way. Hence every Fatou component can be located by a sequence of angles that is of the form $(\frac{j_1}{2^{i_1}}, ..., \frac{j_l}{2^{i_l}})$.

Definition 2. The sequence of angles $(\frac{j_1}{2^{i_1}}, ..., \frac{j_l}{2^{i_l}})$, in which each $j_k, k > 1$ is a positive odd integer less that 2^{i_k} while j_1 is either 0 or a positive odd integer less that 2^{i_1}, is called the *address* associated to a Fatou component.

Especially, the address of F_0 is denoted $(\varnothing)$, and the address of F_{-1} is (0). With the notion of addresses, we can easily locate a Fatou component and represent its orbit. We can also locate the two inverses images of a Fatou component by computing the two branches of addresses. Then the most natural way of choosing a generating loop is to draw a simple curve starting at t and go past the smallest number of Fatou components by crossing the joint points, and then go around a post critical point in counterclockwise orientation and follow the same path back to t. Hence each such loop is also associated with the same address as that of F_2. We will show in the next chapter that the generating sets obtained in this way is indeed consistent.

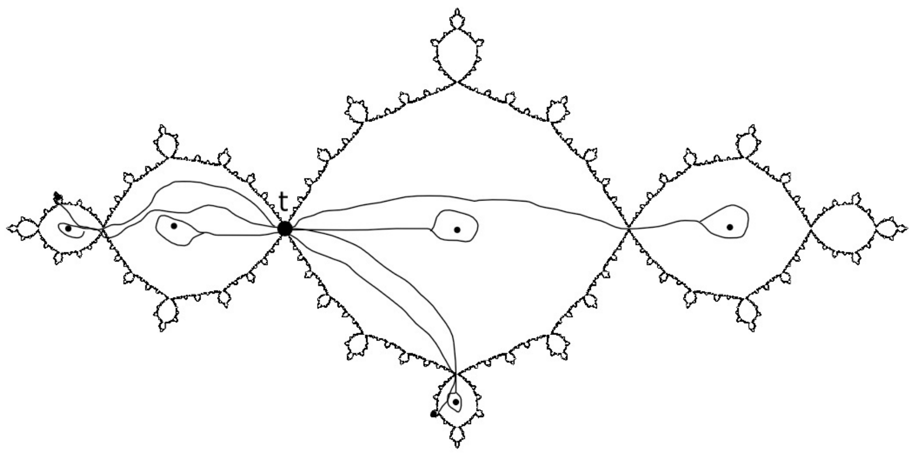

Figure 2.2: An example of generating set: each generating loop is a union of a simple path and a small loop around a puncture. The simple path crosses joint points.

Proposition 2.2. The wreath recursions of IMG(F_2) are of the following form:

$$\begin{aligned} g_2 &= \langle\langle 1, 1\rangle\rangle \\ g_{k+1} &= \langle\langle g_k, 1\rangle\rangle \text{ or } \langle\langle 1, g_k\rangle\rangle, k = 2, ..., m-1 \\ g_{-1} &= \langle\langle 1, g_0\rangle\rangle\sigma \\ g_0 &= \langle\langle g_{-1}, g_m\rangle\rangle \end{aligned} \tag{2.2}$$

Where the presentation of g_{k+1} depends on the address of F_2 (to describe in the next chapter).

Proof. The argument is exactly the same as that of Proposition 2.1. We merely need to note that the only finite critical value of p_{-1} is $-1 = p_{-1}(0)$. □

Comparing (2.1) and (2.2), the difference occurs at g_2: in the above wreath recursion, the lifts of g_2 are trivial loops, while in (2.1), the lifts of g_2 are simple paths. This difference leads to another tool called *captures*.

2.4 Captures on the Basilica Julia set

We have seen that it is possible to choose a generating set for IMG(f_c) such that its wreath recursion is the same, at all generators but g_2, as that of IMG(F_2) for a corresponding F_2. In order to make their wreath recursions look completely the same, we apply capture on F_2

We interpret the description of captures from Section 2.5 of [3] on the polynomial p_{-1}. Recall that an external ray parametrized by an angle $\beta \in \mathbb{R}/\mathbb{Z}$ lands on the root point b of a Fatou component of $\mathcal{B}$ whenever $\beta = \frac{j}{3 \cdot 2^i}$, where j is an integer less than $3 \cdot 2^i$ and does not divide $3 \cdot 2^i$. We also use β to denote this external ray, i.e. we view $\beta : [0,1] \to \widehat{\mathbb{C}}$ as a simple path such that $\beta(0) = \infty$ and $\beta(1) = b$.

Fixing F_2, there are two external rays landing on its root point b_2. These two rays are parametrized by $\frac{k}{3 \cdot 2^i}$ and $\frac{k+2}{3 \cdot 2^i}$, respectively. Let $\beta = \frac{k}{3 \cdot 2^i}$ and let $a_2 \in F_2$ be the pre-period point that is mapped to 0 by p_{-1} as before. Connect the points b_2 and a_2 by a line segment and denote this line segment L_β. The simple path $L_\beta \cup \beta$ is called the *capture path*. Then define a *path homeomorphism* $\sigma_\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\sigma_\beta(\infty) = a_2$ and $\sigma_\beta = Id$ outside a small neighborhood of $L_\beta \cup \beta$. Suppose the neighborhood of $L_\beta \cup \beta$ is small enough that it does not intersect other Fatou components on the orbit of F_2. Then the inverse images of all generators except g_2 under $\sigma_\beta \circ p_{-1}$ are the same as those under p_{-1}. For g_2 we have the following proposition:

Proposition 2.3. The lifts of g_2 under $\sigma_\beta \circ p_{-1}$ are simple paths.

Proof. We need to show that a_2 is a critical point of $\sigma_\beta \circ p_{-1}$. Since σ_β is a homeomorphism, it does have a critical point, and thus ∞ is a critical point of $\sigma_\beta \circ p_{-1}$. It follows that a_2 is a critical value by the definition of σ_β. □

Corollary 2.4. The wreath recursion of the iterated monodromy group of $\sigma_\beta \circ p_{-1}$ is of the form

$$\begin{aligned} g_2 &= \langle\langle h, h^{-1} \rangle\rangle \sigma \\ g_{k+1} &= \langle\langle g_k, 1 \rangle\rangle \text{ or } \langle\langle 1, g_k \rangle\rangle, k = 2, ..., m-1 \\ g_{-1} &= \langle\langle 1, g_0 \rangle\rangle \sigma \\ g_0 &= \langle\langle g_{-1}, g_m \rangle\rangle \end{aligned} \tag{2.3}$$

Where h is a word depending on the address of F_2, and g_{k+1} also depends on the address of F_2 (to describe in the next chapter).

Section 3.2 will serve as the proof of this corollary.

3. ADDRESSES

The remaining problem from the last chapter is to determine the presentations of g_k for $k \neq -1, 0$ (equivalently, $k \neq 1, \infty$ on the reversed Basilica Julia set $\mathcal{RB}$) in (3). This problem can be solved by computing forward orbits of addresses and the two inverses of each address. Since the generating loops g_k can also be represented by addresses, the wreath recursions will be uniquely determined by inverse addresses.

3.1 Forward orbit of a Fatou component represented by forward orbit of addresses

Denote by $D_{-1} := \widehat{\mathbb{C}}$ the dynamic sphere of p_{-1}. For an arbitrary Fatou component $F \neq F_0$, the map $p_{-1} : D_{-1} \to D_{-1}$ induces a homeomorphism of ∂F and $p_{-1}(\partial F)$, and thus it induces an angle map $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, since the boundary of each Fatou component can be parametrized by internal angles.

Definition 3. For an arbitrarily chosen Fatou component F on $\mathcal{B}$ (equivalently on $\mathcal{RB}$), the *forward orbit of* F is a directed graph (V, E) with vertex set $V = \{p_{-1}^{\circ m}(F) | m \in \mathbb{N}\}$ (equivalently $V = \{f_0^{\circ m}(F) | m \in \mathbb{N}\}$), and edge set E defined to be the set of arrows starting at $p_{-1}^{\circ k}(F)$ and ending at $p_{-1}^{\circ(k+1)}(F)$, $k \in \mathbb{N}$.

Proposition 3.1. The forward orbit of F can be represented by an orbit of addresses.

Proof. First, set $F = F_0$. Then the forward orbit of F is the cycle:

$$F_0 \rightleftarrows F_{-1}$$

Hence we always have the cycle

$$(\varnothing) \rightleftarrows (0)$$

Since 0 is a critical point of p_{-1}, the map $p_{-1} : F_0 \to F_{-1}$ induces an angle doubling map of their boundaries $\phi : \partial F_0 \to \partial F_{-1}$ $s.t.$ $\alpha \mapsto 2\alpha$. On the other hand, if $F \neq F_0$, then p_{-1} induces an angle preserving map $\phi : \partial F \to \partial p_{-1}(F)$ $s.t.$ $\alpha \mapsto \alpha$. It follows that all addresses that are of the form $(\frac{j}{2^i})$ for $i > 1$ are mapped to $(0, \frac{j'}{2^{i-1}})$, where $j' \equiv j(\text{mod } 2^{i-1})$; and $(\frac{1}{2})$ is mapped to $(\varnothing)$. By continuity of p_{-1}, all addresses that are of the form $(\frac{j_1}{2^{i_1}}, ..., \frac{j_l}{2^{i_l}})$ for $i_1 > 1$ and will be mapped to $(0, \frac{j_1'}{2^{i_1-1}}, ..., \frac{j_l}{2^{i_l}})$ $(j_1' \equiv j_1(\text{mod } 2^{i_1-1}))$, and then mapped back to $(\frac{j_1'}{2^{i_1-1}}, ..., \frac{j_l}{2^{i_l}})$; while all addresses that are of the form $(\frac{1}{2}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$ will be mapped to $(\frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$. In this way we get an orbit of addresses. □

The proof of the last proposition allows us to make the following definition:

Definition 4. Let $(\frac{j_1}{2^{i_2}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$ be an address. The *forward orbit of this address* is a directed graph satisfying the following rules:

(i) If $\frac{j_1}{2^{i_1}} = 0$, then $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}}) \to (\frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$

(ii) If $\frac{j_1}{2^{i_1}} = \frac{1}{2}$, then $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}}) \to (\frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$

(iii) If $\frac{j_1}{2^{i_1}}$ is such that $i_1 > 1$ and $j_1 \neq 0$, then $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}}) \to (0, \frac{j_1'}{2^{i_1-1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, where $(j_1' \equiv j_1(\text{mod } 2^{i_1-1}))$

(iv) $(\frac{1}{2}) \longrightarrow (\varnothing) \rightleftarrows (0)$

Remark. We use an address to represent a Fatou component F as well as each of the points in F, especially the root point. Also, by our way of choosing generating set, each generating loop can also be represented by the same address as that of a corresponding Fatou component. We also use a set of addresses to represent a simple curve that crosses only the root points. Hence a generating loop can also be represented by a set of addresses. For instance, if the address of a g_k is $(\frac{1}{2^2}, \frac{1}{2^2})$, then it is also represented by the set $\{(\frac{1}{2^2}), (\frac{1}{2^2}, \frac{1}{2^2})\}$.

Example 3.2. Pick the F_2 whose address is $(\frac{1}{2^2}, \frac{1}{2^2})$. Then the forward orbit of F_2 is represented by:

$$(\tfrac{1}{2^2}, \tfrac{1}{2^2}) \longrightarrow (0, \tfrac{1}{2}, \tfrac{1}{2^2}) \longrightarrow (\tfrac{1}{2}, \tfrac{1}{2^2}) \longrightarrow (\tfrac{1}{2^2}) \longrightarrow (0, \tfrac{1}{2}) \longrightarrow (\tfrac{1}{2}) \longrightarrow (\varnothing) \rightleftarrows (0)$$

3.2 Wreath recursions determined by addresses

First, we represent the lifts of each generating loop g_k, for $k \neq 2, 1, \infty$, by branches of the address of the corresponding Fatou component. Divide the dynamic plane D_{-1} into two parts by the union of external rays $\beta_1 = 1/3$ and $\beta_2 = 2/3$, which land on the base point $t = \frac{1-\sqrt{5}}{2}$. Denote respectively by L and R the left part and the right part of D_{-1} divided by $\beta_1 \bigcup \beta_2$ (see Figure 3.1). For an arbitrary Fatou component F, we distinguish between two cases: F being on R and F being on L.

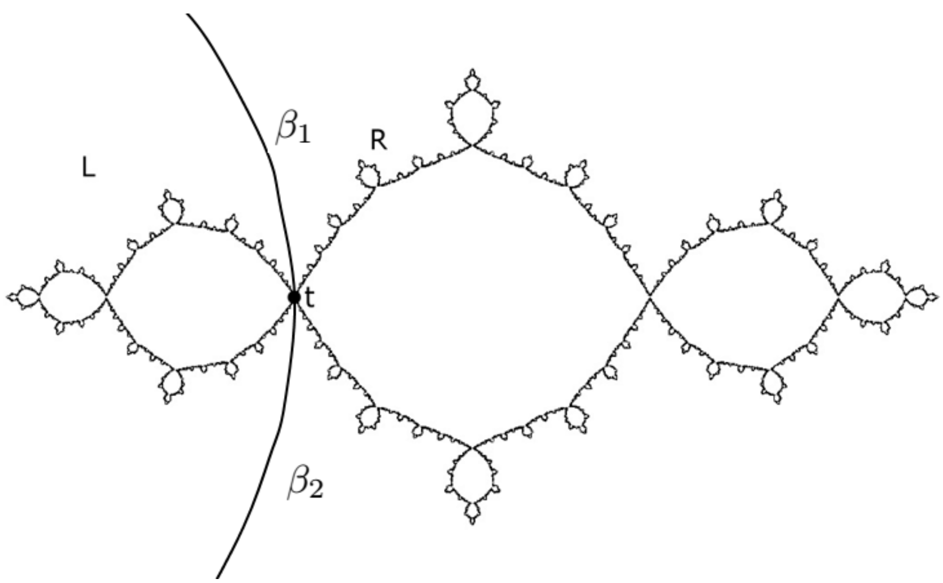

Figure 3.1: Dividing the dynamic plane.

Definition 5. (The first and second branches of an address on R) Let F be a Fatou component on R, then its address is either $(\varnothing)$ or of the form $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, $j_1 \neq 0$. For $(\varnothing)$, the *first branch* is defined to be (0), and the *second branch* is defined to be $(\frac{1}{2})$. For $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, the *first branch* is defined to be $(0, \frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, and the *second branch* is defined to be $(\frac{1}{2}, \frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$.

Remark. Intuitively, given a Fatou component on R, the first inverse of its address is on L and the second inverse is to the right.

It is a little bit tricky to define the two branches of an address on L. Let $\mathbb{H}^+$ denote the upper half plane and $\mathbb{H}^-$ denote the lower half plane. Pick a point $a \in L \bigcap \mathbb{H}^+$. Then a has two inverse images under p_{-1}: one is on $\mathbb{H}^+ \bigcap R$ and the other is on $\mathbb{H}^- \bigcap R$. If this a is a root point, then it is associated with an address that is of the form $(0, \frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, where $j_1 < 2^{i_1-1}$. We can set its first branch to be $(\frac{j_1}{2^{i_1+1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, and second branch to be $(\frac{j_1''}{2^{i_1+1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, where $2^{i_1} < j_1'' < 2^{i_1+1}$ and $j_1'' \equiv j_1 (\text{mod } 2^{i_1})$. However, when a is a root point that is on the x-axis, then its branches are ambiguous: the "first" branch could be on $\mathbb{H}^+$ or $\mathbb{H}^-$, and same applies to the "second" branch. In order to rule out this ambiguity, we "require" a to be on $\mathbb{H}^+$. In other words, if we have an imposed post-critical point $p_{-1}^{\circ k}(a_2)$ inside a Fatou component whose address is $(0, \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$, then we first draw a small loop around $p_{-1}^{\circ k}(a_2)$, and connect the base point t with this loop by a path crossing root points and avoiding $\mathbb{H}^-$. Denote this generating loop g_{k+1}, and its first inverse image will be inside a Fatou component on $\mathbb{H}^-$; its second inverse image will be on $\mathbb{H}^+$. Therefore, we can make the following definition:

Definition 6. (The first and second inverse of an address on L) Let F be a Fatou component on L, then its address is of the form $(0, \frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$. The *first inverse of this address* is defined to be $(\frac{j_1}{2^{i_1+1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$; the *second inverse of this address* is defined to be $(\frac{j_1''}{2^{i_1+1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, where $j_1'' \neq j_1$ and $j_1'' \equiv j_1 (\text{mod } 2^{i_1})$. The two inverses of (0) are equal: they are defined to be $(\varnothing)$

Now we can give a criterion for computing the presentation of g_k, for $k \neq 2, 1, 0, \infty$.

Proposition 3.3. Let F_2 be a Fatou component whose address is $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$. Let $\mathcal{O}$ be the forward orbit of this address. Then each address in $\mathcal{O}$ associated to g_k for $k \neq 2, 1, 0$ (equivalently $k \neq 2, 1, \infty$ on $\mathcal{RB}$) has exactly one branch in $\mathcal{O}$. If the first branch is in $\mathcal{O}$, then $g_k = \langle\langle g_{k-1}, 1 \rangle\rangle$; if the second branch is in $\mathcal{O}$,then $g_k = \langle\langle 1, g_{k-1} \rangle\rangle$. □

Example 3.4. Let F_2 be the same as that in Example 3.2. We exhibit the wreath recursion in the following table:

Address	1st branch	2nd branch	Presentation
$(\frac{1}{2^2}, \frac{1}{2^2})$	$(0, \frac{1}{2^2}, \frac{1}{2^2})$	$(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^2})$	To describe below
$(0, \frac{1}{2}, \frac{1}{2^2})$	$(\frac{1}{2^2}, \frac{1}{2^2})$	$(\frac{3}{2^2}, \frac{1}{2^2})$	$g_3 = \langle\langle g_2, 1\rangle\rangle$
$(\frac{1}{2}, \frac{1}{2^2})$	$(0, \frac{1}{2}, \frac{1}{2^2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2^2})$	$g_4 = \langle\langle g_3, 1\rangle\rangle$
$(\frac{1}{2^2})$	$(0, \frac{1}{2^2})$	$(\frac{1}{2}, \frac{1}{2^2})$	$g_5 = \langle\langle 1, g_4\rangle\rangle$
$(0, \frac{1}{2})$	$(\frac{1}{2^2})$	$(\frac{3}{2^2})$	$g_6 = \langle\langle g_5, 1\rangle\rangle$
$(\frac{1}{2})$	$(0, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$g_7 = \langle\langle g_6, 1\rangle\rangle$
$(\varnothing)$	(0)	$(\frac{1}{2})$	$g_0 = \langle\langle g_{-1}, g_7\rangle\rangle$
(0)	$(\varnothing)$	$(\varnothing)$	$g_{-1} = \langle\langle 1, g_0\rangle\rangle\sigma$

Table 3.1: An example of wreath recursion determined by addresses.

It remains to give the recursion for g_2. We use $\mathcal{RB}$ to exhibit the lifts of g_2 under $\sigma_\beta \circ p_{-1}$. It is worth noting that we are in fact using $\sigma_\beta \circ f_0$ to take the lifts, where σ_β is the capture mapping that maps 0 to a_2 and fixes all other points outside a small neighborhood of the capture path. Nevertheless, since $\mathcal{RB}$ and $\mathcal{B}$ are the same dynamics, the only difference is the coordinate of each point on the dynamic planes, while the addresses are exactly the same under a modified parametrization by $\mathbb{R}/\mathbb{Z}$. The base point on $\mathcal{RB}$ is chosen to be $t = \frac{1+\sqrt{5}}{2}$, which has itself to be the first inverse image and $-t = -\frac{1+\sqrt{5}}{2}$ to the second inverse image. Denote by F_∞ the Fatou component containing ∞, and parameterize its boundary by $\theta : \mathbb{R}/\mathbb{Z} \to \partial F_\infty$ such that $\theta(0) = \theta(1) = t$, and all other points are parameterized counterclockwise. Denote by $(\varnothing)$ the address of F_∞. Then the addresses of all other Fatou components are determined in exactly the same way as those on $\mathcal{B}$.

Denote, respectively, by g_{21} and g_{22} the first and second inverse image of g_2. Then g_{21} is a simple path starting at t ending at $-t$ and crossing all the inverse images of the joint points that g_2 crosses; g_{22} is also a simple path in the opposite direction that coincides with g_{21} outside F_0 and forms a loop with g_{21} on F_0. Hence g_{21} and g_{22} are represented by the same set of addresses. The two connecting paths l_0 and l_1 are showed in the following figure:

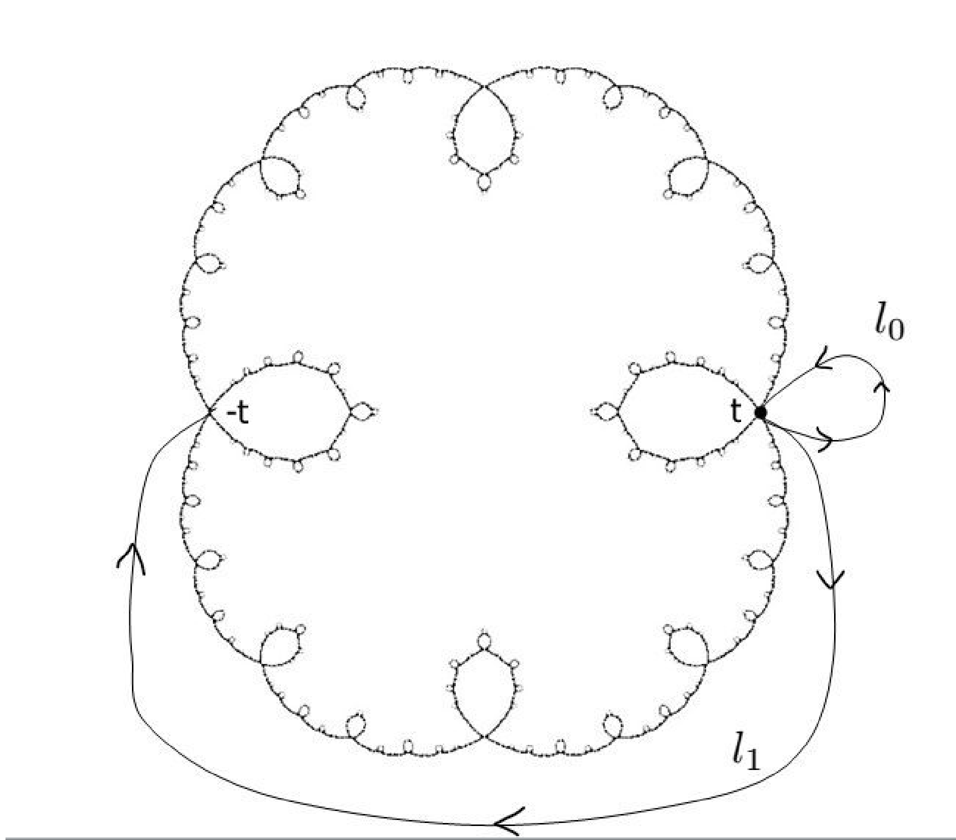

Figure 3.2: Connecting paths.

The above description results in the following lemma:

Lemma 3.5. The presentation of g_2 can be written as $g_2 = \langle\langle l_1^{-1} \cdot g_{21} \cdot l_0, l_0 \cdot g_{22} \cdot l_1 \rangle\rangle \sigma$, where the product notation $\cdot$ is read from right to left. $\square$

Proposition 3.6. Let $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$ be the address of g_2 and $\mathcal{O}$ be the forward orbit of this address. Then $g_2 = \langle\langle h, h^{-1} \rangle\rangle \sigma$, where h is a composition of generating loops whose addresses are enclosed by the loop $l_1^{-1} \cdot g_{21} \cdot l_0$.

Proof. The loops $l_1^{-1} \cdot g_{21} \cdot l_0$ and $l_0 \cdot g_{22} \cdot l_1$ are represented by the same set of addresses as that of g_{21} (as well as g_{22}). Hence they go around exactly the same post critical points but in opposite orientations. It follows that $h := l_1^{-1} \cdot g_{21} \cdot l_0$ and $h^{-1} = l_0 \cdot g_{22} \cdot l_1$. □

Example 3.7. Let F_2 be the same as that in Example 3.2. Then g_2 is represented by $\{(\frac{1}{2^2}), (\frac{1}{2^2}, \frac{1}{2^2})\}$; thus g_{21} and g_{22} are represented by $\{(\frac{1}{2^2}, \frac{1}{2^2}), (0, \frac{1}{2^2}, \frac{1}{2^2}), (0, \frac{1}{2^2}), (\frac{1}{2}, \frac{1}{2^2})\}$. Hence $h = g_4 g_7$ and $h^{-1} = g_7^{-1} g_4^{-1}$. All products are read from right to left. Hence $g_2 = \langle\!\langle g_4 g_7, g_7^{-1} g_4^{-1} \rangle\!\rangle \sigma$.

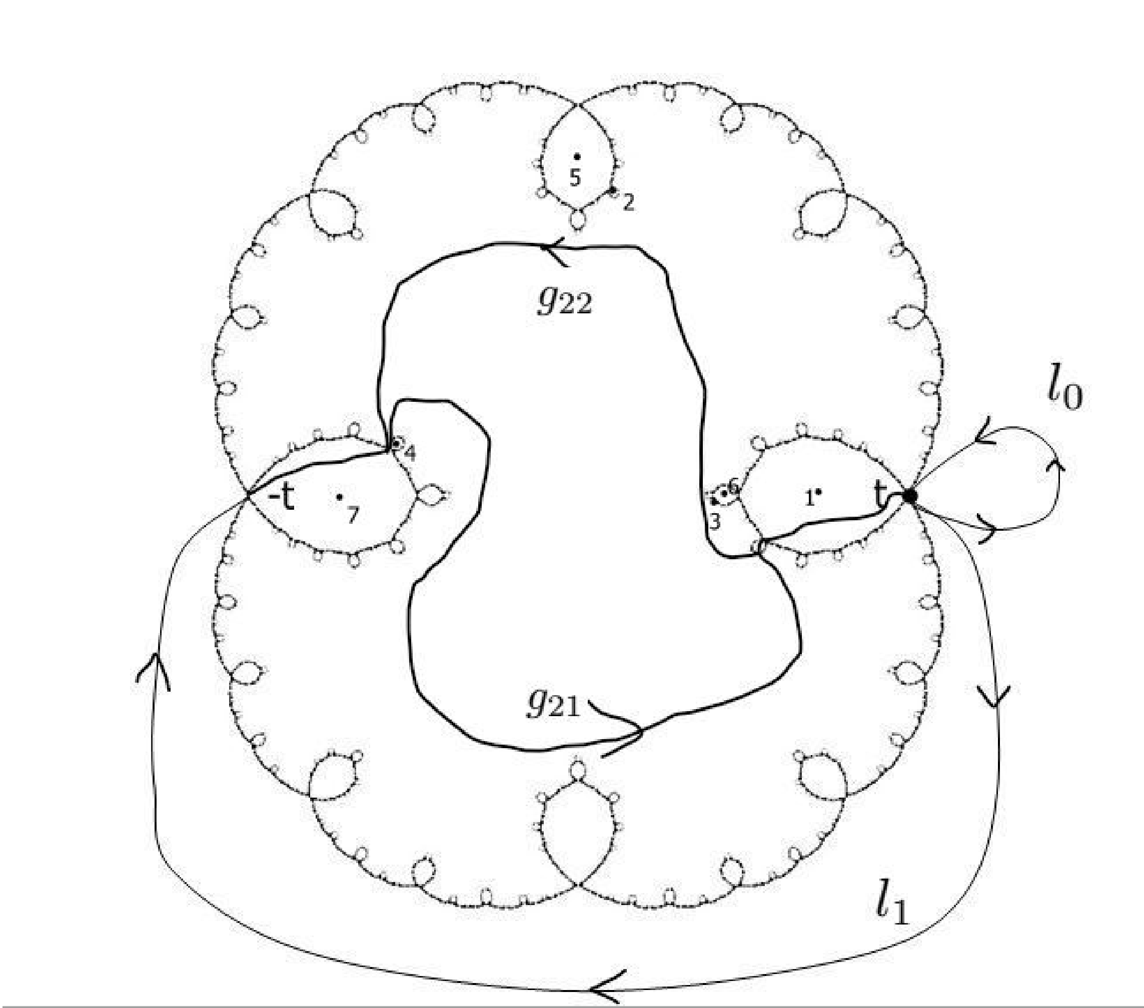

Figure 3.3: Two lifts of g_2. Numbers stand for the subscript of each generator.

4. CONJECTURE AND CONCLUSION

Conjecture. The function $\sigma_\beta \circ p_{-1}$ is topologically equivalent to some post-critically finite rational functions, i.e. it has no Thurston obstructions (see [8]).

The proof of the above conjecture will be given in later research. We merely assume this conjecture in this paper.

Lemma 4.1. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a degree 2 rational mapping with ramification portrait

$$0 \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto \infty \rightleftarrows 1$$

where 0 and ∞ are critical points of f. Then f is of the form $\dfrac{z^2+c}{z^2-1}$, $c \in \mathbb{C}$.

Proof. Write $f = \dfrac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials in $\mathbb{C}[x]$. Since f has degree 2, it follows that either deg$P(z) = 2$ and deg$Q(z) \leq 2$, or deg$P(z) \leq 2$ and deg$Q(z) = 2$. Moreover, since $f(1) = \infty$, it follows that $Q(z) = z^2 - 1$ or $z - 1$.

(1) If $Q(z) = z^2 - 1$, then

$$f'(z) = \frac{P'(z)(z^2-1) - 2zP(z)}{(z^2-1)^2}$$

$f'(0) = 0 \implies P'(0) = 0 \implies$ deg$P(z) = 2$ or 0. Since $f(\infty) = 1$, deg$P(z)$ cannot be 0. It follows that deg$P(z) = 2$, an thus the coefficient of the highest term of $P(z)$ is 1.

Write $P(z) = z^2 + bz + c$, $b, c \in \mathbb{C}$. Then

$$\begin{aligned} f'(z) &= \frac{(2z+b)(z^2-1) - (z^2+bz+c)\cdot 2z}{(z^2-1)^2} \\ &= \frac{-bz^2 - (2+2c)z - b}{z^4 - 2z^2 + 1} \end{aligned}$$

Hence it is trivially true that $f'(\infty) = 0$. Since $f'(0) = 0 = -b$, it follows that $b = 0$. Hence $f(z) = \dfrac{z^2+c}{z^2-1}$.

(2) If $Q(z) = z - 1$, then

$$f'(z) = \frac{P'(z)(z-1) - P(z)}{(z-1)^2}$$

$f'(\infty) = 0 \implies \deg P'(z) = 0 \implies \deg P(z) = 1$. This is a contradiction since we assume f has degree 2, but neither $P(z)$ nor $Q(z)$ has degree 2. Hence $Q(z) \neq z - 1$.

Hence the only form of f is $\dfrac{z^2 + c}{z^2 - 1}$. □

Proposition 4.2. The function $\sigma_\beta \circ p_{-1}$ is topologically equivalent to a post-critically finite f_c.

Proof. The ramification portrait of $\sigma_\beta \circ f_0$ (which is topologically equivalent to $\sigma_\beta \circ p_{-1}$) is exactly that in the above lemma. Since we assume the conjecture, it follows that $\sigma_\beta \circ f_0$ is of the form $\dfrac{z^2 + c}{z^2 - 1} =: f_c$. □

Therefore, we obtained our conclusion:

Theorem 4.3. The wreath recursion of IMG(f_c), in which f_c is post-critically finite, is the same as the wreath recursion of an IMG($\sigma_\beta \circ p_{-1}$). Hence IMG($f_c$) can be represented by the addresses.

Proof. This is because two topological mapping are topologically equivalent if and only if their iterated monodromy groups are the same. □

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APPENDIX A

MATLAB CODE FOR FORWARD ORBIT OF ADDRESSES

```
clc
clear
K=input(prompt);%Input an address
l=length(K);
n=0;
while(length(K) =0)

    if K(1)==0
K(1)=[];
disp(K);
elseif K(1)==1
K(1)=0;
disp(K);
else
K(1)=K(1)-1;
K=[0,K];
disp(K);%Displays each address on this orbit
end
n=n+1;

    end
steps=n-1
```

APPENDIX B

MATLAB CODE FOR BRANCHES OF ADDRESSES

```
clc
clear
A=input(prompt); %Input an address
B=A;
C=A;
if isempty(A)
B(1)=0;
C(1)=1/2;
elseif A(1) =0
B=[0,B];
C=[1/2,C];
else
B(1)=[];
B(1)=B(1)2;
C(1)=[];
[a,b]=numden(sym(C(1)));
C(1)=(a+b)/(2*b);
end
disp('The first branch is:');
B %Displays the first branch of the imputed address
disp('The second branch is:');
C %Displays the second branch of the imputed address
```