ON SOME PROBLEMS IN THE NONLINEAR GEOMETRY OF BANACH SPACES

A Dissertation

by

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ABSTRACT

Two general problems in the nonlinear geometry of Banach spaces are to determine the relationship between uniform and coarse embeddings and to characterize local/asymptotic properties in terms of metric structure. The purpose of this research is to investigate these problems and to contribute to a better overall understanding of the structure of Banach spaces and metric spaces.

First, we investigate the relationship between the small-scale and large-scale structures of $c_0(\kappa)$. In 1994, Jan Pelant proved that a metric property related to the notion of paracompactness called the uniform Stone property characterizes a metric space's uniform embeddability into $c_0(\kappa)$ for some cardinality κ. We show that coarse Lipschitz embeddability of a metric space into $c_0(\kappa)$ can be characterized in a similar manner. We also show that coarse, uniform, and bi-Lipschitz embeddability into $c_0(\kappa)$ are equivalent notions for normed linear spaces.

Next, we investigate the relationship between the small-scale and large-scale structures of superstable Banach spaces. In 1983, Yves Raynaud showed that if a Banach space uniformly embeds into a superstable Banach space, then X must contain an isomorphic copy of ℓ_p, for some $p \in [1, \infty)$. Using similar methods, we show that if a Banach space coarsely embeds into a superstable Banach space, then X has a spreading model isomorphic to ℓ_p, for some $p \in [1, \infty)$. This implies the existence of reflexive Banach spaces that do not coarsely embed into any superstable Banach space.

Lastly, we define a class of graphs, which we call the "bundle graphs", and use this to generalize some known metric characterizations of Banach space properties in terms of graph preclusion. In particular, we generalize the characterizations of superreflexivity within the class of Banach spaces and asymptotic uniform convexifiability within the class of reflexive Banach spaces with unconditional asymptotic structure. For the specific case of L_1, we show that every $\aleph_0$-branching bundle graph bi-Lipschitzly embeds into L_1 with distortion no worse than 2.

DEDICATION

To my wife Emily.

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Thanks to my advisors Drs. Schlumprecht and Baudier for their advice and support throughout the second half of my degree. It should be mentioned that although Florent Baudier is not acknowledged on the title page (due to university policy regarding special appointments to the Graduate Faculty), he had an equal role in my academic development. Thanks to the math department, especially to Dr. Johnson and again Dr. Schlumprecht for the generous support in funding. Thanks to the administrative staff in the math department for always being on the students' side when it came to scheduling, reminders, deadlines, reimbursements, etc.; especially to Monique, without whom the mathematics graduate program would surely fall apart. Thanks to my mother, father, and grandmother for their unconditional love and encouragement. Thanks to my wife and friends for keeping life fun and giving me a way to decompress. And a final thank you to Texas A&M for being the best place in the world to get an education.

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Contributors

This work was supported by a dissertation committee consisting of Professors Thomas Schlumprecht (advisor), Florent Baudier (co-advisor), Bill Johnson, and Michael Anshelevich of the Department of Mathematics and Professor Mohsen Pourahmadi of the Department of Statistics.

Research for Section 2 was conducted in collaboration with Bruno de Mendonça Braga, who was at the time a student of the Department of Mathematics at University of Illinois at Chicago.

All other work conducted for the dissertation was completed by the student independently.

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1. INTRODUCTION

1.1 Motivation and organization of research

A result proved by S. Mazur and S. Ulam [7] says that two Banach spaces are in fact linearly isomorphic to each other if there is an onto isometry between them that maps 0 to 0 (and in fact, each such isometry is itself a linear isomorphism). However, it was shown by M. I. Kadets [5] that any two separable infinite-dimensional Banach spaces are homeomorphic. These two extremes in rigidity make it natural to investigate the extent to which the metric structure of a Banach space determines the space's linear structure. One celebrated result proved by M. Ribe [11] is Ribe's rigidity theorem, which says that two uniformly homeomorphic Banach spaces have the same finite-dimensional subspaces up to linear isomorphism with distortion bounded by some constant. This launched what is now called the "Ribe program", an ongoing effort to characterize local properties of Banach spaces in purely metric terms.

The Ribe program has led to many developments in the nonlinear geometry of Banach spaces. Recently, however, there has emerged a need to investigate the large-scale geometry of Banach spaces more thoroughly. This developed largely from work and observations relating to the Novikov and (coarse) Baum-Connes conjectures by M. Gromov (see, for instance, [3]), and since then by G. Kasparov and G. Yu (for instance, [6] and [14]). One strategy for understanding the large-scale structure of Banach spaces is to determine how it relates to the small-scale structure. For instance, it is an open problem to determine whether for any two Banach spaces X and Y, X is uniformly embeddable into Y if and only if X is coarsely embeddable into Y. This problem motivates the research displayed in Sections 2 and 3. The approach to both sections is the same: Start with a specific embeddability result for small-scale structure and try to determine whether an analogous result holds true for the large-scale structure. As will be seen, the approach can be quite effectively used to obtain new information about the large-scale geometry of Banach spaces. A bonus to the approach is that if a large-scale analogue cannot be found or is weaker than the motivating

small-scale result, then at least a strategy for solving the general problem in the negative can be formed.

In Section 2, we discuss the large-scale structure of $c_0(\kappa)$ spaces. We start by describing some known embeddability results concerning $c_0(\kappa)$ and providing the reader with the necessary background in metric covers and coarse embeddings needed to prove the main result. Particular attention is given to J. Pelant's [9] intrinsic characterization of a metric space's uniform embeddability into some $c_0(\kappa)$ in terms of the "uniform Stone property". We then define the "coarse Stone property", a coarse analogue of the uniform Stone property (itself a uniform analogue of paracompactness), and show that having this property is a necessary condition for a metric space to be coarsely embeddable into some $c_0(\kappa)$. To make the results more quantitative, we define a modulus $\Delta_X^{(c)}$ for each metric space X that can be used to determine whether X has the uniform or coarse Stone property. The main result is that $\Delta_X^{(c)}$ can be used to characterize X's coarse Lipschitz embeddability into some $c_0(\kappa)$. A corollary of this work shows that a Banach space is uniformly embeddable into some $c_0(\kappa)$ if and only if it is coarsely embeddable into $c_0(\kappa)$ if and only if it is bi-Lipschitzly embeddable into $c_0(\kappa)$ if and only if it has the coarse Stone property. We conclude the section by showing directly that certain classes of metric spaces have the coarse Stone property. The contents of Section 2 were originally published in Fundamenta Mathematicae [12] and are included here with permission from the copyright holder.

In Section 3, we discuss the large-scale structure of superstable Banach spaces. We start by giving a short history of a small-scale result by Y. Raynaud [10], which says that any Banach space that is uniformly embeddable into a superstable Banach space contains a linearly isomorphic copy of ℓ_p for some $p \in [1, \infty)$. We then devote some time providing the reader with all the necessary background in asymptotic Banach space geometry and topology needed for the rest of the section. We proceed by carefully defining and developing the required notion of "space of types" associated to a given Banach space. This space of types is a metric space that captures some of the algebra and geometry of the given Banach space, but additionally possesses some nice compactness properties. After much technical work using methods commonly found in proofs of the famous Krivine's

Theorem from Banach space theory, we prove the main result which says that any Banach space that is coarsely embeddable into a superstable Banach space contains an ℓ_p spreading model for some $p \in [1, \infty)$. This is a strong enough analogue of Raynaud's result to derive our concluding corollary which says there exist reflexive Banach spaces that are not coarsely embeddable into any superstable Banach space. The contents of Section 3 were jointly researched with B. M. Braga and can be found in a separate preprint on the arXiv [2].

In Section 4, we provide a contribution to the Ribe program and related asymptotic Ribe program (where the goal is to characterize asymptotic properties of Banach spaces in purely metric terms). We start by listing some known results due to W. B. Johnson and G. Schechtman [4], M. Ostrovskii and B. Randrianantoanina [8], and F. Baudier et al. [1] that contain characterizations of Banach space properties in terms of non-equi-bi-Lipschitz-embeddability of certain classes of graphs. We then rigorously define and vertex-label a new and much larger class of graphs we call the "bundle graphs". Most of the remainder of the section is used to generalize three embedding results to the class of bundle graphs, with simpler proofs arising from usage of the new vertex-labeling. At the end of the section, we show how bundle graphs behave under the graph-theoretic $\oslash$-product and infer generalizations of the previously known characterizations of superreflexivity within the class of Banach spaces and asymptotic uniform convexifiability within the class of reflexive Banach spaces with an unconditional asymptotic structure. The contents of Section 4 will be published in Mathematika [12] and are included here with permission from the copyright holder.

Finally, in Section 5, we give a quick summary of the main results and highlight some open problems that they might be applied to in the future.

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2. ON COARSE LIPSCHITZ EMBEDDABILITY INTO $c_0(\kappa)$*

2.1 Introduction

I. Aharoni showed in 1974 [1] that for any $K > 6$, every separable metric space K-Lipschitzly embeds into c_0^+ (where the positive cone of c_0, denoted c_0^+, is the set $\{(x_i)_{i=1}^\infty \in c_0 \mid x_i \geqslant 0 \text{ for all } i \in \mathbb{N}\}$ with metric inherited from c_0); and also that ℓ_1 does not K-Lipschitzly embed into c_0 for any $K < 2$. In 1978, P. Assouad [3] improved Aharoni's result and showed that for any $K > 3$, every separable metric space K-Lipschitzly embeds into c_0^+. The final improvement for c_0^+ came when Pelant showed in 1994 [8] that every separable metric space 3-Lipschitzly embeds into c_0^+ and that ℓ_1 cannot be K-Lipschitz embedded into c_0^+ for any $K < 3$. This left open the problem of finding the best constant for bi-Lipschitzly embedding a separable metric space into c_0 until N. J. Kalton and G. Lancien showed in 2008 [6] that every separable metric space 2-Lipschitzly embeds into c_0. They do this by showing that every separable metric space has property $\Pi(2)$, property $\Pi(\lambda)$ being a sufficient criterion they define for implying λ-Lipschitz embeddability into c_0 for a separable metric space. Recently, F. Baudier and R. Deville [4] have made a slight improvement to Kalton and Lancien's proof using a related criterion $\pi(\lambda)$ to show that every separable metric space 2-Lipschitzly embeds into c_0 via a special kind of bi-Lipschitz embedding.

It is natural to ask whether a similar result holds for non-separable metric spaces. In particular, does every metric space bi-Lipschitzly embed into $c_0(\kappa)$ for large enough cardinality κ? The answer to this question comes from the theory of uniform spaces. In 1948, A. H. Stone [10] showed that every metric space is paracompact. In 1960 [11], Stone asked whether every uniform cover of a metric space has a locally finite uniform refinement (or equivalently a point-finite uniform refinement). That is, does every metric space possess a uniform analog of paracompactness (a property that has come to be called the uniform Stone property)? The question was answered in the negative by Pelant [7] and E. V. Shchepin [9], who showed that $\ell_\infty(\Gamma)$ fails to have the uniform

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Stone property if Γ has large enough cardinality. Moreover, Pelant [8] has shown that the uniform Stone property characterizes uniform embeddability into $c_0(\kappa)$ for some κ and thus $\ell_\infty(\Gamma)$ does not even uniformly embed into any $c_0(\kappa)$ when Γ has large enough cardinality.

It remains an open problem in the nonlinear theory of Banach spaces to determine whether a Banach space's uniform embeddability into a given Banach space Y is equivalent to its coarse embeddability into Y, and so one is led to ask whether a characterization of coarse embeddability into $c_0(\kappa)$ involving covers also exists. We suggest a natural candidate for such a "coarse Stone property", and show this to be at least a necessary condition for coarse embeddability into $c_0(\kappa)$. Related to this property, however, is a natural modulus $\Delta_X^{(c)}$ that can be defined for any metric space X and whose growth can be used to characterize coarse Lipschitz embeddability (and also bi-Lipschitz embeddability) into $c_0(\kappa)$. The main result is the following theorem.

Theorem 14. *Let (X, d_X) be a metric space with infinite density character κ. If there are $C \in [1, \infty)$ and $D \in [0, \infty)$ such that $\Delta_X^{(c)}(R) \leqslant CR + D$ for all $R < \infty$; then for any $\lambda > 0$, any $K > 2(C + \lambda)$, and any $L > \frac{(C+\lambda)D}{\lambda}$; there exists a coarse Lipschitz embedding $f\colon X \to c_0^+(\kappa)$ such that*

$$d_X(x, y) - L \leqslant \|f(x) - f(y)\|_\infty \leqslant K d_X(x, y)$$

for every $x, y \in X$. If $D = 0$, then it is possible to take $L = 0$.

2.2 Preliminaries and notation

Let (X, d_X) and (Y, d_Y) be metric spaces. Given $x \in X$ and $r \in [0, \infty)$, we will denote by $B_r(x)$ the open ball of radius r centered at x. For a map $f\colon X \to Y$, the *modulus of continuity* (or *modulus of expansion*) of f is the function $\omega_f\colon [0, \infty) \to [0, \infty]$ defined by

$$\omega_f(t) = \sup\{d_Y(f(x_1), f(x_2)) \mid d_X(x_1, x_2) \leqslant t\},$$

and the *modulus of compression* of f is the function $\rho_f\colon [0, \infty) \to [0, \infty]$ defined by

$$\rho_f(t) = \inf\{d_Y(f(x_1), f(x_2)) \mid d_X(x_1, x_2) \geqslant t\}.$$

Note that ω_f and ρ_f are non-decreasing and for all $x_1, x_2 \in X$,

$$\rho_f(d_X(x_1, x_2)) \leqslant d_Y(f(x_1), f(x_2)) \leqslant \omega_f(d_X(x_1, x_2)).$$

A map f is said to be *uniformly continuous* (or simply *uniform*) if $\lim_{t\to 0} \omega_f(t) = 0$ and is called a *uniform embedding* if furthermore $\rho_f(t) > 0$ for all $t > 0$. A map f is said to be *coarse* (or sometimes *coarsely continuous*) if $\omega_f(t) < \infty$ for all $t \in [0, \infty)$ and is called a *coarse embedding* if furthermore $\lim_{t\to\infty} \rho_f(t) = \infty$. A map f is called a *coarse Lipschitz embedding* (or a *quasi-isometric embedding*, especially in the literature of geometric group theory) if there exist $A \geqslant 1$ and $B \geqslant 0$ such that $\omega_f(t) \leqslant At + B$ and $\rho_f(t) \geqslant \frac{1}{A}t - B$ for all t and is called a *bi-Lipschitz embedding* if furthermore B can be taken to be equal to 0. The *Lipschitz constant* of f is defined to be

$$\mathrm{Lip}(f) = \sup\left\{\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \mid x_1 \neq x_2\right\}.$$

A map f is said to be *Lipschitz* if $\mathrm{Lip}(f) < \infty$. If f is injective, the *distortion* of f is defined to be $\mathrm{dist}(f) = \mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1})$. If $\mathrm{dist}(f) \leqslant K$, then f is called a K-Lipschitz embedding.

Given $a, b \in \mathbb{R}^+$, $S \subseteq X$ is called *a-separated* if $d_X(s_1, s_2) \geqslant a$ for all $s_1, s_2 \in S$, *b-dense* in X if $d_X(x, S) \leqslant b$ for all $x \in X$, and an *(a, b)-skeleton* of X if it is a-separated and b-dense in X. Given a skeleton S of X, there is a coarse Lipschitz embedding $f\colon X \to S$ such that $\sup\{d_X(f(x), S)\}_{x\in X} < \infty$ (just map every point of the space to a nearest point in the skeleton), and so questions about coarse embeddings of metric spaces can be reduced to questions about coarse embeddings of uniformly discrete metric spaces. By Zorn's Lemma, every a-separated set can be extended to a maximal (in the sense of set containment) (a, a)-skeleton of X. Note that $|S| \leqslant \mathrm{dens}(X)$ (where $|S|$ denotes the cardinality of S and where $\mathrm{dens}(X)$, the *density character* of X, is the smallest cardinality of a set dense in X) for any skeleton S of X. And if X is a normed linear space, then $X = \overline{\mathrm{span}(S)}$ (the closed linear span of S) for any skeleton of X (or else S is not b-dense in X for any $b \in \mathbb{R}^+$), and so in this case $|S| = \mathrm{dens}(X)$. The following lemma holds.

Lemma 1. *Let $(X, \|\cdot\|_X)$ be a normed linear space and (Y, d_Y) a metric space. If there exists a*

map $f\colon X \to Y$ *such that* $\lim_{t\to\infty} \rho_f(t) = \infty$*, then* $\operatorname{dens}(X) \leqslant \operatorname{dens}(Y)$.

Proof. Let $a > 0$ be such that $\rho_f(a) > 0$, and let S be an (a, a)-skeleton of X. Then $f|_S$ is injective and maps S to a $\rho_f(a)$-separated subset of Y. And so

$$\operatorname{dens}(X) = |S| = |f(S)| \leqslant \operatorname{dens}(Y). \qquad \square$$

A family of sets $\mathcal{U} \subseteq \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X) is called a *cover* of X if $\bigcup_{U\in\mathcal{U}} U = X$. The *diameter* (or *mesh*) of a cover $\mathcal{U}$ of X is

$$\operatorname{diam}(\mathcal{U}) = \sup\{\operatorname{diam}(U) \mid U \in \mathcal{U}\}$$

where for $U \subseteq X$, $\operatorname{diam}(U) = \sup\{d_X(x_1, x_2) \mid x_1, x_2 \in U\}$ is the *diameter* of U. The *Lebesgue number* of a cover $\mathcal{U}$ of X is

$$\begin{aligned}\mathcal{L}(\mathcal{U}) = \sup\{d \in [0, \infty) \mid &\text{For every } E \subseteq X \text{ such that } \operatorname{diam}(E) < d,\\ &\text{there is } U \in \mathcal{U} \text{ such that } E \subseteq U\}.\end{aligned}$$

Note that by definition $\mathcal{L}(\{X\}) = \infty$ and $\mathcal{L}(\{\{x\}\}_{x\in X}) = \inf_{x\neq y} d_X(x, y)$. A cover $\mathcal{U}$ of X is called a *uniform* cover if $\mathcal{L}(\mathcal{U}) > 0$ and is called a *uniformly bounded* (or *coarse*) cover if $\operatorname{diam}(\mathcal{U}) < \infty$. A cover $\mathcal{U}$ of X is called *point-finite* if for all $x \in X$, there are only finitely many $U \in \mathcal{U}$ such that $x \in U$. A cover $\mathcal{V}$ of X is called a *refinement* of the cover $\mathcal{U}$ of X; and in this case, $\mathcal{V}$ is said to *refine* $\mathcal{U}$; if for all $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ such that $V \subseteq U$. We have the following lemma.

Lemma 2. *Let* (X, d_X) *be a metric space with infinite density character* κ*, and let* $\mathcal{U}$ *be a point-finite uniform cover of* X*. There exists* $\mathcal{V} \subseteq \mathcal{U}$ *such that* $|\mathcal{V}| \leqslant \kappa$ *and such that* $\mathcal{V}$ *is a point-finite uniform cover of* X *with* $\mathcal{L}(\mathcal{V}) = \mathcal{L}(\mathcal{U})$.

Proof. Let $\{x_\tau\}_{\tau<\kappa}$ be a dense set in X and let

$$\mathcal{V} = \{U \in \mathcal{U} \mid x_\tau \in U \text{ for some } \tau < \kappa\}.$$

Then $|\mathcal{V}| \leqslant \kappa$ since $\mathcal{U}$ is point-finite. Now take any $A \subseteq X$ such that $\operatorname{diam}(A) < \mathcal{L}(\mathcal{U})$. If $A = \varnothing$, then clearly there is $V \in \mathcal{V}$ such that $A \subseteq V$, so suppose $A \neq \varnothing$ and let $x \in A$. Choose any $0 < r < \mathcal{L}(\mathcal{U}) - \operatorname{diam}(A)$ and let $B = A \cup B_r(x)$. Then $\operatorname{diam}(B) < \mathcal{L}(\mathcal{U})$, and so there is $U \in \mathcal{U}$ such that $B \subseteq U$. But there is $\tau < \kappa$ such that $x_\tau \in B_r(x) \subseteq B \subseteq U$ by the density of $\{x_\tau\}_{\tau<\kappa}$, and so $U \in \mathcal{V}$. Therefore $\mathcal{V}$ is a cover of X such that $\mathcal{L}(\mathcal{V}) \geqslant \mathcal{L}(\mathcal{U})$. Furthermore, $\mathcal{V}$ is point-finite and $\mathcal{L}(\mathcal{V}) = \mathcal{L}(\mathcal{U})$ because $\mathcal{V} \subseteq \mathcal{U}$. □

2.3 Characterizing embeddability

We start by defining the uniform Stone property, which characterizes a metric space's uniform embeddability into some $c_0(\kappa)$. One can view the property as a generalization of having finite (uniform) covering dimension, which is the natural notion of dimension associated with the class of uniform spaces.

Definition 3. A metric space (X, d_X) is said to have the *uniform Stone property* if every uniform cover of X has a point-finite uniform refinement.

The class of coarse spaces has a similar notion of dimension associated with it, called asymptotic dimension. It has become clear in recent years that many ideas in the uniform theory have useful analogues in the coarse theory, and so the motivation behind the following definition is to generalize the property of having finite asymptotic dimension in a manner similar to the way the uniform Stone property generalizes having finite covering dimension.

Definition 4. A metric space (X, d_X) is said to have the *coarse Stone property* if every uniformly bounded cover of X refines a point-finite uniformly bounded cover.

We immediately turn to more quantitative formulations. Given a metric space (X, d_X), define

the functions $\Delta_X^{(u)}, \Delta_X^{(c)}\colon\ [0,\infty) \to [0,\infty]$ by

$$\Delta_X^{(u)}(r) = \sup\{\mathcal{L}(\mathcal{U}) \mid \mathcal{U} \text{ is a point-finite cover of } X \text{ and } \operatorname{diam}(\mathcal{U}) \leqslant r\}$$

and

$$\Delta_X^{(c)}(R) = \inf\{\operatorname{diam}(\mathcal{U}) \mid \mathcal{U} \text{ is a point-finite cover of } X \text{ and } \mathcal{L}(\mathcal{U}) \geqslant R\}.$$

Proposition 5. *Let (X, d_X) be a metric space.*

(i) X has the uniform Stone property if and only if $\Delta_X^{(u)}(r) > 0$ for all $r > 0$.

(ii) X has the coarse Stone Property if and only if $\Delta_X^{(c)}(R) < \infty$ for all $R \in [0,\infty)$.

Proof. (i): Suppose first that X has the uniform Stone property, and take any $r > 0$. Let $\mathcal{U} = \{B_{r/2}(x)\}_{x\in X}$, and note that $\mathcal{U}$ is a uniform cover of X with $\operatorname{diam}(\mathcal{U}) \leqslant r$. By assumption, $\mathcal{U}$ has a point-finite uniform refinement $\mathcal{V}$, and so $\Delta_X^{(u)}(r) \geqslant \mathcal{L}(\mathcal{V}) > 0$. Conversely, suppose $\Delta_X^{(u)}(r) > 0$ for all $r > 0$, and take any uniform cover $\mathcal{U}$ of X. Since $\mathcal{U}$ is uniform, there is $r > 0$ such that $\mathcal{L}(\mathcal{U}) > r$. And by assumption, there is a point-finite cover $\mathcal{V}$ of X such that $0 < \mathcal{L}(\mathcal{V}) \leqslant \Delta_X^{(u)}(r)$ and $\operatorname{diam}(\mathcal{V}) \leqslant r$. But then $\mathcal{V}$ is a point-finite uniform refinement of $\mathcal{U}$, and so X has the uniform Stone property.

(ii): Suppose first that X has the coarse Stone property, and take any $R \in [0,\infty)$. Let $\mathcal{U} = \{B_R(x)\}_{x\in X}$, and note that $\mathcal{U}$ is a uniformly bounded cover of X with $\mathcal{L}(\mathcal{U}) \geqslant R$. By assumption, $\mathcal{U}$ refines a point-finite uniformly bounded cover $\mathcal{V}$, and so $\Delta_X^{(c)}(R) \leqslant \operatorname{diam}(\mathcal{V}) < \infty$. Conversely, suppose $\Delta_X^{(c)}(R) < \infty$ for all $R \in [0,\infty)$, and take any uniformly bounded cover $\mathcal{U}$ of X. Since $\mathcal{U}$ is uniformly bounded, there is $R \in [0,\infty)$ such that $\operatorname{diam}(\mathcal{U}) < R$. And by assumption, there is a point-finite cover $\mathcal{V}$ of X such that $\Delta_X^{(c)}(R) \leqslant \operatorname{diam}(\mathcal{V}) < \infty$ and $\mathcal{L}(\mathcal{V}) \geqslant R$. But then $\mathcal{V}$ is a point-finite uniformly bounded cover refined by $\mathcal{U}$, and so X has the coarse Stone property. $\square$

From this point forward, whenever we write "uniform Stone property" or "coarse Stone property", we are using the equivalent formulations of these properties in terms of $\Delta_X^{(u)}$ and $\Delta_X^{(c)}$, respectively. We have the following lemma.

Lemma 6. *Let (X, d_X) be a metric space and let $r, R \in (0, \infty)$.*

(i) $\Delta_X^{(c)}$, $\Delta_X^{(u)}$ *are non-decreasing functions.*

(ii) If $\Delta_X^{(c)}(R) < \infty$*, then* $\Delta_X^{(u)}(\Delta_X^{(c)}(R) + \varepsilon) \geqslant R$ *for all* $\varepsilon > 0$.

(iii) If $\Delta_X^{(u)}(r) > 0$*, then* $\Delta_X^{(c)}(\Delta_X^{(u)}(r) - \varepsilon) \leqslant r$ *for all* $0 < \varepsilon < \Delta_X^{(u)}(r)$.

(iv) X *has the uniform Stone property if and only if* $\lim_{R \to 0} \Delta_X^{(c)}(R) = 0$.

(v) X *has the coarse Stone property if and only if* $\lim_{r \to \infty} \Delta_X^{(u)}(r) = \infty$.

Proof. (i): This is clear from the definitions.

(ii): If $\Delta_X^{(c)}(R) < \infty$, then there is a point-finite cover $\mathcal{U}$ of X such that $\operatorname{diam}(\mathcal{U}) \leqslant \Delta_X^{(c)}(R) + \varepsilon$ and $\mathcal{L}(\mathcal{U}) \geqslant R$. Thus $\Delta_X^{(u)}(\Delta_X^{(c)}(R) + \varepsilon) \geqslant \mathcal{L}(\mathcal{U}) \geqslant R$.

(iii): If $\Delta_X^{(u)}(r) > 0$, then there is a point-finite cover $\mathcal{U}$ of X such that $\mathcal{L}(\mathcal{U}) \geqslant \Delta_X^{(u)}(r) - \varepsilon$ and $\operatorname{diam}(\mathcal{U}) \leqslant r$. Thus $\Delta_X^{(c)}(\Delta_X^{(u)}(r) - \varepsilon) \leqslant \operatorname{diam}(\mathcal{U}) \leqslant r$.

(iv): Suppose first that X has the uniform Stone property and take any $\varepsilon > 0$. Then by assumption, $\Delta_X^{(u)}(\varepsilon) > 0$, and so $\Delta_X^{(c)}(R) \leqslant \varepsilon$ for any $R < \Delta_X^{(u)}(\varepsilon)$ by parts (i) and (iii). Thus $\lim_{R \to 0} \Delta_X^{(c)}(R) = 0$. Conversely, suppose $\lim_{R \to 0} \Delta_X^{(c)}(R) = 0$ and take any $r > 0$. Let $R > 0$ be such that $\Delta_X^{(c)}(R) < r$. Then $\Delta_X^{(u)}(r) \geqslant R > 0$ by part (ii), and so X has the uniform Stone property.

(v): Suppose first that X has the coarse Stone property and take any $N \in \mathbb{N}$. Then by assumption, $\Delta_X^{(c)}(N) < \infty$, and so $\Delta_X^{(u)}(r) \geqslant N$ for any $r > \Delta_X^{(c)}(N)$ by parts (i) and (ii). Thus $\lim_{r \to \infty} \Delta_X^{(u)}(r) = \infty$. Conversely, suppose $\lim_{r \to \infty} \Delta_X^{(u)}(r) = \infty$ and take any $R \in [0, \infty)$. Let $r \in [0, \infty)$ be such that $\Delta_X^{(u)}(r) > R$. Then $\Delta_X^{(c)}(R) \leqslant r < \infty$ by part (iii), and so X has the coarse Stone property. $\square$

It is clear that if X is a metric space with finite diameter, then there are $C \in [0, 1)$ and $D \in [0, \infty)$ such that $\Delta_X^{(c)}(R) \leqslant CR + D$ for all $R \in [0, \infty)$ (indeed, one may take $C = 0$ and $D = \operatorname{diam}(X)$ in this case). The converse is also true.

Lemma 7. *Let (X, d_X) be a metric space. If $C \in [0,1)$ and $D \in [0,\infty)$ are such that $\Delta_X^{(c)}(R) \leqslant CR + D$ for all $R \in [0,\infty)$, then $\operatorname{diam}(X) \leqslant \frac{D}{1-C}$.*

Proof. Take any $0 < \varepsilon < 1 - C$. Then for any $R > \frac{D}{1-C-\varepsilon}$,

$$\Delta_X^{(c)}(R) \leqslant CR + D < CR + (1 - C - \varepsilon)R = (1 - \varepsilon)R.$$

So suppose there exist $x, y \in X$ such that $d_X(x,y) > \frac{D}{1-C-\varepsilon}$. Then

$$\Delta_X^{(c)}((1+\varepsilon)d_X(x,y)) < (1-\varepsilon)(1+\varepsilon)d_X(x,y) = (1-\varepsilon^2)d_X(x,y).$$

Thus there is a point-finite cover $\mathcal{U}$ of X with $\mathcal{L}(\mathcal{U}) \geqslant (1+\varepsilon)d(x,y) > d(x,y)$ satisfying $\operatorname{diam}(\mathcal{U}) < (1-\varepsilon^2)d(x,y) < d(x,y)$. But this is a contradiction since $\operatorname{diam}(\{x,y\}) = d(x,y)$. Therefore $d(x,y) \leqslant \frac{D}{1-C-\varepsilon}$ for every $x, y \in X$. Since $0 < \varepsilon < 1 - C$ was arbitrary, $\operatorname{diam}(X) \leqslant \frac{D}{1-C}$. □

In some cases it might be more natural to find bounds for $\Delta_X^{(u)}$ rather than $\Delta_X^{(c)}$ or vice versa. Lemma 6 provides a way of switching from one to the other and this is especially easy in the case below.

Lemma 8. *Let (X, d_X) be a metric space. Given $C \in (0,\infty)$, $\Delta_X^{(c)}(R) \leqslant CR$ for all $R \in [0,\infty)$ iff $\Delta_X^{(u)}(r) \geqslant \frac{1}{C}r$ for all $r > 0$.*

Proof. Suppose first that $\Delta_X^{(c)}(R) \leqslant CR$ for all $R \in [0,\infty)$. Take any $r > 0$ and $0 < \varepsilon < r$. Then $\Delta_X^{(c)}\left(\frac{r-\varepsilon}{C}\right) \leqslant r - \varepsilon$ and so by Lemma 6,

$$\Delta_X^{(u)}(r) \geqslant \Delta_X^{(u)}\left(\Delta_X^{(c)}\left(\frac{r-\varepsilon}{C}\right) + \varepsilon\right) \geqslant \frac{1}{C} \cdot (r - \varepsilon).$$

Since $0 < \varepsilon < r$ was arbitrary, $\Delta_X^{(u)}(r) \geqslant \frac{1}{C}r$ for all $r > 0$.

Now suppose $\Delta_X^{(u)}(r) \geqslant \frac{1}{C}r$ for all $r > 0$. Take any $R \in [0,\infty)$ and $\varepsilon > 0$. Then $\Delta_X^{(u)}(C(R+\varepsilon)) \geqslant$

$R+\varepsilon$ and so by Lemma 6,

$$\Delta_X^{(c)}(R) \leqslant \Delta_X^{(c)}(\Delta_X^{(u)}(C(R+\varepsilon))-\varepsilon) \leqslant C(R+\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, $\Delta_X^{(c)}(R) \leqslant CR$ for all $R \in [0,\infty)$. □

Lemma 9. *Let $(X, \|\cdot\|_X)$ be a normed linear space. The following are equivalent:*

(i) $\Delta_X^{(c)}(R) < \infty$ *for some* $R \in (0,\infty)$.

(ii) There is $C \in [0,\infty)$ *such that* $\Delta_X^{(c)}(R) \leqslant CR$ *for all* $R \in [0,\infty)$.

(iii) X *has the coarse Stone property.*

(iv) X *has the uniform Stone property.*

Proof. (i) ⇒ (ii): Let $R \in (0,\infty)$ be such that $\Delta_X^{(c)}(R) < \infty$. Pick any uniformly bounded point-finite cover $\mathcal{U}$ of X such that $\mathcal{L}(\mathcal{U}) \geqslant R$. Simply scaling $\mathcal{U}$ shows that $\Delta_X^{(c)}(R') \leqslant \frac{\operatorname{diam}(\mathcal{U})}{R} R'$ for any $R' \in [0,\infty)$.

(ii) ⇒ (iii): Clear.

(iii) ⇒ (iv): If X has the coarse Stone property, then in particular, $\Delta_X^{(c)}(1) < \infty$. Thus, X has the uniform Stone property by (i) ⇒ (ii) and Lemma 8.

(iv) ⇒ (i): Lemma 6. □

The following two propositions show that the uniform (respectively, coarse) Stone properties is hereditary in the sense that a uniformly (respectively, coarsely) embedded subset of a metric space with the uniform (respectively, coarse) Stone property has the uniform (respectively, coarse) Stone property respectively.

Proposition 10. *Let (X, d_X) be a metric space and (Y, d_Y) a metric space with the uniform Stone property. If there exists a uniform embedding $f\colon X \to Y$, then X has the uniform Stone property. If f is a bi-Lipschitz embedding and if there is $c > 0$ such that $\Delta_Y^{(u)}(r) \geqslant cr$ for all $r > 0$, then $\Delta_X^{(u)}(r) \geqslant \frac{c}{\operatorname{dist}(f)} r$ for all $r > 0$.*

Proof. Fix $r > 0$. Since f is a uniform embedding, $\lim_{t\to 0}\omega_f(t) = 0$ and $\rho_f(t) > 0$ for all $t > 0$. Take any $0 < \varepsilon_1 < \rho_f(r)$ and $0 < \varepsilon_2 < \Delta_Y^{(u)}(\rho_f(r)-\varepsilon_1)$ and let $\mathcal{V}$ be a point-finite cover of Y such that $\operatorname{diam}(\mathcal{V}) \leqslant \rho_f(r)-\varepsilon_1$ and $\mathcal{L}(\mathcal{V}) \geqslant \Delta_Y^{(u)}(\rho_f(r)-\varepsilon_1)-\varepsilon_2$. Let $\mathcal{U} = \{f^{-1}(V)\}_{V\in\mathcal{V}}$. Then $\mathcal{U}$ is a cover of X. Note that $\mathcal{U}$ inherits point-finiteness from $\mathcal{V}$. And for any $V \in \mathcal{V}$, $\operatorname{diam}(f^{-1}(V)) \leqslant r$ since $\operatorname{diam}(V) \leqslant \rho_f(r) - \varepsilon_1$. This means $\operatorname{diam}(\mathcal{U}) \leqslant r$. Thus,

$$\Delta_X^{(u)}(r) \geqslant \mathcal{L}(\mathcal{U}) \geqslant \inf \omega_f^{-1}([\mathcal{L}(\mathcal{V}), \infty]) \geqslant \inf \omega_f^{-1}([\Delta_Y^{(u)}(\rho_f(r) - \varepsilon_1) - \varepsilon_2, \infty]) > 0$$

by definition of $\Delta_X^{(u)}$, the assumptions on ρ_f and ω_f, and since Y has the uniform Stone property. Thus, X has the uniform Stone property. The special case follows by bounding ω_f, ρ_f, and $\Delta_Y^{(u)}$ with linear functions and letting $\varepsilon_1, \varepsilon_2 \to 0$. □

Proposition 11. *Let (X, d_X) be a metric space and (Y, d_Y) a metric space with the coarse Stone property. If there exists a coarse embedding $f : X \to Y$, then X has the coarse Stone property. If f is a coarse Lipschitz embedding and there are $C, D \in [0, \infty)$ such that $\Delta_Y^{(c)}(R) \leqslant CR + D$ for all $R \in [0, \infty)$, then there are $C', D' \in [0, \infty)$ such that $\Delta_X^{(c)}(R) \leqslant C'R + D'$ for all $R \in [0, \infty)$. If, in particular, f is a bi-Lipschitz embedding and $D = 0$, then $\Delta_X^{(c)}(R) \leqslant C\operatorname{dist}(f)R$ for all $R \in [0, \infty)$.*

Proof. Fix $R \in [0, \infty)$. Since f is a coarse embedding, $\omega_f(t) < \infty$ for all $t \in [0, \infty)$ and $\lim_{t\to\infty}\rho_f(t) = \infty$. Take any $\varepsilon > 0$ and let $\mathcal{V}$ be a point-finite cover of Y such that $\mathcal{L}(\mathcal{V}) \geqslant \omega_f(R) + \varepsilon$ and $\operatorname{diam}(\mathcal{V}) \leqslant \Delta_Y^{(c)}(\omega_f(R) + \varepsilon) + \varepsilon$. Let $\mathcal{U} = \{f^{-1}(V)\}_{V\in\mathcal{V}}$. Then $\mathcal{U}$ is a cover of X. Note that $\mathcal{U}$ inherits point-finiteness from $\mathcal{V}$. Now take any $A \subseteq X$ such that $\operatorname{diam}(A) < R$. Then $\operatorname{diam}(f(A)) < \omega_f(R) + \varepsilon$, and so $f(A) \subseteq V$ for some $V \in \mathcal{V}$. Therefore $A \subseteq f^{-1}(V) = U$ for some $U \in \mathcal{U}$. Since $A \subseteq X$ was arbitrary, this means $\mathcal{L}(\mathcal{U}) \geqslant R$. Thus,

$$\begin{aligned}\Delta_X^{(c)}(R) \leqslant \operatorname{diam}(\mathcal{U}) &\leqslant \sup \rho_f^{-1}([0, \operatorname{diam}(\mathcal{V})]) \\ &\leqslant \sup \rho_f^{-1}([0, \Delta_Y^{(c)}(\omega_f(R) + \varepsilon) + \varepsilon]) < \infty\end{aligned}$$

by definition of $\Delta_X^{(c)}$, the assumptions on ρ_f and ω_f, and since Y has the coarse Stone property. Thus, X has the coarse Stone property. The special cases follow by bounding ω_f, ρ_f, and $\Delta_Y^{(c)}$ with affine or linear functions and letting $\varepsilon \to 0$. $\square$

Proposition 12. *For any cardinality κ, $\Delta_{c_0^+(\kappa)}^{(c)}(R) = R$ for all $R \in [0, \infty)$.*

Proof. Take any $n \in \mathbb{N}$ and $R \in [0, \infty)$. Given a finite subset M of κ, denote the set $\{x \in (\mathbb{N} \cup \{0\})^\kappa \mid x_\xi = 0 \text{ if } \xi \notin M\}$ by $\mathbb{N}^M$. For each finite subset M of κ and $x \in \mathbb{N}^M$, let

$$U_{M,x} = \left\{ f \in c_0^+(\kappa) \mid f(\xi) \in \frac{x_\xi}{n} + \left[0, 2R + \frac{1}{n}\right) \text{ for all } \xi \in \kappa \right\}.$$

Then for a fixed finite subset M of κ and a fixed $f \in c_0^+(\kappa)$, there are at most $(2n\lceil R\rceil + 1)^{|M|}$ many $x \in \mathbb{N}^M$ such that $f \in U_{M,x}$. Let

$$\mathcal{U} = \left\{ U_{M,x} \mid M \text{ is a finite subset of } \kappa \text{ and } x \in \mathbb{N}^M \right\}.$$

Now take any $f \in c_0^+(\kappa)$. There is a finite subset M of κ such that $f(\xi) < \frac{1}{n}$ if $\xi \notin M$, and in this case there is $x \in \mathbb{N}^M$ such that $B_R(f) \subseteq U_{M,x}$ (simply choose $x_\xi = \lfloor n(f(\xi) - R)\rfloor$ when $\xi \in M$ and $x_\xi = 0$ otherwise). Now suppose $M' \supsetneq M$ and $x' \in \mathbb{N}^{M'}$ is such that $f \in U_{M',x'}$. Then for all $\xi \in M' \backslash M$, $x'_\xi = 0$ (or else $f(\xi) \geqslant \frac{1}{n}$, contradicting the choice of M). Thus $U_{M',x'} = U_{M,y}$ for some $y \in \mathbb{N}^M$. This means that for every $f \in c_0^+(\kappa)$, $f \in U$ for only finitely many $U \in \mathcal{U}$. By the above, $\mathcal{U}$ is a point-finite cover of $c_0^+(\kappa)$ refined by $\{B_R(f)\}_{f \in c_0^+(\kappa)}$ such that $\operatorname{diam}(\mathcal{U}) = 2R + \frac{1}{n}$. Since every $A \subseteq c_0^+(\kappa)$ such that $\operatorname{diam}(A) < 2R$ is contained in a ball of radius R (centered at $\left(\frac{\sup \pi_\tau(A) - \inf \pi_\tau(A)}{2}\right)_{\tau<\kappa}$, where π_τ is the τ-th coordinate functional), this means $\mathcal{L}(\mathcal{U}) \geqslant 2R$. Thus, since $n \in \mathbb{N}$ was arbitrary, $\Delta_{c_0^+(\kappa)}^{(c)}(2R) \leqslant 2R$. And so, by Lemma 7, $\Delta_{c_0^+(\kappa)}^{(c)}(R) = R$ for all $R \in [0, \infty)$. $\square$

Corollary 13. *For any infinite cardinality κ, $\Delta_{c_0(\kappa)}^{(c)}(R) = 2R$ for all $R \in [0, \infty)$.*

Proof. Fix $R \in [0, \infty)$ and suppose $\Delta_{c_0(\kappa)}^{(c)}(R) < 2R$. Then there exists a point-finite cover $\mathcal{U}$ of $c_0(\kappa)$ such that $\mathcal{L}(\mathcal{U}) \geqslant R$ and $\operatorname{diam}(\mathcal{U}) < 2R$. Let $(e_\tau)_{\tau<\kappa}$ be the standard basis for $c_0(\kappa)$. Given

a finite subset M of κ and an $\varepsilon \in \{-1,1\}^M$, let

$$A_{M,\varepsilon} = \left\{ \sum_{\tau \in M} C_\tau e_\tau \mid \varepsilon_\tau C_\tau \in \left(\frac{1}{8}\mathrm{diam}(\mathcal{U}) - \frac{1}{4}R, \frac{1}{4}\mathrm{diam}(\mathcal{U}) + \frac{1}{2}R \right) \right\}.$$

Fix a finite subset M of κ. Note that for any $\varepsilon \in \{-1,1\}^M$, $\mathrm{diam}(A_{M,\varepsilon}) < R$, and so there is $U_{M,\varepsilon} \in \mathcal{U}$ such that $A_{M,\varepsilon} \subseteq U_{M,\varepsilon}$. But $\mathrm{diam}(A_{M,\delta} \cup A_{M,\varepsilon}) > \mathrm{diam}(\mathcal{U})$ whenever $\delta, \varepsilon \in \{-1,1\}^M$ are such that $\delta \neq \varepsilon$, and so in this case $U_{M,\delta} \neq U_{M,\varepsilon}$. Thus, as $0 \in A_{M,\varepsilon}$ for every every $\varepsilon \in \{-1,1\}^M$, there are at least $2^{|M|}$ different $U \in \mathcal{U}$ such that $0 \in U$. But κ is infinite, and so has subsets of arbitrarily large finite cardinality. That is, there are infinitely many $U \in \mathcal{U}$ such that $0 \in U$, contradicting the point-finiteness of $\mathcal{U}$. Therefore $\Delta^{(c)}_{c_0(\kappa)}(R) \geqslant 2R$. Now, given $f \in c_0(\kappa)$, define $g_f \in c_0^+(\kappa)$ by $g(2\xi) = \max\{0, f(\xi)\}$ and $g(2\xi+1) = \max\{0, -f(\xi)\}$ for every $\xi < \kappa$. The map $f \mapsto g_f$ is a 2-Lipschitz embedding, and so by Proposition 11 and Proposition 12, $\Delta^{(c)}_{c_0(\kappa)}(R) \leqslant 2R$. That is, $\Delta^{(c)}_{c_0(\kappa)}(R) = 2R$. □

Note that Proposition 11, Proposition 12, and Corollary 13 together show that the optimal distortion for a bi-Lipschitz embedding of $c_0(\kappa)$ into $c_0^+(\kappa)$ is 2. We now come to the main result. The proof combines techniques from both Pelant and Assouad.

Theorem 14. *Let (X, d_X) be a metric space with infinite density character κ. If there are $C \in [1,\infty)$ and $D \in [0,\infty)$ such that $\Delta^{(c)}_X(R) \leqslant CR + D$ for all $R \in [0,\infty)$; then for any $\lambda > 0$, any $K > 2(C+\lambda)$, and any $L > \frac{(C+\lambda)D}{\lambda}$; there exists a coarse Lipschitz embedding $f\colon X \to c_0^+(\kappa)$ such that*

$$d_X(x,y) - L \leqslant \|f(x) - f(y)\|_\infty \leqslant K d_X(x,y)$$

for every $x, y \in X$. If $D = 0$, then it is possible to take $L = 0$.

Proof. Note that for any $\lambda > 0$, $\Delta^{(c)}_X(R) < (C+\lambda)R$ for every $R \in (\frac{D}{\lambda}, \infty)$. Pick any $t > 1$, any $0 < \varepsilon < 1$, any $\lambda > 0$, and any point $O \in X$. Let $K = \frac{2t(C+\lambda)}{1-\varepsilon}$. Let $A = \left\{ n \in \mathbb{Z} \mid t^n > \frac{D}{\lambda} \right\}$. Then for each $n \in A$ there is a point-finite cover $\mathcal{U}_n = \{U_{n,\tau}\}_{\tau<\kappa}$ of X (one can take $|\mathcal{U}_n| \leqslant \kappa$ by Lemma 2) such that $\mathcal{L}(\mathcal{U}_n) \geqslant t^n$ and $\mathrm{diam}(\mathcal{U}_n) \leqslant (C+\lambda)t^n$. For each $n \in A$ and $\tau < \kappa$, let

$V_{n,\tau} = U_{n,\tau} \backslash B_{(C-1+\lambda)t^n/2}(O)$ and define $f_{n,\tau} \colon X \to \mathbb{R}^+$ by

$$f_{n,\tau}(x) = K \min \left\{ d_X(x, V_{n,\tau}^c), \frac{t^n}{2} \right\}$$

for each $x \in X$. Then for each $n \in A$ and $\tau < \kappa$, $f_{n,\tau}$ is Lipschitz with $\mathrm{Lip}(f) \leqslant K$ and bounded by $\frac{Kt^n}{2}$. Note that if $f_{n,\tau}(x) > 0$, then $x \in V_{n,\tau}$ and so $x \notin B_{(C-1+\lambda)t^n/2}(O)$. Thus $f_{n,\tau}$ is supported on the complement of $B_{(C-1+\lambda)t^n/2}(O)$. Therefore, by the bound on $f_{n,\tau}$ and the point-finiteness of $\mathcal{U}_n$, for fixed $x \in X$ and $\eta > 0$, the set

$$\{(n, \tau) \in A \times \kappa \mid f_{n,\tau}(x) > \eta\}$$

is finite. It follows that the map $f \colon X \to c_0^+(\kappa)$ defined by

$$f(x) = \sum_{(n,\tau) \in A \times \kappa} f_{n,\tau}(x) e_{n,\tau}$$

for every $x \in X$ (where $\{e_{n,\tau}\}_{(n,\tau) \in A \times \kappa}$ is any enumeration of the standard basis of $c_0(\kappa)$) is a well-defined Lipschitz map with $\mathrm{Lip}(f) \leqslant K$. Now fix $x, y \in X$ such that $d_X(x, y) > (C + \lambda) \inf\{t^n \mid n \in A\}$ and $d_X(x, O) \geqslant d_X(y, O)$. Let $n \in A$ be such that $(C + \lambda)t^n < d_X(x, y) \leqslant (C + \lambda)t^{n+1}$. Then by the triangle inequality,

$$d_X(x, O) > \frac{(C + \lambda)t^n}{2} = \frac{t^n}{2} + \frac{(C - 1 + \lambda)t^n}{2}$$

and so $B_{t^n/2}(x) \cap B_{(C-1+\lambda)t^n/2}(O) = \varnothing$. But $\mathcal{L}(\mathcal{U}_n) \geqslant t^n$, and so there is $\tau \in \kappa$ such that $B_{(1-\varepsilon)t^n/2}(x) \subseteq U_{n,\tau}$. Therefore $f_{n,\tau}(x) \geqslant K \frac{(1-\varepsilon)t^n}{2}$. Furthermore,

$$d_X(y, V_{n,\tau}) \geqslant d_X(x, y) - \mathrm{diam}(V_{n,\tau}) > (C + \lambda)t^n - (C + \lambda)t^n = 0$$

and so $f_{n,\tau}(y) = 0$. Thus

$$\begin{aligned}\|f(x) - f(y)\| \geqslant |f_{n,\tau}(x) - f_{n,\tau}(y)| &\geqslant \frac{K(1-\varepsilon)t^n}{2} \\ &= \frac{K(1-\varepsilon)}{2(C+\lambda)t}(C+\lambda)t^{n+1} \geqslant d_X(x,y).\end{aligned}$$

And so, for every $x, y \in X$,

$$d_X(x,y) - (C+\lambda)\inf\{t^n \mid n \in A\} \leqslant \|f(x) - f(y)\| \leqslant K d_X(x,y).$$

Since $t > 1$ and $0 < \varepsilon < 1$ were arbitrary, the theorem follows. □

Corollary 15. *A metric space (X, d_X) is coarse Lipschitzly embeddable into $c_0(\kappa)$ for some cardinality κ if and only if there are $C, D \in [0,\infty)$ such that $\Delta_X^{(c)}(R) \leqslant CR + D$ for all $R \in [0,\infty)$. A metric space (X, d_X) is bi-Lipschitzly embeddable into $c_0(\kappa)$ for some cardinality κ if and only if there is $C \in [0,\infty)$ such that $\Delta_X^{(c)}(R) \leqslant CR$ for all $R \in [0,\infty)$.*

Proof. The case when X is a finite metric space is trivial, so suppose X is an infinite metric space. If X coarse Lipschitzly embeds into $c_0(\kappa)$, then the implication follows from Corollary 13 and Proposition 11.

Conversely, if there are $C, D \in [0,\infty)$ such that $\Delta_X^{(c)}(R) \leqslant CR + D$ for all $R \in [0,\infty)$, then X coarse Lipschitzly embeds (bi-Lipschitzly embeds if $D = 0$) into $c_0^+(\mathrm{dens}(X))$ and hence into $c_0(\mathrm{dens}(X))$ by Theorem 14. □

Compare the above corollary to Pelant [8], who shows the uniform Stone property characterizes uniform embeddability of a metric space into $c_0(\kappa)$ for some κ; and to Baudier and Deville [4], who show property $\pi(\lambda)$ characterizes good-λ-embeddability of a separable metric space into c_0 (see [4] for the definitions). Lemma 6 and Corollary 15 together show that a metric space X's uniform, coarse Lipschitz, and bi-Lipschitz embeddability into $c_0^+(\kappa)$ for some κ can all be determined from the modulus $\Delta_X^{(c)}$. In light of this, it is natural to ask whether a metric space's coarse embeddability

into $c_0(\kappa)$ for some cardinality κ can similarly be determined from $\Delta_X^{(c)}$. Proposition 11 shows that the coarse Stone property is at least a necessary condition.

Corollary 16. *Let X be a normed linear space. The following are equivalent:*

(i) X coarsely embeds into $c_0(\kappa)$.

(ii) X has the coarse Stone property.

(iii) X bi-Lipschitzly embeds into $c_0(\kappa)$.

(iv) X uniformly embeds into $c_0(\kappa)$.

(v) X has the uniform Stone property.

Proof. (i) $\Rightarrow$ (ii): By Lemma 1, $\mathrm{dens}(X) \leqslant \kappa$. By Corollary 13 and Proposition 11, X has the coarse Stone property.

(ii) $\Rightarrow$ (iii): By Lemma 9, there is $C \in [0, \infty)$ such that $\Delta_X^{(c)}(R) \leqslant CR$ for all $R \in [0, \infty)$. And so X bi-Lipschitzly embeds into $c_0(\kappa)$ by Theorem 14.

(iii) $\Rightarrow$ (iv): Clear.

(iv) $\Rightarrow$ (v): By Corollary 13 and Proposition 10, X has the uniform Stone property.

(v) $\Rightarrow$ (i): By Lemma 9, there is $C \in [0, \infty)$ such that $\Delta_X^{(c)}(R) \leqslant CR$ for all $R \in [0, \infty)$. And so X bi-Lipschitzly embeds (and therefore coarsely embeds) into $c_0(\kappa)$ by Theorem 14. $\square$

Kalton [5] has shown that coarse/uniform/Lipschitz embeddability into ℓ_∞ are also equivalent notions for normed linear spaces. So far ℓ_∞ and $c_0(\kappa)$ seem to be the only spaces known to have this property, and given that $c_0(\mathfrak{c})$ (where $\mathfrak{c}$ is the cardinality of the continuum) bi-Lipschitzly embeds into ℓ_∞ (see [2]), one might ask whether the ℓ_∞ case actually follows from the $c_0(\kappa)$ case. That is, can one find a bi-Lipschitz embedding of ℓ_∞ into $c_0(\mathfrak{c})$? Equivalently, does ℓ_∞ have the coarse (or uniform) Stone property? Pelant [7] and Shchepin [9] have shown that $\ell_\infty(\Gamma)$ fails to have the uniform Stone property when $|\Gamma|$ is large enough, but to the author's knowledge, the minimal cardinality is unknown.

2.4 Spaces with the coarse Stone property

In this subsection, we show directly that certain classes of metric spaces have the coarse Stone property. In each of the examples given, $C \in [0, \infty)$ is found such that $\Delta_X^{(c)}(R) \leqslant CR$ for all $R \in [0, \infty)$ and so one can use Theorem 14 to estimate how well X bi-Lipschitzly embeds into some $c_0^+(\kappa)$. Recall that a metric space is called *locally finite* if every bounded set is finite.

Proposition 17. *If (X, d_X) is a locally finite metric space, then $\Delta_X^{(c)}(R) \leqslant R$ for all $R \in [0, \infty)$. Consequently, every locally finite metric space $(2+\varepsilon)$-Lipschitzly embeds into c_0^+ for all $\varepsilon > 0$.*

Proof. Fix $R \in [0, \infty)$. Let $\mathcal{U} = \{U \subseteq X \mid \operatorname{diam}(U) < R\}$. Then $\mathcal{U}$ is a cover of X such that $\mathcal{L}(\mathcal{U}) \geqslant R$ and $\operatorname{diam}(\mathcal{U}) \leqslant R$. Now take any $x \in X$ and suppose $x \in U$. Then $d(x, y) < R$ for all $y \in U$, and so $U \subseteq B_R(x)$. But $|B_R(x)| < \infty$ since X is locally finite. Thus, since there are only $2^{|B_R(x)|} < \infty$ many $U \subseteq X$ such that $U \subseteq B_R(x)$, there are only finitely many $U \in \mathcal{U}$ such that $x \in U$. This means $\mathcal{U}$ is point-finite, and thus, $\Delta_X^{(c)}(R) \leqslant R$. $\square$

Note that Proposition 17 actually recovers the best distortion for embedding the class of locally finite metric spaces into c_0^+ (found by Kalton and Lancien [6]). The author does not know whether the same bound holds for $\Delta_X^{(c)}$ when X is an arbitrary *proper* metric space (that is, a metric space whose balls are all relatively compact).

Proposition 18. *If (X, d_X) is a separable metric space, then $\Delta_X^{(c)}(R) \leqslant 2R$ for all $R \in [0, \infty)$. Consequently, every separable metric space $(4+\varepsilon)$-Lipschitzly embeds into c_0^+ for all $\varepsilon > 0$.*

Proof. Take any $r > 0$ and any $0 < \varepsilon < \frac{r}{2}$. Let $\{x_n\}_{n=1}^{\infty}$ be a dense subset of X. For each $n \in \mathbb{N}$, let $U_n = B_{\frac{r}{2}}(x_n) \setminus \bigcup_{j=1}^{n-1} B_\varepsilon(x_j)$. Then $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$ is a cover of X such that $\operatorname{diam}(\mathcal{U}) \leqslant r$. Now fix $x \in X$ and suppose $n \in \mathbb{N}$ is such that $d_X(x, x_n) < \varepsilon$. If $x \in U_j$, then $j \leqslant n$ by the way $\mathcal{U}$ was defined. Therefore $x \in U_j$ for only finitely many $j \in \mathbb{N}$. Thus, $\mathcal{U}$ is point-finite. Now suppose $A \subseteq X$ is such that $\operatorname{diam}(A) < \frac{r}{2} - \varepsilon$. Let $m = \min\{j \in \mathbb{N} \mid d_X(x_j, A) < \varepsilon\}$. Then for each $y \in A$, $d_X(x_m, y) \leqslant d_X(x_m, A) + \operatorname{diam}(A) < \frac{r}{2}$ and $d_X(x_j, y) \geqslant \varepsilon$ for all $j < m$. Thus $A \subseteq U_m$, and therefore $\mathcal{L}(\mathcal{U}) \geqslant \frac{r}{2} - \varepsilon$. Since $0 < \varepsilon < \frac{r}{2}$ was arbitrary, $\Delta_X^{(u)}(r) \geqslant \frac{1}{2}r$ for all $r > 0$. By Lemma 8, $\Delta_X^{(c)}(R) \leqslant 2R$ for all $R \in [0, \infty)$. $\square$

Note that the 2 in Proposition 18 is optimal by Corollary 13. At this point, it should be remarked that $\Delta_{\ell_p}^{(c)}(R) \geqslant \frac{(2^p+1)^{1/p}}{2} R$ for every $p \in [1, \infty)$ and $R \in [0, \infty)$. This follows from Theorem 14 and Kalton and Lancien [6], who show that the best possible bi-Lipschitz embedding of ℓ_p into c_0^+ has distortion $(2^p + 1)^{1/p}$.

Definition 19. A metric space (T, d_T) is called an $\mathbb{R}$-*tree* if it satisfies the following conditions:

(i) For any $s, t \in T$, there exists a unique isometric embedding $\phi_{s,t}\colon\ [0, d_T(s,t)] \to T$ such that $\phi_{s,t}(0) = s$ and $\phi_{s,t}(d_T(s,t)) = t$.

(ii) Any injective continuous mapping $\varphi\colon [0,1] \to T$ has the same range as $\phi_{\varphi(0),\varphi(1)}$.

A *rooted* $\mathbb{R}$-*tree* is an $\mathbb{R}$-tree T paired with a point $t_0 \in T$, and in this case t_0 is called the *root* of T. Given $t_1, t_2 \in T$, a point $s \in T$ is said to be *between* t_1 and t_2 if $s = \phi_{t_1,t_2}(x)$ for some $x \in [0, d_T(t_1, t_2)]$. Given a nonempty subset A of a rooted $\mathbb{R}$-tree (T, t_0), a point $s \in T$ is called a *common ancestor* of A if s is between t_0 and t for all $t \in A$, and is called the (necessarily unique) *last common ancestor* of A if $s = \phi_{t_0,t}(\max\{x \in [0, d_T(t_0,t)] \mid \phi_{t_0,t}(x) \text{ is a common ancestor of } A\})$ for some $t \in A$. One can think of an $\mathbb{R}$-tree as being a graph-theoretical tree with the edges "filled in".

Proposition 20. *If (T, d_T) is an $\mathbb{R}$-tree (possibly non-separable), then $\Delta_T^{(c)}(R) \leqslant 2R$ for all $R \in [0, \infty)$. Consequently, every $\mathbb{R}$-tree $(4+\varepsilon)$-Lipschitzly embeds into $c_0^+(\kappa)$ for some κ for all $\varepsilon > 0$.*

Proof. Pick any $t_0 \in T$ to be the root. Fix $R \in [0, \infty)$ and take any $n \in \mathbb{N}$. For each $t \in T$, let

$$U_t = \left\{ s \in T \mid t \text{ is between } t_0 \text{ and } s \text{ and } d_T(t,s) \leqslant R + \frac{1}{n} \right\}.$$

For each $m \in \mathbb{N} \cup \{0\}$, let $\mathcal{U}_m = \left\{ U_t \mid d_T(t_0, t) = \frac{m}{n} \right\}$. Let $\mathcal{U} = \bigcup_{m=0}^{\infty} \mathcal{U}_m$. Then for fixed $s \in T$, there are at most $n\lceil R \rceil + 1$ many $U \in \mathcal{U}$ such that $s \in U$. Thus $\mathcal{U}$ is a point-finite cover of T such that $\operatorname{diam}(\mathcal{U}) \leqslant 2(R + \frac{1}{n})$. Now take any $A \subseteq T$ with $\operatorname{diam}(A) < R$. Then if t is the last common ancestor of all the points in A and $t' = \phi_{t_0,t}(\max\{\frac{m}{n} \mid m \in \mathbb{N} \cup \{0\} \text{ and } \frac{m}{n} \leqslant d_T(t_0,t)\})$, then

$A \subseteq U_{t'} \in \mathcal{U}$. This means $\mathcal{L}(\mathcal{U}) \geqslant R$. Thus, since $n \in \mathbb{N}$ was arbitrary, $\Delta_T^{(c)}(R) \leqslant 2R$ for all $R \in [0, \infty)$. $\square$

Proposition 21. *Given $N \in \mathbb{N}$, $\Delta_{\ell_\infty^N}^{(c)}(R) = R$ for all $R \in [0, \infty)$.*

Proof. Take any $n \in \mathbb{N}$. For each $x \in \mathbb{Z}^N$, let

$$U_x = \left\{ f \in \ell_\infty^N \mid f(j) \in \frac{x_j}{n} + \left(-1, 1 + \frac{1}{n}\right) \right\}.$$

Then for fixed $f \in \ell_\infty^N$, there are at most $(2n+1)^N$ many $x \in \mathbb{Z}^N$ such that $f \in U_x$. Let $\mathcal{U} = \{U_x \mid x \in \mathbb{Z}^N\}$. Then by the above, $\mathcal{U}$ is a point-finite cover of ℓ_∞^N refined by $\{B_1(f)\}_{f \in \ell_\infty^N}$ such that $\operatorname{diam}(\mathcal{U}) = 2 + \frac{1}{n}$. Since every $A \subseteq \ell_\infty^N$ such that $\operatorname{diam}(A) < 2$ is contained in a ball of radius 1 (centered at $\left(\frac{\sup(\pi_j(A)) - \inf(\pi_j(A))}{2}\right)_{j=1}^N$, where π_j is the j-th coordinate functional), this means $\mathcal{L}(\mathcal{U}) \geqslant 2$. Thus, since $n \in \mathbb{N}$ was arbitrary, $\Delta_{\ell_\infty^N}^{(c)}(2) \leqslant 2$. By Lemma 7 and (the proof of) Lemma 9, $\Delta_{\ell_\infty^N}^{(c)}(R) = R$ for all $R \in [0, \infty)$. $\square$

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3. COARSE EMBEDDINGS INTO SUPERSTABLE SPACES

3.1 Introduction

D. J. Aldous showed in Theorem 1.1 of [2] that every infinite-dimensional subspace of L_1 contains an isomorphic copy of ℓ_p, for some $p \in [1, \infty)$. In order to generalize Aldous's result, J. L. Krivine and B. Maurey [10] introduced the notion of stable Banach space. A metric space (M, d) is called *stable* if

$$\lim_{i,\mathcal{U}} \lim_{j,\mathcal{V}} d(x_i, y_j) = \lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} d(x_i, y_j)$$

for all bounded sequences $(x_i)_{i=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ in M, and all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over $\mathbb{N}$. A Banach space is called *stable* if it is stable as a metric space with the metric induced by its norm. Stability for general metric spaces seems to have first been defined by D. J. H. Garling [6]. Krivine and Maurey showed in Theorem IV.1 of [10] that every stable Banach space contains an isomorphic copy of ℓ_p for some $p \in [1, \infty)$. As L_p is stable for all $p \in [1, \infty)$ (see Theorem II.2 of [10]), this is a generalization of Aldous's result.

Krivine and Maurey's result can be extended to the nonlinear setting as follows. Let (M, d_M) and (N, d_N) be metric spaces. Given $f\colon M \to N$, define $\omega_f\colon [0, \infty) \to [0, \infty]$ and $\rho_f\colon [0, \infty) \to [0, \infty]$ by

$$\omega_f(t) = \sup\{d_N(f(x), f(y)) \mid d_M(x, y) \leqslant t\} \tag{3.1}$$

and

$$\rho_f(t) = \inf\{d_N(f(x), f(y)) \mid d_M(x, y) \geqslant t\} \tag{3.2}$$

for all $t \in [0, \infty)$. The function f is called a *uniform embedding* (in which case M is said to be *uniformly embeddable* into N) if $\lim_{t \to 0^+} \omega_f(t) = 0$ (i.e., if f is uniformly continuous) and $\rho_f(t) > 0$ for all $t \in [0, \infty)$ (i.e., f^{-1} exists and is uniformly continuous). The function f is called a *uniform equivalence* (in which case M is said to be *uniformly equivalent* to N) if f is a uniform embedding and is surjective. A Banach space X is said to be *superstable* if every Banach space

that is finitely representable in X is also stable. Y. Raynaud showed in the corollary of Theorem 0.2 of [13] that if a Banach space is uniformly embeddable into a superstable Banach space, then X contains an isomorphic copy of ℓ_p for some $p \in [1, \infty)$. As L_p is superstable for all $p \in [1, \infty)$ (see Theorem 0.1 of [13]), this is a generalization of Krivine and Maurey's result.

The interest in Banach spaces as metric spaces and their nonlinear geometric properties has recently increased significantly, hence the question whether analogues of Raynaud's result hold for different kinds of nonlinear embeddings other than uniform embeddings becomes natural. Given metric spaces (M, d_M) and (N, d_N), a function $f\colon M \to N$ is said to be *expanding* if $\lim_{t\to\infty} \rho_f(t) = \infty$ and is said to be *coarse* if $\omega_f(t) < \infty$ for all $t \in [0, \infty)$. If f is both expanding and coarse, then f is called a *coarse embedding* (in which case M is said to be *coarsely embeddable* into N). If f is a coarse embedding and $\sup_{y\in N} d_N(y, f(M)) < \infty$, then f is called a *coarse equivalence* (in which case M is said to be *coarsely equivalent* to N). In Problem 6.6 of [8], N. J. Kalton asked: If a Banach space X is coarsely embeddable into a superstable Banach space, must X contain an isomorphic copy of ℓ_p for some $p \in [1, \infty)$? Although we are not able to answer Kalton's question, we obtain the following result.

Theorem 3.7.4. *If a Banach space X is coarsely embeddable into a superstable Banach space, then X has a basic sequence that generates a spreading model isomorphic to ℓ_p for some $p \in [1, \infty)$.*

Kalton proved in Theorem 2.1 of [8] that every stable metric space is both uniformly and coarsely embeddable into some reflexive Banach space (and this can be witnessed by a single function). In Problem 6.1 of [8], Kalton asked: Is every (separable) reflexive Banach space coarsely (or uniformly) embeddable into a stable metric space? By Raynaud's result, it is clear that there are separable reflexive spaces that are not uniformly embeddable into any superstable Banach space. However, to the best of our knowledge, it was unknown whether every reflexive Banach space is coarsely embeddable into some superstable Banach space. As a corollary of Theorem 3.7.4, we obtain the following.

Corollary 3.7.6. *There are separable reflexive Banach spaces that are not coarsely embeddable*

into any superstable Banach space.

We now describe the organization of this section. In Section 3.2, we recall all the background that will be needed and prove two key lemmas, Lemma 3.2.1 and Lemma 3.2.5, that we could not find elsewhere in the literature. In Section 3.3, we define the space of types that we will deal with, along with its operations of dilation and convolution and the relevant notions of admissibility and symmetry, and then derive some of its basic properties. In Section 3.4, we define conic class and show both the existence of a minimal closed admissible conic class and the existence of a common admissible point of continuity for the family of finitely many applications of dilation and convolution within every closed admissible conic class. In Section 3.5, we discuss how to associate a spreading model to a well-chosen admissible symmetric type and show how inequalities involving the spreading model relate to inequalities involving the type. In Section 3.6, we use what we call "coarse approximating sequences" to derive an inequality that allows long convolutions to be shortened. Finally, in Section 3.7 we define what it means for a type to be a coarse ℓ_p-type and show that every minimal closed admissible conic class must contain such a type. From the work done in preceding sections, the main theorem follows.

The contents of this section were jointly researched with B. M. Braga and can be found in a separate preprint on the arXiv [5].

3.2 Preliminaries

We let $\mathbb{N} = \{n\}_{n=1}^{\infty}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{R}_+ = [0, \infty)$, and $\mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$. The Banach space notation we use is standard, and we refer the reader to [1] if review is necessary. For instance, we denote the closed unit ball of a Banach space X by B_X. Also, given $p \in [1, \infty]$ and $\overline{x} = (x_i)_{i=1}^N \subseteq \mathbb{R}$, we let $\|\overline{x}\|_p = (\sum_{i=1}^N |x_i|^p)^{1/p}$ and $\|\overline{x}\|_\infty = \max\{|x_i| \mid 1 \leqslant i \leqslant N\}$.

We define stability for metric spaces and superstability for Banach spaces as in Section 3.1. By Theorem II.1 of [10] and Theorem 0.1 of [13], both stability and superstability are closed under taking ℓ_p-sums, for $p \in [1, \infty)$. Precisely, given $p \in [1, \infty)$ and a stable (respectively, superstable) Banach space X, $\ell_p(X)$ is also stable (respectively, superstable). We will use this property without mention. In particular, ℓ_p is superstable for every $p \in [1, \infty)$. Note however that c_0 is not even

coarsely or uniformly embeddable into a stable metric space (see [8]).

We say that (M, d_M) is a *pseudometric space* if $d_M \colon M \times M \to \mathbb{R}$ is a *pseudometric*, i.e., if d is a nonnegative symmetric map satisfying the triangle inequality. We define stability for pseudometric spaces analogously to stability for metric spaces. Given pseudometric spaces (M, d_M) and (N, d_N), we define ω_f and ρ_f for a function $f \colon M \to N$ by the formulas given in Equations 3.1 and 3.2, and define uniform and coarse embedding and equivalence for pseudometric spaces analogously to the same terminology for metric spaces found in Section 3.1. Two pseudometrics d_M and d'_M defined for the same set M are said to be *coarsely equivalent* if the identity map $\mathrm{Id} \colon (M, d_M) \to (M, d'_M)$ is a coarse equivalence.

A Banach space S is called a *sequence space* if it is the completion of c_{00} under some norm such that the standard basis vectors $(\zeta_n)_{n=1}^{\infty}$ of c_{00} each have norm one. Let $(X, \|\cdot\|)$ be a Banach space and $(x_n)_{n=1}^{\infty}$ a bounded sequence in X without Cauchy subsequences. Then, after possibly taking a subsequence of $(x_n)_{n=1}^{\infty}$, there exists a sequence space $(S, \|\cdot\|_S)$ such that for all $(\alpha_j)_{j=1}^{k} \subseteq \mathbb{R}$,

$$\left\| \sum_{j=1}^{k} \alpha_j \zeta_j \right\|_S = \lim_{n_1 < \cdots < n_k} \left\| \sum_{j=1}^{k} \alpha_j x_{n_j} \right\|,$$

where for a function $f \colon \mathbb{N}^k \to \mathbb{R}$, $\lim_{n_1 < \cdots < n_k} f(n_1, \ldots, n_k)$, when it exists, denotes the unique $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|f(n_1, \ldots, n_k) - L| < \varepsilon$ whenever $N \leqslant n_1 < \cdots < n_k$. For a proof of this, see Theorem 11.3.7 in [1]. The space S is called a *spreading model* for $(x_n)_{n=1}^{\infty}$. Within a spreading model, the sequence $(\zeta_n)_{n=1}^{\infty}$ is 1-*spreading*, i.e., $(\zeta_n)_{n=1}^{\infty}$ is 1-equivalent to all of its subsequences. Also, the sequence $(\zeta_{2n-1} - \zeta_{2n})_{n=1}^{\infty}$ is 1-unconditional (see Proposition II.3.3 of [7]).

Let $(X, \|\cdot\|)$ be a Banach space, I an index set, and $\mathcal{U}$ a nonprincipal ultrafilter over I. We define the *ultrapower* $X^I/\mathcal{U}$ of X with respect to $\mathcal{U}$ as the set

$$\left\{ (x_i)_{i \in I} \in X^I \mid \sup_{i \in I} \|x_i\| < \infty \right\} / \sim,$$

where $(x_i)_{i \in I} \sim (y_i)_{i \in I}$ if $\lim_{i,\mathcal{U}} \|x_i - y_i\| = 0$, equipped with the norm $\|\cdot\|_{X^I/\mathcal{U}}$ defined by

$\|(x_i)_{i\in I}\|_{X^I/\mathcal{U}} = \lim_{i,\mathcal{U}} \|x_i\|$ for all equivalence classes $(x_i)_{i\in I}$. Every ultrapower $X^I/\mathcal{U}$ of a Banach space X is finitely representable in X (see Proposition 11.1.12(i) of [1]). On the other hand, if a separable Banach space Y is finitely representable in X, then Y is isometrically isomorphically embeddable into some ultrapower of X (see Proposition 11.1.12(ii) of [1]). Therefore a Banach space X is superstable if and only if all of its ultrapowers are stable.

Given a coarse map $f\colon X \to Y$ between Banach spaces, we would like to be able to modify f so that it has the additional property that the difference of the images of two points in X has the same norm as the image of the difference. In the lemma below, we use Markov-Kakutani's fixed-point theorem to show that if we allow ourselves to substitute Y with an ultrapower of the ℓ_1-sum of Y, then such a modification is possible. Precisely, we have the following.

Lemma 3.2.1. *Let X and Y be Banach spaces and $f\colon X \to Y$ a coarse map. Then there exists a nonprincipal ultrafilter $\mathcal{U}$ on an index set I, and a map $F\colon X \to \ell_1(Y)^I/\mathcal{U}$, such that for all $x, y \in X$,*

$$\rho_f(\|x-y\|) \leqslant \|F(x)-F(y)\| = \|F(x-y)\| \leqslant \omega_f(\|x-y\|).$$

Proof. Let

$$C = \prod_{(x,y)\in X\times X} [\rho_f(\|x-y\|), \omega_f(\|x-y\|)].$$

We consider C as a subset of $\mathbb{R}^{X\times X}$ with its pointwise convergence topology. By the assumption that f is coarse and Tychonoff's theorem, C is compact. Let $d\colon X \times X \to \mathbb{R}$ be defined by $d(x,y) = \|f(x)-f(y)\|$ for all $x, y \in X$ and note that $d \in C$.

For each $z \in X$, define $\hat{z}\colon \mathbb{R}^{X\times X} \to \mathbb{R}^{X\times X}$ by $\hat{z}(g)(x,y) = g(x+z, y+z)$ for all $g \in \mathbb{R}^{X\times X}$ and all $x, y \in X$. Let $A = \overline{\mathrm{conv}}\{\hat{z}(d) \mid z \in X\} \subseteq \mathbb{R}^{X\times X}$. Then $A \subseteq C$ by the definition of the pointwise convergence topology on $\mathbb{R}^{X\times X}$. The family $\{\hat{z}\restriction_A\}_{z\in X}$ is easily seen to be a commuting family of continuous, affine self-mappings of the compact convex subset A of $\mathbb{R}^{X\times X}$. By Markov-Kakutani's fixed-point theorem, there exists $D \in A$ such that $\hat{z}(D) = D$ for all $z \in X$.

That is, $D(x+z, y+z) = D(x,y)$ for all $x, y, z \in X$. By the definition of A, there is a set $\{D_i\}_{i\in I} \subseteq \operatorname{conv}\{\hat{z}(d) \mid z \in X\}$ and a nonprincipal ultrafilter $\mathcal{U}$ over I such that $D = \lim_{i\in\mathcal{U}} D_i$. For each $i \in I$, let $(\alpha_{i,j})_{j=1}^{s(i)} \subseteq [0,1]$ and $(z_{i,j})_{j=1}^{s(i)} \subseteq X$ be such that $\sum_{j=1}^{s(i)} \alpha_{i,j} = 1$ and $D_i = \sum_{j=1}^{s(i)} \alpha_{i,j}\hat{z}_{i,j}(d)$. Then for each $i \in I$, define $F_i \colon X \to \ell_1(Y)$ by $F_i(x) = (F_{i,j}(x))_{j=1}^{\infty}$ for each $x \in X$, where

$$F_{i,j}(x) = \begin{cases} \alpha_{i,j}(f(x+z_{i,j}) - f(z_{i,j})) & 1 \leqslant j \leqslant s(i) \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in X$ and $j \in \mathbb{N}$. Finally, define $F \colon X \to \ell_1(Y)^I/\mathcal{U}$ by $F(x) = (F_i(x))_{i\in I}$. As $\sup_{i\in I} \|F_i(x)\|_{\ell_1(Y)} \leqslant \omega_f(\|x\|)$ for all $x \in X$, the map F is well-defined. And by definition of d, F, and the norm on $\ell_1(Y)^I/\mathcal{U}$,

$$\|F(x) - F(y)\|_{\ell_1(Y)^I/\mathcal{U}} = D(x,y)$$

for all $x, y \in X$. Therefore, as $D(x,y) = D(x-y, 0)$ for all $x, y \in X$, and $F(0) = 0$, we are finished. □

Corollary 3.2.2. *Let $(X, \|\cdot\|)$ be a Banach space and Y a superstable Banach space. If $f \colon X \to Y$ is a coarse map, then there exists a translation-invariant stable pseudometric d on X such that $\rho_f(\|x-y\|) \leqslant d(x,y) \leqslant \omega_f(\|x-y\|)$ for all $x, y \in X$. In particular, if X is coarsely embeddable into a superstable Banach space, then there exists a translation-invariant stable pseudometric d on X such that the identity map $\mathrm{Id} \colon (X, \|\cdot\|) \to (X, d)$ is a coarse equivalence.*

Proof. Let $F \colon X \to \ell_1(Y)^I/\mathcal{U}$ be obtained from Lemma 3.2.1 applied to f. Define a map $d \colon X \times X \to \mathbb{R}_+$ by $d(x,y) = \|F(x) - F(y)\|_{\ell_1(Y)^I/\mathcal{U}}$ for all $x, y \in X$. It can easily be seen that d is a translation-invariant pseudometric on X and that $\rho_f(\|x-y\|) \leqslant d(x,y) \leqslant \omega_f(\|x-y\|)$ for all $x, y \in X$. As $\ell_1(Y)^I/\mathcal{U}$ is stable, it follows that d is a stable pseudometric. Furthermore, if f is a coarse embedding, then $\mathrm{Id} \colon (X, \|\cdot\|) \to (X, d)$ is a coarse equivalence since $\rho_f(t) \leqslant \rho_{\mathrm{Id}}(t)$ and $\omega_{\mathrm{Id}}(t) \leqslant \omega_f(t)$ for all $t \in [0, \infty)$. □

The corollary above is analogous to Theorem 0.2 of [13], which says that if a Banach space $(X, \|\cdot\|)$ is uniformly embeddable into a superstable Banach space, then X has a translation-invariant stable metric that is uniformly equivalent to the metric induced by $\|\cdot\|$. However, Raynaud's proof relies on an averaging process that uses the uniform continuity of a given uniform embedding. Through the use of Markov-Kakutani's fixed-point theorem, we have proved something more general, as it can be easily shown using the triangle inequality that uniformly continuous maps between Banach spaces are automatically coarse.

Remark 3.2.3. Although not necessary for the main result, Corollary 3.2.2 can actually be improved to show the existence of a translation-invariant stable *metric* on X coarsely equivalent to the metric induced by the norm. Indeed, it has been shown by B. M. Braga in Theorem 1.6 of [4] that if X and Y are Banach spaces and $f\colon X \to Y$ is a coarse embedding, then there is a coarse embedding $\hat{f}\colon X \to \ell_1(Y)$ with uniformly continuous inverse (meaning $\rho_{\hat{f}}(t) > 0$ whenever $t > 0$). Thus, the same proof as in Corollary 3.2.2 with $\ell_1(Y)$ replacing Y and $\hat{f}$ replacing f will yield that $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ is a coarse embedding with uniformly continuous inverse. In particular, d is a metric because in this case $d(x, y) = 0$ implies $x = y$.

Let X and Y be metrizable topological spaces. Recall that a subset of a topological space is called F_σ if it is the countable union of closed sets, is called G_δ if it is the countable intersection of open sets, and is called *comeager* if it is the countable intersection of sets with dense interiors. A function $f\colon X \to Y$ is called *Baire class 1* if the inverse image of any open subset of Y under f is an F_σ subset of X. If Y is separable, and f is Baire class 1, then the set of points of continuity for f is a comeager G_δ subset of X. If Y is separable and $(f_n\colon X \to Y)_{n=1}^\infty$ is a sequence of Baire class 1 functions, then $(f_n)_{n=1}^\infty\colon X \to Y^{\mathbb{N}}$ is a Baire class 1 function. The pointwise limit of a sequence of continuous functions from X to Y is a Baire class 1 function. The restriction of a Baire class 1 function is a Baire class 1 function. For proofs of these facts and more info about Baire class 1 functions, see [9] and [11].

Lemma 3.2.4. *Let X be a metrizable σ-compact topological space, Y a topological space, and let $f\colon X \times Y \to \mathbb{R}$ be separately continuous. Given a metric d inducing the topology of X and a*

countable family $\mathcal{K}$ of compact subsets of X such that $X = \bigcup_{K \in \mathcal{K}} K$; if there is $\delta > 0$ such that for each $x \in X$, $B_\delta(x) \cap K \neq \varnothing$ for only finitely many $K \in \mathcal{K}$, then f is the pointwise limit of a sequence of continuous functions.

Proof. For each $n \in \mathbb{N}$, let $\{x_{n,i}\}_{i=1}^{\infty}$ be a $\frac{\delta}{2(n+1)}$-dense set in (X, d) such that $\left|\{x_{n,i}\}_{i=1}^{\infty} \cap K\right| < \infty$ for every $K \in \mathcal{K}$. For each $n, i \in \mathbb{N}$, define $g_{n,i} \colon X \to \mathbb{R}_+$ by $g_{n,i}(x) = \max\left\{\frac{\delta}{n+1} - d\left(x_{n,i}, x\right), 0\right\}$ for every $x \in X$. Note that $g_{n,i}$ is continuous and given $x \in X$, $g_{n,i}\restriction_{B_{\delta/2}(x)}$ is a nonzero function for some but only finitely many $i \in \mathbb{N}$. Thus the function $h_{n,i} := \frac{g_{n,i}}{\sum_{j=1}^{\infty} g_{n,j}}$ is well-defined and continuous. For each $n \in \mathbb{N}$, define $f_n \colon X \times Y \to \mathbb{R}$ by

$$f_n(x, y) = \sum_{i=1}^{\infty} f\left(x_{n,i}, y\right) h_{n,i}(x)$$

for every $(x, y) \in X \times Y$ and note that f_n is itself continuous by the separate continuity of f and the observation on $g_{n,i}\restriction_{B_{\delta/2}(x)}$. The sequence $(f_n)_{n=1}^{\infty}$ converges pointwise to f. Indeed, take any $(x, y) \in X \times Y$ and any $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $|f(x, y) - f(x', y)| < \varepsilon$ when $d(x, x') < \frac{\delta}{N}$. Then, for $n \geqslant N$,

$$\begin{aligned}
|f(x, y) - f_n(x, y)| &= \left|\sum_{i=1}^{\infty} \left(f(x, y) - f\left(x_{n,i}, y\right)\right) h_{n,i}(x)\right| \\
&\leqslant \sum_{i=1}^{\infty} \left|f(x, y) - f\left(x_{n,i}, y\right)\right| h_{n,i}(x) \\
&< \varepsilon \cdot \sum_{i=1}^{\infty} h_{n,i}(x) \\
&= \varepsilon.
\end{aligned}$$

□

Given a set X and a family $\mathcal{F}$ of functions from $X \times X$ to X, define the sequence $\left(\mathcal{F}^{[k]}\right)_{k=1}^{\infty}$ of subsets of X^X recursively by

$$\begin{aligned}
\mathcal{F}^{[0]} &= \{x \mapsto x\} \\
\mathcal{F}^{[k+1]} &= \left\{x \mapsto f(x, g(x)) \mid f \in \mathcal{F}, g \in \mathcal{F}^{[k]}\right\}.
\end{aligned}$$

The following lemma will be needed for the proof of Lemma 3.4.5, which is essential for the proof of Theorem 3.7.4.

Lemma 3.2.5. *Let X be a separable metric space and $\mathcal{F}$ a countable family of Baire class 1 functions from $X \times X$ to X. There is a comeager G_δ subset E of X such that g is continuous on E for all $g \in \bigcup_{k=1}^{\infty} \mathcal{F}^{[k]}$.*

Proof. Certainly, g is continuous on $E_0 = X$ for $g \in \mathcal{F}^{[0]}$. Suppose $k \in \mathbb{N}_0$ is such that there is a comeager G_δ subset E_k of X such that g is continuous on E_k for all $g \in \mathcal{F}^{[k]}$. For each $g \in \mathcal{F}^{[k]}$, let $\Gamma_g = \{(x, g(x)) \mid x \in E_k\}$. Since $\mathcal{F}$ is a countable family of Baire class 1 functions with separable codomain X, there is a comeager G_δ subset F_g of Γ_g such that $f\restriction_{\Gamma_g}$ is continuous on F_g for all $f \in \mathcal{F}$. Let $\pi\colon X \times X \to X$ be the first coordinate projection. Consider $U = \Gamma_g \cap V \times W$, where V, W are open subsets of X; and suppose $x \in \pi(U)$, so that $(x, g(x)) \in U$. As W is open and $g(x) \in W$, there is $r_1 > 0$ such that $B_{r_1}(g(x)) \subseteq W$. Since g is continuous on E_k, there is $r_2 > 0$ such that $g(B_{r_2}(x) \cap E_k) \subseteq B_{r_1}(g(x))$. Thus $(V \cap B_{r_2}(x)) \cap E_k$ is an open neighborhood of x in E_k contained in $\pi(U)$. Since $x \in \pi(U)$ was arbitrary, $\pi(U)$ is open in E_k. And U was an arbitrary element in a basis for the topology on Γ_g, so $\pi(U)$ is open in E_k whenever U is open in Γ_g. It follows easily that $\pi(F_g)$ is a comeager G_δ subset of E_k since F_g is a comeager G_δ subset of Γ_g. Let $E_{k+1} = \bigcap_{g \in \mathcal{F}^{[k]}} \pi(F_g)$. Since $\mathcal{F}^{[k]}$ is countable, E_{k+1} is a comeager G_δ subset of E_k, and therefore also of X, since E_k is a comeager G_δ subset of X. Now take any $g \in \mathcal{F}^{[k+1]}$. Then there is $f \in \mathcal{F}$ and $g' \in \mathcal{F}^{[k]}$ such that $g(x) = f(x, g'(x))$ for all $x \in X$. And if $x \in E_{k+1}$, then by construction x is a point of continuity for g' and $(x, g'(x))$ is a point of continuity for $f\restriction_{\Gamma_{g'}}$. Therefore x is a point of continuity for g. Thus, we have constructed a comeager G_δ subset E_{k+1} of E_k such that g is continuous on E_{k+1} for all $g \in \mathcal{F}^{[k+1]}$. And so we can recursively define such E_k for all $k \in \mathbb{N}$. The result follows by taking $E = \bigcap_{k=0}^{\infty} E_k$. $\square$

3.3 Space of types

Following Raynaud, our strategy for proving Theorem 3.7.4 is to first make an appropriate definition for the space of types of a Banach space coarsely embeddable into a superstable Banach

space. This space of types needs to have certain compactness properties and needs to be able to not only reflect the metric structure of the Banach space, but also the algebraic structure. Using compactness and methods commonly employed in the proof of Krivine's theorem, we'll be able to show the existence of a type that satisfies a nice ℓ_p-inequality, and then push this back down onto the Banach space.

To motivate the definition of the space of types, first consider a general metric space (M, d). One may ask whether M can be compactified in a way that preserves the metric structure on M. That is, under what conditions will there exist a compact metrizable space $\mathcal{T}$ such that M homeomorphically maps onto a dense subset of $\mathcal{T}$? Separability is certainly a necessary condition, and given that $\mathrm{Lip}_1(M)$ (the space of all real-valued Lipschitz functions over M with Lipschitz constant less than or equal to 1) is metrizable and closed in $\mathbb{R}^M$ under the pointwise-convergence topology when M is separable, a natural σ-locally compact metrizable $\mathcal{T}$ that contains a dense homeomorphic copy of M is the closure of $\{\overline{x}\}_{x\in M}$ in $\mathbb{R}^M$, where $\overline{x}$ is defined for all $x \in M$ by $\overline{x}(y) = d(x, y)$ for all $y \in M$. If d is a bounded metric, then $\mathcal{T}$ is in fact compact, and since every topology induced by a metric can be induced by a bounded metric, separability is also a sufficient condition.

Supposing now that M is a vector space, and $\lim_{n\to\infty} \overline{x}_n$, $\lim_{n\to\infty} \overline{y}_n$ both exist, one may further ask under what conditions do $\lim_{n\to\infty} \overline{(x_n + y_n)}$ and $\lim_{n\to\infty} \overline{(\alpha x_n)}$ exist, where α is some scalar. Stability of d is enough to show the existence of $\lim_{m\to\infty} \lim_{n\to\infty} \overline{x}_n(z - y_m)$ for any $z \in M$, and if d is also translation-invariant, this means $\lim_{n\to\infty} \overline{(x_n + y_n)}$ exists after taking an appropriate subsequence. If d is induced by a norm then $\lim_{n\to\infty} \overline{(\alpha x_n)}$ certainly exists since $\overline{(\alpha x_n)}(y) = |\alpha|\overline{x}_n(y/\alpha)$. Otherwise, a slight modification needs to be made to $\mathcal{T}$. One must now account for scalars by defining $\mathcal{T}$ to be a subset of $\mathbb{R}^{\mathbb{F}\times M}$, where $\mathbb{F}$ is the field of scalars, and $\overline{x}(\lambda, y) = d(\lambda x, y)$ for all $(\lambda, y) \in \mathbb{F} \times M$. Now, in this setting, $\lim_{n\to\infty} \overline{(\alpha x_n)}$ exists since $\overline{(\alpha x_n)}(\lambda, y) = \overline{x}_n(\lambda\alpha, y)$. With these ideas in mind, we are now ready to explicitly define the space of types we need. For a more complete discussion of some of the ideas above, see [6].

From now on, we consider a separable infinite-dimensional Banach space $(X, \|\cdot\|)$ which

admits a translation-invariant stable pseudometric d coarsely equivalent to the metric induced by $\|\cdot\|$, and the corresponding identity map $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$. By Corollary 3.2.2, such d exists as long as X is coarsely embeddable into a superstable Banach space.

Remark 3.3.1. By Remark 3.2.3, we can actually assume that d is a metric. However, in order to obtain the isomorphism constant in Remark 3.7.5 below, we need to work with d being the pseudometric given by Corollary 3.2.2.

Remark 3.3.2. Our definition of the space of types will be similar to Raynaud's, with a few changes to the proofs resulting from having a metric that is coarsely equivalent rather than uniformly equivalent to the metric induced by the norm on X. Note in particular that, in our case, a sequence may be dense in $(X, \|\cdot\|)$ while not being dense in (X, d). Thus, in order to have metrizability, we must use a countable subset of X to define the space of types.

Let Δ be a countable $\|\cdot\|$-dense $\mathbb{Q}$-vector subspace of X. Given $x \in \Delta$, define the function $\overline{x} \in \mathbb{R}_+^{\mathbb{Q}\times\Delta}$ by $\overline{x}(\lambda, y) = d(\lambda x, y)$ for all $(\lambda, y) \in \mathbb{Q} \times \Delta$. The *space of types* on $(\Delta, d\restriction_{\Delta\times\Delta})$, which we denote by $\mathcal{T}$, is defined to be the closure of $\{\overline{x}\}_{x\in\Delta}$ in $\mathbb{R}^{\mathbb{Q}\times\Delta}$ (with the topology of pointwise convergence). An element σ of $\mathcal{T}$ is called a *type*, and is called a *realized type* if $\sigma = \overline{x}$ for some $x \in \Delta$, in which case σ is also called the type realized by x. The type $\overline{0}$ is called the *null* or *trivial* type.

Note that the countability of $\mathbb{Q} \times \Delta$ implies that $\mathcal{T}$ is metrizable, and so every $\sigma \in \mathcal{T}$ can be expressed as $\lim_{n\to\infty} \overline{x}_n$ for some sequence $(x_n)_{n=1}^{\infty}$ in Δ. Such a sequence is called a *defining sequence* for σ. Note also that in this case $\sigma(\lambda, x) = \lim_{n,\mathcal{U}} d(\lambda x_n, x)$ for every $(\lambda, x) \in \mathbb{Q} \times \Delta$ and every nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$. In particular, $\lim_{n\to\infty} d(x_n, 0)$ exists, and so $(x_n)_{n=1}^{\infty}$ is a d-bounded (and therefore also $\|\cdot\|$-bounded) sequence in Δ.

For every $M \in \mathbb{R}_+$, we let $\mathcal{T}_M = \{\sigma \in \mathcal{T} \mid \sigma(1, 0) \leqslant M\}$. We will need the following lemma.

Lemma 3.3.3. *For all $M \in \mathbb{R}_+$, $\mathcal{T}_M$ is compact.*

Proof. Take any $\sigma \in \mathcal{T}_M$, and let $(x_n)_{n=1}^{\infty}$ is a defining sequence for σ. As $\lim_{n\to\infty} d(x_n, 0) = \sigma(1, 0) \leqslant M$, we may suppose that the defining sequence for σ is contained in the d-ball of radius

$M+1$ around 0. As $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ is expanding, there exists $R < \infty$ such that $t \leqslant R$ whenever $\rho_{\mathrm{Id}}(t) \leqslant M+1$. Then, since $\rho_{\mathrm{Id}}(\|x_n\|) \leqslant d(x_n, 0) \leqslant M+1$ for every $n \in \mathbb{N}$, we have

$$\sigma(\lambda, x) = \lim_n d(\lambda x_n, x) \leqslant \lim_n (d(\lambda x_n, 0) + d(0, x)) \leqslant \omega_{\mathrm{Id}}(|\lambda| R) + d(0, x)$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. That is, we have

$$\mathcal{T}_M \subseteq \prod_{(\lambda, x) \in \mathbb{Q} \times \Delta} [0, \omega(|\lambda| R) + d(x, 0)],$$

since $\sigma \in \mathcal{T}_M$ was arbitrary. By Tychonoff's theorem and the fact that $\mathcal{T}_M$ is closed, we are finished. $\square$

Corollary 3.3.4. *The metric space $\mathcal{T}$ is σ-locally compact.*

The next lemma will allow us to define analogues of scalar multiplication and vector addition in the space of types, capturing some of the algebraic structure of X.

Lemma 3.3.5. *Suppose $\sigma, \tau \in \mathcal{T}$. Then if $(w_n)_{n=1}^\infty, (x_n)_{n=1}^\infty$ are defining sequences for σ and $(y_n)_{n=1}^\infty, (z_n)_{n=1}^\infty$ are defining sequences for τ, then*

(i) The limits $\lim_n \overline{(\alpha w_n)}$ and $\lim_n \overline{(\alpha x_n)}$ exist and are equal for every $\alpha \in \mathbb{Q}$.

(ii) The limits $\lim_n \lim_m \overline{(w_n + y_m)}$ and $\lim_n \lim_m \overline{(x_n + z_m)}$ exist and are equal.

Proof. Item (i) follows immediately from the definitions. By a straightforward argument using the translation-invariance and stability of d, item (ii) also follows. $\square$

Definition 3.3.6. Let $\sigma, \tau \in \mathcal{T}$ and let $(x_n)_{n=1}^\infty, (y_m)_{m=1}^\infty$ be any defining sequences for σ and τ, respectively. We define the *dilation* operation on $\mathcal{T}$ by $(\alpha, \sigma) \in \mathbb{Q} \times \mathcal{T} \mapsto \alpha \cdot \sigma \in \mathcal{T}$, where $\alpha \cdot \sigma := \lim_n \overline{(\alpha x_n)}$. We define the *convolution* operation on $\mathcal{T}$ by $(\sigma, \tau) \in \mathcal{T} \times \mathcal{T} \mapsto \sigma * \tau \in \mathcal{T}$, where $\sigma * \tau := \lim_n \lim_m \overline{(x_n + y_m)}$. By Lemma 3.3.5, both dilation and convolution are well-defined. For $(\sigma_j)_{j=1}^k \subseteq \mathcal{T}$, we define $*_{j=1}^k \sigma_j$ in the obvious way, and we allow dilation to bind more strongly than convolution in our notation, i.e., we write $\alpha \cdot \sigma * \tau$ to mean $(\alpha \cdot \sigma) * \tau$.

It follows easily from the definitions that, given $\sigma \in \mathcal{T}$ and a defining sequence $(x_n)_{n=1}^{\infty}$ for σ, we have $\alpha \cdot \sigma(\lambda, x) = \sigma(\lambda\alpha, x)$ for every $(\lambda, x) \in \mathbb{Q} \times \Delta$ and $\sigma * \tau = \lim_{n\to\infty} \overline{x}_n * \tau$ for every $\tau \in \mathcal{T}$. Furthermore, using the translation-invariance and stability of d, it is easily shown that convolution is associative and commutative, and that dilation distributes over convolution. We now prove some continuity properties of our dilation and convolution maps.

Lemma 3.3.7. *Dilation is a right-continuous map from $\mathbb{Q} \times \mathcal{T}$ to $\mathcal{T}$.*

Proof. Fix $\alpha \in \mathbb{Q}$ and suppose $(\sigma_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{T}$ converging to $\sigma \in \mathcal{T}$. Then $\alpha \cdot \sigma(\lambda, x) = \sigma(\lambda\alpha, x) = \lim_{n\to\infty} \sigma_n(\lambda\alpha, x) = \lim_{n\to\infty} \alpha \cdot \sigma_n(\lambda, x)$ for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. Thus $\alpha \cdot \sigma = \lim_{n\to\infty} \alpha \cdot \sigma_n$. This was for an arbitrary converging sequence in $\mathcal{T}$, so dilation is right continuous. □

Lemma 3.3.8. *Convolution is a separately continuous map from $\mathcal{T} \times \mathcal{T}$ to $\mathcal{T}$.*

Proof. Let D be a metric compatible with the topology on $\mathcal{T}$. Fix $\tau \in \mathcal{T}$ and suppose $(\sigma_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{T}$ converging to $\sigma \in \mathcal{T}$. For each $n \in \mathbb{N}$, let $(x_{n,m})_{m=1}^{\infty}$ be a defining sequence for σ_n, and let $m_n \in \mathbb{N}$ be such that $D(\sigma_n, \overline{x}_{n,m_n}) < \frac{1}{n}$ and $D(\overline{x}_{n,m_n} * \tau, \sigma_n * \tau) < \frac{1}{n}$. Then $(x_{n,m_n})_{n=1}^{\infty}$ is a defining sequence for σ by the triangle inequality; and so, again by triangle inequality, $\sigma * \tau = \lim_n \sigma_n * \tau$. This was for an arbitrary converging sequence in $\mathcal{T}$, so convolution (which is commutative) is separately continuous. □

Corollary 3.3.9. *Convolution is a Baire class 1 map from $\mathcal{T} \times \mathcal{T}$ to $\mathcal{T}$.*

Proof. Given $(\lambda, x) \in \mathbb{Q} \times \Delta$, let $\Phi_{\lambda,x} \colon \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ be defined by $\Phi_{\lambda,x}(\sigma, \tau) = \sigma * \tau(\lambda, x)$ for all $\sigma, \tau \in \mathcal{T}$. Choose a compatible metric D for the topology on $\mathcal{T}$ and note that there is $\delta > 0$ such that $D(\sigma, \tau) \geqslant \delta$ whenever $|\sigma(1, 0) - \tau(1, 0)|$ is large enough. Now, by Lemma 3.3.8 and the topology on $\mathcal{T}$, $\Phi_{\lambda,x}$ is separately continuous; and by Lemma 3.3.3, $\mathcal{T}_M$ is compact for every $M \in \mathbb{R}_+$. Thus; applying Lemma 3.2.4 with $X = Y = \mathcal{T}$, $f = \Phi_{\lambda,x}$, $d = D$, $\mathcal{K} = \{\mathcal{T}_{M+1} \backslash \mathrm{int}(\mathcal{T}_M)\}_{M=0}^{\infty}$, and with δ as above; we have that $\Phi_{\lambda,x}$ is the pointwise limit of a sequence of continuous functions, and is therefore Baire class 1. As this is true for any $(\lambda, x) \in \mathbb{Q} \times \Delta$, convolution is itself Baire class 1. □

The sequence in the statement of our main theorem will be a defining sequence for one of the types in $\mathcal{T}$. We will eventually prove an inequality for the type and then show that a similar inequality holds for the spreading model associated to the sequence, but first we need to know under what circumstances a type's defining sequence even has a spreading model. We already know that a defining sequence $(x_n)_{n=1}^{\infty}$ for a type σ is bounded in norm, but we want to put a condition on σ that guarantees $(x_n)_{n=1}^{\infty}$ is eventually bounded away from zero in norm. This motivates our next definition.

Definition 3.3.10. A type $\sigma \in \mathcal{T}$ is called *admissible* if $\sigma(1,0) > \inf_{t>0} \omega_{\mathrm{Id}}(t)$.

Note that if σ is an admissible type and $(x_n)_{n=1}^{\infty}$ is a defining sequence for σ, then

$$\liminf_n \omega_{\mathrm{Id}}(\|x_n\|) \geqslant \lim_n d(x_n, 0) = \sigma(1,0) > \inf_{t>0} \omega_{\mathrm{Id}}(t).$$

Thus, since ω_{Id} is an increasing function, we can find $\delta > 0$ such that $(x_n)_{n=1}^{\infty}$ is eventually δ-bounded in norm away from zero. From this point forward, we will let $\gamma = \inf_{t>0} \omega_{\mathrm{Id}}(t)$.

Remark 3.3.11. If $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ is uniformly continuous, then $\gamma = 0$. If, in addition, d is a metric, then the inequality in our definition is trivial, and every nontrivial type will be admissible. Given our assumption that d is coarsely equivalent to the metric induced by $\|\cdot\|$, we do not need to place any additional conditions on a type to guarantee its defining sequences to be bounded in norm. Had this not been the case, we would have had to include such a condition in our definition of admissibility. One condition we could use would be to require a type σ to also satisfy the inequality $\sigma(1,0) < \sup_{t<\infty} \rho_{\mathrm{Id}}(t)$ (a trivial inequality in our case). In [13], where the author is concerned with a translation-invariant stable metric d *uniformly* equivalent to the metric induced by $\|\cdot\|$, the author does exactly this.

At this point, we have established a condition to put on a type to guarantee its defining sequences are bounded in norm and eventually bounded away from zero in norm. In our goal to obtain a spreading model, we now need an extra condition that will guarantee that a type's defining sequences contain no norm-Cauchy subsequences.

Definition 3.3.12. A type σ is said to be *symmetric* if $\sigma = (-1) \cdot \sigma$, i.e., if $\sigma(\lambda, x) = \sigma(-\lambda, x)$, for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. We denote $\{\sigma \in \mathcal{T} \mid \sigma \text{ is symmetric}\}$ by $\mathcal{S}$ and $\mathcal{S} \cap \mathcal{T}_M$ by $\mathcal{S}_M$ for each $M \in \mathbb{R}_+$.

Note that by Lemma 3.3.7, $\mathcal{S}$ is closed, and therefore $\mathcal{S}_M$ is compact for all $M \in \mathbb{R}_+$.

Lemma 3.3.13. *Suppose $\sigma \in \mathcal{T}$ is an admissible symmetric type and $(x_n)_{n=1}^{\infty}$ is a defining sequence for σ. Then $(x_n)_{n=1}^{\infty}$ has no $\|\cdot\|$-Cauchy subsequence.*

Proof. Suppose to the contrary, that $(x_n)_{n=1}^{\infty}$ has a $\|\cdot\|$-Cauchy subsequence. After taking this subsequence, we can assume that $(x_n)_{n=1}^{\infty}$ converges in norm to some $x \in X$. Then, as σ is symmetric, we have

$$\begin{aligned}
&\liminf_n d(\lambda x_n, -\lambda x_n) \\
&\qquad = \liminf_n \Big(d(\lambda x_n, -\lambda x_n) - \sigma(\lambda, -\lambda x_n) + \sigma(-\lambda, -\lambda x_n) \Big) \\
&\qquad = \liminf_n \lim_m \Big(d(\lambda x_n, -\lambda x_n) - d(\lambda x_m, -\lambda x_n) + d(-\lambda x_m, -\lambda x_n) \Big) \\
&\qquad \leqslant \liminf_n \lim_m \Big(d(\lambda x_n, \lambda x_m) + d(-\lambda x_m, -\lambda x_n) \Big) \\
&\qquad \leqslant 2 \cdot \liminf_n \liminf_m \omega_{\mathrm{Id}}(|\lambda| \cdot \|x_n - x_m\|) \\
&\qquad = 2\gamma
\end{aligned}$$

for all $\lambda \in \mathbb{Q}$. This implies $\rho_{\mathrm{Id}}(\|\lambda x\|) \leqslant \liminf_n \rho_{\mathrm{Id}}(2\|\lambda x_n\|) \leqslant 2\gamma$ for all $\lambda \in \mathbb{Q}$. As d is coarsely equivalent to the metric induced by the norm of X, this can only happen if $x = 0$. But then the admissibility of σ yields

$$\gamma < \sigma(1, 0) = \lim_n d(x_n, 0) \leqslant \liminf_n \omega_{\mathrm{Id}}(\|x_n\|) = \gamma,$$

a contradiction. $\square$

3.4 Conic classes

To show the existence of a type that satisfies an ℓ_p-inequality, we will use a limiting argument and the existence of a shared point of continuity for every finite combination of convolutions and dilations by a scalar. The definition of conic class below is motivated by the desire to use Lemma 3.2.5 with the Baire category theorem to find a shared point of continuity, and the need for a minimality argument to make sure this point can be used in the limiting argument.

Definition 3.4.1. A nonempty subset $\mathcal{C}$ of $\mathcal{S}$ is called a *conic class* if

(i) $\mathcal{C} \neq \{\overline{0}\}$,

(ii) $\lambda \cdot \sigma \in \mathcal{C}$ for all $\lambda \in \mathbb{Q}$ and $\sigma \in \mathcal{C}$,

(iii) $\sigma * \tau \in \mathcal{C}$ for all $\sigma, \tau \in \mathcal{C}$.

Moreover, $\mathcal{C}$ is called *admissible* if $\mathcal{C}$ contains an admissible type, i.e., if there exists $\sigma \in \mathcal{C}$ such that $\sigma(1,0) > \gamma$.

Lemma 3.4.2. *The set $\mathcal{S}$ is a closed admissible conic class.*

Proof. That $\mathcal{S}$ is closed follows from Lemma 3.3.7. The properties (ii) and (iii) follow easily from the definitions of dilation and convolution and from the translation-invariance of d. All that remains is to show that there is an admissible (and therefore nontrivial) type σ in $\mathcal{S}$. Let $R \in [0, \infty)$ be such that $\rho_{\mathrm{Id}}(t) > \gamma$ whenever $t \geqslant R$. By the infinite-dimensionality of X, there is a bounded R-separated sequence $(x_n)_{n=1}^{\infty}$ in $(X, \|\cdot\|)$. After possibly taking a subsequence, we may suppose that $(x_n)_{n=1}^{\infty}$ is a defining sequence for some $\sigma \in \mathcal{T}$. In this case,

$$(\sigma * (-1) \cdot \sigma)(1,0) = \lim_n \lim_m d(x_n - x_m, 0) \geqslant \inf_{n \neq m} d(x_n - x_m, 0) \geqslant \rho_{\mathrm{Id}}(R) > \gamma.$$

That is, the symmetric type $\sigma * (-1) \cdot \sigma$ is admissible. Therefore $\mathcal{S}$ is a closed admissible conic class. $\square$

Now that the existence of a closed admissible conic class has been shown, we will show the existence of one that is minimal (with respect to set inclusion), using the following lemma.

Lemma 3.4.3. *Let σ be an admissible type. Given any $0 \leqslant r_1 < r_2$, there is $\alpha \in \mathbb{Q}_+$ such that $\rho_{\mathrm{Id}}(r_1) \leqslant \alpha \cdot \sigma(1,0) \leqslant \omega_{\mathrm{Id}}(r_2)$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a defining sequence for σ. The admissibility of σ implies that $(x_n)_{n=1}^{\infty}$ is a $\|\cdot\|$-bounded sequence that is eventually $\|\cdot\|$-bounded away from 0. Thus, we may suppose after possibly taking a subsequence that $\lim_n \|x_n\|$ exists and is nonzero. Let $\alpha \in \mathbb{Q}_+$ be such that $\lim_n \|\alpha x_n\| \in [r_1, r_2]$. As $\alpha \cdot \sigma(1,0) = \lim_n d(\alpha x_n, 0)$, we then have

$$\rho_{\mathrm{Id}}(r_1) \leqslant \alpha \cdot \sigma(1,0) \leqslant \omega_{\mathrm{Id}}(r_2). \qquad \square$$

Lemma 3.4.4. *Every closed admissible conic class contains a minimal closed admissible conic class.*

Proof. Fix a closed admissible conic class $\mathcal{C}$. Let $\mathcal{F}$ be the family of closed admissible conic classes contained in $\mathcal{C}$ ordered by reverse set inclusion and let $\{\mathcal{C}_i\}_{i\in I}$ be some chain in $\mathcal{F}$. We will show that $\bigcap_{i\in I} \mathcal{C}_i$ is a closed admissible conic class.

Certainly, $\bigcap_{i\in I} \mathcal{C}_i \subseteq \mathcal{S}$ is closed and satisfies conditions (ii) and (iii) in the definition of conic class. So we only need to show that $\bigcap_{i\in I} \mathcal{C}_i$ contains an admissible type. For that, fix $R \in [0,\infty)$ such that $\rho_{\mathrm{Id}}(t) > \gamma$ whenever $t \geqslant R$ and let $\mathcal{B}_i = C_i \cap (\mathcal{T}_{\omega_{\mathrm{Id}}(R+1)} \backslash \mathrm{int}(\mathcal{T}_{\rho_{\mathrm{Id}}(R)}))$ for all $i \in I$. By Lemma 3.3.3, $\mathcal{B}_i$ is compact. Given $i \in I$, let $\sigma_i \in \mathcal{C}_i$ be admissible. By the previous lemma, there is $\alpha_i \in \mathbb{Q}_+$ such that $\alpha_i \cdot \sigma_i \in \mathcal{B}_i$, and so $\mathcal{B}_i$ is nonempty. Hence, $\{\mathcal{B}_i\}_{i\in I}$ is a family of compact sets with the finite intersection property, which implies $\bigcap_{i\in I} \mathcal{C}_i \supseteq \bigcap_{i\in I} \mathcal{B}_i$ is nonempty. By our choice of R, $\bigcap_{i\in I} \mathcal{B}_i$ can only contain admissible types, hence $\bigcap_{i\in I} \mathcal{C}_i$ contains an admissible type. Thus, $\bigcap_{i\in I} \mathcal{C}_i$ is a closed admissible conic class, and so is an upper bound for the chain $\{\mathcal{C}_i\}_{i\in I}$ in $\mathcal{F}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element. That is, $\mathcal{C}$ contains a minimal closed admissible conic class. $\square$

We come now to the main result of this section. For Raynaud, it was enough to show that the maps $\sigma \mapsto \sigma * \alpha \cdot \sigma$, where α is any scalar, share a point of continuity. He then uses this to show $\sigma * \alpha \cdot \sigma = (1 + |\alpha|^p)^{1/p} \cdot \sigma$ for some $p \in [1, \infty)$. With this equality, one may then easily show that for any finite sequence of scalars $\overline{\alpha} = (\alpha_j)_{j=1}^N$, one has $\ast_{j=1}^N \alpha_j \cdot \sigma = \|\overline{\alpha}\|_p \cdot \sigma$. In our case however, we will only be able to show that given any $(t_m)_{m=1}^\infty \subseteq \mathbb{Q}$ converging to $\|\overline{\alpha}\|_p$ and $(\lambda, x) \in \mathbb{Q} \times \Delta$, $\limsup_m |\ast_{j=1}^N \alpha_j \cdot \sigma(\lambda, x) - t_m \cdot \sigma(\lambda, x)| \leqslant L$, for some constant L depending on γ. The next lemma will allow us to make sure L does not depend on the length of $\overline{\alpha}$.

Lemma 3.4.5. *Let $\mathcal{C}$ be a closed admissible conic class. Then there is an admissible $\phi \in \mathcal{C}$ such that ϕ is a common point of continuity for the family of functions $\{\sigma \mapsto \ast_{j=1}^m \alpha_j \cdot \sigma \mid (\alpha_j)_{j=1}^m \subseteq \mathbb{Q}\} \subseteq \mathcal{C}^{\mathcal{C}}$.*

Proof. By Lemma 3.2.5 and Corollary 3.3.9 (with $X = \mathcal{C}$ and $\mathcal{F} = \{\sigma \mapsto \alpha \cdot \sigma * \beta \cdot \sigma \mid \alpha, \beta \in \mathbb{Q}\}$), there is a comeager G_δ subset E of $\mathcal{C}$ such that g is continuous on E for all

$$g \in \{\sigma \mapsto \mathop{\ast}_{j=1}^{m} \alpha_j \cdot \sigma \mid (\alpha_j)_{j=1}^m \subseteq \mathbb{Q}\} \subseteq \mathcal{C}^{\mathcal{C}}.$$

But $\mathcal{C}$ is closed, and so is locally compact, by Corollary 3.3.4. Therefore E is dense in $\mathcal{C}$, by the Baire category theorem, and the statement follows from the admissibility of $\mathcal{C}$. $\square$

3.5 Spreading models associated to well-chosen types

Let σ be an admissible symmetric type and $(x_n)_{n=1}^\infty$ a defining sequence for σ. Then $(x_n)_{n=1}^\infty$ is bounded, and by Lemma 3.3.13, has no $\|\cdot\|$-Cauchy subsequence. So let $(S, \|\cdot\|)$ be a spreading model for $(x_n)_{n=1}^\infty$, and let $(\zeta_n)_{n=1}^\infty$ be the standard basis for S. Define $(\xi_n)_{n=1}^\infty$ by $\xi_n = \zeta_{2n-1} - \zeta_{2n}$ for all $n \in \mathbb{N}$. Recall that $(\zeta_n)_{n=1}^\infty$ is 1-spreading and $(\xi_n)_{n=1}^\infty$ is 1-unconditional.

Let $\tau = \sigma * (-1) \cdot \sigma$. As $\sigma = \lim_n \overline{x}_n$, we may assume after taking a subsequence that $\tau = \lim_n \overline{x_{2n-1} - x_{2n}}$. As $(x_n)_{n=1}^\infty$ has no $\|\cdot\|$-Cauchy subsequence, we may further assume after taking another subsequence that $\inf_{n \neq m} \|x_n - x_m\| > 0$. As $\tau(1, 0) = \lim_n d(x_{2n-1} - x_{2n}, 0) \geqslant \rho_{\mathrm{Id}}(\inf_{n \neq m} \|x_n - x_m\|)$, by dilating σ, we can also assume that τ is an admissible type. It is clear that $(\overline{\mathrm{span}}\{\xi_n\}_{n=1}^\infty, \|\cdot\|_S)$ is a spreading model for $(x_{2n-1} - x_{2n})_{n=1}^\infty$.

From this point forward, we fix a minimal closed admissible conic class $\mathcal{C}$ and an admissible $\phi \in \mathcal{C}$ that is a common point of continuity for the family of functions

$$\mathcal{F} = \{\sigma \mapsto \mathop{\ast}_{j=1}^{m} \alpha_j \cdot \sigma \mid (\alpha_j)_{j=1}^m \subseteq \mathbb{Q}\} \subseteq \mathcal{C}^{\mathcal{C}}$$

such that $\psi = \phi * (-1) \cdot \phi$ is also admissible. We also fix a defining sequence $(x_n)_{n=1}^\infty$ for ϕ with spreading model $(S, \|\cdot\|_S)$ such that $(x_{2n-1} - x_{2n})_{n=1}^\infty$ is a defining sequence for ψ. All this is possible by Lemma 3.4.2, Lemma 3.4.4, Lemma 3.4.5, and the discussion above. Remember that we have only defined dilation for rational numbers, and so we will restrict our attention to vectors in S that have rational coefficients with respect to the basis $(\zeta_n)_{n=1}^\infty$. Given a set of vectors V, we will denote the rational linear span of V by $\operatorname{span}_{\mathbb{Q}} V$.

Definition 3.5.1. Given $(\alpha_j)_{j=1}^m \subseteq \mathbb{Q}$, we say that $\sum_{j=1}^m \alpha_j \zeta_j$ *realizes* the type $\ast_{j=1}^m \alpha_j \cdot \phi$.

Note that, if $u = \sum_{j=1}^{m_1} \alpha_j \zeta_j$ realizes the type σ, and $v = \sum_{j=m_1+1}^{m_2} \beta_j \zeta_j$ realizes the type τ, it follows that $u + v$ realizes the type $\sigma * \tau$.

Lemma 3.5.2. *Suppose $u, v \in \operatorname{span}_{\mathbb{Q}}\{\zeta_n\}_{n=1}^\infty$ realize the types σ and τ, respectively. Then for every $(\lambda, x) \in \mathbb{Q} \times \Delta$,*

$$\sup_{0<\varepsilon\leqslant|\lambda|\|u-v\|_S} \rho_{\mathrm{Id}}(|\lambda|\,\|u-v\|_S - \varepsilon) \leqslant \sigma(\lambda, x) + \tau(\lambda, x)$$

and

$$|\sigma(\lambda, x) - \tau(\lambda, x)| \leqslant \inf_{\varepsilon>0} \omega_{\mathrm{Id}}(|\lambda|\|u-v\|_S + \varepsilon).$$

In particular, for each $\delta > 0$, the following hold.

(i) If $\|u\|_S > \delta$, then $\sigma(1, 0) \geqslant \rho_{\mathrm{Id}}(\delta)$.

(ii) If $\sigma(1, 0) > \omega_{\mathrm{Id}}(\delta)$, then $\|u\|_S \geqslant \delta$.

Proof. Let $(\alpha_j)_{j=1}^m, (\beta_j)_{j=1}^m \subseteq \mathbb{Q}$ be such that $u = \sum_{j=1}^m \alpha_j \zeta_j$ and $v = \sum_{j=1}^m \beta_j \zeta_j$. Then

$$\begin{aligned}
\rho_{\mathrm{Id}}(|\lambda| \, \|u - v\|_S - \varepsilon) &\leqslant \limsup_{n_1} \ldots \limsup_{n_m} \rho_{\mathrm{Id}} \left(\left\| \lambda \sum_{j=1}^m (\alpha_j - \beta_j) x_{n_j} \right\| \right) \\
&\leqslant \lim_{n_1} \ldots \lim_{n_m} d \left(\lambda \sum_{j=1}^m (\alpha_j - \beta_j) x_{n_j}, 0 \right) \\
&\leqslant \lim_{n_1} \ldots \lim_{n_m} \left(d \left(\lambda \sum_{j=1}^m \alpha_j x_{n_j}, x \right) + d \left(\lambda \sum_{j=1}^m \beta_j x_{n_j}, x \right) \right) \\
&= \sigma(\lambda, x) + \tau(\lambda, x)
\end{aligned}$$

for all $0 < \varepsilon < |\lambda| \, \|u - v\|_S$. Similarly,

$$\begin{aligned}
|\sigma(\lambda, x) - \tau(\lambda, x)| &= \lim_{n_1} \ldots \lim_{n_m} \left| d \left(\lambda \sum_{j=1}^m \alpha_j x_{n_j}, x \right) - d \left(\lambda \sum_{j=1}^m \beta_j x_{n_j}, x \right) \right| \\
&\leqslant \lim_{n_1} \ldots \lim_{n_m} d \left(\lambda \sum_{j=1}^m (\alpha_j - \beta_j) x_{n_j}, 0 \right) \\
&\leqslant \liminf_{n_1} \ldots \liminf_{n_m} \omega_{\mathrm{Id}} \left(\left\| \lambda \sum_{j=1}^m (\alpha_j - \beta_j) x_{n_j} \right\| \right) \\
&\leqslant \omega_{\mathrm{Id}}(|\lambda| \, \|u - v\|_S + \varepsilon)
\end{aligned}$$

for all $\varepsilon > 0$. The particular case follows by letting $v = 0$ and $\lambda = 1$. $\square$

Let $H = \mathrm{span}_{\mathbb{Q}}\{\xi_n\}_{n=1}^\infty$. Given $\overline{\alpha} = (\alpha_j)_{j=1}^m \subseteq \mathbb{Q}$, we define a bounded linear map $T_{\overline{\alpha}} \colon \overline{H} \to \overline{H}$ as follows. For each $n \in \mathbb{N}$, let $T_{\overline{\alpha}}(\xi_n) = \sum_{j=1}^m \alpha_j \xi_{mn+j-1}$ and extend $T_{\overline{\alpha}}$ linearly to H. As $(\xi_n)_{n=1}^\infty$ is both 1-spreading and 1-unconditional, $\|T_{\overline{\alpha}}(u)\|_S \leqslant \|\alpha\|_1 \, \|u\|_S$ for all $u \in H$. Thus $T_{\overline{\alpha}}$ can be extended to all of $\overline{H}$. If $\overline{\alpha} = (\alpha_1)$ is a sequence of length 1, then $T_{\overline{\alpha}} u$ is just the scaling of u by α_1. We also define the function $\widehat{T}_{\overline{\alpha}} \colon \mathcal{C} \to \mathcal{C}$ by $\widehat{T}_{\overline{\alpha}}(\sigma) = *_{j=1}^m \alpha_j \cdot \sigma$ for all $\sigma \in \mathcal{C}$.

Lemma 3.5.3. *Suppose $\overline{\alpha} = (\alpha_i)_{i=1}^n, \overline{\beta} = (\beta_j)_{j=1}^m \subseteq \mathbb{Q}$. Define $\overline{\gamma} = (\gamma_k)_{k=1}^{nm} \subseteq \mathbb{Q}$ by $\gamma_k = \alpha_i \beta_j$ whenever $k = n(j-1) + i$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Then $T_\alpha \circ T_\beta = T_\gamma$ and $\widehat{T}_\alpha \circ \widehat{T}_\beta = \widehat{T}_\gamma$.*

Proof. For any $k \in \mathbb{N}$,

$$\begin{aligned}
(T_{\overline{\alpha}} \circ T_{\overline{\beta}})(\xi_k) &= T_{\overline{\alpha}}\Big(\sum_{j=1}^{m} \beta_j \xi_{mk+j-1}\Big) \\
&= \sum_{j=1}^{m}\sum_{i=1}^{n} \alpha_i\beta_j \xi_{n(mk+j-1)+i-1} \\
&= \sum_{j=1}^{m}\sum_{i=1}^{n} \alpha_i\beta_j \xi_{nmk+n(j-1)+i-1} \\
&= \sum_{\ell=1}^{nm} \gamma_\ell \xi_{nmk+\ell-1} \\
&= T_{\overline{\gamma}}(\xi_k)
\end{aligned}$$

Therefore $T_{\overline{\alpha}} \circ T_{\overline{\beta}} = T_{\overline{\gamma}}$, by linearity and continuity. Similarly,

$$(\widehat{T}_{\overline{\alpha}} \circ \widehat{T}_{\overline{\beta}})(\sigma) = \widehat{T}_{\alpha}\Big(\mathop{\ast}_{j=1}^{m} \beta_j\sigma\Big) = \mathop{\ast}_{j=1}^{m}\mathop{\ast}_{i=1}^{n} \alpha_i\beta_j\sigma = \mathop{\ast}_{\ell=1}^{nm} \gamma_\ell\sigma = \widehat{T}_{\overline{\gamma}}(\sigma).$$

for all $\sigma \in \mathcal{C}$, and so $\widehat{T}_{\overline{\alpha}} \circ \widehat{T}_{\overline{\beta}} = \widehat{T}_{\overline{\gamma}}$. □

The previous lemma suggests the following notation. For $\overline{\alpha} = (\alpha_i)_{i=1}^n, \overline{\beta} = (\beta_j)_{j=1}^m \subseteq \mathbb{Q}$, we denote by $\overline{\alpha} \circ \overline{\beta}$ the sequence $(\gamma_k)_{k=1}^{nm} \subseteq \mathbb{Q}$ defined by $\gamma_k = \alpha_i\beta_j$ whenever $k = n(j-1)+i$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. We define $\overline{\alpha}^{\circ k}$ recursively by $\overline{\alpha}^{\circ 1} = \overline{\alpha}$ and $\overline{\alpha}^{\circ k+1} = \overline{\alpha} \circ \overline{\alpha}^{\circ k}$ for every $k \in \mathbb{N}$. Note that $\widehat{T}_{\overline{\alpha}}^k = \widehat{T}_{\overline{\alpha}^{\circ k}}$ for all finite length sequences $\overline{\alpha} \subseteq \mathbb{Q}$ and all $k \in \mathbb{N}$.

Lemma 3.5.4. *Fix $\overline{\alpha} = (\alpha_j)_{j=1}^m \subseteq \mathbb{Q}$. Suppose $u \in H$ realizes the type σ. Then $T_{\overline{\alpha}}(u)$ realizes the type $\widehat{T}_{\overline{\alpha}}(\sigma)$.*

Proof. Let $(\lambda_i)_{i=1}^n \subseteq \mathbb{Q}$ be such that $u = \sum_{i=1}^n \lambda_i\xi_i$, which implies $\sigma = \ast_{i=1}^n \lambda_i \cdot \psi$. Then

$$T_{\overline{\alpha}}(u) = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \alpha_j \xi_{mi+j-1} = \sum_{j=1}^{m}\sum_{i=1}^{n} \alpha_j\lambda_i \xi_{mi+j-1},$$

which realizes the type

$$\mathop{\ast}_{i=1}^{m} \mathop{\ast}_{j=1}^{n} \alpha_j \lambda_i \cdot \psi = \mathop{\ast}_{j=1}^{m} \alpha_j \cdot \mathop{\ast}_{i=1}^{n} \lambda_i \cdot \psi = \widehat{T}_{\overline{\alpha}}(\sigma). \qquad \square$$

Lemma 3.5.5. *Fix $N \in \mathbb{N}$ and $(b_i)_{i=1}^N \subseteq \mathbb{Q}$. And for each $1 \leqslant i \leqslant N$, fix $\overline{\alpha}_i, \overline{\beta}_i \subseteq \mathbb{Q}$. Suppose $u_i, v_i \in H$ realize the types σ and τ, respectively, for each $1 \leqslant i \leqslant N$. Then for every $(\lambda, x) \in \mathbb{Q} \times \Delta$,*

$$\left| \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i} \sigma(\lambda, x) - \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\beta}_i} \tau(\lambda, x) \right| \leqslant \inf_{\varepsilon > 0} \omega_{\mathrm{Id}} \left(|\lambda| \sum_{i=1}^{N} |b_i| \cdot \left\| T_{\overline{\alpha}_i} u_i - T_{\overline{\beta}_i} v_i \right\|_S + \varepsilon \right).$$

Proof. For each $m \in \mathbb{N}$, let $s_m \colon H \to H$ be the linear map defined by $s_m(\xi_n) = \xi_{n+m}$ for each $n \in \mathbb{N}$, extended linearly to H. We construct sequences $(u_i')_{i=1}^N, (v_i')_{i=1}^N \subseteq H$ recursively as follows. Let $u_1' = b_1 T_{\overline{\alpha}_1} u_1$ and $v_1' = b_1 T_{\overline{\beta}_1} v_1$. Given u_i', v_i' for $1 \leqslant i < N$, let $m_i = \max\{\mathrm{supp}(u_i') \cup \mathrm{supp}(v_i')\}$ and then let $u_{i+1}' = b_{i+1} s_{m_i}(T_{\overline{\alpha}_{i+1}} u_{i+1})$ and $v_{i+1}' = b_{i+1} s_{m_i}(T_{\overline{\beta}_{i+1}} v_{i+1})$. Clearly, both sequences $(u_i')_{i=1}^N$ and $(v_i')_{i=1}^N$ have disjoint supports. By Lemma 3.5.4 and the remark after Definition 3.5.1, $\sum_{i=1}^N u_i'$ and $\sum_{i=1}^N v_i'$ realize $\ast_{i=1}^N b_i \cdot \widehat{T}_{\overline{\alpha}_i} \sigma$ and $\ast_{i=1}^N b_i \cdot \widehat{T}_{\overline{\beta}_i} \tau$, respectively. Thus, by Lemma 3.5.2 and the fact that $(\xi_n)_{n=1}^\infty$ is 1-spreading,

$$\begin{aligned}
\left| \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i} \sigma(\lambda, x) - \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\beta}_i} \tau(\lambda, x) \right| &\leqslant \inf_{\varepsilon > 0} \omega_{\mathrm{Id}} \left(|\lambda| \cdot \left\| \sum_{i=1}^{N} (u_i' - v_i') \right\|_S + \varepsilon \right) \\
&\leqslant \inf_{\varepsilon > 0} \omega_{\mathrm{Id}} \left(|\lambda| \sum_{i=1}^{N} \| u_i' - v_i' \|_S + \varepsilon \right) \\
&= \inf_{\varepsilon > 0} \omega_{\mathrm{Id}} \left(|\lambda| \sum_{i=1}^{N} |b_i| \cdot \left\| T_{\overline{\alpha}_i} u_i - T_{\overline{\beta}_i} v_i \right\|_S + \varepsilon \right). \qquad \square
\end{aligned}$$

3.6 Coarse approximating sequences

The goal of this section is to show that the type ψ satisfies the conclusion of Lemma 3.6.7 below. For that, we introduce the notion of coarse approximating sequences.

Definition 3.6.1. Given $u = \sum_{i=1}^k \alpha_i \xi_i \in \mathrm{span}\{\xi_n\}_{n=1}^\infty$, a vector $v \in \mathrm{span}\{\xi_n\}_{n=1}^\infty$ is said to be a

spreading of u if $v = \sum_{i=1}^{k} \alpha_i \xi_{n_i}$ for some $n_1 < \ldots < n_k \in \mathbb{N}$.

Definition 3.6.2. Fix $N \in \mathbb{N}$. And for each $1 \leqslant i \leqslant N$, fix $\overline{\alpha}_i \subseteq \mathbb{Q}$ and $\beta_i \in \mathbb{R}_+$. A sequence of types $(\sigma_n)_{n=1}^{\infty} \subseteq \mathcal{C}$ is called a *coarse* $(\overline{\alpha}_i, \beta_i)_{i=1}^{N}$*-approximating sequence* if there exists a sequence $(u_n)_{n=1}^{\infty} \subseteq H$ and sequences $(u_{i,n})_{n=1}^{\infty} \subseteq H$ for each $1 \leqslant i \leqslant N$ such that

(i) u_n realizes σ_n for all $n \in \mathbb{N}$,

(ii) $u_{i,n}$ is a spreading of u_n for all $n \in \mathbb{N}$ and $1 \leqslant i \leqslant N$,

(iii) $\lim_n \|T_{\overline{\alpha}_i}(u_n) - \beta_i u_{i,n}\|_S = 0$ for all $1 \leqslant i \leqslant N$.

Lemma 3.6.3. *Fix $\overline{\alpha} \subseteq \mathbb{Q}$, $\beta \in \mathbb{R}_+$, and $(u_n)_{n=1}^{\infty} \subseteq H$. If there is a spreading $(u'_n)_{n=1}^{\infty}$ of $(u_n)_{n=1}^{\infty}$ such that $\lim_n \|T_{\overline{\alpha}}(u_n) - \beta u'_n\|_S = 0$, then for every $k \in \mathbb{N}$ there is a spreading $(u''_n)_{n=1}^{\infty}$ of $(u_n)_{n=1}^{\infty}$ such that $\lim_n \left\|T_{\overline{\alpha}}^k(u_n) - \beta^k u''_n\right\|_S = 0$.*

Proof. For $k = 1$ the result is trivial. Suppose the result holds for some $k \in \mathbb{N}$. Let $(u''_n)_{n=1}^{\infty}$ be a spreading of $(u_n)_{n=1}^{\infty}$ such that $\lim_n \left\|T_{\overline{\alpha}}^k(u_n) - \beta^k u''_n\right\|_S = 0$. By the definition of $T_{\overline{\alpha}}$, it follows that $(T_{\overline{\alpha}}(u''_n))_{n=1}^{\infty}$ is a spreading of $(T_{\overline{\alpha}}(u_n))_{n=1}^{\infty}$, so there exists a spreading $(u'''_n)_{n=1}^{\infty}$ of $(u_n)_{n=1}^{\infty}$ such that also $(T_{\overline{\alpha}}(u''_n) - \beta u'''_n)_{n=1}^{\infty}$ is a spreading of $(T_{\overline{\alpha}}(u_n) - \beta u'_n)_{n=1}^{\infty}$. Thus, by the fact that $(\xi_n)_{n=1}^{\infty}$ is 1-spreading,

$$\begin{aligned}
\left\|T_{\overline{\alpha}}^{k+1}(u_n) - \beta^{k+1} u'''_n\right\|_S &\leqslant \left\|T_{\overline{\alpha}}^{k+1}(u_n) - T_{\overline{\alpha}}(\beta^k u''_n)\right\|_S + \left\|T_{\overline{\alpha}}(\beta^k u''_n) - \beta^{k+1} u'''_n\right\|_S \\
&= \left\|T_{\overline{\alpha}}(T_{\overline{\alpha}}^k(u_n) - \beta^k u''_n)\right\|_S + \beta^k \left\|T_{\overline{\alpha}}(u''_n) - \beta u'''_n\right\|_S \\
&\leqslant \|T_{\overline{\alpha}}\|_S \cdot \left\|T_{\overline{\alpha}}^k(u_n) - \beta^k u''_n\right\|_S + \beta^k \left\|T_{\overline{\alpha}}(u_n) - \beta u'_n\right\|_S .
\end{aligned}$$

Therefore $\lim_n \left\|T_{\overline{\alpha}}^{k+1}(u_n) - \beta^{k+1} u'''_n\right\|_S = 0$, and so the result holds for $k+1$. By induction, we are finished. □

With the above lemma and Lemma 3.5.3, we have the following corollary.

Corollary 3.6.4. *Fix $N \in \mathbb{N}$. And for each $1 \leqslant i \leqslant N$, fix $\overline{\alpha}_i \subseteq \mathbb{Q}$, $\beta_i \in \mathbb{R}_+$, and $k_i \in \mathbb{N}$. If $(\sigma_n)_{n=1}^{\infty}$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^{N}$-approximating sequence, then it is also a coarse $(\overline{\alpha}_i^{\circ k_i}, \beta_i^{k_i})_{i=1}^{N}$-approximating sequence.*

Lemma 3.6.5. *Fix $N \in \mathbb{N}$. And for each $1 \leqslant i \leqslant N$, fix $\overline{\alpha}_i \subseteq \mathbb{Q}$. Suppose $\overline{\alpha}_i \circ \overline{\alpha}_j = \overline{\alpha}_j \circ \overline{\alpha}_i$ for all $1 \leqslant i, j \leqslant N$. Then there exist $(\beta_i)_{i=1}^{N} \subseteq \mathbb{R}_+$ such that $\|\overline{\alpha}_i\|_\infty \leqslant \beta_i \leqslant \|\overline{\alpha}_i\|_1$ for each $1 \leqslant i \leqslant N$ and $(\sigma_n)_{n=1}^{\infty} \subseteq \mathcal{C}$ such that $(\sigma_n)_{n=1}^{\infty}$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^{N}$-approximating sequence. Moreover, we may choose $(\sigma_n)_{n=1}^{\infty}$ so that for all $n \in \mathbb{N}$, $b_1 \leqslant \sigma_n(1,0) \leqslant b_2$ for some $\gamma < b_1 \leqslant b_2$ not depending on n.*

Proof. For those $\overline{\alpha}_i$'s that are length 1 sequences, the lemma is clear with $(\beta_i) = \overline{\alpha}_i$. So suppose for each $1 \leqslant i \leqslant N$ that $\overline{\alpha}_i$ is a sequence of length at least 2. As the basis $(\xi_n)_{n=1}^{\infty}$ of $\overline{H}$ is 1-unconditional and 1-spreading, $\|\overline{\alpha}_i\|_\infty \|u\|_S \leqslant \|T_{\overline{\alpha}_i}(u)\|_S \leqslant \|\overline{\alpha}_i\|_1 \|u\|_S$, for all $u \in \overline{H}$ and all $1 \leqslant i \leqslant N$. Also, for each $1 \leqslant i \leqslant N$, it is clear from the definition of $T_{\overline{\alpha}_i}$ that $\|T_{\overline{\alpha}_i}(u) - \xi_1\|_S > 0$ for all $u \in \overline{H}$, and so $T_{\overline{\alpha}_i}$ is not invertible. Hence, the spectrum of $T_{\overline{\alpha}_i}$ has a real non-negative boundary point, and so $T_{\overline{\alpha}_i}$ has a real non-negative approximate eigenvalue for each $1 \leqslant i \leqslant N$ (see Proposition IV.1 of [10]). By Lemma 3.5.3, $T_{\overline{\alpha}_i}$ commutes with $T_{\overline{\alpha}_j}$ for all $1 \leqslant i, j \leqslant N$. Thus, there exists $(\beta_i)_{i=1}^{N} \subseteq \mathbb{R}_+$ and a single normalized sequence $(u_n)_{n=1}^{\infty} \subseteq \overline{H}$ such that $\lim_n \|T_{\overline{\alpha}_i} u_n - \beta_i u_n\|_S = 0$ for every $1 \leqslant i \leqslant N$ (see Proposition 12.18 of [3]). As $\|u_n\|_S = 1$ for each $n \in \mathbb{N}$, the bounds above for $\|T_{\overline{\alpha}_i}(u)\|_S$ yield that $\|\overline{\alpha}_i\|_\infty \leqslant \beta_i \leqslant \|\overline{\alpha}_i\|_1$ for each $1 \leqslant i \leqslant N$. By density, one may assume that $(u_n)_{n=1}^{\infty} \subseteq H$ and $1 \leqslant \|u_n\|_S \leqslant 2$ for all $n \in \mathbb{N}$. Finally, let $\delta > 0$ be such that $\rho_{\mathrm{Id}}(\delta/2) > \gamma$ and let σ_n be the type realized by δu_n for each $n \in \mathbb{N}$. The result now follows by letting $b_1 = \rho_{\mathrm{Id}}(\delta)$ and $b_2 = \omega_{\mathrm{Id}}(3\delta)$ (see Lemma 3.5.2). $\square$

Lemma 3.6.6. *Fix $N \in \mathbb{N}$. And for each $1 \leqslant i \leqslant N$, fix $\overline{\alpha}_i \subseteq \mathbb{Q}$. Suppose $\overline{\alpha}_i \circ \overline{\alpha}_j = \overline{\alpha}_j \circ \overline{\alpha}_i$ for all $1 \leqslant i, j \leqslant N$. Then there exists $(\beta_i)_{i=1}^{N} \subseteq \mathbb{R}_+$ such that $\|\overline{\alpha}_i\|_\infty \leqslant \beta_i \leqslant \|\overline{\alpha}_i\|_1$ for each $1 \leqslant i \leqslant N$ and such that every $\sigma \in \mathcal{C}$ is the limit of a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^{N}$-approximating sequence.*

Proof. Let $\gamma < b_1 \leqslant b_2$, $(\beta_i)_{i=1}^{N} \subseteq \mathbb{R}_+$, and $(\sigma_n)_{n=1}^{\infty} \subseteq \mathcal{C}$ be given by Lemma 3.6.5, so that $(\sigma_n)_{n=1}^{\infty}$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^{N}$-approximating sequence and $b_1 \leqslant \sigma_n(1,0) \leqslant b_2$ for every $n \in \mathbb{N}$. Let $\tilde{\mathcal{C}}$ be

the subset of $\mathcal{C}$ consisting of all types of $\mathcal{C}$ which are the limit of a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence. Let $\mathcal{T}_{b_1,b_2}$ be the set $\{\sigma \in \mathcal{T} \mid b_1 \leqslant \sigma(1,0) \leqslant b_2\}$. As $\mathcal{T}_{b_1,b_2}$ is compact and metrizable, $(\sigma_n)_n$ has a converging subsequence which converges to an element $\sigma \in \mathcal{C} \cap \mathcal{T}_{b_1,b_2}$. A subsequence of a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence is still a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence, so $\tilde{\mathcal{C}} \neq \{0\}$, and in particular $\tilde{\mathcal{C}}$ contains an admissible type.

By the minimality of $\mathcal{C}$, it is enough to show that $\tilde{\mathcal{C}}$ is a closed conic class. Suppose $\sigma \in \tilde{\mathcal{C}}$ and $(\sigma_n)_{n=1}^\infty$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence converging to σ. Then, by Lemma 3.3.7, $\lambda \cdot \sigma$ is the limit of $(\lambda \cdot \sigma_n)_{n=1}^\infty$, which is easily seen to be a coarse $(\overline{\alpha}_i, \beta_i)$-approximating sequence for every $\lambda \in \mathbb{Q}$. Thus $\tilde{C}$ is closed under dilation by any $\lambda \in \mathbb{Q}$.

Let D be a metric compatible with the topology of $\mathcal{T}$. Take any $\sigma, \tau \in \tilde{\mathcal{C}}$ and let $(\sigma_n)_{n=1}^\infty$ and $(\tau_n)_{n=1}^\infty$ be coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequences in $\mathcal{C}$ converging to σ and τ, respectively. As convolution is separately continuous, $\lim_k \sigma_k * \tau = \sigma * \tau$ and, for each fixed $k \in \mathbb{N}$, $\lim_n \sigma_k * \tau_n = \sigma_k * \tau$. For each $k \in \mathbb{N}$, let $n(k) \geqslant k$ be such that $D(\sigma_k * \tau_{n(k)}, \sigma_k * \tau) \leqslant 2^{-k}$. Letting $\sigma'_k = \sigma_k * \tau_{n(k)}$ for each $k \in \mathbb{N}$, it follows that $\lim_k \sigma'_k = \sigma * \tau$. To show that $\sigma * \tau \in \tilde{\mathcal{C}}$, it remains to show that $(\sigma'_k)_{k=1}^\infty$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence.

So for each $1 \leqslant i \leqslant N$, let $(u_n)_{n=1}^\infty$, $(u_{i,n})_{n=1}^\infty$, $(v_n)_{n=1}^\infty$ and $(v_{i,n})_{n=1}^\infty$ be sequences realizing $(\sigma_n)_{n=1}^\infty$ and $(\tau_n)_{n=1}^\infty$ respectively, as given by Definition 3.6.2. By translating the supports of $v_{n(k)}$ and $v_{i,n(k)}$, if necessary, we may assume that $\operatorname{supp}(u_k) < \operatorname{supp}(v_{n(k)})$ and $\operatorname{supp}(u_{i,k}) < \operatorname{supp}(v_{i,n(k)})$ for all $1 \leqslant i \leqslant N$ and $k \in \mathbb{N}$. Let $(z_k)_{k=1}^\infty = (u_k + v_{n(k)})_{k=1}^\infty$, so that z_k realizes σ'_k for each $k \in \mathbb{N}$. Let $(z_{i,k})_{k=1}^\infty = (u_{i,k} + v_{i,n(k)})_{k=1}^\infty$ for all $1 \leqslant i \leqslant N$, so that $z_{i,k}$ is a spreading of z_k for each $k \in \mathbb{N}$ and $1 \leqslant i \leqslant N$. This shows that $(\sigma'_k)_{k=1}^\infty$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence. Thus, $\sigma * \tau \in \tilde{\mathcal{C}}$, and so $\tilde{\mathcal{C}}$ is closed under convolution.

It remains to show that $\tilde{\mathcal{C}}$ is closed. Take any $(\sigma_k)_{k=1}^\infty \subseteq \tilde{\mathcal{C}}$ converging to some $\sigma \in \mathcal{C}$. For each $k \in \mathbb{N}$, there exists a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence $(\sigma_{k,n})_{n=1}^\infty$ in $\mathcal{C}$ converging to σ_k. For each $k \in \mathbb{N}$, let $(u_{k,n})_{n=1}^\infty$ be a sequence realizing $(\sigma_{k,n})_{n=1}^\infty$ and let $(u_{k,i,n})_{n=1}^\infty$ be a spreading of $(u_{k,n})_{n=1}^\infty$ for each $1 \leqslant i \leqslant N$ as given by Definition 3.6.2. For each $k \in \mathbb{N}$, choose an integer $n(k) \geqslant k$ such that $D(\sigma_{k,n(k)}, \sigma_k) \leqslant 1/k$ and $\left\| T_{\overline{\alpha}_i}(u_{k,n(k)}) - \beta_i u_{k,i,n(k)} \right\|_S < 1/k$ for each

$1 \leqslant i \leqslant N$. Let $\tau_k = \sigma_{k,n(k)}$ for each $k \in \mathbb{N}$. Then $(\tau_k)_{k=1}^{\infty}$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^{N}$-approximating sequence converging to σ. That is, $\sigma \in \tilde{\mathcal{C}}$. Thus, $\tilde{\mathcal{C}}$ is closed since σ was an arbitrary limit point. By what was shown, $\tilde{\mathcal{C}}$ is a closed admissible conic class contained in $\mathcal{C}$. By the minimality of $\mathcal{C}$, we are finished. $\square$

Lemma 3.6.7. *Fix $N \in \mathbb{N}$. And for each $1 \leqslant i \leqslant N$, fix $\overline{\alpha}_i \subseteq \mathbb{Q}$. Suppose $\overline{\alpha}_i \circ \overline{\alpha}_j = \overline{\alpha}_j \circ \overline{\alpha}_i$ for all $1 \leqslant i, j \leqslant N$. Then there exists $(\beta_i)_{i=1}^{N} \subseteq \mathbb{R}_+$ such that $\|\overline{\alpha}_i\|_\infty \leqslant \beta_i \leqslant \|\overline{\alpha}_i\|_1$ for each $1 \leqslant i \leqslant N$ and such that*

$$\limsup_m \Big| \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i}^{k_i} \psi(\lambda, x) - \mathop{\ast}_{i=1}^{N} b_i \beta_{i,m}^{k_i} \cdot \psi(\lambda, x) \Big| \leqslant \gamma$$

for every $(b_i)_{i=1}^{N} \subseteq \mathbb{Q}$, every $(k_i)_{i=1}^{N} \subseteq \mathbb{N}$, every $(\lambda, x) \in \mathbb{Q} \times \Delta$, and every sequence $(\beta_{i,m})_{m=1}^{\infty} \subseteq \mathbb{Q}_+$ converging to β_i for $1 \leqslant i \leqslant N$.

Proof. Let $(\beta_i)_{i=1}^{N} \subseteq \mathbb{R}_+$ be given by Lemma 3.6.6 and let $(\phi_n)_{n=1}^{\infty}$ be a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^{N}$-approximating sequence converging to ϕ, also given by Lemma 3.6.6. For each $n \in \mathbb{N}$ let $\psi_n = \phi_n * (-1) \cdot \phi_n$. Then, by our choice of ϕ (see Lemma 3.4.5 and the intro to Section 3.5),

$$\lim_n \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i}^{k_i} \psi_n(\lambda, x) = \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i}^{k_i} \psi(\lambda, x)$$

and

$$\lim_n \mathop{\ast}_{i=1}^{N} b_i \beta_{i,m}^{k_i} \cdot \psi_n(\lambda, x) = \mathop{\ast}_{i=1}^{N} b_i \beta_{i,m}^{k_i} \cdot \psi(\lambda, x)$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$ and all $m \in \mathbb{N}$.

By Corollary 3.6.4, $(\phi_n)_{n=1}^{\infty}$ is a coarse $(\overline{\alpha}_i^{\circ k_i}, \beta_i^{k_i})_{i=1}^{N}$-approximating sequence and we can pick a sequence $(u_n)_{n=1}^{\infty}$ realizing $(\phi_n)_{n=1}^{\infty}$ and sequences $(u_{i,n})_{n=1}^{\infty}$ that are spreadings of $(u_n)_{n=1}^{\infty}$ and satisfy $\lim_n \left\| T_{\overline{\alpha}^{\circ k_i}} u_n - \beta_i^{k_i} u_{i,n} \right\|_S = 0$ for every $1 \leqslant i \leqslant N$. For each $n \in \mathbb{N}$, let $u_n' \in H$ have the same basis coordinates as u_n except translated so that $\operatorname{supp}(u_n) < \operatorname{supp}(u_n')$ and $\operatorname{supp}(u_{i,n}) < \operatorname{supp}(u_n')$ for every $1 \leqslant i \leqslant N$. For each $1 \leqslant i \leqslant N$ and $n \in \mathbb{N}$, let $u_{i,n}'$ be a spreading of u_n so that $T_{\overline{\alpha}_i^{\circ k_i}} u_n' - \beta^{k_i} u_{i,n}'$ is a spreading of $T_{\overline{\alpha}_i^{\circ k_i}} u_n - \beta^{k_1} u_{i,n}$ and such that $\operatorname{supp}(u_{i,n}) < \operatorname{supp}(u_{i,n}')$.

Note that both $u_n - u'_n$ and $u_{i,n} - u'_{i,n}$ realize ψ_n. Therefore, by Lemma 3.5.5,

$$\begin{aligned}
&\left|\mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i}^{k_i}\psi_n(\lambda,x) - \mathop{\ast}_{i=1}^{N} b_i\beta_{i,m}^{k_i}\cdot\psi_n(\lambda,x)\right| \\
&\qquad = \left|\mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i^{\circ k_i}}\psi_n(\lambda,x) - \mathop{\ast}_{i=1}^{N} b_i\beta_{i,m}^{k_i}\cdot\psi_n(\lambda,x)\right| \\
&\qquad \leqslant \inf_{\varepsilon>0}\omega_{\mathrm{Id}}\left(|\lambda|\sum_{i=1}^{N}|b_i|\cdot\left\|T_{\overline{\alpha}_i^{\circ k_i}}(u_n-u'_n) - \beta_{i,m}^{k_i}(u_{i,n}-u'_{i,n})\right\|_S + \varepsilon\right) \\
&\qquad \leqslant \inf_{\varepsilon>0}\omega_{\mathrm{Id}}\left(2|\lambda|\sum_{i=1}^{N}|b_i|\cdot\left\|T_{\overline{\alpha}_i^{\circ k_i}}u_n - \beta_{i,m}^{k_i}u_{i,n}\right\|_S + \varepsilon\right) \\
&\qquad \leqslant \inf_{\varepsilon>0}\omega_{\mathrm{Id}}\left(2|\lambda|\left(\sum_{i=1}^{N}|b_i|\cdot\left(\left\|T_{\overline{\alpha}_i^{\circ k_i}}u_n - \beta_i^{k_i}u_n\right\|_S + |\beta_i^{k_i}-\beta_{i,m}^{k_i}|\cdot\|u_n\|_S\right)\right) + \varepsilon\right)
\end{aligned}$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. As the sequence $(u_n)_{n=1}^{\infty}$ is bounded (see Lemma 3.5.2), taking the limit superiors over n and m in the inequality above yields the result. $\square$

3.7 Coarse ℓ_p-types and coarse c_0-types

In this section, we will define what it means for a type to be an ℓ_p-type or c_0-type and use Lemma 3.6.7 to show that ψ is such a type. Finally, we will show that $\overline{H}$ is isomorphic to ℓ_p for some $p \in [1, \infty)$.

Definition 3.7.1. Fix $p \in [1, \infty)$. A type σ is said to be a *coarse ℓ_p-type* if there exists $L > 0$ such that for all $(\lambda, x) \in \mathbb{Q} \times \Delta$ and all $\overline{\alpha} = (\alpha_i)_{i=1}^{N} \subseteq \mathbb{Q}$,

$$\limsup_{m}\left|\mathop{\ast}_{i=1}^{N}\alpha_i\cdot\sigma(\lambda,x) - t_m\cdot\sigma(\lambda,x)\right| \leqslant L$$

for all $(t_m)_{m=1}^{\infty} \subseteq \mathbb{Q}$ converging to $\|\overline{\alpha}\|_p$. A type σ is said to be a *coarse c_0-type* if for all $(\lambda, x) \in \mathbb{Q} \times \Delta$ and all $(\alpha_i)_{i=1}^{N} \subseteq \mathbb{Q}$,

$$\left|\mathop{\ast}_{i=1}^{N}\alpha_i\cdot\sigma(\lambda,x) - \max_{1\leqslant i\leqslant N}|\alpha_i|\cdot\sigma(\lambda,x)\right| \leqslant L.$$

Theorem 3.7.2. *The type ψ is either a coarse c_0-type or a coarse ℓ_p-type for some $p \in [1, \infty)$.*

Proof. Let $\overline{\alpha}_2 = (1,1)$ and $\overline{\alpha}_3 = (1,1,1)$, and note that $\overline{\alpha}_2 \circ \overline{\alpha}_3 = \overline{\alpha}_3 \circ \overline{\alpha}_2$. Let $\beta_2, \beta_3 \in \mathbb{R}$ be given by Lemma 3.6.7 for $\overline{\alpha}_2$ and $\overline{\alpha}_3$, respectively. Let $(\beta_{2,m})_{m=1}^{\infty}, (\beta_{3,m})_{m=1}^{\infty} \subseteq \mathbb{Q}$ be nonzero increasing sequences converging to β_2 and β_3 respectively. By our choice of β_2 and β_3,

$$\limsup_m \left| b \cdot \mathop{\ast}_{i=1}^{j^k} \psi(\lambda, x) - b\beta_{j,m}^k \cdot \psi(\lambda, x) \right| \leqslant \gamma$$

for all $j \in \{2,3\}$, all $b \in \mathbb{Q}$, all $k \in \mathbb{N}$, and all $(\lambda, x) \in \mathbb{Q} \times \Delta$. Let $\ell, k \in \mathbb{N}$ be such that $3^k \leqslant 2^\ell < 3^{k+1}$. As $(\xi_n)_{n=1}^{\infty}$ is 1-unconditional, $\left\|\sum_{i=1}^{3^k} \xi_i\right\|_S \leqslant \left\|\sum_{i=1}^{2^\ell} \xi_i\right\|_S \leqslant \left\|\sum_{i=1}^{3^{k+1}} \xi_i\right\|_S$. Let $a_\ell \in \mathbb{Q}$ be such that $\frac{1}{2}\left\|\sum_{i=1}^{2^\ell} \xi_i\right\|_S \leqslant a_\ell \leqslant \left\|\sum_{i=1}^{2^\ell} \xi_i\right\|_S$. Then, for any $\mu > 0$,

$$\mu \leqslant \left\| \mu \cdot \frac{\sum_{i=1}^{3^{k+1}} \xi_i}{a_\ell} \right\|_S .$$

As $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ is expanding, we can pick $\mu, \eta \in \mathbb{Q}$ such that $\rho_{\mathrm{Id}}(\mu/2) > 2\omega_{\mathrm{Id}}(1) + \gamma$ and $\rho_{\mathrm{Id}}(\eta\|\xi_1\|_S/2) > 2\omega_{\mathrm{Id}}(1) + \gamma$. Let $M \in \mathbb{N}$ be such that

$$\left| \frac{\mu}{a_\ell} \cdot \mathop{\ast}_{i=1}^{3^{k+1}} \psi(1,0) - \frac{\mu\beta_{3,M}^{k+1}}{a_\ell} \cdot \psi(1,0) \right| \leqslant \gamma + \omega_{\mathrm{Id}}(1)$$

and let $M' \geqslant M$ be such that

$$\left| \frac{\eta}{\beta_{2,M}^\ell} \cdot \mathop{\ast}_{i=1}^{2^\ell} \psi(1,0) - \frac{\eta\beta_{2,M'}^\ell}{\beta_{2,M}^\ell} \cdot \psi(1,0) \right| \leqslant \gamma + \omega_{\mathrm{Id}}(1).$$

Then, as $(\mu/a_\ell) \cdot (\sum_{i=1}^{3^{k+1}} \xi_i)$ realizes $(\mu/a_l) \cdot \ast_{i=1}^{3^{k+1}} \psi$, by Lemma 3.5.2(i),

$$2\omega_{\mathrm{Id}}(1) + \gamma < \frac{\mu}{a_\ell} \cdot \mathop{\ast}_{i=1}^{3^{k+1}} \psi(1,0) \leqslant \frac{\mu\beta_{3,M}^{k+1}}{a_\ell} \cdot \psi(1,0) + \gamma + \omega_{\mathrm{Id}}(1).$$

Therefore, as $(\mu\beta_{3,M}^{k+1}/a_\ell) \cdot \xi_1$ realizes $(\mu\beta_{3,M}^{k+1}/a_\ell) \cdot \psi$, by Lemma 3.5.2(ii),

$$1 \leqslant \frac{\beta_{3,M}^{k+1}\mu}{a_\ell} \cdot \|\xi_1\|_S . \tag{3.3}$$

Similarly, by Lemma 3.5.2(i) and the fact that $(\eta\beta^{\ell}_{2,M'}/\beta^{\ell}_{2,M}) \cdot \xi_1$ realizes $(\eta\beta^{\ell}_{2,M'}/\beta^{\ell}_{2,M}) \cdot \psi$,

$$\frac{\eta}{\beta^{\ell}_{2,M}} \cdot \mathop{\ast}_{i=1}^{2^{\ell}} \psi(1,0) \geqslant \frac{\eta\beta^{\ell}_{2,M'}}{\beta^{\ell}_{2,M}} \cdot \psi(1,0) - \gamma - \omega_{\mathrm{Id}}(1) > \omega_{\mathrm{Id}}(1).$$

Thus, as $(\eta/\beta^{\ell}_{2,M}) \cdot (\sum_{i=1}^{2^{\ell}} \xi_i)$ realizes $(\eta/\beta^{\ell}_{2,M}) \cdot \ast_{i=1}^{2^{\ell}} \psi$, by Lemma 3.5.2(ii),

$$\frac{2\eta a_{\ell}}{\beta^{\ell}_{2,M}} \geqslant \left\| \frac{\eta \sum_{i=1}^{2^{\ell}} \xi_i}{\beta^{\ell}_{2,M}} \right\|_S \geqslant 1. \tag{3.4}$$

After combining (3.3) and (3.4), one obtains

$$\frac{\beta^k_3}{\beta^{\ell}_2} = \lim_M \frac{\beta^k_{3,M}}{\beta^{\ell}_{2,M}} \geqslant \frac{1}{2\eta\mu\beta_3 \left\|\xi_1\right\|_S}.$$

The lower bound for β^k_3/β^{ℓ}_2 above does not depend on k or ℓ, so long as $2^{\ell} < 3^{k+1}$. Similarly, a lower bound for β^{ℓ}_2/β^k_3 that also does not depend on k and ℓ can be obtained, so long as $3^k \leqslant 2^{\ell}$. This implies the existence of $a, b > 0$ such that for all k and ℓ satisfying $3^k \leqslant 2^{\ell} < 3^{k+1}$, $a \leqslant \frac{\beta^k_3}{\beta^{\ell}_2} \leqslant b$. Thus, there exists $L \geqslant 0$ such that $\beta_2 = 2^L$ and $\beta_3 = 3^L$. Also, as $\beta_2 \leqslant 2$, it must be the case that $L \in [0,1]$. The same argument as for 2 and 3 works for arbitrary natural numbers. Therefore $\beta_N = N^L$ for all $N \in \mathbb{N}$, where β_N is given by Lemma 3.6.7 for $\overline{\alpha} = (1)^N_{j=1}$.

Suppose first that $L \neq 0$ and let $p = 1/L$. Fix $\overline{\alpha} = (\alpha_i)^N_{i=1} \subseteq \mathbb{Q}$ and a sequence $(t_m)^{\infty}_{m=1} \subseteq \mathbb{Q}$ converging to $\|\overline{\alpha}\|_p$. Take any $\epsilon > 0$ and, for each $1 \leqslant j \leqslant N$, let $r_j \in \mathbb{Q}_+$ be such that $||\alpha_j| - r_j^{1/p}| < \varepsilon$. Let $k \in \mathbb{N}$ be a common denominator so that for each $1 \leqslant j \leqslant N$ there is $n_j \in \mathbb{N}_0$ such that $r_j = n_j/k$. Let $s > 0$ be a rational number such that $|s - (1/k)^{1/p}| < \varepsilon$. For each $1 \leqslant j \leqslant N$, let $(\beta_{j,m})^{\infty}_{m=1} \subseteq \mathbb{Q}$ be a sequence converging to $n_j^{1/p}$and let $(\beta_m)^{\infty}_{m=1} \subseteq \mathbb{Q}$ be a sequence converging to $(\sum_{j=1}^N n_j)^{1/p}$. By Lemma 3.5.5 (and the symmetry of ψ),

$$\left| \mathop{\ast}_{j=1}^{N} \alpha_j \cdot \psi(\lambda, x) - \mathop{\ast}_{j=1}^{N} s\beta_{j,m} \cdot \psi(\lambda, x) \right| \leqslant \omega_{\mathrm{Id}} \left(|\lambda| \sum_{j=1}^{N} ||\alpha_j| - s\beta_{j,m}| \left\|\xi_j\right\|_S + \varepsilon \right)$$

and

$$|s\beta_m \cdot \psi(\lambda, x) - t_m \cdot \psi(\lambda, x)| \leqslant \omega_{\mathrm{Id}}(|\lambda||s\beta_m - t_m| \, \|\xi_1\|_S + \varepsilon)$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. By Lemma 3.6.7 and what was shown above with $L = 1/p$,

$$\limsup_m \left| \mathop{\ast}_{j=1}^{N} s\beta_{j,m} \cdot \psi(\lambda, x) - \mathop{\ast}_{j=1}^{N} s \cdot \mathop{\ast}_{i=1}^{n_j} \psi(\lambda, x) \right| \leqslant \gamma$$

and

$$\limsup_m \left| s \cdot \mathop{\ast}_{j=1}^{N} \mathop{\ast}_{i=1}^{n_j} \psi(\lambda, x) - s\beta_m \cdot \psi(\lambda, x) \right| \leqslant \gamma$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. Combining the four inequalities above with the triangle inequality, taking a limit superior over m, and letting $\epsilon \to 0$, one obtains

$$\limsup_m \left| \mathop{\ast}_{j=1}^{N} \alpha_j \cdot \psi(\lambda, x) - t_m \cdot \psi(\lambda, x) \right| \leqslant 4\gamma$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. Therefore ψ is a coarse ℓ_p-type.

Suppose now that $L = 0$. Fix $\overline{\alpha} = (\alpha_i)_{i=1}^N \subseteq \mathbb{Q}$ such that $\alpha_1 = 1$ and $\alpha_j \leqslant 1$ for $2 \leqslant j \leqslant N$ (the general case will follow from dilation). Using Lemma 3.6.7, find $\beta \geqslant 1$ and a nonzero increasing sequence $(\beta_m)_{m=1}^\infty \subseteq \mathbb{Q}$ converging to β such that

$$\limsup_m |b \cdot \widehat{T}_{\overline{\alpha}}^k \psi(\lambda, x) - b\beta_m^k \cdot \psi(\lambda, x)| \leqslant \gamma$$

for all $b \in \mathbb{Q}$, $k \in \mathbb{N}$ and $(\lambda, x) \in \mathbb{Q} \times \Delta$. Fix $k \in \mathbb{N}$ and note that

$$\widehat{T}_{\overline{\alpha}}^k \psi = \mathop{\ast}_{i_k=1}^{N} \cdots \mathop{\ast}_{i_1=1}^{N} (\prod_{\ell=1}^{k} \alpha_{i_\ell}) \cdot \psi$$

(using the definition of $\widehat{T}_{\overline{\alpha}}$ and the distributivity of dilation over convolution). After combining like terms using the commutativity of convolution, by Lemma 3.6.7 and what was shown above

with $L = 0$,

$$\left| b \cdot \widehat{T}^k_{\overline{\alpha}} \psi(\lambda, x) - b \cdot \mathop{\ast}_{\overline{n} \in F} (\prod_{j=1}^{k} \alpha_j^{n_j}) \cdot \psi(\lambda, x) \right| \leqslant \gamma$$

for every $b \in \mathbb{Q}$ and $(\lambda, x) \in \mathbb{Q} \times \Delta$, where $F = \{\overline{n} = (n_j)_{j=1}^k \subseteq \mathbb{N}_0 \mid \sum_{j=1}^k n_j = k\}$. Now, let $\mu \in \mathbb{Q}$ be such that $\rho_{\mathrm{Id}}(\mu \|\xi_1\|_S /2) > 2\omega_{\mathrm{Id}}(1) + 2\gamma$. Fix $M \in \mathbb{N}$, and let $M' \geqslant M$ be such that

$$|\frac{\mu}{\beta_M^k} \widehat{T}^k_{\overline{\alpha}} \psi(1, 0) - \frac{\mu \beta_{M'}^k}{\beta_M^k} \cdot \psi(1, 0)| \leqslant \gamma + \omega_{\mathrm{Id}}(1).$$

Combining the two inequalities above yields

$$\left| \frac{\mu \beta_{M'}^k}{\beta_M^k} \cdot \psi(1, 0) - \frac{\mu}{\beta_M^k} \cdot \mathop{\ast}_{\overline{n} \in F} (\prod_{j=1}^{k} \alpha_j^{n_j}) \cdot \psi(1, 0) \right| \leqslant 2\gamma + \omega_{\mathrm{Id}}(1).$$

As $(\mu \beta_{M'}^k / \beta_M^k) \xi_1$ realizes $(\mu \beta_{M'}^k / \beta_M^k) \cdot \psi$, by Lemma 3.5.2(i),

$$\frac{\mu}{\beta_M^k} \cdot \mathop{\ast}_{\overline{n} \in F} (\prod_{j=1}^{k} \alpha_j^{n_j}) \cdot \psi(1, 0) \geqslant \omega_{\mathrm{Id}}(1).$$

So, as $\frac{\mu}{\beta_M^k} \sum_{\overline{n} \in F} (\prod_{j=1}^k \alpha_j^{n_j}) \cdot \xi_{I(\overline{n})}$ realizes $\frac{\mu}{\beta_M^k} \cdot \ast_{\overline{n} \in F} (\prod_{j=1}^k \alpha_j^{n_j}) \cdot \psi$ for any injective map $I \colon F \to \mathbb{N}$, by Lemma 3.5.2(ii),

$$1 \leqslant \left\| \frac{\mu}{\beta_M^k} \sum_{\overline{n} \in F} (\prod_{j=1}^{k} \alpha_j^{n_j}) \cdot \xi_{I(\overline{n})} \right\|_S \leqslant \frac{\mu \|\xi_1\|_S}{\beta_M^k} \prod_{\alpha_j < 1} \frac{1}{1 - \alpha_j}.$$

But this was for any $k, M \in \mathbb{N}$, and so it must be the case that $\beta \leqslant 1$. That is, $\beta = 1$. Therefore ψ is a coarse c_0-type. □

Lemma 3.7.3. *Given $p \in [1, \infty)$, if ψ is a coarse ℓ_p-type, then $(\xi_n)_{n=1}^\infty$ is equivalent to the standard basis for ℓ_p. If ψ is a coarse c_0-type, then $(\xi_n)_{n=1}^\infty$ is equivalent to the standard basis for c_0.*

Proof. Suppose that $\psi \in \mathcal{T}$ is a coarse ℓ_p-type (the c_0 case is similar). Let $L > 0$ be such that for

any $(\alpha_j)_{j=1}^N \subseteq \mathbb{Q}$, any $(t_m)_{m=1}^\infty \subseteq \mathbb{Q}$ converging to $\|\overline{\alpha}\|_p$, and any $(\lambda, x) \in \mathbb{Q} \times \Delta$,

$$\limsup_m \left| \mathop{\ast}_{j=1}^N \alpha_j \cdot \psi(\lambda, x) - t_m \cdot \psi(\lambda, x) \right| \leqslant L. \tag{3.5}$$

Let $(e_n)_{n=1}^\infty$ be the standard basis for ℓ_p, and define $T\colon \operatorname{span}\{e_n\}_{n=1}^\infty \to H$ by $Te_n = \xi_n / \|\xi_1\|_S$ for each $n \in \mathbb{N}$, extended linearly. Fix $0 < \varepsilon < 1$ and let $b \in \mathbb{Q}$ be such that $1/\|\xi_1\|_S < b < (1+\varepsilon)/\|\xi_1\|_S$. For each $\overline{\alpha} = (\alpha_i)_{i=1}^N \subseteq \mathbb{Q}$, let $t_{\overline{\alpha}} \in \mathbb{Q}$ be such that $|t_{\overline{\alpha}} - \|\overline{\alpha}\|_p| \leqslant \varepsilon \|\overline{\alpha}\|_p$ and $|\ast_{j=1}^N \alpha_j \cdot \psi(b,0) - t_{\overline{\alpha}} \cdot \psi(b,0)| \leqslant L + \varepsilon$. By (3.5) and Lemma 3.5.2,

$$\begin{aligned}
\rho_{\mathrm{Id}}\left((1-\varepsilon)\left\|\sum_{i=1}^N \alpha_i \frac{\xi_j}{\|\xi_1\|_S}\right\|_S\right) &\leqslant \rho_{\mathrm{Id}}\left((1-\varepsilon)\left\|\sum_{i=1}^N \alpha_i b \xi_j\right\|_S\right) \\
&\leqslant \mathop{\ast}_{i=1}^N \alpha_i \cdot \psi(b,0) \\
&\leqslant t_{\overline{\alpha}} \cdot \psi(b,0) + L + \varepsilon \\
&\leqslant \omega_{\mathrm{Id}}\left(b\|\xi_1\|_S t_{\overline{\alpha}} + \varepsilon\right) + L + \varepsilon \\
&\leqslant \omega_{\mathrm{Id}}\left((1+\varepsilon)^2\|\overline{\alpha}\|_p + \varepsilon\right) + L + \varepsilon,
\end{aligned}$$

for all $\overline{\alpha} = (\alpha_i)_{i=1}^N \subseteq \mathbb{Q}$. Thus, as $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ is a coarse equivalence, there exists $K > 0$ such that $\|\overline{\alpha}\|_p \leqslant 1$ implies $\left\|\sum_{i=1}^N \alpha_i \frac{\xi_i}{\|\xi_1\|_S}\right\|_S \leqslant K$. Therefore T is bounded.

Clearly, T is a bijection, and so has a linear inverse T^{-1}. By (3.5) and Lemma 3.5.2,

$$\begin{aligned}
\rho_{\mathrm{Id}}\left((1-\varepsilon)^2\|\overline{\alpha}\|_p\right) - L - \varepsilon &\leqslant \rho_{\mathrm{Id}}\left((1-\varepsilon) b t_{\overline{\alpha}} \|\xi_1\|_S\right) - L - \varepsilon \\
&\leqslant t_{\overline{\alpha}} \cdot \psi(b,0) - L - \varepsilon \\
&\leqslant \mathop{\ast}_{i=1}^N \alpha_i \cdot \psi(b,0) \\
&\leqslant \omega_{\mathrm{Id}}\left(b\left\|\sum_{i=1}^N \alpha_i \xi_i\right\|_S + \varepsilon\right) \\
&\leqslant \omega_{\mathrm{Id}}\left((1+\varepsilon)\left\|\sum_{i=1}^N \alpha_i \frac{\xi_i}{\|\xi_1\|_S}\right\|_S + \varepsilon\right).
\end{aligned}$$

for all $\overline{\alpha} = (\alpha_i)_{i=1}^N \subseteq \mathbb{Q}$. Thus, as $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ is a coarse equivalence, there exists some

$R > 0$ such that $\left\|\sum_{i=1}^{N} \alpha_i \frac{\xi_i}{\|\xi_1\|_S}\right\|_S \leqslant 1$ implies $\|\overline{\alpha}\|_p < R$. Therefore T^{-1} is bounded. This means T is an isomorphism, and can be extended to an isomorphism between ℓ_p and $\overline{\operatorname{span}}\{\xi_n\}_{n=1}^{\infty}$. $\square$

Theorem 3.7.4. *If a Banach space X is coarsely embeddable into a superstable Banach space, then X has a basic sequence that generates a spreading model isomorphic to ℓ_p for some $p \in [1, \infty)$.*

Proof. By Corollary 3.2.2, if X is coarsely embeddable into a superstable Banach space Y, then there exists a translation-invariant stable pseudometric d on X that is coarsely equivalent to the metric induced by the norm of X. Thus, a space of types $\mathcal{T}$ can be defined as in Section 3.3. Let $\psi \in \mathcal{C}$ be chosen as in the introduction to Section 3.5. By Theorem 3.7.2, ψ is either a c_0-type or an ℓ_p-type for some $p \in [1, \infty)$. Then, by Lemma 3.7.3, X has a spreading model isomorphic to either c_0 or to ℓ_p for some $p \in [1, \infty)$.

Suppose X has a spreading model isomorphic to c_0. In particular, c_0 is finitely representable in X. Thus, c_0 is (isometrically) isomorphic to a subspace of an ultrapower of X. As ultrapowers of X are coarsely embeddable into ultrapowers of Y, this implies that c_0 is coarsely embeddable into an ultrapower of Y, which is a stable space. But this is impossible (see Theorem 2.1 and Theorem 3.6 of [8]). Therefore X contains a spreading model isomorphic to ℓ_p for some $p \in [1, \infty)$.

Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in X without Cauchy subsequences with a spreading model isomorphic to ℓ_p for some $p \in [1, \infty)$. By Rosenthal's ℓ_1 theorem, either $(x_n)_{n=1}^{\infty}$ has a subsequence which is equivalent to the standard basis of ℓ_1, or it has a weakly Cauchy subsequence. In the first case, $(x_n)_{n=1}^{\infty}$ is a basic sequence. So suppose $(x_n)_{n=1}^{\infty}$ is weakly Cauchy. Then $(x_{2n-1} - x_{2n})_{n=1}^{\infty}$ is weakly null and has a spreading model isomorphic to ℓ_p. Thus, after possibly taking a subsequence, we can assume that $(x_{2n-1} - x_{2n})_{n=1}^{\infty}$ is basic. $\square$

Remark 3.7.5. By the last inequality of the ℓ_p case in Theorem 3.7.2, and by following the proof of Lemma 3.7.3, an upper bound of

$$\left(\inf_{\varepsilon>0} \sup \rho_{\mathrm{Id}}^{-1}([0, \omega_{\mathrm{Id}}(1) + 5\gamma + \varepsilon])\right)^2$$

can be found for the Banach-Mazur distance between ℓ_p and the spreading model constructed for X.

Corollary 3.7.6. *There are separable reflexive Banach spaces that are not coarsely embeddable into any superstable Banach space.*

Proof. The original Tsirelson space (see [14]) does not have a spreading model isomorphic to ℓ_p for any $p \in [1, \infty)$, and so provides an example. Another example is the space constructed by E. Odell and Th. Schlumprecht (see Theorem 1.4 of [12]). □

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4. A CODING OF BUNDLE GRAPHS AND THEIR EMBEDDINGS INTO BANACH SPACES*

4.1 Introduction

Recall that given two metric spaces (X, d_X) and (Y, d_Y), X is said to be bi-Lipschitzly embeddable into Y if there is a function $f\colon X \to Y$ and constants $C_1, C_2 > 0$ such that

$$C_1 d_X(x_1, x_2) \leqslant d_Y(f(x_1), f(x_2)) \leqslant C_2 d_X(x_1, x_2) \tag{4.1}$$

for all $x_1, x_2 \in X$, and in this case f is called a bi-Lipschitz embedding. The distortion $\mathrm{dist}(f)$ of a bi-Lipschitz embedding f is the infimum of C_2/C_1 over all constants $C_1, C_2 > 0$ satisfying (4.1). We let $c_Y(X)$ be the infimum of $\mathrm{dist}(f)$ over all bi-Lipschitz embeddings $f\colon X \to Y$. A family of metric spaces $\{X_i\}_{i\in\mathcal{I}}$ is said to be equi-bi-Lipschitzly embeddable into Y if $\sup_{i\in\mathcal{I}} c_Y(X_i) < \infty$.

In [5], J. Bourgain proved that superreflexivity of Banach spaces can be characterized by the non-equi-bi-Lipschitz embeddability of the family of binary trees with finite height. Since then, the non-equi-bi-Lipschitz embeddability of several other families of graphs have also been shown to characterize superreflexivity ([3], [11], [17]). In [4], F. Baudier et al. proved that the non-equi-bi-Lipschitz embeddability of the family of $\aleph_0$-branching diamond graphs characterizes the asymptotic uniform convexifiability of refexive Banach spaces with an unconditional asymptotic structure. They also show that this same family of graphs is equi-bi-Lipschitzly embeddable into L_1.

The families of graphs used in [11], [17], and [4] are all contained in a larger class of graphs, that we call the "bundle graphs". The class of (finitely branching) bundle graphs may be thought of as the class of all bundles (see the seminal paper of A. Gupta, I. Newman, Y. Rabinovich, and A. Sinclair [10]) that have regularity imposed on their branching. The goal of this paper is to gen-

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eralize the results mentioned above to this larger class while providing unified proofs. We order the sections roughly in terms of ease of proof. In Section 4.2, we define what a bundle graph is and provide a natural labeling of the vertices of such graphs. We then derive a formula for the graph metric in terms of this labeling. In Sections 4.3 and 4.4, we generalize two results in [4]. In Section 4.3 we show that every countably-branching bundle graph is bi-Lipschitzly embeddable into any Banach space with a good ℓ_∞-tree with distortion bounded above by a constant depending only on the good ℓ_∞-tree, which implies a more general characterization of asymptotic uniform convexifiability for the class of reflexive Banach spaces with an unconditional asymptotic structure. In Section 4.4 we show that every countably-branching bundle graph is bi-Lipschitzly embeddable into L_1 with distortion bounded above by 2. In Section 4.5 we show that every finitely branching bundle graph is bi-Lipschitzly embeddable into any Banach space containing an equal-signs-additive basic sequence with distortion bounded above by a constant not depending on the branching number (although it will still depend on the bundle graph). However, in Section 4.6, we show that this constant does not increase with $\oslash$-products, and thus generalize the characterizations of superreflexivity found in [11] and [17].

The problem of characterizing superreflexivity in purely metric terms belongs to a more general investigation of metric characterizations of local properties of Banach spaces, called the Ribe program. Surveys of other results in this program can be found in [2] and [15].

4.2 Notation and definitions

We will denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$ and given $n \in \mathbb{N}_0$, we will denote the set $\{i \in \mathbb{N}_0 \mid i \leqslant n\}$ by $[n]$. Given a finite sequence $A = (a_i)_{i=1}^n \subseteq \mathbb{N}_0$, the length of A, denoted by $|A|$, is defined to be n; and the maximum of A, denoted by $\max A$, is defined to be $\max\{a_i\}_{i=1}^n$. If $m \in \mathbb{N}_0$, then we define $A\restriction_m$ by $A\restriction_m = (a_i)_{i=1}^m$ if $m \leqslant n$ and $A\restriction_m = A$ if $m > n$. We write $B \preceq A$ if $B = A\restriction_m$ for some $m \in \mathbb{N}_0$, and write $B \prec A$ if $B \preceq A$ and $B \neq A$. Given another finite sequence B, we denote by $A \wedge B$ the longest sequence C such that $C \preceq A$ and $C \preceq B$, and by $A^\frown B$ the concatenation of A and B. Note that if $A_1 \preceq A_2$ and $A_1 \npreceq B$, then $A_2 \wedge B = A_1 \wedge B$. Note also that if $A_1 \preceq B$ and $A_2 \preceq B$, then either $A_1 \preceq A_2$ or $A_2 \preceq A_1$. We denote the sequence of length 0 (the empty

sequence) by $\varnothing$. Given a set X and $n \in \mathbb{N}_0$, we denote by X^n the set of sequences in X with length equal to n and by $X^{\leqslant n}$ the set of all sequences in X with length at most n.

Given a graph G, we always use the (unweighted) shortest-path metric when discussing the distance between two vertices in G. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$.

Definition 4.2.1. Given a cardinality $\kappa \neq 0$, a graph with two distinguished vertices, one designated the "top", and the other the "bottom", is called a *κ-branching bundle graph* if it can be formed by any (finite) sequence of the following operations:

- (Initialization) Create a path of length 1, with one endpoint designated the top and the other the bottom.
- (Series Composition) Given two κ-branching bundle graphs G_1 and G_2, create a new graph $G_1^\frown G_2$ by identifying the top of G_1 with the bottom of G_2. The bottom of $G_1^\frown G_2$ will be the bottom of G_1 and the top of $G_1^\frown G_2$ will be the top of G_2.
- (Parallel Composition) Given a κ-branching bundle graph G, create a new graph $G^{\|^\kappa}$ by taking κ copies of G and then identifying all the bottoms with each other and all the tops with each other. The bottom of $G^{\|^\kappa}$ will be the bottom of G and the top of $G^{\|^\kappa}$ will be the top of G.

The *height of a bundle graph* is the distance between its bottom and top. The *height of a vertex v* in a bundle graph G is the distance between v and the bottom of G.

Remark 4.2.2. For finite κ, the class of bundle graphs described here is a proper subclass of the class of arbitrary "bundles" found in [10]. Indeed, (finitely branching) bundle graphs are bundles given an unweighted graph metric that have some regularity in the branching that occurs as one travels from bottom to top.

For what follows, two bundle graphs are considered the same if there is a graph isomorphism between them mapping top to top and bottom to bottom. Suppose G is a κ-branching bundle graph

for some cardinality $\kappa \neq 0$. If $\kappa = 1$, then G is a path, and in this case $G^{\|^\kappa} = G$. If κ is finite and $\kappa > 1$, then $G \neq G^{\|^\kappa}$ whenever $|V(G)| \geqslant 3$ by a simple cardinality argument. However, if κ is infinite, then $(G^{\|^\kappa})^{\|^\kappa} = G^{\|^\kappa}$ (since $\kappa \cdot \kappa = \kappa$). But note that for $\kappa > 1$, G could not have been formed in the last step of the sequence of operations by both series composition and parallel composition if $|V(G)| \geqslant 3$. This is because if G is formed in the last step via series composition, then G will have a vertex v that is neither the top nor bottom of G such that no other vertex in G has the same height as v (this v is the glued vertex from the definition). That is, G has a vertex cut consisting of one vertex v. And if G is formed in the last step via parallel composition, then no such vertex will exist. Thus, for infinite κ, $G \neq G^{\|^\kappa}$ implies G was formed in the last step via series composition.

Now, suppose H is another κ-branching bundle graph. Then if G' is a κ-branching bundle graph with the same height as G, and H' is a κ-branching bundle graph, then $G^\frown H = G'^\frown H'$ if and only if $G = G'$ and $H = H'$, by an easy connected component argument. Finally, note that if $u, v \in V(G^{\|^\kappa})$ are adjacent, then they must be copies of vertices that are adjacent in G and must furthermore be contained in the same copy of G. Thus, if $G \neq G^{\|^\kappa}$ and $H \neq H^{\|^\kappa}$, then $G^{\|^\kappa} = H^{\|^\kappa}$ if and only if $G = H$. These observations show that the following definition is well defined.

Definition 4.2.3. Given a vertex v in a κ-branching bundle graph G, the *depth* or *level* of v (with respect to κ) is defined recursively as follows:

- If v is the top or bottom of G, then v has depth 0.
- If v is neither the top nor bottom of G, and G can be constructed in the last step via series composition between two κ-branching bundle graphs G_1 and G_2, then the depth of v in G is the same as its depth in G_1 if $v \in V(G_1)$ or its depth in G_2 if $v \in V(G_2)$.
- If v is neither the top nor bottom of G, and G can be constructed in the last step via parallel composition of a κ-branching bundle graph $G' \neq G$, then the depth of v in G is one more than the depth of v' in G' if v is a copy of $v' \in V(G')$.

It isn't difficult to see that given two κ-branching bundle graphs, G and G', a new κ-branching bundle graph can be created by replacing every edge of G with a copy of G' (where the bottom of G' is placed on the lower endpoint of the edge and the top on the higher). We give a proof of this fact in Section 4.6. Thus the diamond and Laakso graphs used in [11], [17], and [4] are all examples of bundle graphs.

Suppose G is a κ-branching bundle graph with height $M+1$ for some cardinality κ and $M \in \mathbb{N}_0$. From the definitions, every vertex of G at a given height will have the same depth. And if we know the depth associated with each height, we can use Definition 4.2.3 to go backwards to find a sequence of operations from Definition 4.2.1 that can be used to create G. That is, we don't actually need to know the sequence of operations used to create G. All information about G is contained in the branching number κ and the sequence of depths $W = (w_r)_{r=0}^{M+1}$, where w_r is the depth (with respect to κ) associated to height r (we include $w_0 = w_{M+1} = 0$ for convenience).

Suppose $r \in [M+1]$ is such that $w_r > 0$, and let v be a vertex of G with height r. Since $w_r > 0$, v is a copy of some vertex v' in some κ-branching bundle graph G' (that was used in parallel composition in one of the steps to create G). To distinguish v from other copies of v', we label the copies of G' with elements of κ and record the copy in which v was found as a_{w_r}. Now in the graph G', v' has depth $w_r - 1$. If $w_r - 1 > 0$, then we go through this process again for v' to obtain a_{w_r-1}. We repeat this process until we obtain a sequence $A = (a_i)_{i=1}^{w_r}$ that can be used to distinguish v from any other vertex at height r. Doing this for every vertex in G yields a labeling of the vertex set. Actually, we could perform this process on any finite sequence in $\mathbb{N}_0$ beginning and ending in 0 to obtain a κ-branching bundle graph (although in this case, the sequence we start with may not correspond to the sequence of depths for the bundle graph).

With this labeling in mind, we are now in the position to give a non-recursive definition of a bundle graph that is equivalent to Definition 4.2.1. Note that two adjacent vertices u and v of a bundle graph must differ in height by exactly 1. And recall that if u and v are adjacent and were created during parallel composition of a bundle graph G', then u and v must be in the same copy of G'.

Definition 4.2.4. Given a finite sequence $W = (w_r)_{r=0}^{M+1} \subseteq \mathbb{N}_0$ such that $w_0 = w_{M+1} = 0$ and a cardinality κ, the *κ-branching bundle graph associated with* W is $T_{W,\kappa} = (V, E)$, defined by

$$V = \{(r, A) \mid r \in [M+1] \text{ and } A \in \kappa^{w_r}\},$$
$$E = \{\{(r, A), (s, B)\} \subseteq V \mid |r - s| = 1 \text{ and } A \preceq B\}.$$

The vertices $(0, \varnothing)$ and $(M+1, \varnothing)$ in V are called the *bottom* and *top*, respectively, of $T_{W,\kappa}$.

We illustrate in Figure 4.1 below a typical bundle graph with its vertex labeling.

Remark 4.2.5. If we don't specify the branching cardinality κ (or if $\kappa = 1$ or κ is infinite), then many bundle graphs have multiple representations from Definition 4.2.4. For instance, $T_{(0,2,0),2}$ and $T_{(0,1,0,),4}$ are graph isomorphic (both represent a diamond graph of height 2 with 4 midpoints between top and bottom). In the first case we think of the graph as being a 2-branching graph and in the second a 4-branching graph. If we want our graphs to have both a unique branching number and sequence of depths, we could modify Definition 4.2.1 to only allow parallel composition on graphs that could not have been created by parallel composition (that is, we don't allow parallel composition to be performed twice in a row to obtain a bundle graph). Equivalently, we would only allow parallel composition on graphs that contain a vertex v that is neither top nor bottom such that no other vertex has the same height as v. Then we could put requirements on W in addition to $w_0 = w_{M+1} = 0$ to obtain a unique representation of all bundle graphs in Definition 4.2.4 (in this case $T_{(0,1,0,),4}$ would be the canonical representation for our example). However, this is an unnecessary complication for the purpose of this paper.

Remark 4.2.6. The only graphs this paper deals with are bundle graphs. However, in some cases, results concerning other graphs can be recovered. Note for instance that every tree is (isometrically) contained in some bundle graph as a subgraph. Indeed, a κ-branching tree with all leaves having the same finite height can be "doubled" to obtain a bundle graph containing the tree as its lower half. For instance, the binary tree with all leaves having height 3 is contained in $T_{(0,1,2,3,2,1,0),2}$.

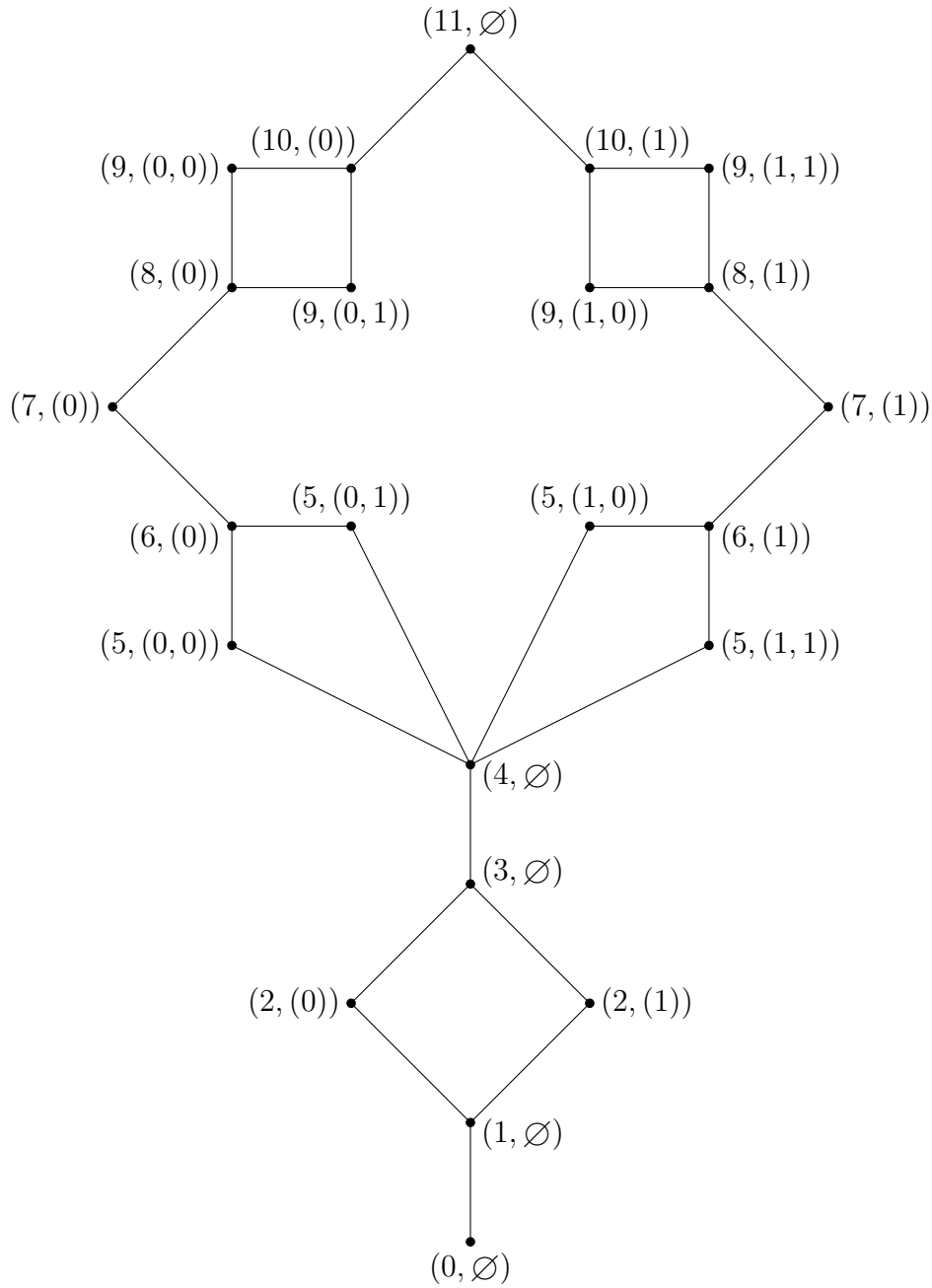

Figure 4.1: $T_{W,\kappa}$ with $W = (0, 0, 1, 0, 0, 2, 1, 1, 1, 2, 1, 0)$ and $\kappa = 2$.

Now that we have a way to represent a bundle graph with Definition 4.2.4, the next order of business is deriving a formula for the shortest-path metric.

Lemma 4.2.7. *Let $T_{W,\kappa} = (V, E)$ be a bundle graph and fix $u = (r, A)$ and $v = (s, B)$ in V. If $(t_i, C_i)_{i=0}^n$ is a path between u and v, then there is $i \in [n]$ such that $C_i \preceq A \wedge B$.*

Proof. Suppose the statement is false and let $(t_i, C_i)_{i=0}^n$ be a path starting at u and ending at v such that $C_i \npreceq A \wedge B$ for all $i \in [n]$. Then in particular, $A = C_0 \npreceq B$. That is, C_0 is such that $C_0 \npreceq B$ and $C_0 \wedge B = A \wedge B$. Take any $i \in [n-1]$ such that $C_i \npreceq B$ and $C_i \wedge B = A \wedge B$. Either $C_{i+1} \preceq C_i$ or $C_i \preceq C_{i+1}$, so suppose first that $C_i \preceq C_{i+1}$. Then $C_{i+1} \npreceq B$ and $C_{i+1} \wedge B = C_i \wedge B = A \wedge B$ because $C_i \npreceq B$.

Suppose now that $C_{i+1} \preceq C_i$. In this case, either $C_{i+1} \preceq B$ or $C_{i+1} \npreceq B$, so suppose first that $C_{i+1} \preceq B$. Then $A \wedge B \prec C_{i+1} = C_{i+1} \wedge B$ because $A \wedge B \preceq B$ and by hypothesis $C_{i+1} \npreceq A \wedge B$. But $C_{i+1} \wedge B \preceq C_i \wedge B = A \wedge B$ since $C_{i+1} \preceq C_i$. This is a contradiction, and so it must be the case that $C_{i+1} \npreceq B$. Therefore $C_{i+1} \wedge B = C_i \wedge B = A \wedge B$ since $C_{i+1} \preceq C_i$.

By induction, it has been shown that for every $i \in [n]$, $C_i \npreceq B$ (and $C_i \wedge B = A \wedge B$). In particular, $B = C_n \npreceq B$, which is impossible. Therefore no such path $(t_i, C_i)_{i=0}^n$ exists as supposed, and so there is $i \in [n]$ such that $C_i \preceq A \wedge B$. $\square$

By Lemma 4.2.7, a path between two vertices $u = (r, A)$ and $v = (s, B)$ in a bundle graph must contain a vertex (t, C) such that both $C \preceq A$ and $C \preceq B$. There are two cases to consider when trying to create a shortest path between u and v: Either such a vertex (t, C) can be found so that t is between r and s, or not. In either case we have $w_t = |C| \leqslant |A \wedge B|$.

We introduce some notation to differentiate these two possibilities. Given two vertices $u = (r, A)$ and $v = (s, B)$ in a bundle graph, we will write $u \Updownarrow v$ to mean that there is $t \in [M + 1]$ between r and s (inclusive) such that $w_t \leqslant |A \wedge B|$. We will write $u \not\Updownarrow v$ to mean the opposite. Note, in particular, that if $A \preceq B$ or $B \preceq A$, then $u \Updownarrow v$, and if $u \not\Updownarrow v$, then $|A \wedge B| < \min\{|A|, |B|\}$.

Definition 4.2.8. Given two vertices $u = (r, A)$ and $v = (s, B)$ in a bundle graph, u is said to be an *ancestor* of v and v is said to be a *descendant* of u if $u \Updownarrow v$ and $r \leqslant s$.

We show in the following proposition that the distance between a vertex in a bundle graph and one of its ancestors or descendants is simply the difference in height between the two. By Lemma 4.2.7, finding the distance between two arbitrary vertices in a bundle graph thus amounts to finding a highest common ancestor or lowest common descendant of the two vertices.

For example, using Figure 4.1, one may see that $(5,(1,1)) - (6,(1)) - (7,(1)) - (8,(1)) - (9,(1,0))$ is a shortest path between $(5,(1,1))$ and $(9,(1,0))$. But a path between $(5,(1,1))$ and $(9,(0,1))$ must first go through either $(4,\varnothing)$ or $(11,\varnothing)$.

Given two vertices $u = (r, A)$ and $v = (s, B)$ in a bundle graph $T_{W,\kappa}$, we define $n(u,v)$ and $m(u,v)$ by

$$n(u,v) = \max\{t \in [M+1] \mid w_t \leqslant |A \wedge B| \text{ and } t \leqslant \min\{r,s\}\},$$
$$m(u,v) = \min\{t \in [M+1] \mid w_t \leqslant |A \wedge B| \text{ and } t \geqslant \max\{r,s\}\}.$$

Following the definitions, one sees that if $u \not\Updownarrow v$, then $\big(n(u,v), A \wedge B\restriction_{w_{n(u,v)}}\big)$ is the highest common ancestor of u and v and $\big(m(u,v), A \wedge B\restriction_{w_{m(u,v)}}\big)$ is the lowest common descendant of u and v.

Proposition 4.2.9. *Let $T_{W,\kappa} = (V,E)$ be a bundle graph with shortest-path metric d, and fix $u = (r,A)$ and $v = (s,B)$ in V. Then*

$$d(u,v) = \begin{cases} |r-s| & u \Updownarrow v \\ \min\{r+s-2n(u,v), 2m(u,v)-(r+s)\} & u \not\Updownarrow v \end{cases}$$

Proof. By the definition of the edge set E, $d(u,v) \geqslant |r-s|$. Suppose first that $A \preceq B$. Recursively construct the sequence $(C_i)_{i=0}^{|r-s|}$ by letting

$$C_0 = \begin{cases} A & r \leqslant s \\ B & s < r \end{cases}$$

and given C_{i-1} for $i \in [|r-s|]\backslash\{0\}$, choosing $C_i \in \kappa^{w_{\min\{r,s\}+i}}$ so that $C_{i-1} \wedge C_i \in \{C_{i-1}, C_i\}$ and $C_i \wedge B \in \{C_i, B\}$. Then $((\min\{r,s\}+i, C_i))_{i=0}^{|r-s|}$ is a path between u and v, and so $d(u,v) \leqslant |r-s|$. That is, $d(u,v) = |r-s|$. The case $B \preceq A$ is similar.

Now, if $u \Updownarrow v$, then there is t between r and s such that $|A \wedge B| \geqslant w_t$. Thus, by what was shown above,

$$\begin{aligned} d(u,v) &\leqslant d\left(u, (t, A \wedge B\restriction_{w_t})\right) + d\left((t, A \wedge B\restriction_{w_t}), v\right) \\ &= |r-t| + |t-s| \\ &= |r-s| \end{aligned}$$

and so $d(u,v) = |r-s|$.

If $u \not\Updownarrow v$, then by Lemma 4.2.7 a shortest path between u and v must contain a vertex that is either a common ancestor or a common descendant of u and v. The result follows from what was shown above by taking the minimum of lengths of paths between u and v that contain either the highest common ancestor or lowest common descendant. $\square$

Continuing our example from Figure 4.1, one may check that $w_7 = 1 \leqslant |(1)| = |(1,1) \wedge (1,0)|$. Thus $(5,(1,1)) \Updownarrow (9,(1,0))$ and so $d((5,(1,1),(9,(1,0))) = 9-5 = 4$ by Proposition 4.2.9. Similarly, there is no $t \in [11]$ between 5 and 9 such that $w_t \leqslant 0 = |\varnothing| = |(1,1) \wedge (0,1)|$, meaning $(5,(1,1)) \not\Updownarrow (9,(1,0))$. One may determine that $n((5,(1,1)),(9,(0,1))) = 4$ and $m((5,(1,1)),(9,(0,1))) = 11$. By Proposition 4.2.9, $d((5,(1,1)),(9,(0,1))) = \min\{5+9-2\cdot 4, 2\cdot 11-(9+5)\} = 6$.

Throughout the next few sections, we describe various bi-Lipschitz embeddings of bundle graphs into Banach spaces. We define now the notation that will regularly be used, and fix for the rest of the paper $W = (w_r)_{r=0}^{M+1} \subseteq \mathbb{N}_0$ such that $w_0 = w_{M+1} = 0$. For $r \in [M+1]$ and $i \in \mathbb{N}_0$, define $x(r,i)$, $y(r,i)$, and $z(r,i)$ by

$$x(r,i) = \begin{cases} 0 & i = 0 \\ \max\{t \in [M+1] \mid w_t < i \text{ and } t \leqslant r\} & i > 0, \end{cases} \tag{4.2}$$

$$y(r,i) = \begin{cases} M+1 & i = 0 \\ \min\{t \in [M+1] \mid w_t < i \text{ and } t \geqslant r\} & i > 0, \end{cases} \tag{4.3}$$

$$z(r,i) = \begin{cases} r & i = 0 \\ \min\{r - x(r,i), y(r,i) - r\} & i > 0. \end{cases} \tag{4.4}$$

Given $r \in [M+1]$ and $i \in \mathbb{N}$, $x(r,i)$ records the last height no greater than r in which the vertices of a bundle graph associated with W have depth less than i. Similarly, $y(r,i)$ records the first height no lesser than r in which the vertices of a bundle graph associated with W have depth less than i. And $z(r,i)$ simply records the distance one would have to travel from height r to get to a vertex with depth less than i in a bundle graph associated with W. Note that $x(r,i) = y(r,i) = r$ and $z(r,i) = 0$ for $i > w_r$.

Consider $u = (r, A)$ and $v = (s, B)$ in $V(T_{W,\kappa})$ for some cardinality κ. If $u \not\parallel v$ (recall this means there is no $t \in [M+1]$ between r and s such that $w_t \leqslant |A \wedge B|$), then a comparison of the definitions of $n(u,v)$ and $m(u,v)$ with notations (4.2) and (4.3), respectively, yields $n(u,v) = x(r, |A \wedge B| + 1) = x(s, |A \wedge B| + 1)$ and $m(u,v) = y(r, |A \wedge B| + 1) = y(s, |A \wedge B| + 1)$. This fact will be used repeatedly in the proofs to follow.

4.3 Embedding into Banach spaces with good ℓ_∞-trees

In this section, we show that for any countable cardinality κ, $T_{W,\kappa}$ is bi-Lipschitzly embeddable into any Banach space with a good ℓ_∞-tree of height $\max W$ with distortion bounded above by a constant depending only on the good ℓ_∞-tree.

Definition 4.3.1. Given $n \in \mathbb{N}$ and $C, D > 0$, a Banach space $(X, \|\cdot\|_X)$ is said to *contain a (C,D)-good ℓ_∞-tree of height n* if given some enumeration $(\sigma_i)_{i=0}^{\infty}$ of $\aleph_0^{\leqslant n}$ such that $i_1 \leqslant i_2$

whenever $\sigma_{i_1} \preceq \sigma_{i_2}$, there is a sequence $(y_{\sigma_i})_{i=0}^{\infty} \subseteq S_X$ such that, given any $(\alpha_i)_{i=0}^{\infty} \subseteq \mathbb{R}$,

(i) $1/C\|(\alpha_i)_{i=0}^n\|_\infty \leqslant \|\sum_{B\preceq A} \alpha_{|B|} y_B\|_X \leqslant C\|(\alpha_i)_{i=0}^n\|_\infty$ for all $A \in \aleph_0^n$,

(ii) $\|\sum_{i=0}^{m_1} \alpha_i y_{\sigma_i}\|_X \leqslant D\,\|\sum_{i=0}^{m_2} \alpha_i y_{\sigma_i}\|_X$ for all $m_1, m_2 \in \mathbb{N}_0$ such that $m_1 \leqslant m_2$.

In this case $(y_{\sigma_i})_{i=0}^{\infty}$ is called a (C, D)-*good* ℓ_∞-*tree of height* n.

The given definition is a finite height analogue of Definition 3.1 in [4]. The first condition states that every "branch" $(y_B)_{B\preceq A}$ of the good ℓ_∞-tree is C^2-equivalent to the unit vector basis of ℓ_∞^{n+1}. The second condition states that the sequence making up the good ℓ_∞-tree is basic with basis constant less than or equal to D. Note that the condition on the enumeration implies $\sigma_0 = \varnothing$.

Theorem 4.3.2 below generalizes Theorem 3.1 in [4]. The proof we show here has the same main idea as the proof in [4], but is much shorter due to the fact that we fix at the beginning a single bundle graph, rather than prove the result for an entire family of bundle graphs. This gets rid of a lengthy induction argument.

Theorem 4.3.2. *Fix a countable cardinality κ and suppose X is a Banach space containing a (C, D)-good ℓ_∞-tree $(y_{\sigma_i})_{i=0}^{\infty}$ of height $\max W$ for some $C, D > 0$. Then there is a bi-Lipschitz embedding $\psi\colon T_{W,\kappa} \to X$ such that for all $u, v \in V(T_{W,\kappa})$,*

$$\frac{1}{3D(1+D)} d(u,v) \leqslant \|\psi(u) - \psi(v)\|_X \leqslant C d(u,v),$$

where d is the shortest-path metric for $T_{W,\kappa}$, and furthermore, $\|\psi(u) - \psi(v)\|_X \geqslant d(u,v)/D$ when $u \Updownarrow v$.

Proof. Define the map $\psi\colon T_{W,\kappa} \to X$ by

$$\psi((r, A)) = \sum_{B\preceq A} z(r, |B|) y_B$$

(see notation (4.4) in the previous section) for every $(r, A) \in V(T_{W,\kappa})$.

Take any $u = (r, A)$ and $v = (s, B)$ in $V(T_{W,\kappa})$ and suppose first that u and v are adjacent with $A \preceq B$ (implying $w_r \leqslant w_s$). Then,

$$\begin{aligned}
\|\psi(u) - \psi(v)\|_X &= \left\| \sum_{E \preceq A} z(r, |E|) y_E - \sum_{E \preceq B} z(s, |E|) y_E \right\|_X \\
&= \left\| \sum_{E \preceq B} (z(r, |E|) - z(s, |E|)) y_E \right\|_X \\
&\leqslant C \max_{0 \leqslant i \leqslant w_s} \{|z(r, i) - z(s, i)|\} \\
&\leqslant C,
\end{aligned}$$

where the second line follows from the assumption that $A \preceq B$ and the fact that $z(r, i) = 0$ for $w_r < i \leqslant w_s$, the third line from property (i) of the ℓ_∞-tree, and the last line from the assumption that $|r - s| = 1$. The triangle inequality applied to shortest paths then shows that $\|\psi(u) - \psi(v)\|_X \leqslant Cd(u, v)$ for all $u, v \in V(T_{W,\kappa})$.

For the left-hand inequality, take any $u = (r, A)$ and $v = (s, B)$ in $V(T_{W,\kappa})$, and suppose first that $u \Updownarrow v$. Then property (ii) of the ℓ_∞-tree yields

$$\|\psi(u) - \psi(v)\|_X \geqslant \frac{1}{D} |z(r, 0) - z(s, 0)| = \frac{1}{D} |r - s| = \frac{1}{D} d(u, v).$$

Suppose now that $u \not\Updownarrow v$. As mentioned in the last paragraph of the previous section, note that $n(u, v) = x(r, |A \wedge B| + 1) = x(s, |A \wedge B| + 1)$ and $m(u, v) = y(r, |A \wedge B| + 1) = y(s, |A \wedge B| + 1)$. Let $n_1 \in \mathbb{N}_0$ be such that $\sigma_{n_1} \preceq A$ and $|\sigma_{n_1}| = |A \wedge B| + 1$. Similarly, let $n_2 \in \mathbb{N}_0$ be such that $\sigma_{n_2} \preceq B$ and $|\sigma_{n_2}| = |A \wedge B| + 1$. For $i \in \mathbb{N}_0$, let

$$\alpha_i = \begin{cases} z(r, |\sigma_i|) - z(s, |\sigma_i|) & \sigma_i \preceq A \wedge B \\ z(r, |\sigma_i|) & A \wedge B \prec \sigma_i \preceq A \\ -z(s, |\sigma_i|) & A \wedge B \prec \sigma_i \preceq B \\ 0 & \text{otherwise.} \end{cases}$$

Then, by property (ii) of the ℓ_∞-tree and Proposition 4.2.9,

$$\begin{aligned}
\|\psi(u)-\psi(v)\|_X &= \left\|\sum_{i=0}^{\infty}\alpha_i y_{\sigma_i}\right\|_X \\
&\geqslant \frac{1}{D}\max\left\{\left\|\sum_{i=0}^{n_1}\alpha_i y_{\sigma_i}\right\|_X, \left\|\sum_{i=0}^{n_2}\alpha_i y_{\sigma_i}\right\|_X, \|\alpha_0 y_\varnothing\|_X\right\} \\
&\geqslant \frac{1}{D(1+D)}\max\{|\alpha_{n_1}|,|\alpha_{n_2}|,|\alpha_0|\} \\
&= \frac{1}{D(1+D)}\max\{z(r,|A\wedge B|+1), z(s,|A\wedge B|+1), |z(r,0)-z(s,0)|\} \\
&= \frac{1}{D(1+D)}\max\{\min\{r-n(u,v), m(u,v)-r\}, \\
&\qquad\qquad \min\{s-n(u,v), m(u,v)-s\}, |r-s|\} \\
&\geqslant \frac{1}{3D(1+D)}d(u,v).
\end{aligned}$$

□

Remark 4.3.3. In the proof above, $1+D$ appears by using the triangle inequality and the fact that D is a monotonicity constant for the sequence $(y_{\sigma_i})_{i=0}^{\infty}$. One may replace $1+D$ with D if D is actually a *bimonotonicity* constant.

It was also shown in [4] (see Theorem 3.2 of that paper) that any reflexive Banach space with an unconditional asymptotic structure that is not asymptotically uniformly convexifiable will contain $(1+\varepsilon, 1+\varepsilon)$-good ℓ_∞-trees of arbitrary height, for any $\varepsilon > 0$. Thus, Theorem 4.3.2 of this paper yields the following corollary.

Corollary 4.3.4. *For any $\varepsilon > 0$, every countably-branching bundle graph is bi-Lipschitzly embeddable with distortion bounded above by $6+\varepsilon$ into any reflexive Banach space with an unconditional asymptotic structure that is not asymptotically uniformly convexifiable.*

Finally, in [4] it was shown (see Theorem 4.1 of that paper) that if a Banach space X is asymptotically midpoint uniformly convexifiable, then no family of bundle graphs with nontrivial (meaning there is a vertex with nonzero depth) $\aleph_0$-branching base graph is equi-bi-Lipschitzly embeddable into X. This fact combined with Corollary 4.3.4 actually shows that the non-equi-bi-Lipschitz embeddability of any family of bundle graphs generated by a nontrivial $\aleph_0$-branching

bundle graph characterizes asymptotic uniform convexifiability within the class of reflexive Banach spaces with an asymptotic unconditional structure. We recall the definition of a family of bundle graphs generated from a base graph in Section 4.6.

4.4 Embedding into L_1

In this section we show that for any countable cardinality κ, $T_{W,\kappa}$ is bi-Lipschitzly embeddable into $L_1([0, M+1])$ (equipped with Lebesgue measure λ) with distortion bounded above by 2. Note however that for finite κ, $T_{W,\kappa}$ belongs to the family of series-parallel graphs and so is already known to be bi-Lipschitzly embeddable into ℓ_1 with distortion bounded above by 2 (see [8]). That this distortion bound is optimal follows from work done in [14]. For each $v \in V(T_{W,\kappa})$, we map v to the characteristic function of some set. To get the distortion we desire, we need to make sure that the symmetric differences of the sets involved are large enough in Lebesgue measure. The construction is somewhat technical, but is essentially done through intersections of supports of independent Bernoulli random variables.

Let D be any common multiple of the numbers in $[M+1]\backslash\{0\}$. Let $\mathcal{F}$ be the family of all finite unions of open subintervals of $[0, M+1]$ such that for each $P \in \mathcal{F}$, there is $N \in \mathbb{N}_0$ such that each maximal (with respect to set containment) subinterval of P is equal to $\left(\frac{qD}{D^{N+1}}, \frac{qD+r}{D^{N+1}}\right)$ for some $q \in [(M+1)D^N - 1]$ and $r \in [D]$; and given $P \in \mathcal{F}$, let $N(P)$ be the minimum of such N associated with P. Note here that if $P, P' \in \mathcal{F}$ are such that $N(P) < N(P')$, then every subinterval of P' is either contained in a subinterval of P or has empty intersection with P.

Let $(\sigma_i)_{i=0}^{\infty}$ be an enumeration of $\aleph_0^{\leqslant \max W}$ and let $(P_i)_{i=0}^{\infty}$ be an enumeration of $\mathcal{F}$ such that $P_0 = \varnothing$ and $N(P_i) \leqslant i$ for all $i \in \mathbb{N}_0$. For each $i, j \in \mathbb{N}_0$, let $\theta(i,j) = 2^i 3^j - N(P_j)$. Define $f\colon \aleph_0^{\leqslant \max W} \times \mathcal{F} \times [D] \to \mathcal{F}$ by

$$f(\sigma_i, P_j, k) = \bigcup_{\ell=0}^{n} \bigcup_{m=0}^{D^{\theta(i,j)}-1} \left(\alpha_\ell + \frac{mD(\beta_\ell - \alpha_\ell)}{D^{\theta(i,j)+1}}, \alpha_\ell + \frac{(mD+k)(\beta_\ell - \alpha_\ell)}{D^{\theta(i,j)+1}}\right)$$

whenever $P_j = \bigsqcup_{\ell=0}^{n} (\alpha_\ell, \beta_\ell)$ (where $\sqcup$ means disjoint union), for all $i, j \in \mathbb{N}_0$ and $k \in [D]$.

We list the properties we need from f in the following lemma, but one may think of $f(A, P, k)$

as the intersection of P with the support of a Bernoulli random variable that has probability of success equal to k/D. And if $B \neq A$, then $f(B, P, k)$ is also such an intersection, but with a Bernoulli random variable that is independent from that used for $f(A, P, k)$.

Lemma 4.4.1. *The following hold for all $i, j \in \mathbb{N}_0$ and $k \in [D]$:*

(i) $f(\sigma_i, P_j, k) \subseteq f(\sigma_i, P_j, k') \subseteq P$ *if* $k' \in [D]$ *is such that* $k \leqslant k'$.

(ii) $2^i 3^j - 1 \leqslant N(f(\sigma_i, P_j, k)) \leqslant 2^i 3^j + 2$ *if* $j \neq 0$ *and* $k \neq 0$.

(iii) $\lambda f(\sigma_i, P_j, k) = \frac{k}{D}\lambda(P_j)$.

(iv) $\lambda(P \cap f(\sigma_i, P_j, k)) = \frac{k}{D}\lambda(P \cap P_j)$ *if* $P \in \mathcal{F}$ *is such that* $N(P) < N(P_j)$.

Proof. (i): This is obvious from the definition of f.

(ii): Every maximal subinterval of P_j has length less than or equal to $1/D^{N(P_j)}$, so by the definition of f, every maximal subinterval of $f(\sigma_i, P_j, k)$ has length less than or equal to

$$\frac{k}{D^{\theta(i,j)+1}} \cdot \frac{1}{D^{N(P_j)}} = \frac{k}{D^{2^i 3^j + 1}} \leqslant \frac{1}{D^{2^i 3^j}}.$$

This means $N(f(\sigma_i, P_j, k)) \geqslant 2^i 3^j - 1$. Similarly, every (nontrivial) maximal subinterval of P_j has length greater than or equal to $1/D^{N(P_j)+1}$, and so every (nontrivial) maximal subinterval of $f(\sigma_i, P_j, k)$ has length greater than or equal to

$$\frac{k}{D^{\theta(i,j)+1}} \cdot \frac{1}{D^{N(P_j)+1}} = \frac{k}{D^{2^i 3^j + 2}} \geqslant \frac{1}{D^{2^i 3^j + 2}}.$$

Therefore $N(f(\sigma_i, P_j, k)) \leqslant 2^i 3^j + 2$.

(iii): Supposing $P_j = \bigsqcup_{\ell=0}^{n} (\alpha_\ell, \beta_\ell)$, then

$$\lambda(f(\sigma_i, P_j, k)) = \sum_{\ell=0}^{n} \sum_{m=0}^{D^{\theta(i,j)}-1} \frac{k(\beta_\ell - \alpha_\ell)}{D^{\theta(i,j)+1}} = \frac{k}{D} \sum_{\ell=0}^{n} (\beta_\ell - \alpha_\ell) = \frac{k}{D}\lambda(P_j).$$

(iv): Since $N(P) < N(P_j)$, every subinterval of P_j is either contained in a subinterval of P or has empty intersection with P. Thus, if $P_j = \bigsqcup_{\ell=0}^{n}(\alpha_\ell, \beta_\ell)$ and $I = \{\ell \in [n] \mid (\alpha_\ell, \beta_\ell) \subseteq P\}$, then

$$\lambda(P \cap f(\sigma_i, P_j, k)) = \sum_{\ell \in I} \sum_{m=0}^{D^{\theta(i,j)}-1} \frac{k(\beta_\ell - \alpha_\ell)}{D^{\theta(i,j)+1}} = \frac{k}{D} \sum_{\ell \in I} (\beta_\ell - \alpha_\ell) = \frac{k}{D} \lambda(P \cap P_j). \qquad \square$$

We are now ready to define the sets needed for our bi-Lipschitz embedding. This is done by recursively defining the sets based on the depths of the vertices in our bundle graph. At any given depth we construct the sets out of subsets of the sets that were defined for the previous depth. If a vertex has height r and depth 0, we assign the set $[0, r]$ to this vertex. Suppose there are vertices at heights $r \leqslant s$ with depth 0 and v is a vertex with depth 1 at height halfway between r and s. We assign a set of measure $r+(s-r)/2$ to v by taking the union of $[0, r]$ with half of the set $[0, s]\backslash[0, r]$. So for instance, we might assign the set $[0, r] \cup [r, (s-r)/2]$ to v. However, there will be another vertex with depth 1 at the same height as v (if $\kappa > 1$). For this vertex, we need to assign a different set of measure $r + (s-r)/2$, so we take half of the set $[0, s]\backslash[0, r]$ in a way that is independent of the way we did it with v. For instance, we might use $[0, r] \cup [r, (s-r)/4] \cup [(s-r)/2, 3(s-r)/4]$. A similar process for all depths is used until every vertex has a subset of $[0, M+1]$ assigned to it with Lebesgue measure equal to its height. We use the function f defined above to take care of the independent selection of sets. At this point we fix for the rest of the section a countable cardinality κ. The formal construction follows (recall notations (4.2) and (4.3) from Section 4.2).

Given $v = (r, A) \in V(T_{W,\kappa})$, define the sets $S_x(v, i)$ and $S_y(v, i)$ in $\mathcal{F}$ for $i \in [w_r]$ recursively by

$$S_x(v, 0) = [0, x(r, 1)]\backslash[x(r, 1)],$$

$$S_y(v, 0) = [0, y(r, 1)]\backslash[y(r, 1)],$$

and

$$S_x(v,i) = f\left(A\restriction_i, S_y(v,i-1)\backslash\mathrm{clos}(S_x(v,i-1)), \frac{x(r,i+1)-x(r,i)}{y(r,i)-x(r,i)}D\right),$$
$$S_y(v,i) = f\left(A\restriction_i, S_y(v,i-1)\backslash\mathrm{clos}(S_x(v,i-1)), \frac{y(r,i+1)-x(r,i)}{y(r,i)-x(r,i)}D\right),$$

for $i \in [w_r]\backslash\{0\}$. Finally, let $S(v) = \mathrm{clos}\left(\bigcup_{i=0}^{w_r} S_x(v,i)\right)$.

Lemma 4.4.2. *Fix $v = (r, A) \in V(T_{W,\kappa})$. The following hold for all $i \in [w_r]$.*

(i) $\lambda(S_x(v,i)) = x(r,i+1) - x(r,i)$.

(ii) $\lambda(S_y(v,i)) = y(r,i+1) - x(r,i)$.

(iii) $\lambda(S_y(v,i)\backslash S_x(v,i)) = y(r,i+1) - x(r,i+1)$.

(iv) $S_x(v,i) \cap S_x(v,i') = \varnothing$ *if* $i' \in [w_r]$ *is such that* $i' \neq i$.

(v) $\lambda(\cup_{k=0}^{i} S_x(v,k)) = x(r,i+1)$.

Proof. (i)-(iii): These statements certainly hold true for $i = 0$. And by (simultaneous) induction and Lemma 4.4.1 (i) and (iii), they hold true for all $i \in [w_r]$.
(iv): By Lemma 4.4.1 (i), $S_x(v, w_r - k) \subseteq S_y(v, w_r - k - 1)\backslash S_x(v, w_r - k - 1)$, and so $S_x(v, w_r - k) \cap S_x(v, w_r - k - 1) = \varnothing$ for all $k \in [w_r - 1]$. In the same way, $S_y(v, w_r - k - 1) \subseteq S_y(v, w_r - k - 2)\backslash S_x(v, w_r - k - 2)$, and so the first set inclusion implies $S_x(v, w_r - k) \cap S_x(v, w_r - k - 2) = \varnothing$ for all $k \in [w_r - 2]$. Inductively, $S_x(v, w_r - k_1) \cap S_x(v, w_r - k_2) = \varnothing$ for all $k_1 < k_2 \in [w_r]$.
(v): This follows from parts (i) and (iv). □

Lemma 4.4.3. *Fix $u = (r, A)$ and $v = (s, B)$ in $V(T_{W,\kappa})$. Then*

$$\lambda(S(u) \cap S(v)) = \begin{cases} \min\{r, s\} & u \Updownarrow v \\ n(u,v) + \frac{(r-n(u,v))(s-n(u,v))}{m(u,v)-n(u,v)} & u \not\Updownarrow v \end{cases}$$

Proof. Let $K = |A \wedge B|$. Suppose first that $u \Updownarrow v$ and $r \leqslant s$. Then $y(r, K+1) \leqslant x(s, K+1)$. Let $n \in [K]$ be such that $x(s,i) < y(r,i)$ (which implies $x(s,i) = x(r,i)$ and $y(s,i) = y(r,i)$) for all $i \in [n]$ while $y(r, n+1) \leqslant x(s, n+1)$. An easy induction argument shows that $S_x(u,i) = S_x(v,i)$ and $S_y(u,i) = S_y(v,i)$ for all $i \in [n-1]$, and $S_y(u,n) \subseteq S_x(v,n)$. Another easy induction argument and Lemma 4.4.1 (i) shows $S_x(u,i) \subseteq \bigcup_{j=0}^{n} S_x(v,j)$ for all $i \in [w_r]$ by , and so Lemma 4.4.2 (v) yields

$$\lambda(S(u) \cap S(v)) = \lambda\left(\bigcup_{i=0}^{w_r} S_x(u,i) \cap \bigcup_{j=0}^{w_s} S_x(v,j)\right) = \lambda\left(\bigcup_{i=0}^{w_r} S_x(u,i)\right) = x(r, w_r+1) = r.$$

Suppose now that $u \not\Updownarrow v$. Then $x(r,i) = x(s,i)$ and $y(r,i) = y(s,i)$ for all $i \in [K+1]$, and so $S_x(u,i) = S_x(v,i)$ and $S_y(u,i) = S_y(v,i)$ for all $i \in [K]$, by an easy induction argument. Note that $x(r, K+1) = x(s, K+1) = n(u,v)$ and $y(r, K+1) = y(s, K+1) = m(u,v)$. For any $i \in [w_r] \backslash [K]$ and $j \in [w_s] \backslash [K]$, repeated applications of Lemma 4.4.1 (ii) and (iv) and then Lemma 4.4.2 (iii) show

$$\begin{aligned}
&\lambda(S_x(u,i) \cap S_x(v,j)) \\
&\qquad = \frac{x(r,i+1) - x(r,i)}{y(r,K+1) - x(r,K+1)} \cdot \frac{x(s,j+1) - x(s,j)}{y(s,K+1) - x(s,K+1)} \cdot \lambda(S_y(u,K) \backslash S_x(u,K)) \\
&\qquad = \frac{(x(r,i+1) - x(r,i))(x(s,j+1) - x(s,j))}{m(u,v) - n(u,v)}.
\end{aligned}$$

This with Lemma 4.4.2 (iv) and (v) implies

$$\begin{aligned}
\lambda(S(u) \cap S(v)) &= \lambda\left(\bigcup_{i=0}^{w_r} S_x(u,i) \cap \bigcup_{j=0}^{w_s} S_x(v,j)\right) \\
&= x(r, K+1) + \sum_{i=K+1}^{w_r} \sum_{j=K+1}^{w_s} \frac{(x(r,i+1) - x(r,i))(x(s,j+1) - x(s,j))}{m(u,v) - n(u,v)} \\
&= x(r, K+1) + \frac{(x(r, w_r+1) - x(r, K+1))(y(s, w_s+1) - y(s, K+1))}{m(u,v) - n(u,v)} \\
&= n(u,v) + \frac{(r - n(u,v))(s - n(u,v))}{m(u,v) - n(u,v)}.
\end{aligned}$$

□

The next theorem generalizes Theorem 3.3 in [4] (and our earlier lemmas likewise generalized Lemma 3.6 and its discussion in [4]). We once again follow the same line of thought in our proof as in the proof in [4], but again our proof is shorter. This time we have still used a recursive process to define the embedding (this has been absorbed in our lemmas), but it has been made simpler by again fixing a single bundle graph at the beginning and performing the recursion over the depths of its vertices.

Theorem 4.4.4. *There is a bi-Lipschitz embedding $\psi\colon T_{W,\kappa} \to L_1$ such that for all $u, v \in V(T_{W,\kappa})$,*

$$\frac{1}{2}d(u,v) \leqslant \|\psi(u) - \psi(v)\|_{L_1} \leqslant d(u,v),$$

where d is the shortest-path metric for $T_{W,\kappa}$, and furthermore $\|\psi(u) - \psi(v)\|_{L_1} = d(u,v)$ when $u \Updownarrow v$.

Proof. Define the map $\psi\colon T_{W,\kappa} \to L_1$ by $\psi(v) = \chi_{S(v)}$ for every $v \in V(T_{W,\kappa})$.

Take any $u = (r, A)$ and $v = (s, B)$ in $V(T_{W,\kappa})$ and suppose first that $u \Updownarrow v$. By Lemma 4.4.2 (v) and Lemma 4.4.3,

$$\|\psi(u)-\psi(v)\|_{L_1} = \lambda(S(u))+\lambda(S(v))-2\lambda(S(u)\cap S(v)) = r+s-2\min\{r,s\} = |r-s| = d(u,v).$$

Lemma 4.2.7 and the triangle inequality applied to shortest paths then shows that $\|\psi(u)-\psi(v)\|_{L_1} \leqslant d(u,v)$ for all $u, v \in V(T_{W,\kappa})$.

Suppose now that $u \not\Updownarrow v$. By Lemma 4.4.2 (v) and Lemma 4.4.3,

$$\begin{aligned}
\|\psi(u) - \psi(v)\|_{L_1} &= \lambda(S(u)) + \lambda(S(v)) - 2\lambda(S(u) \cap S(v)) \\
&= r + s - 2n(u,v) - 2\frac{(r - n(u,v))(s - n(u,v))}{m(u,v) - n(u,v)} \\
&= \alpha + \beta - 2\frac{\alpha\beta}{\gamma},
\end{aligned}$$

where $\alpha = r-n(u,v)$, $\beta = s-n(u,v)$, and $\gamma = m(u,v)-n(u,v)$. Suppose first that $\max\{\alpha, \beta\} \leqslant$

$\frac{1}{2}\gamma$. Then by Proposition 4.2.9 and the above,

$$\|\psi(u) - \psi(v)\|_{L_1} \geqslant \alpha + \beta - \min\{\alpha, \beta\} = \max\{\alpha, \beta\} \geqslant \frac{1}{2}d(u, v).$$

Suppose next that $\min\{\alpha, \beta\} \leqslant \frac{1}{2}\gamma \leqslant \max\{\alpha, \beta\}$. Then by Proposition 4.2.9 and the above,

$$\|\psi(u) - \psi(v)\|_{L_1} = \min\{\alpha, \beta\} + \max\{\alpha, \beta\}\left(1 - 2\frac{\min\{\alpha, \beta\}}{\gamma}\right) \geqslant \frac{1}{2}\gamma \geqslant \frac{1}{2}d(u, v).$$

Finally, suppose $\frac{1}{2}\gamma \leqslant \min\{\alpha, \beta\}$. Then by Proposition 4.2.9 and the above,

$$\begin{aligned}
\|\psi(u) - \psi(v)\|_{L_1} &= \frac{1}{2\gamma}(2\alpha\gamma + 2\beta\gamma - 4\alpha\beta) \\
&= \frac{1}{2\gamma}(\gamma^2 - (2\alpha - \gamma)(2\beta - \gamma)) \\
&\geqslant \frac{1}{2\gamma}(\gamma^2 - (\alpha + \beta - \gamma)\gamma) \\
&= \frac{1}{2}(2\gamma - (\alpha + \beta)) \\
&= \frac{1}{2}d(u, v).
\end{aligned}$$

□

4.5 Embedding into Banach spaces with ESA bases

In this section we show that for any finite cardinality κ, $T_{W,\kappa}$ is bi-Lipschitzly embeddable into any Banach space with an ESA basis with distortion bounded above by a constant depending only on W.

Definition 4.5.1. Let $(X, \|\cdot\|_X)$ be a Banach space.

(i) A sequence $(e_n)_{n=1}^{\infty} \subseteq X$ is said to be *equal-signs-additive (ESA)* if for all $(a)_{n=1}^{\infty} \in c_{00}$ and $k \in \mathbb{N}$ such that $a_k a_{k+1} \geqslant 0$,

$$\left\|\sum_{n=1}^{k-1} a_n e_n + (a_k + a_{k+1})e_k + \sum_{n=k+2}^{\infty} a_n e_n\right\|_X = \left\|\sum_{n=1}^{\infty} a_n e_n\right\|_X.$$

(ii) A sequence $(e_n)_{n=1}^{\infty} \subseteq X$ is said to be *subadditive (SA)* if for all $(a)_{n=1}^{\infty} \in c_{00}$,

$$\left\| \sum_{n=1}^{k-1} a_n e_n + (a_k + a_{k+1}) e_k + \sum_{n=k+2}^{\infty} a_n e_n \right\|_X \leqslant \left\| \sum_{n=1}^{\infty} a_n e_n \right\|_X .$$

(iii) A sequence $(e_n)_{n=1}^{\infty} \subseteq X$ is said to be *invariant under spreading (IS)* if for all $(a)_{n=1}^{\infty} \in c_{00}$ and increasing sequences $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$,

$$\left\| \sum_{n=1}^{\infty} a_n e_{k_n} \right\| = \left\| \sum_{n=1}^{\infty} a_n e_n \right\| .$$

The properties ESA, SA, and IS were first defined and studied by Brunel and Sucheston in [6] and [7]. In [7], they show that a sequence is ESA if and only if it is SA, and that every ESA sequence is also an IS basis for its linear span. We will use these facts without mention. More information about ESA sequences can be found in [7] and [1].

To construct the embedding, we follow roughly the same procedure as used in the previous section. However, with L_1 we were able to subdivide the interval $[0, M+1]$ as finely as needed to accommodate the bundle graph. That is, we could use the existence of infinitely many independent Bernoulli random variables. If instead of L_1, we try to embed into a general Banach space with a basis, we still need independent Bernoulli random variables to choose the support of an embedded vertex, but the random variables are now discrete. In other words, the more vertices in our graph we have to embed, the further down the basis we have to go if we want to mimic the procedure used for L_1. This and the fact that we don't have an explicitly defined norm anymore make the argument more subtle.

We fix now for the rest of this section a finite cardinality κ and let $\mu = \left|\kappa^{\leqslant \max W}\right| \in \mathbb{N}$. We also fix an independent collection $\{Y_i\}_{i=1}^{\mu}$ of Bernoulli random variables defined on $[2^{\mu}] \backslash \{0\}$ (equipped with the uniform probability measure) with probability of success equal to $1/2$. Concretely, for

each $i \in [\mu]\backslash\{0\}$, we may define $Y_i\colon\ [2^\mu]\backslash\{0\} \to \{0,1\}$ by

$$Y_i(j) = \begin{cases} 1 & j \equiv n \pmod{2^{\mu-(i-1)}} \text{ for some } n \in [2^{\mu-i}]\backslash\{0\} \\ 0 & \text{otherwise} \end{cases}$$

for all $j \in [2^\mu]\backslash\{0\}$. For each $j \in \mathbb{N}$, let $I_j = [j(M+1)]\backslash[(j-1)(M+1)]$, and let $\mathcal{F}_j$ be the family of subsets of I_j such that for each $P \in \mathcal{F}_j$, either $P = \varnothing$, or $P \neq \varnothing$ and $|P| = \max(P) - \min(P) + 1$ (which implies P has no "gaps"). That is, we break up the natural number line into blocks of size $M+1$ and let $\mathcal{F}_j$ be the family of intervals contained in the the j-th block.

Let $\{\sigma_i\}_{i=0}^{\mu}$ be an enumeration of $\kappa^{\leqslant \max W}$. For $A \in \kappa^{\leqslant \max W}$ and $i \in [\mu]$, we let $Y_A = Y_i$ if $A = \sigma_i$. Define for each $j \in [2^\mu]\backslash\{0\}$ the function $f_j\colon \kappa^{\leqslant \max W} \times \mathcal{F}_j \times [M] \to \mathcal{F}_j$ by

$$f_j(\sigma_i, P, k) = I_j \cap \{Y_i(j)(\inf P + \ell), (1 - Y_i(j))(\sup P - \ell)\}_{\ell=0}^{k-1}$$

for all $i \in [\mu]$, $P \in \mathcal{F}_j$, and $k \in [M]$.

To summarize what is happening, we assign independent Bernoulli random variables to the elements of $\kappa^{\leqslant \max W}$. Given $A \in \kappa^{\leqslant \max W}$ with its assigned random variable Y_A and an interval P in I_j, $f_j(A, P, k)$ will take the first k elements of P in I_j if $Y_A(j) = 1$ and the last k elements if $Y_A(j) = 0$. Note that $f_j(\sigma_i, P, k) \subseteq P$ if $k \leqslant |P|$. Thus, if P is a union of subintervals of the $I_j's$, we can use the $f_j's$ to simultaneously select k elements from P out of each interval $P \cap I_j$, and these selections will be independent for different elements of $\kappa^{\leqslant \max W}$. This is quite analogous to what happened in the last section. The construction of the supports of our embedded vertices is likewise similar, but in the end we can't just map a vertex to a characteristic function. We have to modify slightly in order to use the ESA property to obtain a good distortion (again, recall notations (4.2) and (4.3) from Section 4.2).

Given $v = (r, A) \in V(T_{W,\kappa})$ and $j \in [2^\mu]\backslash\{0\}$, define the sets $S_{x,j}(v,i)$ and $S_{y,j}(v,i)$ in $\mathcal{F}_j$ for

$i \in [w_r]$ recursively by

$$S_{x,j}(v,0) = [(j-1)(M+1) + x(r,1)] \backslash [(j-1)(M+1)],$$
$$S_{y,j}(v,0) = [(j-1)(M+1) + y(r,1)] \backslash [(j-1)(M+1)],$$

and

$$S_{x,j}(v,i) = f_j\left(A\restriction_i, S_{y,j}(v,i-1) \backslash S_{x,j}(v,i-1), x(r,i+1) - x(r,i)\right),$$
$$S_{y,j}(v,i) = f_j\left(A\restriction_i, S_{y,j}(v,i-1) \backslash S_{x,j}(v,i-1), y(r,i+1) - x(r,i)\right),$$

for $i \in [w_r] \backslash \{0\}$;

and then define

$$S_j(v) = \bigcup_{i=0}^{w_r} S_{x,j}(v,i).$$

Finally, let

$$S_{j,+}(v) = \{(j-1)(M+1) + n \mid n \in S_j(v)\},$$
$$S_{j,-}(v) = \{(3j-1)(M+1) - n \mid n \in S_j(v)\}.$$

$S_{j,+}$ will take a copy of $S_j(V)$ and put it in I_{2j-1}. $S_{j,-}$ will also take a copy of $S_j(V)$ and put it in I_{2j}. The copy for $S_{j,-}$, however, is backwards. That is, $S_{j,-}$ is a reflection of $S_{j,+}$ across the middle of $I_{2j-1} \cup I_{2j}$. The purpose of this is to allow us to take advantage of the subadditivity of an ESA basis later.

Lemma 4.5.2. *Fix $v = (r, A) \in V(T_{W,\kappa})$. The following hold for all $i \in [w_r]$ and $j \in [2^\mu] \backslash \{0\}$,*

(i) $|S_{x,j}(v,i)| = x(r,i+1) - x(r,i)$.

(ii) $|S_{y,j}(v,i)| = y(r,i+1) - x(r,i)$.

(iii) $|S_{y,j}(v,i)\backslash S_{x,j}(v,i)| = y(r,i+1) - x(r,i+1)$.

(iv) $S_{x,j}(v,i) \cap S_{x,j}(v,i') = \varnothing$ *if* $i' \in [w_r]$ *is such that* $i' \neq i$.

(v) $|\bigcup_{k=0}^{i} S_{x,j}(v,k)| = x(r,i+1)$.

(vi) $|S_{j,+}(v)| = |S_{j,-}(v)| = r$.

(vii) $S_{j,+}(v) \subseteq I_{2j-1}$ *and* $S_{j,-}(v) \subseteq I_{2j}$.

(viii) *If* $i' \in [w_r]\backslash[i]$ *and* $Y_{A\restriction_n}(j) = 1$ *for all* $n \in [i']\backslash[i]$, *then*

(a) $\bigcup_{k=i+1}^{i'} S_{x,j}(v,k) = I_j \cap \{\inf S_{y,j}(v,i)\backslash S_{x,j}(v,i) + \ell\}_{\ell=0}^{x(r,i'+1)-x(r,i+1)-1}$.

(b) $S_{y,j}(v,i')\backslash S_{x,j}(v,i') = I_j \cap \{\inf S_{y,j}(v,i)\backslash S_{x,j}(v,i) + \ell\}_{\ell=x(r,i'+1)-x(r,i+1)}^{y(r,i'+1)-x(r,i+1)-1}$

(ix) *If* $i' \in [w_r]\backslash[i]$ *and* $Y_{A\restriction_n}(j) = 0$ *for all* $n \in [i']\backslash[i]$, *then*

(a) $\bigcup_{k=i+1}^{i'} S_{x,j}(v,k) = I_j \cap \{\sup S_{y,j}(v,i)\backslash S_{x,j}(v,i) - \ell\}_{\ell=0}^{x(r,i'+1)-x(r,i+1)-1}$.

(b) $S_{y,j}(v,i')\backslash S_{x,j}(v,i') = I_j \cap \{\sup S_{y,j}(v,i)\backslash S_{x,j}(v,i) - \ell\}_{\ell=x(r,i'+1)-x(r,i+1)}^{y(r,i'+1)-x(r,i+1)-1}$.

Proof. (i)-(iii): These statements certainly hold true for $i = 0$. And by (simultaneous) induction, they hold true for all $i \in [w_r]$.

(iv): We have $S_{x,j}(v, w_r - k) \subseteq S_{y,j}(v, w_r - k - 1)\backslash S_{x,j}(v, w_r - k - 1)$, and so $S_{x,j}(v, w_r - k) \cap S_{x,j}(v, w_r - k - 1) = \varnothing$ for all $k \in [w_r - 1]$. In the same way, $S_{y,j}(v, w_r - k - 1) \subseteq S_{y,j}(v, w_r - k - 2)\backslash S_{x,j}(v, w_r - k - 2)$, and so the first set inclusion implies $S_{x,j}(v, w_r - k) \cap S_{x,j}(v, w_r - k - 2) = \varnothing$ for all $k \in [w_r - 2]$. Inductively, $S_{x,j}(v, w_r - k_1) \cap S_{x,j}(v, w_r - k_2) = \varnothing$ for all $k_1 < k_2 \in [w_r]$.

(v): This follows from parts (i) and (iv).

(vi)-(vii): By definition, $S_j(v) \subseteq I_j$. The rest follows from the definitions, part (v), and the fact that $n \mapsto (j-1)(M+1) + n$ is a bijection from I_j to I_{2j-1} and $n \mapsto (3j-1)(M+1) - n$ is a bijection from I_j to I_{2j} for each $j \in [2^\mu]\backslash\{0\}$.

(viii)-(ix): The statements are true for $i' = i + 1$ by the definitions of $S_{x,j}(v, i+1)$ and f_j, and the statements hold for arbitrary i' by a simple induction. $\square$

At this point we are almost ready to define the embedding, but we need to be able to make sure that for enough $j \in [\mu]\backslash\{0\}$, the symmetric difference of $S_j(u)$ and $S_j(v)$ is large enough when $u \neq v \in V(T_{W,\kappa})$. Unfortunately, the amount of $j \in [\mu]\backslash\{0\}$ we can do this for depends on W. We define the parameter p_W to be the minimum of all $p \in \mathbb{N}$ such that for all $r \in [M+1]$ and $i \in \mathbb{N}_0$,

- $x(r, i+p) \geqslant (x(r,i) + y(r,i))/2$ whenever $r \geqslant (x(r,i) + y(r,i))/2$.
- $y(r, i+p) \leqslant (x(r,i) + y(r,i))/2$ whenever $r \leqslant (x(r,i) + y(r,i))/2$.

One may easily check that $p_W \leqslant \max W + 1$.

Lemma 4.5.3. *Fix $u = (r, A)$ and $v = (s, B)$ in $V(T_{W,\kappa})$ such that $u \not\between v$ and $r \leqslant s$. Let*

$$\mathcal{I}_{u,v} = \{j \in [2^\mu]\backslash\{0\} \mid Y_{B\restriction_{|A\wedge B|+n}}(j) = 1 \textit{ and } Y_{A\restriction_{|A\wedge B|+n}}(j) = 0 \textit{ for all } n \in [p_W]\backslash\{0\}\}.$$

Then for each $j \in \mathcal{I}_{u,v}$, there is $\mathcal{L}_j \subseteq \bigcup_{k=|A\wedge B|+1}^{w_s} S_{x,j}(v,k)$ such that

(i) $|\mathcal{L}_j| \geqslant d(u,v)/2$*, where d is the shortest-path metric for $T_{W,\kappa}$.*

(ii) $\max \mathcal{L}_j < \inf \bigcup_{k=|A\wedge B|+1}^{w_r} S_{x,j}(u,k)$.

Proof. Let $K = |A \wedge B|$. We have $x(r,i) = x(s,i)$ and $y(r,i) = y(s,i)$ for all $i \in [K+1]$, and so $S_{x,j}(u,i) = S_{x,j}(v,i)$ and $S_{y,j}(u,i) = S_{y,j}(v,i)$ for all $i \in [K]$, by an easy induction argument. Note that $x(r, K+1) = x(s, K+1) = n(u,v)$ and $y(r, K+1) = y(s, K+1) = m(u,v)$, which implies $|S_{y,j}(v,K)\backslash S_{x,j}(v,K)| = m(u,v) - n(u,v) > 0$, by Lemma 4.5.2 (iii). Note that $x(r,i) \leqslant x(s,i)$ and $y(r,i) \leqslant y(s,i)$ for all $i \in [M+1]$ by the fact that $r \leqslant s$.

Suppose first that $s \leqslant (m(u,v) + n(u,v))/2$ and let $\mathcal{L}_j = \bigcup_{k=K+1}^{w_s} S_{x,j}(v,k)$ for each $j \in \mathcal{I}_{u,v}$.

Then by definition of p_W,

$$\begin{aligned}
&\min S_{y,j}(v,K)\backslash S_{x,j}(v,K) + y(s,K+p_W+1) - n(u,v) - 1 \\
&\qquad\qquad \leqslant \min S_{y,j}(v,K)\backslash S_{x,j}(v,K) + \frac{m(u,v)-n(u,v)}{2} - 1 \\
&\qquad\qquad \leqslant \max S_{y,j}(u,K)\backslash S_{x,j}(u,K) - \left(\frac{m(u,v)-n(u,v)}{2} - 1\right) \\
&\qquad\qquad \leqslant \max S_{y,j}(u,K)\backslash S_{x,j}(u,K) - (y(r,K+p_W+1) - n(u,v) - 1),
\end{aligned}$$

and so, by Lemma 4.5.2 (viii) and (ix), $\mathcal{L}_j \cap \left(\bigcup_{k=K+1}^{w_r} S_{x,j}(u,k)\right) = \varnothing$ and therefore $\max \mathcal{L}_j < \inf \bigcup_{k=K+1}^{w_r} S_{x,j}(u,k)$ for each $j \in \mathcal{I}_{u,v}$. Moreover, by Lemma 4.5.2 (v) and Proposition 4.2.9,

$$|\mathcal{L}_j| = x(s,w_s+1) - x(s,K+1) = s - n(u,v) \geqslant d(u,v)/2$$

for each $j \in \mathcal{I}_{u,v}$.

Suppose now that $r \geqslant (m(u,v)+n(u,v))/2$ and let

$$\mathcal{L}_j = \left(\bigcup_{k=K+1}^{w_s} S_{x,j}(v,k)\right) \backslash \left(\bigcup_{k=K+1}^{w_r} S_{x,j}(u,k)\right)$$

for each $j \in \mathcal{I}_{u,v}$. Then by definition of p_W,

$$\begin{aligned}
&\min S_{y,j}(v,K)\backslash S_{x,j}(v,K) + x(s,K+p_W+1) - n(u,v) - 1 \\
&\qquad\qquad \geqslant \min S_{y,j}(v,K)\backslash S_{x,j}(v,K) + \frac{m(u,v)-n(u,v)}{2} - 1 \\
&\qquad\qquad \geqslant \max S_{y,j}(u,K)\backslash S_{x,j}(u,K) - \left(\frac{m(u,v)-n(u,v)}{2} - 1\right) \\
&\qquad\qquad \geqslant \max S_{y,j}(u,K)\backslash S_{x,j}(u,K) - (x(r,K+p_W+1) - n(u,v) - 1)\,,
\end{aligned}$$

and so, by Lemma 4.5.2 (viii) and (ix), $\mathcal{L}_j \cup \left(\bigcup_{k=K+1}^{w_r} S_{x,j}(u,k)\right) = S_{y,j}(v,K)\backslash S_{x,j}(v,K)$ and

therefore , by Lemma 4.5.2 (v) and Proposition 4.2.9,

$$\begin{aligned}|\mathcal{L}_j| &= |S_{y,j}(v,i)\backslash S_{x,j}(v,i)| - \left|\left(\bigcup_{k=K+1}^{w_r} S_{x,j}(u,k)\right)\right| \\ &= (m(u,v)-n(u,v)) - (r-n(u,v)) = m(u,v)-r \geqslant d(u,v)/2\end{aligned}$$

for each $j \in \mathcal{I}_{u,v}$. Moreover, $\max \mathcal{L}_j < \inf \bigcup_{k=K+1}^{w_r} S_{x,j}(u,k)$ for each $j \in \mathcal{I}_{u,v}$.

Finally, suppose $r \leqslant (m(u,v)+n(u,v))/2 \leqslant s$ and let

$$\mathcal{L}_j = \{\min S_{y,j}(v,K)\backslash S_{x,K}(v,i) + \ell\}_{\ell=0}^{(m(u,v)-n(u,v))/2-1}$$

for each $j \in \mathcal{I}_{u,v}$. Then by definition of p_W,

$$\begin{aligned}\max \mathcal{L}_j \leqslant \min\{ &\min S_{y,j}(v,K)\backslash S_{x,j}(v,K) + x(s,K+p_W+1) - n(u,v) - 1, \\ &\max S_{y,j}(u,K)\backslash S_{x,j}(u,K) - (y(r,K+p_W+1) - n(u,v) - 1)\},\end{aligned}$$

and so, by Lemma 4.5.2 (viii) and (ix), $\mathcal{L}_j \subseteq \bigcup_{k=K+1}^{w_s} S_{x,j}(v,k)$ and $\max \mathcal{L}_j < \inf \bigcup_{k=K+1}^{w_r} S_{x,j}(u,k)$ for each $j \in \mathcal{I}_{u,v}$. Moreover, by Proposition 4.2.9, $|\mathcal{L}_j| = (m(u,v)-n(u,v))/2 \geqslant d(u,v)/2$ for each $j \in \mathcal{I}_{u,v}$.. □

The next theorem (combined with the two previous lemmas) generalizes the results and procedures found in Section 3 of [17]. As in the previous two sections, the techniques we use are largely inspired from the source, with perhaps the most important differences being the introduction of the parameter p_W and the fixing of a single bundle graph at the beginning. And as before, both the labeling and recursive procedure here are simpler, with the main difficulty being taken care of in Lemma 4.5.3.

Theorem 4.5.4. *Suppose X is a Banach space with an ESA basis $(e_n)_{n=1}^{\infty}$. Then there is a bi-*

Lipschitz embedding $\psi\colon T_{W,\kappa} \to X$ *such that for all* $u, v \in V(T_{W,\kappa})$*,*

$$\frac{1}{2^{2p_W+1}} d(u,v) \leqslant \|\psi(u) - \psi(v)\|_X \leqslant d(u,v),$$

where d *is the shortest-path metric for* $T_{W,\kappa}$*, and furthermore* $\|\psi(u) - \psi(v)\|_X = d(u,v)$ *when* $u \Updownarrow v$*.*

Proof. Let $\eta = \left\| \sum_{j=1}^{2^\mu} e_{2j-1} - e_{2j} \right\|_X$ and define the map $\psi\colon T_{W,\kappa} \to X$ by

$$\psi(v) = 1/\eta \sum_{j=1}^{2^\mu} \left(\sum_{n \in S_{j,+}(v)} e_n - \sum_{n \in S_{j,-}(v)} e_n \right)$$

for every $v \in V(T_{W,\kappa})$.

Take any $u = (r, A)$ and $v = (s, B)$ in $V(T_{W,\kappa})$. Let $K = |A \wedge B|$ and suppose first that $u \Updownarrow v$ and $r \leqslant s$. Then $y(r, K+1) \leqslant x(s, K+1)$. Let $n \in [K]$ be such that $x(s,i) < y(r,i)$ (which implies $x(r,i) = x(s,i)$ and $y(r,i) = y(s,i)$) for all $i \in [n]$ while $y(r, n+1) \leqslant x(s, n+1)$. An easy induction argument shows that $S_{x,j}(u,i) = S_{x,j}(v,i)$ and $S_{y,j}(u,i) = S_{y,j}(v,i)$ for all $i \in [n]$ and $j \in [2^\mu] \backslash \{0\}$; and $S_{y,j}(u,n) \subseteq S_{x,j}(v,n)$ for all $j \in [2^\mu] \backslash \{0\}$. Another easy induction argument shows $S_{x,j}(u,i) \subseteq \bigcup_{k=0}^{n} S_{x,j}(v,k)$ for all $i \in [w_r]$ and $j \in [2^\mu] \backslash \{0\}$ (and so $S_{j,+}(u) \subseteq S_{j,+}(v)$ and $S_{j,-}(u) \subseteq S_{j,-}(v)$ for all $j \in [2^\mu] \backslash \{0\}$). Thus, by Lemma 4.2.9, Lemma 4.5.2 (vi), and the

assumption that the basis is ESA;

$$\begin{aligned}
\|\psi(u)-\psi(v)\|_X &= \frac{1}{\eta}\left\|\sum_{j=1}^{2^\mu}\left(\left(\sum_{n\in S_{j,+}(u)} e_n - \sum_{n\in S_{j,-}(u)} e_n\right) - \left(\sum_{n\in S_{j,+}(v)} e_n - \sum_{n\in S_{j,-}(v)} e_n\right)\right)\right\|_X \\
&= \frac{1}{\eta}\left\|\sum_{j=1}^{2^\mu}\left(\sum_{n\in \cap S_{j,+}(v)\setminus S_{j,+}(u)} e_n - \sum_{n\in S_{j,-}(v)\setminus S_{j,-}(u)} e_n\right)\right\|_X \\
&= \frac{1}{\eta}\left\|\sum_{j=1}^{2^\mu}(s-r)(e_{2j-1}-e_{2j})\right\|_X \\
&= s-r \\
&= d(u,v).
\end{aligned}$$

Lemma 4.2.7 and the triangle inequality applied to shortest paths then shows that $\|\psi(u)-\psi(v)\|_X \leqslant d(u,v)$ for all $u,v \in V(T_{W,\kappa})$.

Suppose now that $u \not\sim v$ and $r \leqslant s$. Define $\mathcal{I}_{u,v}$ as in Lemma 4.5.3 and for each $j \in \mathcal{I}_{u,v}$, let $\mathcal{L}_j$ be chosen as in Lemma 4.5.3. Note that, by independence of the Bernoulli random variables defined at the beginning of this section, $|\mathcal{I}_{u,v}| = 2^{\mu-\nu}$ for some $\nu \in [2p_W]$. Let

$$\begin{aligned}
\mathcal{L}_{j,+} &= \{(j-1)(M+1)+n \mid n \in \mathcal{L}_j\}, \\
\mathcal{L}_{j,+} &= \{(3j-1)(M+1)-n \mid n \in \mathcal{L}_j\}.
\end{aligned}$$

Recall that $n \mapsto (j-1)(M+1)+n$ is a bijection from I_j to I_{2j-1} and $n \mapsto (3j-1)(M+1)-n$ is a bijection from I_j to I_{2j} for each $j \in \mathcal{I}_{u,v}$. Furthermore, the images of the two maps will be reflections of each other across the middle of $I_{2j-1} \cup I_{2j}$ when the maps are applied to the same

set. By Lemma 4.5.2 (vi), Lemma 4.5.3, and the assumption that the basis is ESA;

$$\begin{aligned}
\|\psi(u)-\psi(v)\|_X &= \frac{1}{\eta}\left\|\sum_{j=1}^{2^\mu}\left(\left(\sum_{n\in S_{j,+}(u)} e_n - \sum_{n\in S_{j,-}(u)} e_n\right)-\left(\sum_{n\in S_{j,+}(v)} e_n - \sum_{n\in S_{j,-}(v)} e_n\right)\right)\right\|_X \\
&\geqslant \frac{1}{\eta}\left\|\sum_{j\in\mathcal{I}_{u,v}}\left(\left(\sum_{n\in S_{j,+}(v)} e_n - \sum_{n\in S_{j,+}(u)} e_n\right)-\left(\sum_{n\in S_{j,-}(v)} e_n - \sum_{n\in S_{j,-}(u)} e_n\right)\right)\right\|_X \\
&\geqslant \frac{1}{\eta}\left\|\sum_{j\in\mathcal{I}_{u,v}}\left(\sum_{n\in\mathcal{L}_{j,+}} e_n - \sum_{n\in\mathcal{L}_{j,-}} e_n\right)\right\|_X \\
&\geqslant \frac{d(u,v)}{2\eta}\left\|\sum_{j\in\mathcal{I}_{u,v}}(e_{2j-1}-e_{2j})\right\|_X \\
&= \frac{d(u,v)}{2\eta}\cdot\frac{1}{2^\nu}\cdot\sum_{k=1}^{2^\nu}\left\|\sum_{j=(k-1)2^{\mu-\nu}+1}^{k2^{\mu-\nu}}(e_{2j-1}-e_{2j})\right\|_X \\
&\geqslant \frac{d(u,v)}{2^{\nu+1}\eta}\left\|\sum_{j=1}^{2^\mu}(e_{2j-1}-e_{2j})\right\|_X \\
&\geqslant \frac{1}{2^{2p_W+1}}d(u,v).
\end{aligned}$$

□

We show in the next section that actually the entire family of bundle graphs generated by $T_{W,\kappa}$ is bi-Lipschitzly embeddable with the same distortion bound of 2^{2p_W+1} into a Banach space with an ESA basis.

4.6 The $\oslash$-product

Given two κ-branching bundle graphs G and H, we can define a new κ-branching bundle graph $G\oslash H$ by replacing every edge in G with a copy of H (where the bottom of H is identified with the lower endpoint of the edge H is replacing and the top of H is identified with the higher endpoint). The definition of $\oslash$-product seems to have first been formally introduced by J. R. Lee and P. Raghavendra in [14], although the family of 2-branching diamond graphs (see the comment under Definition 4.6.5 for a definition involving $\oslash$-products) were studied in [10] and [16].

For this section we fix another sequence $W' = \{w'_r\}_{r=0}^{M'+1} \subseteq \mathbb{N}_0$ such that $w'_0 = w'_{M'+1} = 0$. We will show how to determine W'' so that $T_{W,\kappa}\oslash T_{W',\kappa} = T_{W'',\kappa}$. Once W'' is found, we can

use Theorem 4.5.4 to find a bound on the worst distortion for a bi-Lipschitz embedding of $T_{W'',\kappa}$ into a Banach space with an ESA basis. In particular, we show that the distortion bound found in Theorem 4.5.4 is no worse for $T_{W,\kappa} \oslash T_{W,\kappa}$ than it was for $T_{W,\kappa}$, allowing us to generalize the characterizations of superreflexivity found in [11] and [17].

Given a bundle graph $G = (V, E)$ and $n \in \mathbb{N}_0$, let $V_n = \{v \in V \mid \text{height}(v) = n\}$ and let $E_n = \{\{u, v\} \in E \mid u \in V_n \text{ and } v \in V_{n+1}\}$. If another bundle graph H is given, we may create a new bundle graph $G \oslash_n H$ by replacing every edge in E_n with H for some n. Explicitly, if $G = (V, E)$ and $H = (V', E')$, and if b and t are the bottom and top, respectively, of H; then we define $G \oslash_n H = (V'', E'')$ by

$$V'' = V \cup (E_n \times (V' \backslash \{b, t\}))$$

and

$$\begin{aligned} E'' = \{e \in E \mid e \cap (V_n \cup V_{n+1}) = \varnothing\} \\ \cup \{\{u, (e, v)\} \mid e \in E_n,\ u \in V_n \cap e, \text{ and } \{b, v\} \in E'\} \\ \cup \{\{u, (e, v)\} \mid e \in E_n,\ u \in V_{n+1} \cap e, \text{ and } \{v, t\} \in E'\} \\ \cup \{\{(e, u), (e, v)\} \mid e \in E_n \text{ and } (\{u, v\} \backslash \{b, t\}) \in E'\}. \end{aligned}$$

The formal definition of $G \oslash H$ is similar (just remove the subscripts and the first term in the definition of E''). It is clear that $G \oslash H$ can be created by performing $\oslash_n$-products repeatedly until all the edges that were originally in G have been replaced.

Lemma 4.6.1. *Given $n \in [M]$, the graph $T_{W,\kappa} \oslash_n T_{W',\kappa}$ is the same bundle graph as $T_{W'',\kappa}$, where*

$W'' = (w''_r)_{r=0}^{M+M'+1} \subseteq \mathbb{N}_0$ *is defined by*

$$w''_r = \begin{cases} w_r & 0 \leqslant i \leqslant n \\ \max\{w_n, w_{n+1}\} + w'_{i-n} & n < r < n + M' + 1 \\ w_{r-M'} & n + M' + 1 \leqslant r \leqslant M + M' + 1. \end{cases}$$

Proof. We simply provide the graph isomorphism, and leave the details to the reader. Define $F\colon T_{W,\kappa} \oslash T_{W',\kappa} \to T_{W''}$ by

$$F(v) = \begin{cases} (r, A) & v = (r, A) \text{ and } r \leqslant n \\ (n + r, B^\frown C) & v = (\{(n, A), (n + 1, B)\}, (r, C)) \text{ and } A \preceq B \\ (n + r, A^\frown C) & v = (\{(n, A), (n + 1, B)\}, (r, C)) \text{ and } B \preceq A \\ (r + M' + 1, A) & v = (r, A) \text{ and } r > n \end{cases}$$

for each $v \in V(T_{W,\kappa} \oslash_n T_{W',\kappa})$. □

With repeated application of Lemma 4.6.1, we obtain the following formula.

Proposition 4.6.2. *The graph $T_{W,\kappa} \oslash T_{W',\kappa}$ is graph-isomorphic to $T_{W'',\kappa}$ where $W'' = (w''_r)_{r=0}^{(M+1)(M'+1)} \subseteq \mathbb{N}_0$ is defined by $w''_0 = 0$ and*

$$w''_r = \begin{cases} \max\{w_n, w_{n+1}\} + w'_{r-n(M'+1)} & n(M' + 1) < r < (n + 1)(M' + 1) \\ w_{n+1} & r = (n + 1)(M' + 1) \end{cases}$$

for all $n \in [M]$.

Proposition 4.6.2 confirms what is to be expected regarding the depths of vertices in $T_{W,\kappa} \oslash T_{W',\kappa}$. Namely, that vertices in $T_{W,\kappa} \oslash T_{W',\kappa}$ that originated from $T_{W,\kappa}$ will keep the same depth as they originally had, and vertices that arise from a copy of $T_{W',\kappa}$ replacing an edge of $T_{W,\kappa}$ would keep the same depths as they originally had, except that the maximum depth of the endpoints of the

edge being replaced will be added to these depths. We now fix W'' obtained in Proposition 4.6.2. For what follows, we define the functions x' and y' for W', and x'' and y'' for W'', in the same way x and y were defined for W (notations (4.2) and (4.3)) at the end of Section 4.2.

Corollary 4.6.3. *For each $n \in [M]$, let $K_n = \max\{w_n, w_{n+1}\}$. Then for all $n \in [M]$, $r \in [(n+1)(M'+1)] \backslash [n(M'+1)]$, and $i \in \mathbb{N}_0$,*

$$x''(r,i) = \begin{cases} (M'+1)x(n+1,i) & r = (n+1)(M'+1) \\ (M'+1)x(n,i) & r \neq (n+1)(M'+1) \text{ and } i \leqslant K_n \\ n(M'+1) + x'(r - n(M'+1), i - K_n) & \text{otherwise}, \end{cases}$$

$$y''(r,i) = \begin{cases} (M'+1)y(n+1,i) & r = (n+1)(M'+1) \text{ or } i \leqslant K_n \\ n(M'+1) + y'(r - n(M'+1), i - K_n) & \text{otherwise}. \end{cases}$$

Proof. We prove the formula for x''. The proof for y'' is similar. Take any $n \in [M]$ and $r \in [(n+1)(M'+1)] \backslash [n(M'+1)]$. The case $i = 0$ is trivial, so take any $i \in \mathbb{N}$. Let $t \in [(M+1)(M'+1)]$ be such that $t = x''(r,i)$, and let $m \in [M+1]$ be such that $m(M'+1) \leqslant t < (m+1)(M'+1)$.

Suppose first that $r = (n+1)(M'+1)$. If $t \neq r$, then $(m+1)(M'+1) \leqslant r$ by the definition of m and x''. This means that $t = m(M'+1)$ by Proposition 4.6.2 and the definition of x''. Either way, $t = \ell(M'+1)$ for some $\ell \in [M]$, and by Proposition 4.6.2 and the definition of x'', $w_\ell = w''_t < i$. But, by definition of t and x'', $w''_{t'} \geqslant i$ for all $t < t' \leqslant r$, and in particular $w''_{\ell'(M'+1)} \geqslant i$ for all $\ell < \ell' \leqslant n+1$. By Proposition 4.6.2, this means $w_\ell < i \leqslant w_{\ell'}$ for all $\ell < \ell' \leqslant n+1$. And so, by the definition of x, $x(n+1,i) = \ell$. That is, $x''(n+1,i) = t = (M'+1)x(n+1,i)$. The case when $r \neq (n+1)(M'+1)$ and $i \leqslant K_n$ is similar because we may still conclude that $t = \ell(M'+1)$ for some $\ell \in [M]$.

Suppose now that $r \neq (n+1)(M'+1)$ and $i > K_n$. Since $w''_{n(M'+1)} = w_n \leqslant K_n < i$, we must have $m = n$ by the definition of x''. Thus $t = n(M'+1) + \ell$ for some $\ell \in [M']$. And by Proposition 4.6.2 and the definition of x'', $w'_\ell = w''_t - K_n < i - K_n$. But, by definition of t, $w''_{t'} \geqslant i$ for all $t < t' \leqslant r$, and in particular $w''_{n(M'+1)+\ell'} \geqslant i$ for all $\ell < \ell' \leqslant r - n(M'+1)$. By Proposition

4.6.2, this means $w'_\ell < i - K_n \leqslant w'_{\ell'}$ for all $\ell < \ell' \leqslant r - n(M'+1)$. And so, by definition of x', $x'(r-n(M'+1), i-K_n) = \ell$. That is, $x''(n+1, i) = t = n(M'+1)+x'(r-n(M'+1), i-K_n)$. $\square$

In the next lemma we define $p_{W'}$ for W' and $p_{W''}$ for W'' in the same way p_W was defined for W before Lemma 4.5.3 in the last section.

Lemma 4.6.4. *The parameter $p_{W''}$ satisfies the inequality $p_{W''} \leqslant \max\{p_W, p_{W'}\}$.*

Proof. Let $r \in [(M+1)(M'+1)]$ and $i \in \mathbb{N}_0$ be such that $r \geqslant (x''(r,i) + y''(r,i))/2$ and suppose first that $r = (n+1)(M'+1)$ for some $n \in [M]$ (the case $r = 0$ is trivial). Then after using Corollary 4.6.3 and dividing by $M'+1$, we see that $n+1 \geqslant (x(n+1,i) + y(n+1,i))/2$, which, by definition of p_W, implies $x(n+1, i+p_W) \geqslant (x(n+1), i) + y(n+1,i))/2$. By multiplying by $M'+1$ and again using Corollary 4.6.3, we see that $x''(r, i+p_W) \geqslant (x''(r,i) + y''(r,i))/2$.

Suppose now that $n(M'+1) < r < (n+1)(M'+1)$ for some $n \in [M]$ and $i > K_n$ (where $K_n = \max\{w_n, w_{n+1}\}$). Then after using Corollary 4.6.3 and subtracting $n(M'+1)$, we see that $r - n(M'+1) \geqslant (x'(r-n(M'+1), i-K_n) + y'(r-n(M'+1), i-K_n))/2$, which, by definition of $p_{W'}$ implies $x'(r-n(M'+1), i-K_n+p_{W'}) \geqslant (x'(r-n(M'+1), i-K_n) + y'(r-n(M'+1), i-K_n))/2$. By adding $n(M'+1)$ and again using Corollary 4.6.3, we see that $x''(r, i+p_{W'}) \geqslant (x''(r,i) + y''(r,i))/2$.

Suppose finally that $n(M'+1) < r < (n+1)(M'+1)$ for some $n \in [M]$ and $i \leqslant K_n$. Suppose further that $w_n < i \leqslant w_{n+1}$. Then $x(n,i) = n$ and $y(n+1,i) > n+1$. Thus, after applying Corollary 4.6.3 and then dividing by $M'+1$, we have

$$n+1 > r/(M'+1) \geqslant (n + y(n+1,i))/2,$$

which implies $n+1 \geqslant y(n+1,i)$, a contradiction. Therefore, in order for the hypothesis to hold true, either $i \leqslant \min\{w_n, w_{n+1}\}$ or $w_{n+1} < w_n$. In either case, $y(n,i) = y(n+1,i)$. So after applying Corollary 6.3 and dividing by $M'+1$, we have

$$n+1 > (x(n,i) + y(n+1,i))/2 = (x(n,i) + y(n,i))/2.$$

This means $n \geqslant (x(n,i) + y(n,i))/2$ and so, by definition of p_W, $x(n, i + p_W) \geqslant (x(n,i) + y(n,i))/2 = (x(n,i) + y(n+1,i))/2$. After multiplying by $M' + 1$ and again applying Corollary 4.6.3, we have

$$\begin{aligned} x''(r, i + p_W) &\geqslant x''(n(M'+1), i + p_W) \\ &\geqslant (x''(n(M'+1), i) + y''((n+1)(M'+1), i))/2 \\ &= (x''(r,i) + y''(r,i))/2. \end{aligned}$$

We have shown that for all $r \in [(M+1)(M'+1)]$ and $i \in \mathbb{N}_0$, $x''(r, i + \max\{p_W, p_{W'}\}) \geqslant (x''(r,i) + y''(r,i))/2$ whenever $r \geqslant (x''(r,i) + y''(r,i))/2$. Similarly it can be shown $y''(r, i + \max\{p_W, p_{W'}\}) \leqslant (x''(r,i) + y''(r,i))/2$ whenever $r \leqslant (x''(r,i) + y''(r,i))/2$. Therefore $p_{W''} \leqslant \max\{p_W, p_{W'}\}$. $\square$

Definition 4.6.5. Given a bundle graph G, the *family of bundle graphs generated by* G is the set $\left\{G^{\oslash^k}\right\}_{k=1}^{\infty}$, where $G^{\oslash^k}$ is defined recursively by $G^{\oslash^1} = G$ and $G^{\oslash^{k+1}} = G^{\oslash^k} \oslash G$ for all $k \in \mathbb{N}$.

There are a few families of bundle graphs that have earned special names. Given a cardinality κ, the family of κ-branching diamonds is the family generated by $T_{(0,1,0),\kappa}$, the family of κ-branching Laakso graphs is the family generated by $T_{(0,0,1,0,0),\kappa}$, and the family of κ-branching parasol graphs is the family generated by $T_{(0,0,1,0),\kappa}$. Diamond graphs seem to have first been introduced in [10] and [16]. Laakso graphs were first introduced in [13], based on ideas found in [12]. Parasol graphs were first introduced in [9]. Lemma 4.6.4 and Theorem 4.5.4 yield the following corollary.

Corollary 4.6.6. *Given a finitely branching bundle graph G, the family of bundle graphs generated by G is equi-bi-Lipschitzly embeddable into any Banach space with an ESA basis, with distortion bounded above by a constant not depending on the target space or branching number of G. In particular, every finitely branching diamond, Laakso, and parasol graph is bi-Lipschitzly embeddable into any Banach space with an ESA basis with distortion bounded above by* 8.

Now that we've formally defined families of bundle graphs generated by a base graph, we come to the characterizations of Banach space properties via non-equi-bi-Lipschitz embeddability

of families of graphs, as promised in the introduction. Work done by Brunel and Sucheston ([7] and [6], see also Theorem 2.3 in [17]), shows that for every non-reflexive Banach space X, there is a Banach space with an ESA basis that is finitely representable in X; and so every family of bundle graphs generated by a finitely-branching bundle graph is equi-bi-Lipschitzly embeddable into any non-reflexive Banach space by Theorem 4.5.4. Conversely, a consequence of Lemma 1 in Section 4 of [11], says that the family of binary (that is, 2-branching) diamond graphs is not equi-bi-Lipschitzly embeddable into any Banach space with uniformly convex norm. Virtually the same proof shows that, in fact, every family of bundle graphs generated by a nontrivial bundle graph is not equi-bi-Lipschitzly embeddable into a Banach space with uniformly convex norm. Every superreflexive Banach space is uniformly convexifiable, so we have the following characterization(s) of superreflexivity.

Theorem 4.6.7. *Fix a nontrivial finitely branching bundle graph G. Then a Banach space X is superreflexive if and only if the family of bundle graphs generated by G is non-equi-bi-Lipschitzly embeddable into X.*

Remark 4.6.8. Johnson and Schechtman [11] obtained a distortion bound of $16+\varepsilon$ for the equi-bi-Lipschitz embeddability of the family of binary diamond graphs into a non-superreflexive Banach space. Ostrovskii and Randrianantoanina [17] improved and generalized this, obtaining a distortion bound of $8+\varepsilon$ for any family of finitely-branching diamond or Laakso graphs. Corollary 4.6.6 yields a further generalization, and recovers the same distortion bound of $8+\epsilon$ for any family of finitely-branching diamond, Laakso, or parasol graph. It is unknown to the author for which families of bundle graphs, if any, the distortion bound implied by Corollary 4.6.6 is optimal, but a distortion bound of $2+\varepsilon$ was obtained for the 2-branching diamonds by Pisier in [18] (see the proof of Theorem 13.17).

Theorem 3.2 in [4] shows that within the class of reflexive Banach spaces with an unconditional structure, a Banach space that is not asymptotically uniformly convexifiable will contain $(1+\varepsilon, 1+\varepsilon)$-good ℓ_∞ trees of arbitrary height for all $\varepsilon > 0$. Theorem 4.1 in [4] then shows that

every family of bundle graphs generated by a nontrivial infinitely-branching bundle graph is not equi-bi-Lipschitzly embeddable into any Banach space that is asymptotically midpoint uniformly convexifiable. Thus we have the following metric characterization(s) of asymptotic uniform convexifiability within the class of reflexive Banach spaces with an unconditional asymptotic structure.

Theorem 4.6.9. *Fix a nontrivial $\aleph_0$-branching bundle graph G. Then a reflexive Banach space X with an unconditional asymptotic structure is asymptotically uniformly convexifiable if and only if the family of bundle graphs generated by G is non-equi-bi-Lipschitzly embeddable into X.*

4.7 References

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5. SUMMARY

5.1 Possible applications of research

The purpose of this research was to investigate two general problems in the nonlinear geometry of Banach spaces. As evidenced in the preceding sections, these general problems can inspire the formulation of countless interesting subproblems that allow Banach space theorists to test the limits of their methods. There are still many questions that remain unanswered, and the work done here only scratches the surface. However, our results suggest some natural avenues of research that can be pursued in the future.

In Sections 2 and 3, we investigated two specific instances of the following problem, whose solution remains unknown for general X and Y.

Problem 5.1.1. Given two Banach spaces X and Y, determine whether X's uniform embeddability into Y is equivalent to X's is coarse embeddability into Y.

In Section 2, we were able to fully solve Problem 5.1.1 in the positive for $Y = c_0(\kappa)$, given any cardinality κ. Not only that, but we have also shown that both uniform and coarse embeddability into $c_0(\kappa)$ can be characterized by an intrinsic property called the "coarse Stone property". Thus, a useful tool for future research into the coarse and uniform subspace structure of $c_0(\kappa)$ spaces has been provided. In particular, it seems to be an open problem (see [1]) to determine whether ℓ_∞ is uniformly or coarsely embeddable into $c_0(\mathfrak{c})$ (where $\mathfrak{c}$ is the cardinality of the continuum). The coarse Stone property provides a condition that might be checked for ℓ_∞ to solve this problem. The coarse Stone property has already been used (see [2]) to show that, for a large enough cardinality κ, a Banach space with density character equal to κ must have trivial cotype if it is uniformly or (equivalently) coarsely embeddable into $c_0(\kappa)$.

In Section 3, we were able to partially solve Problem 5.1.1 for when Y is a superstable Banach space. Indeed, in this case we have shown that the problem has a positive solution for all X containing no spreading model isomorphic to ℓ_p for some $p \in [1, \infty)$. However, there is a gap

between this condition and the known requirement for uniform embeddability into a superstable Banach space. A Banach space must in fact have a subspace isomorphic to ℓ_p if it is uniformly embeddable into a superstable Banach space. The proof of our result relies heavily on a constant γ being finite. However, the special case of $\gamma = 0$ will only occur if uniform embeddability is assumed, and this will imply the existence of an ℓ_p subspace. Thus, there is some hope that the general solution to Problem 5.1.1 is negative, as it is possible that our result cannot be strengthened to guarantee the existence of an ℓ_p subspace in Banach spaces that are only assumed to be coarsely embeddable into a superstable Banach space. If one wants to try to solve Problem 5.1.1 in the negative, a good strategy to do this now is to pick Y so that Y is superstable and to pick X so that X has a spreading model isomorphic to some ℓ_p, but no subspace isomorphic to any ℓ_p. In particular, it is an open problem to determine whether there exist $p, q, r \in [1, \infty)$ such that r-convexified Tsirelson space is coarsely embeddable into $\ell_p(\ell_q)$.

Finally, in Section 4, we generalized solutions to several specific instances of the following problem.

Problem 5.1.2. Given a local or asymptotic property $\mathcal{P}$ of Banach spaces, find a purely metric characterization of $\mathcal{P}$.

The metric characterizations of interest for us were those stated in terms of graph preclusion. Namely, the goal was to characterize $\mathcal{P}$ (or $\mathcal{P}$'s negation) by the equi-bi-Lipschitz embeddability of some family of graphs. In the primary literature we referenced, the two typical families considered were the diamond graphs and the Laakso graphs. However, although the proofs involved were similar, these families were always treated separately. In Section 4, we defined a larger class of graphs (containing all families of diamond graphs and Laakso graphs) and were able to provide a vertex-labeling that allowed us to generalize some previous results with easier proofs. It is our hope that our vertex-labeling will make proofs involving diamond graphs, Laakso graphs, or other similar families of graphs conceptually simpler in the future. There are still many properties to consider for Problem 5.1.2, including Pisier's property (α) (a local property) and asymptotic uniform smoothifiability (an asymptotic property).

5.2 References

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