THE ARVESON-DOUGLAS CONJECTURE

A Dissertation

by

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ABSTRACT

The Arveson-Douglas Conjecture states that, closures of polynomial ideals in some analytic Hilbert modules, such as the Drury-Arveson module, Bergman module or Hardy module, are essentially normal. The conjecture has connections to multivariable operator theory, index theory and function theory. In this dissertation, we discuss an approach using tools from harmonic analysis and several complex variables. Two methods are introduced, approaching this problem from different aspects. Each method has given some interesting new results. Most of the theories developed in this dissertation concerns the Bergman space.

First, we introduce the Arveson-Douglas Conjecture, its background and applications.

Then we describe the first method. The main subject is a geometric version of the Arveson-Douglas Conjecture. For this method to work we need the variety to have nice properties, such as smoothness or transversality, around the boundary. Two types of varieties are considered.

Next, we develop the second method, which mainly treats principal submodules. This method gives weaker results but is more flexible. As a consequence, we are able to extend our discussion to strongly pseudoconvex domains and to the Hardy space. As an application, we apply this theory to the unit ball and obtain some nontrivial results.

Finally, we end this dissertation with some concluding remarks as well as future plans.

DEDICATION

To Professor Ronald G. Douglas.

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The four years in Texas A&M University are of special meaning for me. My career as a researcher starts here. During this wonderful journey, many people have helped and supported me. To them I owe a debt of gratitude.

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NOMENCLATURE

$\mathbb{D}$	The unit disk
$\mathbb{C}^n$	The n-dimensional complex space
$\mathbb{B}_n$	The unit ball of $\mathbb{C}^n$
$\mathcal{B}(\mathcal{H})$	Bounded linear operators on a Hilbert space $\mathcal{H}$
$L^2_a(\Omega)$	The Bergman space on a domain Ω
$H^2(\Omega)$	The Hardy space on a domain Ω
$\mathcal{T}(L^\infty)$	Toeplitz algebra of L^∞ symbols on the Bergman space
$P_{L^2_a}$	The Bergman projection
$\mathbb{C}[z_1, \cdots, z_n]$	Ploynomial ring of n variables
M_f	Multiplication operator with symbol f
T_μ	Integral operator defined by a Carleson measure μ
Δf	The Laplacian of a function f
∇f	The gradient of a function f
$Z(I)$	Zero variety of an ideal I
$\mathcal{P}_I$	Closure of an ideal I, viewed as a submodule
$\mathcal{Q}_I$	Quotient module of $\mathcal{P}_I$
$\mathcal{P}_M$	Submodule that vanish on M
$\mathcal{Q}_M$	Quotient module of $\mathcal{P}_M$
P, Q	Projection operators onto the corresponding spaces
$A \approx B$	$\exists C > c > 0$ such that $cA \leq B \leq CA$
$A \lesssim B$	$\exists C > 0$ such that $A \leq CB$
$A \gtrsim B$	$\exists c > 0$ such that $A \geq cB$

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1. INTRODUCTION

Denote $\mathbb{C}[z_1, \cdots, z_n]$ the polynomial ring of n variables. Given a commuting n-tuple of operators $(T_1, \cdots, T_n)$ acting on a Hilbert space $\mathcal{H}$, one can define a $\mathbb{C}[z_1, \cdots, z_n]$-module structure on $\mathcal{H}$. A polynomial p acts on $\mathcal{H}$ by $p(T_1, \cdots, T_n)$. This simple observation offers a a new perspective of studying multivariable operator theory using algebraic and geometric methods.

Definition 1.0.1. *Suppose $\mathcal{H}$ is a Hilbert space and*

$$\Phi : \mathbb{C}[z_1, \cdots, z_n] \to \mathcal{B}(\mathcal{H})$$

is a homomorphism from $\mathbb{C}[z_1, \cdots, z_n]$ to the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. The map Φ gives a $\mathbb{C}[z_1, \cdots, z_n]$-module structure on $\mathcal{H}$. We say that $\mathcal{H}$ is a Hilbert Module.

Perhaps the most well-known result in this direction is the generalization of von Neumann's inequality for commuting row contractions. Recall that an operator T on a Hilbert space $\mathcal{H}$ is a contraction if $\|T\| \leq 1$. The famous von Neumann's inequality states that for any polynomial p of one variable,

$$\|p(T)\| \leq \sup_{z \in \mathbb{D}} |p(z)|.$$

An n-tuple $(T_1, \cdots, T_n)$ of mutually commuting operators on $\mathcal{H}$ is called a *row contraction* if

$$\|T_1\xi_1 + \cdots + T_n\xi_n\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_n\|^2$$

for any $\xi_1, \cdots, \xi_n \in \mathcal{H}$. Equivalently, this means that the "row operator"

$$(T_1, \cdots, T_n) : \oplus_{i=1}^n \mathcal{H} \to \mathcal{H}$$

is a contraction. It seems natural to expect a similar inequality for commuting row contractions. To one's surprise, in [2], Arveson showed that there exists row contractions that are not polynomial bounded. Indeed, there is no constant K satisfying

$$\|p(T_1, \cdots, T_n)\| \leq K \sup\{|p(z)| : z \in \mathbb{B}_n\}.$$

Arveson gave an appropriate generalization of the von Neumann's inequality, using the multiplier norm on a special function space — the Drury-Arveson space.

Definition 1.0.2. *The* Drury-Arveson space H_n^2 *is the reproducing kernel Hilbert space on* $\mathbb{B}_n$ *defined by reproducing kernels*

$$K_z(w) = \frac{1}{1 - \langle w, z \rangle}, \quad z, w \in \mathbb{B}_n.$$

The polynomials act on H_n^2 *by multiplication:*

$$\Phi(p) = M_p, \quad M_p(f) = pf.$$

Arveson [2] showed that the following inequality holds for any row contraction.

$$\|p(T_1, \cdots, T_n)\| \leq \|p(M_{z_1}, \cdots, M_{z_n})\|.$$

Definition 1.0.3. *Suppose* $\mathcal{P} \subseteq \mathcal{H}$ *is a closed subspace that is invariant under the module actions, i.e.,*

$$\Phi(z_i)\mathcal{P} \subseteq \mathcal{P}, \quad i = 1, \cdots, n.$$

By defining

$$\Phi' : \mathbb{C}[z_1, \cdots, z_n] \to \mathcal{B}(\mathcal{P}), \quad q \mapsto \Phi(q)|_{\mathcal{P}},$$

one obtains a Hilbert module structure on $\mathcal{P}$*. We say that* $\mathcal{P}$ *is a* submodule *of* $\mathcal{H}$*.*

Similarly, for $\mathcal{Q} := \mathcal{P}^{\perp}$ *we consider*

$$\Phi'' : \mathbb{C}[z_1, \cdots, z_n] \to \mathcal{B}(\mathcal{Q}), \quad q \mapsto Q\Phi(q)|_{\mathcal{Q}},$$

where Q denotes the orthogonal projection onto $\mathcal{Q}$. The Hilbert module $\mathcal{Q}$ is called a quotient module *of the Hilbert module $\mathcal{H}$.*

For a Hilbert module $\mathcal{H}$, we say that $\mathcal{H}$ is *essentially normal* if the cross commutators $[\Phi(z_i), \Phi(z_j)^*]$ belong to the algebra of compact operators $\mathcal{K}(\mathcal{H})$, $\forall i, j = 1, \cdots, n$. For $1 \le p < \infty$, we say that $\mathcal{H}$ is *p-essentially normal* if $[\Phi(z_i), \Phi(z_j)^*]$ belong to the Schatten class C_p, $\forall i, j = 1, \cdots, n$.

Essentially normal Hilbert modules are very important objects in functional analysis. It gives rise to several important invariants which links operator theory with geometry. The following proposition states the relation between essential normality of quotient modules and submodules.

Proposition 1.0.4. *[5, Proposition 4.1] Let $\mathcal{H}$ be a Hilbert module, $\mathcal{P} \subseteq \mathcal{H}$ a submodule, and $\mathcal{Q} = \mathcal{P}^\perp$ the quotient module. Write $A_i = \Phi(z_i)$, $B_i = \Phi'(z_i)$ and $C_i = \Phi''(z_i)$, $i = 1, \cdots, n$. Then*

(1) $[B_i, B_k^*]P = -[P, A_i][P, A_j]^* + P[A_i, A_j^*]P$

(2) $[C_i, C_j^*]Q = [P, A_j]^*[P, A_i] + Q[A_i, A_j^*]Q.$

As a consequence, if $\mathcal{H}$ is p-essentially normal (essentially normal), then the following are equivalent.

(1) $\mathcal{P}$ is p-essentially normal (essentially normal).

(2) $\mathcal{Q}$ is p-essentially normal (essentially normal).

(3) $[P, A_i] \in C_{2p}$ *($\mathcal{K}(\mathcal{H})$), $i = 1, \cdots, n$.*

(4) $[Q, A_i] \in C_{2p}$ *($\mathcal{K}(\mathcal{H})$), $i = 1, \cdots, n$.*

In [4] and [5], Arveson raised a conjecture about module actions on submodules and quotient modules.

Arveson's Conjecture: Let I be a homogeneous ideal in $\mathbb{C}[z_1, \cdots, z_n]$ and let $\mathcal{P}_I$ be its closure on the Drury-Arveson space H_n^2. Then $\mathcal{P}_I$ is p-essentially normal for all $p > n$.

Arveson [4] [5] showed that if $\mathcal{P}_I$ is essentially normal, the module actions give rise to a quantity called the curvature invariant [3], which is stable under compact perturbations. Moreover, an essentially normal Hilbert module gives an exact sequence

$$0 \to \mathcal{K} \to C^*(T_1, \cdots, T_n) + \mathcal{K} \to C(X) \to 0.$$

Here X is some topological space, and $(T_1, \cdots, T_n)$ are the module actions corresponding to the coordinate functions $z_1, \cdots, z_n$. For the submodule $\mathcal{P}_I$, it is known that X always equals $\partial\mathbb{B}_n$. On the other hand, for the quotient module Q_I, $X = Z(I) \cap \partial\mathbb{B}_n$. This property of quotient modules gives rise to very important geometric applications. In [11], Douglas indicates that the exact sequence above for $[Q_I]$ defines an element in the odd K-homology group in $K_1(Z(I) \cap \partial\mathbb{B}_n)$. For almost all known cases, this element proves to be nontrivial, and for some special cases, it was shown to coincide with the one determined by the complex structure or the $spin^c$ structure on the variety. These ideas lead to an analytic version of the Grothendieck-Riemann-Roch Theorem (cf. [13]). Therefore proving essential normality opens the door for formulating and proving a generalization of the Grothendieck-Riemann-Roch Theorem for varieties with singularities.

The Conjecture can also be formulated on other function spaces. In [10], Douglas extended the results of Arveson to a general class of Hilbert modules.

Definition 1.0.5. *Denote $\mathbb{B}_n$ the open unit ball of $\mathbb{C}^n$ and v the normalized Lebesgue measure on $\mathbb{B}_n$, i.e., $v(\mathbb{B}_n) = 1$. The* Bergman space *is the Hilbert space of holomorphic functions on $\mathbb{B}_n$ that are square integrable with respect to the measure v.*

$$L^2_a(\mathbb{B}_n) = \left\{ f \in \mathrm{Hol}(\mathbb{B}_n) : \int_{\mathbb{B}_n} |f(w)|^2 \mathrm{d}v(w) < \infty \right\}.$$

For a positive integer l, the weighted Bergman space $L_{a,l}(\mathbb{B}_n)$ *is defined as*

$$L^2_{a,l}(\mathbb{B}_n) = \left\{ f \in \mathrm{Hol}(\mathbb{B}_n) : \int_{\mathbb{B}_n} |f(w)|^2 (1 - |w|^2)^l \mathrm{d}v(w) < \infty \right\}.$$

Let σ denote the normalized surface measure on the unit sphere $S = \partial\mathbb{B}_n$. The Hardy space *$H^2(\mathbb{B}_n)$ consists of holomorphic functions f on $\mathbb{B}_n$ such that*

$$\sup_{0<r<1}\int_S |f(r\zeta)|^2 \mathrm{d}\sigma(\zeta) < \infty.$$

For a domain $\Omega \subseteq \mathbb{C}^n$, one can also define the Bergman space and Hardy space on Ω (cf. [37]).

The module actions on these spaces are defined by multiplication, $M_q(f) = qf$, $\forall q \in \mathbb{C}[z_1, \cdots, z_n]$.

More generally, for $f \in L^\infty(\Omega)$, one can define the Toeplitz operator T_f on $L^2_a(\Omega)$ such that $T_f(g) = P_{L^2_a(\Omega)}(fg)$. The Toeplitz algebra *with L^∞ symbols $\mathcal{T}(L^\infty)$ is the C^*-algebra generated by the operators T_f where $f \in L^\infty(\Omega)$. For the Hardy space one can define the Toeplitz algebra similarly, the only difference is that $f \in L^\infty(\partial\Omega)$.*

Since our discussions are independent within the sections regarding different spaces, we will use the same notation M_q for multiplication operators on these spaces.

One important and well-known property of these Hilbert modules is that they are p-essentially normal for all $p > n$.

In general, let $\mathcal{H}$ denote one of the Drury-Arveson module, the Bergman module, or the Hardy module. Given an ideal I of $\mathbb{C}[z_1, \cdots, z_n]$, one can consider its closure $\mathcal{P}_I$ in $\mathcal{H}$. A moment of reflection shows that $\mathcal{P}_I$ is a Hilbert submodule of $\mathcal{H}$.

The geometric applications carry over to the cases of the Bergman module or Hardy module. In [11], Douglas gave a few refinements of the Arveson's Conjecture . The following is the most well-known form.

Arveson-Douglas Conjecture Suppose I is a homogeneous ideal of $\mathbb{C}[z_1, \cdots, z_n]$. Write $\mathcal{P}_I$ for the submodule as defined above, and Q_I the corresponding quotient module. Let d be the complex dimension of the variety $Z(I)$. Then for any $p > d$ the quotient module is p-essentially normal.

We remark that, some submodules with non-homogeneous, and even non-algebraic generators, were also shown to derive essentially normal submodules and quotient modules. On the other hand, even for some simple cases, the essential normality is already hard to prove, and will already give us remarkable applications. These results are also generally referred to as the Arveson-Douglas Conjecture.

One can formulate another type of submodules. Suppose $M \subset \mathbb{B}_n$ (or a domain Ω) is a complex zero variety, define

$$\mathcal{P}_M = \{f \in \mathcal{H} : f|_M = 0\}.$$

Clearly $\mathcal{P}_M$ is a submodule of $\mathcal{H}$.

The submodules $\mathcal{P}_I$ and $\mathcal{P}_M$ are connected with each other. In fact, one can show that, when I is homogeneous and radical, we have $\mathcal{P}_I = \mathcal{P}_M$ with $M = Z(I) \cap \mathbb{B}_n$. Here $Z(I)$ is the zero variety of I.

$$Z(I) = \{z \in \mathbb{C}^n : q(z) = 0, \forall q \in I\}.$$

Thus $\mathcal{P}_M$ can be viewed as $\mathcal{P}_I$ without multiplicity. Regarding the second example $\mathcal{P}_M$, a geometric version of the Arveson-Douglas Conjecture was raised and studied.

Geometric Arveson-Douglas Conjecture Suppose $M \subseteq \mathbb{B}_n$ is a complex zero variety of dimension d. Let the submodule

$$\mathcal{P}_M = \{f \in L_a^2(\mathbb{B}_n) : f|_M = 0\},$$

and let $Q_M := \mathcal{P}_M$ be the quotient module. Then Q_M is p-essentially normal for all $p > d$.

In this dissertation, we will focus our study on the Bergman space (and Hardy space in Section 3.3). One reason for doing so is that, for the geometric applications, one can work on either space. Moreover, for homogeneous ideals, one can show that the conjecture is equivalent among different spaces. Another reason is on the technical side, unlike the Drury-Arveson norm, the Bergman norm is defined by a measure. Therefore it is easier to do harmonic analysis on the Bergman space.

There is another aspect of this conjecture that is worth mentioning. Take the Bergman module $L_a^2(\mathbb{B}_n)$ for example. Recall that from Proposition 1.0.4, in order to obtain essential normality of a submodule or a quotient module, one needs to show that $[P, M_{z_i}]$ is compact for any $i = 1, \cdots, n$. Notice that

$$[P, M_{z_i}]^* = (PM_{z_i} - PM_{z_i}P)^* = QM_{z_i}^*P = P_{L_a^2}(I - P)M_{\overline{z_i}}P.$$

The last expression makes sense when we consider P as the projection operator from $L^2(\mathbb{B}_n)$ to $\mathcal{P}$. The form of the operator $(1-P)M_{\overline{z_i}}P$ resembles that of a Hankel operator $H_{\overline{z_i}}$. Indeed, the fact that $L_a^2(\mathbb{B}_n)$ is essentially normal is just the well-known fact that the Hankel operators with symbols $\overline{z_i}$ are compact. Therefore the Arveson-Douglas Conjecture can be viewed as a generalization of this fact to submodules and quotient modules.

The Arveson-Douglas Conjecture has been studied by many mathematicians for almost 20 years. It originated from an operator theoretical problem, but its later development has involved index theory, algebraic geometry, harmonic analysis and several complex variables. In his paper [5], Arveson showed that the conjecture is true for an ideal generated by a monomial. Later, Guo and Wang [25] proved the conjecture for the cases $n \leq 3$, $\dim_{\mathbb{C}} Z(I) \leq 1$, or I is principal. In 2011, Douglas and Wang [14] made a breakthrough, showing that the conjecture holds on the Bergman space $L_a^2(\mathbb{B}_n)$, for any principal ideal, not necessarily homogeneous. They introduced harmonic analysis into the discussion of the conjecture. A geometrical version of the conjecture was raised and discussed in [27] by Kennedy and Shalit. In this paper, they considered decomposition of algebraic varieties. Shalit [34] has also obtained results in the case that the ideal I has the stable division property. Almost symutaneously, two breakthroughs were made on the Geometric Arveson-Douglas Conjecture. Using the Boutet de Monvel calculus, Englis and Eschmeier [18] showed that the conjecture holds for homogeneous varieties smooth away from the origin, and also for smooth varieties that are transversal with the boundary. Douglas, Tang and Yu [13] introduced tools from index theory to study the case when the variety is a complete intersection space, smooth on $\mathbb{B}_n$ and transeversal to $\partial\mathbb{B}_n$. They also found out explicitly the K-homology element for the quotient module.

The work discussed in this dissertation contains several joint works of the author with Professor Ronald G. Douglas, Professor Kunyu Guo and Professor Jingbo Xia. In [16], Douglas and the author used harmonic analysis methods and showed the Geometric Arveson-Douglas Conjecture for varieties smooth on $\partial\mathbb{B}_n$ and transversal with $\partial\mathbb{B}_n$. In [17] they showed the conjecture for unions of the above varieties intersecting cleanly on $\partial\mathbb{B}_n$. In [38], Xia and the author filled a remaining gap in [16] and [17]. They showed that the quotient module is p-essentially normal for $p > \dim M$, as stated in the conjecture. Along another line, in [12], Douglas, Guo and the author proved for the Bergman space on bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary that any principal ideals, generated by a holomorphic function defined on a neighborhood of the closure, is p-essentially normal for $p > n$. Later, Xia and the author showed that the same holds for Hardy space on such domains [39] .

2. ESSENTIAL NORMALITY AND HOLOMORPHIC EXTENSIONS*

In this chapter, we discuss the following form of the Arveson-Douglas Conjecture.

Geometric Arveson-Douglas Conjecture *Suppose $M \subseteq \mathbb{B}_n$ is a complex zero variety of dimension d. Let the submodule*

$$\mathcal{P}_M = \{f \in L_a^2(\mathbb{B}_n) : f|_M = 0\}$$

and let $Q_M := \mathcal{P}_M$ be the quotient module. Then Q_M is p-essentially normal for all $p > d$.

As explained in the previous chapter, to obtain essential normality of a submodule or quotient module, one needs to have a good understanding of the projection operator onto it. In the case of the Geometric Arveson-Douglas Conjecture, the quotient module actually live on the variety M. Let us explain in detail.

Proposition 2.0.6. *Let Q_M be defined as above, then*

$$Q_M = \overline{span}\{K_z : z \in M\}.$$

Proof. The proof is essentially bookkeeping. From the definition, it is easy to see that $\overline{\text{span}}\{K_z : z \in M\} \subseteq Q_M$. On the other hand, if $f \perp K_z$ for any $z \in M$, by definition, $f \in \mathcal{P}_M = Q_M^{\perp}$. Therefore the two spaces are equal. This completes the proof. □

From Proposition 2.0.6, we see that Q_M coincides with the reproducing kernel Hilbert space on M determined by $\{K_z|_M : z \in M\}$ (cf. [1]). Another type of reproducing kernel Hilbert spaces that live on M are the weighted Bergman spaces. Let $\tilde{M}$ be an d-dimensional complex submanifold of a neighborhood of the closed unit ball $\overline{\mathbb{B}_n}$. For $z \in M$ write $\delta(z)$ for the distance from z to the boundary of M with respect to the Riemannian metric on M. For $s > 0$, we denote by $L_{a,s}^2(M)$ the

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space of holomorphic functions on M that are square integrable with respect to the measure $\delta^s \mathrm{d}v_M$. Here v_M denotes the volume measure on M.

$$L^2_{a,s}(M) = \{f \in \mathrm{Hol}(M) : \int_M |f(z)|^2 \delta^s(z) \mathrm{d}v_M(z) < \infty\}.$$

The following example shows some connection between quotient spaces Q_M and weighted Bergman spaces $L^2_{a,s}(M)$.

Example 2.0.7. *Suppose M is the intersection of a d-dimensional linear subspace with the open unit ball $\mathbb{B}_n$, where $d < n$. We can identify M with the unit ball in $\mathbb{C}^d$. Let $\mathcal{P}_M$ and Q_M be defined as above.*

Let ρ be the weighted Bergman measure on M: $d\rho = c(1-|z|^2)^{n-d} dm_d$, where m_d is the Lebesgue measure on M and $c > 0$ is chosen such that $\rho(M) = 1$. It is well-known that the weighted Bergman space on M defined by

$$L^2_{a,n-d}(M) = \{f \text{ is an analytic function on } M : \int_M |f(w)|^2 d\rho(w) < \infty\}$$

is a reproducing kernel Hilbert space on M with reproducing kernels

$$\{K_z^{n-d}(w) = \frac{1}{(1 - \langle w,\ z\rangle)^{n+1}} :\ z \in M\}.$$

We have shown that Q_M is also a reproducing kernel Hilbert space on M with the same reproducing kernels. By [1], the two spaces are isometric. In fact, a simple calculation shows that the restriction map

$$R : Q_M \to L^2_{a,n-d}(M), f \mapsto f|_M$$

is an isometry.

An immediate question is that, is it true that Q_M is a weighted Bergman space? The answer is, unfortunately, no in general.

Example 2.0.8. *Suppose $M = \{a_1, \cdots, a_m\} \subset \mathbb{B}_n$, then there exists a positive measure μ on M such that $\forall f \in \mathcal{Q}_M$,*

$$\|f\|^2 = \int_M |f|^2 d\mu,$$

if and only if $m = 1$.

Proof. The "if" part is obvious, we prove the "only if" part.

Suppose μ is supported on M such that the equation holds, then $\mu = \sum_{i=1}^{m} c_i\delta_i$, where $c_i \geq 0$ and δ_i are the point masses at a_i, $i = 1, \cdots, m$. For any $x_1, \cdots, x_m \in \mathbb{C}$, let $x = \sum_{i=1}^{m} x_i k_{a_i} \in \mathcal{Q}_M$, then

$$\|x\|^2 = \sum_{i,j} x_i\overline{x_j}\langle k_{a_i}, k_{a_j}\rangle.$$

On the other hand,

$$\begin{aligned}\int_M |x(w)|^2 d\mu(w) &= \sum_{i=1}^{m} c_i|x(a_i)|^2 \\ &= \sum_{i=1}^{m} c_i(1-|a_i|^2)^{-(n+1)}|\sum_{j=1}^{m} x_j\langle k_{a_j}, k_{a_i}\rangle|^2.\end{aligned}$$

Let G be the $m \times m$ matrix $(\langle k_{a_i}, k_{a_j}\rangle)_{ij}$, then

$$\|x\|^2 = \begin{pmatrix} x_1 & \dots & x_m \end{pmatrix} G \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_m} \end{pmatrix}$$

and

$$\int_M |x(w)|^2 d\mu(w) = \begin{pmatrix} x_1 & \dots & x_m \end{pmatrix} G \begin{pmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_m \end{pmatrix} G^* \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_m} \end{pmatrix}$$

where $d_i = c_i(1 - |a_i|^2)^{-(n+1)}$. Since x_i are arbitrary, we have

$$G = G \begin{pmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_m \end{pmatrix} G^*.$$

This only holds when G is diagonal, which implies $m = 1$. □

Now let us turn our attention to another problem related to complex varieties - the problem of holomorphic extensions. This problem has been studied extensively by function theorists. Given a complex variety in a domain, one asks whether a holomorphic function on that variety can be extended to a holomorphic function on the entire domain. Norm estimates of the extension maps are also studied. The work of Frank Beatrous [6] is mostly related to our topic.

Definition 2.0.9. *Let Y be a manifold and let X, Z be two submanifolds of Y. We say that the submanifolds X and Z are* transversal *if $\forall x \in X \cap Z$, $T_x(X) + T_x(Z) = T_x(Y)$.*

Theorem 2.0.10 (Beatrous)**.** *Let $\tilde{M}$ be an d-dimensional complex submanifold of a neighborhood of the closed unit ball $\overline{\mathbb{B}_n}$ which intersects $\partial\mathbb{B}_n$ transversally. Let $M = \tilde{M} \cap \mathbb{B}_n$ and $s = n - d$, then there is a bounded linear operator*

$$E : L^2_{a,s}(M) \to L^2_a(\mathbb{B}_n)$$

such that $Ef|_M = f$ for any $f \in L^2_{a,s}(M)$.

Douglas, Tang and Yu [13] extended this result to complete intersection spaces intersecting $\partial\mathbb{B}_n$ transversally, with possibly finite singularities inside the unit ball.

In Section 2.2 we will build a connection between the Geometric Arveson-Douglas Conjecture and the problem of holomorphic extensions. These two problems might seem unrelated at first sight. At second thought, however, one realizes that they have something in common. Denote by Q_M the orthogonal projection onto the quotient module $\mathcal{Q}_M$. Given a function $f \in L^2_a(\mathbb{B}_n)$, its image under Q_M depends only on the restriction $f|_M$. In other words, the operator Q_M sends the restricted function $f|_M$ to the function $Q_M(f)$, which is defined on the unit ball $\mathbb{B}_n$. Therefore Q_M can be viewed as an extension operator from the variety M to $\mathbb{B}_n$. This shows some connection between the two problems. In order to explain this in detail, we need to introduce some tools from harmonic analysis.

2.1 Carleson Measures and Integral Operators

This section serves as a crash course on harmonic analysis regarding the Bergman space $L^2_a(\mathbb{B}_n)$. We have no intention in giving a full description over the material. Instead, we will just list the parts that are related to our topic. The interested readers can refer to [33], [40] and [36]for more details.

For $z \in \mathbb{B}_n$, write P_z for the orthogonal projection onto the complex line $\mathbb{C}z$ and $Q_z = I - P_z$. The map

$$\varphi_z(w) = \frac{z - P_z(w) - (1-|z|^2)^{1/2}Q_z(w)}{1 - \langle w, z\rangle}$$

is the (unique) automorphism of $\mathbb{B}_n$ that satisfies $\varphi_z \circ \varphi_z = id$ and $\varphi_z(0) = z$.

The following lemma can be verified by direct computation, for a proof see [33].

Lemma 2.1.1. *Suppose a, z, $w \in \mathbb{B}_n$, then*

(1)

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1-\langle a,a\rangle)(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)}.$$

(2) As a consequence of (1),

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}.$$

(3) The Jacobian of the automorphism φ_z is

$$(J\varphi_z(w)) = \frac{(1-|z|^2)^{n+1}}{|1-\langle w,z\rangle|^{2(n+1)}}.$$

Definition 2.1.2. *The* pseudo-hyperbolic metric *on* $\mathbb{B}_n$ *is defined by*

$$\rho(z, w) = |\varphi_z(w)|.$$

And the hyperbolic metric *is defined by*

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

Thus $\rho(z, w) = \tanh \beta(z, w)$. *The metric* β *is sometimes also called the Bergman metric. It is well known that both metrics are invariant under actions of* $Aut(\mathbb{B}_n)$*, the group of holomorphic automorphisms of* $\mathbb{B}_n$*. That is, given* $\psi \in Aut(\mathbb{B}_n)$*,*

$$\rho(z, w) = \rho(\psi(z), \psi(w)),$$

and

$$\beta(z, w) = \beta(\psi(z), \psi(w))$$

for all z*,* $w \in \mathbb{B}_n$*. For* $r > 0$ *and* $z \in \mathbb{B}_n$*, write*

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(w, z) < r\} = \{w \in \mathbb{B}_n : \rho(w, z) < s_r\},$$

where $s_r = \tanh r$*.*

We will use the notation $D_d(z', r)$ to denote the hyperbolic ball in $\mathbb{B}_d$. The notation $B(z, \tau)$ is used to denote the Euclidean ball with center z and radius τ.

Lemma 2.1.3. *For* $z \in \mathbb{B}_n$*,* $r > 0$*, the hyperbolic ball* $D(z, r)$ *consists of all* w *that satisfy:*

$$\frac{|Pw - c|^2}{s_r^2 \rho^2} + \frac{|Qw|^2}{s_r^2 \rho} < 1,$$

where $P = P_z$, $Q = Q_z$, and

$$c = \frac{(1 - s_r^2)z}{1 - s_r^2|z|^2}, \quad \rho = \frac{1 - |z|^2}{1 - s_r^2|z|^2}.$$

Thus $D(z, r)$ is an ellipsoid with center c, radius $s_r\rho$ in the z direction and $s_r\sqrt{\rho}$ in the directions perpendicular to z. Therefore the Lebesgue measure of $D(z, r)$ is

$$v_n(D(z, r)) = C s_r^{2n} \rho^{n+1},$$

where $C > 0$ is a constant depending only on n. Note that when we fix r, ρ is comparable with $1 - |z|^2$. Hence $v(D(z, r))$ is comparable with $(1 - |z|^2)^{n+1}$.

One of the reasons that the hyperbolic metric is important is that it matches the analytic structure on the unit ball. From the properties of the map φ_z, it is easy to deduce

Lemma 2.1.4. [40] *Given any $0 < r < \infty$, there exists a constant $0 < C_r < \infty$ such that for any $z, w \in \mathbb{B}_n$ satisfying $\beta(z, w) < r$ and any $\lambda \in \mathbb{B}_n$,*

(1) $C_r^{-1} \leq \frac{1-|z|^2}{1-|w|^2} \leq C_r$,

(2) $C_r^{-1} \leq \frac{|1-\langle \lambda, z\rangle|}{|1-\langle \lambda, w\rangle|} \leq C_r$.

Suppose ν is a positive, finite, regular, Borel measure. The operator

$$T_\nu f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1}} d\nu(w)$$

defines an analytic function for every $f \in H^\infty(\mathbb{B}_n)$. The following lemma is implied by the proof of Lemma 2.1 in [36].

Lemma 2.1.5. *Let ν be a positive, finite, regular, Borel measure on $\mathbb{B}_n$ and $r > 0$, then the following are equivalent. When one of these conditions holds, ν is called a* Carleson measure *(for $L_a^2(\mathbb{B}_n)$).*

(1) $\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1}}{|1-\langle w, z\rangle|^{2(n+1)}} d\nu(w) < \infty$,

(2) $\exists C > 0 : \int |f|^2 d\nu \leq C \int |f|^2 dv$ *for all* $f \in L_a^2(\mathbb{B}_n)$,

(3) $\sup_{z\in\mathbb{B}_n} \frac{\nu(D(z,\ r))}{\nu_n(D(z,\ r))} < \infty$,

(4) T_ν *extends to a bounded linear operator on* $L^2_a(\mathbb{B}_n)$.

Suppose ν is a Carleson measure, by Fubini's Theorem, we have:

$$\langle T_\nu f, g\rangle = \int_{\mathbb{B}_n} f(w)\overline{g(w)}d\nu(w), \quad \forall f, g \in L^2_a(\mathbb{B}_n).$$

Carleson measures and the integral operators T_ν play an crucial role in the discussion of this chapter. A particularly important property is the following corollary of Theorem 7.3 in [36].

Lemma 2.1.6. *Suppose* ν *is a Carleson measure, then the operator* T_ν *belongs to the Toeplitz algebra of* L^∞ *symbols* $\mathcal{T}(L^\infty)$.

The following lemma is crucial to our proof of essential normality. One can find a proof in [31, Proposition 1.4].

Lemma 2.1.7. *If* $f \in C(\overline{\mathbb{B}_n})$*, then* T_f *essentially commutes with every operator in the Toeplitz algebra* $\mathcal{T}(L^\infty)$.

We will use the following lemma frequently in calculation. One can find a proof in [33, Proposition 1.4.10].

Lemma 2.1.8. *For* $z \in \mathbb{B}_n$*, c real,* $t > -1$*, define*

$$I_c(z) = \int_S \frac{d\sigma(\zeta)}{|1 - \langle z, \zeta\rangle|^{n+c}}$$

and

$$J_{c,t}(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t d\nu(w)}{|1 - \langle z, w\rangle|^{n+1+t+c}}.$$

When $c < 0$*, then* I_c *and* $J_{c,t}$ *are bounded in* $\mathbb{B}_n$*. When* $c > 0$*, then*

$$I_c(z) \approx (1 - |z|^2)^{-c} \approx J_{c,t}(z).$$

Finally,

$$I_0(z) \approx \log \frac{1}{1-|z|^2} \approx J_{0,t}(z).$$

2.2 Equivalent Measures

In this section we establish some connections between the Geometric Arveson-Douglas Conjecture and the holomorphic extension theory. As illustrated by the examples in the beginning of Chapter 2, the quotient module $\mathcal{Q}_M$ seem to be connected with a weighted Bergman space but does not necessarily be one. Using the language of harmonic analysis we can make this precise.

Theorem 2.2.1. *Suppose M is a subset of $\mathbb{B}_n$. Let $\mathcal{Q}_M$ be the quotient module in the Bergman space defined as in the Geometric Arveson-Douglas Conjecture. If there exists a positive, finite, regular, Borel measure μ on M such that for some constants $C, c > 0$ and for any $f \in \mathcal{Q}_M$,*

$$c\|f\|^2 \leq \int_M |f(w)|^2 \mathrm{d}\mu(w) \leq C\|f\|^2,$$

then the projection operator Q_M is in the Toeplitz algebra $\mathcal{T}(L^\infty)$. As a consequence, the quotient module $\mathcal{Q}_M$ is essentially normal.

Proof. First, we prove that the measure μ is a Carleson measure. From the assumption, we have for any $z \in \mathbb{B}_n$,

$$\begin{aligned}
\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1}}{|1-\langle w,\ z\rangle|^{2(n+1)}} d\mu(w) &= \int_M |k_z(w)|^2 d\mu(w) \\
&= \int_M |Q_M k_z(w)|^2 d\mu(w) \\
&\leq C\|Q_M k_z\|^2 \\
&\leq C.
\end{aligned}$$

The second equality is because $k_z - Q_M k_z \in \mathcal{P}_M$, therefore $k_z(w) = Q_M k_z(w), \forall w \in M$. By Lemma 2.1.5, μ is a Carleson measure.

Next we show that the projection operator Q_M is a continuous function calculus of T_μ and therefore is in the Toeplitz algebra. From the equation

$$\langle T_\mu f,\ f\rangle = \int_M |f(w)|^2 d\mu(w), \quad \forall f \in L_a^2(\mathbb{B}_n)$$

we see that T_μ is positive and vanishes on $\mathcal{P}_M$. Also, the equivalence of $L^2(\mu)$-norm and Bergman norm on Q_M implies that T_μ is bounded below on Q_M. Therefore 0 is isolated in $\sigma(T_\mu)$ and $\mathcal{P}_M = \ker T_\mu$. Take any continuous function f on $\mathbb{R}$ that vanishes at 0 and equals 1 on the rest of the spectrum, then $Q_M = f(T_\mu)$.

Finally, by Lemma 2.1.6, Q_M is in the Toeplitz algebra. By Lemma 2.1.7, Q_M essentially commutes with the operators M_{z_i}, $i = 1, \cdots, n$. Therefore the quotient module Q_M is essentially normal. This completes the proof. □

We are now ready to discuss the connection between the Geometric Arveson-Douglas Conjecture and the problem of holomorphic extensions.

Remark 2.2.2. *First let us remark that Theorem 2.2.1 together with Theorem 2.0.10 give us an immediate proof of the Geometric Arveson-Douglas Conjecture for smooth varieties.*

Proposition 2.2.3. *Let $\tilde{M}$ be an d-dimensional complex submanifold of a neighborhood of the closed unit ball $\overline{\mathbb{B}_n}$ which intersects $\partial\mathbb{B}_n$ transversally. Let $M = \tilde{M} \cap \mathbb{B}_n$, then the quotient module Q_M is essentially normal.*

Proof. Let $s = n - d$ and the extension operator

$$E : L_{a,s}^2(M) \to L_a^2(\mathbb{B}_n)$$

be as in Theorem 2.0.10, i.e., $RE = Id$ where R is the restriction operator. Therefore Range$(R) = L_{a,s}^2(M)$ and by the closed graph theorem, R is bounded. For every $f \in Q_M$,

$$\int_M |f(z)|^2 \delta^s(z) \mathrm{d}v_M(z) = \|Rf\|^2 \leq \|R\|^2 \|f\|^2.$$

On the other hand,

$$\|f\|^2 \le \|ERf\|^2 \le \|E\|^2\|Rf\|^2 = \|E\|^2 \int_M |f(z)|^2\delta^2(z)\mathrm{d}v_M(z).$$

The result follows by Theorem 2.2.1. $\square$

From this we see that existence of a bounded extension operator implies essential normality. On the other hand, suppose μ satisfies the assumption of Theorem 2.2.1, then the map

$$E : \mathrm{Range}(R) \to Q_M \subset L_a^2(\mathbb{B}_n), Rf \mapsto f \in Q_M$$

is an extension operator. So this also provides a way to solve the holomorphic extension problem.

2.3 Varieties with Smooth Boundaries

In this section, we will show that, for certain type of varieties, equivalent measures exist. We will achieve this by doing local analysis on the variety. Tools from harmonic analysis allow us to achieve global estimates from local ones. Before stating the theorem, let us give a few definitions.

Definition 2.3.1. *Let Ω be a complex manifold. A set $A \subset \Omega$ is called a* (complex) analytic subset *of Ω if for each point $a \in \Omega$ there are a neighborhood $U \ni a$ and functions $f_1, \cdots, f_N$ holomorphic in this neighborhood such that*

$$A \cap U = \{z \in U : f_1(z) = \cdots = f_N(z) = 0\}.$$

A point $a \in A$ is called regular *if there is a neighborhood $U \ni a$ in Ω such that $A \cap U$ is a complex submanifold of Ω. A point $a \in A$ is called a* singular point *of A if it is not regular.*

The main goal of this section is to prove the following theorem.

Theorem 2.3.2. *Suppose $\tilde{M}$ is a complex analytic subset of an open neighborhood of $\overline{\mathbb{B}_n}$ satisfying the following conditions:*

(1) $\tilde{M}$ intersects $\partial\mathbb{B}_n$ transversally.

(2) $\tilde{M}$ has no singular point on $\partial\mathbb{B}_n$.

Let $M = \tilde{M} \cap \mathbb{B}_n$ and let $\mathcal{P}_M = \{f \in L_a^2(\mathbb{B}_n) : f|_M = 0\}$, $\mathcal{Q}_M = \mathcal{P}_M^{\perp}$, then there exists a measure μ on M satisfying the hypotheses of Theorem 2.2.1. Consequently, the quotient module $\mathcal{Q}_M$ is essentially normal.

Note that in this case, condition (1) is equivalent to that $\tilde{M}$ is not tangent to $\partial\mathbb{B}_n$ at every point of $\tilde{M} \cap \partial\mathbb{B}_n$. Condition (2) implies that $\tilde{M}$ has only finitely many singular points inside $\mathbb{B}_n$ [8].

In order to prove Theorem 2.3.2, we need to establish a few lemmas.

Lemma 2.3.3. *Let α be the intersection of a d-dimensional affine space and $\mathbb{B}_n$. Then α is a d-dimensional ball. Let r be the radius of α and v be the volume measure on α. Then for any function f holomorphic on α and any $t > 0$, $z \in \alpha$,*

$$\int_{\alpha \cap D(z,r)} f(w) \frac{(1 - |w|^2)^{n-d}}{(1 - \langle z, w \rangle)^{n+1}} dv(w) = r^{-2} C_t f(z).$$

where

$$C_t = \int_{D_d(0,t)} (1 - |w|^2)^{n-d} dv(w).$$

Here $D_d(0, t)$ means the hyperbolic ball in $\mathbb{B}_d$ centered at 0 with radius t and v is the volume measure on $\mathbb{B}_d$.

Proof. Let z_0 be the center of α and let

$$\phi : \alpha \to \beta = \frac{1}{t}(\alpha - z_0), \; z \mapsto \frac{1}{t}(z - z_0).$$

The affine space β is the intersection of a hyperplane and $\mathbb{B}_n$, therefore can be identified with $\mathbb{B}_d$. Clearly ϕ is biholomorphic. For $z \in \alpha$, consider the map

$$\varphi_z \phi^{-1} : \beta \to \gamma = \varphi_z(\alpha).$$

By [33, Proposition 2.4.2], γ lies in an affine space containing 0. Hence γ can also be identified with $\mathbb{B}_d$. So $\varphi_z\phi^{-1}$ is an automorphism of $\mathbb{B}_d$ and therefore preserves the hyperbolic metric. We get

$$\begin{aligned}\phi(D(z,t)\cap\alpha) &= \phi\varphi_z^{-1}\varphi_z(D(z,t)\cap\alpha) = \phi\varphi_z^{-1}(D(0,t)\cap\gamma)\\ &= \phi\varphi_z^{-1}(D_d(0,t)) = D_d(\phi(z),t).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{\alpha\cap D(z,t)} f(w)\frac{(1-|w|^2)^{n-d}}{(1-\langle z,w\rangle)^{n+1}}d\nu(w)\\ =\ &\int_{D_d(\phi(z),t)} f\phi^{-1}(\eta)\frac{(1-|\phi^{-1}(\eta)|^2)^{n-d}}{(1-\langle z,\phi^{-1}(\eta)\rangle)^{n+1}}d\nu(\phi^{-1}(\eta))\\ =\ &\int_{D_d(\phi(z),t)} f\phi^{-1}(\eta)\frac{(r^2-r^2|\eta|^2)^{n-d}}{(r^2-r^2\langle\phi(z),\eta\rangle)^{n+1}}r^{2d}d\nu(\eta)\\ =\ &r^{-2}\int_{D_d(\phi(z),t)} f\phi^{-1}(\eta)\frac{(1-|\eta|^2)^{n-d}}{(1-\langle\phi(z),\eta\rangle)^{n+1}}d\nu(\eta)\\ =\ &r^{-2}C_tf(z).\end{aligned}$$

The last equation comes from the following argument.

In general, if g is holomorphic on $\mathbb{B}_d$, for $r>0$ and $\xi\in\mathbb{B}_d$,

$$\begin{aligned}&\int_{D_d(\xi,t)} g(\eta)\frac{(1-|\eta|^2)^{n-d}}{(1-\langle\xi,\eta\rangle)^{n+1}}d\nu(\eta)\\ =\ &\int_{D_d(0,t)} g\varphi_\xi(w)\frac{(1-|\varphi_\xi(w)|^2)^{n-d}}{(1-\langle\xi,\varphi_\xi(w)\rangle)^{n+1}}\frac{(1-|\xi|^2)^{d+1}}{(1-\langle w,\xi\rangle)^{2(d+1)}}d\nu(w)\\ =\ &\int_{D_d(0,t)} g\varphi_\xi(w)\frac{(1-\langle\xi,w\rangle)^{n+1}(1-|\xi|^2)^{n-d}(1-|w|^2)^{n-d}(1-|\xi|^2)^{d+1}}{(1-|\xi|^2)^{n+1}|1-\langle\xi,w\rangle|^{2(n-d)}|1-\langle w,\xi\rangle|^{2(d+1)}}d\nu(w)\\ =\ &\int_{D_d(0,t)} g\varphi_\xi(w)\frac{(1-|w|^2)^{n-d}}{(1-\langle w,\xi\rangle)^{n+1}}d\nu(w)\\ =\ &C_tg(\xi).\end{aligned}$$

This completes the proof. □

Lemma 2.3.4. *For $t > 0$, we have*

$$\lim_{r\to 1-}\sup_{z\in\mathbb{B}_d}\int_{w\in\mathbb{B}_d: r<|w|<1}\frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{d+1}}d\nu(w)=0.$$

Proof. Let

$$I(z)=\int_S\frac{1}{|1-\langle z,\zeta\rangle|^{d+1}}d\sigma(\zeta).$$

Where S is the unit sphere in $\mathbb{C}^d$ and σ is the volume measure on S. By Lemma 2.1.8, there exists $C>0$ such that

$$I(z)\le C(1-|z|^2)^{-1}.$$

Hence

$$\begin{aligned}\int_{r<|w|<1}\frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{d+1}}d\nu(w) &= \int_r^1\int_S\frac{(1-s^2)^t}{|1-\langle z,s\zeta\rangle|^{d+1}}s^{2d-1}d\sigma(\zeta)ds\\ &= \int_r^1(1-s^2)^t s^{2d-1}I(sz)ds\\ &\le C\int_r^1(1-s^2)^t(1-|sz|^2)^{-1}ds\\ &\le C\int_r^1(1-s^2)^{t-1}ds\to 0. \qquad (r\to 1-)\end{aligned}$$

This completes the proof. □

Suppose $\tilde{M}$ is as in Theorem 2.3.2. We first assume that $\tilde{M}$ is connected. For $0\le s<t\le 1$, define

$$M_s^t=\{z\in M|\ s\le|z|<t\}.$$

Write $M_s=M_s^1, M^t=M_0^t$. Since $\tilde{M}$ has no singular point on $\partial\mathbb{B}_n$, we can cover $\partial\mathbb{B}_n\cap\tilde{M}$ with finitely many open sets $\{U_i\}$, $U_i\subset\tilde{M}$ such that:

(1) For each i, we can find $n-1$ of the canonical basis vectors of $\mathbb{C}^n$, denoted $e_{i_1},\cdots,e_{i_{n-1}}$ such that for any $z\in U_i$, the n vectors $\{z,e_{i_1},\cdots,e_{i_{n-1}}\}$ span $\mathbb{C}^n$.

(2) $\tilde{M}$ has local coordinates on each U_i, i.e., there exists open set $\Omega_i \subset \mathbb{C}^d$ and $\varphi_i : \Omega_i \to U_i$ which is one to one and holomorphic.

Fix $z \in U_i$, apply the Gram-Schmidt process to $\{z, e_{i_1}, \cdots, e_{i_{n-1}}\}$ to obtain a new basis $\{f_1^z, \cdots, f_n^z\}$, then z has coordinates $(z_1, 0, \cdots, 0)$, where $z_1 = |z|$, under this basis. Let

$$G^z : \Omega_i \to U_i, \lambda = (\lambda_1, \cdots, \lambda_d) \mapsto (g_1^z(\lambda), \cdots, g_n^z(\lambda))$$

be the expression of φ_i under the new basis. Note that the new basis and expression depend continuously on z.

Since $\tilde{M}$ intersects $\partial\mathbb{B}_n$ transversally in $\partial\mathbb{B}_n$, the tangent space of $\tilde{M}$ at a point $z \in \partial\mathbb{B}_n \cap \tilde{M}$ can not be orthogonal to $\mathbb{C}z$. Therefore the first entry of the vectors $\{\frac{\partial G^z}{\partial \lambda_i}\}_{i=1}^d$ can not vanish simultaneously. By continuity, this is true for $z \in \tilde{M} \cap \mathbb{B}_n$ when z is close enough to $\partial\mathbb{B}_n$. Therefore by possibly refining the cover $\{U_i\}$, we can assume that for each U_i, $\forall z \in U_i$, $(\frac{\partial g_1^z}{\partial \lambda_1}, \cdots, \frac{\partial g_1^z}{\partial \lambda_d})$ is non-zero at z. Since the matrix $[\frac{\partial g_i^z}{\partial \lambda_j}(z)]_{1 \le i \le n, 1 \le j \le d}$ has rank d, by possibly refining $\{U_i\}$ again we could get $2 \le k_1, \cdots, k_{d-1} \le n$ for each U_i, such that the determinant $\frac{\partial(g_1^z, g_{k_1}^z, \cdots, g_{k_{d-1}}^z)}{\partial(\lambda_1, \cdots, \lambda_d)}|_{\varphi_i^{-1}(z)} \neq 0$, $\forall z \in U_i$. Let ϵ be the Lebesgue number of the cover $\{U_i\}$ and let $V_i = \{z \in U_i | \ d(z, \partial U_i) > \frac{1}{2}\epsilon\}$, then $\partial\mathbb{B}_n \cap \tilde{M} \subset \cup V_i$. The function $\frac{\partial(g_1^z, g_{k_1}^z, \cdots, g_{k_{d-1}}^z)}{\partial(z_1, \cdots, z_d)}(w)$ is uniformly continuous on $\{(z, w) | z \in \bar{V}_i, w \in U_i\}$. Therefore $\exists \tau > 0$ such that $\forall z \in V_i$, $\forall w \in B(z, \tau)$, $\frac{\partial(g_1^z, g_{k_1}^z, \cdots, g_{k_{d-1}}^z)}{\partial(\lambda_1, \cdots, \lambda_d)}(\varphi_i^{-1}(w)) \neq 0$. By the implicit function theorem, we have:

Lemma 2.3.5. *There exists a finite open cover $\{V_i\}$ of $\partial\mathbb{B}_n \cap \tilde{M}$ and $\tau > 0$ such that for any fixed V_i, we can pick $d-1$ numbers out of $\{2, \cdots, n\}$, assume they are $\{2, \cdots, d\}$ without loss of generality, such that $\forall z \in \bar{V}_i$ and $\forall w \in B(z, \tau)$,*

$$w = (w_1, \cdots, w_d, F_{d+1}^z(w'), \cdots, F_n^z(w'))$$

under the basis $\{f_1^z, \cdots, f_n^z\}$, where $w' = (w_1, \cdots, w_d)$. The functions $F_i^z(w')$ are holomorphic in w' and depend continuously on z.

In the later discussion, whenever we fix a $z \in V_i$, we will discuss under the new basis $\{f_i^z\}_{i=1}^n$ and the new expression $(w', F_{d+1}^z, \cdots, F_n^z)$ and we will omit the superscript "z" for convenience. Moreover, we will denote any constant that depends only on M by C as long as it does not cause confusion. So C may refer to different constant in different places.

By Proposition 1 in [8, Page 31], the assumptions in Theorem 2.3.2 imply that $\tilde{M}$ has only finitely many singular points in $\mathbb{B}_n$. Let $\Sigma = \{z_1, \cdots, z_m\}$ be the set of all singular points of $\tilde{M}$ inside $\mathbb{B}_n$. Take $0 < s_1 < 1$ such that $\Sigma \cap M_{s_1} = \emptyset$. The volume measure v_d is well-defined on M_{s_1}. In local coordinates, v_d corresponds to the volume form $E(w)dx_1 \wedge dy_1 \wedge \cdots \wedge dy_d$, where $E(w)$ is the square root of the absolute value of the determinant of the matrix representation of the metric tensor on M_{s_1}. Note that $E(w)$ is uniformly continuous on z and w. Let

$$\delta = \sum_{i=1}^{m} (1 - |z_i|^2)^{n+1} \delta_{z_i},$$

where δ_{z_i} is the point mass at z_i. For $s_1 < s < 1$, let

$$d\mu_s = (1 - |w|^2)^{n-d} dv_d|_{M_s} + d\delta.$$

We will prove that for s sufficiently close to 1, μ_s satisfies the assumption of Theorem 2.2.1, therefore Theorem 2.3.2 holds.

Fix $z \in V_i$ (and the basis depending on z), and define the map

$$p_z : \tilde{M} \cap B(z, \tau) \to T\tilde{M}|_z$$

$$(w', F_{d+1}(w'), \cdots, F_n(w')) \mapsto (w', \sum_{i=1}^{d} \frac{\partial F_{d+1}}{\partial w_i}(z')(w_i - z_i), \cdots, \sum_{i=1}^{d} \frac{\partial F_n}{\partial w_i}(z')(w_i - z_i))$$

Here $T\tilde{M}|_z$ is the tangent space of $\tilde{M}$ at z. Note that by construction, $F_i(z') = 0, i = d+1, \cdots, n$. Clearly, p_z is one to one and holomorphic, $p_z(w) - w \perp z$ and

$$|p_z(w) - w| = O(|w' - z'|^2).$$

Lemma 2.3.6. *Fix $r > 0$, then there exists $1 > s_2 > s_1$, such that*

(1) $\forall z \in M_{s_2}$, $D(z,r) \subset B(z,\tau)$.

(2) $\forall z \in M_{s_2}$, $\forall w \in D(z,r)$, $p_z(w) \in \mathbb{B}_n$.

(3) $\sup_{w\in D(z,r)} |\frac{1-|p_z(w)|^2}{1-|w|^2} - 1| \to 0, \quad |z| \to 1.$

(4) $\sup_{w\in D(z,r)} \beta(p_z(w), w) \to 0, \quad |z| \to 1.$

Proof. By Lemma 2.1.3, it is easy to see that (1) holds as long as we take s_2 sufficiently close to 1.

To prove (2), we notice first that Lemma 2.1.3 also implies

$$\sup_{w\in D(z,r)} |w - z| = O((1 - |z|^2)^{\frac{1}{2}}).$$

Therefore

$$\sup_{w\in D(z,r)} |p_z(w) - w| = O(1 - |z|^2).$$

Since $\langle z, p_z(w)\rangle = \langle z, w\rangle \neq 0$, $\varphi_z(p_z(w))$ is well defined. It is easy to verify that

$$\varphi_z(\xi) \in \mathbb{B}_n \text{ if and only if } \xi \in \mathbb{B}_n.$$

So we only need to make sure that $\varphi_z(p_z(w)) \in \mathbb{B}_n$. Since

$$|\varphi_z(p_z(w)) - \varphi_z(w)| = \frac{(1 - |z|^2)^{\frac{1}{2}}}{|1 - \langle w, z\rangle|} O(1 - |z|^2) = O((1 - |z|^2)^{\frac{1}{2}})$$

and

$$|\varphi_z(w)| \le s_r,$$

when we take s_2 sufficiently close to 1, we have $|\varphi_z(p_z(w))| < 1$. Therefore (2) is proved.

We prove (4) first. Take s_2 so close to 1 that $\forall z \in M_{s_2}$, $\forall w \in D(z,r)$, $\varphi_z(p_z(w)) \in D(0, 2r)$.

On $D(0, 2r)$, the hyperbolic distance and Euclidean distance are equivalent. Hence

$$\beta(p_z(w), w) = \beta(\varphi_z(p_z(w)), \varphi_z(w)) \leq C|\varphi_z(p_z(w)) - \varphi_z(w)| \to 0,$$

as $|z| \to 1$.

Finally, since $p_z(w) = \varphi_w\varphi_w(p_z(w))$ and $|\varphi_w(p_z(w))| \to 0$, apply Lemma 2.1.1 (2), we have

$$\frac{1 - |p_z(w)|^2}{1 - |w|^2} = \frac{1 - |\varphi_w((p_z(w)))|^2}{|1 - \langle\varphi_w(p_z(w)), w\rangle|^2}.$$

Notice that $|\varphi_w(p_z(w))| = \rho(w, p_z(w))$ tends to 0 uniformly. We have (3). This completes the proof. □

Lemma 2.3.7. *For* $1 > s > s_1$*, the measure*

$$d\mu_s = (1 - |w|^2)^{n-d} dv_d|_{M_s} + \sum_{i=1}^{m} (1 - |z_i|^2)^{n+1} \delta_{z_i}$$

is a Carleson measure.

Proof. Fix $r > 0$, by Lemma 2.1.5, we only need to prove that

$$\int_{D(z,r)\cap M_s} (1 - |w|^2)^{n-d} dv_d(w) \leq C(1 - |z|^2)^{n+1}$$

for some constant $C > 0$. Since

$$\frac{1 - |w|^2}{1 - |z|^2} = \frac{1 - |\varphi_z(w)|^2}{|1 - \langle\varphi_z(w), z\rangle|^2} \leq C,$$

it suffices to show

$$v_d(D(z, r)) \leq C(1 - |z|^2)^{d+1}.$$

Now

$$v_d(D(z, r)) = \int_{\{w' : w\in D(z,r)\}} E(w') dv(w') \leq C \int_{\{w' : w\in D(z,r)\}} dv(w').$$

By definition, $|\varphi_{z'}(w')| \le |\varphi_z(w)|$, so

$$\{w' : w \in D(z,r)\} \subset D_d(z', r).$$

Therefore

$$v_d(D(z,r)) \le C(1-|z|^2)^{d+1}.$$

This completes the proof. □

Lemma 2.3.8. *There exists a constant $C > 0$ such that*

$$T_\delta^3 > CT_\delta.$$

Proof. The lemma follows from the fact that T_δ has closed range and is positive. □

Lemma 2.3.9. *For any $\epsilon > 0$, there exists $1 > s_3 > s_1$ and $r > 0$ such that,*

(1)

$$\sup_{z\in M_{s_3}} \int_{M_{s_3}} \frac{(1-|z|^2)^{\frac{n-d}{2}}(1-|w|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) < \infty.$$

(2) $\forall z \in M_{s_3}$,

$$\int_{M_{s_3}\setminus D(z,r)} \frac{(1-|z|^2)^{\frac{n-d}{2}}(1-|w|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) < \epsilon.$$

Proof. We prove (1) and (2) together. For $z \in \bar{V}_i \cap M_{s_1}$ and $r > 0$,

$$\begin{aligned} &\int_{M_{s_3}} \frac{(1-|z|^2)^{\frac{n-d}{2}}(1-|w|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \\ \le\ &\int_{B(z,\tau)\cap M_{s_3}} \frac{(1-|z|^2)^{\frac{n-d}{2}}(1-|w|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \\ +\ &\int_{M_{s_3}\setminus B(z,\tau)} \frac{(1-|z|^2)^{\frac{n-d}{2}}(1-|w|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w). \end{aligned}$$

For the second part, the integrand is smaller than $C\tau^{-2(n+1)}(1-s_3^2)^{n-d}$ because

$$|1-\langle z,w\rangle| \geq (1-Re\langle z,w\rangle) \geq \frac{1}{2}(|z|^2+|w|^2-2Re\langle z,w\rangle) = \frac{1}{2}|z-w|^2.$$

So when s_3 is close to 1, the second part will be smaller than $\frac{1}{2}\epsilon$.

For the first part,

$$\begin{aligned}
& \int_{B(z,\tau)\cap M_{s_3}} \frac{(1-|z|^2)^{\frac{n-d}{2}}(1-|w|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \\
= & \int_{\{w':w\in B(z,\tau)\cap M_{s_3}\}} \frac{(1-|z'|^2)^{\frac{n-d}{2}}(1-|w|^2)^{\frac{n-d}{2}}}{|1-\langle z',w'\rangle|^{n+1}} E(w')dv(w') \\
\leq & \ C\int_{\mathbb{B}_d} \frac{(1-|z'|^2)^{\frac{n-d}{2}}(1-|w'|^2)^{\frac{n-d}{2}}}{|1-\langle z',w'\rangle|^{n+1}} dv(w') \\
= & \ C\int_{\mathbb{B}_d} \frac{(1-|z'|^2)^{\frac{n-d}{2}}(1-|\varphi_{z'}(\eta')|^2)^{\frac{n-d}{2}}}{|1-\langle z',\varphi_{z'}(\eta')\rangle|^{n+1}} \frac{(1-|z'|^2)^{d+1}}{|1-\langle z',\eta'\rangle|^{2(d+1)}} dv(\eta') \\
= & \ C\int_{\mathbb{B}_d} \frac{(1-|\eta'|^2)^{\frac{n-d}{2}}}{|1-\langle z',\eta'\rangle|^{d+1}} dv(\eta').
\end{aligned}$$

Where the second equality from the bottom is by change of variable $w' = \varphi_{z'}(\eta')$. By the proof of Lemma 2.3.4, the integral above is uniformly bounded, which proves (1).

The above argument also gives

$$\begin{aligned}
& \int_{M_{s_3}\cap B(z,\tau)\backslash D(z,r)} \frac{(1-|z|^2)^{\frac{n-d}{2}}(1-|w|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \\
\leq & \int_{\{\varphi_{z'}(w'):w\in B(z,\tau)\cap M_{s_3}\backslash D(z,r)\}} \frac{(1-|\eta'|^2)^{\frac{n-d}{2}}}{|1-\langle z',\eta'\rangle|^{d+1}} dv(\eta').
\end{aligned}$$

Claim: There exists $c > 0$ such that for any $r > 0$,

$$\{\varphi_{z'}(w') : w \in B(z,\tau)\cap M_{s_3}\backslash D(z,r)\} \cap cs_r\mathbb{B}_d = \emptyset.$$

Assume the claim, then (2) follows from Lemma 2.3.4.

Now we prove the claim. For $z \in M_{s_1}$, $w \in B(z,\tau)$, let $\eta = \varphi_z(w)$, η' be the first d entries of η.

Then $\eta' = \varphi_{z'}(w')$.

$$\begin{aligned}
|\eta|^2 - |\eta'|^2 &= \frac{1-|z|^2}{|1-\langle w, z\rangle|^2} \sum_{i=d+1}^{n} |f_i^z(w')|^2 \\
&\leq C\frac{1-|z'|^2}{|1-\langle w', z'\rangle|^2}|w'-z'|^2 \\
&\leq C\Big(\frac{1}{|1-\langle w', z'\rangle|^2}|z_1-w_1|^2 + \sum_{i=2}^{d} \frac{1-|z'|^2}{|1-\langle w', z'\rangle|^2}|w_i|^2\Big) \\
&= C|\eta'|^2.
\end{aligned}$$

The last inequality above follows from the facts that $1-|z'|^2 < 1$ and that $z_2 = \cdots = z_d = 0$.

Thus

$$|\eta|^2 \leq (C+1)|\eta'|^2.$$

If $w \notin D(z, r)$, then $|\eta| = |\varphi_z(w)| \geq s_r$. Therefore $|\eta'| \geq \frac{1}{\sqrt{C+1}} s_r$. Take $c = \frac{1}{\sqrt{C+1}}$ and the proof is complete. □

Proof of Theorem 2.3.2. First, we prove the theorem under the assumption that M is connected. In this case the dimension of M at every regular point is the same. Let $0 < d < n$ be the dimension.

Let $\epsilon > 0$ be determined later. Let $r > 0$ and s_3, s_2 be as in Lemma 2.3.9 and Lemma 2.3.6. Let $s = \max\{s_2, s_3\}$ and $1 > s' > s$ be such that $\forall z \in M_{s'}$, $D(z, 2r) \cap M \subset M_s$. We may enlarge s (and the associated s') in the proof and still denote it by s (s').

We will prove that $T_{\mu_s}^3 \geq cT_{\mu_s}$ for some $c > 0$. Since T_{μ_s} is self-adjoint and $\ker T_{\mu_s} = \mathcal{P}_M$, T_{μ_s} is bounded below on $\mathcal{Q}_M$. This will give us the desired result, by Theorem 2.2.1.

Denote $\mathcal{P}(\mu_s)$ to be the closure of the restriction of all analytic polynomials to M in $L^2(\mu_s)$. Clearly Range(R) $\subset \mathcal{P}(\mu_s)$. Suppose $s' < t < 1$, for every $z \in M_{s'}^t$, there is an open neighborhood $U \ni z$ contained in M_s and not touching $\partial\mathbb{B}_n$ such that M has local coordinates on U. It is easy to prove that for a compact set $V \subset U$, there is a constant $C > 0$ such that $\forall p \in \mathbb{C}[z_1, \cdots, z_n]$, $\forall z \in V$,

$$|p(z)|^2 \leq C\int_U |p(w)|^2 dv_d(w) \leq C' \int_M |p(w)|^2 d\mu_s(w).$$

Clearly the same is true for $z = z_i, i = 1, \cdots, m$. Suppose $K \subset M$ is compact, then K is contained in M^t for some $t < 1$. Assume $t > s'$, we can cover $\overline{M^t_{s'}}$ with finite compact neighborhoods V_i as above. So there is a constant $C > 0$ such that for any analytic polynomial p, $\forall z \in M^t_{s'} \cup \Sigma$,

$$|p(z)|^2 \leq C \int_M |p(w)|^2 d\mu_s(w).$$

For $z \in M^{s'} \backslash \Sigma$, using the maximum modulus principle, we have

$$|p(z)|^2 \leq \sup_{w \in M^t_{s'} \cup \Sigma} |p(w)|^2 \leq C \int_M |p(w)|^2 d\mu_s(w).$$

This means the evaluation at every point in M is bounded on $\mathcal{P}(\mu_s)$. Therefore we can think of $f \in \mathcal{P}(\mu_s)$ as a pointwise defined function on M (instead of an equivalence class in $L^2(\mu_s)$). Also, it is easy to prove that under this definition, $\forall f \in L^2_a, \forall z \in M, Rf(z) = f(z)$.

In conclusion, the space $\mathcal{P}(\mu_s)$ is a reproducing kernel Hilbert space on M, and the reproducing kernels on any compact subset are uniformly bounded.

Consider the operator

$$T : \mathcal{P}(\mu_s) \to L^2(\mu_s),\ Tf = f\chi_{M^{s'}_s}.$$

The operator T is compact: suppose $\{f_k\} \subset \mathcal{P}(\mu_s)$ and f_k weakly converges to 0, then f_k converges to 0 pointwise and are uniformly bounded on $M^{s'}_s$. By the above argument and the dominated convergence theorem,

$$\|Tf_k\|^2 = \int_{M^{s'}_s} |f(w)|^2 d\mu_s(w) \to 0.$$

So T is compact, therefore $|T|$ is compact. Also $\|Tf\| = \||T|f\|$, $\forall f \in \mathcal{P}(\mu_s)$. Using the spectral decomposition of $|T|$, we see that for any $0 < a < 1$, there exists a finite codimensional subspace $L \subset \mathcal{P}(\mu_s)$, such that $\forall f \in L$,

$$\int_{M^{s'}_s} |f|^2 d\mu_s \leq (1-a) \int_M |f|^2 d\mu_s,$$

so

$$\int_M |f|^2 d\mu_{s'} \geq a \int_M |f|^2 d\mu_s.$$

We will use this in the last part of our proof.

Define the operator

$$\tilde{T}_{\mu_s} : \mathcal{P}(\mu_s) \to \mathcal{P}(\mu_s)$$

$$\tilde{T}_{\mu_s} f(z) = \int_M f(w) \frac{1}{(1 - \langle z, w \rangle)^{n+1}} d\mu_s(w), \ \forall z \in M.$$

By definition, $\forall F \in L^2_a$, $\tilde{T}_{\mu_s} RF = RT_{\mu_s} F$. Since

$$\|RT_{\mu_s} F\|^2_{\mu_s} = \langle T^3_{\mu_s} F, F \rangle \leq \|T_{\mu_s}\|^2 \langle T_{\mu_s} F, F \rangle = \|T_{\mu_s}\|^2 \|RF\|^2_{\mu_s},$$

$\tilde{T}_{\mu_s}$ is bounded on $\mathcal{P}(\mu_s)$. We will show that $\tilde{T}_{\mu_s}$ is bounded below.

For $z \in M_{s'}$, $f = RF \in \mathcal{P}(\mu_s)$, $F \in L^2_a$,

$$\tilde{T}_{\mu_s} f(z) = \int_\Sigma f(w) K_w(z) d\mu_s(w) + \int_{M_s} f(w) \frac{(1 - |w|^2)^{n-d}}{(1 - \langle z, w \rangle)^{n+1}} dv_d(w).$$

Consider the map $p_z : D(z, 2r) \cap M \to TM|_z$ defined before Lemma 2.3.6; by (4) of Lemma 2.3.6, by enlarging s, we could assume $\beta(p_z(w), w) < \frac{1}{2} r$, $\forall w \in D(z, 2r)$. Therefore

$$p_z(D(z, 2r) \cap M) \supset D(z, \frac{3}{2} r) \cap TM|_z$$

and

$$p_z^{-1}(D(z, \frac{3}{2} r) \cap TM|_z) \supset D(z, r) \cap M.$$

Define

$$I(z) = \begin{cases} \int_\Sigma f(w) K_w(z) d\mu_s(w) & z \in \Sigma \\ \int_{p_z^{-1}(D(z, \frac{3}{2} r) \cap TM|_z)} f(w) \frac{(1-|w|^2)^{n-d}}{(1-\langle z, w \rangle)^{n+1}} dv_d(w) & z \in M_{s'} \end{cases}$$

and

$$II(z) = \tilde{T}_{\mu_s} f(z) - I(z),$$

then $I(z) + II(z) = \tilde{T}_{\mu_s} f(z)$, $\forall z \in M_{s'} \cup \Sigma$.

For $I(z)$,

$$\int_M |I(z)|^2 d\mu_{s'} = \langle T_\delta^3 F, F\rangle + \int_{M_{s'}} |I(z)|^2 (1-|z|^2)^{n-d} dv_d(z).$$

By Lemma 2.3.8, the first part is greater than $c\langle T_\delta F, F\rangle = c\int_M |f(w)|^2 d\delta$.

If $z \in M_{s'}$,

$$\begin{aligned} I(z) &= \int_{p_z^{-1}(D(z,\frac{3}{2}r)\cap TM|_z)} f(w)\frac{(1-|w|^2)^{n-d}}{(1-\langle z,w\rangle)^{n+1}} dv_d(w) \\ &= \int_{D(z,\frac{3}{2}r)\cap TM|_z} f p_z^{-1}(\eta)\frac{(1-|p_z^{-1}(\eta)|^2)^{n-d}}{(1-\langle z,\eta\rangle)^{n+1}}\frac{E(w')}{E(z')} dv(\eta) \\ &= \int_{D(z,\frac{3}{2}r)\cap TM|_z} f p_z^{-1}(\eta)\frac{(1-|\eta|^2)^{n-d}}{(1-\langle z,\eta\rangle)^{n+1}} g(\eta) dv(\eta) \end{aligned}$$

where

$$g(\eta) = \frac{(1-|p_z^{-1}(\eta)|^2)^{n-d}}{(1-|\eta|^2)^{n-d}}\frac{E(w')}{E(z')}.$$

By Lemma 2.3.6 (3) and the absolute continuity of E, we could enlarge s (so that the Euclidean size of $D(z, 2r)$ is small enough) so that $g(\eta)$ is sufficiently close to 1 and

$$|g(\eta) - 1| \le \epsilon g(\eta).$$

By Lemma 2.3.3,

$$\int_{D(z,\frac{3}{2}r)\cap TM|_z} f p_z^{-1}(\eta)\frac{(1-|\eta|^2)^{n-d}}{(1-\langle z,\eta\rangle)^{n+1}} dv(\eta) = C_z f(z),$$

where $C_z \geq C_{\frac{3}{2}r}$. Furthermore,

$$\begin{aligned}
& |\int_{D(z,\frac{3}{2}r)\cap TM|z} f p_z^{-1}(\eta) \frac{(1-|\eta|^2)^{n-d}}{(1-\langle z,\eta\rangle)^{n+1}}(g(\eta)-1)dv(\eta)| \\
\leq & \ \epsilon \int_{D(z,\frac{3}{2}r)\cap TM|_z} |f p_z^{-1}(\eta)| \frac{(1-|\eta|^2)^{n-d}}{|1-\langle z,\eta\rangle|^{n+1}} g(\eta) dv(\eta) \\
\leq & \ \epsilon \int_{M_s} |f(w)| \frac{(1-|w|^2)^{n-d}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w).
\end{aligned}$$

So

$$\begin{aligned}
& \int_{M_{s'}} |I(z)|^2 (1-|z|^2)^{n-d} dv_d(z) \\
\geq & \ \frac{1}{2} C_{\frac{3}{2}r}^2 \int_{M_{s'}} |f(z)|^2 (1-|z|^2)^{n-d} dv_d(z) \\
& -\epsilon^2 \int_{M_{s'}} \Big(\int_{M_s} |f(w)| \frac{(1-|w|^2)^{n-d}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \Big)^2 (1-|z|^2)^{n-d} dv_d(z).
\end{aligned}$$

Using Holder's inequality and Lemma 2.3.9 (1), the second part is smaller than

$$\begin{aligned}
& \epsilon^2 \int_{M_{s'}} \Big(\int_{M_s} \frac{(1-|w|^2)^{\frac{n-d}{2}} (1-|z|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \Big) \\
& \cdot \Big(\int_{M_s} |f(w)|^2 \frac{(1-|w|^2)^{\frac{3(n-d)}{2}} (1-|z|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \Big) dv_d(z) \\
\leq & \ C\epsilon^2 \int_{M_s} \int_{M_{s'}} \frac{(1-|w|^2)^{\frac{n-d}{2}} (1-|z|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(z) \\
& \quad |f(w)|^2 (1-|w|^2)^{n-d} dv_d(w) \\
\leq & \ C^2\epsilon^2 \int_{M_s} |f(w)|^2 (1-|w|^2)^{n-d} dv_d(w) \\
\leq & \ C^2\epsilon^2 \int_M |f|^2 d\mu_s.
\end{aligned}$$

The above estimation is inspired from [26]. We will use the same kind of argument in the estimation of $II(z)$. Combining the above, we have

$$\int_M |I(z)|^2 d\mu_{s'} \geq C_1 \int_M |f|^2 d\mu_{s'} - C_2 \epsilon^2 \int_M |f|^2 d\mu_s.$$

Next we estimate $II(z)$.

$$
\begin{aligned}
&\int_M |II(z)|^2 d\mu_{s'}(z) \\
= \quad &\sum_{i=1}^m \Big| \int_{M_s} f(w) \frac{(1-|w|^2)^{n-d}}{(1-\langle z_i, w\rangle)^{n+1}} dv_d(w) \Big|^2 (1-|z_i|^2)^{n+1} + \int_{M_{s'}} \Big| T_\delta F(z) + \\
&\int_{M_s \setminus p_z^{-1}(D(z, \frac{3}{2}r)\cap TM|_z)} f(w) \frac{(1-|w|^2)^{n-d}}{(1-\langle z, w\rangle)^{n+1}} dv_d(w) \Big|^2 (1-|z|^2)^{n-d} dv_d(z) \\
\leq \quad & A + 2B + 2C.
\end{aligned}
$$

Here

$$
A = \sum_{i=1}^m \Big| \int_{M_s} f(w) \frac{(1-|w|^2)^{n-d}}{(1-\langle z_i, w\rangle)^{n+1}} dv_d(w) \Big|^2 (1-|z_i|^2)^{n+1},
$$

$$
B = \int_{M_{s'}} |T_\delta f(z)|^2 (1-|z|^2)^{n-d} dv_d(z),
$$

and

$$
C = \int_{M_{s'}} \Big| \int_{M_s \setminus p_z^{-1}(D(z, \frac{3}{2}r)\cap TM|_z)} f(w) \frac{(1-|w|^2)^{n-d}}{(1-\langle z, w\rangle)^{n+1}} dv_d(w) \Big|^2 (1-|z|^2)^{n-d} dv_d(z).
$$

Let $a = d(\Sigma, M_s)$, we have

$$
\begin{aligned}
A \quad &\leq \quad (1/2a^2)^{-2(n+1)} \sum_{i=1}^m (1-|z_i|^2)^{n+1} \Big(\int_{M_s} |f(w)|(1-|w|^2)^{n-d} dv_d(w) \Big)^2 \\
&\leq \quad C \Big(\int_{M_s} |f(w)|^2 (1-|w|^2)^{n-d} dv_d(w) \Big) \Big(\int_{M_s} (1-|w|^2)^{n-d} dv_d(w) \Big) \\
&\leq \quad C(1-s^2)^{n-d} \int_M |f(w)|^2 d\mu_s(w)
\end{aligned}
$$

where the first inequality holds because

$$
|1-\langle z_i, w\rangle| \geq 1 - Re\langle z_i, w\rangle \geq 1/2(|z_i|^2 + |w|^2 - Re\langle z_i, w\rangle) = 1/2|z_i - w|^2
$$

and the second inequality is by Holder's inequality. By taking s closer to 1, we could make

$$A \leq \epsilon^2 \int_M |f(w)|^2 d\mu_s(w).$$

Similar argument will give us

$$B \leq \epsilon^2 \int_M |f(w)|^2 d\mu_s(w).$$

Now we estimate C.

$$\begin{aligned}
C &\leq \int_{M_{s'}} \left| \int_{M_s \setminus D(z,r)} f(w) \frac{(1-|w|^2)^{n-d}}{(1-\langle z,w\rangle)^{n+1}} dv_d(w) \right|^2 (1-|z|^2)^{n-d} dv_d(z) \\
&\leq \int_{M_{s'}} \Big(\int_{M_s \setminus D(z,r)} \frac{(1-|w|^2)^{\frac{n-d}{2}} (1-|z|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \Big) \\
&\quad \cdot \Big(\int_{M_s \setminus D(z,r)} |f(w)|^2 \frac{(1-|w|^2)^{\frac{3(n-d)}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(w) \Big) (1-|z|^2)^{\frac{n-d}{2}} dv_d(z) \\
&\leq \epsilon \int_{M_s} \Big(\int_{M_{s'} \setminus D(w,r)} \frac{(1-|w|^2)^{\frac{n-d}{2}} (1-|z|^2)^{\frac{n-d}{2}}}{|1-\langle z,w\rangle|^{n+1}} dv_d(z) \Big) \\
&\quad \cdot |f(w)|^2 (1-|w|^2)^{n-d} dv_d(w) \\
&\leq \epsilon^2 \int_{M_s} |f(w)|^2 (1-|w|^2)^{n-d} dv_d(w).
\end{aligned}$$

Combining the three inequalities, we get

$$\int_M |II(z)|^2 d\mu_{s'}(z) \leq 5\epsilon^2 \int_M |f|^2 d\mu_s.$$

Finally, we have

$$\begin{aligned}
\int_M |\tilde{T}_{\mu_s} f(z)|^2 d\mu_s(z) &\geq \int_M |\tilde{T}_{\mu_s} f(z)|^2 d\mu_{s'}(z) \\
&\geq \frac{1}{2} \int_M |I(z)|^2 d\mu_{s'}(z) - \int_M |II(z)|^2 d\mu_{s'}(z) \\
&\geq C \int_M |f|^2 d\mu_{s'} - C'\epsilon^2 \int_M |f|^2 d\mu_s.
\end{aligned}$$

This holds for all $f \in \mathcal{P}(\mu_s)$. From the argument in the beginning, we can find a finite codimensional space $L \subset \mathcal{P}(\mu_s)$ such that $\forall f \in L$,

$$\int_M |f|^2 d\mu_{s'} > \frac{1}{2} \int_M |f|^2 d\mu_s.$$

Therefore $\forall f \in L$,

$$\int_M |\tilde{T}_{\mu_s} f(z)|^2 d\mu_s(z) \geq (\frac{1}{2}C - C'\epsilon^2) \int_M |f|^2 d\mu_s.$$

Take $\epsilon > 0$ such that $\alpha = \frac{1}{2}C - C'\epsilon^2 > 0$, then

$$||\tilde{T}_{\mu_s} f||_{\mu_s}^2 \geq \alpha ||f||_{\mu_s}^2, \quad \forall f \in L.$$

Next we show that $\ker \tilde{T}_{\mu_s} = \{0\}$. Consider the commuting diagram

$$\begin{array}{ccc} Q_M & \xrightarrow{T_{\mu_s}} & Q_M \\ \downarrow R & & \downarrow R \\ \mathcal{P}(\mu_s) & \xrightarrow{\tilde{T}_{\mu_s}} & \mathcal{P}(\mu_s) \end{array}$$

Since $\tilde{T}_{\mu_s}$ is positive, it suffices to show that Range($\tilde{T}_{\mu_s}$) is dense in $\mathcal{P}(\mu_s)$. We already know that Range(T_{μ_s}) is dense in Q_M (since $\ker T_{\mu_s} = \{0\}$). Therefore $RT_{\mu_s}(Q_M)$ is dense in $R(Q_M)$, which is dense in $\mathcal{P}(\mu_s)$. So Range($\tilde{T}_{\mu_s}$) $\supset RT_{\mu_s}(Q_M)$ is dense in $\mathcal{P}(\mu_s)$. Hence $\ker \tilde{T}_{\mu_s} = \{0\}$.

Now suppose $\tilde{T}_{\mu_s}$ is not bounded below, then there exists a pairwise orthogonal sequence $\{f_n\} \subset \mathcal{P}(\mu_s)$, $||f_n||_{\mu_s} = 1$ such that $||\tilde{T}_{\mu_s}(f_n)||_{\mu_s} \to 0$, $n \to \infty$. Since L is finite codimensional,

$$||f_n - Lf_n||_{\mu_s} \to 0, n \to \infty.$$

But

$$||\tilde{T}_{\mu_s} L f_n||_{\mu_s} \leq ||\tilde{T}_{\mu_s} f_n||_{\mu_s} + ||\tilde{T}_{\mu_s}(f_n - Lf_n)||_{\mu_s} \to 0, \quad n \to \infty,$$

a contradiction. So $\tilde{T}_{\mu_s}$ is bounded below.

Suppose

$$\|\tilde{T}_{\mu_s} f\|_{\mu_s}^2 \geq c\|f\|_{\mu_s}^2, \quad \forall f \in \mathcal{P}(\mu_s),$$

then $\forall F \in L_a^2$,

$$\langle T_{\mu_s}^3 F, F\rangle = \|\tilde{T}_{\mu_s} RF\|_{\mu_s}^2 \geq c\|RF\|_{\mu_s}^2 = c\langle T_{\mu_s} F, F\rangle.$$

This means $T_{\mu_s}^3 \geq cT_{\mu_s}$, which implies that T_{μ_s} is bounded below on Q_M. Therefore $\|\cdot\|_{\mu_s}$ and $\|\cdot\|$ are equivalent on Q_M. This completes the proof when $\tilde{M}$ is connected.

If $\tilde{M}$ is not connected, then by the theorem in [8, Page 52], after restricting it to a smaller neighborhood of $\overline{\mathbb{B}_n}$, we can divide $\tilde{M}$ into finitely many connected components, each two having positive Euclidean distance (although they may have different dimensions). We can divide $II(z)$ into more parts, the rest of the proof remains unchanged. □

2.4 Unions of Varieties

The purpose of this section is to construct more examples of essentially normal submodules based on Theorem 2.3.2. Given two analytic subsets, their union is another analytic subset. It is natural to ask that, given two analytic subsets having equivalent measures, when does their union have an equivalent measure? A byproduct of the proof of Theorem 2.2.1 is that the projection operator to the quotient module is in the Toeplitz algebra. We are going to take advantage of this fact and combine it with the theory of localization of Toeplitz operators developed by Suárez [35] [36].

2.4.1 Suárez's Method

Let A be the algebra of bounded functions on $\mathbb{B}_n$ which are uniformly continuous in the hyperbolic metric, equipped with the supremum norm. It is easy to see that A is a commutative C^* algebra. Let M_A be its maximal ideal space, then the unit ball $\mathbb{B}_n$ is naturally contained in M_A as evaluations. The algebra A can be used to study the properties of the Toeplitz operators (cf. [35] [36]).

Definition 2.4.1. *A sequence* $\{z_m\} \subseteq \mathbb{B}_n$ *is said to be* separated *if there exists* $\delta > 0$ *such that*

$\rho(z_k, z_l) \geq \delta$ for $k \neq l$.

If $x, y \in M_A$, define

$$\rho(x, y) = \sup \rho(\mathcal{S}, \mathcal{T}),$$

where $\mathcal{S}$, $\mathcal{T}$ run over all separated sequences in $\mathbb{B}_n$ so that $x \in \overline{\mathcal{S}}^A$ and $y \in \overline{\mathcal{T}}^A$. Here

$$\rho(\mathcal{S}, \mathcal{T}) = \inf\{\rho(z, w) : \; z \in \mathcal{S}, w \in \mathcal{T}\}$$

and $\overline{\mathcal{S}}^A$ denotes the closure in M_A. Define

$$\beta(x, y) = \frac{1}{2} \log \frac{1 + \rho(x, y)}{1 - \rho(x, y)}.$$

For $x \in M_A$ and any net $\{z_\alpha\}$ that converges to it, there is a map $\varphi_x : \mathbb{B}_n \to M_A$ such that $a \circ \varphi_x \in A$ and $a \circ \varphi_{z_\alpha} \to a \circ \varphi_x$ uniformly on compact sets of $\mathbb{B}_n$, for all $a \in A$ (cf. [35]).

The following lemma was proved in [35], Section 3.2 for the unit disc and the same proof works for the $\mathbb{B}_n$ case verbatimly.

Lemma 2.4.2. *Let $x, y \in M_A \backslash \mathbb{B}_n$, then*

(1) $\rho(x, y) = a < 1$ if and only if $y = \varphi_x(w)$ for some w with $|w| = a$.

(2) $y = \varphi_x(\xi)$ with $\xi \in \mathbb{B}_n$ if and only if every separated sequences $\mathcal{S}$, $\mathcal{T}$ such that $x \in \overline{\mathcal{S}}^A$ and $y \in \overline{\mathcal{T}}^A$ satisfy $\rho(\mathcal{T}, \{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}) = 0$.

(3) $\rho(\varphi_x(\xi_1), \varphi_x(\xi_2)) = \rho(\xi_1, \xi_2)$ for every ξ_1, $\xi_2 \in \mathbb{B}_n$.

(4) β is a $[0, +\infty]$-valued metric on M_A.

Definition 2.4.3. *For $z \in \mathbb{B}_n$, define $U_z : L^2_a(\mathbb{B}_n) \to L^2_a(\mathbb{B}_n)$ to be*

$$U_z(f) = f \circ \varphi_z \cdot k_z.$$

Here k_z is the normalized reproducing kernel.

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \frac{(1-|z|^2)^{(n+1)/2}}{(1-\langle w, z\rangle)^{n+1}}.$$

Then U_z is an unitary operator on $L^2_a(\mathbb{B}_n)$ with $U_z^* = U_z$.

The following lemmas can be found in Section 8 and 10 of [36].

Lemma 2.4.4. *If $S \in \mathcal{T}(L^\infty)$, then the map $\Psi_S : \mathbb{B}_n \to (\mathcal{B}(L^2_a(\mathbb{B}_n)), SOT)$, $z \mapsto S_z := U_z S U_z$ extends continuously to M_A. We write S_x for the operator $\Psi_S(x)$.*

Here SOT denotes the strong operator topology on $\mathcal{B}(L^2_a(\mathbb{B}_n))$.

Lemma 2.4.5. *$x \in M_A$, $S, T \in \mathcal{T}(L^\infty)$, then*

$$(ST)_x = S_x T_x, (S_x)^* = (S^*)_x, \|S_x\| \leq \|S\|.$$

From the lemma, we see that for any normal Toeplitz operator S and any $f \in C(\sigma(S))$, $f(S)_x = f(S_x)$. The localized operators S_x completely determine the essential norm of $S \in \mathcal{T}(L^\infty)$. In [36], Suárez proved the following lemma.

Lemma 2.4.6. *$S \in \mathcal{T}(L^\infty)$, then*

$$\|S\|_e = \sup_{x \in M_A \setminus \mathbb{B}_n} \|S_x\|.$$

2.4.2 Angles of Subspaces

Suppose H is a Hilbert space and H_1, H_2 are subspaces of H. When is $H_1 + H_2$ closed? Write $H_3 = H_1 \cap H_2$, then $H_1 + H_2 = (H_1 \ominus H_3 + H_2 \ominus H_3) \oplus H_3$. Therefore $H_1 + H_2$ is closed if and only if $H_1 \ominus H_3 + H_2 \ominus H_3$ is closed. In the case when $H_1 \cap H_2 = \{0\}$, by open mapping Theorem we know that $H_1 + H_2$ is closed if and only if the norm on $H_1 + H_2$ is equivalent to the norm on $H_1 \oplus H_2$.

Definition 2.4.7. *Suppose H_1, H_2 are subspaces of a Hilbert space H. And write $H_3 = H_1 \cap H_2$. We define the* angle *of H_1 and H_2 to be*

$$\arccos \sup \left\{ \frac{|\langle u, v\rangle|}{||u||||v||} : u \in H_1 \ominus H_3, v \in H_2 \ominus H_3 \right\}.$$

This definition appeared in [23, Part II, 1.1], where the author used it to study pairs of subspaces. In [23], it is called the "smallest angle", we abbreviate it into "angle" for convenience.

In [22], the author studied closedness of sums of subspaces. The following lemma is elementary. Part of it comes from [22, Proposition 2.1]. We provide a proof for completeness.

Lemma 2.4.8. *The following quantities are comparable with each other and with the angle of H_1 and H_2, i.e.,* (1) $\approx$ (2) $\approx$ (3) $\approx$ *angle of H_1 and H_2. Here $A \approx B$ means that there exists $C > c > 0$ such that $cA \leq B \leq CA$.*

(1) $\inf\{\frac{||u-v||^2}{||u||^2+||v||^2} : u \in H_1 \ominus H_3, v \in H_2 \ominus H_3\}$,

(2) $1 - ||H_2 H_1 - H_3||$,

(3) $1 - ||H_1 H_2 H_1 - H_3||$.

Proof. For the relation between the angle and (1), take v by $-v$ in the previous equality. We get

$$\frac{||u - v||^2}{||u||^2 + ||v||^2} = 1 - \frac{2Re\langle u, v\rangle}{||u||^2 + ||v||^2}.$$

Therefore

$$\begin{aligned} & \inf\{\frac{||u - v||^2}{||u||^2 + ||v||^2} : u \in H_1 \ominus H_3, v \in H_2 \ominus H_3\} \\ = \quad & 1 - \sup\{\frac{2Re\langle u, v\rangle}{||u||^2 + ||v||^2} : u \in H_1 \ominus H_3, v \in H_2 \ominus H_3\} \\ = \quad & 1 - \sup\{\frac{2|\langle u, v\rangle|}{||u||^2 + ||v||^2} : u \in H_1 \ominus H_3, v \in H_2 \ominus H_3\} \end{aligned}$$

Since

$$\frac{|\langle u, v\rangle|}{||u||^2 + ||v||^2} \leq \frac{|\langle u, v\rangle|}{2||u||||v||} = \frac{|\langle au, a^{-1}v\rangle|}{||au||^2 + ||a^{-1}v||^2},$$

where $a = \sqrt{\frac{||v||}{||u||}}$, we have

$$\begin{aligned} &\inf\{\frac{||u-v||^2}{||u||^2 + ||v||^2} : u \in H_1 \ominus H_3, v \in H_2 \ominus H_3\} \\ = \quad & 1 - \sup\{\frac{|\langle u, v\rangle|}{||u||||v||} : u \in H_1 \ominus H_3, v \in H_2 \ominus H_3\}. \end{aligned}$$

Also, since $(H_2H_1 - H_3)^*(H_2H_1 - H_3) = H_1H_2H_1 - H_3$, the quantities (2) and (3) are comparable. Now we show that (1) and (2) are comparable. Fix $u \in H_1 \ominus H_3$, for any $v \in H_2 \ominus H_3$,

$$\begin{aligned} & \frac{||u-v||^2}{||u||^2 + ||v||^2} \geq \frac{||u-v||^2}{2||u||^2 + ||u-v||^2} \\ \geq \quad & \frac{||u - H_2u||^2}{2||u||^2 + ||u - H_2u||^2} \geq \frac{||u - H_2u||^2}{3||u||^2 + 3||H_2u||^2}. \end{aligned}$$

Also,

$$\frac{||u - H_2u||^2}{||u||^2 + ||H_2u||^2} \approx \frac{||u - H_2u||^2}{||u||^2}.$$

The first inequality shows that the infimum in (1) is obtained (modulo a constant) by taking $v = H_2u$. The second inequality shows that (1) and (2) are comparable. This completes the proof. □

From our discussion before Definition 2.4.7 and Lemma 2.4.8(1), the following corollary is immediate.

Corollary 2.4.9. *$H_1 + H_2$ is closed if and only if their angle is non-zero.*

As a consequence of Lemma 2.4.8, when the projection operators to subspaces H_1, H_2 and H_3 all change continuously, the quantities in (3) change continuously, therefore the angles have a uniform positive lower bound on any compact set of parameters. This fact will be used in the proof of our main theorem.

2.4.3 Unions via Localization

We begin this section with an example.

Example 2.4.10. *Suppose* $\tilde{M}_1$ *and* $\tilde{M}_2$ *are two linear subspaces of* $\mathbb{C}^n$. $\tilde{M}_3 = \tilde{M}_1 \cap \tilde{M}_2$. *Let* $M_i = \tilde{M}_i \cap \mathbb{B}_n$ *and* $\mathcal{Q}_i := \mathcal{Q}_{M_i} = \overline{\text{span}}\{K_\lambda | \lambda \in M_i\} \subseteq L^2_a$, $i = 1, 2, 3$. *We are going to study the angle between* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$.

Proposition 2.4.11. *Under the setting of Example 2.4.10, we have*

$$\|Q_2 Q_1 Q_2 - Q_3\| < 1.$$

The norm depends on the angle between $\tilde{M}_1$ *and* $\tilde{M}_2$. *More precisely, For any* $\delta > 0$ *there exists an* $0 < a < 1$ *such that whenever the angle between* $\tilde{M}_1$ *and* $\tilde{M}_2$ *is greater than* δ, *we have* $\|Q_2 Q_1 Q_2 - Q_3\| < a$. *As a consequence,* $\mathcal{Q}_1 + \mathcal{Q}_2$ *is closed and* $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \mathcal{Q}_3$.

Proof. As usual we use Q_i to denote the projection operator onto $\mathcal{Q}_i$. For simplicity the notation M_i denotes both the spaces and the projection operators.

Let $\epsilon > 0$ be determined later. Choose $k \in \mathbb{N}$ (depending on ϵ and δ) so that $\forall v \in M_2 \ominus M_3$,

$$|(M_2 M_1)^k v| \leq \epsilon |v|.$$

Clearly the operator $Q_2 Q_1 Q_2 - Q_3$ vanishes on $\mathcal{Q}_2^\perp$ and $\mathcal{Q}_3$. For any $f \in \mathcal{Q}_2 \ominus \mathcal{Q}_3$ with $\|f\| = 1$, since

$$(Q_2 Q_1 Q_2 - Q_3)^k f = (Q_2 Q_1)^k f,$$

it suffices to prove that

$$\|(Q_2 Q_1)^k f\| \leq a$$

for some fixed integer k and fixed constant $a < 1$. Let $d = \dim M_2$, by Example 3.3 in [16], the measure $\mu = c(1-|z|^2)^{n-d} dv_{M_2}$ with a suitable normalizing constant c has the property that

$$\|Q_2 g\|^2 = \int_{M_2} |g|^2 d\mu, \quad \forall g \in L_a^2.$$

Here v_{M_2} is the volume measure on M_2.

It is easy to see that $Q_i f(z) = f(M_i(z)), i = 1, 2, 3$. Here we use M_i to denote the projection operators to $\tilde{M}_i$. Now for any $z \in M_2$,

$$\begin{aligned}(Q_2Q_1)^k f(z) &= Q_1(Q_2Q_1)^{k-1} f(z)\\ &= (Q_2Q_1)^{k-1} f(M_1 z)\\ &= (Q_2Q_1)^{k-1} f(M_2M_1 z)\\ &= \cdots\\ &= f((M_2M_1)^k z).\end{aligned}$$

$$\begin{aligned}(M_2M_1)^k z &= (M_2M_1)^k M_3 z + (M_2M_1)^k (1-M_3) z\\ &= M_3 z + (M_2M_1)^k (1-M_3) z.\end{aligned}$$

By the choice of k,

$$|(M_2M_1)^k (1-M_3) z|^2 \le \epsilon^2 |(1-M_3)z|^2 \le \epsilon^2 (1-|M_3 z|^2).$$

Therefore the pseudo-hyperbolic metric

$$\rho((M_2M_1)^k z, M_3 z) \le r_\epsilon,$$

where $r_\epsilon \to 0$ when $\epsilon \to 0$.

Before continuing, we need the following lemma.

Lemma 2.4.12. *There exists a constant $C > 0$, such that for any $g \in Hol(\mathbb{B}_d)$ and any $z, w \in \mathbb{B}_d$ satisfying $\beta(z, w) < 1/2$,*

$$|g(z) - g(w)|^2 \leq C \frac{\rho(z, w)^2}{(1 - |w|^2)^{d+1}} \int_{D(w)} |g(\lambda)|^2 dv(\lambda),$$

where $D(w) = \{z | \beta(z, w) < 1\}$.

Proof. Using a reproducing kernel argument, it is easy to show that for $g \in Hol(\mathbb{B}_d)$ and $|\lambda| \in D(0, 1/2)$,

$$|g(\lambda) - g(0)|^2 \leq C|\lambda|^2 \int_{D(0)} |g(\eta)|^2 dv(\eta).$$

So if $\beta(z, w) < 1/2$,

$$\begin{aligned}
|g(z) - g(w)|^2 &= |g\varphi_w(\varphi_w(z)) - g\varphi_w(0)|^2 \\
&\leq C|\varphi_w(z)|^2 \int_{D(0)} |g\varphi_w(\eta)|^2 dv(\eta) \\
&= C\rho(z, w)^2 \int_{D(w)} |g(\lambda)|^2 \frac{(1 - |w|^2)^{d+1}}{|1 - \langle \lambda, w \rangle|^{2(d+1)}} dv(\lambda) \\
&\leq C \frac{\rho(z, w)^2}{(1 - |w|^2)^{d+1}} \int_{D(w)} |g(\lambda)|^2 dv(\lambda)
\end{aligned}$$

This completes the proof of lemma. ◻

From the lemma and previous argument,

$$\begin{aligned}
& |f((M_2M_1)^k z)| \\
= \; & |f((M_2M_1)^k z) - f(M_3 z)| \\
\leq \; & C r_\epsilon \frac{1}{(1 - |M_3 z|^2)^{d+1}} \int_{D(M_3 z)} |f(\eta)|^2 dv_{M_2}(\eta).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \|(Q_2Q_1)^k f\|^2 = \int_{M_2} |(Q_2Q_1)^k f(z)|^2(1-|z|^2)^{n-d} dv_{M_2}(z) \\
= & \int_{M_2} |f((M_2M_1)^k z)|^2 (1-|z|^2)^{n-d} dv_{M_2}(z) \\
\leq & \; Cr_\epsilon^2 \int_{M_2} \frac{1}{(1-|M_3 z|^2)^{d+1}} \int_{D(M_3 z)} |f(\eta)|^2 dv_{M_2}(\eta)(1-|z|^2)^{n-d} dv_{M_2}(z) \\
= & \; Cr_\epsilon^2 \int_{M_2} |f(\eta)|^2 \int_{\{z \in M_2 : M_3 z \in D(\eta)\}} \frac{(1-|z|^2)^{n-d}}{(1-|M_3 z|^2)^{d+1}} dv_{M_2}(z) dv_{M_2}(\eta).
\end{aligned}$$

Since the second integral is for $M_3 z \in D(\eta)$, the term $1 - |M_3 z|^2$ in the denominator is comparable to $1 - |\eta|^2$. Therefore

$$\begin{aligned}
& \int_{\{z \in M_2 : M_3 z \in D(\eta)\}} \frac{(1-|z|^2)^{n-d}}{(1-|M_3 z|^2)^{d+1}} dv_{M_2}(z) \\
\leq & \; C(1-|\eta|^2)^{-d-1} \int_{\{z \in M_2 : M_3 z \in D(\eta)\}} (1-|z|^2)^{n-d} dv_{M_2}(z).
\end{aligned}$$

We claim that

$$\int_{\{z \in M_2 : M_3 z \in D(\eta)\}} (1-|z|^2)^{n-d} dv_{M_2}(z) \leq C(1-|\eta|^2)^{n+1}.$$

For $z \in M_2$, write temporarily $z = (z', z'')$ where z' corresponds to the coordinates in M_3.

$$\begin{aligned}
& \int_{\{z \in M_2 : M_3 z \in D(\eta)\}} (1-|z|^2)^{n-d} dv_{M_2}(z) \\
= & \; C \int_{z' \in D(\eta)} (1-|z'|^2)^{n-d+d-d_3} \int_{\lambda \in \mathbb{B}_{d-d_3}} (1-|\lambda|^2)^{n-d} dv(\lambda) dv(z') \\
\leq & \; C(1-|\eta|^2)^{n-d_3}(1-|\eta|^2)^{d_3+1} \\
= & \; C(1-|\eta|^2)^{n+1}.
\end{aligned}$$

This proves the claim. Hence

$$\begin{aligned}
\|(Q_2Q_1)^k f\|^2 &= \int_{M_2} |(Q_2Q_1)^k f(z)|^2(1-|z|^2)^{n-d} dv_{M_2}(z)\\
\leq \; & Cr_\epsilon^2 \int_{M_2} |f(\eta)|^2 \int_{\{z\in M_2: M_3 z\in D(\eta)\}} \frac{(1-|z|^2)^{n-d}}{(1-|M_3 z|^2)^{d+1}} dv_{M_2}(z) dv_{M_2}(\eta)\\
\leq \; & Cr_\epsilon^2 \int_{M_2} |f(\eta)|^2 (1-|\eta|^2)^{n-d}\\
\leq \; & Cr_\epsilon^2 \|f\|^2.
\end{aligned}$$

Therefore

$$\|(Q_2Q_1)^k f\|^2 \leq Cr_\epsilon^2 \|f\|^2.$$

Take $\epsilon > 0$ such that $Cr_\epsilon^2 < 1$, let $a = Cr_\epsilon^2$ and k as in the previous construction, then by the argument in the beginning of our proof, we have

$$\|Q_2Q_1Q_2 - Q_3\| < 1.$$

Therefore by Lemma 2.4.8 and Corollary 2.4.9 we know that $\mathcal{Q}_1 + \mathcal{Q}_2$ is closed. The last assertion follows immediately from the proof. This completes the proof. □

Remark 2.4.13. *From Lemma 2.4.12, it is easy to see that*

$$\|k_z - k_w\| \leq C\rho(z, w)$$

when $\rho(z, w)$ is small. This tells us that the inverse of Example 2.4.10 is also true: if the angle between M_1 and M_2 is small, then so is the angle between $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Take $z_1 \in M_1$, $z_2 \in M_2$ such that $z_i \perp M_3$, then $Q_3 k_{z_i} = 0, i = 1, 2$ and $\|k_{z_1} - k_{z_2}\| \leq C\rho(z_1, z_2)$. When the angle of M_1 and M_2 is small we can take such z_i so that $\rho(z_1, z_2)$ is small. Therefore by Lemma 2.4.8, the angle between $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is small.

Example 2.4.14. *Suppose $\tilde{M}_1$ and $\tilde{M}_2$ are two affine spaces, $\emptyset \neq \tilde{M}_1 \cap \tilde{M}_2 \cap \overline{\mathbb{B}_n} \subseteq \partial\mathbb{B}_n$. Let*

$M_i = \tilde{M}_i \cap \mathbb{B}_n$, $Q_i = \overline{\text{span}}\{K_\lambda : \lambda \in M_i\}$. *Then* $Q_1 \cap Q_2 = \{0\}$ *and* $Q_1 + Q_2$ *is not closed.*

Proof. Since $Q_1 \cap Q_2$ is the orthogonal space of a polynomial ideal with generators of degree one, and has no zero points inside $\mathbb{B}_n$ and only one on $\partial\mathbb{B}_n$, we have $Q_1 \cap Q_2 = \{0\}$.

For $z \in \mathbb{B}_n$, recall that in Lemma 2.4.4 we defined a representation of $\mathcal{T}(L^\infty)$ using the unitary operator U_z. It is easy to prove that $Q_{iz} := (Q_i)_z$ is the projection to the space $\overline{\text{span}}\{K_\lambda : \lambda \in \varphi_z(M_i)\}$. Therefore, without loss of generality, we assume that $\tilde{M}_1$ is a linear subspace.

We claim that $\rho(M_1, M_2) = 0$. To prove this, take $z \in M_1 \cap M_2 \cap \partial\mathbb{B}_n$, then $rz \in M_1, \forall 0 < r < 1$. Change coordinates so that $z = (z_1, 0, \dots, 0)$. Since M_2 is an affine space that intersects $\mathbb{B}_n$, after possibly changing the order of basis, M_2 has expression

$$w = (w', L(w')) + (z_1, 0, \dots, 0), w \in M_2,$$

where L is a linear function of $w' = (w_1, \dots, w_d)$, $d = \dim M_2$. Take $w_r = ((r-1)z', L((r-1)z')) + z$, then

$$\varphi_{rz}(w_r) = (\varphi_{rz'}(rz'), \frac{(1-r^2)^{1/2}}{1-r^2} \cdot (r-1)L(z')) = (0, O((1-r^2)^{1/2})).$$

Therefore $\rho(rz, w_r) \to 0, r \to 1$. This proves the claim. Now the statement in Example 2.4.14 can be proved using Lemma 2.4.8. In fact, since $k_{rz} \in Q_1$ and $k_{w_r} \in Q_2$, by Remark 2.4.13, $\|k_{rz} - k_{w_r}\| \le C\rho(rz, w_r) \to 0$. Now we can apply Lemma 2.4.8(1) to see that the angle between Q_1 and Q_2 is zero. This implies that $Q_1 + Q_2$ is not closed. This completes the proof. □

Next we discuss the more general case: suppose M_1 and M_2 are two varieties and $M_3 = M_1 \cap M_2$. Let $Q_i = \overline{\text{span}}\{K_\lambda : \lambda \in M_i\}$, $1 = 1, 2, 3$. When do we know that $Q_1 + Q_2$ is closed? This question is important because when it holds, the essential normality of $Q = Q_1 + Q_2$ follows from the essential normality of each Q_1 and Q_2 (cf. [27, Lemma 3.3, Proposition 3.4]). In fact, such a result gives us the following exact sequence of Hilbert modules.

$$0 \to Q_1 \cap Q_2 \to Q_1 \oplus Q_2 \to Q_1 + Q_2 \to 0.$$

Here we define the first map to be the embedding on each direct summand and second map to be the difference of two entries. In general, given such a short exact sequence and given that the sum on the right side is closed, then the essential normality of the two modules imply the essential normality of both their sum and their intersection (cf. [15]).

The main purpose of this paper is to establish a sufficient condition for the above result to hold. As preparation for our main result, we establish a few lemmas.

Lemma 2.4.15. *Suppose $x, y \in M_A \backslash \mathbb{B}_n$ and $\rho(x, y) < 1$, then there exists a unitary operator U such that for any $S \in \mathcal{T}(L^\infty)$,*

$$S_y = U^* S_x U.$$

Proof. Suppose $z_\alpha \to x$, $\{z_\alpha\} \subseteq \mathbb{B}_n$. Since $\rho(x, y) < 1$, by Lemma 2.4.2, $\exists \lambda \in \mathbb{B}_n$ such that $w_\alpha := \varphi_{z_\alpha}(\lambda) \to y$.

$$\begin{aligned} U_{w_\alpha} f(z) &= f \circ \varphi_{w_\alpha}(z) \frac{(1 - |w_\alpha|^2)^{(n+1)/2}}{(1 - \langle z, w_\alpha \rangle)^{n+1}} \\ &= f \circ U_\alpha \circ \varphi_\lambda \circ \varphi_{z_\alpha}(z) a_\alpha \frac{(1 - |z_\alpha|^2)^{(n+1)/2}}{(1 - \langle z, z_\alpha \rangle)^{n+1}} \frac{(1 - |\lambda|^2)^{(n+1)/2}}{(1 - \langle \varphi_{z_\alpha}(z), \lambda \rangle)^{n+1}} \\ &= a_\alpha U_{z_\alpha} U_\lambda (f \circ U_\alpha)(z). \end{aligned}$$

Here

$$a_\alpha = \frac{(1 - \langle z_\alpha, \lambda \rangle)^{n+1}}{|1 - \langle z_\alpha, \lambda \rangle|^{n+1}}$$

is a number of absolute value 1 and U_α is a unitary operator on $\mathbb{C}^n$ such that $\varphi_{w_\alpha} = U_\alpha \circ \varphi_\lambda \circ \varphi_{z_\alpha}$ (see the proof of Lemma 6.2 in [36] for existence of such U_α). Now we can take a subnet such that $U_\alpha \to U'$. Here U' is a unitary operator on $\mathbb{C}^n$.

Therefore, for $f, g \in \mathbb{C}[z_1, \ldots, z_n]$,

$$\begin{aligned}
\langle S_{w_\alpha} f, g\rangle &= \langle S U_{w_\alpha} f, U_{w_\alpha} g\rangle \\
&= \langle S U_{z_\alpha} U_\lambda(f \circ U_\alpha), U_{z_\alpha} U_\lambda(g \circ U_\alpha)\rangle \\
&= \langle S_{z_\alpha} U_\lambda(f \circ U_\alpha), U_\lambda(g \circ U_\alpha)\rangle \\
&= \langle S_{z_\alpha} U_\lambda(f \circ U_\alpha - f \circ U'), U_\lambda(g \circ U_\alpha))\rangle \\
&\quad +\langle S_{z_\alpha} U_\lambda(f \circ U'), U_\lambda(g \circ U_\alpha - g \circ U')\rangle \\
&\quad +\langle S_{z_\alpha} U_\lambda(f \circ U'), U_\lambda(g \circ U')\rangle.
\end{aligned}$$

Note that $f \circ U_\alpha$ tends to $f \circ U'$ in norm, and that $\|S_\alpha\| \leq \|S\|$, by standard argument, the first two terms converge to 0. Define $Uf = U_\lambda(f \circ U')$. Taking limit, we see that for any polynomial f, g,

$$\langle S_y f, g\rangle = \langle S_x U f, U g\rangle.$$

Therefore $S_y = U^* S_x U$. This completes the proof. □

Lemma 2.4.8 and 2.4.9 turn the problem of obtaining closedness of the sum of two spaces into norm estimates. Here is a refined version.

Lemma 2.4.16. *Suppose Q_1, Q_2, Q_3 are closed linear subspaces of $L_a^2(\mathbb{B}_n)$, $Q_3 \subseteq Q_1 \cap Q_2$, the projection operators $Q_i \in \mathcal{T}(L^\infty)$. Then the following are equivalent.*

(1) $Q_1 + Q_2$ *is closed and* $Q_1 \cap Q_2 / Q_3$ *is finite dimensional.*

(2) $\|Q_2 Q_1 Q_2 - Q_3\|_e < 1$.

(3) $\exists 0 < a < 1$ *such that* $\forall x \in M_A \backslash \mathbb{B}_n$,

$$\|Q_{2x} Q_{1x} Q_{2x} - Q_{3x}\| < a.$$

Proof. Assume (1), since $Q_1 \cap Q_2 / Q_3$ is finite dimensional, the projection operators to $Q_4 := Q_1 \cap Q_2$

and Q_3 differ by a finite rank operator. Thus the essential norm in (2) equals $\|Q_2Q_1Q_2 - Q_4\|_e$. Since $Q_1 + Q_2$ is closed, they have positive angle. By Lemma 2.4.8, $\|Q_2Q_1Q_2 - Q_4\| < 1$. Therefore

$$\|Q_2Q_1Q_2 - Q_3\|_e = \|Q_2Q_1Q_2 - Q_4\|_e \leq \|Q_2Q_1Q_2 - Q_4\| < 1.$$

Thus (1) implies (2). Now assume (2), then it is easy to see that the point 1 is either outside $\sigma(Q_2Q_1Q_2 - Q_3)$ or an isolated point in $\sigma(Q_2Q_1Q_2 - Q_3)$ of finite multiplicity. Since the eigenvector space of $Q_2Q_1Q_2 - Q_3$ at 1 is exactly the space $Q_4 \ominus Q_3$, we know that $Q_4 \ominus Q_3$ is finite dimensional and $\|Q_2Q_1Q_2 - Q_4\| < 1$. By Lemma 2.4.8, this is exactly statement (1). Therefore (1) and (2) are equivalent.

The equivalence of (3) and (2) is by Lemma 2.4.6 and Lemma 2.4.5. This completes the proof. □

Let us take a closer look at the localized projection operators.

Lemma 2.4.17. *Suppose $\tilde{M}$ is an analytic subset of an open neighborhood of $\overline{\mathbb{B}_n}$. $\tilde{M}$ is smooth on $\partial\mathbb{B}_n$ and transversal with $\partial\mathbb{B}_n$. Let $M = \tilde{M} \cap \mathbb{B}_n$ and*

$$Q = \overline{\operatorname{span}}\{K_\lambda | \lambda \in M\}.$$

Then for $x \in M_A \backslash \mathbb{B}_n$,

If $\rho(x, \overline{M}^A) = 1$, then $Q_x = 0$;

If $x \in \overline{M}^A$, then Q_x is the projection operator onto $\overline{\operatorname{span}}\{K_\lambda | \lambda \in M_x\}$, where $M_x = \tilde{M}_x \cap \mathbb{B}_n$ and

$$\tilde{M}_x = \{v \in T\tilde{M}|_{\hat{x}} : v \perp \hat{x}\} + \mathbb{C}\hat{x}.$$

Here $\hat{x} \in \partial\mathbb{B}_n$ is obtained by evaluating x at each index function z_i;

If $\rho(x, \overline{M}^A) < 1$, then $\exists y \in \overline{M}^A$ such that Q_x is unitary equivalent to Q_y.

Here $\overline{M}^A$ denotes the closure of M in M_A.

Proof. By the proof in [16], there exists an "equivalent measure" μ on M such that 0 is isolated in $\sigma(T_\mu)$ and $Q = \text{Range}(T_\mu)$. In other words, if we take a continuous function f on $\mathbb{R}$ such that $f(0) = 0$ and f takes value 1 on $\sigma(T_\mu\backslash\{0\})$, then the projection operator $Q = f(T_\mu)$. Therefore $Q_x = f((T_\mu)_x)$. Suppose $z_\alpha \to x$, $z_\alpha \in \mathbb{B}_n$. The operators $(T_\mu)_{z_\alpha}$ tend to $(T_\mu)_x$ in the strong operator topology. Since

$$(T_\mu)_{z_\alpha} = T_{\mu_{z_\alpha}},$$

where the positive measure μ_{z_α} is defined by

$$\int g d\mu_{z_\alpha} = \int g \circ \varphi_{z_\alpha} |k_{z_\alpha}|^2 d\mu.$$

From the definition,

$$\|\mu_{z_\alpha}\| = \int d\mu_{z_\alpha} = \int |k_{z_\alpha}|^2 d\mu \le C\|k_{z_\alpha}\| = C$$

since μ is a Carleson measure. Therefore

$$\|\mu_{z_\alpha}\| \le C.$$

The net $\{\mu_{z_\alpha}\}$ has a subnet that converges to some measure μ_x in the *weak** topology. Hence

$$\langle (T_\mu)_x g, h\rangle = \int g\bar{h} d\mu_x, \quad \forall g, h \in \mathbb{C}[z_1, \ldots, z_n].$$

So μ_x is a Carleson measure and $(T_\mu)_x = T_{\mu_x}$.

From our construction, we see that Q_x is the projection operator onto $\text{Range}(T_{\mu_x}) = \overline{\text{span}}\{K_\lambda | \lambda \in \text{supp}\mu_x\}$.

Next we discuss about $\text{supp}\mu_x$. We claim that

$$\begin{aligned} \text{supp}\mu_x &= M_x := \{w \in \mathbb{B}_n | \rho(\varphi_{z_\alpha}(w), M) \to 0\} \\ &= \{w \in \mathbb{B}_n | \rho(w, \varphi_{z_\alpha}(M)) \to 0\}. \end{aligned}$$

For any $w \in \mathbb{B}_n$ and $0 < r < 1$,

$$\begin{aligned}\mu_{z_\alpha}(D(w,r)) &= \int \chi_{D(w,r)} d\mu_{z_\alpha} \\ &= \int \chi_{D(\varphi_{z_\alpha}(w),r)} |k_{z_\alpha}|^2 d\mu \\ &\approx \mu(D(\varphi_{z_\alpha}(w),r)) \frac{|1-\langle w, z_\alpha\rangle|^{2(n+1)}}{(1-|z_\alpha|^2)^{n+1}},\end{aligned}$$

where the approximate equality holds because for $\lambda \in D(\varphi_{z_\alpha}(w), r)$,

$$\begin{aligned}|k_{z_\alpha}(\lambda)|^2 &= \frac{(1-|z_\alpha|^2)^{n+1}}{|1-\langle\lambda, z_\alpha\rangle|^{2(n+1)}} \\ &\approx \frac{(1-|z_\alpha|^2)^{n+1}}{|1-\langle\varphi_{z_\alpha}(w), z_\alpha\rangle|^{2(n+1)}} \\ &= \frac{|1-\langle w, z_\alpha\rangle|^{2(n+1)}}{(1-|z_\alpha|^2)^{n+1}}.\end{aligned}$$

If $\rho(\varphi_{z_\alpha}(w), M) \to 0$, for any $0 < r < 1$, we have

$$\mu(D(\varphi_{z_\alpha}(w), r)) \approx (1-|\varphi_{z_\alpha}(w)|^2)^{n+1}.$$

So

$$\begin{aligned}\mu_{z_\alpha}(D(w,r)) &\approx (1-|\varphi_{z_\alpha}(w)|^2)^{n+1} \frac{|1-\langle w, z_\alpha\rangle|^{2(n+1)}}{(1-|z_\alpha|^2)^{n+1}} \\ &= (1-|w|^2)^{n+1}.\end{aligned}$$

Therefore $\mu_x(D(w,r)) > 0$ for any $0 < r < 1$, i.e., $w \in \operatorname{supp}\mu_x$.

On the other hand, if $\rho(\varphi_{z_\alpha}(w), M) \nrightarrow 0$, by taking a subnet we can assume that $\rho(\varphi_{z_\alpha}(w), M) > \epsilon > 0$. Take $r < \epsilon$ in the proof and it is easy to see that $\mu_x(D(w,r)) = 0$. Therefore w is not in the support. This completes the proof of our claim.

Now we study the set M_x.

First, suppose $\rho(x, \overline{M}^A) = 1$. By definition, this means that for any $0 < r < 1$, there exists a net

$z_\alpha \to x$, such that $\rho(\{z_\alpha\}, M) > r$. Therefore for any $0 < r' < 1$, choose $r > r'$, then from the proof above it is easy to see that $\mu_x(D(0, r')) = 0$, which implies $\mu_x = 0$. Therefore $Q_x = 0$.

Second, the case $\rho(x, \overline{M}^A) < 1$ is by Lemma 2.4.15.

Finally, when $x \in \overline{M}^A$, suppose $\tilde{M}$ has local expression

$$w = (w_1, \dots, w_d, F_{d+1}(w'), \dots, F_n(w')), w \in \tilde{M} \cap B(z, \delta),$$

where $w' = (w_1, \dots, w_d)$. Note that we are using the same kind of expression as in [16], where the basis and functions change continuously with z. And the point z always has expression $(z_1, 0, \dots, 0)$.

Suppose $w \in M_x$ and suppose $z_\alpha \in M$, $z_\alpha \to x$. By definition, $\exists \lambda_\alpha \in M$ such that

$$\rho(w, \varphi_{z_\alpha}(\lambda_\alpha)) \to 0.$$

which is equivalent to $|w - \varphi_{z_\alpha}(\lambda_\alpha)| \to 0$. Take any $\epsilon > 0$ such that $|w| + \epsilon < 1$, then for some subnet we have $\rho(\lambda_\alpha, z_\alpha) = |\varphi_{z_\alpha}(\lambda_\alpha)| < |w| + \epsilon < 1$. Therefore

$$1 - \langle \lambda_\alpha, z_\alpha \rangle \approx 1 - |z_\alpha|^2.$$

Since $\lambda_\alpha = (\lambda'_\alpha, F_\alpha(\lambda'_\alpha))$ under the basis determined by z_α.

$$\varphi_{z_\alpha}(\lambda_\alpha) = (\eta_{\alpha,1}, \eta_{\alpha,2}, \dots, \eta_{\alpha,d}, \dots, \eta_{\alpha,n})$$

where

$$\eta_{\alpha,1} = \frac{z_{\alpha,1} - \lambda_{\alpha,1}}{1 - \langle \lambda_\alpha, z_\alpha \rangle}, \quad \eta_{\alpha,i} = -\frac{(1 - |z_\alpha|^2)^{1/2}}{1 - \langle \lambda_\alpha, z_\alpha \rangle} \lambda_{\alpha,i}, \quad i = 2, \dots, d.$$

and

$$\eta_{\alpha,i} = -\frac{(1 - |z_\alpha|^2)^{1/2}}{1 - \langle \lambda_\alpha, z_\alpha \rangle} F_{\alpha,i}(\lambda'_\alpha), \quad i = d+1, \dots, n.$$

For simplicity we omit the subscript α.

Now

$$F_i(\lambda') = L_i(\lambda') + O(|z - \lambda|^2) = L_i(\lambda') + O(1 - |z|^2).$$

Here L_i is the linear part of F_i:

$$L_i(\lambda') = \sum_{j=1}^{d} A_j(\lambda_j - z_j) = A_1(\lambda_1 - z_1) + \sum_{j=2}^{d} A_j\lambda_j.$$

Hence

$$\eta_i = (1 - |z|^2)^{1/2} A_1\eta_1 + \sum_{j=2}^{d} A_j\eta_j + O((1 - |z|^2)^{1/2}), \quad j = d + 1, \ldots, n.$$

Since $\eta \to w$ as $z \to x$ and the coefficients A_i converge to the corresponding value at $\hat{x}$. We see that

$$w_i = \sum_{j=2}^{d} A_j w_j, i = d + 1, \ldots, n.$$

Also if w is of this form, the argument above also implies that $w \in M_x$.

To write more explicitly, $M_x = \tilde{M}_x \cap \mathbb{B}_n$ and

$$\tilde{M}_x = \{v \in T\tilde{M}|_{\hat{x}} : v \perp \hat{x}\} + \mathbb{C}\hat{x}.$$

□

Now suppose M_1 and M_2 are as in Lemma 2.4.17. Let $M_3 = M_1 \cap M_2$ and let $\mathcal{Q}_i = \overline{\text{span}}\{K_\lambda | \lambda \in M_i\}$. We want to find a suitable condition to ensure that $\mathcal{Q}_1 + \mathcal{Q}_2$ is closed. From Lemma 2.4.16 a sufficient condition is that $\|Q_2Q_1Q_2 - Q_3\|_e < 1$. From Theorem 2.3.2, the projections Q_1 and Q_2 are already in $\mathcal{T}(L^\infty)$. Assume M_3 is also as in Lemma 2.4.17, then by Lemma 2.4.16, we only need to show that the operators $Q_{2x}Q_{1x}Q_{2x} - Q_{3x}$, $x \in M_A \backslash \mathbb{B}_n$ have norms uniformly bounded away from 1. We will explain this in full detail. Before that, let us give a definition.

Definition 2.4.18. *Let K and L be embedded submanifolds of a manifold M and suppose that their intersection $K \cap L$ is also an embedded submanifold of M. $K \cap L$ is said to have* clean intersection *if for each* $p \in K \cap L$ *we have*

$$T_p(K \cap L) = T_pK \cap T_pL.$$

We claim that the condition of clean intersection, plus nice conditions on M_1, M_2 and M_3 are sufficient for $\mathcal{Q}_1 + \mathcal{Q}_2$ to be closed. We are ready to state our main theorem.

Theorem 2.4.19. *Suppose* $\tilde{M}_1$ *and* $\tilde{M}_2$ *are two analytic subsets of an open neighborhood of* $\overline{\mathbb{B}_n}$. *Let* $\tilde{M}_3 = \tilde{M}_1 \cap \tilde{M}_2$. *Assume that*

(i) $\tilde{M}_1$ *and* $\tilde{M}_2$ *intersect transversely with* $\partial\mathbb{B}_n$ *and have no singular points on* $\partial\mathbb{B}_n$.

(ii) $\tilde{M}_3$ *also intersects transversely with* $\partial\mathbb{B}_n$ *and has no singular points on* $\partial\mathbb{B}_n$.

(iii) $\tilde{M}_1$ *and* $\tilde{M}_2$ *intersect cleanly on* $\partial\mathbb{B}_n$.

Let $M_i = \tilde{M}_i \cap \mathbb{B}_n$ *and* $\mathcal{Q}_i = \overline{\text{span}}\{K_\lambda : \lambda \in M_i\}$, $i = 1, 2, 3$. *Then* $\mathcal{Q}_1 \cap \mathcal{Q}_2/\mathcal{Q}_3$ *is finite dimensional and* $\mathcal{Q}_1 + \mathcal{Q}_2$ *is closed. As a consequence,* $\mathcal{Q}_1 + \mathcal{Q}_2$ *is essentially normal.*

Before proving the theorem, let us explain the ideas. From Lemma 2.4.17 we know Q_{1x}, Q_{2x} and Q_{3x} are projections to quotient modules corresponding to linear varieties M_{ix}. We list the cases that are possible:

(1) $\rho(x, \overline{M_1}^A) < 1, \rho(x, \overline{M_2}^A) < 1$. In this case, there are two possibilities for M_3:

(1-a) $\rho(x, \overline{M_3}^A) < 1$ or

(1-b) $\rho(x, \overline{M_3}^A) = 1$.

(2) $\rho(x, \overline{M_1}^A) = 1, \rho(x, \overline{M_2}^A) < 1$, then $\rho(x, \overline{M_3}^A) = 1$.

(3) $\rho(x, \overline{M_1}^A) < 1, \rho(x, \overline{M_2}^A) = 1$, then $\rho(x, \overline{M_3}^A) = 1$.

(4) $\rho(x, \overline{M_1}^A) = 1, \rho(x, \overline{M_2}^A) = 1$, then $\rho(x, \overline{M_3}^A) = 1$.

The case (1-b) corresponds to that of Example 2.4.14, which we want to avoid. In fact, we can show the following lemma.

Lemma 2.4.20. *Assume the same conditions as Theorem 2.4.19, then If* $x \in M_A \backslash \mathbb{B}_n$ *and* $\rho(x, \overline{M_1}^A) < 1$, $\rho(x, \overline{M_2}^A) < 1$, *then* $\rho(x, \overline{M_3}^A) < 1$.

We postpone the proof of Lemma 2.4.20 to the end of the section. Now assuming Lemma 2.4.20, we are ready to prove Theorem 2.4.19.

Proof of Theorem 2.4.19. By Theorem 2.3.2, conditions (i) and (ii) ensure that the projections Q_1, Q_2 and Q_3 are all in the Toeplitz algebra $\mathcal{T}(L^\infty)$. By applying Lemma 2.4.16 we see that it suffices to show condition (3) in Lemma 2.4.16, i.e.,

$$\sup_{x \in M_A \backslash \mathbb{B}_n} \|Q_{2x} Q_{1x} Q_{2x} - Q_{3x}\| < 1.$$

Now Lemma 2.4.17 has already given descriptions for the operators $Q_{ix} := (Q_i)_x$. Let me refer to the cases listed before Lemma 2.4.20. What Lemma 2.4.20 tells us is that case (1-b) does not occur under our assumption. For cases (2)-(4), either Q_{1x} or Q_{2x} is zero, and Q_{3x} is always zero. This means $Q_{2x} Q_{1x} Q_{2x} - Q_{3x} = 0$. Hence we are left with case (1-a). In this case, since $\rho(x, \overline{M_3}^A) < 1$, by Lemma 2.4.17, there exists $y \in \overline{M_3}^A$ such that $Q_{2x} Q_{1x} Q_{2x} - Q_{3x} = (Q_2 Q_1 Q_2 - Q_3)_x$ is unitary equivalent to $(Q_2 Q_1 Q_2 - Q_3)_y = Q_{2y} Q_{1y} Q_{2y} - Q_{3y}$. So without generality we may assume $x \in \overline{M_3}^A$. Again we refer to Lemma 2.4.17, then Q_{ix} is the projection operator to

$$\overline{\text{span}}\{K_\lambda | \lambda \in M_{ix}\}, \quad i = 1, 2, 3.$$

Here the M_{ix}, as defined as in Lemma 2.4.17, is the intersection of a linear subspace $\tilde{M}_{ix}$ with $\mathbb{B}_n$. By the description of M_{ix} and the condition that M_1 intersects cleanly with M_2, one immediately see that $M_{3x} = M_{1x} \cap M_{2x}$. Therefore we could apply Proposition 2.4.11, the norms $\|Q_{2x} Q_{1x} Q_{2x} - Q_{3x}\|$ depend on the angles between linear spaces $\tilde{M}_{1x}$ and $\tilde{M}_{2x}$. The only thing left to show is that the angles between $\tilde{M}_{1x}$ and $\tilde{M}_{2x}$ are uniformly bounded away from 0 for all $x \in \overline{M_3}^A$. Since these subspaces change continuously with $\hat{x} \in M_3$, the projection operators onto $\tilde{M}_{ix}$ change continuously. Therefore by Lemma 2.4.8 and compactness of $\tilde{M}_3 \cap \partial \mathbb{B}_n$, their angles must have a lower bound.

Therefore by Proposition 2.4.11,

$$\sup_{x\in M_A\backslash\mathbb{B}_n}\|Q_{2x}Q_{1x}Q_{2x}-Q_{3x}\|<1.$$

By Lemma 2.4.16, Q_1+Q_2 is closed and $Q_1\cap Q_2/Q_3$ is finite dimensional.

The last assertion in Theorem 2.4.19 can be obtained from Proposition 3.4 in [27]. This completes our proof. □

The only thing left for us to verify is Lemma 2.4.20. We break the proof into several lemmas.

Lemma 2.4.21. *Suppose $z=(z_1,0,\cdots,0)\in\mathbb{B}_n$, then*

$$\frac{\partial}{\partial w_1}|\varphi_z(w)|^2(0)=\bar{z}_1(|z_1|^2-1)$$

and

$$\frac{\partial}{\partial w_i}|\varphi_z(w)|^2(0)=0,\quad i=2,3,\cdots,n.$$

As a consequence, if M is any complex manifold passing through 0 and obtains its minimal hyperbolic distance to z at the point 0, then z must be orthogonal to the tangent space $TM|_0$.

Proof. The two formulas are obtained by direct computation. To prove the last statement, one only need to observe that the derivative of $|\varphi_z(w)|^2$ in u direction is

$$\frac{\partial|\varphi_z(w)|^2}{\partial u}(0)=\sum_{i=1}^{n}u_i\frac{\partial|\varphi_z(w)|^2}{\partial w_i}(0)=\langle u,z\rangle(|z_1|^2-1).$$

Since the minimal value of $|\varphi_z(w)|^2$ is obtained at 0, the derivative of $|\varphi_z(w)|^2$ along all directions in $TM|_0$ must be 0. Therefore z is orthogonal to $TM|_0$. This completes the proof. □

Lemma 2.4.22. *Suppose M satisfies the hypotheses of Theorem 2.3.2, and suppose $\{z_\alpha\},\{w_\alpha\}\subseteq M$ are two separated nets such that , viewed as points in M_A, z_α tends to a point $x\in M_A\backslash\mathbb{B}_n$, viewed as points in $\overline{\mathbb{B}_n}$, w_α tends to $\hat{x}$. Then any limit point of the net $\{\varphi_{z_\alpha}(w_\alpha)\}$ is in $\overline{M_x}\subseteq\overline{\mathbb{B}_n}$.*

Proof. For convenience we omit the subscript α. Using the same convention as before, we take the basis at each z, so

$$z = (z_1, 0, \cdots, 0), \quad w = (w', F(w')),$$

where $w' = (w_1, \cdots, w_d)$ and $F = (f_{d+1}, \cdots, F_n)$ is the expression of $\tilde{M}$ depending continuously on z. Same as in the proof of Lemma 2.4.17, we have

$$\varphi_z(w) = (\eta_1, \cdots, \eta_n),$$

where

$$\eta_1 = \frac{z_1 - w_1}{1 - \langle w, z\rangle}, \quad \eta_i = -\frac{(1-|z|^2)^{1/2} w_i}{1 - \langle w, z\rangle}, i = 2, \cdots, d$$

and

$$\eta_j = -\frac{(1-|z|^2)^{1/2} F_j(w')}{1 - \langle w, z\rangle}, j = d+1, \cdots, n.$$

We write $F_j(w') = L_j(w') + O(|w-z|^2)$, where L is the linear part of F. Since

$$|w-z|^2 = |w|^2 + |z|^2 - 2Re\langle w, z\rangle \leq 2(1 - Re\langle w, z\rangle) \leq 2|1 - \langle w, z\rangle|,$$

for $j = d+1, \cdots, n$,

$$\eta_j + \frac{(1-|z|^2)^{1/2} L_j(w')}{1 - \langle w, z\rangle} = \frac{(1-|z|^2)^{1/2}}{1 - \langle z, w\rangle} O(|1 - \langle w, z\rangle|) \to 0, \;\; z \to \hat{x}.$$

The rest of the proof is as in Lemma 2.4.17(3). □

Proof of Lemma 2.4.20. Suppose $x \in M_A \backslash \mathbb{B}_n$ and $\rho(x, \overline{M_1}^A) < 1, \rho(x, \overline{M_2}^A) < 1$, we will show that $\rho(x, \overline{M_3}^A) < 1$. Clearly, $\hat{x} \in \tilde{M}_1 \cap \tilde{M}_2 = \tilde{M}_3$. By Lemma 2.4.17, without loss of generality, we assume $x \in \overline{M_1}^A$.

Let $\{z_\alpha\} \subseteq M_1$ such that $z_\alpha \to x$. Let $w_\alpha \in M_2$ and $\lambda_\alpha \in M_3$ such that $\rho(z_\alpha, w_\alpha) = \rho(z_\alpha, M_2)$ and $\rho(z_\alpha, \lambda_\alpha) = \rho(z_\alpha, M_3)$. Take subnets (using the same notation) such that both nets converge in M_A. Suppose $w_\alpha \to y \in M_A$ and $\lambda_\alpha \to \xi \in M_A$. Clearly $\hat{y} = \hat{\xi} = \hat{x}$. For convenience we omit the subscript α in the sequel.

Since $\rho(\varphi_\lambda(z), 0) = \rho(\varphi_\lambda(z), \varphi_\lambda(M_3))$, by Lemma 2.4.21, $\varphi_\lambda(z) \perp \varphi_\lambda(M_3)$. The latter tends uniformly to $M_{3\xi}$ while the first has a subnet that converges to some point a in $\partial\mathbb{B}_n$ by compactness. Therefore $a \perp M_{3\xi}$.

On the other hand, $\rho(\varphi_\lambda(z), \varphi_\lambda(w)) = \rho(z, w) \to \rho(x, \overline{M_2}^A) < 1$. Since $|\varphi_\lambda(z)| = \rho(\lambda, z) \to 1$, we have the Euclidean distance $|\varphi_\lambda(z) - \varphi_\lambda(w)| \to 0$. Therefore $\varphi_\lambda(w) \to a$. By Lemma 2.4.22, $a \in M_{1\xi} \cap M_{2\xi}$ which equals $M_{3\xi}$ by the clean intersection condition and the expression of $M_{i\xi}$ in Lemma 2.4.17. So a is a vector of length 1 which both belongs to $M_{3\xi}$ and is perpendicular to $M_{3\xi}$. A contradiction. Therefore such x does not exist. This completes the proof. □

2.5 p-Essential Normality

In the previous sections, we have obtained essential normality for quotient modules corresponding to varieties and unions of varieties under certain assumptions (Theorem 2.3.2 and Theorem 2.4.19). The original conjecture states that the quotient modules are p-essentially normal for all p greater than the complex dimension of the variety. In this section we fill that gap. Let us begin with a lemma about commutators.

Lemma 2.5.1. *Suppose that $\mathcal{H}$ is a Hilbert space. Let A, B be bounded operators on $\mathcal{H}$, and let Q be an orthogonal projection on $\mathcal{H}$. Define $S = QAQ$ and $T = QBQ$. Then*

$$[S, T] = [Q, B](1 - Q)[A, Q] - [Q, A](1 - Q)[B, Q] + Q[A, B]Q.$$

As a consequence, if $[A, B] = 0$ and if $[Q, A]$, $[Q, B] \in C_{2p}$ for some $1 \le p < \infty$, then $[S, T] \in C_p$.

Proof. Since $Q(1 - Q) = 0$ and $(1 - Q)Q = 0$, simple algebra yields

$$\begin{aligned}[S, T] &= QAQBQ - QBQAQ \\ &= QB(1 - Q)AQ - QA(1 - Q)BQ + Q[A, B]Q \\ &= [Q, B](1 - Q)[A, Q] - [Q, A](1 - Q)[B, Q] + Q[A, B]Q.\end{aligned}$$

This completes the proof. ◻

Consider the case where $\mathcal{H} = L^2(\mathbb{B}_n)$, $\mathcal{Q}$ is the quotient module in Notation 1.2, and $Q : L^2(\mathbb{B}_n) \to \mathcal{Q}$ is the orthogonal projection. Let $\hat{M}_{z_i}$ be the operator of multiplication by the coordinate function z_i on the big space $L^2(\mathbb{B}_n)$, $i = 1, \ldots, n$. For $p > n$, since $L_a^2(\mathbb{B}_n)$ is p-essentially normal, if we know that every $[Q, \hat{M}_{z_i}]$ is in the Schatten class $\mathcal{C}_{2p}$, then by Proposition 4.1 in [5] we can conclude that the quotient module $\mathcal{Q}$ is p-essentially normal. But since the essential normality of the Bergman module $L_a^2(\mathbb{B}_n)$ is involved in this argument, it does not cover the case $p \leq n$. That is where Lemma 2.5.1 comes in.

The advantage of Lemma 2.5.1 is that it allows us to bypass the Bergman module $L_a^2(\mathbb{B}_n)$. More to the point, it allows us to bypass Proposition 1.0.4. For any $1 \leq p < \infty$, Lemma 2.5.1 tells us that if we know that $[Q, \hat{M}_{z_i}] \in \mathcal{C}_{2p}$ for every $i \in \{1, \ldots, n\}$, then we can conclude that the quotient module $\mathcal{Q}$ is p-essentially normal.

Recall that, for each $1 \leq p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k \geq 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}} \tag{2.1}$$

defines a symmetric norm for operators, where $s_1(A), \ldots, s_k(A), \ldots$ are the s-numbers of A. On a Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

is a norm ideal. See Sections III.2 and III.14 in [24].

It is well known that $\mathcal{C}_p^+$ contains the Schatten class $\mathcal{C}_p$ and that $\mathcal{C}_p^+ \neq \mathcal{C}_p$. Moreover, we have $\mathcal{C}_p^+ \subset \mathcal{C}_{p'}$ for all $1 \leq p < p' < \infty$. A property of $\mathcal{C}_p^+$ that does not concern us in this paper, but is nonetheless interesting, it that this ideal is not separable with respect to the norm $\|\cdot\|_p^+$.

The reason for introducing $\mathcal{C}_p^+$ is that the norm $\|\cdot\|_p^+$ is particularly easy to handle in the essential normality problems for modules, as was demonstrated in [20]. Estimates in this paper will further show that the norm $\|\cdot\|_p^+$ is user-friendly indeed.

Lemma 2.5.2. *Suppose T is in the weak operator closure of a set of operators $\{T_\alpha\}_{\alpha\in I}$. Assume $T_\alpha \in C_p^+$ and*

$$\sup_{\alpha\in I} \|T_\alpha\|_p^+ \le C < \infty.$$

Then $T \in C_p^+$ and $\|T\|_p^+ \le C$.

Proof. Let us denote $\sigma_k(T) = s_1(T) + \cdots + s_k(T)$. It is well known that

$$\sigma_k(T) = \sup\{|\mathrm{tr}(TA_k)| : \|A_k\| \le 1 \text{ and } \mathrm{rank}(A_k) = k\}.$$

For each A_k, since its rank equals $k < \infty$, there is a sequence $\{\alpha_m\}$ in I such that $\mathrm{tr}(T_{\alpha_m}A_k) \to \mathrm{tr}(TA_k)$ as $m \to \infty$. Therefore

$$|\mathrm{tr}(TA_k)| = \lim_{m\to\infty} |\mathrm{tr}(T_{\alpha_m}A_k)| \le \sup_{\alpha\in I} \sigma_k(T_\alpha) \le C(1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}).$$

Taking supremum over all such A_k, we obtain

$$\sigma_k(T) \le C(1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}).$$

By (2.1), we have

$$\|T\|_p^+ = \sup_k \frac{\sigma_k(T)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}} \le C.$$

This completes the proof. □

The following lemma provides a key estimate.

Lemma 2.5.3. *Given any positive numbers $0 < a \le b < \infty$, there is a constant $0 < B(a,b) < \infty$ such that the following holds true: Let $\mathcal{H}$ be a Hilbert space, and suppose that $F_0, F_1, \ldots, F_k, \ldots$ are operators on $\mathcal{H}$ such that the following two conditions are satisfied for every k:*

(1) $\|F_k\| \le 2^{-ak}$,

(2) $\mathrm{rank}(F_k) \le 2^{bk}$.

Then the operator $F = \sum_{k=0}^{\infty} F_k$ satisfies the estimate $\|F\|_{b/a}^{+} \leq B(a,b)$. In particular, $F \in C_{b/a}^{+}$.

Proof. Recall from [24] that for any bounded operator A and any $i \geq 1$,

$$s_i(A) = \inf\{\|A + K\| : \operatorname{rank}(K) \leq i - 1\}.$$

Obviously, condition (1) implies that F is a bounded linear operator on $\mathcal{H}$. By condition (2),

$$\operatorname{rank}\left(\sum_{j=0}^{k} F_j\right) \leq \sum_{j=0}^{k} 2^{bj} \leq C_1 2^{bk}, \tag{2.2}$$

where $C_1 = (1 - 2^{-b})^{-1}$. For any integer $m > C_1$, let $k \geq 0$ be such that

$$C_1 2^{bk} < m \leq C_1 2^{b(k+1)}.$$

From (2.2) we obtain

$$s_m(F) \leq \left\| \sum_{j=k+1}^{\infty} F_j \right\| \leq \sum_{j=k+1}^{\infty} 2^{-aj} \leq C_2 2^{-ak},$$

where $C_2 = (1 - 2^{-a})^{-1}$. Therefore

$$s_m(F) m^{a/b} \leq C_2 2^{-ak} \cdot (C_1 2^{b(k+1)})^{a/b} = 2^a C_2 C_1^{a/b}.$$

Set $B(a,b) = 2^a C_2 C_1^{a/b}$. Then the above translates to

$$s_m(F) \leq B(a,b) m^{-a/b}$$

for every $m > C_1$. On the other hand, since $\|F\| \leq C_2$, for $m \leq C_1$ we have

$$s_m(F) \leq C_2 = C_2 m^{a/b} m^{-a/b} \leq C_2 C_1^{a/b} m^{-a/b} \leq B(a,b) m^{-a/b}.$$

Combining these two estimates, we see that $s_m(F) \leq B(a,b) m^{-a/b}$ for every $m \geq 1$. By (2.1), this means $\|F\|_{b/a}^{+} \leq B(a,b)$. □

The following lemma can be found in Appendix C to [**?**, Chapter IV].

Lemma 2.5.4. *Suppose $p \geq 1$, S, T are bounded linear operators on a Hilbert space $\mathcal{H}$ and $[S,T] \in C_p$. If S is self-adjoint and if f is a C^∞ function on the spectrum of S, one has $[f(S),T] \in C_p$.*

We will also need the well-known Schur test for boundedness:

Lemma 2.5.5. *Let $(X, d\mu)$ be a measure space and $R(x,y)$ a non-negative, measurable function on $X \times X$. Suppose that there exist a positive, measurable function h function on X and positive numbers C_1, C_2 such that*

$$\int_X R(x,y)h(y)d\mu(y) \leq C_1 h(x) \quad \text{for } \mu\text{-a.e. } x$$

and

$$\int_X R(x,y)h(x)d\mu(x) \leq C_2 h(y) \quad \text{for } \mu\text{-a.e. } y.$$

Then

$$(Tf)(x) = \int_X R(x,y)f(y)d\mu(y)$$

defines a bounded operator on $L^2(X, d\mu)$ with $\|T\| \leq (C_1C_2)^{1/2}$.

Suppose μ is a Carleson measure supported on M. Let $\hat{T}_\mu$ denote the operator on $L^2(\mathbb{B}_n)$ that sends $L_a^2(\mathbb{B}_n)^\perp$ to $\{0\}$ and coincides with T_μ on $L_a^2(\mathbb{B}_n)$. Our first observation is that we have the integral representation

$$\hat{T}_\mu = \int K_w \otimes K_w d\mu(w).$$

This is verified by direct calculation: for $f \in L^2(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$,

$$\int \langle f, K_w\rangle K_w(z)d\mu(w) = \int \frac{(Pf)(w)}{(1-\langle z,w\rangle)^{n+1}}d\mu(w) = (\hat{T}_\mu f)(z),$$

where $P : L^2(\mathbb{B}_n) \to L^2_a(\mathbb{B}_n)$ is the orthogonal projection.

For each $\varphi \in L^\infty(\mathbb{B}_n)$, let $\hat{M}_\varphi$ denote the operator of multiplication by φ on $L^2(\mathbb{B}_n)$. That is,

$$\hat{M}_\varphi f = \varphi f, \quad f \in L^2(\mathbb{B}_n).$$

The following theorem is the main step in the proof of Theorem 1.4.

Theorem 2.5.6. *Let μ be a Carleson measure supported on M, then for every $j \in \{1, \ldots, n\}$ and every $p > 2d$, we have $[\hat{T}_\mu, \hat{M}_{z_j}] \in C_p^+$. As a consequence, $[\hat{T}_\mu, \hat{M}_{z_j}] \in C_p$ for every $j \in \{1, \ldots, n\}$ and every $p > 2d$.*

First, let us give the outline of our proof. The main idea is to approximate the operator $\hat{T}_\mu$ by a certain kind of discrete sums. Then we estimate the C_p^+ norms of commutators of these discrete sums with $\hat{M}_{z_j}$. We break the commutators into parts and estimate the ranks and norms of these parts. Finally, an application of Lemma 2.5.3 will end the proof.

Now let us construct the discrete sums. Choose a subset $\mathcal{L} \subset M$ that is maximal with respect to the property that

$$D(z, 1) \cap D(w, 1) = \emptyset \quad \text{for all } z \neq w \text{ in } \mathcal{L}. \tag{2.3}$$

Obviously, such an $\mathcal{L}$ is countable, which allows us to write $\mathcal{L} = \{z_i\}_{i=1}^\infty$. It follows from the maximality of $\mathcal{L}$ that

$$\bigcup_{i=1}^\infty D(z_i, 2) \supset M.$$

There exist Borel sets $\Delta_1, \Delta_2, \ldots, \Delta_i, \ldots$ in $\mathbb{B}_n$ satisfying the following three requirements:

(1) $D(z_i, 1) \subset \Delta_i \subset D(z_i, 2)$ for every i.

(2) $\Delta_i \cap \Delta_{i'} = \emptyset$ for $i \neq i'$.

(3) $\cup_{i=1}^\infty \Delta_i = \cup_{i=1}^\infty D(z_i, 2) \supset M$.

The construction of these sets is standard. In fact, obviously there are pairwise disjoint Borel subsets $E_1, E_2, \ldots, E_i, \ldots$ of $\{\cup_{i=1}^{\infty} D(z_i, 2)\}\backslash\{\cup_{i=1}^{\infty} D(z_i, 1)\}$ such that

$$E_1 \cup E_2 \cup \cdots \cup E_i \cup \cdots = \{\cup_{i=1}^{\infty} D(z_i, 2)\}\backslash\{\cup_{i=1}^{\infty} D(z_i, 1)\}$$

and $E_i \subset D(z_i, 2)$ for every i. The sets $\Delta_i = D(z_i, 1) \cup E_i$, $i = 1, 2, 3, \ldots$, satisfy requirements (1)-(3) above.

Let μ be a Carleson measure supported on M. By Lemma 2.1.4,

$$c_i := \int_{\Delta_i} (1 - |w|^2)^{-(n+1)} d\mu(w) \lesssim (1 - |z_i|^2)^{-(n+1)} \mu(\Delta_i) \lesssim \frac{\mu(D(z_i, 2))}{v(D(z_i, 2))}.$$

By Lemma 2.1.5, there is a constant $0 < C < \infty$ such that $c_i \leq C$ for every i.

Define $N = \{i \in \mathbb{N} : \mu(\Delta_i) \neq 0\} = \{i \in \mathbb{N} : c_i > 0\}$. For each $i \in N$, we define the measure $d\mu_i$ to be the restriction of the measure $c_i^{-1}(1 - |w|^2)^{-(n+1)} d\mu$ to the set Δ_i. Obviously, $\mu_i(\Delta_i) = 1$. Observe that

$$\begin{aligned} \hat{T}_\mu &= \int K_w \otimes K_w d\mu(w) = \sum_{i=1}^{\infty} \int_{\Delta_i} K_w \otimes K_w d\mu(w) \\ &= \sum_{i \in N} c_i \int_{\Delta_i} k_w \otimes k_w c_i^{-1} (1 - |w|^2)^{-(n+1)} d\mu(w) = \sum_{i \in N} c_i \int_{\Delta_i} k_w \otimes k_w d\mu_i(w), \end{aligned}$$

where $k_w = K_w / \|K_w\|$ is the normalized reproducing kernel. Since μ is a Carleson measure, the positive operator $\hat{T}_\mu$ is bounded. By the monotone convergence theorem, the above sums converge in the strong operator topology.

Since μ is supported on M, each probability measure μ_i can be approximated in the weak-* topology by measures of the form $\frac{1}{k} \sum_{j=1}^{k} \delta_{w_j}$, where $w_j \in \Delta_i \cap M$. Therefore each operator $\int_{\Delta_i} k_w \otimes k_w d\mu_i(w)$ can be approximated in the weak operator topology by operators of the form

$$\frac{1}{k} \sum_{j=1}^{k} k_{w_j} \otimes k_{w_j}, \quad w_j \in \Delta_i \cap M.$$

Hence $\hat{T}_\mu$ can be weakly approximated by operators of the form

$$\sum_{i\in F} c_i \frac{1}{k} \sum_{j=1}^{k} k_{w_{i,j}} \otimes k_{w_{i,j}} = \frac{1}{k} \sum_{j=1}^{k} \sum_{i\in F} c_i k_{w_{i,j}} \otimes k_{w_{i,j}},$$

where F is a finite subset of N, $k \in \mathbb{N}$, and $w_{i,j} \in \Delta_i \cap M$. We summarize the above arguments in the following lemma.

Lemma 2.5.7. *The operator $\hat{T}_\mu$ is in the weak closure of the convex hull of operators of the form*

$$\sum_{i\in F} c_i k_{w_i} \otimes k_{w_i}, \tag{2.4}$$

where F is any finite subset of N, $w_i \in \Delta_i \cap M$ and $0 < c_i \leq C$. Moreover, the finite bound C depends only on the Carleson measure μ on M.

It follows immediately that for every $1 \leq m \leq n$, the commutator $[\hat{T}_\mu, \hat{M}_{z_m}]$ is in the weak closure of the convex hull of operators of the form

$$\sum_{i\in F} c_i [k_{w_i} \otimes k_{w_i}, \hat{M}_{z_m}].$$

Thus to estimate $\|[\hat{T}_\mu, \hat{M}_{z_m}]\|_p^+$, it suffices to estimate the C_p^+ norms of operators of the above form. To estimate the latter, we use Lemma 2.5.3. Conditions (1) and (2) in Lemma 2.5.3 will be verified in the following steps.

Let v_M denote the natural volume measure on the smooth part of $\tilde{M}$.

Lemma 2.5.8. *For $0 < s < t < 1$, define*

$$M_s^t = \{z \in \tilde{M} : s < |z| \leq t\},$$

then for r sufficiently close to 1 and $r < s < t < 1$, we have $v_M(M_s^t) \lesssim t - s$.

Proof. Let $r(z) = |z|$ be the radius function. By Assumption 1.1, $\tilde{M}$ intersects $\partial\mathbb{B}_n$ transversely. Thus for each point $\zeta \in \tilde{M} \cap \partial\mathbb{B}_n$, $\tilde{M}$ has a real local coordinate system of the form $\Phi = (\phi_1, \ldots, \phi_{2d-1}, r(z))$ defined on a neighborhood $U_\zeta \cap \tilde{M}$, where U_ζ is an open set containing ζ in $\mathbb{C}^n$. Therefore the volume form locally can be expressed as $dv_M = gd\phi_1 \wedge \ldots \wedge d\phi_{2d-1} \wedge dr$. If we shrink the neighborhood U_ζ we can also assume that g is bounded and Φ maps $U_\zeta \cap \tilde{M}$ to a bounded set in $\mathbb{R}^{2d}$. By the compactness of $\tilde{M} \cap \partial\mathbb{B}_n$, it can be covered by finitely many such open sets U_{ζ_j}, $j = 1, \ldots, m$. Thus it suffices to show that

$$v_M(M_s^t \cap U_{\zeta_j}) \lesssim t - s$$

for each j and $s < t$ sufficiently close to 1. By direct computation,

$$v_M(M_s^t \cap U_{\zeta_j}) \lesssim \int_s^t 1dr \lesssim t - s.$$

This completes the proof. □

Lemma 2.5.9. *There exists a* $0 < r < 1$ *such that* $v_M(D(z, 1) \cap M) \gtrsim (1 - |z|^2)^{d+1}$ *for* $z \in M$ *satisfying the condition* $r < |z| < 1$.

Proof. There is a $0 < r < 1$ such that for each $z \in M$, $|z| > r$, there is a smooth map

$$p_z : M \cap D(z, 2) \mapsto TM|_z$$

defined on page 1513 in [16]. Using the formula for p_z given there and the property

$$\sup_{w \in D(z,2)} \beta(p_z(w), w) \to 0 \quad \text{as } |z| \to 1,$$

it is straightforward to verify that $p_z(D(z, 1) \cap M) \supset D(z, 1/2) \cap TM|_z$ when $|z|$ is close enough to 1. Therefore, writing v_d for the volume measure on $TM|_z = \mathbb{C}^d$, we have

$$v_M(D(z, 1) \cap M) \gtrsim v_d(D(z, 1/2) \cap TM|_z) \approx (1 - |z|^2)^{d+1}.$$

This completes the proof. □

Proposition 2.5.10. *Given any* $0 < \epsilon < 1/2$*, there is a* $0 < C' < \infty$ *such that the following estimate holds: Let* F *be any finite subset of* N*. Suppose that for every* $i \in F$*,* $w_i \in \Delta_i \cap M$ *and* $0 \le c_i \le C$*, where* C *is the constant in Lemma 2.5.7. Define* $\nu = \sum_{i \in F} c_i(1 - |w_i|^2)^{n+1}\delta_{w_i}$ *and*

$$\hat{T}_\nu = \sum_{i \in F} c_i k_{w_i} \otimes k_{w_i}.$$

Then we have $\|[\hat{T}_\nu, \hat{M}_{z_m}]\|^+_{2d/(1-2\epsilon)} \le C'$ *for every* $m \in \{1, \dots, n\}$.

Proof. Let $0 < \epsilon < 1/2$ be given. For each $k \ge 0$, define

$$M_k = \{z \in M : 1 - 2^{-2k} \le |z| < 1 - 2^{-2(k+1)}\} \tag{2.5}$$

and

$$\nu_k = \nu|_{M_k} = \sum_{i \in F, w_i \in M_k} c_i(1 - |w_i|^2)^{n+1}\delta_{w_i}.$$

Also, write

$$F_k = [\hat{T}_{\nu_k}, \hat{M}_{z_m}] = \sum_{i \in F, w_i \in M_k} c_i[k_{w_i} \otimes k_{w_i}, \hat{M}_{z_m}]$$

for $k \ge 0$. We will show that there are constants C_1 and C_2 such that

$$\|F_k\| \le C_1 2^{-(1-2\epsilon)k} \tag{2.6}$$

and

$$\text{rank}(F_k) \le C_2 2^{2dk} \tag{2.7}$$

for every $k \ge 0$. Since $\sum_{k=0}^{\infty} F_k = [\hat{T}_\nu, \hat{M}_{z_m}]$, it follows from these estimates and Lemma 2.5.3 that

$$\|[\hat{T}_\nu, \hat{M}_{z_m}]\|^+_{2d/(1-2\epsilon)} \le C_1(1 + C_2)B(1 - 2\epsilon, 2d).$$

That is, the proposition holds for $C' = C_1(1 + C_2)B(1 - 2\epsilon, 2d)$ provided that we find constants C_1

and C_2 such that (2.6) and (2.7) hold.

To find C_1, note that for any $f \in L^2(\mathbb{B}_n)$,

$$
\begin{aligned}
&([K_{w_i} \otimes K_{w_i}, \hat{M}_{z_m}]f)(z) \\
&\quad= \int_{\mathbb{B}_n} \lambda_m f(\lambda) K_\lambda(w_i) dv(\lambda) K_{w_i}(z) - z_m \int_{\mathbb{B}_n} f(\lambda) K_\lambda(w_i) dv(\lambda) K_{w_i}(z) \\
&\quad= \int_{\mathbb{B}_n} (\lambda_m - w_{i,m}) f(\lambda) K_\lambda(w_i) dv(\lambda) K_{w_i}(z) + \int_{\mathbb{B}_n} (w_{i,m} - z_m) f(\lambda) K_\lambda(w_i) dv(\lambda) K_{w_i}(z),
\end{aligned}
$$

where $w_{i,m}$ denotes the m-th component of w_i. Since

$$
F_k = \sum_{i \in F, w_i \in M_k} c_i (1 - |w_i|^2)^{n+1} [K_{w_i} \otimes K_{w_i}, \hat{M}_{z_m}],
$$

we have

$$
\begin{aligned}
|(F_k f)(z)| \le& \sum_{i \in F, w_i \in M_k} c_i (1 - |w_i|^2)^{n+1} \int_{\mathbb{B}_n} |\lambda - w_i| |f(\lambda)| |K_\lambda(w_i)| dv(\lambda) |K_{w_i}(z)| \\
&+ \sum_{i \in F, w_i \in M_k} c_i (1 - |w_i|^2)^{n+1} \int_{\mathbb{B}_n} |w_i - z| |f(\lambda)| |K_\lambda(w_i)| dv(\lambda) |K_{w_i}(z)|.
\end{aligned}
$$

Recalling the definition of v, we have

$$
\begin{aligned}
|(F_k f)(z)| \le& \int_{M_k} \int_{\mathbb{B}_n} |\lambda - w| |f(\lambda)| |K_\lambda(w)| |K_w(z)| dv(\lambda) dv(w) \\
&+ \int_{M_k} \int_{\mathbb{B}_n} |w - z| |f(\lambda)| |K_\lambda(w)| |K_w(z)| dv(\lambda) dv(w) \\
=& \int_{\mathbb{B}_n} |f(\lambda)| \int_{M_k} |\lambda - w| |K_\lambda(w)| |K_w(z)| dv(w) dv(\lambda) \\
&+ \int_{\mathbb{B}_n} |f(\lambda)| \int_{M_k} |w - z| |K_\lambda(w)| |K_w(z)| dv(w) dv(\lambda) \\
=& \int_{\mathbb{B}_n} |f(\lambda)| G_k(z, \lambda) dv(\lambda) + \int_{\mathbb{B}_n} |f(\lambda)| H_k(z, \lambda) dv(\lambda).
\end{aligned}
$$

Here,

$$G_k(z,\lambda) = \int_{M_k} |\lambda - w||K_\lambda(w)||K_w(z)|d\nu(w) \quad \text{and}$$
$$H_k(z,\lambda) = \int_{M_k} |w - z||K_\lambda(w)||K_w(z)|d\nu(w).$$

To estimate $\|F_k\|$, we apply the Schur test. Let $h(\lambda) = (1-|\lambda|^2)^{-1/2}$, then

$$\begin{aligned}
\int_{\mathbb{B}_n} G_k(z,\lambda)h(\lambda)d\nu(\lambda) &= \int_{M_k}\int_{\mathbb{B}_n} |\lambda - w||K_\lambda(w)|h(\lambda)d\nu(\lambda)|K_w(z)|d\nu(w) \\
&\lesssim \int_{M_k}\int_{\mathbb{B}_n} \frac{(1-|\lambda|^2)^{-1/2}}{|1-\langle w,\lambda\rangle|^{n+1/2}}d\nu(\lambda)\frac{1}{|1-\langle z,w\rangle|^{n+1}}d\nu(w) \\
&\lesssim \int_{M_k}\left(\log\frac{1}{1-|w|^2}\right)\frac{1}{|1-\langle z,w\rangle|^{n+1}}d\nu(w) \\
&\lesssim \int_{M_k}\frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}d\nu(w) \le C\sum_{i\in F, w_i\in M_k}\frac{(1-|w_i|^2)^{-\epsilon}}{|1-\langle z,w_i\rangle|^{n+1}}(1-|w_i|^2)^{n+1},
\end{aligned}$$

where, as we recall, C is the constant in Lemma 2.5.7. By Lemma 2.1.4,

$$\frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}} \approx \frac{(1-|w_i|^2)^{-\epsilon}}{|1-\langle z,w_i\rangle|^{n+1}}$$

for any $z \in \mathbb{B}_n$ and $w \in \Delta_i \subset D(z_i,2) \subset D(w_i,4)$. Recall that $\Delta_i \supset D(z_i,1)$. Therefore the integral above is bounded, up to a constant, by

$$\sum_{i\in F, w_i\in M_k}\int_{\Delta_i}\frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}d\nu(w) = \int_{\bigcup_{i\in F, w_i\in M_k}\Delta_i}\frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}d\nu(w).$$

By Lemma 2.1.4, there is a constant $0 < A < \infty$ such that $\bigcup_{i\in F, w_i\in M_k}\Delta_i \subset W_k$, where

$$W_k = \{w \in \mathbb{B}_n : |w| \ge 1 - 2^{-2(k-A)}\}.$$

Therefore

$$\begin{aligned}
\int_{\mathbb{B}_n} G_k(z,\lambda)h(\lambda)dv(\lambda) &\lesssim \int_{\bigcup_{i\in F, w_i\in M_k}\Delta_i} \frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}dv(w) \le \int_{W_k} \frac{(1-|w|^2)^{-\epsilon}}{|1-\langle z,w\rangle|^{n+1}}dv(w)\\
&\lesssim \int_{\max\{1-2^{-2(k-A)},0\}}^1 (1-r^2)^{-\epsilon}\int_{\partial\mathbb{B}_n}\frac{1}{|1-\langle rz,\zeta\rangle|^{n+1}}d\sigma(\zeta)dr\\
&\lesssim \int_{\max\{1-2^{-2(k-A)},0\}}^1 (1-r^2)^{-\epsilon}(1-|rz|^2)^{-1}dr\\
&\lesssim \int_{\max\{1-2^{-2(k-A)},0\}}^1 (1-r^2)^{-\epsilon-(1/2)}(1-|z|^2)^{-1/2}dr\\
&\lesssim \{1-(1-2^{-2(k-A)})\}^{(1/2)-\epsilon}h(z) \lesssim 2^{-(1-2\epsilon)k}h(z).
\end{aligned}$$

On the other hand, using the same method, we have

$$\begin{aligned}
\int_{\mathbb{B}_n} G_k(z,\lambda)h(z)dv(z) &\lesssim \int_{\mathbb{B}_n}\int_{M_k}\frac{1}{|1-\langle w,\lambda\rangle|^{n+1/2}}\frac{1}{|1-\langle z,w\rangle|^{n+1}}dv(w)(1-|z|^2)^{-1/2}dv(z)\\
&\lesssim \int_{M_k}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{-1/2}}{|1-\langle z,w\rangle|^{n+1}}dv(z)\frac{1}{|1-\langle w,\lambda\rangle|^{n+1/2}}dv(w)\\
&\lesssim \int_{M_k}\frac{(1-|w|^2)^{-1/2}}{|1-\langle w,\lambda\rangle|^{n+1/2}}dv(w) \lesssim \int_{W_k}\frac{(1-|w|^2)^{-1/2}}{|1-\langle w,\lambda\rangle|^{n+1/2}}dv(w)\\
&\lesssim \int_{\max\{1-2^{-2(k-A)},0\}}^1 (1-r^2)^{-1/2}(1-|r\lambda|^2)^{-1/2}dr\\
&\lesssim 2^{-k}h(\lambda) \le 2^{-(1-2\epsilon)k}h(\lambda).
\end{aligned}$$

Combining the last two estimates with Lemma 2.5.5, we conclude that G_k defines an integral operator on $L^2(\mathbb{B}_n)$ whose norm is bounded by $B2^{-(1-2\epsilon)k}$, where the constant B depends only on ϵ, the complex dimension n and the bound $c_i \le C$ in Lemma 2.5.7. Obviously, the same conclusion holds for H_k. Thus we have shown that there is a C_1 such that (2.6) holds.

Next we estimate the rank of F_k. Notice that $\text{rank}([k_{w_i}\otimes k_{w_i}, \hat{M}_{z_m}]) \le 2$. Therefore

$$\text{rank}(F_k) \le 2\text{card}\{w_i : i\in F,\ w_i\in\Delta_i\cap M_k\}.$$

Since $\Delta_i \supset D(z_i,1)$, by Lemma 2.5.9, $v_M(\Delta_i\cap M) \gtrsim (1-|z_i|^2)^{d+1}$. For $w_i\in\Delta_i\cap M_k$, Lemma 2.1.4

gives us $1 - |z_i|^2 \approx 1 - |w_i|^2 \approx 2^{-2k}$. Consequently $v_M(\Delta_i \cap M) \gtrsim 2^{-2(d+1)k}$ if $w_i \in \Delta_i \cap M_k$. On the other hand, we saw in the above that if $w_i \in M_k$, then

$$\Delta_i \cap M \subset \{w \in M : 1 - 2^{-2(k-A)} \leq |w| < 1\}.$$

It follows from Lemma 2.5.8 that $v_M(\{w \in M : 1 - 2^{-2(k-A)} \leq |w| < 1\}) \lesssim 2^{-2k}$. Since $\Delta_i \cap \Delta_{i'} = \emptyset$ for $i \neq i'$, we conclude that

$$\text{card}\{w_i : i \in F,\ w_i \in \Delta_i \cap M_k\} \lesssim \frac{2^{-2k}}{2^{-2(d+1)k}} = 2^{2kd}.$$

Thus we have shown that $\text{rank}(F_k) \lesssim 2^{2dk}$, i.e., (2.7) holds for some C_2 that depends only on n and the analytic set $\tilde{M}$. This completes the proof. □

Proof of Theorem 2.5.6. By Lemma 2.5.7, the commutator $[\hat{T}_\mu, \hat{M}_{z_m}]$ is in the weak operator closure of the convex hull of operators of the form $[\hat{T}_\nu, \hat{M}_{z_m}]$, where ν is a discrete measure as in Proposition 2.5.10. Given any $p > 2d$, let $0 < \epsilon < 1/2$ be such that $2d/(1 - 2\epsilon) < p$. Now Proposition 2.5.10 provides the bound $\|[\hat{T}_\nu, \hat{M}_{z_m}]\|^+_{2d/(1-2\epsilon)} \leq C'$ for all such ν. From this we obtain $\|[\hat{T}_\mu, \hat{M}_{z_m}]\|^+_{2d/(1-2\epsilon)} \leq C'$ by applying Lemma 2.5.2. Thus $[\hat{T}_\mu, \hat{M}_{z_m}] \in C^+_{2d/(1-2\epsilon)} \subset C_p$ as promised. □

Theorem 2.5.11. *We have $[Q, \hat{M}_{z_j}] \in C_p$ for all $p > 2d$ and $j \in \{1, \ldots, n\}$.*

Proof. By Theorem 1.9, there exist a Carleson measure μ supported on M and $0 < c \leq C < \infty$ such that

$$c\|f\|^2 \leq \int_M |f(w)|^2 d\mu(w) \leq C\|f\|^2$$

for every $f \in Q$. If $w \in M$, then $K_w \in Q$. Thus the above inequality implies

$$c\|Qg\|^2 \leq \int_M |\langle g, K_w\rangle|^2 d\mu(w) \leq C\|Qg\|^2$$

for every $g \in L^2(\mathbb{B}_n)$. This translates to the operator inequality $cQ \leq \hat{T}_\mu \leq CQ$ on $L^2(\mathbb{B}_n)$. Thus, by the spectral theory of self-adjoint operators, there is a C^∞ function h such that $Q = h(\hat{T}_\mu)$. Now the membership $[Q, \hat{M}_{z_j}] \in \mathcal{C}_p$, $p > 2d$, follows from Lemma 2.5.4 and Theorem 2.5.6. □

Proof of Theorem 1.4. The point is that on the big space $L^2(\mathbb{B}_n)$, we have $\hat{M}_{z_i}^* = \hat{M}_{\bar{z}_i}$, consequently $[\hat{M}_{z_i}^*, \hat{M}_{z_j}] = 0$. Applying Lemma 2.5.1 and Theorem 2.5.11, we have $[Z_{\mathcal{Q},i}^*, Z_{\mathcal{Q},j}] \in \mathcal{C}_p$ for $p > d$. □

The authors of [17] proved that for two varieties satisfying nice conditions, their union defines a essentially normal quotient module:

Theorem 2.5.12. *Suppose $\tilde{M}_1$ and $\tilde{M}_2$ are two analytic subsets of an open neighborhood of $\overline{\mathbb{B}_n}$. Let $\tilde{M}_3 = \tilde{M}_1 \cap \tilde{M}_2$. Assume that*

(i) $\tilde{M}_1$ and $\tilde{M}_2$ intersect transversely with $\partial\mathbb{B}_n$ and have no singular points on $\partial\mathbb{B}_n$.

(ii) $\tilde{M}_3$ also intersects transversely with $\partial\mathbb{B}_n$ and has no singular points on $\partial\mathbb{B}_n$.

(iii) $\tilde{M}_1$ and $\tilde{M}_2$ intersect cleanly on $\partial\mathbb{B}_n$.

Let $M_i = \tilde{M}_i \cap \mathbb{B}_n$ and $\mathcal{Q}_i = \overline{\operatorname{span}}\{K_\lambda : \lambda \in M_i\}$ for $i = 1, 2, 3$, $M = M_1 \cup M_2$, and $\mathcal{Q} = \overline{\operatorname{span}}\{K_\lambda : \lambda \in M\}$. Then $\mathcal{Q}_1 \cap \mathcal{Q}_2 / \mathcal{Q}_3$ is finite dimensional and $\mathcal{Q}_1 + \mathcal{Q}_2$ is closed. As a consequence, $\mathcal{Q}$ is p-essentially normal for $p > 2d$, where $d = \dim_{\mathbb{C}} M = \max\{\dim_{\mathbb{C}} M_1, \dim_{\mathbb{C}} M_2\}$.

As a consequence of the improved essential normality in Theorem 1.4, the essential normality in Theorem 2.5.12 can be improved accordingly.

Corollary 2.5.13. *Under the same assumption as in Theorem 2.5.12, the quotient module $\mathcal{Q}$ is p-essentially normal for all $p > d$.*

Once we know that $\mathcal{Q}_1 + \mathcal{Q}_2$ is closed from Theorem 2.5.12, we have $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. Thus Corollary 2.5.13 follows from Theorem 2.5.11 and [27, Lemma 3.3].

3. ESSENTIAL NORMALITY FOR PRINCIPAL SUBMODULES

In this chapter, we take a different approach to obtain essential normality. Unlike in Chapter 2, this time we deal with submodules instead of quotient modules. The disadvantage of doing so is that we loose the possibility of obtaining the full Arveson-Douglas Conjecture, which says the quotient module have p-essential normality for $p < n$. However, the techniques in this chapter rely essentially on one single inequality (inequality 3.4). Once we have derived this inequality, things can be very flexible. This allows us to expand our discussion to a wider class of domains in $\mathbb{C}^n$ — bounded strongly pseudoconvex domains with smooth boundary.

The next section presents some preliminaries in several complex variables.

3.1 Bounded Strongly Pseudoconvex Domains

Definition 3.1.1. *For Ω a bounded domain in $\mathbb{C}^n$ with smooth boundary, we call $r(z)$ a* defining function *for Ω provided*

(1) $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$ *and* $r(z) \in C^\infty(\mathbb{C}^n)$.

(2) $|grad\ r(z)| \neq 0$ *for all* $z \in \partial\Omega$.

For Ω a bounded strongly pseudoconvex domain with smooth boundary *we mean that there are a defining function $r \in C^\infty(\mathbb{C}^n)$ and a constant k such that*

$$\sum_{i,j=1}^{n} \frac{\partial^2 r(p)}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j \geq k|\xi|^2$$

for all $p \in \partial\Omega$ and $\xi \in \mathbb{C}^n$.

For a point $p \in \partial\Omega$, the *complex tangent space* (cf. [30]) at p is defined by

$$T_p^{\mathbb{C}}(\partial\Omega) = \left\{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r(p)}{\partial z_j} \xi_j = 0\right\}.$$

For Ω a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, there is a $\delta > 0$ such that if $z \in \Omega_\delta := \{z \in \Omega : d(z, \partial\Omega) < \delta\}$, then there exists a unique point $\pi(z)$ in $\partial\Omega$ with $d(z, \pi(z)) = d(z, \partial\Omega)$. The complex normal (tangent) direction at z means the corresponding direction at $\pi(z)$. For $z \in \Omega_\delta$, we let $P_z(r_1, r_2)$ denote the polydisc centered at z with radius r_1 in the complex normal direction and radius r_2 in each complex tangential direction.

Notations: For a point $z \in \Omega$, denote $\delta(z) = d(z, \partial\Omega)$, where d is the Euclidean distance. In the case when Ω is the unit ball $\mathbb{B}_n$, $\delta(z)$ is just $1 - |z|$. We use the notation $\mathbb{D}$ for the open unit disc in $\mathbb{C}$ and $\Delta(\lambda, r)$ for the 1-dimensional disc centered at λ with radius r. We use $B(z, r)$ for the higher dimensional Euclidean ball centered at z with radius r. For positive integer k, we use v_k to denote the Lebesgue measure on $\mathbb{C}^k$.

Let us recall the definition of Bergman and weighted Bergman spaces on Ω.

Definition 3.1.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with smooth boundary. The* Bergman space $L^2_a(\Omega)$ *consists of all holomorphic functions on Ω which are square integrable with respect to the Lebesgue measure v_n.*

$$L^2_a(\Omega) = \{f \in Hol(\Omega) : \int_\Omega |f(z)|^2 dv_n(z) < \infty\}.$$

For a positive integer l, one defines the weighted Bergman space $L^2_{a,l}(\Omega)$ *in a similar way.*

$$L^2_{a,l}(\Omega) = \{f \in Hol(\Omega) : \int_\Omega |f(z)|^2 |r(z)|^l dv_n(z) < \infty\}.$$

Lemma 3.1.3. *[30, Lemma 8] Let Ω be a bounded strongly pseudoconvex domain with smooth boundary. Fix any defining function r, then for z in a neighborhood of $\overline{\Omega}$ we have*

$$|r(z)| \approx \delta(z).$$

For this reason, in most of our discussions, using either $|r(z)|$ or $\delta(z)$ does not make a difference. We will choose whichever is more convenient.

In this chapter we use $K(z,w)$ and $K_l(z,w)$ to denote their reproducing kernels, i.e.,

$$f(z) = \int_\Omega f(w)K(z,w)dv_n(w), \quad \forall f \in L^2_a(\Omega)$$

and

$$f(z) = \int_\Omega f(w)K_l(z,w)|r(w)|^l dv_n(w), \quad \forall f \in L^2_{a,l}(\Omega).$$

Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary, $p \in \Omega$ and $\xi \in \mathbb{C}^n$, the *infinitesimal Kobayashi metric* (cf. [30] [28] [29]) of Ω is defined by

$$F_K(p,\xi) = \inf\{\alpha > 0 : \ \exists f \in \mathbb{D}(\Omega) \text{ with } f(0) = p \text{ and } f'(0) = \xi/\alpha\},$$

here $\mathbb{D}(\Omega)$ denotes the set of all holomorphic mappings from the open unit disc $\mathbb{D}$ to Ω. For any C^1 curve $\gamma(t) : [0,1] \to \Omega$, we define the *Kobayashi length* of $\gamma(t)$ as

$$L_K(\gamma) = \int_0^1 F_K(\gamma(t),\gamma'(t))dt.$$

If $p,q \in \Omega$, the *Kobayashi metric* $\beta(p,q)$ equals $\inf\{L_K(\gamma)\}$ where the infimum is taken over all C^1 curves with $\gamma(0) = p$ and $\gamma(1) = q$. One can show that $\beta(p,q)$ is a complete metric and gives the usual topology on Ω.

For $w \in \Omega$ and $r > 0$, denote $E(w,r)$ to be the Kobayashi ball

$$E(w,r) = \{z \in \Omega : \ \beta(z,w) < r\}.$$

In the case $\Omega = \mathbb{B}_n$, the Kobayashi metric coincides with the Bergman metric β. Similarly, the Kobayashi balls can be approximated by polydiscs.

Lemma 3.1.4. *[30, Lemma 6] Let Ω be a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, $r > 0$. If $z \in \Omega_\delta := \{z \in \Omega : \ d(z,\partial\Omega) < \delta\}$ and δ is small enough, then there are* *constants a_i and b_i, $i = 1, 2$ only depending on r and Ω such that*

$$P_w(a_1\delta(w), b_1\delta(w)^{1/2}) \subseteq E(w, r) \subseteq P_w(a_2\delta(w), b_2\delta(w)^{1/2}).$$

Here d is the Euclidean distance. In particular, $v_n(E(w, r)) \approx \delta(w)^{n+1}$.

Fix some defining function $r(z)$ of Ω. Let

$$X(z, w) = -r(z) - \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j}(w_j - z_j) \tag{3.1}$$

$$-1/2 \sum \frac{\partial^2 r(z)}{\partial z_j \partial z_k}(w_j - z_j)(w_k - z_k). \tag{3.2}$$

And

$$F(z, w) = |r(z)| + |r(w)| + |ImX(z, w)| + |z - w|^2. \tag{3.3}$$

Let

$$\rho(z, w) = |z - w|^2 + \left| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j}(w_j - z_j) \right|.$$

Take $\mathbb{B}_n$ for example, then $X(z, w) = 1 - \langle w, z \rangle$. For points sufficiently close to $\partial\Omega$, several quantities are equivalent.

Lemma 3.1.5 ([21] [30])**.** *Let Ω be a bounded strongly pseudoconvex domain with smooth boundary, then*

$$|X(z, w)| \approx |r(z)| + |r(w)| + \rho(z, w) \approx F(z, w)$$

in a region

$$R_\delta := \{(z, w) \in \bar{\Omega} \times \bar{\Omega} : \ |r(z)| + |r(w)| + |z - w| < \delta\},$$

for some $\delta > 0$.

The following lemmas generalize the case of $\mathbb{B}_n$.

Lemma 3.1.6 ([32] Theorem 2.3)**.** *Let Ω be a bounded strongly pseudoconvex domain with smooth* *boundary in* $\mathbb{C}^n$. *Write* $\Gamma = \{(z,z) : z \in \partial\Omega\}$. *Let* l *be any positive integer, then there exists a kernel* $G_l(z,w)$ *such that:*

(i) $G_l \in C^\infty(\overline{\Omega} \times \overline{\Omega} \backslash \Gamma)$, G_l *is holomorphic in* z.

(ii) G_l *reproduces the holomorphic functions in* $L^2_{a,l}(\Omega)$; *i.e., for* $f \in L^2_{a,l}(\Omega)$,

$$f(z) = \int_\Omega G_l(z,w) f(w) |r(w)|^l dv_n(w).$$

(iii) $|G_l(z,w)| \approx |X(z,w)|^{-(n+1+l)}$ *for* $(z,w) \in R_\delta$ *for some* $\delta > 0$.

Lemma 3.1.7 ([30])**.** *There exists a* $\delta > 0$ *such that when* $(z,w) \in R_\delta$,

$$|K(z,w)| \approx |X(z,w)|^{-(n+1)}.$$

Moreover, $K(z,w) \in C^\infty(\overline{\Omega} \times \overline{\Omega} \backslash \Gamma)$

In particular, $|K(z,w)|$ and $|G_l(z,w)|$ are uniformly bounded for $(z,w) \notin K_\delta$, for any $\delta > 0$, since K_δ is a neighborhood of Γ. Let $\delta > 0$ be so small that Lemma 3.1.5, 3.1.6 and 3.1.7 hold on K_δ. Notice that the function $F(z,w)$ is continuous and non-zero off the set Γ, we have

$$|K(z,w)| \lesssim F(z,w)^{-(n+1)}$$

and

$$|G_l(z,w)| \lesssim F(z,w)^{-(n+1+l)}$$

for all pairs $(z,w) \in \Omega \times \Omega$.

Lemma 3.1.8. *There exists* $\delta > 0$ *such that for* $(z,w) \in R_\delta$,

$$\rho(z,w) \approx \rho(w,z)$$

and

$$|X(z,w)| \approx |X(w,z)|.$$

Proof. By definition,

$$|\rho(z,w) - \rho(w,z)| \leq |\sum(\frac{\partial r(z)}{\partial z_j} - \frac{\partial r(w)}{\partial z_j})(w_j - z_j)| \lesssim |w-z|^2 \leq \min\{\rho(z,w), \rho(w,z)\}.$$

From this it is easy to see that

$$\rho(z,w) \approx \rho(w,z).$$

The estimate for X follows immediately from this and Lemma 3.1.5. □

By definition, the functions X, F and ρ depend on the defining function. However, for different defining functions, they are equivalent.

Lemma 3.1.9. *Suppose $r(z)$ and $r'(z)$ are two defining functions for Ω and let X and X' be defined as in (3.1) for $r(z)$ and $r'(z)$, then there exists $\delta > 0$ such that for $(z,w) \in K_\delta$,*

$$|X(z,w)| \approx |X'(z,w)|.$$

Proof. By Lemma 3.1.5, there exists $\delta > 0$ so that when $(z,w) \in K_\delta$,

$$|X(z,w)| \approx |r(z)| + |r(w)| + \Big|\frac{\sum_{j=1}^n \partial r(z)}{\partial z_j}(w_j - z_j)\Big| + |z-w|^2.$$

Since $|r(z)| \approx \delta(z) \approx |r'(z)|$, the only part we need to take care of is

$$\Big|\sum_{j=1}^n \frac{\partial r(z)}{\partial z_j}(w_j - z_j)\Big|$$

and the corresponding one for r'. Notice that

$$\Big|\sum_{j=1}^n \frac{\partial r(z)}{\partial z_j}(w_j - z_j) - \sum_{j=1}^n \frac{\partial r(\pi(z))}{\partial z_j}(w_j - z_j)\Big| \lesssim \delta(z).$$

We can replace the derivatives at z by those at $\pi(z)$. But since r and r' are both defining functions for Ω, their gradients on the boundary points vary by a constant multiple with absolute value uniformly bounded above and away from 0 (this follows from the compactness of $\partial\Omega$). From this it is easy to see that the quantities above are equivalent. □

Lemma 3.1.10. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary and Φ is a biholomorphic map on a neighborhood of $\overline{\Omega}$, $\Phi(\Omega) = \Omega'$. Let X and X' be the functions defined in (3.1) for Ω and Ω' respectively. Then there exists $\delta > 0$ such that for $(z, w) \in K_\delta$,*

$$|X(z, w)| \approx |X'(\Phi(z), \Phi(w))|.$$

Proof. Fix a defining function $r(z)$, then $r \circ \Phi^{-1}$ is a defining function for Ω'. By Lemma 3.1.5 and Lemma 3.1.3, there exists $\delta > 0$ so that for $(z, w) \in K_\delta$,

$$|X(z, w)| \approx \delta(z) + \delta(w) + |z - w|^2 + \left| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j}(w_j - z_j) \right|.$$

Similarly,

$$\begin{aligned} & |X'(\Phi(z), \Phi(w))| \\ \approx\; & \delta(\Phi(z)) + \delta(\Phi(w)) + |\Phi(z) - \Phi(w)|^2 + \left| \sum_{j=1}^{n} \frac{\partial r \circ \Phi^{-1}(\Phi(z))}{\partial z_j}(\Phi_j(w) - \Phi_j(z)) \right| \end{aligned}$$

The first three parts for X: $\delta(z)$, $\delta(w)$ and $|z - w|^2$ are each equivalent to the corresponding ones for X' since Φ preserves distances up to a constant. That is, both Φ and Φ^{-1} are Lipschitz. We look at the last one. Let Φ be as in the assumption. It is elementary to check that $r \circ \Phi^{-1}$ is a defining function for $\Phi(\Omega)$. By Lemma 3.1.9, we only need to prove the result using this defining function.

Now

$$
\begin{aligned}
& \sum_{j=1}^{n} \frac{\partial r \circ \Phi^{-1}(\Phi(z))}{\partial z_j}(\Phi_j(w) - \Phi_j(z)) \\
= & \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial r(z)}{\partial z_i} \frac{\partial \Phi_i^{-1}(\Phi(z))}{\partial z_j} \Big(\sum_{k=1}^{n} \frac{\partial \Phi_j(z)}{\partial z_k}(w_k - z_k) + O(|w-z|^2) \Big) \\
= & \sum_{i=1}^{n} \frac{\partial r(z)}{\partial z_i} \sum_{k=1}^{n} \Big(\sum_{j=1}^{n} \frac{\partial \Phi_i^{-1}(\Phi(z))}{\partial z_j} \frac{\partial \Phi_j(z)}{\partial z_k} \Big)(w_k - z_k) + O(|w-z|^2) \\
= & \sum_{i=1}^{n} \frac{\partial r(z)}{\partial z_i} \sum_{k=1}^{n} \delta_{ik}(w_k - z_k) + O(|w-z|^2) \\
= & \sum_{i=1}^{n} \frac{\partial r(z)}{\partial z_i}(w_i - z_i) + O(|w-z|^2).
\end{aligned}
$$

From this it is clear that $|X(z,w)| \approx |X'(\Phi(z), \Phi(w))|$. This completes the proof. □

In analogous with Lemma 2.1.4, we have

Lemma 3.1.11. *[30, Theorem 12] Fix some $r > 0$, then for $z, w \in \Omega$ and $\beta(z,w) < r$,*

$$|r(z)| \approx |r(w)|.$$

Lemma 3.1.12. *Fix some $r > 0$, then there exists $\delta > 0$, for $z, w, \lambda \in \Omega$ such that $(z,\lambda), (w,\lambda) \in K_\delta$ and $\beta(z,w) < r$,*

$$|X(z,\lambda)| \approx |X(w,\lambda)|.$$

As a consequence, $F(z,\lambda) \approx F(w,\lambda)$ for all $w, z, \lambda \in \Omega$ and $\beta(z,w) < r$.

Proof. First, by Lemma 3.1.5, there exists $\delta > 0$ such that

$$|X(z,\lambda)| \approx |r(z)| + |r(\lambda)| + \Big| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j}(\lambda_j - z_j) \Big| + |z - \lambda|^2$$

and

$$|X(w,\lambda)| \approx |r(w)| + |r(\lambda)| + \Big| \sum_{j=1}^{n} \frac{\partial r(w)}{\partial w_j}(\lambda_j - w_j) \Big| + |w - \lambda|^2.$$

for pairs $(z,\lambda), (w,\lambda) \in K_\delta$. By Lemma 3.1.4 and Lemma 3.1.11,

$$|z-w|^2 \lesssim |r(w)| \approx |r(z)|.$$

So

$$|z-\lambda|^2 \le \Big(|z-w| + |w-\lambda|\Big)^2 \lesssim |z-w|^2 + |w-\lambda|^2 \lesssim |r(w)| + |w-\lambda|^2.$$

Therefore, we have

$$\begin{aligned}
& \Big|\sum_{j=1}^n \frac{\partial r(z)}{\partial z_j}(z_j-\lambda_j)\Big| \\
\lesssim\ & \Big|\sum_{j=1}^n \frac{\partial r(w)}{\partial z_j}(w_j-\lambda_j)\Big| + \Big|\Big(\sum_{j=1}^n \frac{\partial r(z)}{\partial z_j} - \frac{\partial r(w)}{\partial z_j}\Big)(z_j-\lambda_j)\Big| + \Big|\sum_{j=1}^n \frac{\partial r(w)}{\partial z_j}(z_j-w_j)\Big| \\
\lesssim\ & \Big|\sum_{j=1}^n \frac{\partial r(w)}{\partial z_j}(w_j-\lambda_j)\Big| + |z-w||z-\lambda| + |r(w)| \\
\lesssim\ & \Big|\sum_{j=1}^n \frac{\partial r(w)}{\partial z_j}(w_j-\lambda_j)\Big| + |z-w|^2 + |z-\lambda|^2 + |r(w)| \\
\lesssim\ & \Big|\sum_{j=1}^n \frac{\partial r(w)}{\partial z_j}(w_j-\lambda_j)\Big| + |r(w)| + |w-\lambda|^2 \\
\lesssim\ & |X(w,\lambda)|.
\end{aligned}$$

Altogether we have

$$|X(z,\lambda)| \lesssim |X(w,\lambda)|.$$

Since the role of z and w are symmetric, we get

$$|X(z,\lambda)| \approx |X(w,\lambda)|.$$

This completes the proof. □

Lemma 3.1.13. *[32, Lemma 2.7] Let Ω be a bounded strongly pseudoconvex domain with smooth* *boundary. Let* $a \in \mathbb{R}$, $\nu > -1$, *then*

$$\int_\Omega \frac{|r(w)|^\nu}{F(z,w)^{n+1+\nu+a}} dv_n(w) \approx \begin{cases} 1 & \text{if } a < 0 \\ \log |r(z)|^{-1} & \text{if } a = 0. \\ |r(z)|^{-a} & \text{if } a > 0 \end{cases}$$

3.2 The Bergman Space

This section is devoted to proving the following inequality (3.4). In fact, with inequality (3.4), the essential normality is obtained by a routine argument.

Theorem 3.2.1. *Suppose* $\Omega \subseteq \mathbb{C}^n$ *is a bounded strongly pseudoconvex domain with smooth boundary,* $\zeta \in \partial\Omega$, *h is a holomorphic function defined in a neighborhood U of* ζ. *Then there exist a neighborhood V of* ζ *and constants* $\delta > 0$, $N > 0$ *such that* $\forall w \in V \cap \Omega$, $\forall z \in B(w,\delta) \cap \Omega$ *and* $\forall f \in Hol(E(w,1))$,

$$|h(z)f(w)| \lesssim \frac{|X(z,w)|^N}{|r(w)|^{N+n+1}} \int_{E(w,1)} |h(\lambda)||f(\lambda)| dv_n(\lambda). \tag{3.4}$$

Remark 3.2.2. *It turns out that from the proof of Theorem 3.2.1, the requirements that z being close to w and that w being close to the boundary are not essential. In fact, from the proof of Theorem 3.2.1, one obtains the following.*

Theorem 3.2.3. *Suppose* $\Omega \subseteq \mathbb{C}^n$ *is a bounded strongly pseudoconvex domain with smooth boundary, h is a holomorphic function defined on a neighborhood of* $\overline{\Omega}$. *Then there exists a constant* $N > 0$ *such that* $\forall w, z \in \Omega$ *and* $\forall f \in Hol(\Omega)$,

$$|h(z)f(w)| \lesssim \frac{F(z,w)^N}{|r(w)|^{N+n+1}} \int_{E(w,1)} |h(\lambda)||f(\lambda)| dv_n(\lambda).$$

Before proving Theorem 3.2.1 and Theorem 3.2.3, we establish a few lemmas.

Roughly speaking, our approach to Theorem 3.2.1 is to prove a slightly stronger result about logarithms of absolute values of the functions and then apply the Jensen's inequality. This will allow us to separate the part involving the function f in the theorem and concentrate on estimates about h. As a first step, we consider the case when our domain is just the unit disc $\mathbb{D}$ in $\mathbb{C}$ and $h(z) = z - a$ for some $a \in \mathbb{C}$. We show the following is true.

Lemma 3.2.4. *There exists a constant $C > 0$, such that for any $0 < c < 1$, $z, w \in \mathbb{D}$ and $a \in \mathbb{C}$, we have*

$$\log \frac{|z-a|}{|w-z| + \delta(w)} \leq \frac{1}{v_1(\Delta(w, c\delta(w)))} \int_{\Delta(w,c\delta(w))} \log|\lambda - a| dv_1(\lambda) - \log \delta(w) - \log c + C.$$

Proof. Note that since $0 < c < 1$, $\delta(w) = 1 - |w|$, the disc $\Delta(w, c\delta(w))$ is contained in the unit disk $\mathbb{D}$. We split the proof into two cases.

Case 1: $a \notin \Delta(w, c\delta(w))$, then the function $\log|\lambda - a|$ is harmonic in the disc $\Delta(w, c\delta(w))$. Therefore

$$\frac{1}{v_1(\Delta(w, c\delta(w)))} \int_{\Delta(w,c\delta(w))} \log|\lambda - a| dv_1(\lambda) = \log|w - a|.$$

For $z \in \mathbb{D}$,

$$\begin{aligned} & \log \frac{|z-a|}{|w-z| + \delta(w)} - \frac{1}{v_1(\Delta(w, c\delta(w)))} \int_{\Delta(w,c\delta(w))} \log|\lambda - a| dv_1(\lambda) + \log \delta(w) \\ = \;& \log \frac{|z-a|}{|w-z| + \delta(w)} - \log|w-a| + \log \delta(w) \\ = \;& \log \frac{|z-a|\delta(w)}{(|w-z| + \delta(w))|w-a|}. \end{aligned}$$

Let m, M be the minimal and maximal of the two numbers $|w - z| + \delta(w)$ and $|w - a|$. Since $|z - a| \leq |z - w| + |w - a| \leq M + m$, we have $M \geq 1/2|z - a|$. Also, since $|w - a| \geq c\delta(w)$ by our assumption and $|z - w| + \delta(w) \geq \delta(w) > c\delta(w)$, we have $m \geq c\delta(w)$. Therefore

$$\frac{|z-a|\delta(w)}{(|w-z| + \delta(w))|w-a|} \leq 2/c.$$

Hence

$$\begin{aligned}&\log\frac{|z-a|}{|w-z|+\delta(w)} - \frac{1}{v_1(\Delta(w,c\delta(w)))}\int_{\Delta(w,c\delta(w))}\log|\lambda-a|dv_1(\lambda)+\log\delta(w)\\ \leq\;& \log 2-\log c.\end{aligned}$$

This completes the proof for case 1.

Case 2: $a \in \Delta(w, c\delta(w))$. First, we make a change of variable. It is easy to verify that

$$\begin{aligned}&\frac{1}{v_1(\Delta(w,c\delta(w)))}\int_{\Delta(w,c\delta(w))}\log|\lambda-a|dv_1(\lambda)\\ =\;& \frac{1}{\pi}\int_{\mathbb{D}}\log\left|\eta-\frac{a-w}{c\delta(w)}\right|dv_1(\eta)+\log\delta(w)+\log c.\end{aligned}$$

In general, for $a \in \mathbb{D}$,

$$\begin{aligned}&\int_{\mathbb{D}}\log|\eta-a|dv_1(\eta)\\ \geq\;& \int_{\{\eta\in\mathbb{C}:|\eta-a|<1\}}\log|\eta-a|dv_1(\eta)\\ =\;& \int_{\mathbb{D}}\log|\eta|dv_1(\eta)\\ \geq\;& -\frac{\pi}{2}.\end{aligned}$$

Therefore

$$\frac{1}{\pi}\int_{\mathbb{D}}\log|\eta-a|dv_1(\eta)\geq -1/2.$$

So

$$\frac{1}{v_1(\Delta(w,c\delta(w)))}\int_{\Delta(w,c\delta(w))}\log|\lambda-a|dv_1(\lambda)\geq\log\delta(w)+\log c-1/2.$$

On the other hand,

$$\log\frac{|z-a|}{|w-z|+\delta(w)}\leq\log\frac{|z-a|}{|w-z|+|w-a|}\leq 0.$$

So

$$\begin{aligned}\log \frac{|z-a|}{|w-z|+\delta(w)} &\le \frac{1}{v_1(\Delta(w, c\delta(w)))} \int_{\Delta(w,c\delta(w))} \log|\lambda - a| dv_1(\lambda) \\ &\quad - \log \delta(w) - \log c + 1/2.\end{aligned}$$

Taking $C = \log 2 + 1/2$ will complete the proof. □

More generally, for a polynomial of one variable, it is easy to prove the following.

Lemma 3.2.5. *There exists $C > 0$ such that for any polynomial $p \in \mathbb{C}[z]$ of degree d, $0 < c < 1$ and any $z, w \in \mathbb{D}$, we have*

$$\begin{aligned}\log \frac{|p(z)|}{(|z-w|+\delta(w))^d} &\le \frac{1}{v_1(\Delta(w, c\delta(w)))} \int_{\Delta(w,c\delta(w))} \log|p(\lambda)| dv_1(\lambda) \\ &\quad -d \log \delta(w) - d \log c + dC.\end{aligned}$$

Proof. The proof is immediate once we write $p(z) = a_0(z-a_1)\cdots(z-a_d)$ and apply Lemma 3.2.4. □

Remark 3.2.6. *In general, if a polynomial p of degree d is defined on a disc $\Delta(\alpha, r)$, then the polynomial*

$$f(z) = p(rz + \alpha)$$

has the same degree with p and is defined on $\mathbb{D}$. For $z \in \Delta(\alpha, r)$ and $\Delta(w, s) \subseteq \Delta(\alpha, r)$, $\frac{z-\alpha}{r} \in \mathbb{D}$, $\Delta(\frac{w-\alpha}{r}, s/r) \subseteq \mathbb{D}$. Apply Lemma 3.2.5 to f, we get

$$\begin{aligned}& \log \frac{|p(z)|}{(|z-w|/r + d(w, \partial\Delta(\alpha, r))/r)^d} \\ = \; & \log \frac{|f(\frac{z-\alpha}{r})|}{(|\frac{z-\alpha}{r} - \frac{w-\alpha}{r}| + \delta(\frac{w-\alpha}{r}))^d} \\ \le \; & \frac{1}{v_1(\Delta(\frac{w-\alpha}{r}, s/r))} \int_{\Delta(\frac{w-\alpha}{r}, s/r)} \log|f(\lambda)| dv_1(\lambda) - d \log s/r + dC \\ = \; & \frac{1}{v_1(\Delta(w, s))} \int_{\Delta(w,s)} \log|p(\lambda)| dv_1(\lambda) - d \log s/r + dC.\end{aligned}$$

We will use Lemma 3.2.5 in this form in the proof of Theorem 3.2.1.

Note also that Lemma 3.2.5 holds trivially for $p \equiv 0$ *with any positive number* d *since the left side is* $-\infty$.

We are proving Theorem 3.2.1 by induction on the dimension. In order to do the induction, we need to choose the basis carefully.

Lemma 3.2.7. *Suppose* $h \in Hol(U)$*, where* U *is an open neighborhood of* $0 \in \mathbb{C}^n$*,* $h \not\equiv 0$*, then there exists an orthonormal basis* $\{e_1, \ldots, e_n\}$ *of* $\mathbb{C}^n$*, such that for every* $i \in \{1, \cdots, n\}$*,* h *is not identically zero on* $\mathbb{C}e_i \cap U$.

Lemma 3.2.7 can be implied by [8, Lemma 2, Page 33]. To avoid employing more terminologies we give a straightforward proof. We thank Hui Dan for suggesting this proof to us.

Proof. In the case when $n = 1$, the conclusion is obvious. In the case when $n = 2$, notice that $\langle (z_1, z_2), (\bar{z}_2, -\bar{z}_1) \rangle = 0$ for all pairs (z_1, z_2). Let

$$f(z_1, z_2) = h(z_1, z_2)\overline{h(\bar{z}_2, -\bar{z}_1)},$$

then f is a product of two non-zero holomorphic functions. Thus f is not identically 0. Pick any $(z_1, z_2) \neq 0$ so that $f(z) \neq 0$ and normalize $\{(z_1, z_2), (\bar{z}_2, -\bar{z}_1)\}$ into an orthonormal basis. This will satisfy our condition.

Next we prove the general case by induction, suppose we have shown the result for $U \subseteq \mathbb{C}^{n-1}$. Now for $U \subset \mathbb{C}^n$, pick $z \neq 0$ so that $h(z) \neq 0$. Pick a two dimensional subspace $L \subset \mathbb{C}^n$ containing z, then $h|_L \not\equiv 0$. Since $\dim L = 2$, by the previous argument we have orthonormal v_1 and $v_2 \in L$ so that h is not identically 0 on $\mathbb{C}v_1$ and $\mathbb{C}v_2$. Now consider $L' = v_1^{\perp}$, since $v_2 \in L'$, $h|_{L'} \not\equiv 0$. By induction, we have orthonormal $\{e_2, \ldots, e_n\} \subseteq L'$ such that h is not identically 0 on $\mathbb{C}e_i$, $i = 2, \ldots, n$. The set $\{v_1, e_2, \ldots, e_n\}$ is the desired basis. This completes the proof. □

Notations: Under the setting of Theorem 3.2.1, and assume further that h is not identically 0 on the complex $n - 1$ dimensional affine space passing through ζ and tangent to Ω at ζ. Applying

Lemma 3.2.7 on this $n-1$ dimensional affine space we get $n-1$ vectors $\{e_1^\zeta, \dots, e_{n-1}^\zeta\}$ such that together with the unit normal vector at ζ, they form an orthonormal basis for $\mathbb{C}^n$, and that h is not identically 0 on each complex line $\zeta + \mathbb{C}e_i^\zeta$, $i = 1, \dots, n-1$. We denote e_n^ζ to be the unit normal vector at ζ.

For any w in a sufficiently small neighborhood of ζ, let e_n^w be the unit normal vector at $\pi(w)$, then e_n^w depend continuously on w and the definition is consistent at the point ζ. Fix e_n^w, use the Gram-Schmidt method on $\{e_1^\zeta, \dots, e_{n-1}^\zeta\}$ to obtain a new orthonormal basis, denoted by $\{e_1^w, \dots, e_n^w\}$. For any n-tuple of complex numbers $\xi = (\xi_1, \dots, \xi_n)$, use ξ_w to denote the point in $\mathbb{C}^n$ having coordinate ξ under the basis $\{e_1^w, \dots, e_n^w\}$, i.e., $\xi_w = \sum_{i=1}^n \xi_i e_i^w$.

In the case when h is identically 0 on the $n-1$ dimensional affine space at ζ tangent to Ω at ζ, we can factor a polynomial out of h, and the rest is not identically 0 on the affine space. Indeed, assume for the moment that the normal vector at ζ is $(0, \dots, 0, 1)$, and that $\zeta = 0$, then $h(z) = z_n^m h'(z)$ for some positive integer m, in a neighborhood of ζ, where h' satisfies our assumption.

Lemma 3.2.8. *Under the assumptions of Theorem 3.2.1, and assume further that h is not identically* 0 *along the normal direction at ζ, there exist a neighborhood V of ζ and constants $\delta > 0$, $0 < m < M$ and $k > 0$ such that for any $i = 1, \dots, n$ and any $w \in V$, we have $\delta(w) < \delta^2$. Also, whenever $|\xi_w - w| < \delta$, we have decompositions*

$$h(\xi_w) = W_i^w(\xi)\varphi_i^w(\xi).$$

Here the functions W_i^w and φ_i^w are such that when we fix all variables but ξ_i, the function W_i^w is either identically 0 *or a polynomial in ξ_i of degree less than k. Moreover, $m < |\varphi_i^w(\xi)| < M$.*

We write $d(w, i, \xi)$ for the degree of the polynomial $W_i^w(\xi)$. If we let $d(w, i, \xi) = 0$ when the polynomial is identically 0, then Lemma 3.2.8 says $d(w, i, \xi) \leq k$. In the subsequent discussion, when no confusion is caused, we simply write d for $d(w, i, \xi)$.

Proof. First, notice that the condition $\delta(w) < \delta^2$ can be easily satisfied by simply shrinking the neighborhood V. We show the rest can also be achieved.

According to our discussion proceeding Lemma 3.2.8, h can be written as a product of a polynomial with some h' such that h' is not identically 0 on the complex $n-1$ dimensional affine space tangent to Ω at ζ. So we get a set of parameterized basis $\{e_i^w\}_{i=1}^n$ where w ranges over a small neighborhood V_1 of ζ.

We prove the lemma for h', and the result for h follows immediately. For simplicity, write h for h'. The case when $h(\zeta) \neq 0$ is obvious, we assume $h(\zeta) = 0$.

By our construction, h is not identically 0 on each complex line $\zeta + \mathbb{C}e_i^\zeta$. The proofs for all i's are the same. For convenience, we only prove the case $i = 1$.

Without loss of generality, assume $\zeta = 0$. Since zero points in dimension 1 are isolated, we can take $r > 0$ small enough so that the function $h((z_1, 0')_\zeta)$ has no zero points other than $z_1 = 0$ on the closed disc $\{(z_1, 0')_\zeta : |z_1| \leq r\}$. Denote m_1 for its degree. By continuity, there exists $\epsilon > 0$ such that whenever $|(\xi_2, \ldots, \xi_n)| < \epsilon$ and $|w - \zeta| < \epsilon$, $h((z_1, \xi_2, \ldots, \xi_n)_w)$ has no zeros on the closed ring $\{(z_1, \xi_2, \ldots, \xi_n)_w : r/2 \leq |z_1| \leq r\}$. By Rouché's Theorem, the function has exactly m_1 zeros (counting multiplicity) in the disc $\{(z_1, \xi_2, \ldots, \xi_n)_w : |z_1| < r/2\}$.

Therefore, for such $\xi' := (\xi_2, \ldots, \xi_n)$ and w we have decomposition

$$h((z_1, \xi')_w) = W_1^w(z_1, \xi')\varphi_1^w(z_1, \xi').$$

Here W_1^w is a monic polynomial of degree m_1 in z_1, with zeros inside $\{z_1 : |z_1| < r/2\}$ and φ_1^w is holomorphic in z_1 and zero-free on $\{z_1 : |z_1| \leq r\}$. In fact, for $|z_1| = r$,

$$|\varphi_1^w(z_1, \xi')| = \frac{|h((z_1, \xi')_w)|}{|W_1^w(z_1, \xi')|}.$$

Since

$$(\frac{1}{2}r)^{m_1} \leq |W_1^w(z_1, \xi')| \leq (\frac{3}{2}r)^{m_1}$$

and $|h((z_1, \xi')_w)|$ can be taken uniformly bounded and bounded away from 0 for all ξ' and w, by possibly shrinking V_1 and ϵ. So there exists $0 < m < M$ such that $m \leq |\varphi_1^w(z_1, \xi')| \leq M$ on the circle

$\{z_1 : |z_1| = r\}$. By the Maximum Principle, it also holds for $|z_1| < r$.

Shrink ϵ to make $\epsilon < r$. Now for $|\xi_1| < r$, $|\xi'| < \epsilon$ and $|w - \xi| < \epsilon$, we have the above decomposition and φ_i^w has the above estimate. Now take $V = B(\zeta, \epsilon/2)$ and $\delta = \epsilon/2$, if $|\xi_w - w| < \delta$ and $w \in V$,

$$|\xi| = |\xi_w| \leq |\xi_w - w| + |w| < \epsilon < r.$$

This completes our proof when $h = h'$ and $i = 1$. For the general case, one easily sees that by modifying the constants the decomposition works for all i. We remind the reader that from h' to h, when we multiply a polynomial to W_i^w, the resulting polynomial might be zero on certain complex lines. But this will not influence our final estimate. □

Proof of Theorem 3.2.1. First, we could replace U by a smaller neighborhood so that $(z, w) \in K_\delta$ for any $z, w \in U \cap \Omega$ for some $\delta > 0$ so that the all the previous lemmas involving K_δ hold.

Note that $v_n(E(w, 1)) \approx \delta(w)^{n+1} \approx |r(w)|^{n+1}$, it is sufficient to show that

$$|h(z)f(w)| \lesssim \frac{|X(z, w)|^N}{|r(w)|^N} \frac{1}{v_n(E(w, 1))} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv_n(\lambda).$$

First, consider the case where $h(\zeta) \neq 0$. Since $|X(z, w)| \gtrsim \delta(w) \gtrsim |r(w)|$, we only need to show

$$|h(z)f(w)| \lesssim \frac{1}{v_n(E(w, 1))} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv_n(\lambda).$$

Take a neighborhood $V_1 \subseteq U$ of ζ so that $0 < m < |h| < M$ on V_1, for some constant m, M. Take $V \subseteq V_1$ and $\delta > 0$ so that $B(w, \delta) \subseteq V_1$ for $w \in V$. By Lemma 3.1.4, the size of $E(w, 1)$ tends to 0 as w approaches $\partial\Omega$, so we can shrink V so that $E(w, 1) \subseteq V_1$ whenever $w \in V \cap \Omega$. For $w \in V \cap \Omega$,

$z \in B(w,\delta) \cap \Omega$,

$$\begin{aligned}
|f(w)| &\leq \frac{1}{v_n(P_w(a_1\delta(w), b_1\delta(w)^{1/2}))} \int_{P_w(a_1\delta(w), b_1\delta(w)^{1/2})} |f(\lambda)| dv_n(\lambda) \\
&\leq \frac{v_n(E(w,1))}{v_n(P_w(a_1\delta(w), b_1\delta(w)^{1/2}))} \frac{1}{v_n(E(w,1))} \int_{E(w,1)} |f(\lambda)| dv_n(\lambda) \\
&\lesssim \frac{1}{v_n(E(w,1))} \int_{E(w,1)} |f(\lambda)| dv_n(\lambda).
\end{aligned}$$

Since $0 < m < |h(z)| < M$ for $z \in V_1$, we have

$$\begin{aligned}
|h(z)f(w)| &\leq M|f(w)| \\
&\lesssim \frac{M}{v_n(E(w,1))} \int_{E(w,1)} |f(\lambda)| dv_n(\lambda) \\
&\leq \frac{M}{m v_n(E(w,1))} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv_n(\lambda) \\
&\lesssim \frac{1}{v_n(E(w,1))} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv_n(\lambda).
\end{aligned}$$

This completes the proof for the case $h(\zeta) \neq 0$.

Now assume $h(\zeta) = 0$. First, we show that we could assume h to be not identically 0 along the normal direction at ζ.

Claim: There is a biholomorphic map Φ defined on a neighborhood of $\overline{\Omega}$ such that $h \circ \Phi$ is not identically 0 along the complex normal direction of $\Phi(\Omega)$ at the point $\Phi(\zeta)$.

Assume the claim and suppose we have proved the theorem in the case when h is not identically 0 along the complex normal direction. Then the result holds for the function $h' = h \circ \Phi^{-1}$ defined in a neighborhood $U' = \Phi(U)$ of $\zeta' = \Phi(\zeta)$, for the domain $\Omega' = \Phi(\Omega)$. Thus we have $V' \subseteq U'$, $\delta' > 0$ and $N > 0$ as stated in the theorem. Let $V = \Phi^{-1}(V')$, then we can find $\delta > 0$ so that $\Phi(B(w,\delta)) \subseteq B(\Phi(w), \delta')$ for any $w \in V$. For $f \in Hol(E(w,1))$, since biholomorphic maps preserve the Kobayashi distance, $f' = f \circ \Phi^{-1} \in Hol(E(\Phi(w),1)$. So for any $w \in V \cap \Omega$ and $z \in B(w,\delta) \cap \Omega$ we have

$$|h'(\Phi(z))f'(\Phi(w))| \lesssim \frac{|X'(\Phi(z), \Phi(w))|^N}{|r \circ \Phi^{-1}(\Phi(w))|^{N+n+1}} \int_{E(\Phi(w),1)} |h'(\lambda)f'(\lambda)| dv_n(\lambda).$$

Since Φ is biholomorphic in a neighborhood of $\overline{\Omega}$, the absolute value of its real Jacobian is both bounded above and away from 0. Combining this with Lemma 3.1.10, we get

$$|h(z)f(w)| = |h'(\Phi(z))f'(\Phi(w))| \lesssim \frac{|X(z,w)|^N}{|r(w)|^{N+n+1}} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv_n(\lambda).$$

This is our desired result.

Now we prove the claim. For any $r > 0$ one can take a ball B in $\mathbb{C}^n$ of radius r that is tangent to Ω at the point ζ. If we make r small enough we can also assume that the center of B is contained in $\Omega \cap U$. By doing a translation and an invertible linear transformation (which are biholomorphic maps) we can assume that B is the unit ball in $\mathbb{C}^n$. Now $0 \in \Omega \cap U$, so h is defined in a neighborhood of 0. Since h is not identically 0, it is not identically 0 in any open set. Since Ω is bounded, we can find an α close enough to 0 so that $h(\alpha) \neq 0$ and the automorphism of $\mathbb{B}_n$ defined by

$$\varphi_\alpha(z) = \frac{\alpha - P_\alpha(z) - (1 - |\alpha|^2)^{1/2} Q_\alpha(z)}{1 - \langle z, \alpha \rangle}$$

is defined and biholomorphic in a neighborhood of $\overline{\Omega}$. Recall that φ_α has properties $\varphi_\alpha(0) = \alpha$ and $\varphi_\alpha^2 = id$ (cf. [40]). It is easy to show that the domains $\varphi_\alpha(\Omega)$ and $\varphi_\alpha(\mathbb{B}_n) = \mathbb{B}_n$ are tangent at $\varphi_\alpha(\zeta)$. Therefore they have the same complex normal direction at $\varphi_\alpha(\zeta)$, which is just the one determined by the points 0 and $\varphi_\alpha(\zeta)$. Since $h \circ \varphi_\alpha^{-1}(0) = h(\alpha) \neq 0$, $h \circ \varphi_\alpha$ is not identically 0 along the complex normal direction. This proves the claim.

Now we prove the theorem assuming $h(\zeta) = 0$ and h is not identically 0 along the complex normal direction. At this point, we could apply Lemma 3.2.8 to get decompositions

$$h(\xi_w) = W_i^w(\xi) \varphi_i^w(\xi)$$

with stated properties.

By Lemma 3.1.4, there exist $a_1, a_2, b_1, b_2 > 0$ for w close enough to $\partial\Omega$, such that

$$P_w(a_1\delta(w), b_1\delta(w)^{1/2}) \subseteq E(w,1) \subseteq P_w(a_2\delta(w), b_2\delta(w)^{1/2}).$$

Since these sets have comparable volume measures, we only need to prove Theorem 3.2.1 with $E(w,1)$ replaced by the polydisk above on the left.

We will use induction to prove the lemma.

Let V, δ, k, m, M be defined as in Lemma 3.2.8. Let $a = \min\{a_1, b_1, 1\}$. Fix $w \in V \cap \Omega$, in the rest of the proof, we will always use the orthonormal basis $\{e_i^w\}$ instead of the canonical one. To simplify notation, we omit any w in the subscript or superscript. Therefore ξ means ξ_w and W_i means W_i^w, etc.. We could also do a translation to make $w = 0$.

For $z \in B(w, \frac{a\delta}{4n})$, suppose $z = (z_1, \ldots, z_n)$. Since $|z-w| < \delta$, the polynomial in λ, $W_1(\lambda, z_2, \ldots, z_n)$ is well defined. Since $|z_1| \leq |z| = |z-w|$, the point z_1 is in the disc $\Delta(0, |z-w| + \delta(w)^{1/2})$. Also. $\Delta(0, \frac{a}{4n}\delta(w)^{1/2}) \subseteq \Delta(0, |z-w| + \delta(w)^{1/2})$. By Lemma 3.2.5 and the Remark after it,

$$\begin{aligned}
& \log \frac{|W_1(z_1, \ldots, z_n)|}{(|z_1|/(|z-w| + \delta(w)^{1/2}) + 1)^d} \\
\leq\ & \frac{1}{v_1(\Delta(0, \frac{a}{4n}\delta(w)^{1/2}))} \int_{\Delta(0, \frac{a}{4n}\delta(w)^{1/2})} \log |W_1(\lambda_1, z_2, \ldots, z_n)| dv_1(\lambda_1) \\
& -d \log \frac{\frac{a}{4n}\delta(w)^{1/2}}{|z-w| + \delta(w)^{1/2}} + dC.
\end{aligned}$$

Since the denominator on the left side is greater than 1 and since $|z-w| + \delta(w)^{1/2} \lesssim |X(z,w)|^{1/2}$, by changing the constant C we have

$$\begin{aligned}
& \log |W_1(z_1, \ldots, z_n)| \\
\leq\ & \frac{1}{v_1(\Delta(0, \frac{a}{4n}\delta(w)^{1/2}))} \int_{\Delta(0, \frac{a}{4n}\delta(w)^{1/2})} \log |W_1(\lambda_1, z_2, \ldots, z_n)| dv_1(\lambda_1) \\
& -d/2 \log \frac{\delta(w)}{|X(z,w)|} + dC.
\end{aligned}$$

Now for $\lambda_1 \in \Delta(0, \frac{a}{4n}\delta(w)^{1/2})$,

$$|(\lambda_1, z_2, \ldots, z_n) - w| \le |\lambda_1| + |z - w| \le \frac{\delta(w)^{1/2}}{4n} + \frac{\delta}{4n} < \frac{\delta}{2n} < \delta.$$

So $\varphi_1(\lambda_1, z_2, \ldots, z_n)$ is well defined and bounded below and above by $0 < m < M$. Therefore

$$\begin{aligned}
& \log|h(z_1, \ldots, z_n)| \\
= \; & \log|W_1(z_1, \ldots, z_n)| + \log|\varphi_1(z_1, \ldots, z_n)| \\
\le \; & \frac{1}{v_1(\Delta(0, \frac{a}{4n}\delta(w)^{1/2}))} \int_{\Delta(0, \frac{a}{4n}\delta(w)^{1/2})} \log|W_1(\lambda_1, z_2, \ldots, z_n)| dv_1(\lambda_1) \\
& -d/2 \log \frac{\delta(w)}{|X(z, w)|} + dC + \log M \\
\le \; & \frac{1}{v_1(\Delta(0, \frac{a}{4n}\delta(w)^{1/2}))} \int_{\Delta(0, \frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda_1, z_2, \ldots, z_n)| dv_1(\lambda_1) \\
& -d/2 \log \frac{\delta(w)}{|X(z, w)|} + dC + \log \frac{M}{m} \\
\le \; & \frac{1}{v_1(\Delta(0, \frac{a}{4n}\delta(w)^{1/2}))} \int_{\Delta(0, \frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda_1, z_2, \ldots, z_n)| dv_1(\lambda_1) \\
& -k/2 \log \frac{\delta(w)}{|X(z, w)|} + kC + \log \frac{M}{m}.
\end{aligned}$$

Here the last inequality is because $d = d(w, 1, z) \le k$. Since $\frac{|X(z,w)|}{\delta(w)} \gtrsim 1$, $\frac{M}{m} \ge 1$, therefore by enlarging the constant C we could make the sum of the coefficients of d positive. Since $d \le k$, we have the last inequality.

Now, for $\lambda_1 \in \Delta(0, \frac{a}{4n}\delta(w)^{1/2})$, we have shown that $|(\lambda_1, z_2, \ldots, z_n) - w| < \frac{\delta}{2n} < \delta$, also notice that for $\lambda_2 \in \Delta(0, \frac{a}{4n}\delta(w)^{1/2})$,

$$|(\lambda_1, \lambda_2, z_3, \ldots, z_n) - w| \le |\lambda_1| + |\lambda_2| + |z - w| \le \frac{2\delta(w)^{1/2}}{4n} + \frac{\delta}{4n} \le \frac{3\delta}{4n} < \delta,$$

This means that we could replace z by $(\lambda_1, z_2, \ldots, z_n)$ and repeat the above argument on the second index. We get

$$\begin{aligned}
&\log|h(\lambda_1, z_2, \dots, z_n)| \\
\leq\ & \frac{1}{v_1(\Delta(0, \frac{a}{4n}\delta(w)^{1/2}))}\int_{\Delta(0,\frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda_1, \lambda_2, z_3, \dots, z_n)| dv_1(\lambda_2) \\
& -k/2 \log\frac{\delta(w)}{|X(z,w)|} + kC + \log\frac{M}{m}.
\end{aligned}$$

In general, for $\lambda_i \in \Delta(0, \frac{a}{4n}\delta(w)^{1/2})$, $i = 1, \dots, n-1$,

$$|(\lambda_1, \dots, \lambda_i, z_{i+1}, \dots, z_n) - w| < \frac{(i+1)\delta}{4n} < \delta/4,$$

So we can repeat the arguments above for each of the first $n-1$ indices to get

$$\begin{aligned}
&\log|h(\lambda_1, \dots, \lambda_i, z_{i+1}, \dots, z_n)| \\
\leq\ & \frac{1}{v_1(\Delta(0, \frac{a}{4n}\delta(w)^{1/2}))}\int_{\Delta(0,\frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda_1, \dots, \lambda_{i+1}, z_{i+2}, \dots, z_n)| dv_1(\lambda_{i+1}) \\
& -k/2 \log\frac{\delta(w)}{|X(z,w)|} + kC + \log\frac{M}{m}.
\end{aligned}$$

Combining the inequalities in each step, we get

$$\begin{aligned}
&\log|h(z_1, \dots, z_n)| \\
\leq\ & \frac{1}{v_{n-1}(\Delta(0, \frac{a}{4n}\delta(w)^{1/2})^{n-1})}\int_{\Delta(0,\frac{a}{4n}\delta(w)^{1/2})^{n-1}} \log|h(\lambda', z_n)| dv_{n-1}(\lambda') \\
& -\frac{k(n-1)}{2} \log\frac{\delta(w)}{|X(z,w)|} + k(n-1)C + (n-1)\log\frac{M}{m}.
\end{aligned}$$

The $n-th$ index represents the normal direction at w, we handle it a little differently.

We have already showed that $|(\lambda', z_n) - w| < \frac{\delta}{4}$. So the decomposition in Lemma 3.2.8 still makes sense. For the polynomial $W_n(\lambda', \lambda_n)$, apply Lemma 3.2.5 on the disc $\Delta(1, 1 + |z_n| + \delta(w))$, taking average on $\Delta(0, \frac{a}{4}\delta(w))$. Clearly $z_n \in \Delta(1, 1 + |z_n| + \delta(w))$. We get

$$
\begin{aligned}
& \log \frac{|W_n(\lambda', z_n)|}{(2|z_n| + \delta(w))^d} \\
\leq \; & \frac{1}{v_1(\Delta(0, \frac{a}{4}\delta(w)))} \int_{\Delta(0, \frac{a}{4}\delta(w))} \log |W_n(\lambda', \lambda_n)| dv_1(\lambda_n) - d \log(\frac{a}{4}\delta(w)) + dC \\
= \; & \frac{1}{v_1(\Delta(0, \frac{a}{4}\delta(w)))} \int_{\Delta(0, \frac{a}{4}\delta(w))} \log |W_n(\lambda', \lambda_n)| dv_1(\lambda_n) - d \log \delta(w) + dC.
\end{aligned}
$$

Note that the constant C has changed in the process.

Again, since $2|z_n| + \delta(w) \lesssim |X(z, w)|$, we have

$$
\begin{aligned}
& \log |W_n(\lambda', z_n)| \\
\leq \; & \frac{1}{v_1(\Delta(0, \frac{a}{4}\delta(w)))} \int_{\Delta(0, \frac{a}{4}\delta(w))} \log |W_n(\lambda', \lambda_n)| dv_1(\lambda_n) \\
& - d \log \delta(w) + d \log |X(z, w)| + dC \\
\leq \; & \frac{1}{v_1(\Delta(0, \frac{a}{4}\delta(w)))} \int_{\Delta(0, \frac{a}{4}\delta(w))} \log |W_n(\lambda', \lambda_n)| dv_1(\lambda_n) \\
& + k \log \frac{|X(z, w)|}{\delta(w)} + kC.
\end{aligned}
$$

And therefore

$$
\begin{aligned}
& \log |h(\lambda', z_n)| \\
\leq \; & \frac{1}{v_1(\Delta(0, \frac{a}{4}\delta(w)))} \int_{\Delta(0, \frac{a}{4}\delta(w))} \log |h(\lambda', \lambda_n)| dv_1(\lambda_n) \\
& + k \log \frac{|X(z, w)|}{\delta(w)} + kC + \log \frac{M}{m}.
\end{aligned}
$$

Again, substituting it into the previous estimate we get

$$
\begin{aligned}
& \log |h(z)| \\
\leq \; & \frac{1}{v_n(\Delta(0, \frac{a}{4n}\delta(w)^{1/2})^{n-1} \times \Delta(0, \frac{a}{4}\delta(w)))} \int_{\Delta(0, \frac{a}{4n}\delta(w)^{1/2})^{n-1} \times \Delta(0, \frac{a}{4}\delta(w))} \log |h(\lambda)| dv_n(\lambda) \\
& + (k + \frac{k(n-1)}{2}) \log \frac{|X(z, w)|}{\delta(w)} + knC + n \log \frac{M}{m}.
\end{aligned}
$$

Note that $\Delta(0, \frac{a}{4n}\delta(w)^{1/2})^{n-1} \times \Delta(0, \frac{a}{4}\delta(w)) = P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})$. Combining the constants, we get

$$\begin{aligned} \log|h(z)| &\leq \frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda)| dv_n(\lambda) && (3.5) \\ &\quad + N \log \frac{|X(z,w)|}{\delta(w)} + C. && (3.6) \end{aligned}$$

Here $N = (k + \frac{k(n-1)}{2})$. The polydisc $P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})$ is obviously contained in the polydisc $P_w(a_1\delta(w), b_1\delta(w)^{1/2})$. Since $\log|f|$ is pluri-subharmonic, we have

$$\log|f(w)| \leq \frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log|f(\lambda)| dv_n(\lambda).$$

Adding them up we have

$$\begin{aligned} &\log|h(z)f(w)| \\ \leq\ & \frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda)f(\lambda)| dv_n(\lambda) \\ & + N \log \frac{|X(z,w)|}{\delta(w)} + C. \end{aligned}$$

Apply the Jensen's inequality, we get

$$|h(z)f(w)| \lesssim \frac{|X(z,w)|^N}{\delta(w)^N} \frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} |h(\lambda)f(\lambda)| dv_n(\lambda).$$

Finally, since $P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}) \subseteq E(w,1)$ and $v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})) \approx v_n(E(w,1)) \approx \delta(w)^{n+1}$, we have for any $w \in V$ and $z \in B(w, \frac{a\delta}{16n})$

$$\begin{aligned} &|h(z)f(w)| \\ \lesssim\ & \frac{|X(z,w)|^N}{\delta(w)^{N+n+1}} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv_n(\lambda) \\ \approx\ & \frac{|X(z,w)|^N}{|r(w)|^{N+n+1}} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv_n(\lambda). \end{aligned}$$

This completes the proof. ☐

Proof of Theorem 3.2.3. As usual, we will not keep track of the constant C in the estimate. So the notation C may denote different constants in the proof.

A key step in the proof of Theorem 3.2.1 is to obtain inequality 3.5. We could apply the proof of Lemma 3.2.1 to every point $\zeta \in \partial\Omega$. Since Ω is bounded, $\partial\Omega$ is compact. Thus we get a finite cover $\{V_i\}_{i=1}^m$ of $\partial\Omega$ where each V_i corresponds to some point ζ_i in $\partial\Omega$. It is easy to see that one can adjust so that the same set of constants work for all points. That is to say, there exist a neighborhood $V = \cup V_i$ of $\partial\Omega$ and constants $\delta > 0$, $N > 0$, $C > 0$ such that $\forall w \in V$, $|z - w| < \delta$,

$$\begin{aligned}\log|h(z)| \quad \leq \quad & \frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda)| dv_n(\lambda) \\ & + N \log \frac{|X(z,w)|}{\delta(w)} + C.\end{aligned}$$

Note that this includes pairs $(z, w) \in K_{\delta'}$ for some $\delta' > 0$. We are left with the case when $(z, w) \notin K_{\delta'}$. For such pairs, $F(z, w)$ is bounded below and above.

Fix finite number of points $z_1, \ldots, z_k \in \Omega$ so that $h(z_i) \neq 0$ and for any $w \in V$ there exists some z_i so that $|z_i - w| < \delta$. Also, $|X(z, w)|$ is bounded above for all $z, w \in \Omega$. Therefore for any $w \in V$,

$$\begin{aligned} & \frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda)| dv_n(\lambda) \\ & + N \log \frac{1}{\delta(w)} + C \\ \geq \quad & \frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda)| dv_n(\lambda) \\ & + N \log \frac{|X(z_i,w)|}{\delta(w)} + C \\ \geq \quad & \log|h(z_i)| \geq C.\end{aligned}$$

By compactness, for $w \in \Omega \backslash V$,

$$\frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log|h(\lambda)| dv_n(\lambda) + N \log \frac{1}{\delta(w)} > C$$

for some constant C. For $(z,w) \notin K_{\delta'}$, $F(z,w)$ is bounded below, Thus

$$\frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log |h(\lambda)| dv_n(\lambda) + N \log \frac{F(z,w)}{\delta(w)} > C$$

for some constant C. Since h is bounded above on Ω, there is some constant C such that

$$\begin{aligned} & \frac{1}{v_n(P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2}))} \int_{P_w(\frac{a}{4}\delta(w), \frac{a}{4n}\delta(w)^{1/2})} \log |h(\lambda)| dv_n(\lambda) + N \log \frac{F(z,w)}{\delta(w)} \\ \geq \quad & \log |h(z)| - C. \end{aligned}$$

Therefore the inequality above holds for all $z, w \in \Omega$.

The rest of the proof is as in the last part of Theorem 3.2.1. This completes the proof. □

We are ready to prove the main theorem of this section.

Theorem 3.2.9. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary, $h \in Hol(\overline{\Omega})$, then the principal submodule of the Bergman module $L^2_a(\Omega)$ generated by h is p-essentially normal for all $p > n$.*

The proof of the following lemma is the same as [16, Lemma 5.1].

Lemma 3.2.10. *Suppose $2 \leq p < \infty$ and $G(z,w)$ is a measurable function in $\Omega \times \Omega$. Let A_G be the integral operator on $L^2(\Omega)$ defined by*

$$A_G f(z) = \int_\Omega \frac{G(z,w)}{F(z,w)^{n+1}} f(w) dv_n(w).$$

If

$$\int_\Omega \int_\Omega \frac{|G(z,w)|^p}{F(z,w)^{2(n+1)}} dv_n(z) dv_n(w) < \infty,$$

then the operator A_G is in the Schatten p class $\mathcal{S}^p$.

It is usually easier to do estimations in spaces of higher weight. The following lemma allows us to replace $M^*_{z_i}$ with the ones with higher weight.

Lemma 3.2.11. *Let $M_{z_i}^*$ be the adjoint of multiplication operator on the Bergman space $L_a^2(\Omega)$, $i = 1, \ldots, n$. Let l be a positive integer and G_l be as in Lemma 3.1.6. Let G_i be the operator on $L_a^2(\Omega)$ defined by*

$$G_i f(z) = \int_\Omega \bar{w}_i f(w) G_l(z, w) |r(w)|^l dv_n(w),$$

then the operator G_i is a bounded operator on $L_a^2(\Omega)$ and $G_i - M_{z_i}^*$ is in the Schatten p class $\mathcal{S}^p$ on $L_a^2(\Omega)$, for any $p > 2n$.

Proof. The fact that G_i is bounded on $L_a^2(\Omega)$ can be obtained by Schur's test. By Lemma 3.1.6,

$$|G_i f(z)| \lesssim \int_\Omega \frac{|f(w)||r(w)|^l}{F(z, w)^{n+1+l}} dv_n(w).$$

Let $h(w) = |r(w)|^{-1/2}$, by Lemma 3.1.13 and Lemma 3.1.8,

$$\int_\Omega \frac{|r(w)|^l}{F(z, w)^{n+1+l}} h(w) dv_n(w) \lesssim h(z)$$

and

$$\int_\Omega \frac{|r(z)|^l}{F(z, w)^{n+1+l}} h(z) dv_n(w) \lesssim h(w).$$

By Schur's test, G_l defines a bounded operator on $L_a^2(\Omega)$.

Now for any $f \in L_a^2(\Omega) \subseteq L_{a,l}^2(\Omega)$,

$$\begin{aligned} & (G_i - M_{z_i}^*) f(z) \\ = & \int_\Omega \bar{w}_i f(w) (|r(w)|^l G_l(z, w) - K(z, w)) dv_n(w) \\ = & \int_\Omega (\bar{w}_i - \bar{z}_i) f(w) (|r(w)|^l G_l(z, w) - K(z, w)) dv_n(w). \end{aligned}$$

Since $|r(w)| \leq F(z, w)$, we have

$$\begin{aligned} & \left|(G_i - M_{z_i}^*)f(z)\right| \\ \lesssim & \int_\Omega |w - z||f(w)| \frac{1}{F(z, w)^{n+1}} dv_n(w). \end{aligned}$$

Now write $G(z, w) = |z - w|$ and apply Lemma 3.2.10, by Lemma 3.1.13, for any $2n < p < 2(n+1)$

$$\begin{aligned} & \int_\Omega \int_\Omega \frac{|z - w|^p}{F(z, w)^{2(n+1)}} \\ \lesssim & \int_\Omega \int_\Omega \frac{1}{F(z, w)^{2n+2-p/2}} dv_n(w) dv_n(z) \\ \lesssim & \int_\Omega |r(w)|^{p/2-n-1} dv_n(w) \\ < & \infty. \end{aligned}$$

Therefore $G_i - M_{z_i}^* \in \mathcal{S}^p$ for $2n < p < 2(n+1)$. If $p \geq 2(n+1)$, $\mathcal{S}^{2n+1} \subseteq \mathcal{S}^p$, we also have $G_i - M_{z_i}^* \in \mathcal{S}^p$. This completes the proof. □

Proof of Theorem 3.2.9. The fact that $L_a^2(\Omega)$ itself is p essentially normal for $p > n$ follows from

$$\begin{aligned} & (M_{z_i}^* M_{z_i} - M_{z_i} M_{z_i}^*) f(z) \\ = & \int_\Omega (|w_i|^2 - z_i \bar{w}_i) f(w) K(z, w) \\ = & \int_\Omega \bar{w}_i (w_i - z_i) f(w) K(z, w) \\ = & \int_\Omega (\bar{w}_i - \bar{z}_i)(w_i - z_i) f(w) K(z, w) \\ = & \int_\Omega |w_i - z_i|^2 f(w) K(z, w) \end{aligned}$$

and a similar argument as in the proof of Lemma 3.2.11. By Proposition 4.1 in [5], we only need to show that the commutator

$$[P, M_{z_i}] = PM_{z_i} - M_{z_i}P = PM_{z_i} - PM_{z_i}P = PM_{z_i}P^\perp$$

is in $\mathcal{S}^p$ for $p > 2n$. Here P is the orthogonal projection onto the principal submodule generated by h. This is equivalent to $P^\perp M_{z_i}^* P$ being in the same class, which, by Lemma 3.2.11, is equivalent to $P^\perp G_i P$ being in the same class. Functions of the form hf where $f \in L_a^2(\Omega)$ are dense in the submodule generated by h. Notice that

$$\|P^\perp G_i P(hf)\| \leq \|G_i(hf) - M_h G_i f\|.$$

We only need to estimate the norm on the right side. Using a similar trick as above, we get

$$\begin{aligned}
& G_i(hf) - M_h G_i f(z) \\
= & \int_\Omega \bar{w}_i (h(w) - h(z)) f(w) G_l(z,w) |r(w)|^l dv_n(w) \\
= & \int_\Omega (\bar{w}_i - \bar{z}_i)(h(w) - h(z)) f(w) G_l(z,w) |r(w)|^l dv_n(w) \\
= & \int_\Omega (\bar{w}_i - \bar{z}_i) h(w) f(w) G_l(z,w) |r(w)|^l dv_n(w) \\
& - \int_\Omega (\bar{w}_i - \bar{z}_i) h(z) f(w) G_l(z,w) |r(w)|^l dv_n(w).
\end{aligned}$$

So

$$\begin{aligned}
& \left| G_i(hf) - M_h G_i f(z) \right| \\
\leq & \int_\Omega |h(w) f(w)| \frac{1}{F(z,w)^{n+1/2}} dv_n(w) + \int_\Omega |h(z) f(w)| \frac{|w-z||r(w)|^l}{F(z,w)^{n+1+l}} dv_n(w) \\
= & I(hf) + II(hf).
\end{aligned}$$

We look at the second part, By Lemma 3.2.3,

$$\begin{aligned} II(hf)(z) &\le \int_\Omega \int_{E(w,1)} |h(\lambda)f(\lambda)| \frac{F(z,w)^N}{|r(w)|^{n+1+N}} \frac{|w-z||r(w)|^l}{F(z,w)^{n+1+l}} dv_n(\lambda) dv_n(w) \\ &\lesssim \int_\Omega \int_{E(w,1)} |h(\lambda)f(\lambda)| \frac{|r(w)|^{l-N-n-1}}{F(z,w)^{n+1/2+l-N}} dv_n(\lambda) dv_n(w) \\ &= \int_\Omega \int_{E(\lambda,1)} \frac{|r(w)|^{l-N-n-1}}{F(z,w)^{n+1/2+l-N}} dv_n(w) |h(\lambda)f(\lambda)| dv_n(\lambda) \\ &\lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+1/2+l-N}} |h(\lambda)f(\lambda)| dv_n(\lambda). \end{aligned}$$

where the last inequality comes from Lemma 3.1.12 and Lemma 3.1.4. We could take $l > N$ in the beginning. Since $|r(\lambda)| \le F(z,\lambda)$, we get

$$II(hf)(z) \lesssim \int_\Omega \frac{1}{F(z,\lambda)^{n+1/2}} |h(\lambda)f(\lambda)| dv_n(\lambda).$$

Altogether we have

$$\left| G_i(hf) - M_h G_i f(z) \right| \lesssim \int_\Omega |h(w)f(w)| \frac{1}{F(z,w)^{n+1/2}} dv_n(w).$$

Take $G(z,w) = F(z,w)^{1/2}$ and apply Lemma 3.2.10 as in the proof of Lemma 3.2.11, we get our desired result. This completes the proof. □

3.3 The Hardy Space

Up until now our discussion are restricted to the Bergman space. As can be seen from the proofs, our method in this chapter depend essentially on inequality (3.4). With a little adjustment we are now ready to treat the Hardy space.

Recall that for a function f on $\mathbb{C}^n$ and $z \in \mathbb{C}^n$, the Laplacian $\Delta f(z) = 4\sum_{i=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_i} f(z)$. The gradient $\nabla f(z)$ is the vector

$$\Big(\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z), \frac{\partial f}{\partial \bar{z}_1}(z), \cdots, \frac{\partial f}{\partial \bar{z}_n}(z)\Big).$$

The following two lemmas are elementary, we list them for future reference.

Lemma 3.3.1 (Green's second identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary, φ and ψ are twice continuously differentiable in a neighborhood of $\overline{\Omega}$, then*

$$\int_{\Omega} (\psi \Delta \varphi - \varphi \Delta \psi) dv = \int_{\partial\Omega} (\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n}) ds.$$

Lemma 3.3.2. *For any C^2 function f, g,*

$$\Delta(fg) = (\Delta f)g + f(\Delta g) + 4\langle \nabla f, \nabla \bar{g} \rangle.$$

For holomorphic functions we have the following estimate.

Lemma 3.3.3. *There exists a constant $C > 0$ such that for any $f \in Hol(\Omega)$ and $z \in \Omega$,*

$$|\nabla f(z)| \leq C \frac{1}{|r(z)|^{n+2}} \int_{E(z,1)} |f(w)| dv(w).$$

As a consequence, for any $l \geq 0$,

$$\|\nabla f\|_{L^2_{a,l+2}(\Omega)} \lesssim \|f - f_\Omega\|_{L^2_{a,l}(\Omega)}$$

where $f_\Omega = \int_\Omega f dv$.

Proof. By Lemma 3.1.4, $E(z, 1) \supseteq P_z(a|r(z)|, b|r(z)|^{1/2})$ for some $a, b > 0$. Let us temporarily use a local coordinate system $(w_1, \ldots, w_n)$ such that the first coordinate represents the complex normal direction at z. Let

$$\tilde{f}(w) = f(z + (a|r(z)|w_1, b|r(z)|^{1/2} w'))$$

where $w' = (w_2, \ldots, w_n)$, then $\tilde{f}$ is defined in $\mathbb{B}_n$ and

$$\begin{aligned}
|\nabla \tilde{f}(0)| &\lesssim \int_{\mathbb{B}_n} |\tilde{f}(w)| dv(w) \\
&= a^2 b^{2(n-1)} |r(z)|^{n+1} \int_{P_z(a|r(z)|, b|r(z)|^{1/2})} |f(\lambda)| dv(\lambda) \\
&\lesssim |r(z)|^{-(n+1)} \int_{E(z,1)} |f(\lambda)| dv(\lambda).
\end{aligned}$$

On the other hand,

$$|\nabla \tilde{f}(0)| \gtrsim |r(z)| |\nabla f(z)|.$$

Therefore

$$|\nabla f(z)| \lesssim |r(z)|^{-(n+2)} \int_{E(z,1)} |f(\lambda)| dv(\lambda).$$

This proves the first assertion. For the second assertion, assume without loss of generality that $f_\Omega = 0$.

$$\begin{aligned}
& \int_\Omega |\nabla f(z)|^2 |r(z)|^{l+2} dv(z) \\
\lesssim\ & \int_\Omega |r(z)|^{-2(n+2)+l+2} \Big(\int_{E(z,1)} |f(\lambda)| dv(\lambda) \Big)^2 dv(z) \\
\leq\ & \int_\Omega |r(z)|^{l-2(n+1)} \int_{E(z,1)} |f(\lambda)|^2 dv(\lambda) \cdot v(E(z,1)) dv(z) \\
\lesssim\ & \int_\Omega \int_{E(z,1)} |r(z)|^{l-(n+1)} |f(\lambda)|^2 dv(\lambda) \\
=\ & \int_\Omega \int_{E(\lambda,1)} |r(z)|^{l-(n+1)} dv(z) |f(\lambda)|^2 dv(\lambda) \\
\lesssim\ & \int_\Omega |f(\lambda)|^2 |r(\lambda)|^l dv(\lambda).
\end{aligned}$$

This completes the proof. □

As a consequence of Lemma 3.3.3 and Theorem 3.2.3, we obtain the following inequality.

Lemma 3.3.4. *Assume the same as Theorem 3.2.3, then there exists an $M > 0$ such that $\forall w, z \in \Omega$*

and $\forall f \in Hol(\Omega)$,

$$|\nabla h(z) f(w)| \lesssim \frac{F(z,w)^M}{|r(w)|^{n+2+M}} \int_{E(w,2)} |h(\lambda) f(\lambda)| dv(\lambda).$$

Proof. Since the entries of ∇h are all holomorphic and defined in a neighborhood of $\overline{\Omega}$, we can apply Theorem 3.2.3 to (the entries of) ∇h. There exists some $M > 0$ such that

$$h(z) f(w)| \lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,1)} |\nabla h(\xi) f(\xi)| dv(\xi).$$

Therefore,

$$\begin{aligned}
|\nabla h(z) f(w)| &\lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,1)} |\nabla h(\xi) f(\xi)| dv(\xi) \\
&\lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,1)} \frac{1}{|r(\xi)|^{n+2}} \int_{E(\xi,1)} |h(\eta)| dv(\eta) |f(\xi)| dv(\xi) \\
&\lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,1)} \int_{E(\xi,1)} \frac{1}{|r(\xi)|^{n+2}} \frac{F(\eta,\xi)^N}{|r(\xi)|^{n+1+N}} \\
&\quad \cdot \int_{E(\xi,1)} |h(\lambda) f(\lambda)| dv(\lambda) dv(\eta) dv(\xi) \\
&\leq \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,2)} \int_{E(\lambda,1)} \int_{E(\lambda,2)} \frac{F(\eta,\xi)^N}{|r(\xi)|^{2n+3+N}} dv(\eta) dv(\xi) |h(\lambda) f(\lambda)| dv(\lambda) \\
&\lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,2)} \frac{1}{|r(\lambda)|} |h(\lambda) f(\lambda)| dv(\lambda) \\
&\lesssim \frac{F(z,w)^M}{|r(w)|^{n+2+M}} \int_{E(w,2)} |h(\lambda) f(\lambda)| dv(\lambda).
\end{aligned}$$

□

In Section 3.2 we have used some properties of the weighted Bergman kernel. In fact, we have a stronger estimate.

Lemma 3.3.5. *For any nonnegative integer l, there exist* C^∞ *functions* G_l, H_l *on* $\overline{\Omega} \times \overline{\Omega}$ *such that*

$$K_l(z,w) = G_l(z,w)(X(z,w))^{-n-1-l} + H_l(z,w) \log(X(z,w)).$$

As a consequence,

$$|K_l(z,w)| \lesssim \frac{1}{F(z,w)^{n+1+l}}$$

$$|\nabla_z K_l(z,w)| \lesssim \frac{1}{F(z,w)^{n+2+l}}$$

Proof. The two inequalities are straightforward once the description of $K_l(z,w)$ is obtained. To prove this equation, notice that the case $l = 0$ is exactly Corollary 1.7 in [9]. To prove the general case, consider the domain

$$\Omega_l = \{(z,\xi) \in \mathbb{C}^n \times \mathbb{C}^l : \; r(z) + |\xi|^2 < 0\}.$$

The reproducing kernel of the unweighted Bergman space on Ω_l has the form

$$K_{\Omega_l}((z,\xi),(w,\eta)) = G_l'((z,\xi),(w,\eta))X_l((z,\xi),(w,\eta))^{n+1+l} + H_l'((z,\xi),(w,\eta))\log(X_l((z,\xi),(w,\eta))).$$

By the same argument as in the proof of Lemma 2.2 in [32], the function

$$K_l(z,w) = K_{\Omega_l}((z,0),(w,0))$$

gives exactly the reproducing kernel of $L^2_{a,l}(\Omega)$. It is easy to check that

$$X(z,w) = X_l((z,0),(w,0)).$$

The rest of the lemma follows by straight forward computation. This completes the proof. □

Boas and Straube [7] proved the following improved version of Poincare inequality.

Theorem 3.3.6. *Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary is locally the graph of a Hölder continuous function of exponent α, where $0 \leq \alpha \leq 1$, and suppose $1 \leq p < \infty$. Let H be a cone in $W^{1,p}_{loc}(\Omega)$ such that the closure of $H \cap W^{1,p}(\Omega,\alpha)$ in $W^{1,p}(\Omega,\alpha)$ contains no nonzero constant* *function, then there is a constant C such that*

$$||u||_p \leq C||\delta^\alpha \nabla u||$$

for every function u in H, where δ denotes the distance to the boundary of Ω.

Corollary 3.3.7. *Let* Ω *be a bounded strongly pseudoconvex domain in* $\mathbb{C}^n$ *with* C^∞ *boundary, then,*

$$||f - f_\Omega||_{L^2_a(\Omega)} \approx ||\nabla f||_{L^2_{a,2}(\Omega)}.$$

Proof. We have proved one side of inequality in Lemma 3.3.3. Now let $H = \{f - f_\Omega : f \in L^2_{a,l}\}$ and apply Theorem 3.3.6, we get the other half of inequality. This completes the proof. □

Proposition 3.3.8. *Let* Ω *be a bounded domain in* $\mathbb{C}^n$ *with* C^∞ *boundary, then*

$$||f - f_\Omega||_{H^2(\Omega)} \approx ||\nabla f||_{L^2_{a,1}(\Omega)}.$$

Proof. Assume without generality that f is holomorphic in a neighborhood of $\overline{\Omega}$. Also, since we can replace f with $f - f_\Omega$, we assume $f_\Omega = 0$.

Applying Lemma 3.3.1 for $\psi = |f|^2$ and $\varphi = r$, note that $r = 0$ on $\partial\Omega$, we have

$$\int_\Omega |f|^2 \Delta r dv + \int_\Omega (-r)\Delta |f|^2 dv = \int_{\partial\Omega} |f|^2 \frac{\partial r}{\partial n} d\sigma. \tag{3.7}$$

Notice that $r < 0$ on Ω, we have $|r| = -r$ on Ω. Applying Lemma 3.3.2, we get $\Delta|f|^2 = 2|\nabla f|^2$. Thus the second term on the left hand side of equation 3.7 is exactly $2||\nabla f||^2_{L^2_{a,1}(\Omega)}$.

Since r is the defining function of Ω, $\frac{\partial r}{\partial n} > 0$ on $\partial\Omega$, by compactness, there exists $0 < c < C$ such that $\frac{\partial r}{\partial n} > c$ on $\partial\Omega$. Therefore

$$||f||^2_{H^2(\Omega)} \approx \int_{\partial\Omega} |f|^2 \frac{\partial r}{\partial n} d\sigma.$$

Also, since r is defined and C^∞ on $\mathbb{C}^n$, by compactness, there exists $C > 0$ such that $|\Delta r| < C$

on Ω. Hence

$$\int_\Omega |f|^2 \Delta r dv \lesssim ||f||^2_{L^2_a(\Omega)}.$$

Since we have assumed $f_\Omega = 0$, we can apply Theorem 3.3.6. This gives us

$$||f||^2_{L^2_a(\Omega)} \lesssim ||\nabla f||^2_{L^2_{a,2}(\Omega)}.$$

Combining the above inequalities, we get

$$||f||_{H^2(\Omega)} \lesssim ||\nabla f||_{L^2_{a,2}(\Omega)} + ||\nabla f||_{L^2_{a,1}(\Omega)} \lesssim ||\nabla f||_{L^2_{a,1}(\Omega)}. \tag{3.8}$$

On the other hand, moving the first term of equation 3.7 to the right and then applying the above inequality we get

$$||\nabla f||_{L^2_{a,1}(\Omega)} \lesssim ||f||_{L^2_a(\Omega)} + ||f||_{H^2(\Omega)} \lesssim ||f||_{H^2(\Omega)}. \tag{3.9}$$

This completes the proof.

□

The following result is well-known on the unit ball [19], we give a proof for the general case for future reference.

Proposition 3.3.9. *Suppose T is a bounded linear operator from a closed subspace L of $H^2(\Omega)$ to $H^2(\Omega)$. If there exists a constant C such that for any $f \in L$,*

$$||Tf||_{H^2(\Omega)} \leq C||f||_{L^2_a(\Omega)},$$

then the operator TP_L belongs to the Schatten class C_p for any $p > 2n$.

Proof. Let us first prove the analogues statement on the Bergman spaces, i.e., suppose $T : L \to L^2_{a,l}(\Omega)$ is bounded and moreover,

$$||Tf||_{L^2_{a,l}(\Omega)} \lesssim ||f||_{L^2_{a,l+1}(\Omega)}, \quad \forall f \in L,$$

then $TP_L \in C_p$ for any $p > 2n$. Here P_L is the orthogonal projection onto L. By assumption, for any $f \in L$,

$$\|Tf\|_{L^2_{a,l}(\Omega)} \lesssim \|f\|_{L^2_{a,l+1}(\Omega)} = \||r|^{1/2} f\|_{L^2(\Omega,|r|^l dv)} = \|Rf\|_{L^2(\Omega,|r|^l dv)}$$

where

$$Rf(z) = \int_\Omega |r(z)|^{1/2} K_l(z,w)|r(w)|^l f(w) dv(w), f \in L^2(\Omega, |r|^l dv).$$

It is easy to construct a bounded operator A such that $TP_L = ARP_L$. We will show that $R \in C_p$, $\forall p > 2n$. Thus the assertion for TP_L follows.

Using the same interpolation technique as in the proof of Lemma 5.1 in [16], we know that it suffices to show that for $p > 2n$,

$$I_p := \int_\Omega \int_\Omega |r(z)|^{p/2} |K_l(z,w)|^2 |r(w)|^l |r(z)|^l dv(w) dv(z) < \infty.$$

By Lemma 3.3.5,

$$\begin{aligned} I_p &\lesssim \int_\Omega \int_\Omega \frac{|r(z)|^{p/2+l} |r(w)|^l}{F(z,w)^{2(n+1+l)}} dv(w) dv(z) \\ &\lesssim \int_\Omega |r(z)|^{p/2-n-1} dv(z) \\ &< \infty. \end{aligned}$$

This completes the prove for $L^2_{a,l}(\Omega)$. For $H^2(\Omega)$, let $H^2_0(\Omega)$ be the subspace of $H^2(\Omega)$ consisting of functions f with $f_\Omega = 0$. Proposition 3.3.8 tells us that the operator ∇ sends the subspace $H^2_0(\Omega)$ to a closed subspace of $L^2_{a,1}(\Omega) \otimes \mathbb{C}^n$. Therefore $\nabla^* \nabla = V + F$ where V is invertible and $\mathrm{rank} F = 1$. Consider the operator $\tilde{T}$ from ∇L to $L^2_{a,1}(\Omega) \otimes \mathbb{C}^n$ defined by

$$\tilde{T}(\nabla f) = \nabla T f, \quad \forall f \in L.$$

Then

$$\|\tilde{T} \nabla f\|_{L^2_{a,1}(\Omega) \otimes \mathbb{C}^n} \lesssim \|Tf\|_{H^2(\Omega)} \lesssim \|f\|_{H^2(\Omega)} \lesssim \|\nabla f\|_{L^2_{a,1}(\Omega) \otimes \mathbb{C}^n}.$$

Moreover

$$\|\tilde{T}\nabla f\|_{L^2_{a,1}(\Omega)\otimes\mathbb{C}^n} \lesssim \|Tf\|_{H^2(\Omega)} \lesssim \|f\|_{L^2_a(\Omega)} \lesssim \|\nabla f\|_{L^2_{a,2}(\Omega)\otimes\mathbb{C}^n}.$$

Therefore the previous arguments apply to $\tilde{T}$. We have $\tilde{T}P_{\nabla L} \in C_p$, $\forall p > 2n$. Notice that $\nabla^*\tilde{T}P_{\nabla L}\nabla P_L = \nabla^*\nabla TP_L$, we obtain that $T \in C_p$ for any $p > 2n$. This completes the proof. □

Theorem 3.3.10. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary and h is a holomorphic function defined in a neighborhood of $\overline{\Omega}$. Then for any $p > n$, the principal submodule $[h] \subseteq H^2(\Omega)$ is p-essentially normal.*

Proof. By Proposition 4.1 in [5], it suffices to show that the commutators $[P, M^*_{z_i}]$ belong to C_p for all $p > 2n$. Here P is the orthogonal projection onto the principal submodule $[h]$. It is easy to show that

$$[P, M^*_{z_i}] = -(I - P)M^*_{z_i}P.$$

For $f \in \mathcal{O}(\overline{\Omega})$, we want to find a function in P that is close to $M^*_{z_i}(hf)$. We are going to show that, for sufficiently large l, the function $hT_i(f)$, where

$$T_i(f)(z) = \int_\Omega \overline{w_i} f(w)K_l(z,w)|r(w)|^l dv(w)$$

is close enough to $M^*_{z_i}(hf)$. First, it is easy to show by Schur's test that the integral kernel $\overline{w_i}K_l(z,w)|r(w)|^l$ defines a bounded operator on $L^2_a(\Omega)$. Therefore when $f \in \mathcal{O}(\overline{\Omega}) \subseteq L^2_a(\Omega)$, we have

$$T_i f \in L^2_a(\Omega) \subseteq H^2(\Omega).$$

Therefore $hT_i(f) \in [h]$. Next let us estimate the norm of $M^*_{z_i}(hf) - hT_i(f)$. let us denote

$$S(hf)(z) = \overline{z_i}hf - hT_i(f).$$

For $\epsilon > 0$, recall that

$$\Omega_\epsilon = \{z \in \Omega : \ r(z) < -\epsilon\}.$$

By the Green's second identity,

$$\begin{aligned}
&\int_{\partial\Omega_\epsilon} |S(hf)(z)|^2 \frac{\partial r}{\partial n}(z) d\sigma_\epsilon(z) \\
= \;& \int_{\partial\Omega_\epsilon} \frac{\partial |S(hf)|^2}{\partial n}(z) r(z) d\sigma_\epsilon(z) + \int_{\Omega_\epsilon} |S(hf)(z)|^2 \Delta r(z) dv(z) \\
& - \int_{\Omega_\epsilon} \Delta(|S(hf)|^2)(z) r(z) dv(z) \\
\lesssim \;& I + II + III.
\end{aligned}$$

Here

$$I = \epsilon \int_{\partial\Omega_\epsilon} \left|\nabla |S(hf)|^2\right| d\sigma_\epsilon(z),$$

$$II = \int_{\Omega} |S(hf)(z)|^2 dv(z)$$

and

$$III(z) = \int_{\Omega} \left|\Delta(|S(hf)|^2)(z)\right| |r(z)| dv(z).$$

Since

$$\left|\nabla |S(hf)|^2\right| = \left|\overline{S(hf)} \nabla S(hf) + S(hf) \nabla \overline{S(hf)}\right| \lesssim |S(hf)| \left|\nabla S(hf)\right|,$$

we have

$$I \lesssim \epsilon \int_{\partial\Omega_\epsilon} |S(hf)(z)| |\nabla S(hf)(z)| d\sigma_\epsilon(z).$$

Also,

$$\Delta |S(hf)|^2 = \overline{S(hf)} \Delta S(hf) + S(hf) \Delta \overline{S(hf)} + 4 |\nabla S(hf)|^2.$$

By definition,

$$\Delta S(hf) = \Delta(\overline{z_i} hf) = hf \Delta(\overline{z_i}) + \overline{z_i} \Delta(hf) + \langle \nabla(hf), \nabla z_i \rangle = \partial_i(hf).$$

Similarly, $\Delta\overline{S(hf)} = \overline{\partial_i(hf)}$. Hence,

$$\Delta|S(hf)|^2 \lesssim |\nabla S(hf)|^2 + |S(hf)||\partial_i(hf)|$$

and therefore

$$\begin{aligned}
III &\lesssim \int_\Omega |\nabla S(hf)(z)|^2 |r(z)| dv(z) + \int_\Omega |S(hf)(z)||\partial_i(hf)(z)||r(z)| \mathrm{d}v(z) \\
&\leq \int_\Omega |\nabla S(hf)(z)|^2 |r(z)| dv(z) + \Big(\int_\Omega |S(hf)|^2 \mathrm{d}v(z)\Big)^{1/2} \cdot \Big(\int_\Omega |\partial_i(hf)(z)|^2 |r(z)|^2 \mathrm{d}v(z)\Big) \\
&\lesssim \int_\Omega |\nabla S(hf)(z)|^2 |r(z)| dv(z) + II^{1/2} \|hf\|_{L^2_a(\Omega)}.
\end{aligned}$$

To estimate the three parts, let us first estimate the two terms that appear in the integrands. By definition,

$$\begin{aligned}
S(hf)(z) &= \overline{z_i} h(z) f(z) - h(z) \int_\Omega \overline{w_i} f(w) K_l(z,w) |r(w)|^l dv(w) \\
&= \int_\Omega (\overline{z_i} - \overline{w_i}) h(z) f(w) K_l(z,w) |r(w)|^l dv(w).
\end{aligned}$$

Therefore

$$\begin{aligned}
|S(hf)(z)| &\leq \int_\Omega |z-w||h(z)f(w)||K_l(z,w)||r(w)|^l dv(w) \\
&\lesssim \int_\Omega |h(z)f(w)| \frac{|r(w)|^l}{F(z,w)^{n+1/2+l}} dv(w) \\
&\lesssim \int_\Omega \frac{F(z,w)^N}{|r(w)|^{n+1+N}} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv(\lambda) \frac{|r(w)|^l}{F(z,w)^{n+1/2+l}} dv(w) \\
&= \int_\Omega \int_{E(\lambda,1)} \frac{|r(w)|^{l-N-n-1}}{F(z,w)^{n+1/2+l-N}} dv(w) |h(\lambda)f(\lambda)| dv(\lambda) \\
&\lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+1/2+l-N}} |h(\lambda)f(\lambda)| dv(\lambda).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla S(hf)(z) &= \int_\Omega \nabla_z \overline{(z_i - w_i)} h(z) f(w) K_l(z,w) |r(w)|^l dv(w) \\
&\quad + \int_\Omega \overline{(z_i - w_i)} \nabla h(z) f(w) K_l(z,w) |r(w)|^l dv(w) \\
&\quad + \int_\Omega \overline{(z_i - w_i)} h(z) f(w) \nabla_z K_l(z,w) |r(w)|^l dv(w) \\
&= A + B + C.
\end{aligned}$$

By similar estimate,

$$|A| \lesssim \int_\Omega |h(z) f(w)| \frac{|r(w)|^l}{F(z,w)^{n+1+l}} dv(w) \lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+1+l-N}} |h(\lambda) f(\lambda)| dv(\lambda).$$

$$\begin{aligned}
|B| &\lesssim \int_\Omega \frac{F(z,w)^N}{|r(w)^{n+2+N}|} \int_{E(w,2)} |h(\lambda) f(\lambda)| dv(\lambda) \frac{|r(w)|^l}{F(z,w)^{n+1/2+l}} dv(w) \\
&= \int_\Omega \int_{E(\lambda,2)} \frac{|r(w)|^{l-N-n-2}}{F(z,w)^{n+1/2+l-N}} dv(w) |h(\lambda) f(\lambda)| dv(\lambda) \\
&\lesssim \int_\Omega \frac{|r(\lambda)|^{l-N-1}}{F(z,\lambda)^{n+1/2+l-N}} |h(\lambda) f(\lambda)| dv(\lambda).
\end{aligned}$$

Finally,

$$\begin{aligned}
|C| &\lesssim \int_\Omega |h(z) f(w)| \frac{|r(w)|^l}{F(z,w)^{n+3/2+l}} dv(\lambda) \\
&\lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+3/2+l-N}} |h(\lambda) f(\lambda)| dv(\lambda).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\nabla S(hf)(z)| &\lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+1+l-N}} |h(\lambda) f(\lambda)| dv(\lambda) \\
&\quad + \int_\Omega \frac{|r(\lambda)|^{l-N}}{|r(z)| F(z,\lambda)^{n+1/2+l-N}} |h(\lambda) f(\lambda)| dv(\lambda).
\end{aligned}$$

Since $\frac{|r(\lambda)|}{F(z,\lambda)} \leq 1$, altogether we have

$$|\nabla S(hf)(z)| \lesssim \int_{\Omega} \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+3/2+K}} |h(\lambda)f(\lambda)| dv(\lambda)$$

and

$$|S(hf)(z)| \lesssim \int_{\Omega} \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+1/2+K}} |h(\lambda)f(\lambda)| dv(\lambda).$$

Here $K = l - N - 1 > 0$. Now let us estimate I. We have obtained

$$I \lesssim \epsilon \int_{\partial\Omega_\epsilon} |S(hf)(z)||\nabla S(hf)(z)| d\sigma_\epsilon(z).$$

By Holder's inequality,

$$I \lesssim \epsilon \Big(\int_{\partial\Omega_\epsilon} |S(hf)(z)|^2 d\sigma_\epsilon(z) \Big)^{1/2} \Big(\int_{\partial\Omega_\epsilon} |\nabla S(hf)(z)|^2 d\sigma_\epsilon(z) \Big)^{1/2}.$$

Also

$$\begin{aligned}
& \int_{\partial\Omega_\epsilon} |S(hf)(z)|^2 d\sigma_\epsilon(z) \\
\lesssim\ & \int_{\partial\Omega_\epsilon} \Big(\int_{\Omega} \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+1/2+K}} dv(\lambda) \Big)^2 d\sigma_\epsilon(z) \\
\leq\ & \int_{\partial\Omega_\epsilon} \int_{\Omega} \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+1/2+K}} |r(\lambda)|^{-1/3} dv(\lambda) \\
& \cdot \int_{\Omega} \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+1/2+K}} |r(\lambda)|^{1/3} |h(\lambda)f(\lambda)|^2 dv(\lambda) d\sigma_\epsilon(z) \\
\lesssim\ & \int_{\partial\Omega_\epsilon} \int_{\Omega} \frac{|r(\lambda)|^{K+1/3}}{F(z,\lambda)^{n+1/2+K}} |h(\lambda)f(\lambda)|^2 dv(\lambda) d\sigma_\epsilon(z) \\
=\ & \int_{\Omega} \int_{\partial\Omega_\epsilon} \frac{|r(\lambda)|^{K+1/3}}{F(z,\lambda)^{n+1/2+K}} d\sigma_\epsilon(z) |h(\lambda)f(\lambda)|^2 dv(\lambda) \\
\lesssim\ & \int_{\Omega} |r(\lambda)|^{-1/6} |h(\lambda)f(\lambda)|^2 dv(\lambda) \\
\lesssim\ & \|hf\|^2_{H^2(\Omega)}.
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{\partial\Omega_\epsilon} |\nabla S(hf)(z)|^2 d\sigma_\epsilon(z)\\
\lesssim\ & \int_{\partial\Omega_\epsilon} \Big(\int_\Omega \frac{|r(\lambda)|^K}{F(z,w)^{n+3/2+K}} |h(\lambda)f(\lambda)| dv(\lambda) \Big)^2 d\sigma_\epsilon(z)\\
\leq\ & \int_{\partial\Omega_\epsilon} \int_\Omega \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+3/2+K}} |r(\lambda)|^{-1} dv(\lambda)\\
& \cdot \int_\Omega \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+3/2+K}} |r(\lambda)||h(\lambda)f(\lambda)|^2 dv(\lambda) d\sigma_\epsilon(z)\\
\lesssim\ & \int_{\partial\Omega_\epsilon} |r(z)|^{-3/2} \int_\Omega \frac{|r(\lambda)|^{K+1}}{F(z,\lambda)^{n+3/2+K}} |h(\lambda)f(\lambda)|^2 dv(\lambda) d\sigma_\epsilon(z)\\
=\ & \int_\Omega \int_{\partial\Omega_\epsilon} \frac{|r(z)|^{-3/2}|r(\lambda)|^{K+1}}{F(z,\lambda)^{n+3/2+K}} d\sigma_\epsilon(z) |h(\lambda)f(\lambda)|^2 dv(\lambda)\\
\lesssim\ & \epsilon^{-3/2} \int_\Omega |r(\lambda)|^{-1/2} |h(\lambda)f(\lambda)|^2 dv(\lambda)\\
\lesssim\ & \epsilon^{-3/2} \|hf\|^2_{H^2(\Omega)}.
\end{aligned}
$$

Therefore

$$
I \lesssim \epsilon^{1/4} \|hf\|^2_{H^2(\Omega)} \to 0, \ \ \epsilon \to 0.
$$

Next, by Schur's test, it is easy to show that

$$
\begin{aligned}
II \ &\lesssim \ \int_\Omega \Big(\int_\Omega \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+1/2+K}} |h(\lambda)f(\lambda)| \Big)^2 dv(z)\\
&\lesssim \ \|hf\|^2_{L^2_a(\Omega)}.
\end{aligned}
$$

Finally,

$$
\int_\Omega \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+3/2+K}} |r(\lambda)|^{-1/2} dv(\lambda) \lesssim |r(z)|^{-1},
$$

and

$$
\int_\Omega \frac{|r(\lambda)|^K}{F(z,\lambda)^{n+3/2+K}} |r(z)|^{-1} |r(z)| dv(z) \lesssim |r(\lambda)|^{-1/2},
$$

by Schur's test,

$$
III \lesssim \|hf\|^2_{L^2_a(\Omega)}.
$$

Now let $\epsilon \to 0$, we get

$$\|S(hf)\|_{L^2(\Omega)} \lesssim \|hf\|_{L^2_a(\Omega)}.$$

Therefore

$$\|(I-P)M_{z_i}^*P(hf)\|_{H^2(\Omega)} \leq \|\overline{z_i}hf - hT_i(f)\|_{L^2(\partial\Omega)} = \|S(hf)\| \lesssim \|hf\|_{L^2_a(\Omega)}.$$

By Proposition 3.3.9, $(I-P)M_{z_i}^*P \in C_p$, $\forall p > 2n$. This completes the proof. □

3.4 Further Results on the Unit Ball

The case that Ω is the unit ball receive special attention. Due to extra properties we have stronger results on $\mathbb{B}_n$. This section is devoted to some further results on principal submodules of the Bergman module $L^2_a(\mathbb{B}_n)$.

For convenience and future reference, we restate the two main results in the previous sections for the unit ball.

Theorem 3.4.1. *Suppose h is a holomorphic function defined on a neighborhood of the closed unit ball $\overline{\mathbb{B}_n}$, then there exists a constant N such that for any function $f \in Hol(\mathbb{B}_n)$ and any $z, w \in \mathbb{B}_n$,*

$$|h(z)f(w)| \lesssim \frac{|1-\langle w,z\rangle|^N}{(1-|w|^2)^{N+n+1}} \int_{E(w,1)} |h(\lambda)f(\lambda)|dv_n(\lambda).$$

Theorem 3.4.2. *Suppose h is a holomorphic function defined on a neighborhood of $\overline{\mathbb{B}_n}$, then the principal submodule*

$$[h] := \overline{\{hf : f \in L^2_a(\mathbb{B}_n)\}}$$

is p-essentially normal for $p > n$.

3.4.1 About Principal Submodules

Besides their main result on p-essential normality, the first author and K. Wang [14] also obtained a characterization of functions in the principal submodule $[p] \in L^2_a(\mathbb{B}_n)$ generated by a polynomial p. We show the same is true for any holomorphic function h defined in a neighborhood of $\overline{\mathbb{B}_n}$.

For a generator h as in Theorem 3.4.2, we are going to get a description of functions in the submodule $[h]$. We will provide proofs for the Bergman module $L^2_a(\mathbb{B}_n)$.

Take the measure $d\mu_h = |h|^2 dv_n$. Let $L^2(\mu_h)$ be the space of functions that are square integrable under this measure. Let $L^2_a(\mu_h)$ be the weighted Bergman space consisting of holomorphic functions in $L^2(\mu_h)$.

Lemma 3.4.3. *The weighted Bergman space $L^2_a(\mu_h)$ is a complete reproducing kernel Hilbert space.*

Proof. First, let us show that evaluation at any point $z \in \mathbb{B}_n$ defines a bounded linear functional on $L^2_a(\mu_h)$. If $h(z) \neq 0$, then by definition, if $f \in L^2_a(\mu_h)$, $fh \in L^2_a(\mathbb{B}_n)$. Therefore

$$
\begin{aligned}
|f(z)| = \frac{1}{|h(z)|}|fh(z)| &\leq \frac{1}{|h(z)|(1-|z|^2)^{(n+1)/2}}\|fh\|_{L^2_a(\mathbb{B}_n)} \\
&= \frac{1}{|h(z)|(1-|z|^2)^{(n+1)/2}}\|f\|_{L^2_a(\mu_h)}.
\end{aligned}
$$

If $h(z) = 0$, choose a complex line L through z such that h is not identically 0 on L, then z is an isolated zero point of h in L. Choose $r > 0$ so that the circle $C_r := \{w \in L : |w - z| = r\}$ does not intersect the zero set of h and is contained in $\mathbb{B}_n$. It is easy to see that evaluations at points in C_r are uniformly bounded. By the Maximum Principal,

$$
|f(z)| \leq \max\{|f(w)| : w \in C_r\}.
$$

Therefore evaluation at z is also bounded. This proves that $L^2_a(\mu_h)$ is a reproducing kernel Hilbert space.

Now we show that $L^2_a(\mu_h)$ is complete, or equivalently, $L^2_a(\mu_h)$ is closed in $L^2(\mu_h)$. Suppose $\{f_n\} \subseteq L^2_a(\mu_h)$ and f_n converges to $f \in L^2(\mu_h)$. From the arguments above, it is easy to see that given a compact subset K of $\mathbb{B}_n$, the evaluation functionals at points in K are uniformly bounded.

Therefore there exists $C > 0$ such that

$$\sup_{z\in K} |f_n(z) - f_m(z)| \le C\|f_n - f_m\|_{L^2(\mu_h)}.$$

Hence the sequence of holomorphic functions $\{f_n(z)\}$ converges uniformly on compact subsets to a holomorphic function $\tilde{f}$ on $\mathbb{B}_n$. Since $f_n \to f$, f_n converges to f in measure. Therefore $f = \tilde{f}$ almost everywhere. This shows that $f \in L^2_a(\mu_h)$. $L^2_a(\mu_h)$ is complete. □

Lemma 3.4.4. *Suppose h is as in Theorem 3.4.1, $f \in L^2_a(\mu_h)$. For $0 < r < 1$ and $z \in \mathbb{B}_n$, write $f_r(z) = f(rz)$, f_r is defined in a neighborhood of $\overline{\mathbb{B}_n}$. We have*

$$\int_{\mathbb{B}_n} |h(z)f_r(z)|^2 dv_n(z) \lesssim \int_{\mathbb{B}_n} |h(z)f(z)|^2 dv_n(z).$$

As a consequence, the set of holomorphic functions defined in a neighborhood of $\overline{\mathbb{B}_n}$ is dense in $L^2_a(\mu_h)$.

Proof. Apply Theorem 3.4.1 for $w = rz$, we get

$$\begin{aligned} |h(z)f(rz)| &\lesssim \frac{(1-r|z|^2)^N}{(1-r^2|z|^2)^{N+n+1}} \int_{E(rz,1)} |h(\lambda)f(\lambda)| dv_n(\lambda) \\ &\le \frac{1}{(1-r^2|z|^2)^{n+1}} \int_{E(rz,1)} |h(\lambda)f(\lambda)| dv_n(\lambda). \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\mathbb{B}_n} |h(z)f_r(z)|^2 dv_n(z) &\lesssim \int_{\mathbb{B}_n} \frac{1}{(1-r^2|z|^2)^{2(n+1)}} \left| \int_{E(rz,1)} |h(\lambda)f(\lambda)| dv_n(\lambda) \right|^2 dv_n(z) \\ &\le \int_{\mathbb{B}_n} \frac{1}{(1-r^2|z|^2)^{2(n+1)}} \int_{E(rz,1)} |h(\lambda)f(\lambda)|^2 dv_n(\lambda) v_n(E(rz,1)) dv_n(z) \\ &\lesssim \int_{\mathbb{B}_n} \frac{1}{(1-r^2|z|^2)^{n+1}} \int_{E(rz,1)} |h(\lambda)f(\lambda)|^2 dv_n(\lambda) dv_n(z). \end{aligned}$$

By the Fubini's Theorem, the last integral is equal to

$$\begin{aligned}
&\int_{\mathbb{B}_n}\int_{\{z:\ rz\in E(\lambda,1)\}}\frac{1}{(1-r^2|z|^2)^{n+1}}dv_n(z)|h(\lambda)f(\lambda)|^2dv_n(\lambda)\\
=\ &\int_{\mathbb{B}_n}\int_{E(\lambda,1)}\frac{1}{(1-|\eta|^2)^{n+1}}\frac{1}{r^{2n}}dv_n(\eta)|h(\lambda)f(\lambda)|^2dv_n(\lambda)\\
\lesssim\ &\int_{\mathbb{B}_n}|h(\lambda)f(\lambda)|^2dv_n(\lambda).
\end{aligned}$$

Here we used the fact that $v_n(E(w,1)) \approx (1-|w|^2)^{n+1}$ and that $1-|\eta|^2 \approx 1-|\lambda|^2$ whenever $\eta \in E(\lambda, 1)$ (cf. [40]).

We have proved the inequality. It remains to show that functions defined in a neighborhood of $\overline{\mathbb{B}_n}$ are dense. For any $f \in L^2_a(\mu_h)$, let $f_m := f_{1-\frac{1}{m+1}}$. The sequence of functions $\{f_m\}$ are defined in a neighborhood of $\overline{\mathbb{B}_n}$. By the previous argument, this is a bounded sequence in $L^2_a(\mu_h)$. Therefore there exists a subsequence that converges weakly. Since $f_m \to f$ pointwise, the weak limit must be f. Thus f lies in the weak closure of the subspace of function defined in a neighborhood of $\overline{\mathbb{B}_n}$. By the Hahn-Banach Theorem, f also belongs to the norm closure. This completes the proof. □

Proposition 3.4.5. *Suppose h is a holomorphic function defined on a neighborhood of $\overline{\mathbb{B}_n}$, then the principal submodule $[h]$ consists of functions of the form $hf \in L^2_a(\mathbb{B}_n)$ where f is a holomorphic function on $\mathbb{B}_n$, i.e.,*

$$[h] = \{fh:\ fh \in L^2_a(\mathbb{B}_n), f \in Hol(\mathbb{B}_n)\}.$$

Proof. Define the operator

$$\mathcal{I}: L^2_a(\mu_h) \to L^2_a(\mathbb{B}_n),\quad f \mapsto fh.$$

Then $\mathcal{I}$ is an isomorphism. Clearly $Ran(\mathcal{I})$ is closed and contains $[h]$. By Lemma 3.4.4, functions that are holomorphic in a neighborhood of $\overline{\mathbb{B}_n}$ are dense in $L^2_a(\mu_h)$. Therefore the image of these functions are dense in $Ran(\mathcal{I})$. But these images are in $[h]$. This proves that $Ran(\mathcal{I}) = [h]$. Therefore

$$[h] = \{fh:\ fh \in L^2_a(\mathbb{B}_n), f \in Hol(\mathbb{B}_n)\}.$$

This completes the proof. □

3.4.2 The Geometric Arveson-Douglas Conjecture

Now let us discuss a special case of the Geometric Arveson-Douglas Conjecture. Recall that we defined the notion of analytic subset in Chapter 2.

Definition 3.4.6. *Let A be an analytic subset of Ω. The* dimension *of A at an arbitrary point $a \in A$ is the number*

$$\dim_a A := \overline{\lim_{z\to a, z \text{ regular}}} \dim_z A.$$

The dimension of A is, by definition, the maximum of its dimensions at points:

$$\dim A := \max_{z\in A} \dim_z A.$$

A is said to be pure *if its dimensions at all points coincide.*

Pure analytic subsets of codimension 1 has some very important properties.

Lemma 3.4.7. *[8, Corollary 1, Page 26] Every pure $(n-1)$ dimensional analytic subset on an n-dimensional complex manifold is locally principal, i.e., for any $a \in A$ there exist open neighborhood U of a in Ω and holomorphic function f on U such that $A \cap U = \{z \in U : f(z) = 0\}$.*

Let A be a principal analytic subset of Ω, i.e., $A = \{z \in \Omega : f(z) = 0\}$ for a ceratin holomorphic function f. The function f is called a *minimal defining function* of A if for every open set $U \subseteq \Omega$ and every $g \in Hol(U)$ such that $g|_{A\cap U} = 0$, there exists an $h \in Hol(U)$ such that $g = fh$ in U.

Lemma 3.4.8. *[8, Proposition 1, Page 27] Every pure $(n-1)$-dimensional analytic subset on an n-dimensional complex manifold locally has a minimal defining function.*

Now suppose V is a pure $(n-1)$-dimensional analytic subset of an open neighborhood of $\overline{\mathbb{B}_n}$. Choose $r > 1$ so that V is defined in a neighborhood of $r\overline{\mathbb{B}_n}$. By Lemma 3.4.8 and compactness, there is a finite open cover $\{U_i\}$ of $r\mathbb{B}_n$ and a minimal defining function h_i on U_i. By definition, if

$U_i \cap U_j \neq \emptyset$, the function $g_{ij} = h_i/h_j$ is holomorphic and non-vanishing on $U_i \cap U_j$. They satisfy

$$g_{ij} \cdot g_{ji} = 1 \text{ on } U_i \cap U_j$$

and

$$g_{ij} \cdot g_{jk} \cdot g_{ki} = 1 \text{ on } U_i \cap U_j \cap U_k.$$

Such a set of functions is called a second Cousin data. By [28], the second Cousin problem is solvable on $r\mathbb{B}_n$. That means, there exists non-vanishing $f_i \in Hol(U_i \cap r\mathbb{B}_n)$ such that $g_{ij} = f_i/f_j$. If we define $f = h_i/f_i$ on $U_i \cap r\mathbb{B}_n$, then one easily checks that f is well defined and becomes a global minimal defining function for V in $r\mathbb{B}_n$.

Suppose $f \in L^2_a(\mathbb{B}_n)$, $f|_{V\cap\mathbb{B}_n} = 0$, then $f = gh$ for some $g \in Hol(\mathbb{B}_n)$. From last subsection we know that this means $g \in L^2_a(\mu_h)$, or $f \in [h]$. The other side of inclusion is obvious: $[h] \subseteq \{f \in L^2_a(\mathbb{B}_n) : f|_{V\cap\mathbb{B}_n} = 0\}$. To sum up, we have obtained the following theorem.

Theorem 3.4.9. *Suppose V is a pure $(n-1)$-dimensional analytic subset of an open neighborhood of $\overline{\mathbb{B}_n}$, then V has a minimal defining function h on an open neighborhood of $\overline{\mathbb{B}_n}$. Moreover,*

$$\mathcal{P}_V := \{f \in L^2_a(\mathbb{B}_n) : f|_{V\cap\mathbb{B}_n} = 0\} = [h].$$

Therefore the submodule $\mathcal{P}_V$ is p-essentially normal for all $p > n$.

4. SUMMARY

In this dissertation, we have discussed some results on the Arveson-Douglas Conjecture. The tools come from operator theory, harmonic analysis and several complex variables. The Arveson-Douglas Conjecture concerns essential normality of submodules or quotient modules of certain analytic function spaces. A good understanding of the projection operators onto these submodules or quotient modules is essential in tackling the problem. We have developed the theory along two lines, one of which deals with the quotient module, the other with the submodule. These are contained in Chapter 2 and Chapter 3, respectively.

For radical ideals, the corresponding quotient module "lives" on the variety of the ideal, in the sense that, the quotient module is the unique reproducing kernel Hilbert space determined by the restriction of the reproducing kernel on the variety. The theory in Chapter 2 gives another interpretation. When an equivalent measure exists, the quotient module is similar to a closed subspace of an L^2 space living on the variety. The theory builds a deep connection between the L^2-extension problem in several complex variables and the Arveson-Douglas Conjecture.

Chapter 3 deals with principal submodules. Our theory shows that the proof essentially relies on inequality (3.4). This allows some flexibility in the proof. As is shown in Chapter 3, we are able to generalize the result to Bergman and Hardy spaces on bounded strongly pseudoconvex domains with smooth boundary, with generator being any holomorphic function defined in a neighborhood of the closure of the domain.

For the next step, we plan to build a theory that deals with both cases. Here are some rough ideas. Complex analytic subsets behave relatively nice locally. Using these local properties, we plan to construct local decompositions of the ideal, into sums of principal ideals. Tools from harmonic analysis will allow us to obtain a global decomposition from local ones. This can lead to a universal proof that works for all the existing results and a generalization of them.

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