TENSOR POWERS OF DRINFELD MODULES AND ZETA VALUES

A Dissertation

by

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ABSTRACT

We study tensor powers of rank 1 sign-normalized Drinfeld $\mathbf{A}$-modules, where $\mathbf{A}$ is the coordinate ring of an elliptic curve over a finite field. Using the theory of $\mathbf{A}$-motives, we find explicit formulas for the $\mathbf{A}$-action of these modules. Then, by developing the theory of vector-valued Anderson generating functions, we give formulas for the period lattice of the associated exponential function. We then give formulas for the coefficients of the logarithm and exponential functions associated to these $\mathbf{A}$-modules. Finally, we show that there exists a vector whose bottom coordinate contains a Goss zeta value, whose evaluation under the exponential function is defined over the Hilbert class field. This allows us to prove the transcendence of certain Goss zeta values and periods of Drinfeld modules as well as the transcendence of certain ratios of those quantities.

DEDICATION

This dissertation is dedicated to my wife, Lisa, and to our daughter, Natalie, who was born during its creation.

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1. INTRODUCTION

1.1 Introduction

The Carlitz module and its tensor powers are well understood. We have explicit formulas for multiplication maps of both the Carlitz module and for its tensor powers (see [13] for the Carlitz module and [33, §3] for tensor powers of the Carlitz module). Further, we have a nice product formula for $\widetilde{\pi}$, the Carlitz period, and a formula for the bottom coordinate of the fundamental period associated with tensor powers of the Carlitz module (see [5, §2.5]).

In his work towards the Langland's program, Drinfeld introduced the notion of Drinfeld modules (see also [24], [30] or [46] for a thorough account of Drinfeld modules), which are a generalization of the Carlitz module. Since their introduction, many researchers have worked to develop an explicit theory for Drinfeld modules which parallels that for the Carlitz module, notably Goss in [22] and [23], Anderson in [2] and [3], Thakur in [44] and [45], Dummit and Hayes in [18], and Hayes in [29]. To discuss the results of the present thesis, we first recall a few basic facts about rank 1 sign-normalized Drinfeld $\mathbf{A}$-modules over rings $\mathbf{A}$, where $\mathbf{A}$ is the affine coordinate ring of an elliptic curve $E/\mathbb{F}_q$ (see §2.1 for a more thorough review of Drinfeld modules). Define $\mathbf{A} = \mathbb{F}_q[t, y]$, where t and y are related via a cubic Weierstrass equation for E. Also define an isomorphic copy of $\mathbf{A}$, which we denote $A = \mathbb{F}_q[\theta, \eta]$, where θ and η satisfy the same cubic Weierstrass equation as t and y. Let K be the fraction field of A, let K_∞ be the completion of K at its infinite place, and let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞. Let H be the Hilbert class field of K, which can be taken to be a subfield of K_∞. A rank 1 sign-normalized Drinfeld module is an $\mathbb{F}_q$-algebra homomorphism

$$\rho : \mathbf{A} \to L[\tau]$$

satisfying certain naturally defined conditions, where $L \subset \mathbb{C}_\infty$ is some algebraically closed field containing H and $L[\tau]$ is the ring of twisted polynomials in the qth power Frobenius endomorphism τ (see §2.1 for definitions). Associated to this Drinfeld module there is a point $V \in E(H)$ called the Drinfeld divisor, satisfying the equation with respect to the group law on E

$$V^{(1)} - V + \Xi = \infty,$$

where $\Xi = (\theta, \eta) \in E(K)$ and $V^{(1)}$ is the image of V under the qth power Frobenius isogeny. We specify that V be in the formal group of E at the infinite place of K, so that V is uniquely determined by the above equation. We define the shtuka function $f \in H(t, y)$ associated to E to have

$$\operatorname{div}(f) = (V^{(1)}) - (V) + (\Xi) - (\infty),$$

and require that the sign of f equals 1 so that f is uniquely determined (see §1.2 for the definition of sign).

Generalizing the Carlitz module further, Anderson introduced the notion of tensor products of Drinfeld modules in [1], which provide higher dimensional analogues of (1-dimensional) Drinfeld modules. Then, in the remarkable paper [5], Anderson and Thakur develop much of the explicit theory for the arithmetic of the nth tensor power of the Carlitz module, including the aforementioned formula for the bottom coordinate of the fundamental period of the exponential function. In a more recent paper, Papanikolas [33] uses hyperderivatives to give extremely explicit formulas for multiplication maps and the fundamental period of tensor powers of the Carlitz module, along with with remarkable log-algebraicity theorems. Both Anderson and Thakur's and Papanikolas's techniques allow them to connect the logarithm function to function field zeta values.

The goal of this thesis is to give a detailed account of tensor powers of rank 1 sign normalized $\mathbf{A}$-modules and their applications to zeta values. The notion of using Drinfeld modules to study L-functions, zeta functions, and their special values over functions fields has been pursued vigorously in the last few years and has born much fruit (see [6], [25], [31], [35], [38] and [42]).

The main focus of this thesis is the study of tensor powers of rank 1 sign-normalized Drinfeld modules over the affine coordinate ring of an elliptic curve. These modules provide a further generalization of the Carlitz module and are an example of Anderson $\mathbf{A}$-modules. An n-dimensional Anderson $\mathbf{A}$-module is an $\mathbf{A}$-module homomorphism

$$\rho : \mathbf{A} \to \mathrm{Mat}_n(L)[\tau]$$

satisfying certain naturally defined conditions, where $\mathrm{Mat}_n(L)[\tau]$ is the ring of twisted polynomials in the qth power Frobenius endomorphism τ, which extends to matrices entry-wise (see §2.2 for the full definition of Anderson $\mathbf{A}$-modules).

The main theorems of this thesis include the following. We give formulas for the $\mathbf{A}$-action of tensor powers of rank 1 sign-normalized $\mathbf{A}$-Drinfeld modules, as well as for the fundamental period of the exponential function associated to this module. This generalizes both the work of Papanikolas and the author on Drinfeld modules in [26] as well as that of Anderson and Thakur on tensor powers of the Carlitz module in [5]. One of the main new aspects of this work, which distinguishes it from that of Anderson and Thakur, is that we prove many of our results in a vector-valued setting. In particular, we define and study vector-valued Anderson generating functions (see (3.15)), and define new operators which act on these vector-valued functions (see §3.1). We also give explicit formulas for the coefficients of the exponential and the logarithm function associated to tensor powers of rank 1 sign-normalized Drinfeld modules, and show that evaluating the exponential function at a special vector with a zeta value in its bottom coordinate gives a vector in H^n. We remark that our techniques only allow us to study small zeta values. As an application of the main theorems we use techniques of Yu from [48] to show that these zeta values and the periods connected to the Drinfeld module are transcendental over $\overline{K}$. This generalizes both the work of Thakur on Drinfeld modules and zeta values in [45] as well as that of Anderson and Thakur on tensor powers of the Carlitz module in [5].

The methods which Anderson and Thakur apply to obtain formulas for the coefficients for the exponential and logarithm functions for tensor powers of the Carlitz module involve recursive matrix calculations, which allow them to analyze a particular coordinate of those coefficients. In the case of tensor powers of Drinfeld modules, however, the matrices involved are much more complicated and do not give clean formulas as they do in the Carlitz case. We develop new techniques to analyze the coefficients of the logarithm and exponential function inspired partially by work of Papanikolas and the author in [26] and partially by ideas of Sinha in [41]. Further, Anderson and Thakur use special polynomials (called Anderson-Thakur polynomials) in [5] to relate evaluations of the logarithm function to zeta values. It is not yet clear how to generalize these Anderson-Thakur polynomials to tensor powers of Drinfeld modules, and so instead we use a generalization of techniques developed by Papanikolas and the author in [26] to prove formulas for zeta values. We comment that this technique allows us to study zeta values only for $1 \leq s \leq q-1$; developing techniques to study zeta values for all $n \geq 1$ is a topic of ongoing study (see Remark 5.1.1).

After setting out the notation and background in §1.2, in §2.1 we begin by defining $\mathbf{A}$-motives and dual $\mathbf{A}$-motives, which are tensor powers of 1-dimensional motives. We realize these $\mathbf{A}$-motives and dual $\mathbf{A}$-motives as spaces of functions

$$M = \Gamma(U, \mathcal{O}_E(nV)), \quad N = \Gamma(U, \mathcal{O}_E(-nV^{(1)})),$$

respectively, where $U = \operatorname{Spec} L[t,y]$ is the affine curve $(L \times_{\mathbb{F}_q} E) \setminus \{\infty\}$. The spaces M and N are generated as a free $L[\tau]$-module and a free $L[\sigma]$-module by the sets of functions

$$\{g_1, \dots, g_n\} \subset M, \quad \{h_1, \dots, h_n\} \subset N, \tag{1.1}$$

respectively, where $g_i, h_i \in L(t,y)$ are naturally defined (see (2.12) and (2.13) for specific definitions). The functions g_i and h_i appear repeatedly throughout this thesis, and one can think of them as a generalization of the shtuka function to the n-dimensional setting.

To ease notation throughout the thesis, for a fixed dimension n, we define

$$N_i \in \operatorname{Mat}_n(\mathbb{F}_q) \tag{1.2}$$

for an integer $i \geq 1$ to be the matrix with 1's along the ith super-diagonal and 0's elsewhere and define N_i for $i \leq -1$ to be the matrix with 1's along the ith sub-diagonal and 0's elsewhere. We also define E_1 to be the matrix with a single 1 in the lower left corner and zeros elsewhere and in general define E_i to be N_{i-n}. We also define $N_i(\alpha_1, \ldots, \alpha_{n-i})$ to be the matrix with the entries $\alpha_1, \alpha_2, \ldots, \alpha_{n-i}$ along the ith super diagonal and similarly for $N_{i-n}(\alpha_1, \ldots, \alpha_{n-i})$ and $E_i(\alpha_1, \ldots, \alpha_i)$. Also let $*^\top$ denote the transpose of a matrix.

Using M and N, in §2.2 we define an Anderson $\mathbf{A}$-motive $\rho^{\otimes n}$, which is the nth tensor power of a (1-dimensional) rank 1 sign-normalized Drinfeld module ρ, and analyze the structure of $\rho_t^{\otimes n}$ and $\rho_y^{\otimes n}$. We find that

$$\rho_t^{\otimes n} = (\theta I + N_1(a_1, \ldots, a_{n-1}) + N_2) + (a_n E_1 + E_2)\tau, \tag{1.3}$$

where a_i are naturally defined constants in H (see (2.21) and Corollary (3.1.5)), and that $\rho_y^{\otimes n}$ is defined similarly (see (2.22)). By way of comparison, recall that for the nth tensor power of the Carlitz module (see Example 2.2.1), we can write

$$C_t^{\otimes n} = (\theta I + N_1) + E_1 \tau.$$

We denote the exponential and logarithm functions associated to $\rho^{\otimes n}$ as

$$\mathrm{Exp}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)}, \quad \mathrm{Log}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} P_i \mathbf{z}^{(i)},$$

where $Q_i, P_i \in \mathrm{Mat}_n(H)$ and denote the period lattice of $\mathrm{Exp}_\rho^{\otimes n}$ as $\Lambda_\rho^{\otimes n}$.

For $g = \sum c_{j,k} t^j y^k \in L[t, y]$, let $g^{(1)}$ denote the Frobenius twist of g, which is defined as

$$g^{(1)} = \sum c_{j,k}^q t^j y^k, \tag{1.4}$$

and let $g^{(i)}$ denote the ith iteration of twisting. In §3.1 we define an $\mathbf{A}$-module of rigid analytic functions Ω_0 which vanish under the operator $\tau - f^n$, where τ acts by twisting. We then proceed to define an n-dimensional "vector version" of the operator $\tau - f^n$ which we denote

$$G - E_1\tau \in \mathrm{Mat}_n(H(t,y))[\tau],$$

which acts on vectors of rigid analytic functions, and in Lemma 3.1.3 we solidify the connection between these two operators. These vector operators allow us in §3.2 to connect the fundamental period Π_n of $\mathrm{Exp}_\rho^{\otimes n}$ with the space Ω_0 and obtain formulas for Π_n. To state the main theorem on periods, we begin by recalling the function

$$\omega_\rho = \xi^{1/(q-1)} \prod_{i=0}^{\infty} \frac{\xi^{q^i}}{f^{(i)}},$$

from [26, §4], where $\xi = -(m + \beta/\alpha)$ (see (2.3) for the definition of m). We also define vector valued Anderson generating functions,

$$E_{\mathbf{u}}^{\otimes n}(t) = \sum_{i=0}^{\infty} \mathrm{Exp}_\rho^{\otimes n}\left(d[\theta]^{-i-1}\mathbf{u}\right) t^i \in \mathbb{T}^n,$$

where $\mathbf{u} \in \mathbb{C}_\infty$ and $\mathbb{T}$ is a Tate algebra (see (1.10) for the definition of $\mathbb{T}$), and prove several properties about them. We relate the function ω_ρ^n to $E_{\mathbf{u}}^{\otimes n}$ using the vector operator $G - E_1\tau$ from §3.1. Using these techniques, we get the following information about the period lattice.

Theorem 3.2.7. *If we denote*

$$\Pi_n = -\begin{pmatrix} \mathrm{Res}_\Xi(\omega_\rho^n g_1 \lambda) \\ \vdots \\ \mathrm{Res}_\Xi(\omega_\rho^n g_n \lambda) \end{pmatrix},$$

where g_i are the functions from (1.1) *and λ is an (suitably normalized) invariant differential on E, then the structure of the period lattice of $\mathrm{Exp}_\rho^{\otimes n}$ is given by*

$$\Lambda_\rho^{\otimes n} = \{d[a]\Pi_n \mid a \in \mathbf{A}\},$$

where $d[a]$ *is the constant term of* $\rho_a^{\otimes n}$*. Further, if* π_ρ *is a fundamental period of the exponential function associated to the (1-dimensional) Drinfeld module* ρ*, then the last coordinate of* $\Pi_n \in \mathbb{C}_\infty^n$ *is*

$$\frac{g_1(\Xi)}{a_1 a_2 \dots a_{n-1}} \cdot \pi_\rho^n,$$

where the constants a_i *are the same as in* (1.3)*.*

In section §4.1 we move on to analyzing the coefficients of the exponential function $\mathrm{Exp}_\rho^{\otimes n}$ associated to tensor powers of rank 1 Drinfeld $\mathbf{A}$-modules. First, we define functions for $1 \leq \ell \leq n$ and $i \geq 1$

$$\gamma_{i,\ell} = \frac{g_\ell}{(ff^{(1)} \dots f^{(i-1)})^n}$$

and find that there is a unique expression for $\gamma_{i,\ell}$ of the form

$$\gamma_{i,\ell} = c_{\ell,1} g_1^{(i)} + c_{\ell,2} g_2^{(i)} + \dots c_{\ell,n} g_n^{(i)} + \sum_{j,k} d_{j,k} \alpha_{j,k},$$

for $c_{\ell,m}, d_{j,k} \in H$, where the functions $\alpha_{j,k} \in H(t,y)$ satisfy naturally defined conditions given in §4.1. We denote $C_i = \langle c_{j,k} \rangle$, and we obtain our first main theorem about the coefficients of the exponential function.

Theorem 4.1.1. *For dimension* $n \geq 2$ *and* $\mathbf{z} \in \mathbb{C}_\infty$*, if we write*

$$\mathrm{Exp}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)},$$

then for $i \geq 0$*, the exponential coefficients* $Q_i = C_i$ *and* $Q_i \in \mathrm{Mat}_n(H)$*.*

We prove this theorem by observing a recursive matrix equation which uniquely identifies the coefficients of the exponential function (see Lemma 4.1.3), and then proving that the matrices C_i satisfy the recursive equation. After a bit more analysis, we obtain more exact formulas for the first column of Q_i.

Corollary 4.1.4. *For $z \in \mathbb{C}_\infty$ we have the expression*

$$\operatorname{Exp}_\rho^{\otimes n}\begin{pmatrix} z \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{i=0}^{\infty} \frac{z^{q^i}}{g_1^{(i)}(ff^{(1)}\dots f^{(i-1)})^n} \cdot \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}\Bigg|_{\Xi^{(i)}}.$$

Next, we transition to studying the coefficients of the logarithm function in §4.2. Our main technique in this section involves proving the commutativity of diagram (4.15), which is inspired by work of Sinha in [41]. We then define a single variable function which, using the machinery from the diagram, allows us to recover the logarithm function. This gives formulas for the logarithm coefficients in terms of residues of quotients of the functions g_i, h_i and f.

Theorem 4.2.4. *For $\mathbf{z}$ inside the radius of convegence of $\operatorname{Log}_\rho^{\otimes n}$, if we let*

$$\operatorname{Log}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} P_i \mathbf{z}^{(i)}$$

for $n \geq 2$ and let λ be an (suitably normalized) invariant differential on E, then $P_i \in \operatorname{Mat}_n(H)$ for $i \geq 0$ and

$$P_i = \left\langle \operatorname{Res}_\Xi\left(\frac{g_j h_{n-k+1}^{(i)}}{(ff^{(1)}\dots f^{(i)})^n}\lambda\right)\right\rangle_{1\leq j,k\leq n}.$$

With a little further analysis we obtain cleaner formulas for the bottom row of the logarithm coefficients.

Corollary 4.2.6. *For the coefficients P_i of the function $\operatorname{Log}_\rho^{\otimes n}$, the bottom row of P_i, for $i \geq 0$, can be written as*

$$\left\langle \frac{h_{n-k+1}^{(i)}}{h_1(f^{(1)}\dots f^{(i)})^n}\Bigg|_\Xi \right\rangle_{1\leq k\leq n}.$$

In section §5.1 we show that evaluating the exponential function at a special vector with a Goss zeta value in its bottom coordinate is in H^n. To state our results, we recall the extension of a rank 1 sign-normalized Drinfeld module ρ to integral ideals $\mathfrak{a} \subset A$ due to Hayes [29] (see §5.1), which maps $\mathfrak{a} \mapsto \rho_\mathfrak{a} \in H[\tau]$. We define $\partial(\rho_\mathfrak{a})$ to be the constant term of $\rho_\mathfrak{a}$ with respect to τ and let $\phi_\mathfrak{a} \in \mathrm{Gal}(H/K)$ denote the Artin automorphism associated to $\mathfrak{a}$, and let the B be the integral closure of A in H. We define a zeta function associated to ρ twisted by the parameter $b \in B$ to be

$$\zeta_\rho(b; s) := \sum_{\mathfrak{a} \subseteq A} \frac{b^{\phi_\mathfrak{a}}}{\partial(\rho_\mathfrak{a})^s}.$$

Theorem 5.1.2. *For $b \in B$ and for $n \leq q-1$, there exists a constant $C \in H$ and a vector $(*, \ldots, *, C\zeta_\rho(b; n))^\top \in \mathbb{C}_\infty^n$ such that*

$$\mathbf{d} := \mathrm{Exp}_\rho^{\otimes n} \begin{pmatrix} * \\ \vdots \\ * \\ C\zeta_\rho(b; n) \end{pmatrix} \in H^n,$$

where $C \in H$ and $\mathbf{d} \in H^n$ are explicitly computable as outlined in the proof.

In §5.2 we discuss the transcendence implications of theorem 5.1.2. Using techniques similar to Yu's in [48] we prove the following theorem.

Theorem 5.2.1. *Let ρ be a rank 1 sign-normalized Drinfeld module, let π_ρ be a fundamental period of the exponential function associated to ρ and define $\zeta_\rho(b; n)$ as above. Then*

$$\dim_{\overline{K}} \mathrm{Span}_{\overline{K}}\{\zeta_\rho(b; 1), \ldots, \zeta_\rho(b; q-1), 1, \pi_\rho, \ldots, \pi_\rho^{q-2}\} = 2(q-1).$$

From Theorem 5.2.1 we get a corollary which relates to a theorem of Goss (see [22, Thm. 2.10]).

Corollary 5.2.4. *For $1 \leq i \leq q-1$, the quantities $\zeta_\rho(b; i)$ are transcendental. Further, for $0 \leq j \leq q-1$ the ratio $\zeta_\rho(b; i)/\pi_\rho^j \in \overline{K}$ if and only if $i = j = q-1$.*

Finally in §6.1 we give examples of the constructions in our main theorems.

1.2 Background and notation

We require much of the same notation as [26, §2] and we use similar exposition in this section. Let p be a prime and $q = p^r$ for some integer $r > 0$ and let $\mathbb{F}_q$ be the field with q elements. Define the elliptic curve E over $\mathbb{F}_q$ with Weierstrass equation

$$E : y^2 + c_1ty + c_3y = t^3 + c_2t^2 + c_4t + c_6, \quad c_i \in \mathbb{F}_q, \tag{1.5}$$

with the point at infinity designated as ∞. Let $\mathbf{A} = \mathbb{F}_q[t, y]$ be the affine coordinate ring of E, the functions on E regular away from ∞, and let $\mathbf{K} = \mathbb{F}_q(t, y)$ be its fraction field. Let

$$\lambda = \frac{dt}{2y + c_1t + c_3} \tag{1.6}$$

be a fixed invariant differential on E. Also define isomorphic copies of $\mathbf{A}$ and $\mathbf{K}$ with an independent set of variables θ and η, which also satisfy (1.5), which we label

$$A = \mathbb{F}_q[\theta, \eta], \quad \text{and} \quad K = \mathbb{F}_q(\theta, \eta).$$

Define the canonical isomorphisms

$$\iota : \mathbf{K} \to K, \quad \chi : K \to \mathbf{K} \tag{1.7}$$

such that $\iota(t) = \theta$ and $\iota(y) = \eta$ and so on. We remark that the maps ι and χ extend to finite algebraic extensions of $\mathbf{K}$ and K respectively.

Let ord_∞ be the valuation of K at the infinite place, and let $\deg := -\mathrm{ord}_\infty$, both normalized so that

$$\deg(\theta) = 2, \quad \deg(\eta) = 3.$$

Define an absolute value on K by setting $|g| = q^{\deg(g)}$ for $g \in K$. Also define ord_∞, $\deg$ and $|\cdot|$ on $\mathbf{K}$ similarly. Let K_∞ be the completion of K at the infinite place, and let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞. Designate the point $\Xi = (\theta, \eta) \in E(K)$.

Extend the absolute value on $\mathbb{C}_\infty$ to a seminorm on $M = \langle m_{i,j} \rangle \in \mathrm{Mat}_{\ell \times m}(\mathbb{C}_\infty)$ as in [33, §2.2] by defining

$$|M| = \max_{i,j}(|m_{i,j}|).$$

Note for $c \in \mathbb{C}_\infty$ and $M, N \in \mathrm{Mat}_{\ell \times m}(\mathbb{C}_\infty)$ that

$$|cM| = |c| \cdot |M|, \quad |M + N| \leq |M| + |N|,$$

and for matrices $M \in \mathrm{Mat}_{k \times \ell}(\mathbb{C}_\infty)$ and $N \in \mathrm{Mat}_{\ell \times m}(\mathbb{C}_\infty)$ that

$$|MN| \leq |M| \cdot |N|,$$

but that the seminorm is not multiplicative in general.

In order to define a sign function, we first note that as an $\mathbb{F}_q$-vector space, $\mathbf{A}$ has a basis $\{t^i, t^j y\}$, for $i, j \geq 0$ where each term has a unique degree. Thus, when expressed in this basis, an element $a \in \mathbf{A}$ has a leading term which allows us to define

$$\mathrm{sgn} : \mathbf{A} \setminus \{0\} \to \mathbb{F}_q^\times,$$

by letting $\mathrm{sgn}(a) \in \mathbb{F}_q^\times$ be the coefficient of the leading term of $a \in \mathbf{A} \setminus \{0\}$. This sign function extends naturally to $\mathbf{K}^\times$. Define a sign function analogously for A and K, which we also call sgn. Then, for any field extension $L/\mathbb{F}_q$, the coordinate ring of E over L is $L[t, y] = L \otimes_{\mathbb{F}_q} \mathbf{A}$, and using the same notion of leading term, we define a group homomorphism

$$\widetilde{\mathrm{sgn}} : L(t, y)^\times \to L^\times,$$

which extends the function sgn on $\mathbf{K}^\times$.

Now, let $L/\mathbb{F}_q$ be an algebraically closed extension of fields containing A. Define $\tau : L \to L$

to be the qth power Frobenius map and define $L[\tau]$ as the ring of twisted polynomials in τ, subject to the relation for $c \in L$

$$\tau c = c^q \tau.$$

For $g = \sum c_{j,k} t^j y^k \in L[t, y]$, let $g^{(1)}$ denote the Frobenius twist of g, which is defined as

$$g^{(1)} = \sum c_{j,k}^q t^j y^k, \tag{1.8}$$

and let $g^{(i)}$ denote the ith iteration of twisting. The twisting operation also extends naturally to matrices in $\mathrm{Mat}_{\ell \times m}(L(t, y))$ by twisting entry-wise. We use this notion of twisting to define the ring $\mathrm{Mat}_n(L)[\tau]$ as the non-commutative ring of polynomials in τ subject to the relation $\tau M = M^{(1)} \tau$ for $M \in \mathrm{Mat}_n(L)$. In the setting of Anderson $\mathbf{A}$-modules, we view $\mathrm{Mat}_n(L)[\tau]$ as a ring of operators acting on L^n for $n \geq 1$ via twisting, i.e. for $\Delta = \sum M_i \tau^i$, with $M_i \in \mathrm{Mat}_n(L)$ and $\mathbf{a} \in L^n$,

$$\Delta(\mathbf{a}) = \sum M_i \mathbf{a}^{(i)}. \tag{1.9}$$

Further, for $X \in E(L)$, we define $X^{(1)} = \mathrm{Fr}(X)$, where $\mathrm{Fr} : E \to E$ is the qth power Frobenius isogeny. We extend twisting to divisors in the obvious way, noting that for $g \in L(t, y)$

$$\mathrm{div}(g^{(1)}) = \mathrm{div}(g)^{(1)}.$$

We define the Tate algebra for $c \in \mathbb{C}_\infty$,

$$\mathbb{T}_c = \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[[t]] \,\middle|\, \left|c^i b_i\right| \to 0 \right\}, \tag{1.10}$$

where $\mathbb{T}_c$ is the set of power series which converge on the closed disk of radius $|c|$. For convenience, we set $\mathbb{T} := \mathbb{T}_1$. Define the Gauss norm $\|\cdot\|_c$ for vectors of functions $\mathbf{h} = \sum \mathbf{d}_i t^i \in \mathbb{T}_c^n$ for some fixed dimension $n > 0$ with $\mathbf{d}_i \in \mathbb{C}_\infty^n$ by setting

$$\|\mathbf{h}\|_c = \max_i |c^i \mathbf{d}_i|,$$

where $|\cdot|$ is the seminorm described above. Extend this norm to $\mathbb{T}_c[y]^n$ for $\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{T}_c^n$ by setting $\|\mathbf{h}_1 + y\mathbf{h}_2\|_c = \max(\|\mathbf{h}_1\|_c, \|\eta\mathbf{h}_2\|_c)$. Note that each of these algebras are complete under their respective norms. Using the definition given from [20, Chs. 3–4], we note that the two rings $\mathbb{T}[y]$ and $\mathbb{T}_\theta[y]$ are affinoid algebras corresponding to rigid analytic affinoid subspaces of $E/\mathbb{C}_\infty$. Let $\mathcal{E}$ be the rigid analytic variety associated to E and let $\mathcal{U} \subset \mathcal{E}$ be the inverse image under t of the closed disk of radius $|\theta|$ in $\mathbb{C}_\infty$ centered at 0. Then observe that $\mathcal{U}$ is the affinoid subvariety of $\mathcal{E}$ associated to $\mathbb{T}_\theta[y]$, and that Frobenius twisting extends to $\mathbb{T}$ and $\mathbb{T}[y]$ and their fraction fields. As proved in [34, Lem. 3.3.2], $\mathbb{T}$ and $\mathbb{T}[y]$ have $\mathbf{A}$ and $\mathbb{F}_q[t]$ as their fixed rings under twisting, respectively.

We extend the action of $\mathrm{Mat}_n(L)[\tau]$ on L^n described in (1.9) to an action of $\mathrm{Mat}_n(\mathbb{T}[y])[\tau]$ on $\mathbb{T}[y]^n$ in the natural way.

2. A-MOTIVES AND A-MODULES

2.1 Tensor powers of A-motives

We briefly review the theory of $\mathbf{A}$-motives and dual $\mathbf{A}$-motives corresponding to rank 1 sign-normalized Drinfeld-Hayes modules as set out in [26, §3]. First note that we can pick a unique point V in $E(H)$ whose coordinates have positive degree (see the discussion preceding [26, (13)]) such that V satisfies the equation on E

$$(1 - \mathrm{Fr})(V) = V - V^{(1)} = \Xi, \tag{2.1}$$

If we set $V = (\alpha, \beta)$, then $\deg(\alpha) = 2$ and $\deg(\beta) = 3$ and $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta) = 1$. Define H to be the Hilbert class field of K, which equals $H = K(\alpha, \beta)$. There is a unique function in $H(t, y)$, called the shtuka function, with $\widetilde{\mathrm{sgn}}(f) = 1$ and with divisor

$$\mathrm{div}(f) = (V^{(1)}) - (V) + (\Xi) - (\infty). \tag{2.2}$$

We can write

$$f = \frac{\nu(t,y)}{\delta(t)} = \frac{y - \eta - m(t-\theta)}{t - \alpha} = \frac{y + \beta + c_1\alpha + c_3 - m(t-\alpha)}{t-\alpha}, \tag{2.3}$$

where m is the slope between the collinear points $V^{(1)}, -V$ and Ξ, and $\deg(m) = q$. We see

$$\mathrm{div}(\nu) = (V^{(1)}) + (-V) + (\Xi) - 3(\infty), \tag{2.4}$$

$$\mathrm{div}(\delta) = (V) + (-V) - 2(\infty). \tag{2.5}$$

Let L/K be an algebraically closed field, and let $U = \mathrm{Spec}\, L[t, y]$ be the affine curve $(L \times_{\mathbb{F}_q} E) \setminus \{\infty\}$.

We let

$$M_0 = \Gamma(U, \mathcal{O}_E(V)) = \bigcup_{i \geq 0} \mathcal{L}((V) + i(\infty)),$$

where $\mathcal{L}((V) + i(\infty))$ is the L-vector space of functions g on E with $\operatorname{div}(g) \geq -(V) - i(\infty)$. We make M_0 into a left $L[t, y, \tau]$-module by letting τ act by

$$\tau g = f g^{(1)}, \quad g \in M_0,$$

and letting $L[t, y]$ act by left multiplication. We find that M_0 is a projective $L[t, y]$-module of rank 1 as well as a free $L[\tau]$-module of rank 1 with basis $\{1\}$. Define the dual $\mathbf{A}$-motive

$$N_0 = \Gamma\bigl(U, \mathcal{O}_E(-(V^{(1)}))\bigr) \subseteq L[t, y]. \tag{2.6}$$

If we let $\sigma = \tau^{-1}$, then we can define a left $L[t, y, \sigma]$-module structure on N_0 by setting

$$\sigma h = f h^{(-1)}.$$

With this action N_0 is a dual $\mathbf{A}$-motive in the sense of Anderson (see [4]), and we note that N_0 is an ideal of $L[t, y]$ and that it is a free left $L[\sigma]$-module of rank 1 generated by $\delta^{(1)}$ (see [26, §3] for proofs of these facts).

A Drinfeld $\mathbf{A}$-module over L is an $\mathbb{F}_q$-algebra homomorphism

$$\rho : \mathbf{A} \to L[\tau],$$

such that for all $a \in \mathbf{A}$,

$$\rho_a = \iota(a) + b_1 \tau + \cdots + b_n \tau^n.$$

The rank r of ρ is the unique integer such that $n = r \deg a$ for all a. Thus, a rank 1 sign-normalized Drinfeld module has $r = 1$ and that $b_n = \operatorname{sgn}(a)$.

For a Drinfeld $\mathbf{A}$-module ρ, we denote the exponential and logarithm function as

$$\exp_\rho(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}, \quad \log_\rho(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{\ell_i} \in H[[z]], \quad d_0 = \ell_0 = 1.$$

Formulas for the coefficients of $\exp_\rho$ and $\log_\rho$ are given in [26, Thm. 3.4 and Cor. 3.5] as

$$\exp_\rho(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{(ff^{(1)}\cdots f^{(i-1)})|_{\Xi^{(i)}}}, \tag{2.7}$$

$$\log_\rho(z) = \sum_{i=0}^{\infty} \mathrm{Res}_\Xi\left(\frac{\widetilde{\lambda}^{(i+1)}}{ff^{(1)}\cdots f^{(i)}}\right) z^{q^i} = \sum_{i=0}^{\infty} \left(\frac{\delta^{(i+1)}}{\delta^{(1)} f^{(1)}\cdots f^{(i)}}\bigg|_\Xi\right) z^{q^i}, \tag{2.8}$$

where $\widetilde{\lambda} \in \Omega^1_{E/H}(-(V) + 2(\infty))$ is the unique differential 1-form such that $\mathrm{Res}_\Xi(\widetilde{\lambda}^{(1)}/f) = 1$. Denote the period lattice of $\exp_\rho$ as Λ_ρ. Theorem 4.6 from [26] states that Λ_ρ is a rank 1 free A-module and is generated by the fundamental period

$$\pi_\rho = -\frac{\xi^{q/(q-1)}}{\theta^q - \alpha} \prod_{i=1}^{\infty} \left(\frac{1 - \dfrac{\theta}{\alpha^{q^i}}}{1 - \left(\dfrac{m}{m\theta - \eta}\right)^{q^i} \cdot \theta + \left(\dfrac{1}{m\theta - \eta}\right)^{q^i} \cdot \eta} \right), \tag{2.9}$$

where $\xi = -(m + \beta/\alpha)$.

We now proceed to developing the theory for n-dimensional tensor powers of $\mathbf{A}$-motives and dual $\mathbf{A}$-motives. This generalizes the theory for the n-dimensional t-motives for the Carlitz module (see [33, §3.6] for the Carlitz module and [27] for Drinfeld modules). For a fixed dimension $n \geq 1$, we define the n-fold tensor power of M_0,

$$M_0^{\otimes n} = M_0 \otimes_{L[t,y]} \cdots \otimes_{L[t,y]} M_0,$$

and similarly for $N_0^{\otimes n}$. We wish to analyze $M_0^{\otimes n}$ and $N_0^{\otimes n}$ and identify them as a spaces of functions over U.

Proposition 2.1.1. *For $n \geq 1$, we have the following $L[t, y]$-module isomporphisms*

$$M_0^{\otimes n} \cong \Gamma(U, \mathcal{O}_E(nV)) \quad \textit{and} \quad N_0^{\otimes n} \cong \Gamma(U, \mathcal{O}_E(-nV^{(1)})).$$

Proof. Define the map

$$\psi : M_0 \otimes_{L[t,y]} \cdots \otimes_{L[t,y]} M_0 \to \Gamma(U, \mathcal{O}_E(nV))$$

on simple tensors for $a_i \in M_0$ as

$$a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdots a_n.$$

Looking at divisors, one quickly sees that $a_1 \cdots a_n$ is indeed in $\Gamma(U, \mathcal{O}_E(nV))$ as desired. Then it follows quickly from Proposition 5.2 of [28] that the map ψ is an $L[t, y]$-module isomorphism. The proof that $N_0^{\otimes n} \cong \Gamma(U, \mathcal{O}_E(-nV^{(1)}))$ follows similarly. □

From here on forward, we will denote

$$M := M_0^{\otimes n} = \Gamma(U, \mathcal{O}_E(nV)), \quad N := N_0^{\otimes n} = \Gamma(U, \mathcal{O}_E(-nV^{(1)})). \tag{2.10}$$

We turn M into an $L[t, y, \tau]$-module and N into an $L[t, y, \sigma]$-module by defining the action for $a \in M$ and $b \in N$ as

$$\tau a = f^n a^{(1)} \quad \text{and} \quad \sigma b = f^n b^{(-1)}. \tag{2.11}$$

Remark 2.1.2. The τ action defined on M in (2.11) is the same as the diagonal action on $M_0^{\otimes n}$, namely for $a_i \in M_0$

$$\psi(\tau(a_1 \otimes \cdots \otimes a_n)) = \psi(\tau a_1 \otimes \cdots \otimes \tau a_n) = \psi(f a_1 \otimes \cdots \otimes f a_n) = f^n \psi(a_1 \otimes \cdots \otimes a_n).$$

Thus the map ψ from Proposition 2.1.1 is actually an $L[t, y, \tau]$-module isomorphism.

For a fixed dimension $n \geq 2$, we define a set of functions which generate M as a free $L[\tau]$-module and define a second set of functions which generate N as a free $L[\sigma]$-module. We remark that for the case of $n = 1$, the present considerations do reduce to those detailed in [26, §3] for motives attached to rank 1 Drinfeld modules, but for ease of exposition we assume that $n \geq 2$. Let $[n]$ denote the multiplication-by-n map on E. Define a sequence of functions $g_i \in M$ for $1 \leq i \leq n$ with $\widetilde{\mathrm{sgn}}(g_i) = 1$ and with divisors

$$\begin{aligned}
\mathrm{div}(g_1) &= -n(V) + (n-1)(\infty) + ([n]V) \\
\mathrm{div}(g_2) &= -n(V) + (n-2)(\infty) + (\Xi) + (V^{(1)} + [n-1]V) \\
\mathrm{div}(g_3) &= -n(V) + (n-3)(\infty) + 2(\Xi) + ([2]V^{(1)} + [n-2]V) \\
&\vdots \\
\mathrm{div}(g_{n-1}) &= -n(V) + (\infty) + (n-2)(\Xi) + ([n-2]V^{(1)} + [2]V) \\
\mathrm{div}(g_n) &= -n(V) + (n-1)(\Xi) + ([n-1]V^{(1)} + V),
\end{aligned}$$

and define functions $h_i \in N$ with $\widetilde{\mathrm{sgn}}(h_i) = 1$ and with divisors

$$\begin{aligned}
\mathrm{div}(h_1) &= n(V^{(1)}) - (n+1)(\infty) + (-[n]V^{(1)}) \\
\mathrm{div}(h_2) &= n(V^{(1)}) - (n+2)(\infty) + (\Xi) + (-[n-1]V^{(1)} - V) \\
\mathrm{div}(h_3) &= n(V^{(1)}) - (n+3)(\infty) + 2(\Xi) + (-[n-2]V^{(1)} - [2]V) \\
&\vdots \\
\mathrm{div}(h_{n-1}) &= n(V^{(1)}) - (2n-1)(\infty) + (n-2)(\Xi) + (-[2]V^{(1)} - [n-2]V) \\
\mathrm{div}(h_n) &= n(V^{(1)}) - (2n)(\infty) + (n-1)(\Xi) + (-V^{(1)} - [n-1]V).
\end{aligned}$$

For ease of reference later on, we succinctly state that

$$\mathrm{div}(g_j) = -n(V) + (n-j)(\infty) + (j-1)(\Xi) + ([j-1]V^{(1)} + [n-(j-1)]V), \qquad (2.12)$$

$$\mathrm{div}(h_j) = n(V^{(1)}) - (n+j)(\infty) + (j-1)(\Xi) + (-[n-(j-1)]V^{(1)} - [j-1]V). \quad (2.13)$$

Recall that a divisor on E is principal if and only if the sum of the divisor is trivial on E [40, Cor. III.3.5] (we will use this fact implicitly going forward), and thus the divisors in (2.12) and (2.13) are principal by (2.1). Also note that the functions g_i and h_i are uniquely defined because of the $\widetilde{\mathrm{sgn}}$ condition and note that $g_i, h_i \in H(t, y)$.

Proposition 2.1.3. *For $n \geq 2$, the set of functions $\{g_i\}_{i=1}^n$ are a basis for M as a free $L[\tau]$-module and the set of functions $\{h_i\}_{i=1}^n$ are a basis for N as a free $L[\sigma]$-module.*

Proof. First observe that by the definition of the action of τ from (2.11) that the L-vector space generated by the functions $\tau^j g_i$ for $1 \leq i \leq n$ and $j \geq 0$ is contained in M. Then observe that each of the functions g_i lives in the 1-dimensional Riemann-Roch space

$$g_i \in \mathcal{L}(n(V) - (n-i)(\infty) - (i-1)(\Xi)).$$

Further, by the Riemann-Roch theorem

$$\mathcal{L}(n(V)) = \bigcup_{j=1}^{n} \mathcal{L}(n(V) - (n-j)(\infty) - (j-1)(\Xi)),$$

so that $\mathcal{L}(n(V))$ is equal to the L-span of the functions g_i. Finally, observe that

$$\deg(\tau^j g_i) = \deg((ff^{(1)} \dots f^{(j-1)})^n g_i^{(j)}) = (j-1)n + i,$$

so that the degree of each $\tau^j g_i$ is unique and that these degrees includes each nonnegative integer, thus

$$M = \bigcup_{i=1}^{\infty} \mathcal{L}(n(V) + i(\infty))$$

is equal to the L-span of the set $\{\tau^j g_i\}$ for $1 \leq i \leq n$ and $j \geq 0$. The proof for the σ-basis of the dual $\mathbf{A}$-motive N follows similarly, once we note that each h_i belongs to a 1-dimensional

Riemann-Roch space

$$h_i \in \mathcal{L}(-n(V^{(1)}) + (n+j)(\infty) - (j-1)(\Xi)).$$

We leave the details of this case to the reader. $\square$

When it is convenient, we will extend the definitions of the functions g_i and h_i for $i > n$ by writing $i = jn + k$, where $1 \leq k \leq n$, and then denoting,

$$g_i := \tau^j(g_k) = (ff^{(1)} \dots f^{(j-1)})^n g_k^{(j)} \quad \text{and} \quad h_i := \sigma^j(h_k) = (ff^{(-1)} \dots f^{(1-j)})^n h_k^{(-j)}. \tag{2.14}$$

For ease of notation later on, we also define (where $*^\top$ denotes the transpose)

$$\mathbf{g} := (g_1, \dots, g_n)^\top. \tag{2.15}$$

Remark 2.1.4. The $\mathbf{A}$-motive N is dual to the $\mathbf{A}$-motive M in a precise sense as outlined in [27, Prop. 4.3]. But, as we do not need this for the rest of the thesis, we omit the details. We do, however, record a lemma about the relationship between the functions g_i and h_i which we will need later.

Lemma 2.1.5. *We obtain the following identities of functions for* $1 \leq j \leq n-1$

$$g_1 h_1^{(-1)} = t - t([n]V),$$

$$g_{j+1} h_{n-(j-1)} = f^n \cdot (t - t([j]V^{(1)} + [n-j]V)).$$

Proof. The first identity is proved trivially, simply by comparing divisors from (2.12) and (2.13), and noting that

$$\widetilde{\operatorname{sgn}}(g_1) = \widetilde{\operatorname{sgn}}(h_1) = \widetilde{\operatorname{sgn}}(t) = 1.$$

The second follows similarly, noting that for $1 \leq j \leq n-1$

$$\operatorname{div}(g_{j+1}h_{n-(j-1)}) = \operatorname{div}(f^n \cdot (t - t([j]V^{(1)} + [n-j]V))),$$

and thus the two sides are equal up to a multiplicative constant. Then, since

$$\widetilde{\operatorname{sgn}}(g_{j+1}h_{n-(j-1)}) = \widetilde{\operatorname{sgn}}(f^n \cdot (t - t([j]V^{(1)} + [n-j]V))) = 1,$$

the equality of functions follows. □

Remark 2.1.6. The $L[t, y, \tau]$-module N and the $L[t, y, \sigma]$-module M with the actions described in (2.11) is an $\mathbf{A}$-motive and a dual $\mathbf{A}$-motive, respectively, in the precise sense described by Anderson (see [27, §4]). Because we do not require this fact going forward in the present thesis, we omit the details.

2.2 Anderson A-modules

In this section we show how to construct an Anderson $\mathbf{A}$-module from the $\mathbf{A}$-motive M of the previous section. An n-dimensional Anderson $\mathbf{A}$-module is an $\mathbb{F}_q$-algebra homomorphism $\rho : \mathbf{A} \to \operatorname{Mat}_n(L)[\tau]$, such that for $a \in \mathbf{A}$

$$\rho_a = d[a] + A_1\tau + \cdots + A_m\tau^m,$$

where $d[a] = \iota(a)I + N$ for some nilpotent matrix $N \in \operatorname{Mat}_n(L)$, and we remark that $d : \mathbf{A} \to \operatorname{Mat}_n(L)$ is a ring homomorphism. The map $\rho^{\otimes n}$ describes an action of $\mathbf{A}$ on the underlying space L^n in the sense defined in (1.9), allowing us to view L^n as an $\mathbf{A}$-module. Anderson $\mathbf{A}$-modules are a generalization of the t-modules introduced by Anderson in [1]; they are studied thoroughly in [27, §5].

Example 2.2.1 (Tensor Powers of the Carlitz Module)**.** For $\mathbf{A} = \mathbb{F}_q[t]$, define an n-dimensional

Anderson $\mathbf{A}$-module $C^{\otimes n} : \mathbb{F}_q[t] \to \mathrm{Mat}_n(\mathbb{F}_q[\theta])[\tau]$ (with the normalization $\deg(t) = 1$) by setting

$$C_t^{\otimes n} = (\theta I + N_1) + E_1 \tau.$$

Thus, for $\mathbf{z} \in L^n$,

$$C_t^{\otimes n}(\mathbf{z}) = (\theta I + N_1)\mathbf{z} + E_1 \mathbf{z}^{(1)},$$

and we extend $C^{\otimes n}$ to all of $\mathbf{A}$ by setting $C_{t^m} = C_t^m$ and using $\mathbb{F}_q$-linearity. The map $C^{\otimes n}$ is an Anderson $\mathbf{A}$-module and is called the nth tensor power of the Carlitz module.

Work by Anderson in [1, Thm. 3] for the $\mathbf{A} = \mathbb{F}_q[t]$ case, then later by Böckle and Hartl in [9, §8.6] for the more general rings $\mathbf{A}$, shows that associated to every Anderson $\mathbf{A}$-module, there is a unique, $\mathbb{F}_q$-linear power series, which we label

$$\mathrm{Exp}_\rho(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)} \in \mathrm{Mat}_{n\times 1}(\mathbb{C}_\infty[[\mathbf{z}]]),$$

where $\mathbf{z} = (z_1, \dots, z_n)^\top$, defined so that $Q_0 = I$ and that for all $a \in \mathbf{A}$ and $\mathbf{z} \in \mathbb{C}_\infty^n$

$$\mathrm{Exp}_\rho(d[a]\mathbf{z}) = \rho_a(\mathrm{Exp}_\rho(\mathbf{z})). \tag{2.16}$$

We call Exp_ρ the exponential function associated to ρ, and note that it is entire on $\mathbb{C}_\infty^n$. We also define the logarithm function associated to $\mathbf{A}$ to be the formal inverse of Exp_ρ. We label its coefficients

$$\mathrm{Log}_\rho(\mathbf{z}) = \sum_{i=0}^{\infty} P_i \mathbf{z}^{(i)} \in \mathrm{Mat}_{n\times 1}(\mathbb{C}_\infty[[\mathbf{z}]]),$$

and note that Log_ρ also satisfies a functional equation for each $a \in \mathbf{A}$

$$\mathrm{Log}_\rho(\rho_a(\mathbf{z})) = d[a] \, \mathrm{Log}_\rho(\mathbf{z}). \tag{2.17}$$

The function Log_ρ has a finite radius of convergence in $\mathbb{C}_\infty^n$, which we denote r_L.

Given the $\mathbf{A}$-motive M and the dual $\mathbf{A}$-motive N defined in (2.1.1), we now describe how to use these motives to define an Anderson $\mathbf{A}$-module. This method generalizes a technique of Thakur [45, 0.3.5] for Drinfeld modules, and also has roots in unpublished work of Anderson (see [27, §5.2]). We also refer the reader to [11] for a thorough account of the functoriality of this process in the case of t-modules. We begin by defining the t- and y-action of the $\mathbf{A}$-module, from which the rest of the action of A can be defined. These actions are defined in terms of constants coming from the functions g_i and h_i from (2.12).

Proposition 2.2.2. *There exist constants $a_i, b_i, y_i, y_i' \in H$ such that we can write for $1 \leq i \leq n$*

$$\begin{aligned} tg_i &= \theta g_i + a_i g_{i+1} + g_{i+2}, \\ yg_i &= \eta g_i + y_i g_{i+1} + y_i' g_{i+2} + g_{i+3}, \\ th_i &= \theta h_i + b_i h_{i+1} + h_{i+2}, \end{aligned}$$

where we recall the definitions of g_i and h_i for $i > n$ from (2.14).

Proof. Note that $tg_i \in M$, and hence we can write

$$tg_i = c_1 g_1 + c_2 g_2 + \cdots + c_m g_m,$$

for $c_i \in \mathbb{C}_\infty$. Examining the order of vanishing at ∞ of g_j from (2.12) and recalling that t has a pole of order 2 at ∞, we see that $c_j = 0$ for $j < i$ and $j > i + 2$. So

$$tg_i = c_i g_i + c_{i+1} g_{i+1} + c_{i+2} g_{i+2}.$$

Then, noting that $\widetilde{\operatorname{sgn}}(g_i) = \widetilde{\operatorname{sgn}}(t) = 1$ and evaluating both sides at Ξ shows that $c_{i+2} = 1$ and that $c_i = \theta$, respectively. Further, all the functions g_i are in $H(t, y)$, and so the constants c_i are as well, which finishes the proof of the first equation. The proofs of the other two equations follow similarly; we leave the details to the reader. $\square$

Given the relationship between the basis elements g_i and h_j described in Lemma 2.1.5, we also expect the coefficients a_i and b_j to be related.

Proposition 2.2.3. *For the coefficients defined in Proposition 2.2.2, for $j \leq n-1$,*

$$a_j = b_{n-j} \quad \textit{and} \quad a_n = b_n^q.$$

Proof. From Proposition 2.2.2 we calculate that

$$0 = (\theta - t)\left(\frac{g_j}{g_{j+2}} - \frac{h_{n-j}}{h_{n-j+2}}\right) + a_j \frac{g_{j+1}}{g_{j+2}} - b_{n-j}\frac{h_{n-j+1}}{h_{n-j+2}}. \tag{2.18}$$

Then using Lemma 2.1.5 yields the equality of functions

$$\frac{h_{n-j}}{h_{n-j+2}} = \frac{t - t\left([j+1]V^{(1)} + [n-(j+1)]V\right)}{t - t\left([j-1]V^{(1)} + [n-(j-1)]V\right)} \cdot \frac{g_j}{g_{j+2}},$$

and so (2.18) becomes

$$(\theta - t)\left(\frac{g_j}{g_{j+2}}\right)\left(1 - \frac{t - t\left([j+1]V^{(1)} + [n-(j+1)]V\right)}{t - t\left([j-1]V^{(1)} + [n-(j-1)]V\right)}\right) = -a_j\frac{g_{j+1}}{g_{j+2}} + b_{n-j}\frac{h_{n-j+1}}{h_{n-j+2}}.$$

From (2.12) and (2.13) we quickly see that

$$\deg_t\left((\theta - t)\left(\frac{g_j}{g_{j+2}}\right)\left(1 - \frac{t - t\left([j+1]V^{(1)} + [n-(j+1)]V\right)}{t - t\left([j-1]V^{(1)} + [n-(j-1)]V\right)}\right)\right) = 0,$$

whereas

$$\deg_t\left(a_j\frac{g_{j+1}}{g_{j+2}}\right) = \deg_t\left(b_{n-j}\frac{h_{n-j+1}}{h_{n-j+2}}\right) = -1.$$

Then, since $\widetilde{\operatorname{sgn}}(g_i) = \widetilde{\operatorname{sgn}}(h_i) = 1$, in order for the degree on the left hand side to match the degree on the right hand side, we must have that $a_j = b_{n-j}$ for $j \leq n-1$. To get the equality

$a_n = b_n^q$, we again use Proposition 2.2.2 to write

$$0 = \frac{(\theta - t)g_n}{f^n g_{n+2}} + a_n \cdot \frac{g_{n+1}}{g_{n+2}} + 1,$$

$$0 = \frac{(\theta^q - t)h_n^{(1)}}{(f^{(1)})^n h_{n+2}^{(1)}} + b_n^q \cdot \frac{h_{n+1}^{(1)}}{h_{n+2}^{(1)}} + 1.$$

Subtract these two equations and recall for $n+1 \leq k \leq 2n$ that $h_k = f^n h_{k-n}^{(-1)}$ and $g_k = f^n g_{k-n}^{(1)}$ to get

$$0 = \frac{(\theta - t)g_n}{f^n g_2^{(1)}} - \frac{(\theta^q - t)h_n^{(1)}}{(f^{(1)})^n h_2} + a_n \cdot \frac{g_1^{(1)}}{g_2^{(1)}} - b_n^q \cdot \frac{h_1}{h_2}. \tag{2.19}$$

Again, using Lemma 2.1.5 implies that

$$\frac{(\theta - t)g_n}{f^n g_2^{(1)}} \cdot \frac{(t - \theta^q)(t - t(V^{(2)} + [n-1]V^{(1)}))}{(t - \theta)(t - t([n-1]V^{(1)} + V))} = \frac{(\theta^q - t)h_n^{(1)}}{(f^{(1)})^n h_2},$$

so equation (2.19) turns into

$$\frac{(\theta - t)g_n}{f^n g_2^{(1)}} \left(1 - \frac{(t - \theta^q)(t - t(V^{(2)} + [n-1]V^{(1)}))}{(t - \theta)(t - t([n-1]V^{(1)} + V))}\right) = -a_n \cdot \frac{g_1^{(1)}}{g_2^{(1)}} + b_n^q \cdot \frac{h_1}{h_2}. \tag{2.20}$$

Again, we have

$$\deg_t \left(\frac{(\theta - t)g_n}{f^n g_2^{(1)}} \left(1 - \frac{(t - \theta^q)(t - t(V^{(2)} + [n-1]V^{(1)}))}{(t - \theta)(t - t([n-1]V^{(1)} + V))}\right)\right) = 0$$

whereas

$$\deg_t \left(-a_n \cdot \frac{g_1^{(1)}}{g_2^{(1)}} \right) = \deg_t \left(b_n^q \cdot \frac{h_1}{h_2} \right) = -1.$$

Then, since $\widetilde{\operatorname{sgn}}(g_i) = \widetilde{\operatorname{sgn}}(h_i) = 1$, in order for the degree on the left hand side to match the degree on the right hand side of (2.20), we must have that $a_n = b_n^q$. $\square$

We begin defining the Anderson $\mathbf{A}$-module associated to M, which is the nth tensor power of the Drinfeld module ρ associated to M_0, by defining

$$\rho_t^{\otimes n} := d[\theta] + E_\theta \tau := \begin{pmatrix} \theta & a_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & \theta & a_2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \theta & a_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \theta & a_{n-2} & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \theta & a_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ a_n & 1 & 0 & \dots & 0 \end{pmatrix} \tau \quad (2.21)$$

and

$$\rho_y^{\otimes n} := d[\eta] + E_\eta \tau := \begin{pmatrix} \eta & y_1 & z_1 & 1 & \dots & 0 & 0 & 0 \\ 0 & \eta & y_2 & z_2 & \dots & 0 & 0 & 0 \\ 0 & 0 & \eta & y_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \eta & y_{n-2} & z_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & \eta & y_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \eta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ z_{n-1} & 1 & 0 & 0 & \dots & 0 \\ y_n & z_n & 1 & 0 & \dots & 0 \end{pmatrix} \tau, \quad (2.22)$$

where a_i, y_i and z_i are given in Proposition 2.2.2.

To simplify notation later, we define strictly upper triangular matrices N_θ and N_η by

$$N_\theta = d[\theta] - \theta I \quad \text{and} \quad N_\eta = d[\eta] - \eta I. \quad (2.23)$$

With the definitions of $\rho_t^{\otimes n}$ and $\rho_y^{\otimes n}$, we define the $\mathbb{F}_q$-linear map

$$\rho_a^{\otimes n} : \mathbf{A} \to \mathrm{Mat}_n(H[\tau])$$

for any $a \in \mathbf{A}$ by writing $a = \sum c_i t^i + y \sum d_i t^i$ with $c_i, d_i \in \mathbb{F}_q$, and extending using linearity and the composition of maps $\rho_{t^a}^{\otimes n} = (\rho_t^{\otimes n})^a$. A priori, the map ρ is just an $\mathbb{F}_q$-linear map, but we will shortly show that it actually is an $\mathbb{F}_q$-algebra homormophism and defines an Anderson $\mathbf{A}$-module.

Remark 2.2.4. In general the coefficients a_i, y_i and z_i are not integral over H, which could lead to our chosen model for $\rho^{\otimes n}$ having bad reduction over certain places of A. We suspect that it is possible to choose a normalization which has everywhere good reduction, but this would come at the expense of having more complicated formulas e.g. not having 1's across the last non-zero super diagonals of $\rho_t^{\otimes n}$ and $\rho_y^{\otimes n}$.

Our main strategy for showing that the map $\rho^{\otimes n}$ is actually an Anderson $\mathbf{A}$-module involves constructing a second Anderson $\mathbf{A}$-module ρ' using techniques of Hartl and Juschka, then showing that the maps $\rho^{\otimes n}$ and ρ' align. In what follows, for convenience, we fix the algebraically closed field L from §2.1 to be $\mathbb{C}_\infty$. For $g \in N = \Gamma(U, \mathcal{O}_E(-nV^{(1)}))$, define the map

$$\varepsilon : N \to \mathbb{C}_\infty^n,$$

by writing g in the basis for the dual $\mathbf{A}$-motive arranged as

$$\begin{aligned} g =& d_{1,0}h_1 + d_{1,1}h_1^{(-1)} f^n + \cdots + d_{1,m}h_1^{(-m)}(ff^{(-1)}\cdots f^{(-m+1)})^n \\ &+ d_{2,0}h_2 + d_{2,1}h_2^{(-1)} f^n + \cdots + d_{2,m}h_2^{(-m)}(ff^{(-1)}\cdots f^{(-m+1)})^n \\ &\ \vdots \\ &+ d_{n,0}h_n + d_{n,1}h_n^{(-1)} f^n + \cdots + d_{n,m}h_n^{(-m)}(ff^{(-1)}\cdots f^{(-m+1)})^n, \end{aligned} \tag{2.24}$$

where $d_{i,j} \in \mathbb{C}_\infty$ and at least one of the $d_{i,m}$ is non-zero, then defining

$$\varepsilon(g) = \begin{pmatrix} d_{n,0} \\ d_{n-1,0} \\ \vdots \\ d_{1,0} \end{pmatrix} + \begin{pmatrix} d_{n,1} \\ d_{n-1,1} \\ \vdots \\ d_{1,1} \end{pmatrix}^{(1)} + \cdots + \begin{pmatrix} d_{n,m} \\ d_{n-1,m} \\ \vdots \\ d_{1,m} \end{pmatrix}^{(m)}. \tag{2.25}$$

Note that the map ε is a special case of the map δ_1 defined in [27, Prop. 5.6]. One observes immediately from the definition that ε is $\mathbb{F}_q$-linear. We then obtain a proposition similar to Lemma 3.6 from [26].

Proposition 2.2.5. *The map $\varepsilon : N \to \mathbb{C}_\infty^n$ is surjective and*

$$\ker(\varepsilon) = (1-\sigma)N = \left\{ g \in N \mid g = h^{(1)} - f^n h \text{ for some } h \in \Gamma(U, \mathcal{O}_E(-n(V))) \right\}.$$

Proof. This proposition is a special case of [27, Prop. 5.6] (note that our map ε is called δ_1 in loc. cit.), and so we encourage the reader to look there for full details. Because it is useful for certain computational examples, we briefly sketch a direct proof of Proposition 2.2.5. For $h \in \Gamma(U, \mathcal{O}_E(-n(V)))$, we have $h^{(1)} \in N$ and $\sigma(h^{(1)}) = f^n h$, so the two objects on the right are the same. Also, if we write $h^{(1)}$ using the basis and notation from (2.24), after a short calculation we find that $\varepsilon(h^{(1)}) = \varepsilon(f^n h)$, and thus $(1-\sigma)N \subseteq \ker(\varepsilon)$. To show that $\ker(\varepsilon) \subseteq (1-\sigma)N$, we note that by the proof of Proposition 2.1.3 each function on the right hand side of (2.24) has unique degree. Then for $g \in \ker(\varepsilon)$, we can construct a function $h \in \Gamma(U, \mathcal{O}_E(-n(V)))$ satisfying $g = h^{(1)} - f^n h$ through the following process. We first note that degree considerations force $\deg(h) = \deg(g) - n$, then we observe that $h^{(1)} \in N$ and so we can write $h^{(1)}$ in terms of the same basis used in (2.24) with coefficients $d'_{i,j} \in \mathbb{C}_\infty$. Next, we set $g = h^{(1)} - f^n h$ and compare coefficients of equal degree terms on each side. The fact that $g \in \ker(\varepsilon)$ allows us to solve for the coefficients $d'_{i,j}$ uniquely in terms of the coefficients of g, which proves that such a function $h \in \Gamma(U, \mathcal{O}_E(-n(V)))$ exists. $\square$

We then combine Proposition 2.2.5 with a theorem of Hartl and Juschka [27, Proposition 5.6] to obtain the following proposition.

Proposition 2.2.6. *The map $\rho^{\otimes n}$ is an Anderson* $\mathbf{A}$*-module.*

Proof. Since N is free of rank n and finitely generated as a $\mathbb{C}_\infty[\sigma]$-module, the quotient module $N/(1-\sigma)N$ is isomorphic as a $\mathbb{C}_\infty$-vector space to $\mathbb{C}_\infty^n$. We choose a basis for $N/(1-\sigma)N$

consisting of the functions $\overline{h_i}$, the images of h_i under the quotient map, then observe by Proposition 2.2.5 that this isomorphism is given by ε. This gives rise to the following commutative diagram,

$$\begin{array}{ccc} N/(1-\sigma)N & \xrightarrow{\varepsilon} & \mathbb{C}_\infty^n \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \rho'_a} \\ N/(1-\sigma)N & \xrightarrow{\varepsilon} & \mathbb{C}_\infty^n \end{array} \tag{2.26}$$

where the vertical map on the left is multiplication by $a \in \mathbf{A}$ and the vertical map on the right is the map induced by multiplication by a under the isomorphism ε. This diagram describes an action of $\mathbf{A}$ on the space $\mathbb{C}_\infty^n$, and a priori, the induced action ρ'_a is in $\mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty^n)$. However, Proposition 5.6 of Hartl and Juschka [27] shows that ρ'_a is actually in $\mathrm{Mat}_n(\mathbb{C}_\infty[\sigma])$ and that it defines an Anderson $\mathbf{A}$-module. To write down the action of ρ'_a, we only need to analyze the action of a on the basis elements h_i (we drop the overline notation, since there is no confusion), and since $\mathbf{A}$ is generated as an algebra by t and y, we only need to consider the action of t and y on the basis elements. We first note that for $1 \le i \le n-2$ and $d_i \in \mathbb{C}_\infty$ by Proposition 2.2.2 and by the definition of ε in (2.25)

$$\varepsilon(td_{n-i+1}h_i) = \varepsilon(d_{n-i+1}(\theta h_i + b_i h_{i+1} + h_{i+2})) = d_{n-i+1}(0,\dots,0,1,b_i,\theta,0,\dots,0)^\top$$

while we also have

$$\varepsilon(td_2h_{n-1}) = \varepsilon(d_2(\theta h_{n-1} + b_{n-1}h_n + \sigma(h_1))) = d_2(b_i,\theta,0,\dots,0)^\top + d_{n-1}^q(0,\dots,0,1)^\top$$

$$\varepsilon(td_1h_n) = \varepsilon(d_1(\theta h_n + b_n\sigma(h_1) + \sigma(h_2))) = d_1(\theta,0,\dots,0)^\top + d_n^q(0,\dots,0,1,b_n^q)^\top.$$

Using the identities from Proposition 2.2.3, and piecing this all together, yields

$$\varepsilon(t(d_nh_1 + \cdots + d_1h_n)) = (d[\theta] + E_\theta\tau)(d_1,\dots,d_n)^\top = \rho_t^{\otimes n}(d_1,\dots,d_n)^\top.$$

Similar analysis gives

$$\varepsilon(y(d_n h_1 + \cdots + d_n h_n)) = \rho_y^{\otimes n}(d_1, \ldots, d_n)^\top.$$

Therefore, the operators $\rho'_t = \rho_t^{\otimes n}$ and $\rho'_y = \rho_y^{\otimes n}$, and we see that the map ρ defined in (2.21) is actually an $\mathbf{A}$-module homomorphism and defines an Anderson $\mathbf{A}$-module. □

Remark 2.2.7. We comment that it is likely possible to prove that ρ is an Anderson $\mathbf{A}$-module by appealing to Mumford's work in [32] as does Thakur in [45], however, we prefer the approach inspired by Hartl and Juschka in [27].

Having proved that $\rho^{\otimes n}$ is an Anderson $\mathbf{A}$-module, we will label the exponential and logarithm function associated to $\rho^{\otimes n}$ as

$$\operatorname{Exp}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)} \in \operatorname{Mat}_{n\times 1}(\mathbb{C}_\infty[[\mathbf{z}]]) \tag{2.27}$$

and

$$\operatorname{Log}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} P_i \mathbf{z}^{(i)} \in \operatorname{Mat}_{n\times 1}(\mathbb{C}_\infty[[\mathbf{z}]]). \tag{2.28}$$

3. ANDERSON GENERATING FUNCTIONS AND PERIODS

3.1 Operators and the space Ω_0

In [5, §2.5] Anderson and Thakur define an $\mathbb{F}_q[t]$-module of functions for the Carlitz module, which they call Ω_n (our notation for this module is Ω_0), which vanish under the operator $\tau-(t-\theta)^n$ (we remark that the shtuka function for the Carlitz module is $(t-\theta)$). They then connect this space of functions to the period lattice of the exponential function by expressing a function $h \in \Omega_n$ in terms of $t-\theta$, then analyzing the principal part h in this expansion. Of particular note, they construct an ancillary vector-valued function $\tilde{h}$ which they use to aid their calculations in the proof of their period formulas. In the case of tensor powers of Drinfeld $\mathbf{A}$-modules, we apply similar techniques using a space of functions Ω_0 which vanish under the operator $\tau - f^n$. However, we found it necessary to rely entirely upon the equivalent version of $\tilde{h}$, rather than using it as an ancillary tool. Because of this, in this section we develop a vector setting in which we can embed the space Ω_0 and analyze vector-valued operators on it.

For a fixed a dimension n define

$$\mathbb{B} := \Gamma\big(\mathcal{U}, \mathcal{O}_E(-n(V)+n(\Xi))\big)$$

where $\mathcal{U}$ is the inverse image under t of the closed disk in $\mathbb{C}_\infty$ of radius $|\theta|$ centered at 0 defined in §1.2. Define the $\mathbf{A}$-module

$$\Omega = \{h \in \mathbb{B} \mid h^{(1)} - f^n h \in N\}, \tag{3.1}$$

where we recall the definition of N from §2.1. Also define a submodule of Ω as

$$\Omega_0 = \{h \in \mathbb{B} \mid h^{(1)} - f^n h = 0\}. \tag{3.2}$$

For a function $h(t,y) \in \Omega$, define the map $T : \Omega \to \mathbb{T}[y]^n$ by

$$T(h(t,y)) = \begin{pmatrix} h(t,y) \cdot g_1 \\ h(t,y) \cdot g_2 \\ \vdots \\ h(t,y) \cdot g_n \end{pmatrix}, \tag{3.3}$$

where the functions g_i are the basis elements defined in (2.12). We observe immediately that T is $\mathbb{F}_q$-linear and injective.

Remark 3.1.1. The map T can be viewed as a generalization of the $\widetilde{\square}$ operator defined by Anderson and Thakur in the proof of 2.5.5 of [5], where they define for $h(t) \in \mathbb{T}$,

$$\tilde{h}(t) = \begin{pmatrix} h(t) \cdot 1 \\ h(t) \cdot (t-\theta) \\ \vdots \\ h(t) \cdot (t-\theta)^{n-1} \end{pmatrix}.$$

Note that the function $t-\theta$, aside from being a uniformizer at Ξ, is also the shtuka function for the Carlitz module, and that it shows up in the τ-basis for the $\mathbf{A}$-motive associated to the nth tensor power of the Carlitz module (see [33, §3.6]). It is not immediately obvious which of these notions leads to the correct generalization of $\widetilde{\square}$ for Anderson $\mathbf{A}$-modules. After noticing properties such as Lemma 3.1.2 and Theorem 3.2.7, however, it seems clear that the definition of $T(\cdot)$ is the correct generalization for the present concerns.

Define operators on the space $\mathbb{T}[y]^n$ which act in the sense defined in §1.2 by setting

$$D_t := \rho_t^{\otimes n} - t, \quad \text{and} \quad D_y = \rho_y^{\otimes n} - y.$$

Lemma 3.1.2. *For* $h \in \Omega_0$,

$$D_t(T(h)) = D_y(T(h)) = 0.$$

Proof. Using (2.2.2) and the fact that $h \in \Omega_0$, observe that

$$t \cdot T(h) = \begin{pmatrix} th(t,y) \cdot g_1 \\ th(t,y) \cdot g_2 \\ \vdots \\ th(t,y) \cdot g_n \end{pmatrix} = \begin{pmatrix} h(t,y) \cdot (\theta g_1 + a_1 g_2 + g_3) \\ h(t,y) \cdot (\theta g_2 + a_2 g_3 + g_4) \\ \vdots \\ h(t,y) \cdot (\theta g_n + a_n g_1^{(1)} f^n + g_2^{(1)} f^n) \end{pmatrix} = d[\theta]T(h) + E \cdot T(h)^{(1)}.$$

Thus we see that $\rho_t^{\otimes n}(\tilde{h}) = t \cdot T(h)$ and so $D_t(T(h)) = 0$. A similar argument shows that $D_y(T(h)) = 0$. $\square$

Define an additional operator on $\mathbb{T}[y]^n$,

$$G - E_1\tau := \begin{pmatrix} g_2/g_1 & -1 & 0 & \dots & 0 \\ 0 & g_3/g_2 & -1 & \dots & 0 \\ 0 & 0 & g_4/g_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_1^{(1)} f^n / g_n \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \tau. \tag{3.4}$$

A quick calculation shows that for any $h \in \Omega_0$

$$[G - E_1\tau](T(h)) = 0,$$

and thus the operator $G - E_1\tau$ can be viewed as a vector version of the operator $\tau - f^n$. In fact, the relationship is even stronger, as proved in the following lemma.

Lemma 3.1.3. *A vector $J(t,y) \in \mathbb{T}[y]^n$ satisfies $(G - E_1\tau)(J) = 0$ if and only if there exists some function $h(t,y) \in \Omega_0$ such that*

$$J(t,y) = T(h(t,y)).$$

Proof. We have already seen above that $(G - E_1\tau)(T(h)) = 0$ for all $h \in \Omega_0$. For the other direction, suppose that $J(t,y) \in \mathbb{T}[y]^n$ satisfies $(G - E_1\tau)(J) = 0$. Then, if we denote $J =$

$(j_1, \ldots, j_n)^\top$, writing out the action of $G - E_1\tau$ on each coordinate gives equations

$$\begin{aligned}
j_1 \frac{g_2}{g_1} - j_2 &= 0 \\
j_2 \frac{g_3}{g_2} - j_3 &= 0 \\
&\vdots \\
j_{n-1} \frac{g_n}{g_{n-1}} - j_n &= 0 \\
j_n \frac{g_1^{(1)} f^n}{g_n} - j_1^{(1)} &= 0.
\end{aligned} \tag{3.5}$$

Solving the first equation for j_2 and then substituting it into the second, and so on, gives the equality of vectors

$$\begin{pmatrix} j_1 \\ j_2 \\ \vdots \\ j_n \end{pmatrix} = \begin{pmatrix} j_1 \\ j_1 \cdot g_2/g_1 \\ \vdots \\ j_1 \cdot g_n/g_1 \end{pmatrix}.$$

From this, we also get the equality $(\tau - f^n)(j_1/g_1) = 0$, so we see that $J = T(j_1/g_1)$ with $j_1/g_1 \in \Omega_0$ as desired. $\square$

We use the quotient functions g_{k+1}/g_k frequently throughout this section, so we briefly describe some of their properties. Using the notation for $k > n$ for g_k from (2.14), the quotients have divisors

$$\operatorname{div}(g_{k+1}/g_k) = (\Xi) - (\infty) + ([k]V^{(1)} + [n-k]V) - ([k-1]V^{(1)} + [n-(k-1)]V), \tag{3.6}$$

for $1 \leq k \leq n$. Thus we can write these functions as a quotient of a linear function of degree 3 and a linear function of degree 2, which we label

$$\frac{\nu_k(t,y)}{\delta_k(t)} := \frac{y - \eta - m_k(t-\theta)}{t - t([k-1]V^{(1)} + [n-(k-1)]V)} = \frac{g_{k+1}}{g_k}, \tag{3.7}$$

for $1 \leq k \leq n$, where m_k is the slope between the points $[k]V^{(1)} + [n-k]V$ and $[-(k-1)]V^{(1)} - [n-(k-1)]V$.

Remark 3.1.4. The functions g_{k+1}/g_k share many similarities with the shtuka funciton f, and the vector $(g_2/g_1, \dots, g_{n_1}/g_n)^\top$ can be viewed as a vector version of the shtuka function; in fact, the divisor of g_{k+1}/g_k matches with the divisor of the shtuka function, except that the points $V^{(1)}$ and V in $\operatorname{div}(f)$ from (2.2) are shifted by $([k-1]V^{(1)} + [n-k]V)$.

With the above analysis we are now equipped to give explicit formulas for the coefficients a_i from Proposition 2.2.2, which determine the action of $\rho_t^{\otimes n}$.

Corollary 3.1.5. *The coefficients a_i from Proposition 2.2.2 are given by*

$$a_i = \frac{2\eta + c_1\theta + c_3}{\theta - t([i]V^{(1)} + [n-i]V)}$$

Proof. Dividing both sides of the first equation from Proposition 2.2.2 by g_{i+1} and evaluating at the point $-\Xi$ gives

$$a_i = -\frac{g_{i+2}}{g_{i+1}}\bigg|_{-\Xi}.$$

Using expression (3.7) for $k = i+1$ we find

$$-\frac{g_{i+2}}{g_{i+1}}\bigg|_{-\Xi} = \frac{2\eta + c_1\theta + c_3}{\theta - t([i]V^{(1)} + [n-i]V)}.$$

□

Remark 3.1.6. In order to get formulas for y_i and z_i one can equate the coordinates on both sides of the identity

$$\rho^{\otimes n}_{\eta^2 + c_1\eta\theta + c_3\eta} = \rho^{\otimes n}_{\theta^3 + c_2\theta^2 + c_4\theta + c_6}$$

and solve for the coefficients y_i and z_i in terms of a_i. We do not use this fact going forward, and thus we omit the details.

Define the operator

$$M_\tau := N_1 + E_1\tau,$$

where we recall the definition of the matrices N_i and E_i from (1.2). Denote the diagonal matrix

$$M_m := \operatorname{diag}(z_1 - a_2, z_2 - a_3, \ldots, z_{n-1} - a_n, z_n - a_1^{(1)}), \tag{3.8}$$

where a_i and z_i are the constants from Proposition 2.2.2 and denote the diagonal matrix of functions in $H[t, y]$

$$M_\delta := \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n). \tag{3.9}$$

Proposition 3.1.7. *We have the operator decomposition*

$$(G - E_1\tau) = M_\delta^{-1}(D_y - (M_\tau + M_m)D_t).$$

Proof. We first compute using the definitions (2.21) and (2.22) and the definitions given above that

$$D_y - M_\tau D_t - M_m D_t = M', \tag{3.10}$$

where

$$\begin{aligned}
M' &:= M_1' + M_2'\tau \\
&:= \begin{pmatrix}
\eta-y-(\theta-t)(z_1-a_2) & y_1-(\theta-t)-a_1(z_1-a_2) & 0 & \ldots & 0 \\
0 & \eta-y-(\theta-t)(z_2-a_3) & y_2-(\theta-t)-a_2(z_2-a_3) & \ldots & 0 \\
0 & 0 & \eta-y-(\theta-t)(z_3-a_4) & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & y_{n-1}-(\theta-t)-a_{n-1}(z_{n-1}-a_n) \\
0 & 0 & 0 & \ldots & \eta-y-(\theta-t)(z_n-a_1^{(1)})
\end{pmatrix} \\
&\quad + \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_n-(\theta-t)-a_n(z_n-a_1^{(1)}) & 0 & 0 & \ldots & 0
\end{pmatrix}\tau.
\end{aligned}$$

If we define $\mathbf{g} := (g_1, \ldots, g_n)^\top$, then by Proposition 2.2.2 we observe that

$$d[\theta]\mathbf{g} + E_\theta f^n \mathbf{g}^{(1)} = 0 \quad \text{and} \quad d[\eta]\mathbf{g} + E_\eta f^n \mathbf{g}^{(1)}.$$

Then from (3.10), we observe that

$$M_1'\mathbf{g} + M_2' f^n \mathbf{g}^{(1)} = 0. \tag{3.11}$$

Examining the coordinates of the above equation gives the equations for $1 \leq k \leq n-1$

$$\frac{g_{k+1}}{g_k} = \frac{y - \eta - (\theta - t)(z_k - a_{k+1})}{t - \theta + y_k - a_k(z_k - a_{k+1})}, \tag{3.12}$$

and

$$\frac{f^n g_1^{(1)}}{g_n} = \frac{y - \eta - (\theta - t)(z_n - a_1^{(1)})}{t - \theta + y_n - a_n(z_n - a_1^{(1)})}.$$

Comparing these formulas with the notation established in (3.7) shows that for $1 \leq k \leq n-1$

$$m_k = z_k - a_{k+1} \quad \text{and} \quad \delta_k = t - \theta + y_k - a_k m_k$$

and

$$m_n = z_n - a_1^{(1)} \quad \text{and} \quad \delta_n = t - \theta + y_n - a_n m_n.$$

With these observations, we then identify

$$M' = M_\delta(G - E_1\tau),$$

so that

$$(G - E_1\tau) = M_\delta^{-1}(D_y - (M_\tau + M_m)D_t).$$

$\square$

Remark 3.1.8. Note the similarity of this decomposition to that in [26, Prop. 4.1].

Corollary 3.1.9. *Define the following matrices*

$$M_1 = M_1'\big|_{t=0,y=0} \quad \textit{and} \quad M_2 = M_2'\big|_{t=0,y=0},$$

with M_1' and M_2' as in (3.10). *Then*

$$\rho_y^{\otimes n} - (M_\tau + M_m)\rho_t^{\otimes n} = M_1 + M_2\tau.$$

Proof. After multiplying both sides by M_δ, the matrices in Proposition 3.1.7 have coefficients in $\overline{K}[t,y]$, and equating the constant terms gives the corollary. $\square$

Define the function

$$\omega_\rho = \xi^{1/(q-1)} \prod_{i=0}^{\infty} \frac{\xi^{q^i}}{f^{(i)}}, \quad \xi = -\frac{m\theta - \eta}{\alpha} = -\left(m + \frac{\beta}{\alpha}\right), \tag{3.13}$$

where m, α, and β are given in §2.1 and recall that $\omega_\rho \in \mathbb{T}[y]^\times$ (see [26, §4], for details of convergence). Note that

$$(\omega_\rho^n)^{(1)} = f^n \omega_\rho^n, \tag{3.14}$$

and thus $\omega_\rho^n \in \Omega_0$. The idea behind the function ω_ρ comes originally from a similar function ω_C defined for tensor powers of the Carlitz module by Anderson and Thakur in [5, §2.5]. Papanikolas and the author genrealized the function ω_C to Drinfeld modules in [26]. Angl'es, Pellarin and Tavares Ribeiro also used this function in [8].

Proposition 3.1.10. *The function ω_ρ^n generates Ω_0 as a free $\mathbf{A}$-module.*

Proof. The proof follows similarly to the proof of [26, Prop. 4.3]. Since all of the zeros and poles of ω_ρ^n lie outside the inverse image under t of the closed unit disk in $\mathbb{C}_\infty$ the function $\omega_\rho^n \in \mathbb{T}[y]^\times$. Then, for any $h \in \Omega_0$ the quotient h/ω_ρ^n is fixed under twisting and thus is in $\mathbf{A}$, and we see that $h = a\omega_\rho^n$ for some $a \in \mathbf{A}$. $\square$

3.2 Anderson generating functions and periods

Anderson and Thakur studied the period lattice of the n-fold tensor power of the Carlitz module in [5], where they find succinct formulas for the last coordinate of a fundamental period. On the other hand, Gekeler, Goss, Thakur, Papanikolas and Lutes and Papanikolas and Chang have studied the fundamental period associated to (1-dimensional) Drinfeld modules (see [21, §III], [24, §7.10], [31, Ex. 4.15], [43, §3], [14] and [15]). More recently, Papanikolas and the author studied periods of rank 1 sign-normalized Drinfeld modules in [26] using Anderson generating functions. This section generalizes the work of both Anderson and Thakur and of Papanikolas and the author; we develop the theory of periods of n-fold tensor powers of rank 1 sign-normalized Drinfeld modules. We remark that because of the additional complexity arising from generalizing in both these directions, our methods required several new ideas, distinct from the works mentioned above. In particular, while the residue formula presented in Proposition 3.2.5 is nearly trivial in the 1-dimensional case, its proof for the n-dimensional case required several new technical insights to account for the higher order poles present in vector-valued Anderson generating functions.

We now define and study vector-valued Andreson generating functions in dimension n. Such functions are used in the proof of Theorem 2.5.5 in [5] for the case of tensor powers of the Carlitz module; here we define them for Anderson $\mathbf{A}$-modules. For $\mathbf{u} = (u_1, ..., u_n)^\top \in \mathbb{C}_\infty^n$ define

$$E_{\mathbf{u}}^{\otimes n}(t) := \begin{pmatrix} e_1(t) \\ \vdots \\ e_n(t) \end{pmatrix} := \sum_{i=0}^{\infty} \operatorname{Exp}_\rho^{\otimes n}\left(d[\theta]^{-i-1}\mathbf{u}\right) t^i, \tag{3.15}$$

then define

$$G_{\mathbf{u}}^{\otimes n}(t,y) := E_{d[\eta]\mathbf{u}}^{\otimes n}(t) + (y + c_1 t + c_3) E_{\mathbf{u}}^{\otimes n}(t). \tag{3.16}$$

We will shortly discuss the convergence of $E_{\mathbf{u}}^{\otimes n}$ and $G_{\mathbf{u}}^{\otimes n}$ as functions in Tate algebras, but before proceeding we require two brief lemmas.

Lemma 3.2.1. *Given an upper triangular matrix $M \in \operatorname{Mat}_n(\mathbb{T})$ with eigenvalues $\lambda_i \in \mathbb{T}$, the* *series*

$$\sum_{i=0}^{\infty} M^i$$

converges with respect to $\|\cdot\|$ *and equals* $(I-M)^{-1}$ *if and only if* $|\lambda_i| < 1$ *for all* $1 \leq i \leq n$.

Proof. This is essentially a standard result from linear algebra, so we only sketch the proof. We write $M = D + N$ where D is the diagonal matrix consisting of eigenvalues and N is a strictly upper triangular matrix. Then we write $M^i = (D+N)^i$ and expand $(D+N)^i$ to find that any term with n or more copies of N vanishes. Thus $\|M^i\| \to 0$ as $i \to 0$ if and only if $|\lambda_i| < 1$. □

Lemma 3.2.2. *The coordinates of the matrix*

$$(d[\eta] - y)\,(d[\theta] - t)^{-1},$$

are regular at Ξ, *where* $d : \mathbf{A} \to \mathrm{Mat}_n(H)$ *is the ring homomorphism from §2.2.*

Proof. For ease of exposition in this proof we will assume that the elliptic curve E has the simplified Weierstrass equation $E: \ y^2 = t^3 + At + B$ for $A, B \in \mathbb{F}_q$. The lemma holds for the more general Weierstrass equation (1.5) and we leave the extra details to the reader. Observe using the simplified Weierstrass equation together with the fact that $d : \mathbf{A} \to \mathrm{Mat}_n(H)$ is a (commutative) ring homomorphism that

$$\begin{aligned}
(d[\eta] - y)\,(d[\theta] - t)^{-1} &= (d[\eta] - y)(d[\eta] + y)(d[\eta] + y)^{-1}\,(d[\theta] - t)^{-1} \\
&= (d[\eta^2] - y^2)(d[\eta] + y)^{-1}\,(d[\theta] - t)^{-1} \\
&= ((d[\theta^3] - t^3) + A(d[\theta] - t))(d[\eta] + y)^{-1}\,(d[\theta] - t)^{-1} \\
&= ((d[\theta^2] + td[\theta] - t^2) + A)(d[\eta] + y)^{-1},
\end{aligned}$$

where in the last equality we factored out $(d[\theta] - t)$ and canceled. Note that $(d[\eta] + y)^{-1}$ and $(d[\theta^2] + td[\theta] - t^2) + A$ are coordinate-wise regular at Ξ and thus so is $(d[\eta] - y)\,(d[\theta] - t)^{-1}$. □

For the case of $n = 1$ and $A = \mathbb{F}_q[\theta]$, El-Guindy and Papanikolas give a detailed proof that

Anderson generating functions are in $\mathbb{T}$ and that they have a meromorphic continuation to $\mathbb{C}_\infty$ in [19] - the original result is due to Anderson. We give a similar theorem for $E_{\mathbf{u}}^{\otimes n}$ and $G_{\mathbf{u}}^{\otimes n}$.

Proposition 3.2.3. *For $\mathbf{u} \in \mathbb{C}_\infty^n$, the function $E_{\mathbf{u}}^{\otimes n} \in \mathbb{T}^n$ and we have the following identity of functions in $\mathbb{T}^n$*

$$E_{\mathbf{u}}^{\otimes n}(t) = \sum_{j=0}^{\infty} Q_j \left(d[\theta]^{(j)} - tI\right)^{-1} \mathbf{u}^{(j)},$$

where Q_i are the coefficients of $\operatorname{Exp}_\rho^{\otimes n}$ from (2.27)*. Further, the function $G_{\mathbf{u}}^{\otimes n}$ extends to a meromorphic function on $U = (\mathbb{C}_\infty \times_{\mathbb{F}_q} E) \setminus \{\infty\}$ with poles in each coordinate only at the points $\Xi^{(i)}$ for $i \geq 0$.*

Proof. Writing in the definition of $\operatorname{Exp}_\rho^{\otimes n}$ from (2.27) and expanding gives the sum

$$E_{\mathbf{u}}^{\otimes n}(t) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} Q_j \left(d[\theta]^{-i-1}\mathbf{u}\right)^{(j)} \right) t^i. \tag{3.17}$$

Recall from (2.23) that $d[\theta] = \theta I + N_\theta$ where N_θ is nilpotent with order n, so we can write

$$\begin{aligned}
\left(d[\theta]^{-i-1}\right) &= \left((\theta I + N_\theta)^{-i-1}\right) \\
&= \left(\left(\frac{1}{\theta} I - \frac{1}{\theta^2} N_\theta + \cdots + \frac{(-1)^{n-1}}{\theta^n} N_\theta^{n-1} \right)^{i+1} \right) \\
&= \left(\left[\sum_{k_1 + \cdots + k_n = i+1} \binom{i+1}{k_1, \ldots, k_n} \prod_{s=1}^{n} \left(\frac{1}{\theta^s} N_\theta^{s-1} \right)^{k_s} \right] \right) \\
&= \left(\left(\frac{1}{\theta^{i+1}} I + d_1 \frac{1}{\theta^{i+2}} N_\theta + \cdots + d_{n-1} \frac{1}{\theta^{i+n}} N_\theta^{n-1} \right) \right),
\end{aligned} \tag{3.18}$$

where in the last two lines we used the multinomial theorem then collected like terms using some constants $d_i \in \mathbb{F}_q$. Using the last line of (3.18) we find that

$$\left| \sum_{j=0}^{\infty} Q_j \left(d[\theta]^{-i-1}\mathbf{u}\right)^{(j)} \right| \leq \max_j \left\{ |Q_j| \cdot |\theta|^{-iq^j} \max_{1 \leq k \leq n} \left\{ \left| \frac{1}{\theta^k} N_\theta^{k-1} \right| \right\}^{q^j} \cdot |\mathbf{u}|^{q^j} \right\}, \tag{3.19}$$

where $|\cdot|$ is the matrix seminorm defined in §1.2. Let us denote

$$N_0 = \max_{1\leq k\leq n}\left\{\left|\frac{1}{\theta^k}N_\theta^{k-1}\right|\right\},$$

which equals some constant independent of i and j. Then, the fact that $\operatorname{Exp}_\rho^{\otimes n}$ is an entire function on $\mathbb{C}_\infty^n$, implies that the factor

$$|Q_j|\cdot\max_{1\leq k\leq n}\left\{\left|\frac{1}{\theta^k}N_\theta^{k-1}\right|\right\}^{q^j}\cdot|\mathbf{u}|^{q^j}$$

goes to zero as $j\to\infty$, and thus is bounded independent of j. Thus by (3.19)

$$\left|\sum_{j=0}^{\infty}Q_j\left(d[\theta]^{-i-1}\mathbf{u}\right)^{(j)}\right|$$

goes to zero as $i\to\infty$, which proves that $E_{\mathbf{u}}^{\otimes n}\in\mathbb{T}^n$. Further, using the above analysis, we find that

$$\left|Q_j\left(d[\theta]^{-i-1}\mathbf{u}\right)^{(j)}\right|\to 0,$$

as $\max(i,j)\to 0$, and thus we are allowed to rearrange the terms of the double sum (3.17) and maintain convergence in $\mathbb{T}^n$ (see [39, §1.2]).

Next, observe that the eigenvalues of the matrix $d[\theta]^{-1}t$ are all equal to t/θ, and that $\|t/\theta\|<1$, and hence by Lemma 3.2.1 we have the geometric series identity in $\mathbb{T}^n$

$$\sum_{i=0}^{\infty}d[\theta]^{-i-1}t^i=\left(d[\theta]^{(j)}-tI\right)^{-1}.$$

Using this we rearrange the terms of $E_{\mathbf{u}}^{\otimes n}$ to get the equality in $\mathbb{T}^n$

$$\begin{aligned} E_{\mathbf{u}}^{\otimes n} &= \sum_{j=0}^{\infty} Q_j \left(\sum_{i=0}^{\infty} d[\theta]^{-i-1} t^i \right)^{(j)} \mathbf{u}^{(j)} \\ &= \sum_{j=0}^{\infty} Q_j \left(d[\theta]^{(j)} - tI \right)^{-1} \mathbf{u}^{(j)}. \end{aligned}$$

Using the above equation, we see that

$$G_{\mathbf{u}}^{\otimes n} = \sum_{j=0}^{\infty} Q_j \left(d[\theta]^{(j)} - tI \right)^{-1} \left(d[\eta]^{(j)} + (y + c_1 t + c_3) I \right) \mathbf{u}^{(j)} \in \mathbb{T}[y]^n. \tag{3.20}$$

We then observe, using analysis similar to that in (3.18), that for any $m \geq 0$ the sum

$$\begin{aligned} &\sum_{j=0}^{\infty} Q_j \left(d[\theta]^{(j)} - tI \right)^{-1} \left(d[\eta]^{(j)} + (y + c_1 t + c_3) I \right) \mathbf{u}^{(j)} \\ &\qquad - \sum_{j=0}^{m} Q_j \left(d[\theta]^{(j)} - tI \right)^{-1} \left(d[\eta]^{(j)} + (y + c_1 t + c_3) I \right) \mathbf{u}^{(j)} \end{aligned}$$

converges for any point $(t, y) \in U$ with $|t| < |\theta|^{m+1}$, providing a meromorphic continuation of $G_{\mathbf{u}}^{\otimes n}$ to U. We also observe that the only possible poles in each coordinate of

$$\sum_{j=0}^{m} Q_j \left(d[\theta]^{(j)} - tI \right)^{-1} \left(d[\eta]^{(j)} + (y + c_1 t + c_3) I \right) \mathbf{u}^{(j)} \in H(t, y)^n \tag{3.21}$$

occur at $\pm \Xi^{(i)}$ for $i \leq m$. We calculate that each coordinate of $G_{\mathbf{u}}^{\otimes n}$ does actually have poles at the positive twists of Ξ (see the proof of Proposition 3.2.5 for more details). On the other hand, under the substitution given by negation on E, namely $(t, y) \mapsto (t, -y - c_1 t - c_3)$ we see that

$$\left(d[\theta]^{(j)} - tI \right) \left(d[\eta]^{(j)} + (y + c_1 t + c_3) I \right) \mapsto \left(d[\theta]^{(j)} - tI \right) \left(d[\eta]^{(j)} - yI \right),$$

and so by Lemma 3.2.2 we see that each coordinate of (3.21) is regular at $-\Xi^{(j)}$ for $j \geq 0$. Thus the meromorphic continuation described above has the correct properties. $\square$

Lemma 3.2.4. *For* $\mathbf{u} \in \mathbb{C}_\infty^n$, *we obtain two identities*

$$\begin{aligned}
&\text{(a) } D_t(G_{\mathbf{u}}^{\otimes n}) = \operatorname{Exp}_\rho^{\otimes n}(d[\eta]\mathbf{u}) + (y + c_1 t + c_3) \operatorname{Exp}_\rho^{\otimes n}(\mathbf{u}) \\
&\text{(b) } D_y(G_{\mathbf{u}}^{\otimes n}) = - c_1 \operatorname{Exp}_\rho^{\otimes n}(d[\eta]\mathbf{u}) + \operatorname{Exp}_\rho^{\otimes n}(d[\theta^2]\mathbf{u}) + (t + c_2) \operatorname{Exp}_\rho^{\otimes n}(d[\theta]\mathbf{u}) \\
&\qquad\qquad + (t^2 + c_2 t + c_4) \operatorname{Exp}_\rho^{\otimes n}(\mathbf{u}).
\end{aligned}$$

Proof. First observe that

$$E_{d[\theta]\mathbf{u}}^{\otimes n} = \rho_t^{\otimes n}(E_{\mathbf{u}}^{\otimes n}) = \sum_{i=0}^{\infty} \rho_t^{\otimes n} \left(\operatorname{Exp}_\rho^{\otimes n} \left(d[\theta]^{-i-1}\mathbf{u} \right) \right) t^i = \operatorname{Exp}_\rho^{\otimes n}(\mathbf{u}) + tE_{\mathbf{u}}^{\otimes n}, \tag{3.22}$$

and thus

$$D_t(E_{\mathbf{u}}^{\otimes n}) = \operatorname{Exp}_\rho^{\otimes n}(\mathbf{u}).$$

Part (a) of the lemma follows directly from this. For part (b), observe that

$$\rho_y^{\otimes n}(E_{\mathbf{u}}^{\otimes n}) = \sum_{i=0}^{\infty} \left(\operatorname{Exp}_\rho^{\otimes n} \left(d[\eta] d[\theta]^{-i-1}\mathbf{u} \right) \right) t^i = E_{d[\eta]\mathbf{u}}^{\otimes n},$$

and so using (1.5)

$$\begin{aligned}
\rho_y^{\otimes n}(E_{d[\eta]\mathbf{u}}^{\otimes n}) &= E_{d[\eta^2]\mathbf{u}}^{\otimes n} \\
&= E_{d[\theta^3 + c_2\theta^2 + c_4\theta + c_6 - c_1\theta\eta - c_3\eta]\mathbf{u}} \\
&= E_{d[\theta^3]\mathbf{u}} + c_2 E_{d[\theta^2]\mathbf{u}} + c_4 E_{d[\theta]\mathbf{u}} + c_6 E_{\mathbf{u}} - c_1 E_{d[\theta\eta]\mathbf{u}} - c_3 E_{d[\eta]\mathbf{u}}
\end{aligned}$$

Then substituting in the above equation, canceling and using (1.5) we write

$$\begin{aligned}
D_y(G_{\mathbf{u}}^{\otimes n}) &= \rho_y(E_{d[\eta]\mathbf{u}}^{\otimes n}) - yE_{d[\eta]\mathbf{u}}^{\otimes n} + (y + c_1 t + c_3) E_{d[\eta]\mathbf{u}}^{\otimes n} - (y^2 + c_1 t y + c_3 y) E_{\mathbf{u}}^{\otimes n} \\
&= E_{d[\theta^3]\mathbf{u}} + c_2 E_{d[\theta^2]\mathbf{u}} + c_4 E_{d[\theta]\mathbf{u}} + c_6 E_{\mathbf{u}} - c_1 E_{d[\theta\eta]\mathbf{u}} - c_3 E_{d[\eta]\mathbf{u}} \\
&+ (c_1 t + c_3) E_{d[\eta]\mathbf{u}}^{\otimes n} - (t^3 + c_2 t^2 + c_4 t + c_6) E_{\mathbf{u}}^{\otimes n}
\end{aligned}$$

We then use (3.22) to get part (b) of the lemma. ☐

Define $\mathcal{M}$ to be the subring of $\mathbb{T}[y]$ consisting of all elements in $\mathbb{T}[y]$ which have a meromorphic continuation to all of U (see [20]). Now define the map

$$\mathrm{RES}_\Xi : \mathcal{M}^n \to \mathbb{C}_\infty^n,$$

for a vector of functions $(z_1(t,y), ..., z_n(t,y))^\top \in \mathcal{M}^n$ as

$$\mathrm{RES}_\Xi \begin{pmatrix} z_1(t,y) \\ \vdots \\ z_n(t,y) \end{pmatrix} = \begin{pmatrix} \mathrm{Res}_\Xi(z_1(t,y)\lambda) \\ \vdots \\ \mathrm{Res}_\Xi(z_n(t,y)\lambda) \end{pmatrix}, \tag{3.23}$$

where λ is the invariant differential of E from (1.6). We remark that in defining the maps T and $\mathrm{RES}_{\Xi^{(i)}}$, we were partially inspired by ideas of Sinha in [41, §4.6.6]. We now analyze the residues of the Anderson generating function $G_{\mathbf{u}}^{\otimes n}$ under the map RES_Ξ.

Proposition 3.2.5. *If we write* $\mathbf{u} = (u_1, ..., u_n)^\top \in \mathbb{C}_\infty^n$*, then*

$$\mathrm{RES}_\Xi(G_{\mathbf{u}}^{\otimes n}) = -(u_1, ..., u_n)^\top.$$

Proof. Again, for ease of exposition in this proof we will assume that the elliptic curve E has the simplified Weierstrass equation $E:\ y^2 = t^3 + At + B$ for $A, B \in \mathbb{F}_q$. The proposition holds for the more general Weierstrass equation (1.5) and we leave the extra details to the reader. Equation (3.20) gives

$$G_{\mathbf{u}}^{\otimes n} = \sum_{j=0}^{\infty} Q_j \left(d[\theta]^{(j)} - tI\right)^{-1} \left(d[\eta]^{(j)} + yI\right) \mathbf{u}^{(j)},$$

so when we calculate $\mathrm{RES}_\Xi(G_{\mathbf{u}}^{\otimes n}\lambda)$, we find that the only possible contributions to the residues come from the $j = 0$ term, since $\left(d[\theta]^{(j)} - tI\right)^{-1}$ is regular at Ξ in each coordinate for $j \geq 1$. In particular, we find that

$$\mathrm{RES}_\Xi(G_{\mathbf{u}}^{\otimes n}) = \mathrm{RES}_\Xi\left((d[\eta]+yI)\,(d[\theta]-tI)^{-1}\,\mathbf{u}\right),$$

and further that

$$\begin{aligned}(d[\eta]+yI)\,(d[\theta]-tI)^{-1}\,\lambda &= (d[\eta]+yI)\,(d[\theta]-tI)^{-1}\cdot\frac{dt}{2y}\\ &= \frac{1}{2}\,(2d[\eta]-(d[\eta]-y))\,(d[\theta]-tI)^{-1}\left(\frac{1}{y}-d[\eta]^{-1}+d[\eta]^{-1}\right)dt\\ &= \frac{1}{2}\,(2d[\eta]-(d[\eta]-y))\,(d[\theta]-tI)^{-1}\left(\frac{d[\eta]^{-1}}{y}(d[\eta]-yI)+d[\eta]^{-1}\right)dt\end{aligned} \tag{3.24}$$

After multiplying out the factors in the last line of (3.24), using Lemma 3.2.2 we find that the only term whose coordinates have poles at Ξ is $(d[\theta]-t)^{-1}$. Thus we see that

$$(d[\eta]+yI)\,(d[\theta]-tI)^{-1}\,\lambda = (d[\theta]-t)^{-1}dt + \mathbf{r}(t,y)dt,$$

where $\mathbf{r}(t,y) \in H(t,y)^n$ is some function which is regular at Ξ in each coordinate. Recall the definition of the matrix

$$d[\theta]-tI = \begin{pmatrix} (\theta-t) & a_1 & 1 & \dots & 0\\ 0 & (\theta-t) & a_2 & \dots & 0\\ 0 & 0 & (\theta-t) & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & (\theta-t)\end{pmatrix},$$

where the constants $a_i \in H$ are from Proposition 2.2.2. Because the matrix is upper triangular, we see immediately that the inverse matrix has the form

$$(d[\theta]-tI)^{-1}=\begin{pmatrix} \frac{1}{\theta-t} & * & * & \dots & * \\ 0 & \frac{1}{\theta-t} & * & \dots & * \\ 0 & 0 & \frac{1}{\theta-t} & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\theta-t} \end{pmatrix},$$

where each off diagonal entry denoted by $*$ is a sum of the form

$$\sum_{k=k_0}^{n}\frac{d_k}{(t-\theta)^k}$$

for $k_0 \geq 0$ and some (possibly zero) constants $d_k \in H$. Using the cofactor expansion of the inverse, we find that $k_0 \geq 2$ for each coordinate, and thus the off diagonal entries will not contribute to the residue. Thus, since $t-\theta$ is a uniformizer at Ξ, for some functions $r_i(t) \in H(t,y)$ which have no residue at Ξ we find that

$$\mathrm{RES}_\Xi(G_{\mathbf{u}}^{\otimes n})=\begin{pmatrix} \mathrm{Res}_\Xi\left((\frac{u_1}{\theta-t}+r_1(t))dt\right) \\ \vdots \\ \mathrm{Res}_\Xi\left((\frac{u_n}{\theta-t}+r_n(t))dt\right) \end{pmatrix}=-\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}. \tag{3.25}$$

□

Proposition 3.2.6. *The composition of maps*

$$\mathrm{RES}_\Xi \circ T : \Omega_0 \to \mathbb{C}_\infty^n$$

is an injective $\mathbf{A}$*-module homomorphism, where* $\mathbf{A}$ *acts on* Ω_0 *by multiplication and on* $\mathbb{C}_\infty^n$ *by* $\rho^{\otimes n}$*, and its image is* $\lambda_\rho^{\otimes n}=\ker(\mathrm{Exp}_\rho^{\otimes n})$.

Proof. The proof follows similarly to the proof of [26, Thm. 4.5]. For an arbitrary $h \in \Omega^n$, each coordinate of $T(h)$ is in $\mathbb{T}[y]$, so we can write

$$T(h) = \sum_{i=0}^{\infty} \mathbf{b}_{i+1} t^i + (y + c_1 t + c_3) \sum_{i=0}^{\infty} \mathbf{c}_{i+1} t^i$$

uniquely for $\mathbf{b}_i, \mathbf{c}_i \in \mathbb{C}_\infty^n$. Then using Lemma 3.1.2, we observe that

$$\begin{aligned}
\sum_{i=0}^{\infty} \rho_t^{\otimes n}(\mathbf{b}_{i+1}) t^i + (y + c_1 t + c_3) \sum_{i=0}^{\infty} \rho_t^{\otimes n}(\mathbf{c}_{i+1}) t^i &= \rho_t^{\otimes n}(T(h)) \\
&= tT(h) \\
&= \sum_{i=0}^{\infty} \mathbf{b}_{i+1} t^{i+1} + (y + c_1 t + c_3) \sum_{i=0}^{\infty} \mathbf{c}_{i+1} t^{i+1},
\end{aligned}$$

from which we see that if we set $\mathbf{b}_0 = \mathbf{c}_0 = 0$, then for $i \geq 0$

$$\rho_t^{\otimes n}(\mathbf{b}_{i+1}) = \mathbf{b}_i, \quad \rho_t^{\otimes n}(\mathbf{c}_{i+1}) = \mathbf{c}_i. \tag{3.26}$$

Similarly we find that for $i \geq 0$

$$\rho_y^{\otimes n}(\mathbf{c}_i) = \mathbf{b}_i. \tag{3.27}$$

Since $|\mathbf{b}_i|$, $|\mathbf{c}_i| \to 0$ as $i \to \infty$, there is some $i_0 > 0$ such that $\mathbf{b}_{i+1}$ and $\mathbf{c}_{i+1}$ both lie within the radius of convergence of $\operatorname{Log}_\rho^{\otimes n}$ for $i > i_0$. Thus by (2.17) and (3.26), for $i > i_0$ we have

$$d[\theta^i] \operatorname{Log}_\rho^{\otimes n}(\mathbf{b}_i) = d[\theta^{i+1}] \operatorname{Log}_\rho^{\otimes n}(\mathbf{b}_{i+1}), \quad d[\theta^i] \operatorname{Log}_\rho^{\otimes n}(\mathbf{c}_i) = d[\theta^{i+1}] \operatorname{Log}_\rho^{\otimes n}(\mathbf{c}_{i+1}),$$

and we note that these two quantities are independent of i. We set

$$\Pi_n := d[\theta^i] \operatorname{Log}_\rho^{\otimes n}(\mathbf{c}_i),$$

for some $i > i_0$, and note that

$$d[\eta]\Pi_n = d[\eta] d[\theta^i] \operatorname{Log}_\rho^{\otimes n}(\mathbf{c}_i) = d[\theta^i] d[\eta] \operatorname{Log}_\rho^{\otimes n}(\mathbf{c}_i) = d[\theta^i] \operatorname{Log}_\rho^{\otimes n}(\rho_y^{\otimes n}(\mathbf{c}_i)) = d[\theta^i] \operatorname{Log}_\rho^{\otimes n}(\mathbf{b}_i).$$

Using (2.16) together with the above discussion we see that

$$\mathrm{Exp}_\rho^{\otimes n}(\Pi_n) = \mathrm{Exp}_\rho^{\otimes n}(d[\theta^i] \, \mathrm{Log}_\rho^{\otimes n}(\mathbf{c}_i)) = \rho_{t^i}^{\otimes n}(\mathbf{c}_i) = \rho_t^{\otimes n}(\mathbf{c}_1) = \mathbf{c}_0 = 0,$$

which implies that $\Pi_n \in \lambda_\rho^{\otimes n}$. Further, we see that

$$\mathbf{b}_i = \mathrm{Exp}_\rho^{\otimes n}\left(d[\eta]d[\theta^{-i}]\Pi_n\right), \quad \mathbf{c}_i = \mathrm{Exp}_\rho^{\otimes n}\left(d[\theta^{-i}]\Pi_n\right),$$

and thus

$$T(h) = G_{\Pi_n}^{\otimes n} = E_{d[\eta]\Pi_n}^{\otimes n} + (y + c_1 t + c_3) E_{\Pi_n}^{\otimes n}.$$

By Proposition 3.2.5, we see that $\mathrm{RES}_\Xi(T(h)) = -\Pi_n$, and thus $\mathrm{RES}_\Xi(T(\Omega_0)) \subseteq \lambda_\rho^{\otimes n}$. Since $G_{\Pi_n}^{\otimes n} = G_{\Pi_n'}^{\otimes n}$ if and only if $\Pi_n = \Pi_n'$, the map $\mathrm{RES}_\Xi \circ T$ is injective. Finally, let $\Pi_n' \in \lambda_\rho^{\otimes n}$, so that Lemma 3.2.4 shows that

$$D_t(G_{\Pi_n'}^{\otimes n}) = D_y(G_{\Pi_n'}^{\otimes n}) = 0.$$

Thus, using Proposition 3.1.7 we find that

$$(G - E_1\tau)(G_{\Pi_n'}^{\otimes n}) = 0,$$

and hence by Lemma 3.1.3 $G_{\Pi_n'}^{\otimes n} = T(h)$ for some function $h \in \Omega_0$. Finally, by Proposition 3.2.5

$$\mathrm{RES}_\Xi(T(h)) = \mathrm{RES}_\Xi(G_{\Pi_n'}^{\otimes n}) = \Pi_n'$$

which shows that $\lambda_\rho^{\otimes n} \subset \mathrm{RES}_\Xi(T(\Omega_0))$. To see that $\mathrm{RES}_\Xi \circ T$ is an $\mathbf{A}$-module homomorphism, for $h \in \Omega_0$, using the above discussion we find that

$$\mathrm{RES}_\Xi(T(th)) = \mathrm{RES}_\Xi(tG_{\Pi_n'}^{\otimes n}),$$

for some $\Pi'_n \in \Lambda_\rho^{\otimes n}$ and using analysis similar to that in the proof of Proposition 3.2.5 that

$$\mathrm{RES}_\Xi(tG_{\Pi'_n}^{\otimes n}) = \mathrm{RES}_\Xi((t - d[\theta])G_{\Pi'_n}^{\otimes n} + d[\theta]G_{\Pi'_n}^{\otimes n}) = d[\theta]\,\mathrm{RES}_\Xi(G_{\Pi'_n}^{\otimes n}) = d[\theta]\,\mathrm{RES}_\Xi(T(h)).$$

It follows similarly that $\mathrm{RES}_\Xi(T(yh)) = d[\eta]\,\mathrm{RES}_\Xi(T(h))$, which finishes the proof. $\square$

Theorem 3.2.7. *If we denote*

$$\Pi_n = -\,\mathrm{RES}_\Xi(T(\omega_\rho^n)),$$

then $T(\omega_\rho^n) = G_{\Pi_n}^{\otimes n}$ *and* $\lambda_\rho^{\otimes n} = \{d[a]\Pi_n \mid a \in \mathbf{A}\}$. *Further, if* π_ρ *is a fundamental period of the (1-dimensional) Drinfeld exponential function* $\exp_\rho$ *from* (2.9)*, then the last coordinate of* $\Pi_n \in \mathbb{C}_\infty^n$ *is*

$$\frac{g_1(\Xi)}{a_1 a_2 \ldots a_{n-1}} \cdot \pi_\rho^n,$$

where the constants a_i *are from Proposition 2.2.2.*

Proof. The first two statements follow immediately from Propositions 3.1.10 and 3.2.6. Then recall from [26] that $\pi_\rho = -\mathrm{Res}_\Xi(\omega_\rho\lambda)$, whereupon the last statement follows by noting that the last coordinate of $-\,\mathrm{RES}_\Xi(T(\omega_\rho^n))$ equals

$$-\,\mathrm{Res}_\Xi(\omega_\rho^n g_n \lambda) = -\,\mathrm{Res}_\Xi\left((t-\theta)^{n-1}\omega_\rho^n\lambda\right) \cdot \left(\left.\frac{g_n}{(t-\theta)^{n-1}}\right|_\Xi\right) = \pi_\rho^n \cdot \left(\left.\frac{g_n}{(t-\theta)^{n-1}}\right|_\Xi\right),$$

since $(t-\theta)^{n-1}\omega_\rho^n$ has a simple pole at Ξ and since $g_n/(t-\theta)^{n-1}$ is regular at Ξ. The formula then follows by dividing the first equation of Proposition 2.2.2 through by g_{i+1} then evaluating at Ξ to get

$$\left.\frac{(t-\theta)g_i}{g_{i+1}}\right|_\Xi = a_i.$$

$\square$

4. COEFFICIENTS OF EXP AND LOG

4.1 Coefficients of the exponential function

The coefficients of the exponential function for rank 1 sign-normalized Drinfeld modules are well understood (see (2.7)). Further, the coefficients for the exponential function of the nth tensor power of the Carlitz module are also well understood. These coefficients were first studied by Anderson and Thakur in [5, §2.2], and have recently been written down explicitly using hyperderivatives by Papanikolas in [33, 4.3.6]. In this section we give explicit formulas for the coefficients of the exponential function for the nth tensor power of a rank 1 sign-normalized Drinfeld module.

In order to write down a formula for the coefficients of $\operatorname{Exp}_\rho^{\otimes n}$ we must first analyze certain functions which arise when calculating residues of the vector-valued Anderson generating functions $G_{\mathbf{u}}^{\otimes n}$. For a fixed dimension n, for $1 \leq \ell \leq n$ and for $i \geq 0$, define the functions

$$\gamma_{i,\ell} = \frac{g_\ell}{(ff^{(1)} \dots f^{(i-1)})^n}, \tag{4.1}$$

where for $i = 0$ we understand $\gamma_{0,\ell} = g_\ell$. Using (2.2) and (2.12) we see that the polar part of the divisor of $\gamma_{i,\ell}$ equals

$$-n(V^{(i)}) - n(\Xi^{(i-1)}) - n(\Xi^{(i-2)}) - \dots - (n - (\ell - 1))(\Xi).$$

We temporarily fix an index ℓ. Using the Riemann-Roch theorem, we observe that we can find unique functions $\alpha_{j,k}$ with $\widetilde{\operatorname{sgn}}(\alpha_{j,k}) = 1$ in each of the following 1-dimensional spaces, Further, using the Riemann-Roch theorem we observe that we can find functions each of the following

1-dimensional spaces, which we denote

$$\alpha_{1,1} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + (\Xi))$$
$$\alpha_{1,2} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + 2(\Xi) - (\infty))$$
$$\alpha_{1,3} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + 3(\Xi) - 2(\infty))$$
$$\vdots$$
$$\alpha_{1,n} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + n(\Xi) - (n-1)(\infty))$$
$$\alpha_{2,1} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + (\Xi^{(1)}) + n(\Xi) - (n)(\infty))$$
$$\alpha_{2,2} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + 2(\Xi^{(1)}) + n(\Xi) - (n+1)(\infty))$$
$$\vdots$$
$$\alpha_{2,n} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + n(\Xi^{(1)}) + n(\Xi) - (2n-1)(\infty))$$
$$\vdots$$
$$\alpha_{i,1} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + (\Xi^{(i-1)}) + \cdots + n(\Xi^{(1)}) + n(\Xi) - ((i-1)n)(\infty))$$
$$\alpha_{i,2} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + 2(\Xi^{(i-1)}) + \cdots + n(\Xi^{(1)}) + n(\Xi) - ((i-1)n+1)(\infty))$$
$$\vdots$$
$$\alpha_{i,n} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + n(\Xi^{(i-1)}) + \cdots + n(\Xi^{(1)}) + n(\Xi) - (in-1)(\infty)).$$

More succinctly we could write for $1 \le j \le i$ and $1 \le k \le n$

$$\alpha_{j,k} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + k(\Xi^{(j-1)}) + n(\Xi^{(j-2)}) + \cdots + n(\Xi^{(1)}) + n(\Xi) - (n(j-1)+k-1)(\infty)).$$

Then, for appropriate constants $d_{j,k} \in H$ we subtract off the principal part of the power series expansion of $g_\ell/(ff^{(1)}\dots f^{(i-1)})^n$ at $\Xi^{(m)}$, for $1 \le m \le j-1$, to find that

$$\gamma_{i,\ell} - \sum_{j,k} d_{j,k}\alpha_{j,k} \in \mathcal{L}\left(n(V^{(i)})\right) = \mathrm{Span}_H(g_1^{(i)}, g_2^{(i)}, \dots, g_n^{(i)}).$$

So for further constants $c_{\ell,1}, \cdots, c_{\ell,n} \in H$ we can write

$$\gamma_{i,\ell} = c_{\ell,1} g_1^{(i)} + c_{\ell,2} g_2^{(i)} + \dots c_{\ell,n} g_n^{(i)} + \sum_{j,k} d_{j,k} \alpha_{j,k}, \tag{4.2}$$

where we note that each of the functions $\alpha_{j,k}$ vanishes with order n at $\Xi^{(i)}$ and that the coefficients $c_{\ell,k}$ are implicitly dependent on i. To ease notation, for each $1 \leq \ell \leq n$ we will write $\alpha_\ell := \sum_{j,k} d_{j,k} \alpha_{j,k}$ and write the equations from (4.2) for $1 \leq \ell \leq n$ in matrix form as

$$\begin{pmatrix} \gamma_{i,1} \\ \gamma_{i,2} \\ \vdots \\ \gamma_{i,n} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n} \\ c_{2,1} & c_{2,2} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \dots & c_{n,n} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}^{(i)} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{4.3}$$

with

$$\boldsymbol{\gamma}_i = \begin{pmatrix} \gamma_{i,1} \\ \gamma_{i,2} \\ \vdots \\ \gamma_{i,n} \end{pmatrix}, \quad C_i = \langle c_{j,k} \rangle, \quad \text{and} \quad \boldsymbol{\alpha}_i = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},$$

so that

$$\boldsymbol{\gamma}_i = C_i \mathbf{g}^{(i)} + \boldsymbol{\alpha}_i. \tag{4.4}$$

Theorem 4.1.1. *With the notation as above, for dimension $n \geq 2$ and $\mathbf{z} \in \mathbb{C}_\infty$, if we write*

$$\operatorname{Exp}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)},$$

then for $i \geq 0$, the exponential coefficients $Q_i = C_i$ and $Q_i \in \operatorname{Mat}_n(H)$.

Remark 4.1.2. We remark that in the case for $n = 1$, if one interprets the empty divisors in (2.12) correctly, then Theorem 4.1.1 still holds. However, for clarity of exposition, we restrict to $n \geq 2$.

Before giving the proof of Theorem 4.1.1, we require a lemma about the coefficients of the exponential function.

Lemma 4.1.3. *Given a sequence of matrices $Q_i \in Mat_n(H)$ for $i \geq 0$ with $Q_0 = I$, the matrices Q_i are the coefficients of $\mathrm{Exp}_\rho^{\otimes n}$ if and only if they satisfy the recurrence relation*

$$M_2 Q_{i-1}^{(1)} + E_1 Q_{i-1}^{(1)} d[\theta]^{(i)} = Q_i d[\eta]^{(i)} - (N_1 + M_m) Q_i d[\theta]^{(i)} - M_1 Q_i, \quad i \geq 1. \tag{4.5}$$

where M_1 and M_2 are defined in Corollary 3.1.9 and M_m is defined in (3.8). *Further, the coefficients $Q_i \in \mathrm{Mat}_n(H)$.*

Proof. First note that by (2.16)

$$(\rho_y^{\otimes n} - (M_\tau + M_m)\rho_t^{\otimes n})(\mathrm{Exp}_\rho^{\otimes n}(z)) = \mathrm{Exp}_\rho^{\otimes n}(d[\eta]z) - (M_\tau + M_m)\,\mathrm{Exp}_\rho^{\otimes n}(d[\theta]z).$$

Then, using Corollary 3.1.9,

$$(M_1 + M_2\tau)(\mathrm{Exp}_\rho^{\otimes n}(z)) = \mathrm{Exp}_\rho^{\otimes n}(d[\eta]z) - (M_\tau + M_m)\,\mathrm{Exp}_\rho^{\otimes n}(d[\theta]z),$$

and expanding $\mathrm{Exp}_\rho^{\otimes n}$ on both sides in terms of its coefficients Q_i and equating like terms gives the equality

$$M_2 Q_{i-1}^{(1)} + E_1 Q_{i-1}^{(1)} d[\theta]^{(i)} = Q_i d[\eta]^{(i)} - (N_1 + M_m) Q_i d[\theta]^{(i)} - M_1 Q_i.$$

Thus the coefficients of the exponential function satisfy the recurrence relation (4.5). Next, for $j \geq 0$, let $\{Q'_j\} \subset \mathrm{Mat}_n(H)$ be a sequence of matrices satisfying recurrence relation (4.5). We will show that $\{Q'_j\}$ is uniquely determined by Q_0, and thus if we fix $Q_0 = I$, the matrices $\{Q'_j\}$ will be the coefficients of $\mathrm{Exp}_\rho^{\otimes n}$. Given a term Q'_{i-1} of the sequence $\{Q'_j\}$ for $i \geq 1$, define

$$W_i = M_2 (Q'_{i-1})^{(1)} + E_1 (Q'_{i-1})^{(1)} d[\theta]^{(i)},$$

so that by (4.5)

$$W_i = Q_i' d[\eta]^{(i)} - (M_m + N_1) Q_i' d[\theta]^{(i)} - M_1 Q_i'. \tag{4.6}$$

Then, denote $M_1 = M_d + M_n$, where M_d is the diagonal part of M_1 and M_n is the nilpotent (super-diagonal) part. Then collect the diagonal and off-diagonal terms of (4.6) to obtain

$$W_i = (\eta^{q^i} I - \theta^{q^i} M_m - M_d) Q_i' + Q_i' N_\eta^{(i)} - \theta^{q^i} N_1 Q_i' - M_m Q_i' N_\theta^{(i)} - N_1 Q_i' N_\theta^{(i)} - M_n Q_i', \tag{4.7}$$

where we recall the definition of N_θ and N_η from (2.23). Next, we denote the matrix $M_D = \eta^{q^i} I - \theta^{q^i} M_m - M_d$, and note that it is diagonal and invertible. Define

$$\beta_i : \mathrm{Mat}_n(H) \to \mathrm{Mat}_n(H)$$

to be the $\mathbb{F}_q$-linear map given for $Y \in \mathrm{Mat}_n(H)$ by

$$Y \mapsto M_D^{-1}(Y N_\eta^{(i)} - \theta^{q^i} N_1 Y - M_m Y N_\theta^{(i)} - N_1 Y N_\theta^{(i)} - M_n Y). \tag{4.8}$$

Note that β_i is a nilpotent map with order at most $2n-1$, since each matrix in definition (4.8), except M_D, is strictly upper triangular, and thus each term of β_i^{2n-1} will have at least n strictly upper triangular matrices on either the left or the right of each matrix Y. Then, using the map β_i and rearranging slightly we can rewrite (4.7) as

$$Q_i' + \beta_i(Q_i') = M_D^{-1} W_i. \tag{4.9}$$

Applying β_i^j to (4.9), multiplying by $(-1)^j$, then adding these together for $j \geq 1$ gives a telescoping sum. Since β_i is nilpotent with order at most $2n-1$, we find

$$Q_i' = \sum_{j=0}^{2n-1} (-1)^j \beta_i^j (M_D^{-1} W_i). \tag{4.10}$$

Thus we have determined Q'_i uniquely in terms of Q'_{i-1}, and so each element in the sequence $\{Q'_j\}$ is determined by Q_0. If we require that $Q_0 = I$, then the matrices $\{Q'_j\}$ are the coefficients of $\mathrm{Exp}_\rho^{\otimes n}$. Further, since M_D and each matrix in the definition of β_i is in $\mathrm{Mat}_n(H)$, we see that the exponential function coefficients $Q_i \in \mathrm{Mat}_n(H)$. $\square$

We now return to the proof of Theorem (4.1.1).

Proof of Theorem (4.1.1). We first recall that $\gamma_{0,\ell} = g_\ell$ and hence by (4.2) we have $C_0 = I = Q_0$, so that the theorem is true trivially for $i = 0$. We then show that the sequence of matrices $\{C_i\}$ satisfies the recurrence in Lemma 4.1.3 for $i \geq 1$. First observe that by Proposition 2.2.2

$$d[\theta]\mathbf{g} = t\mathbf{g} - f^n E_\theta \mathbf{g}^{(1)} \quad \text{and} \quad d[\eta]\mathbf{g} = y\mathbf{g} - f^n E_\eta \mathbf{g}^{(1)}, \tag{4.11}$$

with g defined as in (2.15). Using (4.11), we write

$$\begin{aligned}
&\left(M_2 C_{i-1}^{(1)} + E_1 C_{i-1}^{(1)} d[\theta]^{(i)} - C_i d[\eta]^{(i)} + (N_1 + M_m) C_i d[\theta]^{(i)} + M_1 C_i\right) \mathbf{g}^{(i)} \\
&= \left(M_2 C_{i-1}^{(1)} + t E_1 C_{i-1}^{(1)} - y C_i + t(N_1 + M_m) C_i + M_1 C_i\right) \mathbf{g}^{(i)} \\
&\quad - \left(E_1 C_{i-1}^{(1)} E_\theta^{(i)} - C_i E_\eta^{(i)} + (N_1 + M_m) C_i E_\theta^{(i)}\right) f^n \mathbf{g}^{(i)}.
\end{aligned}$$

We examine the first term in the right hand side of the above equation, which we denote

$$T_1 = \left(M_2 C_{i-1}^{(1)} + t E_1 C_{i-1}^{(1)} - y C_i + t(N_1 + M_m) C_i + M_1 C_i\right) \mathbf{g}^{(i)}, \tag{4.12}$$

and the second term, which we denote

$$T_2 = \left(E_1 C_{i-1}^{(1)} E_\theta^{(i)} - C_i E_\eta^{(i)} + (N_1 + M_m) C_i E_\theta^{(i)}\right) f^n \mathbf{g}^{(i)}, \tag{4.13}$$

separately. By the discussion immediately following (4.4) we see that (4.12) equals

$$\begin{aligned}T_1 &= (M_2 + tE_1)\boldsymbol{\gamma}_{i-1}^{(1)} + (-yI + t(M_m + N_1) + M_1)\boldsymbol{\gamma}_i + \boldsymbol{\alpha}_{i-1}^{(1)} + \boldsymbol{\alpha}_i \\ &= M_2'\boldsymbol{\gamma}_{i-1}^{(1)} + M_1'\boldsymbol{\gamma}_i + \boldsymbol{\alpha}_{i-1}^{(1)} + \boldsymbol{\alpha}_i,\end{aligned}$$

with M_1' and M_2' as given in (3.10). Then, writing out the coordinates of $\boldsymbol{\gamma}$ using the functions $\gamma_{i,\ell}$ from (4.1) and finding a common denominator gives

$$T_1 = \frac{1}{(ff^{(1)}\dots f^{(i-1)})^n}\left(M_1'\mathbf{g} + M_2'f^n\mathbf{g}^{(1)} + \boldsymbol{\alpha}_{i-1}^{(1)} + \boldsymbol{\alpha}_i\right) = \frac{1}{(ff^{(1)}\dots f^{(i-1)})^n}\left(\boldsymbol{\alpha}_{i-1}^{(1)} + \boldsymbol{\alpha}_i\right),$$

since $M_1'\mathbf{g} + M_2'f^n\mathbf{g}^{(1)} = 0$ by (3.11). Thus T_1 vanishes coordinate-wise with order at least n at $\Xi^{(i)}$, because the functions α_ℓ from (4.4) each vanish with order at least n at $\Xi^{(i)}$. Further, the presence of the factored-out $f^n\mathbf{g}^{(i)}$ shows that T_2 from (4.13) also vanishes coordinate-wise with order at least n at $\Xi^{(i)}$. Thus we see that

$$\left(M_2C_{i-1}^{(1)} + E_1C_{i-1}^{(1)}d[\theta]^{(i)} - C_id[\eta]^{(i)} + (N_1 + M_m)C_id[\theta]^{(i)} + M_1C_i\right)\mathbf{g}^{(i)}$$

consists of a constant matrix in $\mathrm{Mat}_n(H)$ multiplied by $\mathbf{g}^{(i)}$, and equals a vector of functions which vanishes coordinate-wise with order at least n at $\Xi^{(i)}$. However, recall from (2.12) that $\mathrm{ord}_{\Xi^{(i)}}(g_j^{(i)}) = j - 1$, and thus

$$\left(M_2C_{i-1}^{(1)} + E_1C_{i-1}^{(1)}d[\theta]^{(i)} - C_id[\eta]^{(i)} + (N_1 + M_m)C_id[\theta]^{(i)} + M_1C_i\right) = 0$$

identically, which proves that $\{C_i\}$ satisfies the recursion equation (4.5) and proves the proposition. $\square$

Corollary 4.1.4. *For $z \in \mathbb{C}_\infty$ we have the formal expression*

$$\operatorname{Exp}_\rho^{\otimes n}\begin{pmatrix} z \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{i=0}^{\infty} \frac{z^{q^i}}{g_1^{(i)}(ff^{(1)}\dots f^{(i-1)})^n} \cdot \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}\Bigg|_{\Xi^{(i)}}.$$

Proof. This follows from Theorem 4.1.1 by evaluating (4.2) at $\Xi^{(i)}$, noticing that $g_j^{(i)}(\Xi^{(i)})$ vanishes for $j \geq 2$, then solving for $c_{\ell,1}$. $\square$

Remark 4.1.5. Theorem 4.1.1 and Corollary 4.1.4 should be considered generalizations Proposition 2.2.5 of [5] and of the remark that follow it.

Remark 4.1.6. The formulas for the coefficients of $\operatorname{Exp}_\rho^{\otimes n}$ in Theorem (4.1.1) may at first seem quite mysterious and unmotivated. Here we provide an explanation of their origin. From the calculations in Proposition 3.2.5, one quickly finds that

$$\operatorname{RES}_{\Xi^{(i)}}\left(G_{\Pi_n}^{\otimes n}\right) = -Q_i \Pi_n^{(i)}.$$

On the other hand, by Theorem 3.2.7, we can write

$$\operatorname{RES}_{\Xi^{(i)}}\left(G_{\Pi_n}^{\otimes n}\right) = \operatorname{RES}_{\Xi^{(i)}}\begin{pmatrix} \omega_\rho^n g_1 \\ \omega_\rho^n g_2 \\ \vdots \\ \omega_\rho^n g_n \end{pmatrix} = \operatorname{RES}_{\Xi^{(i)}}\begin{pmatrix} (\omega_\rho^n)^{(i)} g_1/(ff^{(1)}\dots f^{(i-1)})^n \\ (\omega_\rho^n)^{(i)} g_2/(ff^{(1)}\dots f^{(i-1)})^n \\ \vdots \\ (\omega_\rho^n)^{(i)} g_n/(ff^{(1)}\dots f^{(i-1)})^n \end{pmatrix}, \tag{4.14}$$

where in the second equality we have used (3.14) i times. We then take the expression for the $\gamma_{\ell,i}$

functions from (4.2) to obtain

$$-Q_i\Pi_n^{(i)} = \mathrm{RES}_{\Xi^{(i)}} \begin{pmatrix} (\omega_\rho^n)^{(i)}(c_{1,1}g_1^{(i)} + c_{1,2}g_2^{(i)} + \dots c_{1,n}g_n^{(i)} + \alpha_1) \\ (\omega_\rho^n)^{(i)}(c_{2,1}g_1^{(i)} + c_{2,2}g_2^{(i)} + \dots c_{2,n}g_n^{(i)} + \alpha_2) \\ \vdots \\ (\omega_\rho^n)^{(i)}(c_{n,1}g_1^{(i)} + c_{n,2}g_2^{(i)} + \dots c_{n,n}g_n^{(i)} + \alpha_n) \end{pmatrix}.$$

Since $(\omega_\rho^n)^{(i)}$ has a pole of order n at $\Xi^{(i)}$ and since the functions α_ℓ vanish with order at least n at $\Xi^{(i)}$, the α_ℓ functions do not factor into the residue calculation and we obtain

$$-Q_i\Pi_n^{(i)} = \mathrm{RES}_{\Xi^{(i)}} \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n} \\ c_{2,1} & c_{2,2} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \dots & c_{n,n} \end{pmatrix} \begin{pmatrix} (\omega_\rho^n)^{(i)}g_1^{(i)} \\ (\omega_\rho^n)^{(i)}g_2^{(i)} \\ \vdots \\ (\omega_\rho^n)^{(i)}g_n^{(i)} \end{pmatrix} = -\begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n} \\ c_{2,1} & c_{2,2} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \dots & c_{n,n} \end{pmatrix} \cdot \Pi_n^{(i)}.$$

The functions f and g_i are all defined over H and thus so are the coefficients $c_{j,k}$. So, if we knew that the coordinates of the Π_n were linearly independent over H, then we would have the equality $Q_i = \langle c_{j,k} \rangle$. Such results of linear independence, however, are in general quite difficult, and to the knowledge of the author, this particular result is not yet known. Thus this line of reasoning motivates the formulas for Q_i, but we must prove it using the methods given above.

Remark 4.1.7. Note that for $n = 1$, the τ-basis from Proposition 2.1.3 consists of the single constant function $\{1\}$. Thus we can apply the discussion from Remark 4.1.6 to the case for 1-dimensional Drinfeld **A**-modules with the notation outlined in §2.1 and we find that

$$\mathrm{Res}_{\Xi^{(i)}}(G_{\pi_\rho}) = -\frac{1}{d_i}\pi_\rho^{(i)},$$

while on the other hand

$$\mathrm{Res}_{\Xi^{(i)}}(G_{\pi_\rho}) = \mathrm{Res}_{\Xi^{(i)}}(\omega_\rho) = \mathrm{Res}_{\Xi^{(i)}}\left(\frac{\omega_\rho^{(i)}}{ff^{(1)}\dots f^{(i-1)}}\right).$$

Then, because $G_{\pi_\rho} = \omega_\rho$ has simple poles at the twists of Ξ,

$$\operatorname{Res}_{\Xi^{(i)}}\left(\frac{\omega_\rho^{(i)}}{ff^{(1)}\dots f^{(i-1)}}\right) = \operatorname{Res}_{\Xi^{(i)}}(\omega_\rho^{(i)}) \cdot \frac{1}{ff^{(1)}\dots f^{(i-1)}}\bigg|_{\Xi^{(i)}} = -\pi_\rho^{(i)} \cdot \frac{1}{ff^{(1)}\dots f^{(i-1)}}\bigg|_{\Xi^{(i)}}.$$

Since $n = 1$, there is no issue of linear independence of the coordinates of π_ρ, and we recover (2.7) without resorting to the methods set out in the proof of Theorem 4.1.1.

4.2 Coefficients of the logarithm function

The coefficients for the logarithm function associated to a rank 1 sign-normalized Drinfeld module were first studied by Anderson (see [45, Prop. 0.3.8]) and are described in (2.8). The coefficients for the logarithm associated to the nth tensor power of the Carlitz module were studied by Anderson and Thakur, who give formulas for the lower right entry of these matrix coefficients in [5, §2.1]. Recently, Papanikolas has written down explicit formulas using hyperderivatives in [33, 4.3.1 and Prop. 4.3.6(a)]. In this section we develop new techniques to write down explicit formulas for the coefficients of the logarithm function $\operatorname{Log}_\rho^{\otimes n}$ associated to the nth tensor power of rank 1 sign-normalized Drinfeld modules. Our method was inspired by ideas of Sinha from [41] (see in particular his “main diagram" in section 4.2.3). However, where Sinha uses homological constructions to prove the commutativity of his diagram, we take a more direct approach using Anderson generating functions for ours.

We define the following diagram of maps, where we recall the definition of $\mathcal{M}$ from (3.23) and of Ω from (3.1)

$$\begin{tikzcd} \Omega \arrow[r, "\tau - f^n"] \arrow[d, "T"] & N \arrow[r, "\varepsilon"] & \mathbb{C}_\infty^n \\ \mathcal{M}^n \arrow[r, "-\operatorname{RES}_\Xi"] & \mathbb{C}_\infty^n \arrow[ur, "\operatorname{Exp}_\rho^{\otimes n}"'] & \end{tikzcd} \tag{4.15}$$

and where the maps ε, T and RES_Ξ are defined in (2.25), (3.3) and (3.23) respectively. We remark that using the operator $\tau - f^n$ one quickly sees that $\Omega \subset \mathcal{M}$.

One of the main goals of this section is to prove that the diagram commutes. Before we prove this, however, observe that if $\mathbf{u} \in \mathbb{C}_\infty^n$ is not a period of $\operatorname{Exp}_\rho^{\otimes n}$, then $G_{\mathbf{u}}^{\otimes n} \in \mathcal{M}^n$ is not in the image of Ω under T in diagram 4.15. We require a preliminary result which allows us to modify $G_{\mathbf{u}}^{\otimes n}$ to be in the image of T. For $\mathbf{u} \in \mathbb{C}_\infty^n$, write the coordinates of $G_{\mathbf{u}}^{\otimes n}$ from (3.16) as

$$G_{\mathbf{u}}^{\otimes n}(t,y) = (k_1(t,y), k_2(t,y), \ldots, k_n(t,y))^\top,$$

and then define the vector

$$\mathbf{k} = (k_1([n]V), k_2(V^{(1)} + [n-1]V), k_3([2]V^{(1)} + [n-2]V), \ldots, k_n([n-1]V^{(1)} + V))^\top.$$

Next we define the vector valued function

$$J_{\mathbf{u}}^{\otimes n} := (j_1(t,y), j_2(t,y), \ldots, j_n(t,y))^\top := G_{\mathbf{u}}^{\otimes n} - \mathbf{k}, \tag{4.16}$$

and note that j_k vanishes at the point $[k-1]V^{(1)} + [n-k+1]V$. Also denote

$$\mathbf{w} := (w_1(t,y), w_2(t,y), \ldots, w_n(t,y))^\top := (G - E_1\tau)(J_{\mathbf{u}}^{\otimes n}) \in \mathbb{T}(y)^n,$$

where $G - E_1\tau$ is the operator defined in (3.4), and let $\mathbf{z} := \operatorname{Exp}_\rho^{\otimes n}(\mathbf{u})$ and denote its coordinates $\mathbf{z} := (z_1, z_2, \ldots, z_n)^\top$.

Proposition 4.2.1. *The vector* $\mathbf{w}$ *is in* $H[t,y]^n$ *and equals*

$$\mathbf{w} = \begin{pmatrix} z_1 \cdot (t - t(V^{(1)} + [n-1]V)) \\ z_2 \cdot (t - t([2]V^{(1)} + [n-2]V)) \\ \vdots \\ z_{n-1} \cdot (t - t([n-1]V^{(1)} + [1]V)) \\ z_n \cdot (t - t([n]V^{(1)})) \end{pmatrix}.$$

Proof. By Proposition 3.1.7, Proposition 3.2.4 and (2.16) we write

$$\begin{aligned}\mathbf{w}' :=&(w_1'(t,y), w_2'(t,y), \dots, w_n'(t,y))^\top := (G - E_1\tau)(G_{\mathbf{u}}^{\otimes n}) \\ =&M_\delta^{-1}\big[- c_1\rho_y^{\otimes n}(\mathbf{z}) + \rho_{t^2}^{\otimes n}(\mathbf{z}) + (t + c_2)\rho_t^{\otimes n}(\mathbf{z}) \\ &+ (t^2 + c_2 t + c_4)\mathbf{z} - (M_\tau + M_m)(\rho_y^{\otimes n}(\mathbf{z}) + (y + c_1 t + c_3)\mathbf{z})\big].\end{aligned} \tag{4.17}$$

In particular, from the last line of the above equation we see that $\mathbf{w}'$ is a vector of rational functions in the space $H(t,y)$. Further, for each rational function w_i', the highest degree term in the numerator is $z_k t^2$ and the highest degree term in the denominator is t (coming from the matrix M_δ^{-1}). Thus each w_i' is a rational function in $H(t,y)$ of degree 2 (recall the $\deg(t) = 2$) with $\widetilde{\mathrm{sgn}}(w_i') = z_k$. We also observe that

$$(G - E_1\tau)(\mathbf{k}) \in H(t,y)$$

and that each coordinate has degree 1. This implies that each w_i is in $H(t,y)$ and has degree 2 with $\widetilde{\mathrm{sgn}}(w_i) = z_k$. Writing out the action of $G - E_1\tau$ on the coordinates of $J_{\mathbf{u}}^{\otimes n}$ we obtain equations for $1 \le m \le n$

$$j_m \frac{g_{m+1}}{g_m} - j_{m+1} = w_m. \tag{4.18}$$

From (3.4), (3.7) and (4.17) we see that the only points at which w_k might have poles are the zeros of δ_k, namely the points

$$[k-1]V^{(1)} + [n-k+1]V \quad \text{and} \quad [-(k-1)]V^{(1)} - [n-k+1]V.$$

We remark that this shows that the coordinates of $\mathbf{w}$ are regular at $\Xi^{(i)}$ for $i \ge 0$, even though the coordinates of $J_u^{\otimes n}$ themselves have poles at $\Xi^{(i)}$. Recall from Proposition 3.2.3 that the only poles of j_k occur at ∞ and $\Xi^{(i)}$ for $i \ge 0$ and from (4.16) that j_k vanishes at $[k-1]V^{(1)} + [n-k+1]V$, while from (3.6) we observe that g_{k+1}/g_k is regular away from infinity except for a simple pole at $[k-1]V^{(1)} + [n-k+1]V$. Therefore, the equations in (4.18) show that each coordinate w_k is regular at the points $[k-1]V^{(1)} + [n-k+1]V$ and $[-(k-1)]V^{(1)} - [n-k+1]V$. Thus, the coordinates w_k, being rational functions of degree 2 in $H(t,y)$, which are regular away from ∞, are actually in $H[t,y]$. Further, we see from (4.18) that each function w_k vanishes at the point $[k]V^{(1)} + [n-k]V$. Since we know that $\widetilde{\mathrm{sgn}}(w_i) = z_k$, and since we've identified one of the zeros of w_i, we find using the Riemann-Roch theorem that $w_k = z_k(t - t([k]V^{(1)} + [n-k]V)$. $\square$

Theorem 4.2.2. *Diagram* (4.15) *commutes. In other words, for $h \in \Omega$, if we let*

$$(\tau - f^n)(h) = g \in N$$

and let $-\mathrm{RES}_\Xi(T(h)) = \mathbf{u}$, *then we have* $\varepsilon(g) = \mathrm{Exp}_\rho^{\otimes n}(\mathbf{u})$.

Proof. First observe that the case for $n = 1$ is proved in Theorem 5.1 of [26]. For the rest of the proof, assume $n \geq 2$. Write $\deg(g) = mn + b$ with $0 \leq b \leq q-1$ and write g in the σ-basis for N described in Proposition 2.1.3 with coefficients $b_{j,i}^{(-i)} \in H$ as

$$g = \sum_{i=0}^{m} \sum_{j=1}^{n} b_{j,i}^{(-i)} (ff^{(-1)} \dots f^{(1-i)})^n h_{n-j+1}^{(-i)}, \tag{4.19}$$

where we denote $\mathbf{b}_i = (b_{1,i}, b_{2,i}, \dots, b_{n,i})^\top$, and note that

$$\varepsilon(g) = \mathbf{b}_0 + \mathbf{b}_1 + \cdots + \mathbf{b}_m. \tag{4.20}$$

For $0 \leq i \leq m$ let $\mathbf{u}_i$ be any element in $\mathbb{C}_\infty$ such that

$$\mathrm{Exp}_\rho^{\otimes n}(\mathbf{u}_i) = \mathbf{b}_i, \tag{4.21}$$

The main method for the proof of Theorem 4.2.2 is to write $T(h)$ in terms of Anderson generating functions. To do this we compare the result of $T(h)$ under the operator $G - E_1\tau$ with the result of $J_{\mathbf{u}_i}^{\otimes n}$ under $G - E_1\tau$ for $0 \leq i \leq m$.

By the definition (3.4) we see that for any $\gamma \in \mathbb{C}_\infty(t, y)$

$$(G - E_1\tau)(T(\gamma)) = (0, \ldots, 0, g_1^{(1)}(f^n\gamma - \gamma^{(1)}))^\top. \tag{4.22}$$

Since $f^n h - h^{(1)} = g$, using the notation of (4.19) we can write

$$(G - E_1\tau)(T(h)) = (0, \ldots, 0, g_1^{(1)} \sum_{i=0}^{m} \sum_{j=1}^{n} b_{j,n-i+1}^{(-i)} (ff^{(-1)} \ldots f^{(1-j)})^n h_i^{(-j)})^\top. \tag{4.23}$$

Next, we analyze $(G - E_1\tau)(J_{\mathbf{u}_i})$ for $0 \leq i \leq m$. For the equations in (4.18), if we set $i = 1$, then we can solve for j_2. We then substitute that into the equation for $i = 2$, then solve that for j_3, and so on to get equations for $2 \leq m \leq n$

$$j_1 \frac{g_{m+1}}{g_1} - j_{m+1} = w_m + w_{m-1} \frac{g_{m+1}}{g_n} + w_{n-2} \frac{g_{m+1}}{g_{n-1}} + \cdots + w_1 \frac{g_{m+1}}{g_2}, \tag{4.24}$$

where we understand $j_{n+1} = j_1^{(1)}$. We note that the functions j_k and w_k depend implicitly on $\mathbf{u}_i$. Using these equations we find that

$$J_{\mathbf{u}_i}^{\otimes n} + \left(0, w_1, w_2 + w_1 \frac{g_3}{g_2}, \ldots, w_{n-1} + w_{n-2} \frac{g_n}{g_2} + \cdots + w_1 \frac{g_n}{g_{n-1}}\right)^\top = T\left(j_1/g_1\right). \tag{4.25}$$

In general we will call $I_{\mathbf{u}_i}^{\otimes n} := T\left(j_1/g_1\right)$, noting the implicit dependence on $\mathbf{u}_i$. Then by (4.22) and by (4.24) with $m = n$ we find

$$(G - E_1\tau)(I_{\mathbf{u}_i}^{\otimes n}) = \left(0, \ldots, 0, w_n + w_{n-1} \frac{g_1^{(1)} f^n}{g_n} + w_{n-2} \frac{g_1^{(1)} f^n}{g_{n-1}} + \cdots + w_1 \frac{g_1^{(1)} f^n}{g_2}\right)^\top. \tag{4.26}$$

Denote the entry in the nth coordinate of the last equation as

$$\ell_{\mathbf{u}_i} := w_n + w_{n-1} \frac{g_1^{(1)} f^n}{g_n} + w_{n-2} \frac{g_1^{(1)} f^n}{g_{n-1}} + \cdots + w_1 \frac{g_1^{(1)} f^n}{g_2},$$

so that we can restate (4.24) with $m = n$ as

$$\frac{j_1 g_1^{(1)} f^n}{g_1} = j_1^{(1)} + \ell_{\mathbf{u}_i}. \tag{4.27}$$

Observe then by Lemma 2.1.5 and by Proposition 4.2.1 for $1 \leq k \leq n$ that

$$w_{n-k+1} \frac{g_1^{(1)} f^n}{g_{n-k+2}} = b_{n-k+1,i} g_1^{(1)} h_k,$$

so (4.26) becomes

$$(G - E_1\tau)(I_{\mathbf{u}_i}^{\otimes n}) = \Big(0, \ldots, 0, g_1^{(1)}(b_{n,i}h_1 + b_{n-1,i}h_2 + \cdots + b_{1,i}h_n)\Big)^{\top}$$

For the vector $\mathbf{u}_i$ from (4.21), denote

$$h_{\mathbf{u}_i} = b_{n,i}h_1 + b_{n-1,i}h_2 + \cdots + b_{1,i}h_n, \tag{4.28}$$

and notice that $\ell_{\mathbf{u}_i} = g_1^{(1)} h_{\mathbf{u}_i}$. Specializing the above discussion to $i = 0$, we see that the nth coordinate of $(G - E_1\tau)(I_{\mathbf{u}_0}^{\otimes n})$ matches up with the first n terms of the nth coordinate of $(G - E_1\tau)(T(h))$ from (4.23).

In general for $i > 0$ we find that

$$(f^{(-1)} f^{(-2)} \ldots f^{(-k)})^n \mathrm{diag}\left(\frac{g_1}{g_1^{(-k)}}, \ldots, \frac{g_n}{g_n^{(-k)}}\right)(I_{\mathbf{u}_i}^{\otimes n})^{(-k)} = T\left(\left(\frac{(f f^{(1)} \ldots f^{(k-1)})^n j_1}{g_1}\right)^{(-k)}\right),$$

and to ease notation, for $k \geq 1$ let us denote the matrix

$$R_k := (f^{(-1)} f^{(-2)} \ldots f^{(-k)})^n \mathrm{diag}\left(\frac{g_1}{g_1^{(-k)}}, \ldots, \frac{g_n}{g_n^{(-k)}}\right).$$

Then we use (4.27) k times and apply the fact that T is linear to obtain

$$\begin{aligned} R_k(I_{\mathbf{u}_i}^{\otimes n})^{(-k)} &= T\left(\left(\frac{(ff^{(1)}\dots f^{(k)})^n j_1}{g_1}\right)^{(-k)}\right) \\ &= I_{\mathbf{u}_i}^{\otimes n} + T\left(\frac{\ell_{\mathbf{u}_i}^{(-1)}}{g_1}\right) + \dots + T\left(\frac{(f^{(1)}\dots f^{(-1)})^n \ell_{\mathbf{u}_i}^{(-k)}}{g_1^{(1-k)}}\right). \end{aligned} \tag{4.29}$$

Then, if we let the operator $(G - E_1\tau)$ act on $R_k(I_{\mathbf{u}_i}^{\otimes n})^{(-k)}$, applying (4.22) to the last line of (4.29) we obtain a telescoping sum, and find that

$$(G - E_1\tau)(R_k(I_{\mathbf{u}_i}^{\otimes n})^{(-k)}) = \left(0, \dots, 0, g_1^{(1)}(ff^{(-1)}\dots f^{(1-k)})^n h_{\mathbf{u}_i}^{(-k)}\right)^\top,$$

for $h_{\mathbf{u}_i}$ defined in (4.28). Note again that the terms in the last coordinate of the above vector are exactly the $in+1$ through $(i+1)n$ terms of the last coordinate of (4.23).

Also, note that each term in the last line in (4.29) is coordinate-wise regular at Ξ except $I_{\mathbf{u}_i}^{\otimes n}$, so

$$\mathrm{RES}_\Xi(R_k(I_{\mathbf{u}_i}^{\otimes n})^{(-k)}) = \mathrm{RES}_\Xi(I_{\mathbf{u}_i}^{\otimes n}).$$

Then, recalling that each function w_k and each quotient j_{k+m}/j_k for $1 \le k, m \le n$ is regular at Ξ, using definitions (4.16) and (4.25) together with Proposition 3.2.5 we see that

$$\mathrm{RES}_\Xi(I_{\mathbf{u}_i}^{\otimes n}) = \mathrm{RES}_\Xi(J_{\mathbf{u}_i}^{\otimes n}) = \mathrm{RES}_\Xi(G_{\mathbf{u}_i}^{\otimes n}) = -\mathbf{u}_i. \tag{4.30}$$

Next, define

$$\mathbf{I} = I_{\mathbf{u}_0}^{\otimes n} + R_1 I_{\mathbf{u}_1}^{\otimes n} + \dots + R_m I_{\mathbf{u}_m}^{\otimes n},$$

and observe by the above discussion that

$$(G - E_1\tau)(T(h) - \mathbf{I}) = 0.$$

Further, for $h' \in \Omega$, by Lemma 3.1.3 $(G - E_1\tau)(T(h')) = 0$ if and only if $h' \in \Omega_0$. Since $\mathbf{I}$ is the sum of elements in the image of the map T, we see that $T(h) - \mathbf{I}$ is itself in the image of the map T. Thus there is some $h' \in \Omega_0$ such that $T(h') = T(h) - \mathbf{I}$. Then, Proposition 3.1.10 together with Theorem 3.2.7 implies that for some $b \in \mathbb{F}_q[t, y]$

$$T(h) - \mathbf{I} = T(h') = bG_{\Pi_n}^{\otimes n}.$$

Finally, by (4.30), we calculate that

$$\mathbf{u} = -\operatorname{RES}_{\Xi}(T(h)) = -\operatorname{RES}_{\Xi}(\mathbf{I} + bG_{\Pi_n}^{\otimes n}) = \mathbf{u}_0 + \cdots + \mathbf{u}_m + b\Pi_n,$$

and thus by (4.20) and (4.21) we obtain

$$\operatorname{Exp}_\rho^{\otimes n}(\mathbf{u}) = \operatorname{Exp}_\rho^{\otimes n}(\mathbf{u}_0 + \cdots + \mathbf{u}_m + b\Pi_n) = \mathbf{b}_0 + \cdots + \mathbf{b}_m = \varepsilon(g).$$

□

Having proven that diagram (4.15) commutes, we now apply the maps from the diagram to write down formulas for the coefficients of $\operatorname{Log}_\rho^{\otimes n}$. First, for $d_j \in \mathbb{C}_\infty$ define the function

$$c(t, y) = d_n h_1 + \cdots + d_1 h_n \in N \subset \mathbf{A}, \tag{4.31}$$

where h_j are from Proposition 2.1.3. Then define the formal sum

$$B(t, y; \mathbf{d}) = -\sum_{i=0}^{\infty} \frac{c^{(i)}}{(ff^{(1)}f^{(2)} \dots f^{(i)})^n} \tag{4.32}$$

for the vector $\mathbf{d} = (d_1, \dots d_n)^\top \in \mathbb{C}_\infty^n$. We remark that $B(t, y; \mathbf{d})$ is similar to the function $L_\alpha(t)$ defined by Papanikolas in [34, §6.1] (see also [16, 3.1.2]).

Lemma 4.2.3. *There exists a constant $C_0 > 0$ such that for $|d_j| \le C_0$, the function B is a rigid analytic function in $\Gamma(\mathcal{U}, \mathcal{O}_E(n(\Xi)))$, the space of rigid analytic functions on $\mathcal{U}$ with at most a pole* *of order n at Ξ.*

Proof. Using (2.3) together with the facts that $\deg(\theta) = \deg(\alpha) = 2$, $\deg(\eta) = 3$ and $\deg(m) = q$, for $k \geq 1$ we find that $f^{(k)} \in \mathbb{T}_\theta[y]$ and

$$\left\| f^{(k)} \right\|_\theta = q^{q^{k+1}}.$$

This implies that

$$\left\| \frac{c(t,y)^{(i)}}{f^{(1)} \dots f^{(i)}} \right\|_\theta = \|c(t,y)\|_\theta^{q^i} \cdot q^{\left(-n(q^{i+2}-q^2)/(q-1)\right)}. \tag{4.33}$$

Since each $h_i \in \mathbf{A}$, we see that $\|h_i\|_\theta$ is finite, and thus we can choose $C_0 > 0$ small enough such that for all $d_j \in \mathbb{C}_\infty$ with $|d_j| \leq C_0$ the norm

$$\left\| \frac{c(t,y)^{(i)}}{f^{(1)} \dots f^{(i)}} \right\|_\theta \to 0$$

as $i \to \infty$. This guarantees that for such d_j, the function

$$\sum_{i=0}^{\infty} \frac{c^{(i)}}{(f^{(1)} f^{(2)} \dots f^{(i)})^n} \in \mathbb{T}_\theta[y].$$

To finish the proof, we simply note that

$$B = -\frac{1}{f^n} \cdot \sum_{i=0}^{\infty} \frac{c^{(i)}}{(f^{(1)} f^{(2)} \dots f^{(i)})^n}.$$

$\square$

Theorem 4.2.4. *For $\mathbf{z} \in \mathbb{C}_\infty^n$ inside the radius of convegence of $\mathrm{Log}_\rho^{\otimes n}$, if we write*

$$\mathrm{Log}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} P_i \mathbf{z}^{(i)},$$

for $n \geq 2$, then for λ the invariant differential defined in (1.6)

$$P_i = \left\langle \mathrm{Res}_\Xi \left(\frac{g_j h_{n-k+1}^{(i)}}{(ff^{(1)} \dots f^{(i)})^n} \lambda \right) \right\rangle_{1 \leq j,k \leq n} \tag{4.34}$$

and $P_i \in \mathrm{Mat}_n(H)$ for $i \geq 0$.

Remark 4.2.5. As for Theorem 4.1.1, we remark that the above theorem holds for $n = 1$, but again for ease of exposition in the proof we restrict to the case of $n \geq 2$.

Proof. One quickly observes from the definition of B, that $(\tau - f^n)(B) = c(t, y)$, and thus $B \in \Omega$. Denote $\mathbf{u} := -\mathrm{RES}_\Xi(T(B))$, so that by Theorem 4.2.2 combined with the definition of the map ε in (4.20) and (4.31)

$$\mathrm{Exp}_\rho^{\otimes n}(\mathbf{u}) = \varepsilon(c(t, y)) = (d_1, d_2, \dots, d_n)^\top.$$

We wish to switch our viewpoint to thinking about $-\mathrm{RES}_\Xi(T(B))$ as a vector-valued function with input $(d_1, \dots, d_n)^\top$, $|d_i| < C_0$, where C_0 is the constant defined in Lemma 4.2.3. For D_0 the hyper-disk in $\mathbb{C}_\infty^n$ of radius C_0, we define $\tilde{B} : D_0 \to \mathbb{C}_\infty^n$, for $\mathbf{d} \in D_0$, as

$$\tilde{B}(\mathbf{d}) = -\mathrm{RES}_\Xi(T(B(t, y; \mathbf{d})).$$

From the above discussion, we find that

$$\mathrm{Exp}_\rho^{\otimes n} \circ \tilde{B} : D_0 \to \mathbb{C}_\infty^n$$

is the identity function. Writing out the definition for $\tilde{B}$ gives

$$\tilde{B} = -\begin{pmatrix} \mathrm{Res}_\Xi(Bg_1\lambda) \\ \vdots \\ \mathrm{Res}_\Xi(Bg_n\lambda) \end{pmatrix} = \begin{pmatrix} \mathrm{Res}_\Xi(\sum_{i=0}^\infty \sum_{j=1}^n \frac{(d_j h_{n-j+1})^{(i)}}{(ff^{(1)}f^{(2)} \dots f^{(i)})^n} g_1 \lambda) \\ \vdots \\ \mathrm{Res}_\Xi(\sum_{i=0}^\infty \sum_{j=1}^n \frac{(d_j h_{n-j+1})^{(i)}}{(ff^{(1)}f^{(2)} \dots f^{(i)})^n} g_n \lambda) \end{pmatrix}, \tag{4.35}$$

which we can express as an $\mathbb{F}_q$-linear power series with matrix coefficients

$$\tilde{B} = \sum_{i=0}^{\infty} \left\langle \mathrm{Res}_{\Xi}\left(\frac{g_j h_{n-k+1}^{(i)}}{(ff^{(1)}\dots f^{(i)})^n}\lambda\right)\right\rangle_{1\le j,k\le n} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}^{(i)} .$$

We conclude that $\mathrm{Exp}_{\rho}^{\otimes n} \circ \tilde{B}$ is an $\mathbb{F}_q$-linear power series which as a function on D_0 is the identity. Recall that $\mathrm{Log}_{\rho}^{\otimes n}$ is the functional inverse of $\mathrm{Exp}_{\rho}^{\otimes n}$ on the disk with radius r_L. Thus, on the disk with radius $\min(C_0, r_L)$ we have the functional identity

$$\tilde{B} = \mathrm{Log}_{\rho}^{\otimes n} .$$

Comparing the coefficients of the above expression, and recalling that f, g_i and h_i are defined over H finishes the proof. $\square$

Corollary 4.2.6. *For the coefficients P_i for $i \ge 0$ of the function $\mathrm{Log}_{\rho}^{\otimes n}$, the bottom row of P_i can be written as*

$$\left\langle \left.\frac{h_{n-k+1}^{(i)}}{h_1(f^{(1)}\dots f^{(i)})^n}\right|_{\Xi} \right\rangle_{1\le k\le n} . \tag{4.36}$$

Proof. Recall from (2.12) and (2.13) that $\mathrm{ord}_{\Xi}(g_j) = \mathrm{ord}_{\Xi}(h_j) = j-1$ and from (2.2) that $\mathrm{ord}_{\Xi}(f) = 1$. This implies that, for $i = 0$, each coordinate of the bottom row of the matrix (4.34) is regular at Ξ except the last coordinate, which equals

$$\mathrm{Res}\left(\frac{g_n h_1}{f^n}\lambda\right) = h_1(\Xi)\cdot \mathrm{Res}_{\Xi}\left(\frac{g_n}{f^n}\lambda\right).$$

Using Lemma 2.1.5, and observing that h_1 is regular at Ξ and that $t-\theta$ is a uniformizer at Ξ, a short calculation gives

$$\mathrm{Res}_{\Xi}\left(\frac{g_n}{f^n}\lambda\right) = \mathrm{Res}_{\Xi}\left(\frac{\delta_n}{h_2}\lambda\right) = \mathrm{Res}_{\Xi}\left(-\frac{\nu_n\circ[-1]}{h_1(t-\theta)}\lambda\right) = -\frac{\nu_n(-\Xi)}{h_1(\Xi)}\cdot\frac{1}{2\eta + c_1\theta + c_3},$$

where $[-1] : E \to E$ denotes negation on E. Finally, one calculates from the definition of ν_n from (3.7) that $\nu_n(-\Xi) = -2\eta - c_1\theta - c_3$, which implies that

$$\operatorname{Res}_\Xi\left(\frac{g_n}{f^n}\lambda\right) = \frac{1}{h_1(\Xi)}. \tag{4.37}$$

Thus, for $i = 0$, the bottom row of (4.34) equals $(0, \dots, 0, 1)$, which is the bottom row of $Q_0 = I$.

Then, for $i \geq 1$ note that the only functions in the bottom row of (4.34) which have zeros or poles at Ξ are g_n and f^n, and that the quotient g_n/f^n has a simple pole at Ξ, thus

$$\operatorname{Res}_\Xi\left(\frac{g_n h_{n-k+1}^{(i)}}{(ff^{(1)}\dots f^{(i)})^n}\lambda\right) = \left.\frac{h_{n-k+1}^{(i)}}{(f^{(1)}\dots f^{(i)})^n}\right|_\Xi \operatorname{Res}_\Xi\left(\frac{g_n}{f^n}\lambda\right),$$

which completes the proof using (4.37). □

Remark 4.2.7. Theorem 4.2.4 and Corollary 4.2.6 should be compared with the middle and last equalities in (2.8), respectively.

Remark 4.2.8. It is natural to ask about the relationship between the coefficients of $\operatorname{Log}_\rho^{\otimes n}$ and the Carlitz polylogarithm as defined by Anderson and Thakur at the end of §2.1 in [5]. Define the mth polylogarithm associated to the Anderson $\mathbf{A}$-module ρ by setting

$$\log_{m,\rho}(z) = z + \sum_{i\geq 1}\frac{1}{\ell_{i,m}}z^{q^i} = z + \sum_{i\geq 1}\left.\frac{1}{(f^{(1)}\dots f^{(i)})^m}\right|_\Xi \cdot z^{q^i}. \tag{4.38}$$

Then, using Corollary 4.2.6 we see that the bottom coordinate of $\operatorname{Log}_\rho^{\otimes n}$ can be written in terms of the nth polylogarithm function as

$$\operatorname{Log}_\rho^{\otimes n}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} * \\ \vdots \\ * \\ \sum_{k=1}^n \left.\frac{\log_{n,\rho}(h_{n-k+1}z_k)}{h_1}\right|_\Xi \end{pmatrix}. \tag{4.39}$$

5. ZETA VALUES

5.1 Zeta values

In [5], Anderson and Thakur analyze the lower right coordinate of the coefficient P_i of the logarithm function for tensor powers of the Carlitz module to obtain formulas similar to the ones we have provided in §4.2. They then define a polylogarithm function and use their formulas to relate this to zeta values,

$$\zeta(n) = \sum_{\substack{a \in \mathbb{F}_q[\theta] \\ \operatorname{sgn}(a)=1}} \frac{1}{a^n},$$

for all $n \geq 1$. In this section, we prove a similar theorem for tensor powers of Drinfeld $\mathbf{A}$-modules, but at the present it is unclear how to generalize the special polynomials which Anderson and Thakur used in their proof (the now eponymous Anderson-Thakur polynomials) to tensor powers of $\mathbf{A}$-modules, and so we developed new techniques. Presently, we only consider values of $n \leq q-1$ because these allow us to appeal to formulas from [26].

Remark 5.1.1. We remark that Anglès, Pellarin, Taveres Ribeiro and Perkins develop a multivariable version of L-series in [7], [8], [36] and [37] and that such considerations could possibly enable one to obtain formulas for all zeta values; this is an area of ongoing study.

To define a zeta function for a rank 1 sign-normalized Drinfeld module $\rho : \mathbf{A} \to H[\tau]$, we first define the left ideal of $H[\tau]$ for an ideal $\mathfrak{a} \subseteq A$ by

$$J_{\mathfrak{a}} = \langle \rho_{\overline{a}} \mid a \in \mathfrak{a} \rangle \subseteq H[\tau],$$

where we recall that $\overline{a} = \chi(a)$ from §1.2. Since $H[\tau]$ is a left principal ideal domain [24, Cor. 1.6.3], there is a unique monic generator $\rho_{\mathfrak{a}} \in J_{\mathfrak{a}}$, and we define $\partial(\rho_{\mathfrak{a}})$ to be the constant term of $\rho_{\mathfrak{a}}$ with respect to τ. Let $\phi_{\mathfrak{a}} \in \operatorname{Gal}(H/K)$ denote the Artin automorphism associated to $\mathfrak{a}$, and let the B be the integral closure of A in H. We define the zeta function associated to ρ twisted by the parameter $b \in B$ to be

$$\zeta_\rho(b; s) := \sum_{\mathfrak{a} \subseteq A} \frac{b^{\phi_{\mathfrak{a}}}}{\partial(\rho_{\mathfrak{a}})^s}, \tag{5.1}$$

Theorem 5.1.2. *For $b \in B$ and for $n \leq q-1$, there exists a vector $(*, \dots, *, C\zeta_\rho(b; n))^\top \in \mathbb{C}_\infty^n$ such that*

$$\mathbf{d} := \operatorname{Exp}_\rho^{\otimes n} \begin{pmatrix} * \\ \vdots \\ * \\ C\zeta_\rho(b; n) \end{pmatrix} \in H^n,$$

where $C = \frac{(-1)^{n+1} h_1(-\Xi)}{\theta - t([n]V^{(1)})} \in H$.

Remark 5.1.3. We remark that the vector $\mathbf{d}$ is explicitly computable as outlined in the proof of Theorem 5.1.2.

Remark 5.1.4. One would like to be able to express the above theorem in terms of evaluating $\operatorname{Log}_\rho^{\otimes n}$ at a special point and then getting a vector with $\zeta_\rho(n)$ as its bottom coordinate, as is done in [5]. However, one discovers that $\mathbf{d}$ is not necessarily within the radius of convergence of $\operatorname{Log}_\rho^{\otimes n}$, and in fact $\mathbf{d}$ can be quite large! It is possible that one could use Thakur's idea from [44, Thm. VI] to decompose $\mathbf{d}$ into small pieces which are each individually inside the radius of convergence of the logarithm for specific examples.

Before giving the proof of Theorem 5.1.2 we require several additional definitions and preliminary results. First, we denote $\mathbf{H}$ as the Hilbert class field of $\mathbf{K}$ (which is the fraction field of $\mathbf{A}$), and denote $\operatorname{Gal}(\mathbf{H}/\mathbf{K})$ as the Galois group of $\mathbf{H}$ over $\mathbf{K}$. Then we observe that elements $\overline{\phi} \in \operatorname{Gal}(\mathbf{H}/\mathbf{K})$ act on elements in the compositum field $H\mathbf{H}$ by applying $\overline{\phi}$ to elements of $\mathbf{H}$ and ignoring elements of H. We also define the (isomorphic) Galois group $\operatorname{Gal}(H/K)$ and observe that elements $\phi \in \operatorname{Gal}(H/K)$ act on the compositum field $H\mathbf{H}$ by applying ϕ to elements of H and ignoring elements of $\mathbf{H}$. Let $\mathfrak{p} \subseteq A$ be a degree 1 prime ideal, to which there is an associated point $P = (t_0, y_0) \in E(\mathbb{F}_q)$ such that $\mathfrak{p} = (\theta - t_0, \eta - y_0)$, and let $\phi = \phi_{\mathfrak{p}} \in \operatorname{Gal}(H/K)$ denote the Artin automorphism associated to $\mathfrak{p}$ via class field theory. Define the power sums

$$S_i(s) = \sum_{a \in A_{i+}} \frac{1}{a^s}, \quad S_{\mathfrak{p},i}(s) = \sum_{a \in \mathfrak{p}_{i+}} \frac{1}{a^s}, \tag{5.2}$$

where A_+ is the set of monic elements of A and A_{i+} is the set of monic, degree i elements of A. Then define the sums

$$\mathcal{Z}_{(1)}(b;s) = b \sum_{i \geq 0} S_i(s) = b \sum_{A+} \frac{1}{a^s}, \quad \mathcal{Z}_{\mathfrak{p}}(b;s) = b^{\phi^{-1}}(-f(P)^{\phi^{-1}})^s \sum_{a \in \mathfrak{p}_+} \frac{1}{a^s}. \tag{5.3}$$

We next prove a proposition which allows us to connect $\zeta_\rho(b;s)$ to the sums given above. Much of our analysis follows similarly to that in [26, §7-8], and we will appeal to it frequently throughout the remainder of the section.

Proposition 5.1.5. *Let $\mathfrak{p}_k$ for $2 \leq k \leq h$ be the degree 1 prime ideals as described above which represent the non-trivial ideal classes of A where h is the class number of A and set $\mathfrak{p}_1 = (1)$. Then, for $s \in \mathbb{Z}$ we can write the zeta function*

$$\zeta_\rho(b;s) = \mathcal{Z}_{\mathfrak{p}_1}(b;s) + \cdots + \mathcal{Z}_{\mathfrak{p}_h}(b;s).$$

Proof. Define the sum

$$\widetilde{\mathcal{Z}}_{\mathfrak{p}_k}(b;s) = \sum_{\mathfrak{a} \sim \mathfrak{p}_k} \frac{b^{\phi_\mathfrak{a}}}{\partial(\rho_\mathfrak{a})^s},$$

where the sum is over integral ideals $\mathfrak{a}$ equivalent to $\mathfrak{p}_k$ in the class group of A, and observe

$$\zeta_\rho = \sum_{k=1}^{h} \widetilde{\mathcal{Z}}_{\mathfrak{p}_k}.$$

Then, for $1 \leq k \leq h$, the fact that $\widetilde{\mathcal{Z}}_{\mathfrak{p}_k}(b;s) = \mathcal{Z}_{\mathfrak{p}_k}(b;s)$ follows from slight modifications to equations (98)-(100) and Lemma 7.10 from [26]. □

Now, we let $\{w_i\}_{i=2}^{\infty}$ (the reader should not confuse these with the coordinates w_i of $\mathbf{w}$ from

§4.2) be the sequence of linear functions with $\widetilde{\operatorname{sgn}}(w_i) = 1$ and divisor

$$\operatorname{div}(w_i) = (V^{(i-1)} - V) + (-V^{(i-1)}) + (V) - 3(\infty) \tag{5.4}$$

and let $\{w_{\mathfrak{p},i}\}_{i=2}^{\infty}$ be the sequence of functions with $\widetilde{\operatorname{sgn}}(w_{\mathfrak{p},i}) = 1$ and divisor

$$\operatorname{div}(w_{\mathfrak{p},i}) = (V^{(i-2)} - V - P) + (-V^{(i-2)}) + (V) + (P) - 4(\infty). \tag{5.5}$$

We now extend Theorem 6.5 from [26] to values $1 \leq s \leq q-1$, where we recall the definition of $\nu(t, y)$ from (2.4).

Proposition 5.1.6. *For* $1 \leq s \leq q-1$ *we find*

$$S_i(s) = \left(\frac{\nu^{(i)}}{w_i^{(1)} \cdot f^{(1)} \cdots f^{(i)}} \right)^s \Bigg|_{\Xi}, \quad S_{\mathfrak{p},i}(s) = \left(\frac{\nu^{(i-1)}}{w_{\mathfrak{p},i}^{(1)} \cdot f^{(1)} \cdots f^{(i-1)}} \right)^s \Bigg|_{\Xi}.$$

Proof. The proof of this proposition involves a minor alteration to the proof given for Proposition 6.5 in [26]. Namely, for the deformation $\mathcal{R}_{i,s}(t, y)$ one sets $s = m$ (rather than $s = q-1$ as is done in [26]) then one solves for $S_i(q-m)$ and sets $s = q-m$ to obtain the formula given above. The proof for $S_{\mathfrak{p},i}(s)$ is similar. $\square$

Using equations (82) and (117) from [26] we see that

$$\frac{\delta^{(1)}}{w_i^{(1)}}\bigg|_{\Xi} = \frac{f}{t-\theta}\bigg|_{V^{(i)}} = \frac{f(V^{(i)})}{-\delta^{(i)}(\Xi)},$$

which inspires the definition

$$\mathcal{G} := \frac{\beta + \overline{\beta} + c_1\overline{\alpha} + c_3}{\alpha - \overline{\alpha}} - \frac{\overline{\beta}^q + \overline{\beta} + c_1\overline{\alpha} + c_3}{\overline{\alpha}^q - \overline{\alpha}}, \tag{5.6}$$

where we recall that $V = (\alpha, \beta)$ from (2.1), that $c_i \in \mathbb{F}_q$ are from (1.5) and for $x \in H$ that

$\overline{x} = \chi(x)$ as in (1.7). Observe by (2.3) that $\mathcal{G}^{(i)}(\Xi) = f(V^{(i)})$ and hence

$$\left.\frac{\delta^{(1)}}{w_i^{(1)}}\right|_{\Xi} = -\left.\left(\frac{\mathcal{G}}{\delta}\right)^{(i)}\right|_{\Xi}. \tag{5.7}$$

Finally, we define

$$\widetilde{\mathcal{G}}_b = \sum_{\overline{\phi}\in \mathrm{Gal}(\mathbf{H}/\mathbf{K})} \overline{b}^{\overline{\phi}}(\mathcal{G}^{\overline{\phi}})^n.$$

Proposition 5.1.7. *We have $f^n\widetilde{\mathcal{G}}_b \in N$, where N is the dual $\mathbf{A}$-motive from* (2.10) *and $f^n\widetilde{\mathcal{G}}_b \in H[t,y]$.*

Proof. Our function $\mathcal{G}$ equals the function $\mathcal{F}$ from [26, (125)] (there they set $\phi = \overline{\alpha}$ and $\psi = \overline{\beta}$), and so our function $\widetilde{\mathcal{G}}_b$ differs from the function g_b from [26, (126)] only by the nth power in our definition. The proof of this theorem follows as in the proof of Theorem 8.7 from [26], replacing $\mathcal{F}$ by $\mathcal{G}^n$ and multiplying the divisors by a factor of n where appropriate. We arrive at the statement that the polar divisor of $\widetilde{\mathcal{G}}_b$ equals $-n(\Xi) - (nq - \deg(b))(\infty)$, and that $\widetilde{\mathcal{G}}_b$ vanishes with degree at least n at V so that $f^n \cdot \widetilde{\mathcal{G}} \in N$ as desired. Finally, since the coefficients of f and $\mathcal{G}$ are all in H, we conclude that $f^n\widetilde{\mathcal{G}}_b \in H[t,y]$. □

We are now equipped to give the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. Our starting point is Proposition 5.1.5,

$$\zeta_\rho(b;s) = \mathcal{Z}_{\mathfrak{p}_1}(b;s) + \cdots + \mathcal{Z}_{\mathfrak{p}_h}(b;s) \tag{5.8}$$

where we recall that for a degree 1 prime ideal $\mathfrak{p}$ and its associated Galois automorphism ϕ

$$\mathcal{Z}_{\mathfrak{p}}(b;n) = b^{\phi^{-1}}(-f(P)^{\phi^{-1}})^n \sum_{a\in\mathfrak{p}_+} \frac{1}{a^n} = b^{\phi^{-1}}(-f(P)^{\phi^{-1}})^n \sum_{i=0}^{\infty} S_{\mathfrak{p},i}(n). \tag{5.9}$$

If we let $[-1]$ denote the negation isogeny on E, by comparing divisors and leading terms of the functions in (2.4) and (2.13) we find

$$(\delta^{(1)})^n = \frac{(-1)^{n+1}(h_1)(h_1 \circ [-1])}{t - t([n]V^{(1)})}. \tag{5.10}$$

We will denote $C = \left.\frac{(-1)^{n+1}(h_1 \circ [-1])}{t - t([n]V^{(1)})}\right|_{\Xi} \in H$. Combining (5.3), Proposition 5.1.6, (5.7) and (5.10) we find

$$\mathcal{Z}_{(1)}(b; n) = \sum_{i=0}^{\infty} \left.\frac{\bar{b}\left((-f\mathcal{G})^{(i)}\right)^n}{C \cdot h_1 \left(f^{(1)} \cdots f^{(i)}\right)^n}\right|_{\Xi}. \tag{5.11}$$

Next, we temporarily fix a prime $\mathfrak{p} = \mathfrak{p}_k$ for $2 \le k \le h$. The combination of equations (86) and (118) and Lemma 7.12 from [26] gives

$$\frac{1}{w^{(1)}_{\mathfrak{p},i+1}} = -\left.\frac{f^{\phi^{-1}}}{t-\theta}\right|_{V^{(i)}} \cdot \frac{1}{\delta^{(1)}(\Xi)} \cdot \frac{1}{f(P)^{\phi^{-1}}} = \left. f^{\phi^{-1}}\right|_{V^{(i)}} \cdot \frac{1}{\delta^{(1)}(\Xi)\delta^{(i)}(\Xi)} \cdot \frac{1}{f(P)^{\phi^{-1}}}, \tag{5.12}$$

since $t - \theta(V^{(i)}) = -\delta^{(i)}(\Xi)$. Then, (5.9) and Proposition 5.1.6 together with (5.12) and the fact that $S_{\mathfrak{p},0} = 0$ gives

$$\mathcal{Z}_{\mathfrak{p}}(b; n) = (-1)^n b^{\phi^{-1}} \sum_{i=0}^{\infty} \left.\left(\frac{f^{(i)}}{\delta^{(1)} f^{(1)} \cdots f^{(i)}}\right)^n\right|_{\Xi} \cdot \left.\left(f^{\phi^{-1}}\right)^n\right|_{V^{(i)}} \tag{5.13}$$

We observe by (2.3) and (5.6) that $f^{\phi^{-1}}\left(V^{(i)}\right) = \left(\mathcal{G}^{\overline{\phi}^{-1}}\right)^{(i)}(\Xi)$ and so by (5.10) this gives

$$\mathcal{Z}_{\mathfrak{p}}(b; n) = \sum_{i=0}^{\infty} \left.\frac{\overline{b}^{\overline{\phi}^{-1}}\left((-f\mathcal{G}^{\overline{\phi}^{-1}})^n\right)^{(i)}}{Ch_1 \left(f^{(1)} \cdots f^{(i)}\right)^n}\right|_{\Xi}. \tag{5.14}$$

Therefore, returning to (5.8) we see by (5.11) and (5.14) that

$$\zeta_\rho(b; n) = \sum_{i=0}^{\infty} \sum_{\overline{\phi} \in \mathrm{Gal}(\mathbf{H}/\mathbf{K})} \left.\frac{\overline{b}^{\overline{\phi}}\left((-f\mathcal{G}^{\overline{\phi}})^n\right)^{(i)}}{Ch_1 \left(f^{(1)} \cdots f^{(i)}\right)^n}\right|_{\Xi} = \sum_{i=0}^{\infty} \left.\frac{\left((-1)^n f^n \widetilde{\mathcal{G}}_b\right)^{(i)}}{Ch_1 \left(f^{(1)} \cdots f^{(i)}\right)^n}\right|_{\Xi}. \tag{5.15}$$

From the proof of Proposition 5.1.7 we see that $\deg(f^n \widetilde{\mathcal{G}}_b) = n(q+1) + \deg(b)$ and from (2.14) that $\deg(\sigma^j(h_k)) = n(j+1)+k$. Let us write $\deg(b) = en + b'$ where $0 \leq b' \leq n-1$ so that $\deg(f^n \widetilde{\mathcal{G}}_b) = n(q+e+1)+b'$. Since $(-1)^n (f\widetilde{\mathcal{G}}_b)^n \in N$ by Proposition 5.1.7, we can express it in terms of the basis from Proposition 2.1.3 with coefficients $d_{k,j} \in \overline{K}$,

$$(-1)^n f^n \widetilde{\mathcal{G}}_b = \sum_{j=0}^{q+e} \sum_{k=1}^{n} d_{k,j} \sigma^j(h_{n-k+1}) = \sum_{j=0}^{q+e} \sum_{k=1}^{n} d_{k,j} (ff^{(-1)} \dots f^{(1-j)})^n h_{n-k+1}^{(-j)}, \tag{5.16}$$

where we comment that $d_{k,q+e} = 0$ for $k > b'$. Since $(-1)^n f^n \widetilde{\mathcal{G}}_b \in H[t,y]$ by Proposition 5.1.7, a short calculation involving evaluating (5.16) at $\Xi^{(k)}$ for $0 \leq k \leq q+e$ shows that $d_{k,j}^{(j)} \in H$. Substituting formula (5.16) into (5.15) and recalling that $f(\Xi) = 0$ gives

$$\zeta_\rho(b;n) = \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{\min(i,q+e)} \sum_{k=1}^{n} d_{k,j}^{(i)} h_{n-k+1}^{(i-j)}}{C \cdot h_1 \left(f^{(1)} \cdots f^{(i-j)}\right)^n} \Bigg|_{\Xi}.$$

We observe that the terms of the above sum are the bottom row of the coefficients P_i for $i \geq 0$ of $\operatorname{Log}_\rho^{\otimes n}$ from Corollary 4.2.6 up to the factor of $d_{k,j}^{(i)}/C$. Then, since $\operatorname{Log}_\rho^{\otimes n}$ is the inverse power series of $\operatorname{Exp}_\rho^{\otimes n}$, if we label $\mathbf{d}_j = (d_{1,j}, \dots, d_{n,j})^\top \in \overline{K}^n$ for $0 \leq j \leq q+e$ and sum over $i \geq 0$, then we find that there exists some vector $(*, \dots, *, C\zeta_\rho(b;n))^\top)$ such that

$$\left(\mathbf{d}_0 + \mathbf{d}_1^{(1)} + \cdots + \mathbf{d}_{q+e}^{(q+e)}\right) = \operatorname{Exp}_\rho^{\otimes n} \begin{pmatrix} * \\ \vdots \\ * \\ C\zeta_\rho(b;n) \end{pmatrix} \in H^n.$$

$\square$

5.2 Transcendence implications

In this section we examine some of the transcendence applications of Theorem 5.1.2. This is in line with Yu's results on transcendence in [47] for the Carlitz module, where he proves that the ratio $\zeta_\rho(n)/\widetilde{\pi}^n$ is transcendental if $q-1 \nmid n$ and rational otherwise. Yu's work builds on Anderson's and Thakur's theorem in [5], where they express Carlitz zeta values as the last coordinate of the logarithm of a special vector in A^n similarly to how we have done in Theorem 5.1.2. In the last couple decades, there has been a surge of research answering transcendence questions about arithmetic quantities in function fields, notably [4], [10], [14], [17], [34] and [48].

Theorem 5.2.1. *Let ρ be a rank 1 sign-normalized Drinfeld $\mathbf{A}$-module, let π_ρ be a fundamental period of $\exp_\rho$ and define $\zeta_\rho(b;n)$ as in* (5.1) *for $b \in B$, the integral closure of A in the Hilbert class field of K. Then*

$$\dim_{\overline{K}} \operatorname{Span}_{\overline{K}}\{\zeta_\rho(b;1), \dots, \zeta_\rho(b;q-1), 1, \pi_\rho, \dots, \pi_\rho^{q-2}\} = 2(q-1).$$

Our main strategy for proving Theorem 5.2.1 is to appeal to techniques Yu develops in [48], where he proves an analogue of Wüstholz's analytic subgroup theorem for function fields. Yu's theorem applies to Anderson $\mathbb{F}_q[t]$-modules (called t-modules), whereas here we deal with $\mathbf{A}$-modules. Thus, we switch our perspective slightly by forgetting the y-action of $\rho^{\otimes n}$ in order to view $\rho^{\otimes n}$ as an $\mathbb{F}_q[t]$-module with extra endomorphisms provided by the y-action. We will denote this $\mathbb{F}_q[t]$-module by $\hat{\rho}^{\otimes n}$. Under the construction given in §2.2, the $\mathbb{F}_q[t]$-module $\hat{\rho}^{\otimes n}$ corresponds to the dual t-motive N when viewed as a $\mathbb{C}_\infty[t,\sigma]$-module (we have forgotten the y-action on N), which we denote by N'. Before giving the proof of Theorem 5.2.1 we require a couple of lemmas which ensure that $\hat{\rho}^{\otimes n}$ satisfies the correct properties as a t-module to apply Yu's theorem.

Lemma 5.2.2. *The Anderson $\mathbb{F}_q[t]$-module $\hat{\rho}^{\otimes n}$ is simple.*

Proof. We recall the explicit functor between t-modules and dual t-motives as given in [27, §5.2]. For a t-module ϕ' with underlying algebraic group $J \subset \mathbb{C}_\infty^n$, define the dual t-motive $N(\phi')$ (note that this is denoted as $\check{M}(\underline{E})$ in [27, §5.2]) as $\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, J)$, the $\mathbb{C}_\infty[t,\sigma]$-module of all $\mathbb{F}_q$-linear homomorphisms of algebraic groups over $\mathbb{C}_\infty$. One defines the $\mathbb{C}_\infty[t,\sigma]$-module structure on $N(\phi')$ by having $\mathbb{C}_\infty$ act by pre-composition with scalar multiplication, σ act as pre-composition with the qth-power Frobenius and t acting by $t \cdot m = \phi'_t m$ for $m \in N(\phi')$. Note that $N(\hat{\rho}^{\otimes n}) = \operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, \mathbb{G}_a^n)$ is naturally isomorphic to $\mathbb{C}_\infty[\tau]^n$ where σ acts for $\mathbf{p}(\tau) \in \mathbb{C}_\infty[\tau]^n$ by $\sigma \cdot \mathbf{p}(\tau) = \mathbf{p}(\tau) \cdot \tau$ and $\mathbb{C}_\infty$ acts by scalar multiplication on the right. To maintain clarity, when we mean $\mathbb{C}_\infty$

with the action described above we will denote it as a $\mathbb{C}'_\infty$. Also note that $N(\hat{\rho}^{\otimes n})$ is isomorphic to $N' = \Gamma(U, \mathcal{O}_E(nV))$ as $\mathbb{C}_\infty[t,\sigma]$-modules.

Now suppose that $J \subset \mathbb{C}_\infty^n$ defines a non-trivial algebraic subgroup of $\mathbb{G}_a^n$, invariant under $\hat{\rho}^{\otimes n}(\mathbb{F}_q[t])$, defined by non-zero $\mathbb{F}_q$-linear polynomials $p_j(x_1,\dots,x_n) \in \overline{K}[x_1,\dots,x_n]$ for $1 \leq j \leq m$. We may assume that one of the polynomials, which we will denote as $p(x_1,\dots,x_n)$ has a non-zero term in x_1. Then note that we have the injection of $\mathbb{C}'_\infty[t,\sigma]$-modules given by inclusion

$$\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, J) \hookrightarrow \operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, \mathbb{G}_a^n),$$

which allows us to view $\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, J)$ as a $\mathbb{C}'_\infty[t,\sigma]$-submodule of $\mathbb{C}'_\infty[\tau]^n$, where the σ-action is given by right multiplication by τ as descrived above. Then observe that the map given induced by the polynomial p

$$p_* : \operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, \mathbb{G}_a^n) \to \operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, \mathbb{G}_a)$$

is a $\mathbb{C}'_\infty$-vector space map, that $\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, \mathbb{G}_a) \cong \mathbb{C}'_\infty[\tau]$ and that $\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, J) \subset \ker(p_*)$. By considering degrees in τ, we see that the $\mathbb{C}'_\infty$-vector subspace $(\mathbb{C}'_\infty[\tau], 0, \dots, 0) \subset \mathbb{C}'_\infty[\tau]^n$ maps to an infinite dimensional $\mathbb{C}'_\infty$-vector subspace of $\mathbb{C}'_\infty[\tau]$ under p_*. This implies that the quotient vector space, $\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, \mathbb{G}_a^n)/\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, J)$, also has infinite dimension over $\mathbb{C}_\infty$.

On the other hand, recall that $N' = \Gamma(U, \mathcal{O}_E(-nV^{(1)}))$ is isomorphic to $N(\hat{\rho}^{\otimes n})$ as $\mathbb{C}_\infty[t,\sigma]$-modules and that N' is an ideal of the ring $\mathbb{C}_\infty[t,y]$. Given a $\mathbb{C}_\infty[t,\sigma]$-submodule $J' \subset N'$ we may choose a non-zero element $h \in J'$, and we claim that $\sigma(h)$ is linearly independent from h over $\mathbb{F}_q[t]$. If not, then we would have

$$\beta h = f^n h^{(-1)} \tag{5.17}$$

for some $\beta \in \mathbb{F}_q(t)$. However, this implies that the rational function $f^n h^{(-1)}/h$ is fixed under the negation isogeny $[-1]$ on E, and in particular, for $i \neq 0$ we have

$$\operatorname{ord}_{\Xi^{(i+1)}}(h) - \operatorname{ord}_{-\Xi^{(i+1)}}(h) + \operatorname{ord}_{-\Xi^{(i)}}(h) - \operatorname{ord}_{\Xi^{(i)}}(h) = 0. \tag{5.18}$$

Since h is a polynomial in t and y, we see that $\mathrm{ord}_{\Xi^{(i)}}(h) - \mathrm{ord}_{-\Xi^{(i)}}(h) = 0$ for $|i| \gg 0$, thus (5.18) shows that $\mathrm{ord}_{\Xi^{(i)}}(h) - \mathrm{ord}_{-\Xi^{(i)}}(h) = 0$ for all i. But from (5.17) we see that

$$\mathrm{ord}_{\Xi}(f^n) + \mathrm{ord}_{\Xi^{(1)}}(h) - \mathrm{ord}_{-\Xi^{(1)}}(h) + \mathrm{ord}_{-\Xi}(h) - \mathrm{ord}_{\Xi}(h) = 0,$$

which is a contradiction, since $\mathrm{ord}_{\Xi}(f^n) = n$. So J' contains a rank 2 $\mathbb{C}_\infty[t]$-submodule and thus J' has finite index in N' as a $\mathbb{C}_\infty$-vector space. We conclude that all the $\mathbb{C}_\infty[t,\sigma]$-submodules of N' have finite index over $\mathbb{C}_\infty$ which contradicts our observation in the preceding paragraph, thus $\hat{\rho}^{\otimes n}$ must be simple as a t-module. $\square$

Lemma 5.2.3. *The Anderson $\mathbb{F}_q[t]$-module $\hat{\rho}^{\otimes n}$ has endomorphism algebra equal to $\mathbf{A}$.*

Proof. Recall that endomorphisms of $\hat{\rho}^{\otimes n}$ are $\mathbb{F}_q$-linear endomorphisms α of $\mathbb{C}_\infty^n$ such that $\alpha\hat{\rho}_a^{\otimes n} = \hat{\rho}_a^{\otimes n}\alpha$ for all $a \in \mathbb{F}_q[t]$. Thus $\mathbf{A}$ is certainly contained in $\mathrm{End}(\hat{\rho}^{\otimes n})$. On the other hand, the t-module $\hat{\rho}^{\otimes n}$ and the $\mathbf{A}$-module $\rho^{\otimes n}$ both have the same exponential function $\mathrm{Exp}_\rho^{\otimes n}$ and same period lattice $\Lambda_\rho^{\otimes n}$ (given in Theorem 3.2.7) associated to them. We note, however, that whereas $\Lambda_\rho^{\otimes n}$ is a rank 1 $\mathbf{A}$-module, when viewed as an $\mathbb{F}_q[t]$-module it is rank 2. If we let $\mathrm{End}^0(\hat{\rho}^{\otimes n}) = \mathrm{End}(\hat{\rho}^{\otimes n}) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t)$ as an $\mathbb{F}_q(t)$-vector space, then [12, Prop. 2.4.3] implies that $[\mathrm{End}^0(\hat{\rho}^{\otimes n}) : \mathbb{F}_q(t)] \leq 2$. Since $\mathbf{A} \subset \mathrm{End}(\hat{\rho}^{\otimes n})$ is a rank 2 $\mathbb{F}_q[t]$-module, we see that $[\mathrm{End}^0(\hat{\rho}^{\otimes n}) : \mathbb{F}_q(t)] = 2$, and thus $\mathrm{End}(\hat{\rho}^{\otimes n})$ is a rank 2 $\mathbb{F}_q[t]$-module containing $\mathbf{A}$. Further, $\mathbf{A} \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t) = K$, and thus $\mathrm{End}^0(\hat{\rho}^{\otimes n}) = K$ as an $\mathbb{F}_q(t)$-vector space. Since $\mathrm{End}(\hat{\rho}^{\otimes n})$ is finitely generated over $\mathbf{A}$, it is also integrally closed over $\mathbf{A}$ and thus $\mathrm{End}(\hat{\rho}^{\otimes n}) = \mathbf{A}$.

$\square$

Proof of Theorem 5.2.1. This proof follows nearly identically to the proof of [48, Prop. 4.1]. First, assume by way of contradiction that

$$\dim_{\overline{K}} \mathrm{Span}_{\overline{K}}\{\zeta_\rho(b;1), \ldots, \zeta_\rho(b;q-1), 1, \pi_\rho, \ldots, \pi_\rho^{q-2}\} < 2(q-1),$$

so that there is a $\overline{K}$-linear relation among the $\zeta_\rho(b;i)$ and π_ρ^j for $1 \leq i \leq q-1$ and $0 \leq j \leq q-2$.

Then, let G_L be the 1-dimensional trivial t-module and set

$$G = G_L \times \left(\prod_{i=1}^{q-1} \hat{\rho}^{\otimes i}\right) \times \left(\prod_{j=1}^{q-2} \hat{\rho}^{\otimes j}\right).$$

For $1 \leq i \leq q-1$ set $\mathbf{z}_i = (*, \ldots, *, C\zeta_\rho(b;i))^\top \in \mathbb{C}_\infty^i$ to be the vector from Theorem 5.1.2 such that $\mathrm{Exp}_\rho^{\otimes i}(\mathbf{z}_i) \in H^i$, where H is the Hilbert class field of K. For $1 \leq j \leq q-2$, let $\Pi_j \in \mathbb{C}_\infty^j$ be a fundamental period of $\mathrm{Exp}_\rho^{\otimes j}$ such that the bottom coordinate of Π_j is an H multiple of π_ρ^j as described in Theorem 3.2.7. Define the vector

$$\mathbf{u} = 1 \times \left(\prod_{i=1}^{q-1} \mathbf{z}_i\right) \times \left(\prod_{j=1}^{q-2} \Pi_j\right) \in G(\mathbb{C}_\infty),$$

and note $\mathrm{Exp}_G(\mathbf{u}) \in G(H)$, where Exp_G is the exponential function on G. Our assumption that there is a $\overline{K}$-linear relation among the $\zeta_\rho(b;i)$ and π_ρ^j implies that $\mathbf{u}$ is contained in a $d\,[\mathbb{F}_q[t]]$-invariant hyperplane of $G(\mathbb{C}_\infty)$ defined over $\overline{K}$. This allows us to apply [48, Thm. 3.3], which says that $\mathbf{u}$ lies in the tangent space to the origin of a proper t-submodule $H \subset G$. Then, Lemmas 5.2.2 and 5.2.3 together with [48, Thm 1.3] imply that there exists a linear relation of the form $a\zeta_\rho(b;j) + b\pi_\rho^j = 0$ for some $a, b \in H$ and $1 \leq j \leq q-2$. Since $\zeta_\rho(b;j) \in K_\infty$ and since $H \subset K_\infty$, this implies that $\pi_\rho^j \in K_\infty$. However, we see from the product expansion for π_ρ in [26, Thm. 4.6 and Rmk. 4.7] that $\pi_\rho^j \in K_\infty$ if and only if $q-1|j$, which cannot happen because $j \leq q-2$. This provides a contradiction, and proves the theorem.

□

Corollary 5.2.4. *For $1 \leq i \leq q-1$, the quantities $\zeta_\rho(b;i)$ are transcendental. Further, for $0 \leq j \leq q-1$ the ratio $\zeta_\rho(b;i)/\pi_\rho^j \in \overline{K}$ if and only if $i = j = q-1$.*

Proof. The transcendence of $\zeta_\rho(b;i)$, as well as the statement that $\zeta_\rho(b;i)/\pi_\rho^j \notin \overline{K}$ for $i, j \neq q-1$ follows directly from Theorem 5.2.1. On the other had, if $i = j = q-1$, then [22, Thm. 2.10] guarantees that $\zeta_\rho(b;i)/\pi_\rho^j \in \overline{K}$. □

6.1 Examples and summary

Example 6.1.1. In the case of tensor powers of the Carlitz module (see [33] for a detailed account on tensor powers of the Carlitz module), the formulas in Theorems 4.1.1 and 4.2.4 for the coefficients of $\mathrm{Exp}_C^{\otimes n}$ and $\mathrm{Log}_C^{\otimes n}$ can be worked out completely explicitly using hyper-derivatives. For instance, we find that $g_i = (t-\theta)^{i-1}$ and that the shtuka function is $f = (t-\theta)$, so the left hand side of (4.1) is

$$\gamma_{\ell,i} = \frac{1}{(t-\theta)^{n-\ell}(t-\theta^q)^n \dots (t-\theta^{q^i})^n}.$$

We can expand $\gamma_{\ell,i}$ in terms of powers of $(t-\theta)$ by using hyper-derivatives, as described in [33, §2.3], namely

$$\gamma_{\ell,i} = \sum_{j=0}^{\infty} \partial_t^j(\gamma_{\ell,i})\bigg|_{t=\theta} \cdot (t-\theta)^j.$$

Using this we recover the coefficients of $\mathrm{Exp}_C^{\otimes n}$ as given in formula (4.3.2) and Proposition 4.3.6(b) from [33]. The formulas for coefficients of the logarithm given in (4.3.4) and Proposition 4.3.6(a) from [33] can be derived similarly using Theorem 4.2.4.

Example 6.1.2. Let $E : y^2 = t^3 - t - 1$ be defined over $\mathbb{F}_3$, and note that $A = \mathbb{F}_q[t, y]$ has class number 1. Then from [45] we find that

$$f = \frac{y - \eta - \eta(t-\theta)}{t - \theta - 1}.$$

The Drinfeld module ρ associated to the coordinate ring of E is detailed in Example 9.1 in [26]. We form the 2-dimensional Anderson $\mathbf{A}$-module $\rho^{\otimes 2}$ as outlined in section §2.2, where we recall from (2.12) that

$$\mathrm{div}(g_1) = -2(V) + (\infty) + ([2]V), \quad \mathrm{div}(g_2) = -2(V) + (\Xi) + (V^{(1)} + V).$$

If we denote T_{-V} as translation by $-V$ on E, then we can quickly write down formulas for g_1 and g_2 by observing that $g_1 \circ T_{-V}$ and $g_2 \circ T_{-V}$ are both polynomials with relatively simple divisors, from which we calculate that

$$g_1 = \frac{\eta^2 + \eta y + t - \theta - 1}{\eta t^2 + \eta t\theta + \eta\theta^2 + \eta t - \eta\theta + \eta},$$

$$g_2 = \frac{\eta^2 t^2 + \eta^2 t\theta + \eta^2\theta^2 + \eta^2 t - \eta^2\theta - \eta^2 + t^2 + t\theta + \theta^2 + \eta y - t + \theta}{\eta^2 t^2 + \eta^2 t\theta + \eta^2\theta^2 + \eta^2 t - \eta^2\theta + \eta^2 + t^2 + t\theta + \theta^2 + t - \theta + 1}.$$

We further compute that

$$h_1 = -\frac{\eta^6 - \eta^3 y - \eta^2 + t - \theta + 1}{\eta^3}$$

$$h_2 = \frac{\eta^4 t - \eta^4\theta - \eta^4 + \eta^2 t^2 + \eta^2 t\theta + \eta^2\theta^2 + \eta^3 y + t^2 + t\theta + \theta^2 + t - \theta}{\eta^2 + 1}.$$

Then using Corollary 3.1.5 we calculate that

$$\rho_t^{\otimes n} = \begin{pmatrix} \theta & \frac{-(\eta^2+1)^2}{\eta^3} \\ 0 & \theta \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \frac{-\eta^3(\eta^4-\eta^2-1)}{(\eta^2+1)^3} & 1 \end{pmatrix} \cdot \tau.$$

We then calculate that the bottom coordinate of Π_2 from Theorem 3.2.7 is

$$\frac{-(\eta^2+1)^2}{(\eta^5 - \eta^3 - \eta)} \cdot \pi_\rho^2.$$

Using these, we calculate the first few terms of the expression from Corollary 4.1.4 as

$$\operatorname{Exp}_\rho^{\otimes n} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\eta^4-\eta^2}{-\eta^6-1} \\ \frac{\eta^7-\eta^5-\eta^3}{-\eta^8-\eta^6-\eta^2-1} \end{pmatrix} z^q + \begin{pmatrix} \frac{\eta^{12}-\eta^6}{-\eta^{22}+\eta^{20}-\eta^{18}-\eta^4+\eta^2-1} \\ \frac{\eta^{15}-\eta^{11}+\eta^9-\eta^7}{-\eta^{24}-\eta^{18}-\eta^6-1} \end{pmatrix} z^{q^2} + O(z^{q^3}),$$

and calculate an example of the vector from Corollary 4.2.6, which is the bottom row of P_1,

$$\left(\frac{h_2^{(1)}}{h_1 f^n}\Big|_\Xi, \frac{h_1^{(1)}}{h_1 f^n}\Big|_\Xi \right) = \left(\frac{\eta^7 - \eta^5 - \eta^3}{\eta^8 + \eta^6 + \eta^2 + 1}, \frac{\eta^8 + \eta^4 - 1}{\eta^8 + \eta^6} \right).$$

We calculate that the function $\mathcal{G}$ from (5.6) is $\mathcal{G} = (\eta + y)/(\theta - t) - y$ and that for $b = 1$ we can express $(-1)^2 f^2 \widetilde{\mathcal{G}}_b = (f\mathcal{G})^2$ in the form given in (5.16) as

$$(f\mathcal{G})^2 = \frac{-\eta^3}{\eta^2+1} h_1 + h_2 + \frac{\eta^{5/3}}{\eta^{2/3}+1} h_1^{(-1)} f^2 + h_2^{(-1)} f^2 + \frac{-\eta^{5/9}+\eta^{1/3}}{\eta^{2/9}+1} h_1^{(-2)} (ff^{(-1)})^2 + h_1^{(-2)} (ff^{(-1)})^2.$$

This allows us to write the formulas in Theorem 5.1.2 as

$$\begin{pmatrix} 1 \\ \frac{-\eta^3}{\eta^2+1} \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{\eta^5}{\eta^2+1} \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{-\eta^5+\eta^3}{\eta^2+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathrm{Exp}_\rho^{\otimes n} \begin{pmatrix} * \\ -\frac{\eta^3}{\eta^2+1} \zeta(2) \end{pmatrix}.$$

Thus the special vector $\mathbf{z} = (*, -\eta^3/(\eta^2+1)\zeta(2))^\top$ is in the period lattice for $\mathrm{Exp}_\rho^{\otimes n}$ which by Theorem 3.2.7 implies that the bottom coordinate of $\mathbf{z}$ is a K-multiple of π_ρ^2, the fundamental period associated to ρ. Hence $\zeta(2)/\pi_\rho^2 \in K$ as implied by Goss's [22, Thm. 2.10].

Example 6.1.3. Now let $q = 4$ and let $E/\mathbb{F}_q$ be defined by $y^2 + y = t^3 + c$, where $c \in \mathbb{F}_4$ is a root of the polynomial $c^2 + c + 1 = 0$. Then we know from [45, §2.3] that $A = \mathbb{F}_q[\theta, \eta]$ has class number 1, that $V = (\theta, \eta + 1)$ and that

$$f = \frac{y + \eta + \theta^4(t + \theta)}{t + \theta}.$$

Setting the dimension $n = 2$ and the parameter $b = 1$, from (5.6) we find that

$$\mathcal{G} = \frac{\eta + y + 1}{\theta + t} + \frac{y^4 + y + 1}{t^4 + t}$$

and that $\widetilde{\mathcal{G}}_1 = \mathcal{G}^2$. Then we compute the expansion from (5.16) as

$$\begin{aligned} f^2 \widetilde{\mathcal{G}}_1 = {} & (\theta^4+\theta)^{-1} h_1 + h_2 + (\theta^4+\theta)^{1/4} h_1^{(-1)} f^2 + (\theta^4+\theta)^{1/2} h_2^{(-1)} f^2 + (\theta^4+\theta)^{3/16} h_1^{(-2)} (ff^{(-1)})^2 \\ & + (\theta^4+\theta)^{1/4} h_2^{(-2)} (ff^{(-1)})^2 + (\theta^4+\theta)^{-1/64} h_1^{(-3)} (ff^{(-1)} f^{(-2)})^2 + h_2^{(-3)} (ff^{(-1)} f^{(-2)})^2, \end{aligned}$$

whereupon Theorem 5.1.2 gives

$$\begin{aligned}\begin{pmatrix} 1 \\ (\theta^4+\theta)^{-1} \end{pmatrix} + \begin{pmatrix} (\theta^4+\theta)^2 \\ (\theta^4+\theta) \end{pmatrix} + \begin{pmatrix} (\theta^4+\theta)^4 \\ (\theta^4+\theta)^3 \end{pmatrix} + \begin{pmatrix} 1 \\ (\theta^4+\theta)^{-1} \end{pmatrix} &= \begin{pmatrix} (\theta^4+\theta)^2 + (\theta^4+\theta)^4 \\ (\theta^4+\theta) + (\theta^4+\theta)^3 \end{pmatrix} \\ &= \operatorname{Exp}_\rho^{\otimes n} \begin{pmatrix} * \\ (\theta^4+\theta)^{-1}\zeta(2) \end{pmatrix}.\end{aligned}$$

Summary. In this dissertation, we gave an explicit description of tensor powers of rank 1 sign-normalized Drinfeld modules, gave a formulas for their periods, gave formulas for the coefficients of the exponential and logarithm functions, related these formulas to zeta values and proved a theorem about their transcendence.

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