# CONVERGENCE OF THE POWERS OF FREE TRIANGULAR ARRAYS TO HIGHER VARIATIONS OF FREE LÉVY PROCESSES

A Thesis

by

ZHICHAO WANG

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

| | |
|---|---|
| Chair of Committee, | Michael Anshelevich |
| Committee Members, | Ken Dykema |
| | Jean-Francois Chamberland |
| Head of Department, | Emil J. Straube |

May 2018

Major Subject: Mathematics

ABSTRACT

We focus on a free triangular array of random variables $\{X_{N,r}\}_{1\leqslant r\leqslant N}$, $N\in\mathbb{N}$, in some non-commutative probability space $(\mathcal{A},\phi)$ such that the random variables are freely independent and identically distributed in each row. For each $k\in\mathbb{N}$, our aim is to find some conditions to ensure the convergence of the sum of the $k$-th power of each row in the triangular array. We also want to know the expression of the limit random variable (the $k$-th variation), denoted by $X^{(k)}$. The motivation of this study comes from the relation between the free stochastic measure of a free Lévy process and higher variations.

First, when $(\mathcal{A},\phi)$ is a plain non-commutative probability space, we find out an equivalent condition for the joint convergence in distribution of all powers of the free triangular array. This condition requires the decay of all moments of the random variables in each row. Moreover, after defining the free stochastic measure of a free triangular array in terms of the convergence in distribution, we prove a free Kailath-Segall formula for centered stationary stochastic processes in our settings to describe the relationship between a stochastic measure and the higher variations.

Second, when $(\mathcal{A},\phi)$ is a $W^*-$probability space, considering all self-adjoint (possibly unbounded) operators affiliated with $\mathcal{A}$, we prove that if the convergence in distribution of a free triangular array to a free Lévy process holds, then we have the convergence in distribution of all powers of the original triangular array towards all higher variations, which are also free Lévy processes. Moreover, the free Lévy-Itô decomposition of each higher variation can be simplified by the Lévy-Itô decomposition of the original Lévy process.

DEDICATION

I dedicate this thesis to my parents, Huixia Wang and Junsheng Wang, for all their unconditional love and support.

ACKNOWLEDGMENTS

I would like to give my sincere thanks to my advisor Michael Anshelevich, who gives me patient guidance in doing research. During graduate studies, I learned a lot from him and had a good time working with him on this research project. It is he who helps me know how to discover new problems and how to do graduate research in mathematics.

# CONTRIBUTORS AND FUNDING SOURCES

## Contributors

This work was supervised by a thesis committee consisting of Professor Michael Anshelevich and Professor Ken Dykema of the Department of Mathematics and Professor Jean-Francois Chamberland of the Department of Electrical and Computer Engineering.

All work for the thesis was completed by the student, under the advisement of Michael Anshelevich of the Department of Mathematics at Texas A&M University.

## Funding Sources

There are no outside funding contributions to acknowledge related to the research and compilation of this document.

TABLE OF CONTENTS

# 1. PRELIMINARIES

## 1.1 Non-commutative Probability Spaces

Since our main results are stated in a free probability setting, we would like to first review some background material on free probability for readers. The notations in the following definitions will appear again later. For more details on free probability theory, readers can refer to A. Nica and R. Speicher's textbook [10].

**Definition 1** (Non-commutative Probability Spaces)**.** *A non-commutative probability space is a pair $(\mathcal{A}, \phi)$, consisting of a unital algebra $\mathcal{A}$ over $\mathbb{C}$ and a state $\phi : \mathcal{A} \to \mathbb{C}$, which is a linear functional such that $\phi(\mathbf{1}_{\mathcal{A}}) = 1$. The elements of $\mathcal{A}$ are called non-commutative random variables.*

For simplicity, we often call a non-commutative random variable $a \in \mathcal{A}$ a random variable in $(\mathcal{A}, \phi)$. Notice that Definition 1 requires few structures on the probability space. Although this plain non-commutative probability space is useful in some cases, we sometimes need more structures on this space for understanding or analyzing some problems. Thus, we introduce two more concrete non-commutative probability spaces. The unital algebra $\mathcal{A}$ is replaced by some concrete algebra. For instance, a unital $C^*-$algebra acting on a Hilbert space $H$ is a subalgebra of $B(H)$ (all bounded operators), which contains the multiplicative unit $\mathbf{1}$ of $B(H)$, and which is closed under the adjoint $*$-operation and topologically closed w.r.t. the operator norm. A von Neumann algebra is a unital $C^*-$algebra acting on a Hilbert space $H$, which is closed in the weak operator topology on $B(H)$, namely, the weak topology on $B(H)$ induced by the linear functionals: $T \mapsto \langle Tx, y\rangle$, for any $x, y \in H$.

**Definition 2** ($C^*-$probability Spaces)**.** *A $C^*-$probability space is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital $C^*-$algebra and $\phi$ is a state on $\mathcal{A}$ such that $\phi(a^*a) \geq 0$ (positive) and $\phi(a^*) = \overline{\phi(a)}$, for any $a \in \mathcal{A}$.*

**Definition 3** ($W^*-$probability Spaces)**.** *A $W^*-$probability space is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a von Neumann algebra acting on a Hilbert space and $\phi$ is a faithful normal tracial state on $\mathcal{A}$ i.e.*

1. $\phi(ab) = \phi(ba)$, *for any* $a, b \in \mathcal{A}$ *(tracial);*

2. $\phi$ *is continuous with respect to the weak operator topology on the unit ball of* $\mathcal{A}$ *(normal);*

3. $\phi(a^*a) > 0$, *for any nonzero* $a \in \mathcal{A}$ *(faithful).*

In this project, we will mostly deal with self-adjoint operators on a plain non-commutative probability space or a $W^*-$probability space. Suppose that $(\mathcal{A}, \phi)$ is a $W^*-$probability space and that $b$ is a self-adjoint operator (i.e. $b^* = b$) in $\mathcal{A}$. Then, the spectrum of operator $b$ is a compact set on the real line. Moreover, we can associate a distribution to $b$: there exists a unique compactly supported probability measure $\mu_b$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, satisfying

$$\int_{\mathbb{R}} t^n \mu_b(dt) = \phi(b^n),$$

for any $n \in \mathbb{N}$. Intuitively, $\phi(b^n)$ can be viewed as the $n-$th moment of random variable $b$ or the probability measure $\mu_b$. So $\phi(b)$ is the expectation of random variable $b$. All the moments can uniquely determine a compactly supported probability measure on the real line. Conversely, any compactly supported probability measure on $\mathbb{R}$ is a distribution of some self-adjoint operator in the non-commutative probability space. In this case, all random variables in the non-commutative probability space have all finite moments.

### 1.2 Unbounded Operators and Affiliated Operators

In some cases, we want to study unbounded operators instead of bounded operators because a bounded operator has all finite moments and compactly supported spectrum, but in many situations, our probability measure (the distribution of some operator) may not have compact support or all finite moments. Therefore, we want to first introduce some unbounded operators which can also have distributions on $\mathbb{R}$. In other words, we shall enlarge our non-commutative probability space to include some unbounded operators such that most of the definitions in a plain non-commutative probability space still make sense in a larger space.

Since we only consider probability measures on the real line, which are the distributions of self-adjoint operators, we need to first define the unbounded self-adjoint operators acting on a Hilbert space $H$. Let $T : \mathcal{D}(T) \to H$ be a densely defined linear operator. We say $T$ is symmetric if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{D}(T)$.

**Definition 4.** *The densely defined linear operator $T$ is self-adjoint if $T$ is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$.*

**Definition 5** (Affiliated Operators)**.** *Let $(\mathcal{A}, \phi)$ be a $W^*-$probability space acting on a Hilbert space $\mathcal{H}$. Let $a : \mathcal{D}(a) \to \mathcal{H}$ be a (possibly unbounded) self-adjoint linear operator defined on a dense subspace $\mathcal{D}(a) \subset \mathcal{H}$. We say that $a$ is affiliated with $\mathcal{A}$, if $f(a) \in \mathcal{A}$ for any bounded Borel function $f$ on $\mathbb{R}$ ($f(a)$ is defined in terms of functional calculus).*

For a self-adjoint operator $a$ affiliated with $(\mathcal{A}, \phi)$, its spectrum is a set on $\mathbb{R}$ (possibly unbounded). So, in this case, there exists a unique probability measure $\mu_a$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\int_{\mathbb{R}} f(t)d\mu_a(t) = \phi(f(a)),$$

for any bounded Borel function on $\mathbb{R}$. Thus, $\mu_a$ is the distribution of the affiliated operator $a$. However, the operator $a$ probably does not have finite moments because we are only able to compute the expectation of $f(a)$, for any bounded Borel function $f$ on $\mathbb{R}$. In fact, Definition 5 is not a general definition of affiliated operators, although it is enough for our research. For a general definition, we refer to section 2.5 of [8].

We denote by $\bar{\mathcal{A}}$ the set of all closed, densely defined operators, which are affiliated with $\mathcal{A}$. It turns out that $\bar{\mathcal{A}}$ is a $*-$algebra equipped with adjoint operation, strong sum and product (cf. [8]). And if the operator $a$ is bounded, then $a$ is affiliated with $\mathcal{A}$ if and only if $a \in \mathcal{A}$.

### 1.3 Free Independence

Free independence is a basic concept in free probability and is an analog of independence from classical probability theory. Meanwhile, one can check that free independence is a genuine non-commutative concept in terms of the following definitions.

**Definition 6.** *Let* $(\mathcal{A}, \phi)$ *be a non-commutative probability space. We say that* $a_1, a_2, ..., a_r \in \mathcal{A}$ *are freely independent with respect to* $\phi$*, if*

$$\phi\left[(f_1(a_{i_1}) - \phi(f_1(a_{i_1}))) \cdot (f_2(a_{i_2}) - \phi(f_2(a_{i_2})))...(f_p(a_{i_p}) - \phi(f_p(a_{i_p})))\right] = 0,$$

*for any* $p \in \mathbb{N}$*, any polynomials* $f_1, f_2, ..., f_p$ *in* $\mathbb{C}[X]$ *and any indices* $i_1, i_2, ..., i_p$ *in* $\{1, 2, ..., r\}$ *satisfying* $i_1 \neq i_2, i_2 \neq i_3, ..., i_{p-1} \neq i_p$.

In general, we can define the free independence of several unital subalgebras in a non-commutative probability space. Moreover, one can define the $*-$free independence in a $C^*-$probability space or a $W^*-$probability space, but we are usually dealing with self-adjoint operators, which means that $*-$free independence is equivalent to free independence for self-adjoint operators. For unbounded operators, we can also define the free independence among self-adjoint operators affiliated with a $W^*-$probability space.

**Definition 7.** *Let* $(\mathcal{A}, \phi)$ *be a* $W^*-$*probability space. Let* $a_1, a_2, ..., a_r \in \mathcal{A}$ *be self-adjoint operators affiliated with* $\mathcal{A}$*. We say that they are freely independent with respect to* $\phi$*, if*

$$\phi\left[(f_1(a_{i_1}) - \phi(f_1(a_{i_1}))) \cdot (f_2(a_{i_2}) - \phi(f_2(a_{i_2})))...(f_p(a_{i_p}) - \phi(f_p(a_{i_p})))\right] = 0,$$

*for any* $p \in \mathbb{N}$*, any bounded Borel functions* $f_1, f_2, ..., f_p$ *on* $\mathbb{R}$ *and any indices* $i_1, i_2, ..., i_p$ *in* $\{1, 2, ..., r\}$ *satisfying* $i_1 \neq i_2, i_2 \neq i_3, ..., i_{p-1} \neq i_p$.

Given the independence, we can consider the sum of two independent random variables and try to find the distribution of this new random variable. In classical probability, it is the convolution of two distributions (probability measures on $\mathbb{R}$). Analogously, we can define a free convolution in free sense. It turns out that the distribution of the sum of two freely independent random variables is uniquely determined by the distributions of these two random variables.

**Definition 8** (Free Convolution)**.** *Let* $a$ *and* $b$ *be two freely independent and self-adjoint operators in* $\bar{\mathcal{A}}$ *with distributions* $\mu_a$ *and* $\mu_b$*. Then the distribution* $\mu_{a+b}$ *of* $a + b$ *is uniquely determined by* $\mu_a$

*and $\mu_b$ and is denoted by $\mu_b \boxplus \mu_a$. We say $\mu_b \boxplus \mu_a$ is the free (additive) convolution of $\mu_a$ and $\mu_b$.*

Free convolution is actually an operation on probability measures defined on $\mathbb{R}$, since any self-adjoint (possibly unbounded) random variable has a probability measure on $\mathbb{R}$ as its distribution and, for any probability measure $\mu$, one can find a self-adjoint operator such that $\mu$ is its distribution. There are several transforms of probability measures on $\mathbb{R}$ introduced to study the free convolutions. We mainly use the Voiculescu transform $\Phi_\mu(z)$ of a probability measure $\mu$ on $\mathbb{R}$. For the definition and properties of Voiculescu transform, see [9, 10]. The most important property of this transform is that

$$\Phi_{\mu\boxplus\nu} = \Phi_\mu + \Phi_\nu, \tag{1.1}$$

on their common domain. This property shows that the free convolution of probability measure $\mu$ and probability measure $\nu$ is uniquely determined by the 'marginal' distributions $\mu$ and $\nu$. Also, we need the fact that a probability measure on $\mathbb{R}$ is determined by its Voiculescu transform.

### 1.4 Convergences in a Free Probability Setting

There are different types of convergence in free probability, so, in this section, we briefly introduce two kinds of convergence in free sense. We shall recall the weak convergence of probability measures at first.

**Definition 9.** *The probability measure $\mu_N$ defined on $\mathbb{R}$ converges weakly to the probability measure $\mu$ when*

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(t)d\mu_N(t) = \int_{\mathbb{R}} f(t)d\mu(t),$$

*for all bounded continuous function $f(x)$ on the real line. We denote this convergence as $\mu_N \overset{w.}{\to} \mu$.*

**Definition 10.** *Let $(\mathcal{A}_N, \phi_N)$ $(N \in \mathbb{N})$ and $(\mathcal{A}, \phi)$ be non-commutative probability spaces. Consider random variables $a_N \in \mathcal{A}_N$, for any $N \in \mathbb{N}$ and $a \in \mathcal{A}$. We say that $a_N$ converges in distribution to $a$ as $N \to \infty$, denoted by $a_N \overset{d.}{\to} a$, if we have that*

$$\lim_{N\to\infty} \phi_N(a_N^n) = \phi(a^n),$$

*for all* $n \in \mathbb{N}$.

Particularly, if $a_N$ and $a$ are self-adjoint and have distributions $\mu_N$ and $\mu$ respectively, which are compactly supported on $\mathbb{R}$, then Definition 10 will imply the weak convergence of probability measure $\mu_N$ on $\mathbb{R}$ towards the probability measure $\mu$. Conversely, $\mu_N$ converging weakly to $\mu$ also implies the convergence in distribution of $a_N$ towards $a$ thanks to the compact supports of $\mu_N$ and $\mu$.

Above, we discuss the convergence in distribution for a single random variable, but we sometimes need the joint convergence in distribution for a family of random variables. Thus, next, we introduce the joint convergence in distribution in free sense.

**Definition 11.** *Let* $(\mathcal{A}_N, \phi_N)$ $(N \in \mathbb{N})$ *and* $(\mathcal{A}, \phi)$ *be non-commutative probability spaces. Let* $I$ *be an index set,* $a_N^{(i)} \in \mathcal{A}_N$ *and* $a_i \in \mathcal{A}$*, for each* $i \in I$. *We say that* $\left(a_N^{(i)}\right)_{i\in I}$ *converges jointly in distribution to* $(a_i)_{i\in I}$ *and denote it by*

$$\left(a_N^{(i)}\right)_{i\in I} \xrightarrow{d.} (a_i)_{i\in I},$$

*if for all* $n \in \mathbb{N}$ *and* $i(1), ..., i(n) \in I$,

$$\lim_{N\to\infty} \phi_N(a_N^{(i(1))} \cdots a_N^{(i(n))}) = \phi(a_{i(1)} \cdots a_{i(n)}).$$

The weak convergence of probability measures is defined for all probability measure on $\mathbb{R}$. Thus, we can employ the weak convergence of probability measures to define the convergence in distribution of self-adjoint operators affiliated with a $W^*-$probability space even if the probability measure is not compactly supported on $\mathbb{R}$.

**Definition 12.** *In a* $W^*-$*probability space* $(\mathcal{A}, \phi)$*, we say that self-adjoint operator* $a_n$ *affiliated with* $\mathcal{A}$ *converges in distribution towards a self-adjoint operator* $a$ *affiliated with* $\mathcal{A}$ *as* $n \to \infty$*, if the distribution* $\mu_n$ *of* $a_n$ *converges weakly to the distribution* $\mu$ *of* $a$.

Definition 12 define the convergence in distribution in terms of weak convergence of probability measures. In free probability, the joint distribution of several random variables does not make sense. Therefore, we can not define the joint convergence in distribution according to Definition 12. However, for self-adjoint operators affiliated with a $W^*-$probability space, we can introduce the convergence in probability besides the convergence in distribution.

**Definition 13.** *([8]) Let $(\mathcal{A}, \phi)$ be a $W^*-$probability space and $(a_n)_{n\in\mathbb{N}}$ be a sequence of operators affiliated with $\mathcal{A}$. We say that $a_n \to a$ in probability if $|a_n - a| \to 0$ in distribution as $n \to \infty$.*

Here, $|a| := \sqrt{a^*a}$, which is self-adjoint. When $a_n$ and $a$ are self-adjoint operators affiliated with $\mathcal{A}$, $a_n \to a$ in probability if and only if $a_n - a$ converges to zero in distribution, i.e. the distribution of $a_n - a$ as a probability measure on $\mathbb{R}$ converges weakly to probability measure $\delta_0$. For more details, see [7, 8].

# 2. MOTIVATION AND LITERATURE REVIEWS

## 2.1 Multiple Stochastic Integrals

Our primary motivation comes from the reference [5], where Florin Avram and Murad Taqqu defined the *iterated integrals* of a Lévy process $X(t)$. Let $A_t = \{(s_1, s_2, .., s_k) : 0 \leqslant s_1 < s_2 < ... < s_k \leqslant t\}$. Denote the disjoint increments of $X(t)$ by $X\left(\frac{i}{N}\right) - X\left(\frac{i-1}{N}\right) = X_{N,i}$, for $1 \leqslant i \leqslant N$ and $N \in \mathbb{N}$. The $k$-th order iterated integral over domain $A_t$ is defined by

$$\int_{A_t} dX_{s_1} dX_{s_2} ... dX_{s_k} = w - \lim_{n\to\infty} \sum_{1\leqslant i_1 < i_2 < ... < i_k \leqslant [Nt]} X_{N,i_1} X_{N,i_2} ... X_{N,i_k}. \tag{2.1}$$

There is a corollary in [5] stating that

$$\int_{A_t} dX_{s_1} dX_{s_2} ... dX_{s_k} = P_k(X(t), X^{(2)}(t), ..., X^{(k)}(t)), \tag{2.2}$$

where $X^{(k)}(t)$ is the $k$-th variation of the Lévy process $X(t)$ defined by

$$X^{(k)}(t) = w - \lim_{N\to\infty} \sum_{1\leqslant i \leqslant [Nt]} X^k_{N,i}$$

and $P_k$ is a polynomial with an explicit formula in [5]. This result gives a method to compute or approximate the multiple stochastic integral by the variations of the Lévy process and helps us understand the multiple stochastic measures. For more properties and motivations of multiple stochastic integrals, we recommend the reference [11]. We focus on exploring an analogous concept in free sense and construct a polynomial to describe the relation between multiple stochastic integrals (stochastic measures) and higher variations of a free stochastic process.

## 2.2 The Variations of a Lévy Process in Classical Probability

In the following two sections, we summarize the results and methods that Florin Avram and Murad Taqqu showed in the paper [5]. They introduced the variations $X^{(k)}(t)$ of a Lévy process

$X(t)$ and proved that the limit random variables related to the triangular array of the $k-$th power of some random variables are equal to the $k-$th variation of a Lévy process, for any $k \in \mathbb{N}$. Consider a Lévy process $X(t)$. By the Lévy-Khintchine representations (cf. [4]), there is a Lévy measure $\rho$ on the real line associated with $X(1)$ s.t. $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\}\rho(dx) < \infty$.

**Definition 14.** *Let $(\Theta, \mathcal{B}, \nu)$ be a $\sigma-$finite measure space and $(\Omega, \mathcal{F}, P)$ be a probability space. Let $N : \Omega \times \Theta \to \mathbb{Z}_+ \cup \{\infty\}$ be such that $\{N(\cdot, A) : A \in \mathcal{B}\}$ are random variables defined on $(\Omega, \mathcal{F}, P)$. Then, $N$ is called a Poisson random measure on $(\Theta, \mathcal{B}, \nu)$ with intensity measure $\nu$ if*

1. *for any mutually disjoint sets $A_1, ..., A_n$ in $\Theta$, the random variables $N(A_1), N(A_2), ..., N(A_n)$ are independent,*
2. *for each $A \in \mathcal{B}$, $N(A)$ is a Poisson distributed random variable with parameter $\nu(A)$ and*
3. *$N(\omega, \cdot)$ is a measure on $\Theta$ almost surely.*

Consider the measure space $\Theta = [0, +\infty) \times (\mathbb{R} \setminus \{0\})$ with intensity measure $\nu = Leb \otimes \rho$, i.e. $d\nu = dt \times d\rho$, which describes the discontinuity of the paths of Lévy processes. Here, $\rho$ is the Lévy measure defined above. Consider the jump process $\Delta X = \{\Delta X(t)\}_{t \geq 0}$ associated with a Lévy process and defined by $\Delta X(t) = X(t) - X(t-)$. Then, for $0 \leq t < \infty$ and $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, we define

$$N(t, A) := \#\{0 \leqslant s \leqslant t; \Delta X(s) \in A\} = \sum_{0 \leqslant s \leqslant t} \chi_A(\Delta X(s)).$$

One can check that the intensity measure $\nu([0, t] \times A) = \mathbb{E}(N(t, A))$ and $N(t, A)$ is a Poisson random measure. To introduce the Lévy-Itô decomposition, we define the compensated Poisson random measure by

$$\widetilde{N}(t, A) := N(t, A) - t\rho(A),$$

for each $t \geq 0$, $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ and $0 \notin \bar{A}$. Meanwhile, given any $\omega \in \Omega$, $N(dt, dx)$ is almost surely a measure on $[0, +\infty) \times (\mathbb{R} \setminus \{0\})$. Thus, we define the Poisson integral of a Borel function $f$ by

$$\int_{[0,t) \times A} f(x) N(dt, dx)(\omega) := \int_A f(x) N(t, dx)(\omega) = \sum_{x \in A} f(x) N(t, \{x\})(\omega).$$

Here, $N(t, A)$ counts the number of jumps of $X(t)$ and $N(t, \{x\}) \neq 0 \Leftrightarrow \exists\, 0 \leqslant s \leqslant t$, such that $\Delta X(s) = x$. Consequently,

$$\int_{[0,t)\times A} f(x)N(dt, dx)(\omega) = \sum_{0\leqslant s\leqslant t} f(\Delta X(s))\chi_A(\Delta X(s)).$$

**Theorem 2.2.1** (The Lévy-Itô Decomposition, see [4])**.** *If $X(t)$ is a Lévy process and $\rho$ is the Lévy measure associated with $X(1)$, then $X(t)$ has a representation*

$$X(t) = at + \sigma W(t) + M^{(\rho)}(t), \tag{2.3}$$

*where $a \in \mathbb{R}$, $\sigma \geq 0$ and $W(t)$ is a Brownian motion. The last term of the right hand side in (2.3) is defined by*

$$M^{(\rho)}(t) := \lim_{\delta \searrow 0} \left[ \int_{]0,t]\times([-1,1]\setminus[-\delta,\delta])} x\widetilde{N}(ds, dx) \right] + \int_{]0,t]\times(\mathbb{R}\setminus[-1,1])} xN(ds, dx). \tag{2.4}$$

*If the Lévy measure satisfies*

$$\int_{-1}^{1} |x|\rho(dx) < \infty, \tag{2.5}$$

*then equation (2.4) can be replaced by*

$$M^{(\rho)}(t) = \lim_{\delta \searrow 0} \int_{]0,t]\times(\mathbb{R}\setminus[-\delta,\delta])} xN(ds, dx). \tag{2.6}$$

$M^{(\rho)}(t)$ is actually the compensated sum of the jumps of a Poisson process. By the definition of Poisson integral mentioned above, the equation (2.3) and equation (2.4) can be represented by

$$M^{(\rho)}(t) = \lim_{\delta \searrow 0} \left[ \sum_{\substack{0\leqslant u\leqslant t \\ |\Delta X(u)|>\delta}} \Delta X(u) - t \int_{\delta\leqslant|x|\leqslant 1} x\rho(dx) \right] \tag{2.7}$$

and

$$M^{(\rho)}(t) = \lim_{\delta \searrow 0} \sum_{\substack{0 \leqslant u \leqslant t \\ |\Delta X(u)| > \delta}} \Delta X(u) \tag{2.8}$$

respectively.

In order to show the meanings of equation (2.7) and equation (2.8), we first explain the term

$$\int_{]0,t] \times (-1,1)} x\widetilde{N}(ds, dx) := \lim_{\delta \searrow 0} \left[ \int_{]0,t] \times ((-1,1) \setminus [-\delta,\delta])} xN(ds, dx) - t \int_{((-1,1) \setminus [-\delta,\delta])} x\rho(dx) \right] \tag{2.9}$$

in equation (2.4). Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be a sequence that decreases monotonically to zero and $\epsilon_1 = 1$. For each $m \in \mathbb{N}$, let $B_m = \{x \in \mathbb{R} : \epsilon_{m+1} \leqslant |x| \leqslant \epsilon_m\}$ and $A_n = \cup_{m=1}^n B_m$. Our task is to show the convergence of (2.9). We define the random process as

$$M(t, A_n) = \int_{]0,t] \times A_n} x\widetilde{N}(ds, dx). \tag{2.10}$$

One can prove that $\int_{]0,t] \times (-1,1)} x\widetilde{N}(du, dx) = \lim_{n \to \infty} M(t, A_n)$ in $L^2$ sense and the limit random process lives in a martingale space (see [4]). When condition (2.5) holds,

$$\lim_{\delta \searrow 0} t \int_{((-1,1) \setminus [-\delta,\delta])} x\rho(dx) = t \int_{(-1,1)} x\rho(dx)$$

is finite. Therefore, we can remove this term and combine it into the drift part of (2.3). Then, the compensated sum of jumps in (2.3) can be represented by equation (2.6) or equation (2.8).

The definition of variations $X^{(2)}(t), X^{(3)}(t), ..., X^{(k)}(t)$ of a Lévy process $X(t)$ in the reference [5] is given by the following formula:

$$X^{(k)}(t) := \lim_{\delta \searrow 0} \sum_{\substack{0 \leqslant u \leqslant t \\ |\Delta X(u)| > \delta}} [\Delta X(u)]^k = \lim_{\delta \searrow 0} \int_{]0,t] \times (\mathbb{R} \setminus [-\delta,\delta])} x^k N(du, dx). \tag{2.11}$$

For any $k > 1$, the measure $\rho_k$, defined by $\rho_k(B) = \rho(B^{1/k})$, for any $B \in \mathcal{B}(\mathbb{R})$, satisfies the condition (2.5), i.e.

$$\int_{-1}^{1} |x| \rho_k(dx) = \int_{-1}^{1} |x|^k \rho(dx) \leqslant \int_{-1}^{1} x^2 \rho(dx) < \infty.$$

Thus, the variation of a Lévy process is well-defined. By definition, $X^{(n)}(t)$ is the $n$-th power of the jumps of the Lévy process, so it is also called the power jump process. Meanwhile, $Y^{(n)}(t) := X^{(n)}(t) - \mathbb{E}(X^{(n)}(t))$ is a martingale, which is called Teugels martingale in literature (cf. [4]). One can prove that all the power jump processes of a Poisson process are the same and equal to the original Poisson process. However, all the power jump processes of a Brownian motion are zero when the orders are strictly greater than one.

### 2.3 A Brief Summary of the Proof and Methods in Classical Probability

Consider the triangular array of random variables $\{X_{n,i}\}$ with identical and independent distribution $F_n$ in the $n$-th row ($\forall\ n \in \mathbb{N}$). We restate the main result, Theorem 1.1 in [5], in the following Theorem 2.3.1 and give a sketch of the proof in this theorem.

**Theorem 2.3.1.** *Suppose that triangular array $\{X_{n,i}\}_{1 \leqslant i \leqslant n}$ are i.i.d with distribution $F_n$, ($n \in \mathbb{N}$), which satisfies*

$$nF_n(\cdot) \to \rho(\cdot), \tag{2.12}$$

*vaguely over $[-\infty, +\infty] \setminus \{0\}$, as $n \to \infty$. Also, suppose that*

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}}\, n \left[ \int_{-\delta}^{\delta} x^2 dF_n \right] = \lim_{\delta \to 0} \underline{\lim_{n \to \infty}}\, n \left[ \int_{-\delta}^{\delta} x^2 dF_n \right] =: \sigma^2. \tag{2.13}$$

*Finally, let $\tau$ be a point of continuity of $\rho$ (WLOG, we assume $\tau = 1$) and suppose that*

$$\lim_{n \to \infty} n \int_{-\tau}^{\tau} x dF_n =: a. \tag{2.14}$$

*Then,*

$$\sum_{i=1}^{[nt]} \left( X_{n,i}, X_{n,i}^2, ..., X_{n,i}^k \right) \xrightarrow{d} \left( X(t), t\sigma^2 + X^{(2)}(t), X^{(3)}(t), ..., X^{(k)}(t) \right), \tag{2.15}$$

*where $X(t)$ is a Lévy process and $X^{(m)}(t)$ is the $m$-th variation of $X(t)$, for $1 \leqslant m \leqslant k$.*

**A Sketch of the Proof:** Consider the function $f(x) = (x, x^2, ..., x^k)$. Set the random vectors $Z_{n,i} = f(X_{n,i})$ with a joint distribution $f(F_n)$. Fix $\delta \in (0, 1)$. When $i, j = 1, ..., k$, we define

$$\sigma_{i,j}(\delta, n) := n\left[\int_{\substack{|x^{(m)}|<\delta^m\\ m=1,...,k}} x^{(i)}x^{(j)}df(F_n) - \left(\int_{\substack{|x^{(m)}|<\delta^m\\ m=1,...,k}} x^{(i)}df(F_n)\right)\left(\int_{\substack{|x^{(m)}|<\delta^m\\ m=1,...,k}} x^{(j)}df(F_n)\right)\right].$$

For vector $\vec{x} = (x^{(1)}, ..., x^{(k)})$, we define $\vec{a}_n := \int_{\substack{|x^{(m)}|<\tau^m\\ m=1,...,k}} \vec{x}df(F_n)$. To prove the theorem, we need to verify the following three results via the assumptions (2.12), (2.13) and (2.14):

1. $$nf(F_n)(\cdot) \stackrel{v}{\to} f(\nu)(\cdot); \tag{2.16}$$

2. $$\sigma_{ij} := \lim_{\delta\to 0}\overline{\lim_{n\to\infty}} = \lim_{\delta\to 0}\underline{\lim_{n\to\infty}} = \begin{cases} \sigma^2, & i = j = 1, \\ 0, & otherwise; \end{cases} \tag{2.17}$$

3. $$\lim_{n\to\infty}[nt]\vec{a}_n = t\left(a, \sigma^2 + \int_{-\tau}^{\tau} x^2 d\nu(x), \int_{-\tau}^{\tau} x^3 d\nu(x), ..., \int_{-\tau}^{\tau} x^k d\nu(x)\right). \tag{2.18}$$

First, (2.16) follows by the assumption (2.12) because, for any bounded Borel set $A \subset \mathbb{R}^k \setminus \{0\}$, $f(F_n)(A) = F_n(f^{-1}(A))$ and $f(\nu)(A) = \nu(f^{-1}(A))$. Second, we prove the relation (2.17). When $i = j = 1$, by (2.13) and (2.14), we know that

$$\int_{-\delta}^{\delta} xdF_n = O(\frac{1}{n}).$$

Therefore, by changing variables in the integral, we can get

$$
\begin{aligned}
\sigma_{11} =& \lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\, n \left[ \int_{\substack{|x^{(m)}|<\delta^m \\ m=1,\dots,k}} (x^{(1)})^2 df(F_n) - \left( \int_{\substack{|x^{(m)}|<\delta^m \\ m=1,\dots,k}} x^{(1)} df(F_n) \right)^2 \right] \\
=& \lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\, n \left[ \int_{\substack{|x^{(m)}|<\delta^m \\ m=1,\dots,k}} (x^{(1)})^2 dF_n(f^{-1}(\vec{x})) - \left( \int_{\substack{|x^{(m)}|<\delta^m \\ m=1,\dots,k}} x^{(1)} dF_n(f^{-1}(\vec{x})) \right)^2 \right] \\
=& \lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\, n \left[ \int_{|x|<\delta} x^2 dF_n(x) - \left( \int_{|x|<\delta} x dF_n(x) \right)^2 \right] \\
=& \sigma^2.
\end{aligned}
$$

When $(i, j) \neq (1, 1)$ and $0 < \delta < 1$, without loss of generality, we assume $i \neq 1$. Then, similarly,

$$
\begin{aligned}
|\sigma_{i,j}(\delta, n)| =& \left| n \int_{|x|<\delta} x^{i+j} dF_n(x) - n \int_{|x|<\delta} x^i dF_n \int_{|x|<\delta} x^j dF_n \right| \\
\leqslant& \left| n \int_{|x|<\delta} \delta x^i dF_n(x) \right| + n \left| \int_{|x|<\delta} x dF_n(x) \right| \cdot \left| \int_{|x|<\delta} x^i dF_n(x) \right| \\
\leqslant& n \left[ \delta + \left| \int_{|x|<\delta} x dF_n(x) \right| \right] \cdot \int_{|x|<\delta} x^2 dF_n(x) \to 0,
\end{aligned}
$$

as $\delta \to 0$ and $n \to \infty$. Finally, we show the relation (2.18) via the following observation. Assumption (2.13) can imply that

$$
\left| n \int_{-\delta}^{\delta} x^m dF_n(x) \right| \leqslant n \int_{-\delta}^{\delta} |x|^m dF_n(x) \leqslant n\delta^{m-2} \int_{-\delta}^{\delta} x^2 dF_n(x) \to 0,
$$

for $m > 2$, as $\delta \to 0$ and $n \to \infty$. So,

$$
\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\, n \left| \int_{-\delta}^{\delta} x^m dF_n(x) \right| = 0, \tag{2.19}
$$

when $m \geqslant 3$. This explains why the limit random process of higher order in (2.15) only has compensated jump sum parts among its Lévy-Itô decomposition.

Suppose $\vec{a}_n = (a_n^{(1)}, a_n^{(2)}, ..., a_n^{(k)})$. Then, by (2.14), we know that $\lim_{n\to\infty}[nt]a_n^{(1)} = at$. Let

$$a_n^{(m)} = n\int_{-\delta}^{\delta} x^m dF_n(x) + n\int_{\delta\leqslant|x|\leqslant\tau} x^m dF_n(x).$$

Then, for the second part,

$$\begin{aligned}
&\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, n\int_{\delta\leqslant|x|\leqslant\tau} x^m dF_n(x)\\
=&\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\int_{\delta\leqslant|x|\leqslant\tau} x^m dnF_n(x)\\
=&\lim_{\delta\to 0}\int_{\delta\leqslant|x|\leqslant\tau} x^m d\rho(x)\\
=&\int_{|x|\leqslant\tau} x^m d\rho(x).
\end{aligned}$$

According to (2.13), $\lim_{n\to\infty}[nt]a_n^{(2)} = \sigma^2 t + t\int_{|x|\leqslant\tau} x^2 d\rho(x)$. Based on (2.19), we conclude that $\lim_{n\to\infty}[nt]a_n^{(m)} = t\int_{|x|\leqslant\tau} x^m d\rho(x)$, when $m > 2$. So we get the relations (2.16), (2.17) and (2.18). Then, in the reference [5], the authors claim that relations (2.16), (2.17) and (2.18) allow us to apply the multidimensional central limit theorem of E. Rvaceva in [12] to get the joint convergence in distribution:

$$\lim_{n\to\infty}\sum_{i=1}^{[nt]}(Z_{n,i} - \vec{a}_n) = \sigma B(t) + M^{(f(\rho))}(t).$$

Furthermore, there is a corollary in [5] claiming that if $X(t)$ is a Lévy process, the triangular array $X_{n,i}$ is i.i.d. and, as $n\to\infty$,

$$\sum_{i=1}^{n} X_{n,i} \xrightarrow{d.} X(1), \tag{2.20}$$

then the assumptions (2.12), (2.13) and (2.14) will hold automatically. Thus, (2.15) holds. For simplicity, we denote the variation of order two $X^{(2)}(t)$ by $t\sigma^2 + X^{(2)}(t)$. Thus, the result is expressed by

$$\left(\sum_{i=1}^{[nt]} X_{n,i}, \sum_{i=1}^{[nt]} X_{n,i}^2, ..., \sum_{i=1}^{[nt]} X_{n,i}^k\right) \xrightarrow{d.} \left(X(t), X^{(2)}(t), ..., X^{(k)}(t)\right),$$

as $n \to \infty$. Meanwhile, this result implies that, for any polynomial $P(x_1, ..., x_k)$,

$$P\left(\sum_{i=1}^{[nt]} X_{n,i}, \sum_{i=1}^{[nt]} X_{n,i}^2, ..., \sum_{i=1}^{[nt]} X_{n,i}^k\right) \xrightarrow{d.} P\left(X(t), X^{(2)}(t), ..., X^{(k)}(t)\right).$$

The last thing left is to use a combinatorial method to find an appropriate polynomial $P$ such that

$$P\left(\sum_{i=1}^{N} X_{N,i}, \sum_{i=1}^{N} X_{N,i}^2, ..., \sum_{i=1}^{N} X_{N,i}^k\right) = \sum_{1 \leqslant i_1 < i_2 < ... < i_k \leqslant [Nt]} X_{N,i_1} X_{N,i_2} ... X_{N,i_k}.$$

Then, comparing with the definition (2.1), we can get the final result (2.2) of [5].

In free sense, we aim at imitating the proof and methods we mentioned above to get the joint convergence in distribution of a free triangular array of the powers of some non-commutative random variables. In short, we want to get the last three formulas we showed above in a free probability setting.

### 2.4 Free Stochastic Measures

Another motivation comes from free probability itself. In the reference [1], Michael Anshelevich defined the *free multiple stochastic measures*. Let $X_I$ be a stationary stochastic process with freely independent increments, which are bounded operators. For any $N \in \mathbb{N}$, let $\{X_{N,i}\}_{1 \leqslant i \leqslant N}$ be freely independent and identically distributed increments of $X_I$ that add up to $X_I$. The stochastic measure of degree $k$ is defined by

$$\psi_k := \lim_{N \to \infty} \sum_{i_1 \neq i_2 \neq ... \neq i_k} X_{N,i_1} X_{N,i_2} ... X_{N,i_k}, \tag{2.21}$$

where the limit is taken in the operator norm and $i_1 \neq i_2 \neq ... \neq i_k$ means that all the indices are distinct. The $k$-th diagonal measure (variation) of the process is defined by

$$\Delta_k := \lim_{N \to \infty} \sum_{i=1}^{N} X_{N,i}^k.$$

Usually, we use the $k$-th variation $X^{(k)}$ instead of the $k$-th diagonal measure $\Delta_k$, but they are actually the same things. Heuristically, if $I = [0,t)$, then $\psi_k(t) = \int_{B_t} dX_{s_1} dX_{s_2} ... dX_{s_k}$, where $B_t = \{0 \leqslant s_1 \neq s_2 \neq ... \neq s_k \leqslant t\}$. This means that all the definitions are analogs of the classical definitions. For more details, see [2].

Moreover, there is a free Kailath-Segall formula in [1],

$$\psi_n = X\psi_{n-1} + \sum_{j=2}^{n} (-1)^{j-1} \sum_{q=0}^{n-j} \binom{n-q-2}{j-2} t^{n-j-q} \Delta_j \psi_q, \tag{2.22}$$

where $X = \Delta_1$ is the stochastic process $X_I$ and $t = \phi(X)$. This shows that the stochastic measure is determined by a polynomial of diagonal measures (variations of the stochastic process).

However, all these definitions are in terms of operator norm. This requires that the norms of all random variables should be finite. Thus, we want to consider a more general case and define the stochastic measure $\psi_k$ and the diagonal measure $\Delta_k$ (or the variation $X^{(k)}$) in a weaker sense or a more general sense. For example, we can modify these definitions for self-adjoint (possibly unbounded) affiliated operators and verify an analogous free Kailath-Segall formula in our new definitions and settings.

# 3. RESULTS FOR FINITE-MOMENT NON-COMMUTATIVE RANDOM VARIABLES

In this chapter, we derive some limit theorems of free triangular arrays in a plain non-commutative probability space. As mentioned above, the moments of all orders in a plain non-commutative probability space are finite. In classical probability, the cumulants $\kappa_n$ of a probability distribution are a sequence of quantities that provide an alternative to its moments. In free sense, we can define the free cumulant based on moments. In a non-commutative probability space $(\mathcal{A}, \phi)$, denote the $n$-th mixed moment of $a_1, a_2, ..., a_n$ by $M_n(a_1, a_2, ..., a_n) := \phi(a_1 \cdot a_2 \cdots a_n)$.

**Definition 15.** *In a non-commutative probability space $(\mathcal{A}, \phi)$, define the $n$-th free cumulant functional $R_n$ implicitly by*

$$M_n(a_1, a_2, ..., a_n) = \sum_{\pi \in \mathcal{NC}(n)} R_\pi(a_1, a_2, ..., a_n),$$

*for any random variables $a_1, a_2, ..., a_n \in \mathcal{A}$. The symbol $\mathcal{NC}(n)$ represents the collection of all non-crossing partitions in the set $\{1, 2, ..., n\}$. The notation $R_\pi$ is denoted by*

$$R_\pi(a_1, a_2, ..., a_n) = \prod_{V \in \pi} R_{|V|}(a_i : i \in V).$$

The $n$-th free cumulant of a single element $a$ is defined by $\kappa_n(a) := R_n(a, a, ..., a)$. For more properties of the free cumulant $\kappa_n$, see Lecture 11 in [10].

## 3.1 Free Limit Theorems for Triangular Arrays of Non-commutative Random Variables

For each $N \in \mathbb{N}$, let $(\mathcal{A}_N, \phi_N)$ be a non-commutative probability space and $I$ be an index set. Consider a free triangular array of random variables, namely, for any $i \in I, N \in \mathbb{N}$ and $1 \leq r \leq N$, each entry of the array is formed by some random variables $a_{N,r}^{(i)} \in \mathcal{A}_N$, and for each fixed $N \in \mathbb{N}$, the sets $\{a_{N,1}^{(i)}\}_{i \in I}, \{a_{N,2}^{(i)}\}_{i \in I}, ..., \{a_{N,N}^{(i)}\}_{i \in I}$ are freely independent and identically distributed in the $N$-th row.

**Lemma 3.1.1.** *([10]) The following two statements are equivalent.*

*(1) For each $n \geq 1$, any $i(1), ..., i(n) \in I$ and any $1 \leqslant r \leqslant N$, the following limit of joint free moments exists:*

$$\lim_{N\to\infty} N \cdot \phi_N \left( a_{N,r}^{(i(1))} ... a_{N,r}^{(i(n))} \right).$$

*(2) For each $n \geq 1$, any $i(1), ..., i(n) \in I$ and any $1 \leqslant r \leqslant N$, the the following limit of joint free cumulants exists:*

$$\lim_{N\to\infty} N \cdot \kappa_n^N \left( a_{N,r}^{(i(1))}, ..., a_{N,r}^{(i(n))} \right).$$

*Furthermore, the corresponding limits are the same.*

*Proof.* $(2) \Rightarrow (1)$: by the free cumulant-moment formula (see Lecture 11 in [10]), we have that

$$\lim_{N\to\infty} N \cdot \phi_N \left( a_{N,r}^{(i(1))} ... a_{N,r}^{(i(n))} \right) = \lim_{N\to\infty} \sum_{\pi \in \mathcal{NC}(n)} N \cdot \kappa_\pi^N [a_{N,r}^{(i(1))} ... a_{N,r}^{(i(n))}].$$

So, for any partition $\pi \in \mathcal{NC}(n)$ with more than one block, $N \cdot \kappa_\pi^N [a_{N,r}^{(i(1))} ... a_{N,r}^{(i(n))}]$ tends to zero as $N \to \infty$, by the assumption (2). Therefore,

$$\lim_{N\to\infty} N \cdot \phi_N \left( a_{N,r}^{(i(1))} ... a_{N,r}^{(i(n))} \right) = \lim_{N\to\infty} N \cdot \kappa_n^N \left( a_{N,r}^{(i(1))}, ..., a_{N,r}^{(i(n))} \right).$$

For the other direction, the proof is analogous by the free moment-cumulant formula. $\square$

**Lemma 3.1.2.** *Suppose that a family of random variables $\{c_{N,i}\}_{i\in I}$ in a non-commutative probability space $(\mathcal{A}_N, \phi_N)$ satisfies the joint convergence in distribution towards a family $\{c_i\}_{i\in I}$ in $(\mathcal{A}, \phi)$, i.e. as $N \to \infty$,*

$$\{c_{N,i}\}_{i\in I} \xrightarrow{d} \{c_i\}_{i\in I}.$$

*The following two statements are equivalent.*

1. *For any $n \in \mathbb{N}$ and any $i(1), i(2), ..., i(n) \in I$, $\lim_{N\to\infty} \phi(c_{N,i(1)}, ..., c_{N,i(n)}) = \phi(c_{i(1)}, ..., c_{i(n)})$.*

2. *For any $n \in \mathbb{N}$ and any $i(1), i(2), ..., i(n) \in I$, $\lim_{N\to\infty} \kappa_n(c_{N,i(1)}, ..., c_{N,i(n)}) = \kappa_n(c_{i(1)}, ..., c_{i(n)})$.*

The proof of the above Lemma 3.1.2 is straightforward by the free cumulant-moment formula and its inverse formula (cf. [10]). Combining two lemmas stated above, we can easily prove the following theorem, which is presented and proved in the Lecture 13 of [10].

**Theorem 3.1.3.** *([10]) Suppose that the free triangular array $\{a_{N,i}\}$ satisfies the free independence and identical distribution in each row. Then, the following are equivalent.*

1. *The sums over the rows of our triangular array converge in distribution, i.e. there is a family of random variables $(b_i)_{i\in I}$ in some non-commutative probability space such that*

$$(a_{N,1}^{(i)} + ... + a_{N,N}^{(i)})_{i\in I} \xrightarrow{d.} (b_i)_{i\in I}.$$

2. *For all $n \geq 1$ and all $i(1), ..., i(n) \in I$, the limits*

$$\lim_{N\to\infty} N \cdot \phi_N \left( a_{N,r}^{(i(1))} ... a_{N,r}^{(i(n))} \right)$$

*exist (which are independent of $r$).*

*Moreover, if these conditions are satisfied, then the joint distribution of the limit family $(b_i)_{i\in I}$ is determined by their free cumulants*

$$\kappa_n(b_{i(1)}, ..., b_{i(n)}) = \lim_{N\to\infty} N \cdot \phi_N \left( a_N^{(i(1))} ... a_N^{(i(n))} \right).$$

*Proof.* For all $n \geq 1$ and all $i(1), ..., i(n) \in I$, consider

$$\begin{aligned} A_N &:= \kappa_n \left( \sum_{r=1}^{N} a_{N,r}^{i(1)}, ..., \sum_{r=1}^{N} a_{N,r}^{i(n)} \right) \\ &= \sum_{r(1)=1}^{N} ... \sum_{r(N)=1}^{N} \kappa(a_{N,r(1)}^{i(1)}, ..., a_{N,r(n)}^{i(n)}). \end{aligned}$$

When $r(k) \neq r(j)$, $a_{N,r(k)}^{i(k)}$ and $a_{N,r(j)}^{i(j)}$ are freely independent. So the joint free cumulant that includes $a_{N,r(k)}^{i(k)}$ and $a_{N,r(j)}^{i(j)}$ must be zero. Then,

$$A_N = \sum_{r=1}^{N} \kappa_n(a_{N,r}^{i(1)}, ..., a_{N,r}^{i(n)}) = N \cdot \kappa_n(a_{N,r}^{i(1)}, ..., a_{N,r}^{i(n)}),$$

because the distributions of $a_{N,r}^{i(j)}$ and $a_{N,s}^{i(j)}$ are the same for any $1 \leqslant r, s \leqslant N$. Then, by the two previous lemmas, it is easy to check the equivalence of the condition 1 and condition 2. $\square$

In our case, the family of random variables in each entry of the free triangular array is $\{X_{N,r}^k\}_{k\in\mathbb{Z}_+}$, for $1 \leq r \leq N$, i.e. all the powers of the free triangular array of our original random variables $\{X_{N,r}\}_{1\leqslant r\leqslant N}$. Therefore, we can get the following corollary.

**Corollary 3.1.3.1.** *For a free triangular array* $\{X_{N,r} : 1 \leq r \leq N, N \in \mathbb{N}\}$ *in* $(\mathcal{A}, \phi)$*, we assume that in each row of the array, the random variables are freely independent and identically distributed. We denote that in the* $N$*-th row,* $X_{N,r}$ *has the same distribution as a random variable* $X_N$*, for any* $1 \leq r \leq N$*. Then, there exists a family of random variables* $\{X^{(k)}\}_{k\in\mathbb{Z}_+}$ *in* $\mathcal{A}$ *such that*

$$\left(X_{N,1}^k + ... + X_{N,N}^k\right)_{k\in\mathbb{Z}_+} \xrightarrow{d.} \left(X^{(k)}\right)_{k\in\mathbb{Z}_+}$$

*if and only if* $\forall n \in \mathbb{Z}_+$ *and* $\forall i_1, ..., i_n \in \mathbb{Z}_+$*, the limit of moments* $\lim_{N\to\infty} N \cdot \phi(X_N^{i_1+...+i_n})$ *is finite. Moreover, the joint free cumulant of* $\{X^{(k)}\}_{k\in\mathbb{Z}_+}$ *is determined by*

$$\kappa_n\left(X^{(i_1)}, ..., X^{(i_n)}\right) = \lim_{N\to\infty} N \cdot \phi(X_N^{i_1+...+i_n}).$$

The above corollary shows that, for any $m \in \mathbb{N}$, the $m$-th moments of $X_N$ must decay at least as fast as $\frac{1}{N}$, so that the free cumulants of limit random variables $X^{(i)}$ exists. Besides, the free joint cumulants of limit random variables $X^{(i)}$ are obtained by the moments of random variables $X_{N,r}$ in the triangular array.

**Lemma 3.1.4.** *Suppose that we have the joint convergence in distribution*

$$\left(a_N^{(i)}\right)_{1\le i\le k} \xrightarrow{d.} \left(b^{(i)}\right)_{1\le i\le k},$$

*as $N\to\infty$. Then, for any polynomial $P_k(x_1,x_2,...,x_k)$ fixed, as $N\to\infty$, we can conclude that*

$$P_k(a_N^{(1)},a_N^{(2)},...,a_N^{(k)}) \xrightarrow{d.} P_k(b^{(1)},b^{(2)},...,b^{(k)}).$$

*Proof.* By Definition 10, it suffices to show that, for any $n\in\mathbb{N}$, as $N\to\infty$,

$$\phi\left((P_k(a_N^{(1)},a_N^{(2)},...,a_N^{(k)}))^n\right) \to \phi\left((P_k(b^{(1)},b^{(2)},...,b^{(k)}))^n\right).$$

Since $(P_k(x_1,x_2,...,x_k))^n$ is still a polynomial with $x_1,x_2,...,x_k$ variables, we can get the result above via the joint convergence in distribution of $\left(a_N^{(i)}\right)$ (recall Definition 11). $\square$

**Corollary 3.1.4.1.** *For a triangular array $\{X_{N,r} : 1\le r\le N, N\in\mathbb{N}\}$ in a non-commutative probability space $(\mathcal{A},\phi)$, assume that in each row of the array, the random variables are freely independent and identically distributed. We denote that, in the $N$-th row, $X_{N,r}$ has the same distribution as a random variable $X_N$, for any $1\le r\le N$. If we have that*

$$X_{N,1}+...+X_{N,N} \xrightarrow{d.} X =: X^{(1)},$$

*then there is a sequence of random variables $\{X^{(k)}\}_{k\in\mathbb{Z}_+}$ such that*

$$\left(X_{N,1}^k+...+X_{N,N}^k\right)_{k\in\mathbb{Z}_+} \xrightarrow{d} \left(X^{(k)}\right)_{k\in\mathbb{Z}_+}.$$

*And for any fixed polynomial $P_k$,*

$$P_k(\sum_{r=1}^N X_{N,r},\sum_{r=1}^N X_{N,r}^2,...,\sum_{r=1}^N X_{N,r}^k) \xrightarrow{d.} P_k(X^{(1)},X^{(2)},...,X^{(k)}).$$

*Moreover, the $n-$th order cumulant of $X^{(k)}$ is equal to the $kn-$th order cumulant of $X^{(1)}$, for any $n, k \in \mathbb{N}$.*

*Proof.* By Corollary 3.1.3.1, we know that the convergence in distribution, $X_{N,1} + ... + X_{N,N} \xrightarrow{d} X^{(1)}$, is equivalent to the condition

$$\kappa_n(X^{(1)}) = \lim_{N\to\infty} N \cdot \phi(X_{N,r}^n),$$

for any $1 \leqslant r \leqslant N$ and any $n \in \mathbb{N}$. Then, given $\forall n \in \mathbb{Z}_+$ and $\forall i_1, ..., i_n \in \mathbb{Z}_+$, $\lim_{N\to\infty} N \cdot \phi(X_{N,r}^{i_1+...+i_n})$ is finite, which, by Corollary 3.1.3.1, implies the joint convergence in distribution, $\left(X_{N,1}^k + ... + X_{N,N}^k\right)_{k\in\mathbb{Z}_+} \xrightarrow{d} \left(X^{(k)}\right)_{k\in\mathbb{Z}_+}$, and $\kappa_n(X^{(k)}) = \kappa_{kn}(X^{(1)})$. For the convergence of polynomials of the triangular array, the proof is based on Lemma 3.1.4. $\square$

This corollary states that if we have the convergence of the sum of each row in the free triangular array $X_{N,r}$, we can conclude the joint convergence of all powers of the triangular array $X_{N,r}$. Moreover, by the Corollary 3.1.4.1, the limit random variables are determined by the moments of $X_{N,r}$.

We now give an improvement of Theorem 3.1.3 considering non-identically distributed free random variables in the same triangular arrays. To make this modification, we consider the weak free central limit theorem, where the identical distribution is replaced by $\sup_{i\in\mathbb{N}} |\phi(a_i^n)| < \infty$, for any $n \in \mathbb{N}$.

**Theorem 3.1.5.** *For a free triangular arrays of non-commutative random variables $\{X_{N,r}\}_{N\in\mathbb{N}}(1 \leqslant r \leqslant N)$, we assume that $X_{N,r}$ are freely independent in each row and there exists constant $C_k$ such that*

$$\lim_{N\to\infty} \sup_{1\leqslant r\leqslant N} \left|N \cdot \phi(X_{N,r}^k) - C_k\right| = 0, \tag{3.1}$$

*for any $k \in \mathbb{N}$. Then, there exist random variables $\{b_i\}_{i\in\mathbb{N}}$ such that*

$$\left(\sum_{r=1}^{N} X_{N,r}^i\right)_{i\in\mathbb{N}} \xrightarrow{d} (b_i)_{i\in\mathbb{N}}.$$

*Proof.* For any $n \in \mathbb{N}$ and $i(1), ...i(n) \in \mathbb{N}$, computing the $n$-th joint free cumulant, we can get

$$\begin{aligned}
&\kappa_n\left(\sum_{r=1}^{N} X_{N,r}^{i(1)}, ..., \sum_{r=1}^{N} X_{N,r}^{i(n)}\right) \\
&= \sum_{r(1)=1}^{N} ... \sum_{r(n)=1}^{N} \kappa_n(X_{N,r(1)}^{i(1)}, ..., X_{N,r(n)}^{i(n)}) \\
&= \sum_{r=1}^{N} \kappa_n(X_{N,r}^{i(1)}, ..., X_{N,r}^{i(n)}).
\end{aligned}$$

Here, we use the properties of free cumulants and free independence. Then, by free cumulant-moment formula,

$$\begin{aligned}
&\left|\sum_{r=1}^{N} \kappa_n(X_{N,r}^{i(1)}, ..., X_{N,r}^{i(n)}) - C_{i(1)+...+i(n)}\right| \\
&= \left|\sum_{r=1}^{N} \sum_{\sigma \in \mathcal{NC}(n)} \prod_{V \in \sigma} \phi(V)[X_{N,r}^{i(1)}, ..., X_{N,r}^{i(n)}]\mu(\sigma, 1_n) - C_{i(1)+...+i(n)}\right| \\
&\leqslant \left|\sum_{r=1}^{N} \phi(X_{N,r}^{i(1)+...+i(n)}) - C_{i(1)+...+i(n)}\right| + \sum_{\substack{\sigma \neq 1_n \\ \sigma \in \mathcal{NC}(n)}} \sup_{1 \leqslant r \leqslant N} N \left|\prod_{V \in \sigma} \phi(V)[X_{N,r}^{i(1)}, ..., X_{N,r}^{i(n)}]\right|.
\end{aligned}$$

By the assumption, when $N$ goes to infinity,

$$\begin{aligned}
&\left|\sum_{r=1}^{N} \phi(X_{N,r}^{i(1)+...+i(n)}) - C_{i(1)+...+i(n)}\right| \\
&\leqslant \lim_{N \to \infty} \sup_{1 \leqslant r \leqslant N} \left|N\phi(X_{N,r}^{i(1)+...+i(n)}) - C_{i(1)+...+i(n)}\right| = 0.
\end{aligned}$$

Meanwhile, by condition (3.1), we know that $\lim_{N\to\infty} \sup_{1\leqslant r\leqslant N} \left|N\phi(X_{N,r}^{k})\right|$ is finite. Therefore, when the non-crossing partition $\sigma$ is not $1_n$, there exists at least two blocks in this partition, which implies

$$\lim_{N \to \infty} N \left|\prod_{V \in \sigma} \phi(V)[X_{N,r}^{i(1)}, ..., X_{N,r}^{i(n)}]\right| = 0.$$

So the joint free cumulant $\kappa_n\left(\sum_{r=1}^{N} X_{N,r}^{i(1)}, ..., \sum_{r=1}^{N} X_{N,r}^{i(n)}\right) = C_{i(1)+...+i(n)}$, which concludes the joint convergence in distribution. $\square$

### 3.2 Free Kailath-Segall Formula

One of our aims is to use some specific polynomial of free random variables to approximate the free stochastic measure. So, in this section, we want to show an explicit formula of the polynomials for the approximation. We have known the free Kailath-Segall formula in [1], which is a recursion formula for stochastic measures, i.e.

$$\psi_n = X\psi_{n-1} + \sum_{j=2}^{n}(-1)^{j-1}\sum_{q=0}^{n-j}\binom{n-q-2}{j-2}t^{n-j-q}\Delta_j\psi_q. \tag{3.2}$$

Here, we consider an infinitely divisible random variable $X = X_{[0,1]}$, i.e. for each $N \in \mathbb{N}$, there is a sequence of freely independent and identically distributed random variables $\{X_{N,r}\}_{1\leqslant r\leqslant N}$ s.t.

$$\sum_{r=1}^{N} X_{N,r} = X.$$

Moreover, the $k$-th diagonal measure of the process $X$ is defined by $\Delta_k = \lim_{N\to\infty}\sum_{r=1}^{N} X_{N,r}^k$. Then, $X = \Delta_1$. We assume that the expectation $t = \phi(X)$ is finite. If the expectation is not finite, the recursion formula (3.2) does not make sense.

**Theorem 3.2.1.** *By the recursion formula (3.2), we can get a general formula for free stochastic measure:*

$$\psi_n = \sum_{\overline{k}\in A_n}\prod_{i=1}^{\infty} C_{k_i} \tag{3.3}$$

*Here, the product is non-commutative and $A_n = \{\overline{k} : \sum_{i=1}^{\infty} k_i = n, k_i \geq 0, k_i \in \mathbb{Z}\}$ is a set of infinite sequences of nonnegative integers with finitely many nonzero elements at the their beginning part of the entries. In addition,*

$$C_k := \begin{cases} X = \Delta_1 & , k = 1 \\ \displaystyle\sum_{j=2}^{k}\binom{k-2}{j-2}t^{k-j}\Delta_j(-1)^{j-1} & , k > 1. \end{cases}$$

*Proof.* We can simplify (9) via the following steps.

$$\begin{aligned}
\psi_n =& X\psi_{n-1} + \sum_{j=2}^{n}(-1)^{j-1}\sum_{q=0}^{n-j}\binom{n-q-2}{j-2}t^{n-j-q}\Delta_j\psi_q \\
=& X\psi_{n-1} + \sum_{q=0}^{n-2}\sum_{j=2}^{n-q}\binom{n-q-2}{j-2}t^{n-j-q}\Delta_j\psi_q \\
=& X\psi_{n-1} + \sum_{q=0}^{n-2}(\sum_{j=2}^{n-q}\binom{n-q-2}{j-2}t^{n-j-q}\Delta_j)\psi_q \\
=& \sum_{q=0}^{n-1}C_{n-q}\psi_q
\end{aligned}$$

Therefore, considering $\psi_0 = 1$, we can get the final result (3.3) by induction. □

This theorem tells us that the $n$-th stochastic measure $\psi_n$ is a polynomial of $\Delta_1, \Delta_2, ..., \Delta_n$ and $t$. However, notice that the free stochastic measure in (3.2) is defined by the operator norm rather than the convergence in distribution. We want to consider a general case when elements in a non-commutative probability space have finite moments but possibly no finite norms. In this case, we can redefine the stochastic measure and the diagonal measure (the higher variation).

**Definition 16.** *Given a triangular array $\{X_{N,i}\}_{1\leqslant i\leqslant N}$ with freely independent and identically distributed entries in each row in a non-commutative probability space $(\mathcal{A}, \phi)$ (usually we call it a free triangular array), we define the* ***k-th variation*** *of the triangular array by*

$$X^{(k)} := w - \lim_{N\to\infty}\sum_{i=1}^{N}X_{N,i}^k$$

*and the* ***k-th stochastic measure*** *of the triangular array by*

$$\psi_k := w - \lim_{N\to\infty}\sum_{i_1\neq i_2, i_2\neq i_3, ..., i_{k-1}\neq i_k}X_{N,i_1}X_{N,i_2}...X_{N,i_k},$$

*where $i_1 \neq i_2, i_2 \neq i_3, ..., i_{k-1} \neq i_k$ means all neighbors are distinct.*

In Definition 16, the weak limit represents the convergence in distribution in the non-commutative probability space $(\mathcal{A}, \phi)$. The definition of free stochastic measure is slightly different from the definition in terms of the operator norm in the formula (2.21). Next, we show the existence of free stochastic measures and higher variations in Definition 16.

**Theorem 3.2.2.** *Given any free triangular array $\{X_{N,r}\}$ in a non-commutative probability space $(\mathcal{A}, \phi)$, i.e. $\{X_{N,r}\}_{1\leqslant r\leqslant N}$ are freely independent and identically distributed in each row, we assume that there exists a random variable $X^{(1)} \in \mathcal{A}$ such that*

$$X^{(1)} = w - \lim_{N\to\infty} \sum_{i=1}^{N} X_{N,i}. \tag{3.4}$$

*Then, we claim that there exist $\{X^{(k)}\}_{k\in\mathbb{N}}$ and $\{\psi_k\}_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that*

$$\left(\sum_{i=1}^{N} X_{N,i}^{k}\right)_{k\in\mathbb{N}} \xrightarrow{d.} \left(X^{(k)}\right)_{k\in\mathbb{N}} \tag{3.5}$$

*and*

$$\sum_{i_1\neq i_2, i_2\neq i_3, \ldots, i_{k-1}\neq i_k} X_{N,i_1} X_{N,i_2} \ldots X_{N,i_k} \xrightarrow{d.} \psi_k, \tag{3.6}$$

*for each $k \in \mathbb{N}$. Moreover, we can conclude the following relationship between $X^{(k)}$ and $\psi_k$:*

$$\psi_n = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{j_1,\ldots,j_k \geq 1 \\ j_1+\ldots+j_k=n}} X^{(j_1)} \ldots X^{(j_k)}, \ (\forall n \in \mathbb{N}). \tag{3.7}$$

*Proof.* The first result (3.5), the joint convergence in distribution, follows directly from the assumption (3.4) and Corollary 3.1.4.1. To prove (3.6) and (3.7), we shall show

$$\sum_{i_1\neq i_2, i_2\neq i_3, \ldots, i_{k-1}\neq i_k} X_{N,i_1} X_{N,i_2} \ldots X_{N,i_k} \xrightarrow{d.} \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{j_1,\ldots,j_k \geq 1 \\ j_1+\ldots+j_k=n}} X^{(j_1)} \ldots X^{(j_k)}, \ (\forall n \in \mathbb{N}).$$

For a fixed $n \in \mathbb{N}$, we define a non-commutative polynomial $Q_n$ by

$$Q_n(x_1, ..., x_N) := \sum_{i_1 \neq i_2, i_2 \neq i_3, ..., i_{n-1} \neq i_n} x_{i_1} x_{i_2} ... x_{i_n}.$$

Meanwhile, denote by $P_n$ a non-commutative polynomial

$$P_n(x_1, ..., x_n) := \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{j_1, ..., j_k \geq 1 \\ j_1 + ... + j_k = n}} x_{j_1} ... x_{j_k}.$$

Let $\Delta_k^N := \sum_{r=1}^{N} X_{N,r}^k$, $(k \in \mathbb{N})$. According to Corollary 3.1.4.1, we have that

$$P_n(\Delta_1^N, ..., \Delta_n^N) \xrightarrow{d.} P_n(X^{(1)}, ..., X^{(n)}),$$

as $N \to \infty$. Thus, it suffices to prove that, for each $N \in \mathbb{N}$,

$$Q_n(X_{N,1}, ..., X_{N,N}) = P_n(\Delta_1^N, ..., \Delta_n^N).$$

At first, we expand the polynomial $P_n$:

$$\begin{aligned} P_n(\Delta_1^N, ..., \Delta_n^N) =& (\Delta_1^N)^n + \sum_{k=1}^{n-1} (-1)^{n-k} \sum_{\substack{j_1, ..., j_k \geq 1 \\ j_1 + ... + j_k = n}} \Delta_{j_1}^N ... \Delta_{j_k}^N \\ =& Q_n(X_{N,1}, ..., X_{N,N}) + \sum_{k=1}^{n-1} \sum_{j_1 + ... + j_k = n} \sum_{1 \leq i_1 \neq ... \neq i_k \leq N} X_{i_1}^{j_1} ... X_{i_k}^{j_k} \\ &+ \sum_{k=1}^{n-1} (-1)^{n-k} \sum_{\substack{j_1, ..., j_k \geq 1 \\ j_1 + ... + j_k = n}} (\sum_{i=1}^{N} X_i^{j_1})(\sum_{i=1}^{N} X_i^{j_2}) ... (\sum_{i=1}^{N} X_i^{j_k}). \end{aligned}$$

So, we shall show that

$$\sum_{k=1}^{n-1} \sum_{j_1 + ... + j_k = n} \sum_{1 \leq i_1 \neq ... \neq i_k \leq N} X_{i_1}^{j_1} ... X_{i_k}^{j_k} + \sum_{k=1}^{n-1} (-1)^{n-k} \sum_{\substack{j_1, ..., j_k \geq 1 \\ j_1 + ... + j_k = n}} (\sum_{i=1}^{N} X_i^{j_1})(\sum_{i=1}^{N} X_i^{j_2}) ... (\sum_{i=1}^{N} X_i^{j_k}) = 0.$$

For any fixed $1 \leq k \leq n-1$ and vector $(j_1, ..., j_k)$ such that $j_i \geq 1$ and $\sum_{i=1}^{k} j_i = n$, we need to compute the coefficient of

$$\sum_{1 \leq i_1 \neq ... \neq i_k \leq N} X_{i_1}^{j_1} ... X_{i_k}^{j_k} \tag{3.8}$$

in the term

$$\sum_{l=1}^{n-1} (-1)^{n-l} \sum_{\substack{j_1,...,j_l \geq 1 \\ j_1+...+j_l=n}} (\sum_{i=1}^{N} X_i^{j_1})(\sum_{i=1}^{N} X_i^{j_2})...(\sum_{i=1}^{N} X_i^{j_l}).$$

Obviously, when $l < k$, there is no term (3.8) in the term

$$(-1)^{n-l} \sum_{\substack{j_1,...,j_l \geq 1 \\ j_1+...+j_l=n}} (\sum_{i=1}^{N} X_i^{j_1})(\sum_{i=1}^{N} X_i^{j_2})...(\sum_{i=1}^{N} X_i^{j_l}).$$

When $n-1 \geq l \geq k$, by combinatorics, we know that the coefficient is $(-1)^{n-l}\binom{n-k}{l-k}$. Then, considering the formula

$$\sum_{l=k}^{n-1} (-1)^{n-l} \binom{n-k}{l-k} = -1,$$

we can conclude that

$$Q_n(X_{N,1}, ..., X_{N,N}) = P_n(\Delta_1^N, ..., \Delta_n^N).$$

□

Theorem 3.2.2 provides a condition to ensure the existence of higher variations and stochastic measures of a free triangular array. In particularly, given a *centered* and stationary stochastic process $X$ with freely independent increments in a non-commutative probability space $(\mathcal{A}, \phi)$, for any $N \in \mathbb{N}$, we take into account the freely independent and identically distributed increments $\{X_{N,i}\}_{1 \leqslant i \leqslant N}$ that add up to $X$ as a free triangular array. Theorem 3.2.2 makes sure the existence of the $k$-th **variation (diagonal measure)** of this centered stochastic process, i.e.

$$X^{(k)} = w - \lim_{N \to \infty} \sum_{i=1}^{N} X_{N,i}^{k}.$$

Also, we have the $k$-th **stochastic measure** of this centered stochastic process

$$\psi_k = w - \lim_{N\to\infty} \sum_{i_1\neq i_2, i_2\neq i_3, \ldots, i_{k-1}\neq i_k} X_{N,i_1} X_{N,i_2} \ldots X_{N,i_k},$$

which follows from Definition 16. Meanwhile, equation (3.7) gives us a **Free Kailath-Segall formula for centered stochastic processes** in a plain non-commutative probability space. In the Free Kailath-Segall formula (3.2) proved in [1], if the process is centered, i.e. $t = 0$, we can get the same formula as (3.7) in the above theorem. This shows that our result is consistent with the previous results in the reference [1], though the definitions are slightly different.

We shall mention that Definition 16 is a generalized concept for any free triangular arrays in a plain non-commutative probability space, while the reference [1] only defined the stochastic measure and higher variations when the triangular array is formed by the freely independent increments of a stationary stochastic process. Formula (3.7) can be viewed as a general Free Kailath-Segall formula in the sense of Definition 16 because this formula describe how higher variations decide a stochastic measure via a polynomial.

# 4. BACKGROUND ON FREE INFINITE DIVISIBILITY AND LÉVY-ITÔ DECOMPOSITION

## 4.1 Free Infinite Divisibility and Free Lévy-Khinchine Representation

**Definition 17** (Free Infinite Divisibility)**.** *A probability measure $\mu$ on $\mathbb{R}$ is called $\boxplus$-infinitely divisible if there exists, for any $n \in \mathbb{N}$, a probability measure $\mu_n$ on $\mathbb{R}$ such that*

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \ldots \boxplus \mu_n}_{n \text{ terms}} =: \mu_n^{\boxplus n}.$$

There is an analogous Lévy-Khinchine formula to describe the infinitely divisible random variables with respect to $\boxplus$-convolution. For the classical results, we refer to [4].

**Theorem 4.1.1.** *([7]) Let $\mu$ be a probability measure on $\mathbb{R}$. $\mu$ is $\boxplus$-infinitely divisible if and only if there exist a finite positive Borel measure $\sigma$ and $\gamma \in \mathbb{R}$ such that its Voiculescu transform is*

$$\Phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t} d\sigma(t). \tag{4.1}$$

**Theorem 4.1.2.** *([6]) Let $\mu$ be a probability measure on $\mathbb{R}$. $\mu$ is $\boxplus$-infinitely divisible if and only if there exist a Lévy measure $\rho$, constant $a \in \mathbb{R}_+$ and $\eta \in \mathbb{R}$ such that its Voiculescu transform is*

$$\Phi_\mu(z) = \eta + \frac{a}{z} + \int_{\mathbb{R}} \left[ \frac{z^2}{z-t} - z - t\mathbf{1}_{[-1,1]}(t) \right] d\rho(t). \tag{4.2}$$

*The Lévy measure $\rho$ satisfies $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\} d\rho(x) < \infty$.*

These two theorems describe what a $\boxplus$-infinitely divisible measure looks like from different point of views. In general, both the triplet $(a, \eta, \rho)$ and pair $(\gamma, \sigma)$ is uniquely determined by the measure and uniquely generate the $\boxplus$-infinitely divisible measure. In fact, the Bercovici-Pata bijection is defined by the bijection between free generating triplet $(a, \eta, \rho)$ of any $\boxplus$-infinitely divisible measure and generating triplet $(a, \eta, \rho)$ in the classical Lévy-Khinchine Representation for the corresponding $*$-infinitely divisible measure (see [8]). Besides, the relationship between the triple and the pair is given in the following equations:

$$\begin{cases} \sigma(dt) = a\delta_0(dt) + \dfrac{t^2}{1+t^2}\rho(dt) \\ \gamma = \eta - \displaystyle\int_{\mathbb{R}} t\left[\mathbf{1}_{[-1,1]}(t) - \dfrac{1}{1+t^2}\right] d\rho(t) \end{cases} \tag{4.3}$$

and, conversely,

$$\begin{cases} a = \sigma(\{0\}) \\ \eta = \gamma + \displaystyle\int_{\mathbb{R}\setminus\{0\}} \dfrac{1+t^2}{t}\left[\mathbf{1}_{[-1,1]}(t) - \dfrac{1}{1+t^2}\right] d\sigma(t) \\ \rho(dt) = \dfrac{1+t^2}{t^2}\mathbf{1}_{\mathbb{R}\setminus\{0\}}(t)\sigma(dt). \end{cases} \tag{4.4}$$

The following theorem, proved by H. Bercovici and V. Pata in [9], will become our most important tool to prove our own results because this theorem provides an equivalent statement of the weak convergence of self-adjoint (possibly unbounded) random variables in a free triangular array. We can employ this theorem to check the weak convergence of the powers of a free triangular array.

**Theorem 4.1.3.** *([3, 9]) For a sequence of probability measure $\mu_n$ and a strictly increasing sequence of positive integers $k_n$, the following assertions are equivalent:*

1. *the sequence of $k_n$-th free convolution $\mu_n^{\boxplus k_n}$ converges weakly to a probability measure $\mu$;*

2. *there exist a finite positive Borel measure $\sigma$ on $\mathbb{R}$ and a real number $\gamma$ such that*

$$k_n \frac{x^2}{x^2+1} d\mu_n(x) \xrightarrow{w} d\sigma(x) \tag{4.5}$$

*and*

$$\lim_{n\to\infty} k_n \int_{\mathbb{R}} \frac{x}{1+x^2} d\mu_n(x) = \gamma. \tag{4.6}$$

*The pair of parameters $(\gamma, \sigma)$ comes form the Voiculescu transform (4.1) of $\mu$. This also implies the $\boxplus$-infinite divisibility of $\mu$.*

So far, we have two types of limit theorems about free triangular array of random variables, namely Theorem 3.1.3 and Theorem 4.1.3. These two theorems view the problem from different angles with different settings. Theorem 3.1.3 gives an equivalent condition for the joint convergence in distribution of the sum of random variables in each row. The condition of Theorem 3.1.3 describes the asymptotic behaviors of joint moments of each row. It is worthwhile to notice that Theorem 3.1.3 makes sense for all random variables in any non-commutative probability space, but it requires all finite moments for each random variable. For Theorem 4.1.3, it only deals with the random variables with distributions on the real line, which means this theorem only considers the self-adjoint elements in a non-commutative probability space. In addition, although Theorem 4.1.3 is unable to deal with the joint convergence in distributions for triangular arrays of random variables, it could handle with unbounded self-adjoint operators, which do not have finite moments in general. Meanwhile, it turns out that the equivalent statement of weak convergence in Theorem 4.1.3 can be successfully employed to address the higher variations of free Lévy processes, which we will define and discuss later.

## 4.2 Free Poisson Random Measures and their Integrations

Similarly with our review of classical theory in Chapter 2, we also need to study the free Lévy processes and their decomposition. So, we first introduce the free Poisson random measures.

**Definition 18** (Free Poisson Random Measures)**.** *Let* $(\Theta, \mathcal{E}, \nu)$ *be a measure space and put* $\mathcal{E}_0 = \{E \in \mathcal{E} : \nu(E) < \infty\}$. *Let further* $(\mathcal{A}, \phi)$ *be a* $W^*-$*probability space and let* $\mathcal{A}_+$ *denote the cone of positive operators in* $\mathcal{A}$*. A free Poisson random measure on* $(\Theta, \mathcal{E}, \nu)$ *with values in* $(\mathcal{A}, \phi)$*, is a mapping* $M : \mathcal{E}_0 \to \mathcal{A}_+$*, with following properties:*

1. *the distribution of M(E) is a free Poisson distribution Poiss*$^{\boxplus}(\nu(E))$;
2. *for any mutually disjoint sets* $A_1, ..., A_n$ *in* $\mathcal{E}_0$*, the random variables* $M(A_1), M(A_2), ..., M(A_n)$ *are freely independent and* $M(\cup_{j=1}^n A_j) = \sum_{j=1}^n M(A_j)$.

Here, the free Poisson distribution $\text{Poiss}^{\boxplus}(\lambda)$ is obtained by the limit in distribution of

$$\left((1-\frac{\lambda}{N})\delta_0+\frac{\lambda}{N}\delta_1\right)^{\boxplus N},$$

as $N \to \infty$ (see Lecture 12 in [10]). The measure $\nu$ is called the intensity measure for the free Poisson random measure. In general, the free random measure is a measure on a measure space with values on a non-commutative space. For the free Poisson random measure, there are more restrictions on this measure, namely, $M(E)$ is a bounded positive operator with a free Poisson distribution, for all $E \in \mathcal{E}_0$. The existence of free Poisson random measures is proved by O.E. Barndorff-Nielsen and Steen Thorbjørnsen in [7]. Given a measure, it is natural to consider the integration with respect to this measure like the integration theory with respect to the classical Poisson random measures.

**Definition 19.** *Let $s$ be a real-valued simple function in $L^1(\Theta,\mathcal{E},\nu)$ in the form $s=\sum_{j=1}^{r}a_j\mathbf{1}_{E_j}$, where $a_j \in \mathbb{R}\setminus\{0\}$ and $E_j$ are disjoint sets from $\mathcal{E}_0$. Then, we define the integral of $s$ with respect to $M$ as follows:*

$$\int_\Theta sdM=\sum_{j=1}^{r}a_jM(E_j)\in\mathcal{A}.$$

Because $M(E)$ are positive in $\mathcal{A}$, the element $\int_\Theta sdM$ is self-adjoint in $\mathcal{A}$, for any real-valued simple function in $L^1(\Theta,\mathcal{E},\nu)$. Next, we can extend this integration for general functions in $L^1(\Theta,\mathcal{E},\nu)$.

**Lemma 4.2.1.** *([7]) Let $f$ be a real-valued function in $L^1(\Theta,\mathcal{E},\nu)$ such that there exists a sequence of real-valued simple functions $(s_n)$ in $L^1(\Theta,\mathcal{E},\nu)$, such that $s_n(\theta)\to f(\theta)$, for all $\theta\in\Theta$. Then, $\int_\Theta sdM$ converges in probability to a self-adjoint (possibly unbounded) operator affiliated with $\mathcal{A}$. Besides this operator is independent of the choice of approximating sequence $(s_n)$. Thus, we denote this operator as $\int_\Theta fdM$.*

Since the concept of convergence in probability is a measure topology in $\bar{\mathcal{A}}$ rather than $\mathcal{A}$, the limit operator $\int_\Theta fdM$ may not be in $\mathcal{A}$. This is the reason why we need to define the affiliated operators.

In fact, we only use a special measure space with a concrete intensity measure in our situation. Let $D = \mathbb{R}_+ \times \mathbb{R}$ and $\mathcal{B}(D)$ be the set of all Borel subsets of $D$. In our case, $(\Theta, \mathcal{E}, \nu) = (D, \mathcal{B}(D), Leb \otimes \rho)$, where $\rho$ is a Lévy measure. The free Poisson random measure $M$ that we will use is defined on $(D, \mathcal{B}(D), Leb \otimes \rho)$ with values in a $W^*-$probability space $(\mathcal{A}, \phi)$. Besides, the integration with respect to this free Poisson measure $M$ we will use is also a special case.

**Lemma 4.2.2.** *Let $\rho$ be a Lévy measure on the real line, and let $M$ be a free Poisson random measure on $(D, \mathcal{B}(D), Leb \otimes \rho)$ with values in the $W^*-$probability space $(\mathcal{A}, \phi)$. Suppose that $p(x)$ is a polynomial without constant term.*

1. *For any $\epsilon, s, t$ in $]0, \infty[$, such that $s < t$, the integral*

$$\int_{]s,t] \times \{\epsilon < |x| \leq n\}} p(x) M(dt, dx)$$

*converges in probability, as $n \to \infty$, to some self-adjoint operator affiliated with $\mathcal{A}$, which is denoted by*

$$\int_{]s,t] \times \{\epsilon < |x| < \infty\}} p(x) M(dt, dx).$$

2. *If $\int_{[-1,1]} |p(x)| \rho(dx) < \infty$, then for any $\epsilon, s, t$ in $]0, \infty[$, such that $s < t$, the integral*

$$\int_{]s,t] \times \{|x| \leq n\}} p(x) M(dt, dx)$$

*converges in probability to some self-adjoint operator affiliated with $\mathcal{A}$, as $n \to \infty$. We denote it by*

$$\int_{]s,t] \times \mathbb{R}} p(x) M(dt, dx)$$

.

The statement of Lemma 4.2.2 is quite similar with Lemma 6.3 of [7]. In the paper [7], the authors only proved the situation when $p(x) = x$ but their methods in Lemma 6.1 and Lemma 6.2

of [7] still work well for Lemma 4.2.2. According to Lemma 6.3 of [7], there are only two things for us to check. Since $\rho$ is a Lévy measure, we have that

$$\int_{]s,t]\times\{\epsilon<|x|\leq n\}} |p(x)| Leb \otimes \rho(du, dx) = (t-s)\int_{\{\epsilon<|x|\leq n\}} |p(x)|\rho(dx) < \infty.$$

If $\int_{[-1,1]} |p(x)|\rho(dx) < \infty$, we have that

$$\int_{]s,t]\times\{|x|\leq n\}} |p(x)| Leb \otimes \rho(du, dx) = (t-s)\left[\int_{\{|x|\leq 1\}} |p(x)|\rho(dx) + \int_{\{1<|x|\leq n\}} |p(x)|\rho(dx)\right] < \infty.$$

Then, we can copy the proof of Lemma 6.3 of [7] and replace the function $f(x) = x$ by the polynomial $p(x)$ directly to prove Lemma 4.2.2. The idea for proving Lemma 6.3 is employing the Bercovici-Pata bijection to transform the statement into classical sense and then using Lebesgue's dominated convergence theorem.

### 4.3 Lévy-Itô Decomposition in Free Probability

**Definition 20.** *Let $(\mathcal{A}, \phi)$ be a $W^*-$probability space. A* ***free Lévy process*** *affiliated with $(\mathcal{A}, \phi)$ is a family $(X_t)_{t\geq 0}$ of self-adjoint operators affiliated with $(\mathcal{A}, \phi)$, such that the following conditions are satisfied:*

1. *any increments $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are freely independent self-adjoint operators affiliated with $\mathcal{A}$, for any $0 \leq t_0 < t_1 ... < t_n$;*

2. *$X_0 = 0$;*

3. *$X_t$ is a stationary stochastic process, i.e. the distribution of $X_{s+t} - X_t$ does not depend on $t$,for any $s, t \geqslant 0$;*

4. *for any $s \in [0, +\infty[$, $X_{s+t} - X_s$ converges in distribution to $\delta_0$ as $t \to 0$.*

The reference [8] shows that there exists a bijection between free Lévy processes and classical Lévy processes. The classical Lévy process is determined by its moment generating function, i.e. the Lévy-Khintchine formula, whereas the non-commutative random variable is determined by its

Voiculescu transform. Thus, to prove that a process is a free Lévy process, we need to verify the free Lévy-Khintchine formula of this process. Besides, in classical theory, we are able to prove a Lévy-Itô decomposition formula via the Lévy-Khintchine formula. The following theorem reveals an analogous result in free probability.

**Theorem 4.3.1** (Free Lévy-Itô Decompositions in [7])**.** *Let $X(t) := X_t$ be a free Lévy process affiliated with a $W^*-$probability space $(\mathcal{A}, \phi)$. Let $(a, \eta, \rho)$ be the triplet appearing in the Lévy-Khintchine formula of $X(1)$. Then, $X(t)$ can be decomposed into three freely independent parts.*

1. *If $\int_{[-1,1]} |x|\rho(dx) < \infty$, then*

$$X(t) \overset{d}{=} \eta t \mathbf{1}_{\mathcal{A}^0} + \sqrt{a} W_t + \int_{]0,t]\times\mathbb{R}} x dM(t, x). \tag{4.7}$$

2. *In general, we have that*

$$X(t) \overset{d}{=} \eta t \mathbf{1}_{\mathcal{A}^0} + \sqrt{a} W_t + \lim_{\epsilon \searrow 0} [\int_{]0,t]\times[|x|>\epsilon]} x dM(t, x) \tag{4.8}$$

$$- \int_{]0,t]\times[\epsilon<|x|\leqslant 1]} x Leb \otimes \rho(dt, dx) \mathbf{1}_{\mathcal{A}^0}]. \tag{4.9}$$

*Here, $\eta \in \mathbb{R}$, $a \geq 0$, $W_t$ is a free Brownian motion in some $W^*$-probability space $(\mathcal{A}^0, \phi^0)$ and $M$ is a free Poisson random measure determined by the Lévy measure $\rho$ on the measure space $(D, \mathcal{B}(D), Leb \otimes \rho)$ with values in the $W^*$-probability space $(\mathcal{A}^0, \phi^0)$.*

Notice that the symbol $\overset{d}{=}$ means that the right-hand side and the left-hand side have the same distribution but it does not mean they are same or in the same non-commutative probability space $\mathcal{A}$. The free Lévy-Itô decomposition will become a good representation for us to express the relationship between two free Lévy processes in the next chapter.

## 5. CONCLUSIONS FOR SELF-ADJOINT UNBOUNDED OPERATORS

We now focus on studying the free triangular array of self-adjoint non-commutative random variables $\{X_{N,r}\}_{1\leqslant r\leqslant N}$ affiliated with a W$^*-$probability space $(\mathcal{A},\phi)$, which are identically distributed and freely independent in each row. Assume that the distribution of each random variable $X_{N,r}$ is a probability measure $\mu_N$ on $\mathbb{R}$ and the distribution of $X_{N,r}^k$ is a probability measure $\mu_N^{(k)}$, for any $k\in\mathbb{N}$. Throughout this chapter, we will use these notations defined above all the time.

### 5.1 The Higher Variations of Free Lévy Processes

**Theorem 5.1.1.** *If there exist a finite Borel measure $\sigma$ and a constant $\gamma$ such that*

$$N\frac{x^2}{x^2+1}d\mu_N(x)\xrightarrow{w}d\sigma(x) \tag{5.1}$$

*and*

$$\lim_{N\to\infty}N\int_{\mathbb{R}}\frac{x}{1+x^2}d\mu_N(x)=\gamma, \tag{5.2}$$

*then there exists a family $\{\mu_t\}_{t\geqslant 0}$ of probability measures on $\mathbb{R}$ such that $\mu_N^{\boxplus[Nt]}\xrightarrow{w}\mu_t$, for any $t\in[0,\infty)$. Each $\mu_t$ is $\boxplus$-infinitely divisible and its Voiculescu transform is $\Phi_{\mu_t}(z)=t\gamma+t\int_{\mathbb{R}}\frac{1+xz}{z-x}d\sigma(x)=t\Phi_\mu(z)$, where $\mu:=\mu_1$ is the distribution of $X(1)$.*

*Moreover, there exists a free Lévy process $\{X(t)\}_{t\geqslant 0}$ such that the distribution of each $X(t)$ is $\mu_t$, for all $t\geqslant 0$.*

*Proof.* By Theorem 4.1.3, we know that if there exist a finite Borel measure $\sigma$ and a constant $\gamma$ such that (5.1) and (5.2) hold, then $\mu_N^{\boxplus N}\xrightarrow{w}\mu_1$. For any $t\in[0,\infty)$, we have that

$$[Nt]\frac{x^2}{x^2+1}d\mu_N(x)\xrightarrow{w}td\sigma(x)=:d\sigma_t(x)$$

and

$$\lim_{N\to\infty}[Nt]\int_{\mathbb{R}}\frac{x}{1+x^2}d\mu_N(x)=t\lim_{N\to\infty}N\int_{\mathbb{R}}\frac{x}{1+x^2}d\mu_N(x)=t\gamma=:\gamma_t.$$

Therefore, for any $t \in [0, \infty)$, there exists a probability measure $\mu_t$ such that $\mu_N^{\boxplus[Nt]} \xrightarrow{w.} \mu_t$. According to Theorem 4.1.3, for any $t \in [0, \infty)$, $\mu_t$ is $\boxplus$-infinitely divisible since the Voiculescu transform of $\mu_t$ is

$$\Phi_{\mu_t}(z) = \gamma_t + \int_{\mathbb{R}} \frac{1 + xz}{z - x} d\sigma_t(x) = t\Phi_\mu(z),$$

where $\mu := \mu_1$. Therefore, $\Phi_{\mu_t} = \Phi_{\mu_{t-s}} + \Phi_{\mu_s}$, when $t > s \geqslant 0$. In other words, $\mu_t = \mu_{t-s} \boxplus \mu_s$. Meanwhile, $\Phi_{\mu_t} \to 0$ when $t \to 0$, which means $\mu_t \xrightarrow{w.} \delta_0$, as $t \to 0$. Then, by Remark 6.7 in [7], we can conclude that there exists a free Lévy process $\{X(t)\}_{t \geqslant 0}$, which is a family of self-adjoint operators affiliated with some W$^*-$probability space $(\mathcal{A}^0, \phi^0)$, such that the distribution of each $X(t)$ is $\mu_t$, for all $t \geqslant 0$. $\square$

Theorem 5.1.1 claims that if (5.1) and (5.2) hold, then there exists a free Lévy process $X(t)$ with free generating pair $(\gamma_t, \nu_t)$ such that

$$\sum_{r=1}^{[Nt]} X_{N,r} \xrightarrow{d.} X(t),$$

for each $t \geqslant 0$. Our aim is to find some reasonable conditions of $\{X_{N,r}\}_{1 \leqslant r \leqslant N}$ that imply the convergence of the sum of the $k$-th power of every entry over each row towards some random variable $X^{(k)}(t)$, i.e.

$$\sum_{r=1}^{[Nt]} X_{N,r}^k \xrightarrow{d.} X^{(k)}(t), (t \geqslant 0, k \geqslant 2).$$

Also, we want to make sure that each $X^{(k)}(t)$ is a free Lévy process. For the sake of simplicity, we continue to use the notation 'variations' to denote $X^{(k)}(t)$. Meanwhile, we want to know how $X(t)$ determines higher variations $X^{(k)}(t)$.

Let $D = \mathbb{R}_+ \times \mathbb{R}$ and $\mathcal{B}(D)$ be the set of all Borel subsets of $D$. According to Chapter 2 and 4, it is natural to ask whether there is an analogous result associated with Theorem 2.3.1 in free sense. The Lévy-Itô decompositions in free probability states that if $X(t)$ is a free Lévy process affiliated with a $W^*-$probability space and the Lévy measure $\rho$, appearing in the free generating triplet of $X(1)$ satisfies $\int_{-1}^{1}|x|d\rho(x)<\infty$, then

$$X(t)\overset{d}{=}\eta t\mathbf{1}_{\mathcal{A}}+\sqrt{a}W_t+\int_{]0,t]\times\mathbb{R}}xM(dt,dx),$$

where $\eta\in\mathbb{R}$, $a\geq 0$, $W_t$ is a free Brownian motion and $M$ is a free Poisson random measure determined by Lévy measure $\rho$ on a Borel measure space $(D,\mathcal{B}(D),Leb\otimes\rho)$ with values in some $W^*-$probability space $\mathcal{A}$. We aim at knowing the Lévy-Itô decomposition of higher variations in terms of the relationship between free generating pair of $X(t)$ and $X^{(k)}(t)$. Our goal is to prove the following formulas for higher variations:

$$X^{(2)}(t)\overset{d}{=}at\mathbf{1}_{\mathcal{A}}+\int_{]0,t]\times\mathbb{R}}x^2M(dt,dx) \tag{5.3}$$

and

$$X^{(k)}(t)\overset{d}{=}\int_{]0,t]\times\mathbb{R}}x^kM(dt,dx),(k>2). \tag{5.4}$$

**Lemma 5.1.2.** *Let $(\mathcal{A},\phi)$ be a $W^*-$probability space with operators in $\mathcal{A}$ acting on the Hilbert space $\mathcal{H}$. If the self-adjoint operator $a$ is affiliated with $\mathcal{A}$ and has a distribution $\mu$, which is a probability measure on $\mathbb{R}$, then the distribution $\mu^{(k)}$ of operator $a^k$, the $k-$th power of operator $a$, can be obtained by the following formula:*

$$\int_{\mathbb{R}}f(t^k)d\mu(t)=\int_{\mathbb{R}}f(t)d\mu^{(k)}(t),$$

*for any bounded Borel function $f:\mathbb{R}\to\mathbb{R}$ and $k\in\mathbb{N}$.*

*Proof.* By Definition 5, we know that, for any bounded Borel function $f:\mathbb{R}\to\mathbb{R}$,

$$\phi(f(a))=\int_{\mathbb{R}}f(t)d\mu(t).$$

Consider $g(x) = x^k$. Then, $f \circ g(x)$ is still a bounded Borel function. Thus,

$$\int_{\mathbb{R}} f(t^k) d\mu(t) = \phi\left(f(a^k)\right) = \int_{\mathbb{R}} f(t) d\mu^{(k)}(t).$$

$\square$

Lemma 5.1.2 shows how to change variables in a free probability setting. In classical case, one can easily prove the formula of changing variables under different probability measures. For simplicity, we would like to denote $\int_{\mathbb{R}} f(y) d\mu(y^{1/k}) := \int_{\mathbb{R}} f(t^k) d\mu(t)$ to represent the change of variables. In fact, we can get a more general formula for changing variables in the integration with respect to a free Poisson random measure.

**Lemma 5.1.3.** *Let $p(x)$ be any real-valued polynomial without constant term. Suppose that $M$ is a free Poisson random measure determined by a Lévy measure $\rho$ on the Borel measure space $(D, \mathcal{B}(D), Leb \otimes \rho)$ with values in some $W^*-$probability space $\mathcal{A}$. If $\rho^p$ is a another measure defined by*

$$\int_{\mathbb{R}} f(x) d\rho^p(x) = \int_{\mathbb{R}} f(p(x)) \mathbf{1}_{\mathbb{R}\setminus\{0\}}(p(x)) d\rho(x), \tag{5.5}$$

*for any bounded Borel function $f(x)$ on $\mathbb{R}$, then $\rho^p$ is a Lévy measure. The free Poisson random measure $M^{(p)}$ defined by $\rho^p$ on $(D, \mathcal{B}(D), Leb \otimes \rho^p)$ has the following relation with $M$:*

$$\int_{]0,t]\times\mathbb{R}} x dM^p(t,x) \stackrel{d}{=} \int_{]0,t]\times\mathbb{R}} p(x) dM(t,x), (\forall t \geqslant 0), \tag{5.6}$$

*provided that $\int_{[-1,1]} |p(x)| d\rho(x) < \infty$.*

*Proof.* Notice that $p(0) = 0$. So there exists an interval $[a,b] \subset [-1,1]$ containing 0 such that $|p(x)| \leqslant 1$ when $x \in [a,b]$. Since $p(x)$ does not have constant term, $p(x) = xq(x)$, where $q(x)$ is another polynomial. So there exists a constant $C > 0$ such that $(p(x))^2 \leqslant Cx^2$ when $x \in [a,b]$. First, we show that $\rho^p$ is a Lévy measure. If $f(x) = \mathbf{1}_{\{0\}}(x)$, then $\rho^p(\{0\}) = \int_{\mathbb{R}} \mathbf{1}_{\{0\}}(x) d\rho^p(x)$ is zero by the definition (5.5). Next, if $f(x) = \min\{1, x^2\}$, then we can get the following conclusion:

$$\begin{aligned}\int_{\mathbb{R}} \min\{1, x^2\} d\rho^p(x) &= \int_{\mathbb{R}} \mathbf{1}_{[-1,1]}(x) x^2 d\rho^p(x) + \int_{\mathbb{R}} \mathbf{1}_{\mathbb{R}\setminus[-1,1]}(x)\rho^p(x) \\ &= \int_{\mathbb{R}} \mathbf{1}_{[-1,1]\setminus\{0\}}(p(x))(p(x))^2 d\rho(x) + \int_{\mathbb{R}} \mathbf{1}_{\mathbb{R}\setminus[-1,1]}(p(x)) d\rho(x) \\ &\leqslant \int_{\{x\in\mathbb{R}:0<|p(x)|\leqslant 1\}\setminus[a,b]} (p(x))^2 d\rho(x) + \int_{[a,b]} (p(x))^2 d\rho(x) + \int_{\mathbb{R}\setminus[a,b]} 1 d\rho(x) \\ &\leqslant \int_{\mathbb{R}\setminus[a,b]} 1 d\rho(x) + \int_{[-1,1]} Cx^2 d\rho(x) + \int_{\mathbb{R}\setminus[a,b]} 1 d\rho(x) < \infty.\end{aligned}$$

Therefore, $\rho^p$ is a Lévy measure.

Second, we show that the relation (5.6) holds. If $\int_{[-1,1]} |p(x)| d\rho(x)$ is finite, then the right-hand side of (5.6) makes sense by Lemma 4.2.2. Similarly,

$$\int_{-1}^{1} |x| d\rho^p(x) = \int_{\mathbb{R}} \mathbf{1}_{[-1,1]\setminus\{0\}}(p(x)) \cdot |p(x)| d\rho(x) < \infty,$$

because $p(x) = xq(x)$ and $\int_{[-1,1]} |p(x)| d\rho(x) < \infty$. In other words, the left-hand side of (5.6) makes sense as well. Thus, according to Lemma 4.2.2, we only need to show that

$$\int_{]0,t]\times\{x:-n\leqslant x<n\}} x dM^p(t,x) \stackrel{d}{=} \int_{]0,t]\times\{x:-n\leqslant|p(x)|<n\}} p(x) dM(t,x),$$

for all $t \geqslant 0$ and $n \in \mathbb{N}$. For any $N \in \mathbb{N}$, consider mutually disjoint intervals $E_m^N = [-n + \frac{2n(m-1)}{N}, -n + \frac{2nm}{N})$, where $1 \leqslant m \leqslant N$ and $m \in \mathbb{N}$. Then, the simple function $s_N(x) = \sum_{m=1}^{N} (-n + \frac{2n(m-1)}{N}) \mathbf{1}_{E_m^N}(x)$ converges to $f(x) = x$, for any $x \in [-n, n)$ as $N \to \infty$. Thus, $\int_{]0,t]\times\{x:-n\leqslant x<n\}} s_N(x) dM^p(t,x)$ converges in probability to $\int_{]0,t]\times\{x:-n\leqslant x<n\}} x dM^p(t,x)$. Let

$$J_m^N = \{x : p(x) \in E_m^N\}, (1 \leqslant m \leqslant N, m \in \mathbb{N}).$$

Then, $\cup_{m=1}^{N} J_m^N = \{x : -n \leqslant |p(x)| < n\}$ and $\{J_m^N\}$ are mutually disjoint. Consider the simple function $h_N(x) = \sum_{m=1}^{N} (-n + \frac{2n(m-1)}{N}) \mathbf{1}_{J_m^N}(x)$. which converges to $p(x)$, for any $x \in \{x : -n \leqslant$

$|p(x)| < n\}$, as $N \to \infty$. Also, the integration $\int_{]0,t]\times\{x:-n\leqslant|p(x)|<n\}} h_N(x)dM(t,x)$ converges in probability to $\int_{]0,t]\times\{x:-n\leqslant|p(x)|<n\}} p(x)dM(t,x)$. Therefore, it suffices to show that the distribution of $\int_{]0,t]\times\{x:-n\leqslant|p(x)|<n\}} h_N(x)dM(t,x)$ is equal to the distribution of $\int_{]0,t]\times\{-n\leqslant x<n\}} s_N(x)dM^p(t,x)$.

Let $F_m^N = [0,t]\times E_m^N$ and $K_m^N = [0,t]\times J_m^N$. By Definition 19, we know that

$$\int_{]0,t]\times\{x:-n\leqslant x<n\}} s_N(x)dM^p(t,x) = \sum_{m=1}^{N}(-n+\frac{2n(m-1)}{N})M^p(F_m^N),$$

and

$$\int_{]0,t]\times\{x:-n\leqslant|p(x)|<n\}} h_N(x)dM(t,x) = \sum_{m=1}^{N}(-n+\frac{2n(m-1)}{N})M(K_m^N).$$

By Definition 18, the distribution of $M(K_m^N)$ is Poiss$^{\boxplus}(t\rho(J_m^N))$ and the distribution of $M^p(F_m^N)$ is Poiss$^{\boxplus}(t\rho^p(E_m^N))$. According to (5.5), we conclude that

$$\begin{aligned}\rho^p(E_m^N) &= \int_{\mathbb{R}} \mathbf{1}_{E_m^N}(x)d\rho^p(x)\\ &= \int_{\mathbb{R}} \mathbf{1}_{E_m^N\setminus\{0\}}(p(x))d\rho(x)\\ &= \int_{\mathbb{R}} \mathbf{1}_{\{x:p(x)\in E_m^N\}}(x)d\rho(x) = \rho(J_m^N).\end{aligned}$$

So, Poiss$^{\boxplus}(t\rho(J_m^N))$=Poiss$^{\boxplus}(t\rho^p(E_m^N))$. Then, we get the final result (5.6). □

Lemma 5.1.3 provides an approach to changing variables in integrals with respect to a free Poisson random measure.

## 5.2 Main Results

In this section, we show our main results. We derive the representation of the higher variations related to the Lévy-Itô decomposition of the free Lévy process. All the following results are based on the theory we established in the former section and Theorem 4.1.3.

**Theorem 5.2.1.** *Let $\mu_N$ be the distribution of the self-adjoint random variable $X_{N,r}$ affiliated with some $W^*-$probability space. $\{k_n\}_{n\in\mathbb{N}}$ is any strictly increasing sequence of positive integers.*

*Suppose that there exists a probability measure $\mu$ such that the distribution $\mu_N$ satisfies*

$$\mu_N^{\boxplus\, k_N} \xrightarrow{w} \mu.$$

*Let probability measure $\mu_N^{(k)}$ be the distribution of $X_{N,r}^k$ $(k \geqslant 2)$. Then,*

$$(\mu_N^{(k)})^{\boxplus k_N} \xrightarrow{w} \mu^{(k)}, \tag{5.7}$$

*where $\mu^{(k)}$ is a $\boxplus-$infinitely divisible probability measure on $\mathbb{R}$. In other words, the sum of $X_{N,r}^k$ $(k \geqslant 2)$ with distribution $\mu_N^{(k)}$ converges in distribution to some random variable $X^{(k)}$ with distribution $\mu^{(k)}$, i.e.*

$$\sum_{r=1}^{k_N} X_{N,r}^k \xrightarrow{d} X^{(k)}. \tag{5.8}$$

*Proof.* In order to prove (5.7), we apply Theorem 4.1.3 to verify there exist constants $\gamma^{(k)}$ and finite Borel measures $\sigma^{(k)}$, for any $k \geqslant 2$, such that the following holds:

$$\begin{cases} k_n \dfrac{x^2}{x^2+1} d\mu_n^{(k)}(x) \xrightarrow{w} d\sigma^{(k)}(x), \\ \lim\limits_{n\to\infty} k_n \displaystyle\int_{\mathbb{R}} \frac{x}{1+x^2} d\mu_n^{(k)}(x) = \gamma^{(k)}. \end{cases}$$

By our assumption, there exists a pair $(\gamma, \sigma)$ generating the distribution $\mu$ satisfying (4.5) and (4.6). Lemma 5.1.2 shows that $\int_{\mathbb{R}} f(x) d\mu_n^{(k)}(x) = \phi(f(X_{n,r}^k)) = \int_{\mathbb{R}} f(x^k) d\mu_n(x)$, for any $f \in Cb(\mathbb{R})$. Denote that $g_k(x) = \frac{x^{k-2}(1+x^2)}{1+x^{2k}}$, which is a continuous bounded function on the whole real line when $k \geqslant 2$. Then, by (4.5), we have that

$$\begin{aligned} \lim_{n\to\infty} k_n \int_{\mathbb{R}} \frac{x}{1+x^2} d\mu_n^{(k)}(x) &= \lim_{n\to\infty} k_n \int_{\mathbb{R}} \frac{x^k}{1+x^{2k}} d\mu_n(x) \\ &= \lim_{n\to\infty} k_n \int_{\mathbb{R}} g_k(x) \frac{x^2}{1+x^2} d\mu_n(x) \\ &= \int_{\mathbb{R}} g_k(x) d\sigma(x) =: \gamma^{(k)}. \end{aligned}$$

Meanwhile, for any $f(x) \in Cb(\mathbb{R})$, the function $f(x^k)$ is a continuous bounded function defined on $\mathbb{R}$, too. And

$$\begin{aligned} k_n \int_{\mathbb{R}} f(x) \frac{x^2}{x^2+1} d\mu_n^{(k)}(x) &= k_n \int_{\mathbb{R}} f(x^k) \frac{x^{2k}}{x^{2k}+1} d\mu_n(x) \\ &= k_n \int_{\mathbb{R}} f(x^k) \frac{x^{2k}}{x^{2k}+1} \frac{x^2}{x^2+1} \frac{x^2+1}{x^2} d\mu_n(x) \\ &\overset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(x^k) \frac{x^{2k}+x^{2k-2}}{x^{2k}+1} d\sigma(x), \end{aligned}$$

since $f(x^k) = \frac{x^{2k}+x^{2k-2}}{x^{2k}+1} \in Cb(\mathbb{R})$ and (4.5) holds. Thus, we proved the existence of $d\sigma^{(k)}(x)$ and $d\sigma^{(k)}(x) := \frac{x^{2-\frac{2}{k}}+x^2}{1+x^2} d\sigma(x^{1/k})$. Here, the notation $\sigma(x^{1/k})$ means that $\int f(x) d\sigma(x^{1/k}) := \int f(x^k) d\sigma(x)$ for any bounded Borel function $f(x)$. So applying Theorem 4.1.3, we get the conclusions (5.8) and (5.7). Moreover, we derive the following relations between two generating pairs for $\mu$ and $\mu^{(k)}$, $(\gamma, \sigma)$ and $(\gamma^{(k)}, \sigma^{(k)})$, namely,

$$\begin{cases} \gamma^{(k)} = \displaystyle\int_{\mathbb{R}} \frac{x^{k-2}}{1+x^{2k}} (1+x^2) d\sigma(x), \\ d\sigma^{(k)}(x) = \dfrac{x^{2-\frac{2}{k}}(1+x^{\frac{2}{k}})}{1+x^2} d\sigma(x^{1/k}). \end{cases} \tag{5.9}$$

□

Next, we employ the Lévy-Itô decomposition formula in free probability, introduced in Chapter 4 and the reference [7], to find a concrete relationship between (free) Lévy processes $X(t)$ and higher variations $X^{(k)}(t)$. To use the Lévy-Itô decomposition, we need to consider the (free) generating triplet $(a, \eta, \rho)$ rather than the pair $(\gamma, \sigma)$, so we should at first compute the triplet $(a^{(k)}, \eta^{(k)}, \rho^{(k)})$ of the variation $X^{(k)}(t)$ via (4.4) and (5.9). Define that the function $\delta_{k,2} = 1$ when $k = 2$, otherwise $\delta_{k,2} = 0$.

**Theorem 5.2.2.** *Assume that $\sum_{r=1}^{[Nt]} X_{N,r}$ converges in distribution to a free Lévy process $X(t)$ and $X(1)$ is generated by the triplet $(a, \eta, \rho)$. Then, for $k \geqslant 2$, there exists a Lévy process $X^{(k)}(t)$ such* *that*

$$\sum_{r=1}^{[Nt]} X_{N,r}^k \xrightarrow{d} X^{(k)}(t), (\forall t \geqslant 0).$$

*As usual, we call $X^{(k)}(t)$ the $k-th$ variation of the Lévy process $X(t)$. Moreover, $X^{(k)}(t)$ has a representation in the form:*

$$X^{(k)}(t) \stackrel{d}{=} at\delta_{k,2}\boldsymbol{1}_{\mathcal{A}} + \int_{]0,t]\times\mathbb{R}} x^k dM(t,x), \tag{5.10}$$

*where $M$ is a free Poisson random measure on $(D, \mathcal{B}(D), Leb \otimes \rho)$ coming from the Lévy-Itô decomposition of the Lévy process $X(t)$ with values in a $W^*-$probability space $\mathcal{A}$.*

*Proof.* For simplicity, denote $X^{(1)}(t) := X(t)$ and $(a, \eta, \rho) =: (a^{(1)}, \eta^{(1)}, \rho^{(1)})$. By Theorem 5.2.1, we get the convergence of $\sum_{r=1}^{[Nt]} X_{N,r}^k$ in distribution considering $k_N = [Nt]$. Next, by Theorem 5.1.1, we conclude that there exists a Lévy process $X^{(k)}(t)$ ($\forall k \in \mathbb{N}$) such that $\sum_{r=1}^{[Nt]} X_{N,r}^k \xrightarrow{d} X^{(k)}(t), (\forall t \geqslant 0)$. So we have the Lévy-Itô decomposition of the Lévy process $X^{(k)}(t)$:

$$\begin{aligned} X(t)^{(k)} \stackrel{d}{=} & \eta^{(k)} t \mathbf{1}_{\mathcal{A}} + \sqrt{a^{(k)}} W_t \\ & + \lim_{\epsilon \searrow 0} \left[ \int_{]0,t]\times[|x|>\epsilon]} x dM^{(k)}(t,x) - \int_{]0,t]\times[\epsilon<|x|\leqslant 1]} x Leb \otimes \rho^{(k)}(dt,dx) \mathbf{1}_{\mathcal{A}} \right], \end{aligned}$$

where $(a^{(k)}, \eta^{(k)}, \rho^{(k)})$ is the free generating triplet of $X^{(k)}(1)$ and $M^{(k)}$ is the free Poisson random measure associated with the Lévy measure $\rho^{(k)}$. To prove (5.10), we need to compute $(a^{(k)}, \eta^{(k)}, \rho^{(k)})$ via the free generating pair $(\gamma, \sigma)$. Considering (4.4) and (5.9), we claim the following results. First, when $k \geqslant 2$, $a^{(k)} = \sigma^{(k)}(\{0\}) = 0$. As for $\eta^{(k)}$, it is easy to check that

$$\begin{aligned}
\eta^{(k)} =& \gamma^{(k)} + \int_{\mathbb{R}\setminus\{0\}} \frac{1+x^2}{x} \left( \mathbf{1}_{[-1,1]}(x) - \frac{1}{1+x^2} \right) d\sigma^{(k)}(x) \\
=& \int_{\{0\}} \frac{x^{k-2}}{1+x^{2k}}(1+x^2) d\sigma(x) + \int_{\mathbb{R}\setminus\{0\}} \frac{x^{k-2}}{1+x^{2k}}(1+x^2) d\sigma(x) \\
&+ \int_{\mathbb{R}\setminus\{0\}} x^{k-2}(1+x^2) \left( \mathbf{1}_{[-1,1]}(x) - \frac{1}{1+x^{2k}} \right) d\sigma(x) \\
=& \delta_{k,2}\sigma(\{0\}) + \int_{\mathbb{R}\setminus\{0\}} \frac{x^{k-2}(1+x^2)}{x^2} \mathbf{1}_{[-1,1]} d\sigma(x) \\
=& a\delta_{k,2} + \int_{-1}^{1} x^k d\rho(x).
\end{aligned}$$

Thirdly, $d\rho^{(k)}(x) = \frac{1+x^2}{x^2}\mathbf{1}_{\mathbb{R}\setminus\{0\}} d\sigma^{(k)}(x) = d\rho(x^{1/k})$. The free Poisson random measure $M^{(k)}$ is defined on $(D, \mathcal{B}(D), Leb \otimes \rho^{(k)})$ with values in a W$^*-$probability space $(\mathcal{A}, \phi)$. Lemma 5.1.3 guarantees (5.5) and (5.6), when $p(x) = x^k$. In short,

$$\begin{cases} a^{(k)} = 0 \\ \eta^{(k)} = \delta_{k,2} + \int_{-1}^{1} x^k d\rho(x) \\ d\rho^{(k)}(x) = d\rho(x^{1/k}). \end{cases} \tag{5.11}$$

By the definition of Lévy measures, the Lévy measures of higher variations have the following property:

$$\int_{-1}^{1} |x| d\rho^{(k)}(x) = \int_{-1}^{1} |x|^k d\rho(x) \leqslant \int_{-1}^{1} |x|^2 d\rho(x) < \infty, (k \geqslant 2). \tag{5.12}$$

According to the Theorem 6.4 in [7] and property (5.12), when $k > 1$, the third part of the Lévy-Itô decomposition of $X^{(k)}(t)$, which represents the integration with respect to the Poisson random measure $M^{(k)}$, can be expressed without the limit, i.e.

$$\begin{aligned}
&\lim_{\epsilon \searrow 0}\left[\int_{]0,t]\times[|x|>\epsilon]} x dM^{(k)}(t,x) - \int_{]0,t]\times[\epsilon<|x|\leqslant 1]} x Leb\otimes\rho^{(k)}(t,x)\mathbf{1}_{\mathcal{A}}\right] \\
&= \int_{]0,t]\times[|x|>0]} x dM^{(k)}(t,x) - \int_{]0,t]\times[0<|x|\leqslant 1]} x Leb\otimes\rho^{(k)}(dt,dx)\mathbf{1}_{\mathcal{A}} \\
&= \int_{]0,t]\times[|x|>0]} x dM^{(k)}(t,x) - t\int_{-1}^{1} x\rho^{(k)}(dx)\mathbf{1}_{\mathcal{A}} \\
&= \int_{]0,t]\times[|x|>0]} x dM^{(k)}(t,x) - t\int_{-1}^{1} x^k\rho(dx)\mathbf{1}_{\mathcal{A}} \\
&= \int_{]0,t]\times[|x|>0]} x^k dM(t,x) - t\int_{-1}^{1} x^k\rho(dx)\mathbf{1}_{\mathcal{A}}.
\end{aligned}$$

Here, we employ Lemma 5.1.3 when $p(x) = x^k$. Then, by the formulas in (5.11) and the above calculations, we obtain (5.10) for the representation of higher variations $X^{(k)}(t)$. □

So far, we proved the limit properties of a free triangular array of the powers of random variables. Naturally, we want to check whether the limit result holds even for a free triangular array of polynomials of some given random variables. The following theorem gives the answer, although we need some restriction on the polynomials.

**Theorem 5.2.3.** *Assume that*

$$\sum_{r=1}^{[Nt]} X_{N,r} \xrightarrow{d} X(t),\ (\forall t \geqslant 0),$$

*where $X(t)$ is a free Lévy process. $X(1)$ is generated by the triplet $(a,\eta,\rho)$. Let $p(x)$ be a real-valued polynomial such that $p(x) = x^2 q(x)$, where $q(x)$ is also a polynomial. Then, there exists a Lévy process $X^p(t)$ such that*

$$\sum_{r=1}^{[Nt]} p\left(X_{N,r}\right) \xrightarrow{d} X^p(t), (\forall t \geqslant 0). \tag{5.13}$$

*In addition, if $X(t)$ has the Lévy-Itô decomposition (4.8), then $X^p(t)$ has a representation in the form:*

$$X^p(t) \stackrel{d}{=} aq(0)t\mathbf{1}_{\mathcal{A}} + \int_{]0,t]\times\mathbb{R}} p(x)dM(t,x), \tag{5.14}$$

*where $M$ is a free Poisson random measure on $(D, \mathcal{B}(D), Leb \otimes \rho)$ coming from the Lévy-Itô decomposition of $X(t)$ with values in a $W^*-$probability space $\mathcal{A}$.*

*Proof.* The proof is straightforward since we can continue to use the approaches in Theorem 5.2.1 and Theorem 5.2.2. Let $\mu_N$ and $\mu_N^p$ be the distributions of $X_{N,r}$ and $p(X_{N,r})$ respectively. We can repeat the argument in Lemma 5.1.2 to get $\int_{\mathbb{R}} f(x) d\mu_N^p(x) = \int_{\mathbb{R}} f(p(x)) d\mu_N(x)$, for any real-valued and bounded Borel function $f(x)$. Therefore,

$$\begin{aligned}
\lim_{N\to\infty} [Nt] \int_{\mathbb{R}} \frac{x}{1+x^2} d\mu_N^p(x) &= t \lim_{N\to\infty} N \int_{\mathbb{R}} \frac{p(x)}{1+p(x)^2} d\mu_N(x) \\
&= t \lim_{N\to\infty} N \int_{\mathbb{R}} g_p(x) \frac{x^2}{1+x^2} d\mu_N(x) \\
&= t \int_{\mathbb{R}} g_p(x) d\sigma(x) =: t\gamma^p,
\end{aligned}$$

where $g_p(x) = \frac{p(x)(1+x^2)}{x^2(1+(p(x))^2)} = \frac{p(x)+q(x)}{1+(p(x))^2} \in Cb(\mathbb{R})$. This is because $p(x)$ does not have linear term and constant term. Also, for any $f(x) \in Cb(\mathbb{R})$

$$\begin{aligned}
[Nt] \int_{\mathbb{R}} f(x) \frac{x^2}{x^2+1} d\mu_N^p(x) &= [Nt] \int_{\mathbb{R}} f(p(x)) \frac{p(x)^2}{p(x)^2+1} d\mu_N(x) \\
&= [Nt] \int_{\mathbb{R}} f(p(x)) \frac{p(x)^2}{p(x)^2+1} \frac{x^2}{x^2+1} \frac{x^2+1}{x^2} d\mu_N(x) \\
&\overset{N\to\infty}{\longrightarrow} t \int_{\mathbb{R}} f(p(x)) \frac{p(x)\,(p(x)+q(x))}{p(x)^2+1} d\sigma(x),
\end{aligned}$$

where the measure $d\sigma(x)$ is defined by (4.5). Let $h_p(x) := \frac{p(x)(p(x)+q(x))}{p(x)^2+1}$, which is a positive bounded Borel function on $\mathbb{R}$. We denote $h_p(x)d\sigma(x)$ by the measure $d\widetilde{\sigma}(x)$. The measure $d\sigma^p(x)$ is defined by $\int_{\mathbb{R}} f(x) d\sigma^p(x) = \int_{\mathbb{R}} f(p(x)) d\widetilde{\sigma}(x)$, for any bounded Borel function $f(x)$. Then, $[Nt] \frac{x^2}{x^2+1} d\mu_N^p(x) \overset{w}{\to} d\sigma^p(x)$, as $N \to \infty$. Since $\sigma$ is a finite positive Borel measure on $\mathbb{R}$, we know that $\sigma^p$ is also a finite positive Borel measure. Thus, the conclusion (5.13) follows immediately from Theorem 4.1.3. By Theorem 5.1.1, we know that $\{X^p(t)\}_{t\geqslant 0}$ can be a free Lévy process affiliated with some $W^*-$probability space. Denote the free generating triplet of $X^p(1)$ by $(a^p, \eta^p, \rho^p)$.

Next, to prove the representation of $X^p(t)$, it is necessary to compute the free generating triplet $(a^p, \eta^p, \rho^p)$ in terms of free generating pair $(\gamma, \sigma)$ or free generating triplet $(a, \eta, \rho)$ of $X(t)$. Firstly, $a^p = \sigma^p(\{0\}) = \int_{\mathbb{R}} \mathbf{1}_{\{0\}}(x) d\sigma^p(x) = \int_{\mathbb{R}} \mathbf{1}_{\{p(x)=0\}}(x) \frac{p(x)(p(x)+q(x))}{p(x)^2+1} d\sigma(x) = 0$. Secondly, for Lévy measure $\rho^p$ and any bounded Borel function $f(x)$, we have that

$$\begin{aligned}
\int_{\mathbb{R}} f(x) d\rho^p(x) &= \int_{\mathbb{R}} f(x) \frac{1+x^2}{x^2} \mathbf{1}_{\mathbb{R}\setminus\{0\}}(x) d\sigma^p(x) \\
&= \int_{\mathbb{R}} f(p(x)) \frac{1+(p(x))^2}{p(x)^2} \mathbf{1}_{\mathbb{R}\setminus\{0\}}(p(x)) \frac{p(x)(p(x)+q(x))}{1+p(x)^2} d\sigma(x) \\
&= \int_{\mathbb{R}} f(p(x)) \mathbf{1}_{\mathbb{R}\setminus\{0\}}(p(x)) \frac{p(x)+q(x)}{p(x)} d\sigma(x) \\
&= \int_{\mathbb{R}} f(p(x)) \mathbf{1}_{\mathbb{R}\setminus\{0\}}(p(x)) \frac{1+x^2}{x^2} \mathbf{1}_{\mathbb{R}\setminus\{0\}}(x) d\sigma(x) \\
&= \int_{\mathbb{R}} f(p(x)) \mathbf{1}_{\mathbb{R}\setminus\{0\}}(p(x)) d\rho(x).
\end{aligned}$$

Therefore, $\rho^p$ is a Lévy measure and satisfies the conclusion in Lemma 5.1.3. Meanwhile, we can check the following fact:

$$\begin{aligned}
\int_{-1}^{1} |x| d\rho^p(x) &= \int_{\mathbb{R}} \mathbf{1}_{[-1,1]}(x) |x| d\rho^p(x) \\
&= \int_{\mathbb{R}} \mathbf{1}_{[-1,1]\setminus\{0\}}(p(x)) |p(x)| d\rho(x) \\
&= \int_{-1}^{1} \mathbf{1}_{[-1,1]\setminus\{0\}}(p(x)) |p(x)| d\rho(x) + \int_{\mathbb{R}\setminus[-1,1]} \mathbf{1}_{[-1,1]\setminus\{0\}}(p(x)) |p(x)| d\rho(x) \\
&\leqslant \|q\|_{C([-1,1])} \int_{-1}^{1} x^2 d\rho(x) + \int_{\mathbb{R}\setminus[-1,1]} \mathbf{1}_{\{x: p(x)\in[-1,1]\setminus\{0\}\}}(x) |p(x)| d\rho(x) \\
&\leqslant C \int_{\mathbb{R}} \min\{1, x^2\} d\rho(x) < \infty.
\end{aligned}$$

In other words, $\int_{-1}^{1} |p(x)| d\rho(x)$ is finite, which means that $\int_{]s,t]\times\mathbb{R}} p(x) dM(t,x)$ is a self-adjoint operator affiliated with $\mathcal{A}$. This property implies that the Lévy-Itô decomposition of $X^p(t)$ has the form (4.7) rather than the form (4.8). Thirdly, by the relation $\eta^p = \gamma^p + \int_{\mathbb{R}\setminus\{0\}} \frac{1+x^2}{x} (\mathbf{1}_{[-1,1]}(x) -$

$\frac{1}{1+x^2})d\sigma^p(x)$, we can deduce that

$$\begin{aligned}
\eta^p &= \int_{\mathbb{R}} \mathbf{1}_{\{p(x)\neq 0\}}(x)\frac{p(x)+q(x)}{1+(p(x))^2}d\sigma(x) + \int_{\mathbb{R}} \mathbf{1}_{\{p(x)=0\}}(x)\frac{p(x)+q(x)}{1+(p(x))^2}d\sigma(x) \\
&+ \int_{\mathbb{R}} \mathbf{1}_{\{p(x)\neq 0\}}(x)(p(x)+q(x))\left(\mathbf{1}_{\{-1\leqslant p(x)\leqslant 1\}}(x) - \frac{1}{1+(p(x))^2}\right)d\sigma(x) \\
&= \int_{\mathbb{R}} \mathbf{1}_{\{p(x)=0\}}(x)\frac{p(x)+q(x)}{1+(p(x))^2}d\sigma(x) + \int_{\mathbb{R}} \mathbf{1}_{\{p(x)\neq 0\}}(p(x)+q(x))\mathbf{1}_{\{-1\leqslant p(x)\leqslant 1\}}(x)d\sigma(x) \\
&= \int_{\mathbb{R}} \mathbf{1}_{\{p(x)=0\}}(x)\frac{p(x)+q(x)}{1+(p(x))^2}d\sigma(x) + \int_{\mathbb{R}} \mathbf{1}_{\{p(x)\neq 0\}}(p(x)+q(x))\mathbf{1}_{\{-1\leqslant p(x)\leqslant 1\}}(x)d\sigma(x) \\
&= aq(0) + \int_{\mathbb{R}} (p(x)+q(x))\mathbf{1}_{\{[-1,1]\setminus\{0\}\}}(p(x))d\sigma(x)
\end{aligned}$$

Here, we use the fact that $a = \sigma(\{0\})$. Finally, let us combine the three results we got above. Recall the Lévy-Itô decomposition of $X^p(t)$ with the free generating triplet $(a^p, \eta^p, \rho^p)$. Let $M^p$ be the free Poisson random measure on $(D, \mathcal{B}(D), Leb \otimes \rho^p)$. Notice that $p(x) = x^2 q(x)$, i.e. $\frac{1+x^2}{x^2} = \frac{p+q}{p}$. Notice that $\int_{-1}^{1} |p(x)| d\rho(x) < \infty$. Therefore, the last part of Lévy-Itô decomposition of $X^p(t)$ with respect to the free Poisson random measure $M^p$ can be simplified by the following computation.

$$\begin{aligned}
&\lim_{\epsilon \searrow 0}\left[\int_{]0,t]\times[|x|>\epsilon]} x dM^p(t,x) - \int_{]0,t]\times[\epsilon<|x|\leqslant 1]} x Leb\otimes\rho^p(t,x)\mathbf{1}_{\mathcal{A}}\right] \\
&= \int_{]0,t]\times[|x|>0]} x dM^p(t,x) - \int_{]0,t]\times[0<|x|\leqslant 1]} x Leb\otimes\rho^p(dt,dx)\mathbf{1}_{\mathcal{A}} \\
&= \int_{]0,t]\times\mathbb{R}} p(x)dM(t,x) - t\int_{\mathbb{R}} x\mathbf{1}_{[-1,1]}(x)d\rho^p(x)\mathbf{1}_{\mathcal{A}} \\
&= \int_{]0,t]\times\mathbb{R}} p(x)dM(t,x) - t\int_{\mathbb{R}} p(x)\mathbf{1}_{[-1,1]}(p(x))d\rho(x)\mathbf{1}_{\mathcal{A}} \\
&= \int_{]0,t]\times\mathbb{R}} p(x)dM(t,x) - t\int_{\mathbb{R}} p(x)\mathbf{1}_{[-1,1]}(p(x))\frac{1+x^2}{x^2}\mathbf{1}_{\mathbb{R}\setminus\{0\}}(x)d\sigma(x)\mathbf{1}_{\mathcal{A}} \\
&= \int_{]0,t]\times\mathbb{R}} p(x)dM(t,x) - t\int_{\mathbb{R}} (p(x)+q(x))\mathbf{1}_{[-1,1]\setminus\{0\}}(p(x))d\sigma(x)\mathbf{1}_{\mathcal{A}}.
\end{aligned}$$

Here, we employ the integration by substitution with respect to free Poisson random measures, which we proved in Lemma 5.1.3. Then, the final result of this theorem, the representation of

$X^p(t)$ in terms of the Lévy-Itô decomposition of $X(t)$, follows from the above computation and the formula of $\eta^p$. $\square$

## REFERENCES


[1] ANSHELEVICH, M. Free stochastic measures via noncrossing partitions. *Advances in Mathematics 155*, 1 (2000), 154–179.

[2] ANSHELEVICH, M. Free stochastic measures via noncrossing partitions II. *Pacific Journal of Mathematics 207*, 1 (2002), 13–30.

[3] ANSHELEVICH, M., AND WILLIAMS, J. D. Limit theorems for monotonic convolution and the chernoff product formula. *International Mathematics Research Notices 2014*, 11 (2013), 2990–3021.

[4] APPLEBAUM, D. *Lévy processes and stochastic calculus*. Cambridge University Press, 2009.

[5] AVRAM, F., AND TAQQU, M. S. Symmetric polynomials of random variables attracted to an infinitely divisible law. *Probability Theory and Related Fields 71*, 4 (1986), 491–500.

[6] BARNDORFF-NIELSEN, O. E., FRANZ, U., GOHM, R., KÜMMERER, B., AND THORBJØRNSEN, S. Quantum independent increment processes II. *Lecture Notes in Math 1866* (2006).

[7] BARNDORFF-NIELSEN, O. E., AND THORBJØRNSEN, S. The Lévy-Itô decomposition in free probability. *Probability Theory and Related Fields 131*, 2 (2005), 197–228.

[8] BARNDORFF-NIELSEN, O. E., THORBJØRNSEN, S., ET AL. Self-decomposability and Lévy processes in free probability. *Bernoulli 8*, 3 (2002), 323–366.

[9] BERCOVICI, H., PATA, V., AND BIANE, P. Stable laws and domains of attraction in free probability theory. *Annals of Mathematics 149* (1999), 1023–1060.

[10] NICA, A., AND SPEICHER, R. *Lectures on the Combinatorics of Free Probability*. No. v. 13 in Lectures on the combinatorics of free probability. Cambridge University Press, 2006.

[11] PECCATI, G., AND TAQQU, M. *Wiener Chaos: Moments, Cumulants and Diagrams: A survey with Computer Implementation*. Bocconi & Springer Series. Springer Milan, 2011.

[12] RVACEVA, E. On domains of attraction of multi-dimensional distributions. *Select. Transl. Math. Statist. and Probability 2* (1961), 183–205.