LOWER ALGEBRAIC K-THEORY OF RINGS

A Thesis

by

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ABSTRACT

This thesis is a first step towards a controlled algebraic K-theory. We give explicit formulas for the proof of Fundamental Theorem of Algebraic K-Theory. As a consequence, we provide explicit estimates on the control of propagation.

The first part of this thesis is an introduction to K_0 and K_1-groups of rings, where we develop necessary background materials.

In the second part of this thesis, we prove the Fundamental Theorem of Algebraic K-Theory by elementary means and give explicit formulas. A detailed discussion of propagation control is given at the end of this part.

In the last part of this thesis, we introduce negative algebraic K-theory and prove its Fundamental Theorem of Algebraic K-Theory.

This work is intended as a first step towards quantitative computations for lower algebraic K-theory.

DEDICATION

To my parents

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1. INTRODUCTION

Algebraic K-theory is an important branch of Mathematics, whose origins may date back to A. Grothendieck's work in reformulation of the Riemann-Roch theorem in algebraic geometry and Whitehead's construction of the Whitehead group in homotopy theory. Algebraic K-theory is the study of K-groups with connections and applications to geometry, topology and number theory. In this thesis, we are concerned with K_0-group, K_1-group and K_{-n}-groups for $n = 1, 2, \cdots$. For a detailed description of the history and ideas of lower algebraic K-theory, one can refer to [1, 2] and references therein.

In this thesis, we investigate the quantitative aspects of algebraic K-theory. This investigation is divided into two steps.

First, we prove the Fundamental Theorem of Algebraic K Theory by elementary means and give explicit formulas in the proof.

Theorem. *There is an isomorphism:*

$$K_1(R[t, t^{-1}]) \cong K_0(R) \oplus K_1(R) \oplus NK_1(R) \oplus NK_1(R)$$

where $R[t, t^{-1}]$ is the localization of the polynomial ring $R[t]$ and $NK_1(R)$ is the kernel of the nature map $K_1(R[t]) \longrightarrow K_1(R)$.

This theorem is of fundamental importance for it connects K_1-group, K_0-group and all negative K-groups. Actually, an explicit proof of the Fundamental Theorem of Algebraic K-Theory allows us to understand the quantitative properties of lower algebraic K-groups, which is important for computations.

We prove this theorem by proving there is a split short exact sequence:

$$0 \longrightarrow K_1(R[t]) \longrightarrow K_1(R[t, t^{-1}]) \xrightarrow{\partial} K_0(\text{Nil } R) \longrightarrow 0$$

where Nil R is the monoid of elements of the form (P, τ), where P is finitely generated projective R-module, and τ is a nilpotent endomorphism of P. The boundary map ∂ is given by

$$K_1(R[t,t^{-1}]) \ni [X] \longrightarrow [(R[t]^n/t^k XR[t]^n, t)] - [(R[t]^n/t^k R[t]^n, t)] \in K_0(\text{Nil } R),$$

where $R[t]^n/t^k XR[t]^n \cong PR^m$ for some idempotent matrix P.

Second, we discuss the propagation control of the boundary map ∂. This is inspired by the work of H. Oyono-Oyono and G. Yu on quantitative operator K-theory (cf. [3]).

By the virtue of filtered algebra, we give the abstract definition of propagation:

Definition. *A filtered algebra over commutative ring R, is a R-algebra A with a family of R-submodules (A_r), $r \in \mathbb{R}$, such that*

(1) $A_r \subseteq A_{r'}$, *if* $r \leq r'$

(2) $A_r A_{r'} \subseteq A_{r+r'}$

(3) $A = \bigcup_r A_r$

where the family (A_r), $r \in \mathbb{R}$ is called a filtration of A. Every elements of A_r is said to have propagation $\leq r$.

If no other specification, we assign the propagation of an element a to be the least number r such that $a \in A_r$.

For group G and ring R, RG carries a natural filtration by defining word length[1] on group G. This treatment also endows $M(RG[t,t^{-1}])$ with a filtration, and therefore when we consider group ring RG, matrices $X, P \in M(RG[t,t^{-1}])$ both have well-defined propagations. Our explicit formula allows us to estimate the propagation of P in terms of the propagation of X.

We have a brief introduction to negative K-theory at the last part of this thesis.

[1]This idea arose from geometric group theory, in which Cayley graph can be endowed with a length function, that gives Cayley graph the similar structure as we constructed.

2. REVIEW OF K_0 AND K_1 OF RINGS

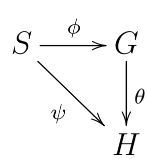
In this part, we are going to review some basic notions and consequences of K_0 and K_1 of rings. Unless specific explanations, in our discussion, all the rings have identities, all the ring homomorphisms are identity-preserving, all modules are unitary left modules, and all ideals are two-sided. With a little abuse of notation, isomorphism classes are always denoted as $[\cdot]$ unless other specification. The concrete meaning of $[\cdot]$ can be derived from context.

2.1 K_0 of Rings

There are many ways to define K_0-groups for rings. We will follow the traditional way, namely the group completion version. This is sufficient to talk about most problems in lower algebraic K-theory. Before the definition, we need some preparations.

2.1.1 Grothendieck Group

Theorem 2.1 (Grothendieck)**.** *For every abelian semigroup S, there is an abelian group $G = G(S)$ (now called Grothendieck group) with the semigroup homomorphism $\phi : S \longrightarrow G$, which satisfies the universal property that for any group H and semigroup homomorphism $\psi : S \longrightarrow H$, there is a unique group homomorphism $\theta : G \longrightarrow H$ such that $\psi = \theta \circ \phi$, or equivalently, the following diagram commutes:*



and if there is another group G' with semigroup homomorphism $\phi' : S \longrightarrow G'$ satisfies the same universal property, then there is an isomorphism $f : G \longrightarrow G'$ such that $\phi' = f \circ \phi$.

Proof. To prove the existence, let F be the free abelian group generated by S, let $\ell : S \longrightarrow F$ be the inclusion map, denote the inclusion image as $\langle x \rangle$ for $x \in S$, then define $G := F/N$, where N is the normal subgroup of F generated by all elements of the form $\langle x \rangle + \langle y \rangle - \langle x + y \rangle$ for $x, y \in S$.

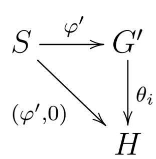
Let $\pi : \langle x \rangle \longmapsto [x]$ be the canonical map from F to G, define $\phi := \pi \circ \ell$. We are going to show G along with ϕ is what we need.

Actually, because F is free abelian group, for any abelian group H and homomorphism $\psi : S \longrightarrow H$, there is a unique homomorphism $\theta' : F \longrightarrow H$ such that $\theta' \circ \ell = \psi$. Because N is obviously contained in the kernel of θ', so an unique homomorphism $\theta : G \longrightarrow H$ such that $\theta \circ \phi = \psi$ is induced.

To prove the uniqueness, if G' with $\phi' : S \longrightarrow G'$ also satisfies this universal property, then there are homomorphisms α, β such that $\phi' = \alpha \circ \phi$ and $\phi = \beta \circ \phi'$, which imply

$$(\alpha \circ \beta) \circ \phi' = \phi'$$
$$(\beta \circ \alpha) \circ \phi = \phi.$$

It follows that $\alpha \circ \beta = 1_{\phi'(S)}$ and $\beta \circ \alpha = 1_{\phi(S)}$. By our construction, $\phi(S)$ generates G, because $\beta \circ \alpha$ is a homomorphism, so $\beta \circ \alpha = 1_G$. Then, we are going to show $\alpha \circ \beta = 1_{G'}$ by proving $\varphi'(S)$ generates G'. To do this, first, let G'' be the normal subgroup of G' generated by $\phi'(S)$. Define $H := G' \oplus (G'/G'')$. Then, there are two homomorphism $\theta_1 = (1, 0)$ and $\theta_2 = (1, q)$, where q is the quotient map, 1 is the identity map, that make the following diagrams commute:



for $i = 1, 2$. By universal property, we must have $\theta_1 = \theta_2$, so $q = 0$. It follows that $G' = G''$, and thus $\phi'(S)$ generates G'. Therefore $\alpha \circ \beta = 1_{G'}$, α is an isomorphism.

□

The Grothendieck group of semigroup S is also called the group completion of S. Actually, it is the way to define the integers from natural numbers.

Example 2.2. *For the semigroup $\mathbb{N}$ of natural number, $G(\mathbb{N}) = \mathbb{Z}$ is the group of integers.*

Corollary 2.3 ([4]). *Let S be an abelian semigroup, then*

(a) Every element of $G(S)$ has the form $[x]-[y]$ for $x, y \in S$.

(b) For any $[x],[y] \in G(S)$, $[x]=[y]$ if and only if $x+z=y+z$ for some $z \in S$.

Proof. (a) By our construction in Theorem 2.1, every element $[z]$ of $G(S)$ can be written as the difference of two finite sums, namely

$$[z]=\sum_{i=1}^{n}[a_i]-\sum_{j=1}^{m}[b_j]$$

where $a_i, b_j \in S$. Because $[a]+[b]=[a+b]$ for $a, b \in S$, let

$$x=\sum_{i=1}^{n} a_i, \qquad y=\sum_{j=1}^{m} b_j$$

therefore

$$[z]=\sum_{i=1}^{n}[a_i]-\sum_{j=1}^{m}[b_j]=[\sum_{i=1}^{n} a_i]-[\sum_{j=1}^{m} b_j]=[x]-[y].$$

(b) If $x+z=y+z$ for $x, y, z \in S$, then $[x]+[z]=[x+z]=[y+z]=[y]+[z]$, so $[x]=[y]$. If $[x]=[y]$, by Theorem 2.1, $\langle x\rangle-\langle y\rangle \in N$. It follows that

$$\langle x\rangle-\langle y\rangle=\sum_{i=1}^{n}(\langle a_i\rangle+\langle b_i\rangle-\langle a_i+b_i\rangle)-\sum_{j=1}^{m}(\langle a_j'\rangle+\langle b_j'\rangle-\langle a_j'+b_j'\rangle).$$

By transplanting negative terms to the other side, we get

$$\langle x\rangle+\sum_{i=1}^{n}\langle a_i+b_i\rangle+\sum_{j=1}^{m}(\langle a_j'\rangle+\langle b_j'\rangle)=\langle y\rangle+\sum_{i=1}^{n}(\langle a_i\rangle+\langle b_i\rangle)+\sum_{j=1}^{m}\langle a_j'+b_j'\rangle.$$

Because presently all the terms lie in the image of inclusion map from S to F, so we have

$$x+\sum_{i=1}^{n}(a_i+b_i)+\sum_{j=1}^{m}(a_j'+b_j')=y+\sum_{i=1}^{n}(a_i+b_i)+\sum_{j=1}^{m}(a_j'+b_j').$$

Let

$$z = \sum_{i=1}^{n}(a_i + b_i) + \sum_{j=1}^{m}(a'_j + b'_j)$$

then $x + z = y + z$.

$\square$

Although we do not need category theory in our discussion, we sometimes use categorical terminologies to simplify our statements.

Proposition 2.4. *G is a covariant functor from the category of abelian semigroup to the category of abelian group.*

Proof. For any seimigroup homomorphism $\alpha : S \longrightarrow S'$, by Theorem 2.1, we get the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & S' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ G(S) & \xrightarrow{\theta} & G(S') \end{array}$$

where θ is the unique homomorphism induced by ϕ and $\phi' \circ \alpha$. Define $G(\alpha) := \theta$. If α is an isomorphism (namely the identity morphism), then $S \cong S'$ and thus $G(S) \cong G(S')$ with isomorphism θ. Also, if there is additional semigroup homomorphism $\beta : S' \longrightarrow S''$, we have the following commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{\alpha} & S' & \xrightarrow{\beta} & S'' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi''} \\ G(S) & \xrightarrow{G(\alpha)} & G(S') & \xrightarrow{G(\beta)} & G(S'') \end{array}$$

where $G(\beta \circ \alpha) = G(\beta) \circ G(\alpha)$ by the uniqueness.

$\square$

2.1.2 Definition and Properties of $K_0(R)$

Definition 2.5. *Define* Proj R *as the abelian monoid of all isomorphism classes of finitely generated projective R-modules, with direct product $\oplus$ as the addition operation and the zero module* 0

as the identity element.

Remark 2.6. Proj R *is indeed a set. It is because for every finitely generated projective* R*-module* P*, there is a finitely generated projective* R*-module* Q *such that* $P \oplus Q \cong R^n$ *for some positive integer* n*, so* P *is isomorphic to a direct summand of* R^n *and thus we can speak of the set of classes of finitely generated* R*-modules with respect to isomorphism (cf. [5], Chapter II, §6.9).*

We are ready to define K_0-group of rings.

Definition 2.7. *For any ring R, define* $K_0(R) := G(\text{Proj } R)$.

Especially, this definition is for rings with identity. Sometimes, we need to define K_0-groups for rings without identity. We will generalized this definition after introducing relative K_0-groups, see Definition 2.19.

Corollary 2.8. *For any* $[A], [B] \in K_0(R)$*,* $[A] = [B]$ *if and only if* $A \oplus R^n \cong B \oplus R^n$ *for some integer* n*.*

Proof. First, we see if $A \oplus R^n \cong B \oplus R^n$ for some integer n, then $[A \oplus R^n] = [B \oplus R^n]$. Because $[A \oplus R^n] = [A] + [R^n]$ and $[B \oplus R^n] = [B] + [R^n]$ so $[A] = [B]$. In the other direction, assume $[A] = [B]$ in $K_0(R)$, by Corollary 2.3, we see $A \oplus P \cong B \oplus P$ for some finitely generated projective R-module P. Assume $P \oplus Q \cong R^n$, then $A \oplus R^n \cong B \oplus R^n$ as desired. □

Example 2.9. *If* R *is a division ring, then we see every finitely generated* R*-module is free with finite basis. However, the dimension of free* R*-module is the only isomorphism invariant*[1]*, which means* Proj $R \cong \mathbb{N}$ *and thus* $K_0(R) \cong \mathbb{Z}$*.*

Proposition 2.10. K_0 *can be defined as a covariant functor from the category of rings to the category of abelian groups.*

Proof. To see this, first, for any ring homomorphism $\varphi : R \longrightarrow R'$, define a homomorphism from Proj R to Proj R' by

$$[P] \longmapsto [R' \otimes_\varphi P],$$

[1] Any two free R-modules are isomorphic if they have same dimension.

where $R' \otimes_\varphi P$ means that, in this tensor product, R' is considered as a right R-module while the scalar multiplication is given by

$$(a, r) \longmapsto \varphi(r)a,$$

for $r \in R$ and $a \in R'$. To verify this map is well-defined, first, because P is a finitely generated projective R-module, so $P \oplus Q \cong R^n$ for some finitely generated R-module Q, and some integer n, then

$$(R' \otimes_\varphi P) \oplus (R' \otimes_\varphi Q) \cong R' \otimes_\varphi (P \oplus Q) \cong R' \otimes_\varphi R^n \cong (R' \otimes_\varphi R)^n \cong (R')^n,$$

so $R' \otimes_\varphi P$ is finitely generated projective R'-module. Assume $[P'] = [P]$ in Proj R, then $P' \cong P$, so $R' \otimes_\varphi P \cong R' \otimes_\varphi P'$, which implies $[R' \otimes_\varphi P] = [R' \otimes_\varphi P']$.

By Theorem 2.1, define $K_0(\varphi) := \varphi_* : K_0(R) \longrightarrow K_0(R')$ to be the unique homomorphism makes the following diagram commutes:

$$\begin{array}{ccc} \text{Proj } R & \longrightarrow & \text{Proj } R' \\ \phi \downarrow & & \downarrow \phi' \\ K_0(R) & \longrightarrow & K_0(R'). \end{array}$$

To check this functor is well-defined, first, if $R \cong R'$, then every finitely generated R-module is also a finitely generated R'-module and vice versa by this isomorphism. So there is a R'-module isomorphism $R' \otimes_\varphi P \cong P$, which means the homomorphism $[P] \longmapsto [R' \otimes_\varphi P]$ is an isomorphism. Because G is a covariant functor as we proved in Proposition 2.4, $K_0(R) \cong K_0(R')$.

Also by Proposition 2.4, we have $K_0(\varphi_1 \circ \varphi_2) = K_0(\varphi_1) \circ K_0(\varphi_2)$.

□

We are in a position to give alternative definition of K_0-group of rings by matrices, which make K_0-theory, to some extent, connect with linear algebra, and endows K_0-theory more computational characteristics.

Definition 2.11. *For ring R, let $M_n(R)$ be the ring of all $n \times n$ matrices on R. Define $M(R)$ as the union of the resulting sequence:*

$$M_1(R) \subset M_2(R) \subset \cdots \subset M_n(R) \subset \cdots$$

by identifying $g \in M_n(R)$ with

$$\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+1}(R).$$

Let $GL_n(R)$ be the group of $n \times n$ matrices on R. Define $GL(R)$ as the union of the resulting sequence:

$$GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_n(R) \subset \cdots$$

by identifying $g \in GL_n(R)$ with

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R).$$

Define $\mathrm{Idem}(R)$ *as the set of all idempotent matrices in $M(R)$, that is, $A \in \mathrm{Idem}(R)$ if and only if $A \in M(R)$ and $A^2 = A$.*

Remark 2.12. *$M(R)$ is also a ring, while $GL(R)$ is also a group. That is because, for example, for any $A, B \in M(R)$, assume A has dimension n, B has dimension m, and $n \geq m$, then B can be embedding into $M_n(R)$. So we can talk about all ring operations of A, B in $M_n(R)$, which implies $M(R)$ is a ring.*

We also say a R-module endomorphism α idempotent, if $\alpha^2 = \alpha$. The definition of idempotent for matrices is a special case of the definition for endomorphisms.

Theorem 2.13 ([6])**.** Proj R *is isomorphic to the monoid of conjugation orbits of $GL(R)$ on* $\mathrm{Idem}(R)$*, with zero matrix as the identity element, and with the semigroup operation induced* *by*

$$(A,B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

This monoid is denoted as $\overline{\text{Idem}(R)}$.

Proof. For any $[P] \in \text{Proj } R$, we have $P \oplus Q \cong R^n$ for some integer n, and for some finitely generated projective R-module Q. Assume this isomorphism is $f : P \oplus Q \longrightarrow R^n$. Consider the idempotent endomorphism $1\oplus 0$ on $P \oplus Q$, we see $f(1\oplus 0)f^{-1}$ is also an idempotent endomorphism on R^n. Because R^n is a free R-module, so there is an idempotent matrix A corresponding to $f(1 \oplus 0)f^{-1}$, then $AR^n \cong P$.

Define a homomorphism $g : \text{Proj } R \longrightarrow \overline{\text{Idem}(R)}$ by $[P] \longmapsto \overline{A}$ such that $AR^n \cong P$. To see this map is well-defined, let $g([Q]) = \overline{B}$, $[Q] = [P]$, we have

$$AR^n \cong P \cong Q \cong BR^m.$$

Assume this isomorphism is $\alpha : AR^n \longrightarrow BR^m$, which induces a homomorphism $\alpha' : R^n \longrightarrow R^m$ because

$$AR^n \oplus (1-A)R^n \cong R^n$$
$$BR^m \oplus (1-B)R^m \cong R^m$$

and by letting $\alpha' = 0$ on $(1-A)R^n$. It follows that there is a $m \times n$ matrix A' corresponding to α'. Similarly, α^{-1} induced a homomorphism $\beta : R^m \longrightarrow R^n$, and there is a corresponding $n \times m$ matrix B'. Under our definition, we see, in $M(R)$, $A'B' = B$, $B'A' = A$, $A' = AA' = A'B$, $B' = BB' = B'A$. Therefore,

$$\begin{pmatrix} 1-A & A' \\ B' & 1-B \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-A & A' \\ B' & 1-B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

where

$$\begin{pmatrix} 1-A & A' \\ B' & 1-B \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies in $M(R)$, A, B are in the same conjugation orbits of $GL(R)$, so $\overline{A} = \overline{B}$.

For any $[P], [Q] \in$ Proj R, $[P] + [Q] = [P \oplus Q]$. By our definition of semigroup operation on $\overline{\text{Idem}(R)}$,

$$g([P] + [Q]) = g([P \oplus Q]) = \begin{pmatrix} g([P]) & 0 \\ 0 & g([Q]) \end{pmatrix} = g([P]) + g([Q]).$$

Therefore g is indeed a semigroup homomorphism.

We are going to see g is an isomorphism by proving it has an inverse $g^{-1} : \overline{\text{Idem}(R)} \longrightarrow$ Proj R, given by $\overline{B} \longmapsto [BR^n]$, where B is an idempotent matrix in $M_n(R)$. To see this map is well-defined, assume $A = U^{-1}BU$, for some $U \in GL_n(R)$, then $AR^n \cong BR^n$ which means $[AR^n] = [BR^n]$. It is obvious the inverse of g. Also by our definition of semigroup operation, g^{-1} is a homomorphism.

□

Corollary 2.14. *$K_0(R) \cong G(\overline{\text{Idem}(R)})$, the Grothendieck group of $\overline{\text{Idem}(R)}$.*

As an applications of this equivalent definition of K_0-groups for rings, we prove the following proposition:

Proposition 2.15. *For rings R_1, R_2, $K_0(R_1 \times R_2) \cong K_0(R_1) \oplus K_0(R_2)$.*

Proof. It is obvious that $GL(R_1 \times R_2) = GL(R_1) \times GL(R_2), \mathrm{Idem}(R_1 \times R_2) = \mathrm{Idem}(R_1) \times \mathrm{Idem}(R_2)$. By Theorem 2.13, we see that Proj R is isomorphic to the monoid of conjugation orbits of $GL(R)$ on Idem(R), and then isomorphic to the monoid of conjugation orbits of $GL(R_1) \times GL(R_2)$ on $\mathrm{Idem}(R_1) \times \mathrm{Idem}(R_2)$, which is $\overline{\mathrm{Idem}(R_1)} \times \overline{\mathrm{Idem}(R_2)}$. Then, take Grothendieck group on both sides. □

2.1.3 Relative K_0-Groups

Definition 2.16. *Let R be a ring, with ideal I, define $D(R, I)$ as the subring of $R \times R$ such that*

$$D(R, I) := \{(x, y) \in R \times R : x - y \in I\}.$$

Define

$$K_0(R, I) := \ker \{(p_1)_* : K_0(D(R, I)) \longrightarrow K_0(R)\}$$

as the relative K_0-group of R and its ideal I, where $(p_1)_* = K_0(p_1)$, and $p_1 : D(R, I) \longrightarrow R$ is the projection onto the first coordinate.

Lemma 2.17. *Let R be a ring, and I an ideal of R. For any $A \in GL(R/I)$, the matrix*

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

can be lift to a matrix on $GL(R)$.

Proof. Actually, we have the decomposition:

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Assume B, C are the liftings of A and A^{-1}, then we see

$$\begin{pmatrix} 2B - BCB & -1 + BC \\ -CB + 1 & C \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -C & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a lifting of

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

because all matrices on right hand side belong to $GL(R)$. $\square$

Theorem 2.18 ([6])**.** *For ring R and ideal $I \subseteq R$, we have short exact sequence:*

$$K_0(R, I) \xrightarrow{(p_2)_*} K_0(R) \xrightarrow{q_*} K_0(R/I)$$

where $p_2 : D(R, I) \longrightarrow R$ is the projection onto the second coordinate, q is the quotient map, and $(p_2)_*$ is $K_0(p_2)$ restricted to $K_0(R, I)$ and $q_* = K_0(q)$.

Proof. For any element $[a] - [b]$ of $K_0(R, I)$, a, b are idempotent matrices on $D(R, I)$, which have the form $a = (a_1, a_2)$, $b = (b_1, b_2)$, where $a_1, a_2, b_1, b_2 \in \text{Idem}(R)$. It follows that

$$(p_2)_*([a] - [b]) = [a_2] - [b_2] \in K_0(R)$$

and

$$q_*([a_2] - [b_2]) = [\overline{a_2}] - [\overline{b_2}] \in K_0(R/I).$$

By definition of $K_0(R, I)$,

$$[a_1] - [b_1] = (p_1)_*([a] - [b]) = 0$$

then

$$[\overline{a_2}] - [\overline{b_2}] = [\overline{a_1}] - [\overline{b_1}] = 0$$

follows, which implies the image of $(p_2)_*$ is contained in the kernel of q_*.

In another direction, assume $[x]-[y] \in K_0(R)$, where x, y are idempotent matrices on R, thus

$$q_*([x]-[y]) = [\overline{x}] - [\overline{y}] = 0.$$

We assume $\overline{x}$ is similar to $\overline{y}$, otherwise, we can replace $\overline{x}$ and $\overline{y}$ by

$$\begin{pmatrix} \overline{x} & 0 \\ 0 & \overline{1_m} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \overline{y} & 0 \\ 0 & \overline{1_m} \end{pmatrix}$$

for some integer m. So, there is a $\overline{z}$ such that $\overline{x} = \overline{zyz^{-1}}$. It follows that

$$\begin{pmatrix} \overline{x} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \overline{z} & 0 \\ 0 & \overline{z^{-1}} \end{pmatrix} \begin{pmatrix} \overline{y} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{z^{-1}} & 0 \\ 0 & \overline{z} \end{pmatrix}.$$

By Lemma 2.17, there is a lifting of

$$\begin{pmatrix} \overline{z} & 0 \\ 0 & \overline{z^{-1}} \end{pmatrix}$$

to a matrix $h \in GL(R)$.

Let

$$s = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad t = h \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} h^{-1}$$

while $[s] = [x]$ and $[t] = [y]$ in $K_0(R)$. Because $\overline{s} = \overline{t}$ on R/I, which means (t, s) is an idempotent matrix on $D(R, I)$, and $[(t, s)] - [(t, t)]$ is the preimage in $K_0(R, I)$ of $[x] - [y] \in K_0(R)$, thus the kernel of q_* is contained in the image of $(p_2)_*$.

□

We sometimes need to handle rings without identity, especially when we handle a nontrivial ideal of a ring.

Definition 2.19. *For any ring R (which may not have identity), define the augmented ring R_+ as*

$R_+ := R \oplus \mathbb{Z}$, where the multiplication is defined as

$$(x, n) \cdot (y, m) = (xy + ny + mx, mn)$$

and the identity is $(0, 1)$.

Define $K_0(R)$ as

$$K_0(R) := \ker\{(p_2)_* : K_0(R_+) \longrightarrow K_0(\mathbb{Z})\},$$

where $p_2 : R_+ \longrightarrow \mathbb{Z}$ is the projection onto the second coordinate, and $K_0(\mathbb{Z}) \cong \mathbb{Z}$.

Remark 2.20. *The verification of the well-definition of R^+ is trivial. To see this definition is consistent with the our original one, let $K_0'(R)$ denoted our original definition of K_0-group of R. We first notice that if R has an identity, then $R_+ \cong R \times \mathbb{Z}$. Actually, there is an isomorphism $\alpha : R_+ \longrightarrow R \times \mathbb{Z}$ given by*

$$(x, n) \longmapsto (x + ne, n).$$

Then, we see that

$$K_0'(R_+) \cong K_0'(R \times \mathbb{Z}) \cong K_0'(R) \oplus \mathbb{Z},$$

where the kernel of the induced homomorphism $\rho_*' : K_0'(R) \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ coincides with $K_0'(R)$ and by definition of K_0, $K_0(R) \cong K_0'(R)$.

Theorem 2.21 (Excision). *Let R be a ring, and I an ideal of R, then $K_0(R, I) \cong K_0(I)$.*

Proof. Define a homomorphism $\gamma : I_+ \longrightarrow D(R, I)$ by

$$(x, n) \longmapsto (n \cdot 1, n \cdot 1 + x)$$

where 1 is the identity of R. Then, we can see there is a commutative diagram

$$\begin{array}{ccc} I_+ & \xrightarrow{\gamma} & D(R, I) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle p_1} \\ \mathbb{Z} & \xrightarrow{\ell} & R \end{array}$$

where ℓ is the inclusion map given by $n \longmapsto n \cdot 1$.

Because K_0 is a covariant functor[2], so above commutative diagram induces a new diagram:

$$\begin{array}{ccc} K_0(I_+) & \xrightarrow{\gamma_*} & K_0(D(R,I)) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle (p_1)_*} \\ \mathbb{Z} & \xrightarrow{\ell_*} & K_0(R). \end{array}$$

It follows that γ_* maps the kernel of ρ_* to the kernel of $(p_1)_*$. However, by our definition, $K_0(I)$ is the kernel of ρ_* and $K_0(R, I)$ is the kernel of $(p_1)_*$. By restricting γ_* to $K_0(I)$, we get a homomorphism $f : K_0(I) \longrightarrow K_0(R, I)$.

f is an isomorphism. The methods of the proof are similar to the methods used in the proof of Theorem 2.18, as we omit here. For details, one can refer to [6], Theorem 1.5.9.

□

[2] Because all the rings in this diagram have identities, K_0 can be used as covariant functor for this diagram as we have proved in Proposition 2.10.

2.2 K_1 of Rings

2.2.1 Definition and Properties of $K_1(R)$

Definition 2.22. *Define $K_1(R) := GL(R)/[GL(R), GL(R)]$, the abelianization of $GL(R)$, where $GL(R)$ is as defined in Definition 2.11 and $[GL(R), GL(R)]$ is the commutator subgroup of $GL(R)$.*

Proposition 2.23. *K_1 can be defined as a covariant functor from the category of rings to the category of abelian groups.*

Proof. To see this, for any ring homomorphism $\varphi : R \longrightarrow S$, define a group homomorphism $\varphi' : GL(R) \longrightarrow GL(S)$ by $A \longmapsto B$ where $b_{ij} = \varphi(a_{ij})$, a_{ij}, b_{ij} are (i, j)-entry of A, B respectively.

To verify φ' is well-defined, assume $A \in GL(R)$, to simplify the notation, denote $D := A^{-1}$, we claim that $\varphi'(A) \in GL(S)$, where the inverse is $\varphi'(D)$. Actually,

$$\begin{aligned}(\varphi'(A)\varphi'(D))_{ij} &= \sum_k \varphi(a_{ik})\varphi(d_{kj}) \\ &= \varphi(\sum_k a_{ik}d_{kj}) \\ &= \varphi((AD)_{ij}).\end{aligned}$$

Because $\varphi(1) = 1$, $\varphi(0) = 0$, so $(\varphi'(A)\varphi'(D))_{ij} = 1$ if $i = j$, otherwise, $(\varphi'(A)\varphi'(D))_{ij} = 0$. so $\varphi'(A)\varphi'(D)$ is the identity matrix. Similarly, $\varphi'(D)\varphi'(A)$ is the identity matrix, which means $\varphi'(A) \in GL(S)$.

To verify φ' is indeed a homomorphism, assume $A, C \in GL_n(R) \subseteq GL(R)$, $B = AC$, then we see

$$b_{ij} = \sum_k a_{ik}c_{kj},$$

therefore

$$\varphi(b_{ij}) = \sum_k f(a_{ik})f(c_{kj})$$

which implies that $\varphi'(AC) = \varphi'(A)\varphi'(C)$.

Then, define $K_1(\varphi) := \varphi_* : K_1(R) \longrightarrow K_1(S)$ to be the homomorphism induced by φ'.

For ring homomorphism $\varphi : R \longrightarrow S$ and $\psi : S \longrightarrow T$, by our definition, $(\psi \circ \varphi)' = \psi' \circ \varphi'$, which induced that $K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi)$. So, K_1 is a covariant functor.

□

Definition 2.24. *For integers $i \neq j$, define elementary matrix $e_{ij}(a) \in GL(R)$ to be the matrix whose entries on diagonal are all 1, the off-diagonal (i, j)-entry is a, and other entries are 0. The subgroup generated by all elementary matrices in $GL_n(R)$ is denoted by $E_n(R)$. The union of all $E_n(R)$ is denoted by $E(R)$, which is a subgroup of $GL(R)$.*

Remark 2.25. *By induction, we see every matrix that has the form*

$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix}$$

belongs to $E(R)$, because they can be decomposed as the product of elementary matrices.

Proposition 2.26 (Whitehead's Lemma)**.** $E(R) = [GL(R), GL(R)]$.

Proof. Because for any $e_{ij}(b) \in E(R)$ we have $e_{ij}(b)^{-1} = e_{ij}(-b)$, so for any $e_{ik}(a) \in E(R)$, we have

$$\begin{aligned} e_{ik}(a) &= e_{ij}(a)e_{jk}(1)e_{ij}(-a)e_{jk}(-1) \\ &= e_{ij}(a)e_{jk}(1)e_{ij}(a)^{-1}e_{jk}(1)^{-1} \end{aligned}$$

so

$$e_{ik}(a) \in [E(R), E(R)] \subseteq [GL(R), GL(R)]$$

which implies $E(R) \subseteq [GL(R), GL(R)]$. We are going to prove $[GL(R), GL(R)] \subseteq E(R)$.

Actually, for any $A, B \in GL(R)$, we have

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

The matrices on the left hand side are all belongs to $E(R)$, which implies

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in E(R)$$

so that $ABA^{-1}B^{-1} \in E(R)$, which implies $[GL(R), GL(R)] \subseteq E(R)$. □

Corollary 2.27. *For* $A \in GL(R)$, $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E(R)$.

Proof. Because we have

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Because all matrices in this decomposition belong to $E(R)$, so

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E(R).$$

□

Remark 2.28. *By Definition 2.22, the product of* $[A], [B] \in K_1(R)$ *is* $[AB]$*, but by Corollary 2.27,* *we see that*

$$[AB] = \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right].$$

Actually, this fact follows immediately from

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$$

where

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \in E(R)$$

and thus vanishes after taking isomorphic class.

2.2.2 Relative K_1-Groups

As we mention before, for any ring R and its ideal I, $D(R, I)$ is defined as

$$D(R, I) := \{(x, y) \in R \times R : x - y \in I\}.$$

We can continue to define $K_1(R, I)$:

Definition 2.29. *Define $K_1(R, I)$ as*

$$K_1(R, I) := \ker \{(p_1)_* : K_1(D(R, I)) \longrightarrow K_1(R)\}$$

where $p_1 : D(R, I) \longrightarrow R$ is the projection onto the first coordinate, $(p_1)_* = K_1(p_1)$.

Theorem 2.30 ([6])**.** *Let R be a ring, and I is an ideal of R, then we have the following exact sequence:*

$$K_1(R, I) \xrightarrow{(p_2)_*} K_1(R) \xrightarrow{q_*} K_1(R/I),$$

where $p_2 : D(R, I) \longrightarrow R$ is the projection onto the second coordinate ,$(p_2)_*$ is $K_1(p_2)$ restricted *to $K_1(R, I)$, q is quotient map, $q_* = K_1(q)$.*

Proof. By definition of $K_1(R, I)$, any element of $K_1(R, I)$ has the form $[(e, B)] \in K_1(R, I)$, where $e \in E(R)$, because we have

$$[(1, Be^{-1})] = [(e, B)][(e^{-1}, e^{-1})]$$

where $[(e^{-1}, e^{-1})] \in E(D(R, I))$. So any element of $K_1(R, I)$ has the form $[(1, B)] \in K_1(R, I)$, which also means $[\overline{1}] = [\overline{B}]$, so $q_*([B]) = [\overline{1}]$. So, the image of $(p_2)_*$ is contained in the kernel of q_*.

In another direction, assume $[B] \in K_0(R)$ and $[\overline{B}] = q_*([B]) = [\overline{1}]$, then $\overline{B} \in E(R/I)$, so $\overline{B}$ can be represented as a product of elementary matrices over R/I. However, because every elementary matrix over R/I can be lift to an elementary matrix over R, so $\overline{B}$ can be lift to a matrix $C \in E(R)$ because C is also a product of elementary matrices over R. At that time, we see $[(1, BC^{-1})] \in K_1(R, I)$, because $\overline{BC^{-1}} = \overline{1}$. Therefore $[(1, BC^{-1})]$ is the preimage of $[B]$. So, the kernel of q_* is contained in the image of $(p_2)_*$. □

Theorem 2.31 ([6]). *Let R be a ring, and I is an ideal of R, then there is an exact sequence:*

$$K_1(R, I) \xrightarrow{(p_2)_*} K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(R, I) \xrightarrow{(p_2)_*} K_0(R) \xrightarrow{q_*} K_0(R/I),$$

where $(p_2)_*$ is $K_0(p_2)$ (or $K_1(p_2)$) restricted to $K_0(R, I)$ (or $K_1(R, I)$), q is quotient map, $q_* = K_0(q)$(or $K_1(q)$), ∂ is the boundary map.

Proof. We are going to define the boundary map and prove the exactness at $K_1(R/I)$ and $K_0(R, I)$, then the conclusion follows by Theorem 2.18 and Theorem 2.30.

For any $\overline{A} \in GL(R/I)$, where A is an n-dimensional matrix on R. Define a $D(R, I)$-module

$$P(\overline{A}) := \{(x, y) \in R^n \times R^n : \overline{y} = \overline{Ax}\}$$

and the scalar multiplication is defined as

$$(r_1, r_2) \cdot (x, y) = (r_1 x, r_2 y).$$

Especially, we see $P(\overline{1}) \cong D(R, I)^n$, where $\overline{1}$ is the identity matrix. More generally, if $A \in GL(R)$, then $P(\overline{A}) \cong D(R, I)^n$, where the isomorphism from $P(\overline{1})$ to $P(\overline{A})$ is given by

$$(x, y) \longmapsto (A^{-1}x, y).$$

Also, for any $\overline{A} \in GL(R/I)$, by Lemma 2.17, the matrix

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

can be lift to some $B \in GL_{2n}(R)$, we have

$$P(\overline{A}) \oplus P(\overline{A^{-1}}) \cong P(\overline{B}) \cong D(R, I)^{2n}$$

which implies $P(\overline{A})$ is projective.

Define the boundary map $\partial : K_1(R/I) \longrightarrow K_0(R, I)$ as

$$\partial([\overline{A}]) := [P(\overline{A})] - [D(R, I)^n]$$

where n is the dimension of A.

One can see $(p_1)_*(\partial([\overline{A}])) = [R^n] - [R^n] = 0$, thus by definition of $K_0(R, I)$, $\partial([\overline{A}]) \in K_0(R, I)$. Also, for any elementary matrix $\overline{B} \in E(R/I)$, we see that

$$P(\overline{BA}) \cong P(\overline{AB}) \cong P(\overline{A}),$$

so the boundary map is well-defined.

To see ∂ is a homomorphism, for any $[\overline{A}], [\overline{B}] \in K_1(R/I)$, let

$$X := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

we have

$$\partial([\overline{A}][\overline{B}]) = \partial(\left[\overline{X}\right]) = [P(\overline{X})] - [D(R,I)^{2n}] = [P(\overline{A})] - [D(R,I)^n] + [P(\overline{B})] - [D(R,I)^n]$$

which means $\partial([\overline{A}][\overline{B}]) = \partial([\overline{A}]) + \partial([\overline{B}])$.

Next, we prove the exactness at $K_1(R/I)$:

For any $[\overline{A}] \in q_*(K_1(R))$, where $A \in GL(R)$, by our previous discussion, $\partial([\overline{A}]) = 0$. So, the image of q_* is contained in the kernel of ∂.

In the other direction, for any $[\overline{A}] \in K_1(R/I)$ such that $\partial([\overline{A}]) = 0$, we have $[P(\overline{A})] = [D(R,I)^n]$. Assume $P(\overline{A}) \cong D(R,I)^n$, otherwise, redefine

$$A := \begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix}$$

then $P(\overline{A}) \cong D(R,I)^{m+n}$.

Let f be an isomorphism from $D(R,I)^n$ to $P(\overline{A})$. Because $D(R,I)^n$ and $P(\overline{A})$ are both finitely generated $D(R,I)$-module, so there is a matrix (B,C) on $D(R,I)$ corresponding to f, namely

$$f(x,y) = (B,C)(x,y)$$

for any $(x,y) \in D(R,I)^n$. It follows from the definition of $P(\overline{A})$ that $\overline{ABx} = \overline{Cy}$. By definition of $D(R,I)$, we have $\overline{x} = \overline{y}$. So, $\overline{AB} = \overline{C}$ by arbitrariness of x. Because B, C are invertible, so $CB^{-1} \in GL(R)$, and $[CB^{-1}] \in K_1(R)$ is the preimage of $[\overline{A}]$. So, the kernel of ∂ is contained in the image of q_*.

Then, we prove the exactness at $K_0(R, I)$:

It is obvious that $(p_2)_*(\partial([\overline{A}])) = [R^n] - [R^n] = 0$ so the image of ∂ is contained in the kernel of $(p_2)_*$.

In the other direction, for any $[Q]-[D(R,I)^n] \in K_0(R,I)$ we have $(p_2)_*([Q]-[D(R,I)^n]) = 0$ where Q is a finitely generated $D(R,I)$-module. Because, by definition of $K_0(R,I)$, $(p_1)_*([Q] - [D(R,I)^n]) = 0$. It follows that

$$[(p_1)_*([Q])] = [R^n]$$
$$[(p_2)_*([Q])] = [R^n].$$

We assume that

$$(p_1)_*([Q]) \cong (p_2)_*([Q]) \cong R^n,$$

otherwise, by the same trick as before, direct summing some finitely generated $D(R,I)$-module on Q. It follows that Q can be represented as $P(\overline{G})$, for some $\overline{G} \in GL(R/I)$. So, $\overline{G}$ is the preimage of $[Q] - [D(R,I)^n]$, which means the kernel of $(p_2)_*$ is contained in the image of ∂.

$\square$

Corollary 2.32. *By Theorem 2.21, we have the following exact sequence*

$$K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(I) \xrightarrow{\ell_*} K_0(R) \xrightarrow{q_*} K_0(R/I),$$

where ℓ is the inclusion map, and $\ell_* = K_0(\ell)$.

In the next section, we will extend this exact sequence to arbitrary long to the right.

3. FUNDAMENTAL THEOREM OF ALGEBRAIC K-THEORY

3.1 Proof of the Fundamental Theorem of Algebraic K-Theory

In this section, we are going to prove the Fundamental Theorem of Algebraic K-Theory. Before giving the proof, we need more structures.

Definition 3.1. *Define* Nil R *as the abelian monoid of isomorphism classes of ordered pairs* (P, τ)*, where* P *are finitely generated projective* R*-modules,* τ *are nilpotent endomorphisms of* P*, and the homomorphisms* $(P, \tau) \longrightarrow (P', \tau')$ *are* R*-module homomorphisms such that the following diagram commutes:*

$$\begin{array}{ccc} P & \longrightarrow & P' \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau'} \\ P & \longrightarrow & P'. \end{array}$$

$[(0,0)]$ *is the identity element, where the first 0 means zero* R*-module, the second 0 means zero homomorphism. The addition operation of this semigroup is defined as*

$$[(P, \tau)] + [(Q, \nu)] = [(P \oplus Q, \tau \oplus \nu)].$$

Remark 3.2. *First,* Nil R *is indeed a set for the similar reason as in Remark 2.6.*

Second, the addition operation is well-defined. To check this, assume $[(P, \tau)] = [(P', \tau')]$ *and* $[(Q, \nu)] = [(Q', \nu')]$*, the following diagram commutes:*

$$\begin{array}{ccc} P \oplus Q & \xrightarrow{\cong} & P' \oplus Q' \\ {\scriptstyle \tau \oplus \nu}\downarrow & & \downarrow{\scriptstyle \tau' \oplus \nu'} \\ P \oplus Q & \xrightarrow{\cong} & P' \oplus Q' \end{array}$$

which implies $[(P \oplus Q, \tau \oplus \nu)] = [(P' \oplus Q', \tau' \oplus \nu')]$*. Also, we have*

$$[(P, \tau)] + [(0,0)] = [(P \oplus 0, \tau \oplus 0)] = [(P, \tau)]$$

because of the following commutative diagram:

$$\begin{array}{ccc} P \oplus 0 & \xrightarrow{p} & P \\ {\scriptstyle \tau\oplus 0}\downarrow & & \downarrow{\scriptstyle \tau} \\ P \oplus 0 & \xrightarrow{p} & P \end{array}$$

where p *is the projection map. Similarly, we get* $[(0,0)] + [(P,\tau)] = [(P,\tau)]$*. The verification that addition is associative is trivial. To see addition is commutative, we claim that* $[(P \oplus Q, \tau \oplus \nu)] = [(Q \oplus P, \nu \oplus \tau)]$ *by the commutative diagram:*

$$\begin{array}{ccc} P \oplus Q & \xrightarrow{\cong} & Q \oplus P \\ {\scriptstyle \tau\oplus \nu}\downarrow & & \downarrow{\scriptstyle \nu\oplus\tau} \\ P \oplus Q & \xrightarrow{\cong} & Q \oplus P. \end{array}$$

Short exact sequences in Nil R do not split in general. To overcome this difficult, we give the following definition of $K_0(\text{Nil } R)$:

Definition 3.3. *Define* $K_0(\text{Nil } R) := F_R/N_R$*, where* F_R *is the free abelian group generated by elements of* Nil R*, and* N_R *is the normal subgroup of* F_R *generated by elements of the form* $[(P_1,\tau_1)] + [(P_3,\tau_3)] - [(P_2,\tau_2)]$*, if there is a short exact sequence:*

$$0 \longrightarrow (P_1,\tau_1) \longrightarrow (P_2,\tau_2) \longrightarrow (P_3,\tau_3) \longrightarrow 0.$$

Remark 3.4. *First, for any* $[(P,\tau)], [(Q,\nu)] \in K_0(\text{Nil } R)$*, since*

$$0 \longrightarrow (P,\tau) \longrightarrow (P \oplus Q, \tau \oplus \nu) \longrightarrow (Q,\nu) \longrightarrow 0$$

is exact, so $[(P,\tau)] + [(Q,\nu)] = [(P \oplus Q, \tau \oplus \nu)]$ *in* $K_0(\text{Nil } R)$*.*

The next Proposition is parallel to Corollary 2.8.

Proposition 3.5. *For any* $[(P_1,\tau)], [(P_2,\tau_2)] \in K_0(\text{Nil } R)$*,* $[(P_1,\tau_1)] = [(P_2,\tau_2)]$ *if and only if* *there are short exact sequences in* Nil R*:*

$$0 \longrightarrow (Q', \nu') \longrightarrow (Q_1, \nu_1) \longrightarrow (Q'', \nu'') \longrightarrow 0$$

$$0 \longrightarrow (Q', \nu') \longrightarrow (Q_2, \nu_2) \longrightarrow (Q'', \nu'') \longrightarrow 0$$

such that $(P_1 \oplus Q_1, \tau_1 \oplus \nu_1) \cong (P_2 \oplus Q_2, \tau_2 \oplus \nu_2)$.

Proof. To prove the sufficiency, by those two short exact sequences, we have $[(Q_1, \nu_1)] = [(Q_2, \nu_2)] \in K_0(\text{Nil } R)$. It follows that

$$\begin{aligned}[(P_1, \tau_1)] &= [(P_1 \oplus Q_1, \tau_1 \oplus \nu_1)] - [(Q_1, \nu_1)] \\ &= [(P_2 \oplus Q_2, \tau_2 \oplus \nu_2)] - [(Q_2, \nu_2)] \\ &= [(P_2, \tau_2)].\end{aligned}$$

To prove the necessity, for any $[(P_1, \tau_1)] = [(P_2, \tau_2)]$ in $K_0(\text{Nil } R)$, we have

$$\begin{aligned}&[(P_1, \tau_1)] + [(D_1', \gamma_1')] + [(D_1'', \gamma_1'')] - [(D_1, \gamma_1)] \\ &= [(P_2, \tau_2)] + [(D_2', \gamma_2')] + [(D_2'', \gamma_2'')] - [(D_2, \gamma_2)]\end{aligned}$$

in the free abelian group F_R, where there are short exact sequences:

$$0 \longrightarrow (D_1', \gamma_1') \longrightarrow (D_1, \gamma_1) \longrightarrow (D_1'', \gamma_1'') \longrightarrow 0$$

$$0 \longrightarrow (D_2', \gamma_2') \longrightarrow (D_2, \gamma_2) \longrightarrow (D_2'', \gamma_2'') \longrightarrow 0$$

and thus

$$[(P_1 \oplus D_1' \oplus D_1'' \oplus D_2, \tau_1 \oplus \gamma_1' \oplus \gamma_1'' \oplus \gamma_2)]$$
$$= [(P_2 \oplus D_2' \oplus D_2'' \oplus D_1, \tau_2 \oplus \gamma_2' \oplus \gamma_2'' \oplus \gamma_1)]$$

in Nil R, so

$$(P_1 \oplus D_1' \oplus D_1'' \oplus D_2, \tau_1 \oplus \gamma_1' \oplus \gamma_1'' \oplus \gamma_2)$$
$$\cong (P_2 \oplus D_2' \oplus D_2'' \oplus D_1, \tau_2 \oplus \gamma_2' \oplus \gamma_2'' \oplus \gamma_1).$$

Let

$$(Q', \nu') = (D_1' \oplus D_2', \gamma_1' \oplus \gamma_2')$$
$$(Q'', \nu'') = (D_1'' \oplus D_2'', \gamma_1'' \oplus \gamma_2'')$$
$$(Q_1, \nu_1) = (D_1' \oplus D_1'' \oplus D_2, \gamma_1' \oplus \gamma_1'' \oplus \gamma_2)$$
$$(Q_2, \nu_2) = (D_2' \oplus D_2'' \oplus D_1, \gamma_2' \oplus \gamma_2'' \oplus \gamma_1)$$

then we see

$$(P_1 \oplus Q_1, \tau_1 \oplus \nu_1) \cong (P_2 \oplus Q_2, \tau_2 \oplus \nu_2)$$

and there are short exact sequences:

$$0 \longrightarrow (Q', \nu') \longrightarrow (Q_1, \nu_1) \longrightarrow (Q'', \nu'') \longrightarrow 0$$
$$0 \longrightarrow (Q', \nu') \longrightarrow (Q_2, \nu_2) \longrightarrow (Q'', \nu'') \longrightarrow 0.$$

□

Corollary 3.6. *$K_0(\text{Nil } R) \cong K_0(R) \oplus \text{Nil}_0(R)$, where $\text{Nil}_0(R)$ is the kernel of the forgetful map $F : K_0(\text{Nil } R) \longrightarrow K_0(R)$, that sends every $[(P, \tau)]$ to $[P]$.*

Proof. This can be done by proving that $K_0(R)$ embeds into $K_0(\text{Nil } R)$ as a direct sum. Define $\upsilon : K_0(R) \longrightarrow K_0(\text{Nil } R)$ as the homomorphism induced by $[P] \longmapsto [(P, 0)]$. Because F is the left inverse of υ, so $K_0(R)$ embeds in $K_0(\text{Nil } R)$ as a direct sum via υ, and $K_0(\text{Nil } R) \cong K_0(R) \oplus \text{Nil}_0(R)$.

□

Proposition 3.7. $\text{Nil}_0(R)$ *is generated by elements of form* $[(R^n, \nu)] - [(R^n, 0)]$.

Proof. First, because $\text{Nil}_0(R)$ is generated by elements of form $[(P_1, \tau_1)] - [(P_2, \tau_2)]$, such that $[P_1] = [P_2]$, so $P_1 \oplus Q \cong P_2 \oplus Q$ for some finitely generated projective R-module Q.

Therefore we have

$$
\begin{aligned}
[(P_1, \tau_1)] - [(P_2, \tau_2)] &= ([(P_1, \tau_1)] + [(Q, 0)]) - ([(P_2, \tau_2)] + [(Q, 0)]) \\
&= [(R^n, \tau_1')] - [(R^n, \tau_2')] \\
&= ([(R^n, \tau_1')] - [(R^n, 0)]) - ([(R^n, \tau_2')] - [(R^n, 0)]),
\end{aligned}
$$

which implies $\text{Nil}_0(R)$ is generated by elements of form $[(R^n, \nu)] - [(R^n, 0)]$. □

Proposition 3.8. *For any finitely generated projective R-module P, there is a natural homomorphism from* $\text{Aut}(P)$ *to* $K_1(R)$*, which send* $\alpha \in \text{Aut}(P)$ *to an element of* $K_1(R)$ *that is induced by* $\alpha \oplus 1 \in \text{Aut}(P \oplus Q)$ *and the isomorphism* $P \oplus Q \cong R^n$ *for some integer* n.

Give an isomorphism $f : P \oplus Q \longrightarrow R^n$*, then the image of this natural homomorphism of* $\alpha \in \text{Aut}(P)$ *can be represented as* $[f(\alpha \oplus 1)f^{-1}] \in K_1(R)$.

Proof. To prove this map is well-defined, first, we prove that this map is independent of choice of the isomorphisms $P \oplus Q \cong R^n$. Assume there are two different isomorphism $f, g : P \oplus Q \longrightarrow R^n$, assume their corresponding natural homomorphism images are $[A], [B]$ respectively, where

$$
\begin{aligned}
A &= f(\alpha \oplus 1_Q)f^{-1} \\
B &= g(\alpha \oplus 1_Q)g^{-1}.
\end{aligned}
$$

It follows that $B = (gf^{-1})A(gf^{-1})^{-1}$. Because $gf^{-1} \in GL(R)$, so in $K_1(R)$, we have $[B] = [A]$.

Second, we prove that if $P \oplus Q$ is replaced by $P \oplus Q \oplus R^j$ then the corresponding image in $K_1(R)$ is the same as $[A] \in K_1(R)$ corresponding to $P \oplus Q$. Actually, the correspondence image of $P \oplus Q \oplus R^j$ is

$$\left[\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right] = [A],$$

where 1 is the identity on R^j.

Third, if there is $P \oplus Q' \cong R^m$, without loss of generosity, assume $m \geq n$, then by the second part, we can replace $P \oplus Q$ by $P \oplus Q \oplus R^{m-n}$ so that

$$P \oplus Q \oplus R^{m-n} \cong P \oplus Q' \cong R^m.$$

Therefore there is an isomorphism $T : P \oplus Q \oplus R^{m-n} \longrightarrow P \oplus Q'$. Assume the corresponding image of $\alpha \oplus 1_Q \oplus 1_{R^{m-n}}$ is A, the corresponding image of $\alpha \oplus 1_{Q'}$ is B, namely, there are isomorphisms f, g such that

$$A = f(\alpha \oplus 1_Q \oplus 1_{R^{m-n}})f^{-1}$$
$$B = g(\alpha \oplus 1_Q)g^{-1}.$$

Because $\alpha \oplus 1_{Q'} = T^{-1}(\alpha \oplus 1_Q \oplus 1_{R^{m-n}})T$, so $B = (fTg^{-1})^{-1}A(fTg^{-1})$. Also, because $fTg^{-1} \in GL(R)$, so $[B] = [A]$.

□

Lemma 3.9. *If α is an automorphism of $R[t, t^{-1}]^n$, which maps $R[t]^n$ into $R[t]^n$, then $R[t]^n/\alpha R[t]^n$ is finite generated projective module over R.*

Proof. Assume the inverse of α is β, then, $t^k\beta$ is an endomorphism on $R[t]^n$ for large enough k. Denote $e_i \in R[t]^n$, as the vector whose i index equals 1, and 0 otherwise, $i = 1, \cdots, n$. Then we have $\beta t^k e_i = t^k \beta e_i \in R[t]^n$, which means $t^k e_i \in \alpha R[t]^n$ for all i. So, the generators of

$R[t]^n/\alpha R[t]^n$ are contained in $\{t^j e_i\}$, $i = 1, \cdots, n$, $j = 1, \cdots, k-1$, and thus $R[t]^n/\alpha R[t]^n$ a finite generated R-module.

To verify $R[t]^n/\alpha R[t]^n$ is projective R-module, we see for n-dimensional elementary matrix $e_{ij}(a) \in E(R[t, t^{-1}])$, we have the short exact sequence:

$$0 \longrightarrow (R[t]^{n-1} + t^k e_{ij}(a)R[t]^n)/t^k e_{ij}(a)R[t]^n \longrightarrow R[t]^n/t^k e_{ij}(a)R[t]^n$$

$$\longrightarrow R[t]^n/(R[t]^{n-1} + t^k e_{ij}(a)R[t]^n) \longrightarrow 0,$$

where $R[t]^{n-1}$ is considered as the embedding image in $R[t]^n$, the homomorphisms[1]

$$(R[t]^{n-1} + t^k e_{ij}(a)R[t]^n)/t^k e_{ij}(a)R[t]^n \longrightarrow R[t]^n/t^k e_{ij}(a)R[t]^n$$

and

$$R[t]^n/t^k e_{ij}(a)R[t]^n \longrightarrow R[t]^n/(R[t]^{n-1} + t^k e_{ij}(a)R[t]^n)$$

are both canonical maps.

Because $R[t] \cong R[t]^n/R[t]^{n-1}$, then the homomorphism given by the composition:

$$R[t]^n \longrightarrow R[t]^n/R[t]^{n-1} \xrightarrow{\cong} R[t]$$

induces an isomorphism $R[t]^n/(R[t]^{n-1} + t^k e_{ij}(a)R[t]^n) \cong R[t]/t^k R[t]$.

In addition, we have isomorphism

$$(R[t]^{n-1} + t^k e_{ij}(a)R[t]^n)/t^k e_{ij}(a)R[t]^n \cong R[t]^{n-1}/(R[t]^{n-1} \cap t^k e_{ij}(a)R[t]^n)$$

induced by projection onto the first $n-1$ coordinates, where

$$R[t]^{n-1}/(R[t]^{n-1} \cap t^k e_{ij}(a)R[t]^n) = R[t]^{n-1}/t^k R[t]^{n-1}.$$

[1]They are not only R-module homomorphisms but also $R[t]$-module homomorphisms. We will use this fact soon.

To sum up, there is a short exact sequence:

$$0 \longrightarrow R[t]/t^k R[t] \longrightarrow R[t]^n/t^k e_{ij}(a)R[t]^n \longrightarrow R[t]^{n-1}/t^k R[t]^{n-1} \longrightarrow 0\ .$$

Because $R[t]^{n-1}/t^k R[t]^{n-1}$ is free and thus projective R-module, so this sequence is split exact, and thus

$$R[t]^n/t^k e_{ij}(a)R[t]^n \cong (R[t]/t^k R[t]) \oplus (R[t]^{n-1}/t^k R[t]^{n-1}) = R[t]^n/t^k R[t]^n.$$

So, by induction, for any $e \in E(R[t, t^{-1}])$, we have

$$[R[t]^n/t^k eR[t]^n] = [R[t]^n/t^k R[t]^n] \tag{3.1}$$

for large enough integer k. The similar result that

$$[R[t]^n/et^k R[t]^n] = [R[t]^n/t^k R[t]^n] \tag{3.2}$$

also holds.

However,

$$(R[t]^n/\alpha R[t]^n) \oplus (R[t]^n/t^k\beta R[t]^n) \cong R[t]^{2n}/(\alpha \oplus t^k\beta)R[t]^{2n}$$

while

$$\begin{pmatrix} t^k & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^k\beta & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & t^k\beta \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

and the matrices on the right hand side except $\begin{pmatrix} \alpha & 0 \\ 0 & t^k\beta \end{pmatrix}$ belong to $E(R[t,t^{-1}])$, so we get

$$[R[t]^{2n}/(\alpha \oplus t^k\beta)R[t]^{2n}] = [R[t]^{2n}/(t^k \oplus -1)R[t]^{2n}],$$

where $R[t]^{2n}/(t^k \oplus -1)R[t]^{2n}$ is obviously free R-module. It follows that

$$(R[t]^{2n}/(\alpha \oplus t^k\beta)R[t]^{2n}) \oplus R^m \cong (R[t]^{2n}/(t^k \oplus -1)R[t]^{2n}) \oplus R^m.$$

So, $R[t]^n/\alpha R[t]^n$ is embedded into a free R-module as a direct sum, $R[t]^n/\alpha R[t]^n$ is projective R-module.

□

The following Lemma is due to H. Bass:

Lemma 3.10 ([6]). *For ring R, we have the following propositions:*

(a) Every matrix X in $GL(R[t])$ can be reduced, modulo $E(R[t])$ and $GL(R)$, to the form $1 + Bt$, where B is a nilpotent matrix on R.

(b) Every matrix X in $GL(R[t,t^{-1}])$ can be reduced, modulo $E(R[t,t^{-1}])$ and $GL(R)$ to the form

$$(1 + A(t-1)) \begin{pmatrix} t^{-k} & 0 \\ 0 & 1 \end{pmatrix}$$

in which A is matrix on R such that $A = P + N$, with idempotent P, nilpotent N such that $PN = NP$.

Proof. (a) We see that $X = X_0 + tX_1 + \cdots + t^n X_n$, where $X_0, \cdots, X_n$ are matrix on R. We claim that X can be reduced, modulo $E(R[t])$ and $GL(R)$ to a matrix polynomial whose degree less than n. Actually, we have

$$\begin{aligned}
[X] &= \left[\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 1 & t^{n-1}X_n \\ 0 & 1 \end{pmatrix}\right] \left[\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}\right] \left[\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} X & t^{n-1}X_n \\ 0 & 1 \end{pmatrix}\right] \left[\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} X - t^n X_n & t^{n-1}X_n \\ -t & 1 \end{pmatrix}\right]
\end{aligned}$$

by modulo $E(R[t])$.

However the last matrix can be represented as a matrix polynomial with degree less than n. So, by induction, we can prove for any X, it can be reduced to the form $B_0 + B_1 t$. If $B_1 = 0$, then the conclusion is obvious, if $B_1 \neq 0$, then because the polynomial $B_0 + B_1 t \in GL(R[t])$, taking $t = 0$, we see $B_0 \in GL(R)$. By factoring out B_0, X can be reduced to $1 + Bt$ where $B = B_0^{-1} B_1$.

Because this matrix is invertible, assume its inverse is $C_0 + \cdots + C_j t^j$, namely

$$(1 + Bt)(C_0 + \cdots + C_j t^j) = (C_0 + \cdots + C_j t^j)(1 + Bt) = 1$$

By straightforward computation and comparing the coefficients of terms, we get

$$\begin{aligned}
&C_0 = 1 \\
&BC_0 + C_1 = C_0 B + C_1 = 0 \\
&\cdots \quad \cdots \\
&BC_{j-1} + C_j = C_{j-1}B + C_j = 0 \\
&BC_j = C_j B = 0
\end{aligned}$$

which implies $C_i = (-B)^i$. Then, because $C_{j+1} = 0$, so $B^{j+1} = 0$, so B is nilpotent.

(b) Similarly, we can write $X \in GL(R[t, t^{-1}])$ as

$$X = (X_0 + X_1 t + X_2 t^2 + \cdots + X_n t^n)t^{-k},$$

while all X_i are matrices on R. By the same trick as in (a), X can be reduced to the form

$$(B_0 + B_1 t)\begin{pmatrix} t^{-k} & 0 \\ 0 & 1 \end{pmatrix} = ((B_0 + B_1) + B_1(t-1))\begin{pmatrix} t^{-k} & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from the fact that X is invertible that $((B_0 + B_1) + B_1(t-1))$ is invertible, then we claim that $B_0 + B_1$ is invertible in $R([t, t^{-1}])$.

To see this, assume the inverse of $(B_0 + B_1) + B_1(t-1)$ is the Laurent polynomial Y, therefore

$$((B_0 + B_1) + B_1(t-1))Y = Y((B_0 + B_1) + B_1(t-1)) = 1$$

then let $t = 1$, we got

$$Y'(B_0 + B_1) = (B_0 + B_1)Y' = 1$$

where Y' is the value of Y when $t = 1$, which implies $B_0 + B_1$ is invertible. Factor out $B_0 + B_1$, X can be reduced to the form

$$(1 + A(t-1))\begin{pmatrix} t^{-k} & 0 \\ 0 & 1 \end{pmatrix}.$$

Assume the inverse of $1 + A(t-1)$ is $C_{-i}t^{-i} + \cdots + C_0 + \cdots + C_j t^j$, therefore

$$(1 + A(t-1))(C_{-i}t^{-i} + \cdots + C_0 + \cdots + C_j t^j) = 1$$

and

$$(C_{-i}t^{-i} + \cdots + C_0 + \cdots + C_j t^j)(1 + A(t-1)) = 1$$

By straightforward computation and comparing the coefficients of terms, we got

$$\begin{aligned}
&(1-A)C_{-i} = C_{-i}(1-A) = 0 \\
&(1-A)C_{-i+1} + AC_{-i} = C_{-i+1}(1-A) + C_{-i}A = 0 \\
&\cdots \quad \cdots \\
&(1-A)C_0 + AC_{-1} = C_0(1-A) + C_{-1}A = 1 \\
&\cdots \quad \cdots \\
&(1-A)C_j + AC_{j-1} = C_j(1-A) + C_{j-1}A = 0 \\
&AC_j = C_jA = 0
\end{aligned}$$

Multiply $1-A$ to the second line both from the left and right, since $1-A$ commutes with A, we got

$$(1-A)^2 C_{-i+1} = C_{-i+1}(1-A)^2 = 0.$$

Continuous this process, we have

$$(1-A)^i C_{-1} = C_{-1}(1-A)^i = 0.$$

Similarly,

$$A^{j+1}C_0 = C_0 A^{j+1} = 0,$$

so

$$0 = (A(1-A))^{i+j+1}((1-A)C_0 + AC_{-1}) = (A(1-A))^{i+j+1}$$

which shows, by induction, that $A(1-A)$ is nilpotent.

To show A can be written as $A = P + N$, where P is idempotent, N nilpotent, assume $A^n(1-A)^n = (A(1-A))^n = 0$, then because x^n and $(1-x)^n$ are relatively prime in $\mathbb{Z}[x]$, so there are polynomials p, q such that $p(x)x^n + q(x)(1-x)^n = 1$. Let[2] $P = p(A)A^n$, $N = A - P$, then we see

$$P^2 - P = P(1-P) = p(A)A^n q(A)(1-A)^n = p(A)q(A)A^n(1-A)^n = 0$$

which means P is idempotent. Because

$$N = A - p(A)A^n = A(1 - p(A)A^{n-1})$$
$$N = -(1-A) + (1-P) = (1-A)(-1 + q(A)(1-A)^{n-1}),$$

also by the fact that x and $1-x$ are relative prime, $N = A(1-A)T(A)$ for some polynomial $T(x)$, which means N is nilpotent as well.

□

Remark 3.11. *If α is an automorphism of $R[t,t^{-1}]^m$, which maps $R[t]^m$ to $R[t]^m$, then by the proof of (b) in Lemma 3.10 and equations (3.1), (3.2),*

$$[R[t]^m/\alpha R[t]^m] = [R[t]^n/(1+(P+N)(t-1))R[t]^n]$$

for integer n, where P is idempotent, N is nilpotent, and P, N commute. We claim that

$$R[t]^n/(1+(P+N)(t-1))R[t]^n \cong PR^n. \tag{3.3}$$

[2]Actually, $p(x), q(x)$ can be represented as

$$p(x) = \sum_{k=0}^{n-1} \binom{2n-1}{k} (1-x)^k x^{n-k-1}$$
$$q(x) = \sum_{k=n}^{2n-1} \binom{2n-1}{k} (1-x)^{k-n} x^{2n-k-1}$$

by considering the binomial expansion of $(1-x+x)^{2n-1}$.

Actually, because

$$1 + (P + N)(t - 1) = P(t + N(t - 1)) + (1 - P)(1 + N(t - 1))$$
$$P(t + N(t - 1))R[t]^n \cap (1 - P)(1 + N(t - 1))R[t]^n = \emptyset$$

so $(1 + (P + N)(t - 1))R[t]^n = P(t + N(t - 1))R[t]^n \oplus (1 - P)(1 + N(t - 1))R[t]^n$.

Similarly, $R[t]^n = PR[t]^n \oplus (1 - P)R[t]^n$*. Because* $1 + N(t - 1)$ *is invertible matrix on* $R[t]$*, so*

$$(1 - P)(1 + N(t - 1))R[t]^n = (1 - P)R[t]^n.$$

It follows that[3]

$$R[t]^n/(1 + (P + N)(t - 1))R[t]^n = PR[t]^n/P(t + N(t - 1))R[t]^n \tag{3.4}$$

where $PR[t]^n/P(t + N(t - 1))R[t]^n \cong P(R[t]^n/(t + N(t - 1))R[t]^n)$.

Also we have $R[t]^n/(t + N(t - 1))R[t]^n \cong R^n$*. To see this, first,*

$$t + N(t - 1) = (1 + N)t - N = (t - N(1 + N)^{-1})(1 + N).$$

It follows that $(t + N(t - 1))R[t]^n = (t - N(1 + N)^{-1})R[t]^n$*, and thus*

$$R[t]^n/(t + N(t - 1))R[t]^n = R[t]^n/(t - N(1 + N)^{-1})R[t]^n. \tag{3.5}$$

Then, we have $R[t]^n/(t - N(1 + N)^{-1})R[t]^n \cong R^n$*. To see this, first, for any* $X(t) \in R[t]^n$*,* $X(t) = X_0 + tX_1 + t^2X_2 + \cdots + t^kX_k$*. Then, we define the evaluation* $R[t]^n \longrightarrow R^n$ *which is given by* $t \longrightarrow N(1 + N)^{-1}$*. Because* R^n *is embedded into* $R[t]^n$*, so the evaluation is an*

[3] With a little abuse of language we still use t to represent tI_n for identity matrix I_n.

epimorphism, and the kernel is $(t - N(1+N)^{-1})R[t]^n$*, therefore*

$$R[t]^n/(t - N(1+N)^{-1})R[t]^n \cong R^n.$$

To sum up, $R[t]^n/(1 + (P+N)(t-1))R[t]^n \cong PR^n$*.*

Definition 3.12. *Define* $NK_i(R)$ *to be the cokernel of the natural map*

$$K_i(R) \longrightarrow K_i(R[t]),$$

where $i = 0, 1$*.*

Remark 3.13. *Because the evaluation*

$$R[t] \overset{t \longrightarrow 1}{\longrightarrow} R$$

induces a splitting of the natural map $K_i(R) \longrightarrow K_i(R[t])$*, so we see*

$$K_i(R[t]) \cong K_i(R) \oplus NK_i(R)$$

where $i = 0, 1$*.*

Lemma 3.14. *There is a surjective boundary map* $\partial : K_1(R[t, t^{-1}]) \longrightarrow K_0(\text{Nil } R)$ *that sends* $[\alpha] \in K_1(R[t, t^{-1}])$ *to*

$$[(R[t]^n/t^k \alpha R[t]^n, t)] - [(R[t]^n/t^k R[t]^n, t)] \in K_0(\text{Nil } R)$$

for large enough k*, where* t *is considered as the homomorphism induced by multiplying* t*,* n *is the dimension of the square matrix* α*, and the right inverse of* ∂ *embeds* $K_0(\text{Nil } R)$ *as a direct sum of* $K_1(R[t, t^{-1}])$*.*

Proof. First, we show ∂ is well defined. By Lemma 3.9, we see $R[t]^n/t^k \alpha R[t]^n$ and $R[t]^n/t^k R[t]^n$

are indeed finitely-generated projective R-modules, and both t are nilpotents. We also claim that ∂ is independent of choice of k. Actually, we have the short exact sequence:

$$0 \longrightarrow t^k\alpha R[t]^n/t^{k+j}\alpha R[t]^n \longrightarrow R[t]^n/t^{k+j}\alpha R[t]^n \longrightarrow R[t]^n/t^k\alpha R[t]^n \longrightarrow 0,$$

where the intermediate two homomorphisms from left to right are canonical map. Because the intermediate two homomorphisms are $R[t]$-module homomorphism, due to fact that they commute with t, and there is a $R[t]$-module isomorphism $R[t]^n/t^jR[t]^n \xrightarrow{t^k\alpha} t^k\alpha R[t]^n/t^{k+j}\alpha R[t]^n$, so there is a commutative diagram with top and bottom row exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R[t]^n/t^jR[t]^n & \longrightarrow & R[t]^n/t^{k+j}\alpha R[t]^n & \longrightarrow & R[t]^n/t^k\alpha R[t]^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \\ 0 & \longrightarrow & R[t]^n/t^jR[t]^n & \longrightarrow & R[t]^n/t^{k+j}\alpha R[t]^n & \longrightarrow & R[t]^n/t^k\alpha R[t]^n & \longrightarrow & 0 \end{array}$$

which implies

$$[(R[t]^n/t^{k+j}\alpha R[t]^n, t)] = [(R[t]^n/t^k\alpha R[t]^n, t)] + [(R[t]^n/t^jR[t]^n, t)].$$

Similarly, we have

$$[(R[t]^n/t^{k+j}R[t]^n, t)] = [(R[t]^n/t^kR[t]^n, t)] + [(R[t]^n/t^jR[t]^n, t)].$$

It follows that

$$[(R[t]^n/t^{k+j}\alpha R[t]^n, t)] - [(R[t]^n/t^{k+j}R[t]^n, t)] = [(R[t]^n/t^k\alpha R[t]^n, t)] - [(R[t]^n/t^kR[t]^n, t)],$$

so ∂ is independent of choice of k(if k is large enough).

Because α is identified with[4] $\alpha \oplus 1$ in $K_1(R[t, t^{-1}])$, we are going to prove the image of $\alpha \oplus 1$

[4]To simplify the notation, for square matrices A, B on ring R, the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is denoted as $A \oplus B$. This is consistent with the notation when we consider A, B as endomorphisms of the finitely generated free R-modules.

is the same as α. Actually, we have the short exact sequence:

$$0 \longrightarrow R[t]^n/t^k\alpha R[t]^n \xrightarrow{\ell} R[t]^{n+j}/t^k(\alpha\oplus 1)R[t]^{n+j} \xrightarrow{p_j} R[t]^j/t^kR[t]^j \longrightarrow 0$$

where ℓ embeds $R[t]^n/t^k\alpha R[t]^n$ into the first n coordinates of $R[t]^{n+j}/t^k(\alpha\oplus 1)R[t]^{n+j}$, and p_j is the projection of $R[t]^{n+j}/t^k(\alpha\oplus 1)R[t]^{n+j}$ onto the last j coordinates.

Also, because ℓ and p_j are $R[t]$-module homomorphism, by the same manner, we get

$$[(R[t]^{n+j}/t^k(\alpha\oplus 1)R[t]^{n+j}, t)] = [(R[t]^n/t^k\alpha R[t]^n, t)] + [(R[t]^j/t^kR[t]^j, t)]$$

and thus

$$\begin{aligned}&[(R[t]^{n+j}/t^k(\alpha\oplus 1)R[t]^{n+j}, t)] - [(R[t]^{n+j}/t^kR[t]^{n+j}, t)]\\ =&[(R[t]^n/t^k\alpha R[t]^n, t)] - [(R[t]^n/t^kR[t]^n, t)].\end{aligned}$$

Also, for any $[\beta], [\gamma] \in K_1(R[t, t^{-1}])$, consider β, γ as the square matrix of large enough dimension n, by embedding them into $GL_n(R[t, t^{-1}])$. We have the short exact sequence:

$$0 \longrightarrow t^k\beta R[t]^n/t^{2k}\beta\gamma R[t]^n \longrightarrow R[t]^n/t^{2k}\beta\gamma R[t]^n \longrightarrow R[t]^n/t^k\beta R[t]^n \longrightarrow 0,$$

so in the same manner as above, we have

$$[(R[t]^n/t^{2k}\beta\gamma R[t]^n, t)] = [(R[t]^n/t^k\beta R[t]^n, t)] + [(R[t]^n/t^k\gamma R[t]^n, t)],$$

which also implies if ∂ is well-defined, then it is a homomorphism, because

$$\begin{aligned}&[(R[t]^n/t^{2k}\beta\gamma R[t]^n, t)] - [(R[t]^n/t^{2k}R[t]^n, t)]\\ =&[(R[t]^n/t^k\beta R[t]^n, t)] + [(R[t]^n/t^k\gamma R[t]^n, t)] - 2[(R[t]^n/t^kR[t]^n, t)].\end{aligned}$$

Also, for n-dimensional elementary matrix $e_{ij}(a) \in E(R[t,t^{-1}])$, similarly as we did in Lemma 3.9, there is a commutative diagram with the top and bottom rows exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R[t]/t^kR[t] & \longrightarrow & R[t]^n/t^k e_{ij}(a)R[t]^n & \longrightarrow & R[t]^{n-1}/t^kR[t]^{n-1} & \longrightarrow & 0 \\ & & \downarrow t & & \downarrow t & & \downarrow t & & \\ 0 & \longrightarrow & R[t]/t^kR[t] & \longrightarrow & R[t]^n/t^k e_{ij}(a)R[t]^n & \longrightarrow & R[t]^{n-1}/t^kR[t]^{n-1} & \longrightarrow & 0. \end{array}$$

It follows that

$$\begin{aligned} [(R[t]^n/t^k e_{ij}(a)R[t]^n, t)] &= [(R[t]/t^kR[t], t)] + [(R[t]^{n-1}/t^kR[t]^{n-1}, t)] \\ &= [(R[t]^n/t^kR[t]^n, t)] \end{aligned}$$

which implies $\partial(e_{ij}(a)\alpha) = \partial(\alpha)$, for any n-dimensional elementary matrix $e_{ij}(a) \in E(R[t,t^{-1}])$. Similarly, $\partial(\alpha e_{ij}(a)) = \partial(\alpha)$. By induction, we see for any $\zeta, \eta \in E(R[t,t^{-1}])$, $\partial(\zeta\alpha\eta) = \partial(\alpha)$. So ∂ is well-defined and a homomorphism.

We are going to prove ∂ is surjective and $K_0(\text{Nil } R)$ is a summand of $K_1(R[t,t^{-1}])$ by proving ∂ has right inverse.

Define a map $\varphi_1 : K_0(R) \longrightarrow K_1(R[t,t^{-1}])$ induced by $[P] \longmapsto [tp+1-p]$, where p is a corresponding idempotent matrix of projective R-module P.

To begin with, we show that this map is well-defined. For $tp+1-p$, it has an inverse $t^{-1}p+1-p$, which implies $tp+1-p \in GL(R[t,t^{-1}])$. Then, for another idempotent matrix p' on R such that $p' = MpM^{-1}$, $M \in GL(R)$ we have

$$[tp'+1-p'] = [M][tp+1-p][M^{-1}] = [M][M]^{-1}[tp+1-p] = [tp+1-p].$$

To see this map is a homomorphism, consider another $[P'] \in K_0(R)$.

Because $[P]+[P']=[P\oplus P']$, so

$$\begin{aligned}
&\varphi_1([P]+[P'])\\
&=\varphi_1([P\oplus P'])\\
&=\left[t\begin{pmatrix} p & 0\\ 0 & p'\end{pmatrix}+\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix} p & 0\\ 0 & p'\end{pmatrix}\right]\\
&=\left[\begin{pmatrix} tp+1-p & 0\\ 0 & tp'+1-p'\end{pmatrix}\right]\\
&=[tp+1-p][tp'+1-p']\\
&=\varphi_1([P])\varphi_1([P']).
\end{aligned}$$

By Remark 3.11, we see φ_1 is the right inverse of $F\circ\partial$, where F is the forgetful map.

Define a homomorphism $\varphi_2 : K_0(\text{Nil } R)\longrightarrow K_1(R[t])$ induced by sending every $[(P,\nu)]\in$ Nil R to the image of the automorphism $1-\nu t\in \text{Aut}(P[t])$ under the natural homomorphism of Proposition 3.8. To see this map is well-defined, we need to check:

(1) If $(P,\nu)\cong(P',\nu')$, then $\varphi_2([(P,\nu)])=\varphi_2([(P',\nu')])$.

(2) If there is a short exact sequence:

$$0\longrightarrow(P_1,\nu_1)\longrightarrow(P_2,\nu_2)\longrightarrow(P_3\nu_3)\longrightarrow 0,$$

then $\varphi_2([(P_2,\nu_2)])=\varphi_2([(P_1,\nu_1)])\varphi_2([(P_3,\nu_3)])$.

For (1), assume h is the isomorphism between (P,ν) and (P',ν'), then we got $\nu'=h\nu h^{-1}$, and thus $1-\nu' t=h(1-\nu t)h^{-1}$. So in the similar manner as we used in the proof of Proposition 3.8, the images of $1-\nu t$ and $1-\nu' t$ under the homomorphism of Proposition 3.8 are the same.

For (2), because $P_2\cong P_1\oplus P_3$, by selecting particular isomorphism, we can write $1-\nu_2 t$ as an upper triangular matrix:

$$1-\nu t=\begin{pmatrix}1-\nu_1 t & \gamma t\\ 0 & 1-\nu_2 t\end{pmatrix}=\begin{pmatrix}1-\nu_1 t & 0\\ 0 & 1-\nu_2 t\end{pmatrix}\begin{pmatrix}1 & \gamma' t\\ 0 & 1\end{pmatrix}$$

which implies $[1-\nu_2 t]=[1-\nu_1 t][1-\nu_3 t]$, by taking natural homomorphism we see $\varphi_2([(P_2,\nu_2)])=\varphi_2([(P_1,\nu_1)])\varphi_2([(P_3,\nu_3)])$.

In addition, we see the image of φ_2 is contained in $NK_1(R)$ and $\varphi_2:(R^n,\nu)\longmapsto[1-\nu t]$.

We define a homomorphism $\psi:NK_1(R)\longrightarrow \text{Nil}_0(R)$ as the composition:

$$NK_1(R)\longrightarrow K_1(R[t])\longrightarrow K_1(R[t,t^{-1}])\longrightarrow K_1(R[s,s^{-1}])\xrightarrow{\partial} K_0(\text{Nil }R)\xrightarrow{p}\text{Nil}_0(R)$$

where the left two homomorphisms are both inclusion map, p is the projection map, the homomorphism from $K_1(R[t,t^{-1}])$ to $K_1(R[s,s^{-1}])$ is induced by identifying t with s^{-1}.

Define $\varphi_2':\text{Nil}_0(R)\longrightarrow K_1(R[s,s^{-1}])$ to be φ_2 restricted on $\text{Nil}_0(R)$. By Proposition 3.7, $\text{Nil}_0(R)$ is generated by elements of form $[(R^n,\nu)]-[(R^n,0)]$. However, we have[5]

$$\partial(\varphi_2'([(R^n,\nu)]-[(R^n,0)]))=\partial([1-\nu s])=[(R^n,\nu)]-[(R^n,0)]$$

which means φ_2' is the right inverse of ∂ composited with projection $p:K_0(\text{Nil }R)\longrightarrow\text{Nil}_0(R)$.

Now, we see, $K_0(\text{Nil }R)\cong K_0(R)\oplus\text{Nil}_0(R)$, and $F:K_0(\text{Nil }R)\longrightarrow K_0(R)$ is the forgetful map, $p:K_0(\text{Nil }R)\longrightarrow\text{Nil}_0(R)$ is the projection map, $F\circ\partial$ is right invertible by φ_1, and $p\circ\partial$ is right invertible by φ_2'. It follows that ∂ is right invertible.

□

Remark 3.15. *By (a) of Lemma 3.10, we see that every element of $NK_1(R)$ can be reduced to $[1-\nu s]$. So, we have $\varphi_2'(\psi([1-\nu s]))=\varphi_2'([(R^n,\nu)]-[(R^n,0)])=[1-\nu s]$, which means ψ is actually the inverse of φ_2', it follows that* $\text{Nil}_0(R)\cong NK_1(R)$.

[5]Actually, $\partial([1-\nu s])=\partial([1-\nu t^{-1}])=\partial([t-\nu][t^{-1}])=\partial([t-\nu])+\partial([t^{-1}])=[(R^n,\nu)]-[(R^n,0)]$.

Proposition 3.16. *The homomorphism $K_1(R[t]) \longrightarrow K_1(R[t,t^{-1}])$, which is induced by embedding $R[t] \hookrightarrow R[t,t^{-1}]$, is injective.*

Proof. First, we claim $K_1(R)$ is embedded in $K_1(R[t,t^{-1}])$ as a direct sum. Actually, the homomorphism from $K_1(R[t,t^{-1}]) \longrightarrow K_1(R)$ induced by $t \longrightarrow 1$ is the left inverse of the natural homomorphism $K_1(R) \longrightarrow K_1(R[t,t^{-1}])$.

Consider the sequence whose composition is ψ as we define in Lemma 3.14:

$$NK_1(R) \longrightarrow K_1(R[t]) \longrightarrow K_1(R[t,t^{-1}]) \longrightarrow \mathrm{Nil}_0(R)$$

Because we have $K_1(R[t]) \cong K_1(R) \oplus NK_1(R)$, and $K_1(R)$ is embedded in $K_1(R[t,t^{-1}])$ as a direct sum, and $K_1(R)$ is contained in the kernel of $K_1(R[t,t^{-1}]) \longrightarrow \mathrm{Nil}_0(R)$, so we have

$$NK_1(R) \cong K_1(R[t])/K_1(R) \longrightarrow K_1(R[t,t^{-1}])/K_1(R) \longrightarrow \mathrm{Nil}_0(R)$$

whose composition is also ψ. It follows from ψ is an isomorphism that

$$NK_1(R) \longrightarrow K_1(R[t,t^{-1}])/K_1(R)$$

is injective, and thus the following homomorphism

$$K_1(R) \oplus NK_1(R) \longrightarrow K_1(R) \oplus K_1(R[t,t^{-1}])/K_1(R)$$

is injective.

Because $K_1(R[t]) \cong K_1(R) \oplus NK_1(R)$, and $K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_1(R[t,t^{-1}])/K_1(R)$, so we get $K_1(R[t]) \longrightarrow K_1(R[t,t^{-1}])$ is injective. □

Proposition 3.17 ([4])**.** *For any $\alpha, \beta \in GL_n(R[t,t^{-1}])$, which map $R[t]^n$ to $R[t]^n$, if*

$$R[t]^n/\alpha R[t]^n \cong R[t]^n/\beta R[t]^n$$

as a $R[t]$-module isomorphism, then $[\alpha][\beta^{-1}]$ lies in the image of $K_1(R[t]) \longrightarrow K_1(R[t,t^{-1}])$.

Proof. Choose a $R[t]$-module isomorphism γ_0 from $R[t]^n/\alpha R[t]^n$ to $R[t]^n/\beta R[t]^n$. Define $M := R[t]^{2n}/(\alpha \oplus \beta)R[t]^{2n}$ then we see

$$\gamma := \begin{pmatrix} 0 & \gamma_0^{-1} \\ \gamma_0 & 0 \end{pmatrix}$$

is an $R[t]$-module automorphism of M whose inverse is itself.

Similarly, as we did in Lemma 2.17, the automorphism

$$\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$$

on $M \oplus M$ can be lift to an $R[t]$-module automorphism γ_1 of $R[t]^{4n}$.

As a result, the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R[t]^{4n} & \xrightarrow{e_1} & R[t]^{4n} & \xrightarrow{\pi_1} & R[t]^n/\alpha R[t]^n & \longrightarrow & 0 \\ & & & & \downarrow{\gamma_1} & & \downarrow{\gamma_0} & & \\ 0 & \longrightarrow & R[t]^{4n} & \xrightarrow{e_2} & R[t]^{4n} & \xrightarrow{\pi_2} & R[t]^n/\beta R[t]^n & \longrightarrow & 0 \end{array}$$

where $e_1 := (\alpha, 1, 1, 1)$, $e_2 := (1, \beta, 1, 1)$, π_1, π_2 are projection map, and the top and bottom rows exact. So there is isomorphism $\gamma_2 : R[t]^{4n} \longrightarrow R[t]^{4n}$ induced, that makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R[t]^{4n} & \xrightarrow{e_1} & R[t]^{4n} & \xrightarrow{\pi_1} & R[t]^n/\alpha R[t]^n & \longrightarrow & 0 \\ & & \downarrow{\gamma_2} & & \downarrow{\gamma_1} & & \downarrow{\gamma_0} & & \\ 0 & \longrightarrow & R[t]^{4n} & \xrightarrow{e_2} & R[t]^{4n} & \xrightarrow{\pi_2} & R[t]^n/\beta R[t]^n & \longrightarrow & 0, \end{array}$$

which implies $[e_1][\gamma_1] = [\gamma_2][e_2]$. So $[\alpha][\beta^{-1}] = [e_1][e_2^{-1}] = [\gamma_2][\gamma_1^{-1}]$ lies in the embedding image of $K_1(R[t])$ into $K_1(R[t,t^{-1}])$.

□

Theorem 3.18 (Fundamental Theorem of Algebraic K-Theory)**.** *There is an isomorphism:*

$$K_1(R[t,t^{-1}]) \cong K_0(R) \oplus K_1(R) \oplus NK_1(R) \oplus NK_1(R).$$

Proof. We are going to prove there is a short exact sequence:

$$0 \longrightarrow K_1(R[t]) \longrightarrow K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(\text{Nil } R) \longrightarrow 0.$$

This sequence is exact on the right by Lemma 3.14, and exact on the left by Proposition 3.16. To verify that it is exact at $K_1(R[t,t^{-1}])$, first we see for any $[\alpha] \in K_1(R[t,t^{-1}])$, where $\alpha \in GL_n(R[t])$, we have $\alpha R[t]^n = R[t]^n$, so $\partial([\alpha]) = 0$.

For the other direction, we notices that if $\partial([\alpha]) = 0$, namely

$$[(R[t]^n/t^k\alpha R[t]^n, t)] = [(R[t]^n/t^k R[t]^n, t)]$$

then by Proposition 3.5, there are short exact sequences

$$0 \longrightarrow (Q', \nu') \longrightarrow (Q_1, \nu_1) \longrightarrow (Q'', \nu'') \longrightarrow 0 \tag{3.6}$$

$$0 \longrightarrow (Q', \nu') \longrightarrow (Q_2, \nu_2) \longrightarrow (Q'', \nu'') \longrightarrow 0 \tag{3.7}$$

such that

$$((R[t]^n/t^k\alpha R[t]^n) \oplus Q_1, t \oplus \nu_1) \cong ((R[t]^n/t^k R[t]^n) \oplus Q_2, t \oplus \nu_2). \tag{3.8}$$

Next, we claim that for any pair (P, τ), where P is finitely generated projective R-module, τ is a nilpotent endomorphism of P, there is an isomorphism

$$(P, \tau) \cong (R[t]^m/\beta R[t]^m, t)$$

for $\beta = 1 + (p + (1-\tau)^{-1}\tau)(t-1)$ where p is an idempotent matrix corresponding to P. Actually, by equations (3.4), (3.5) in Remark 3.11, $R[t]^m/\beta R[t]^m = p(R[t]^m/(t-\tau)R[t]^m)$, but obviously we have $(P,\tau) = (p(R[t]^m/(t-\tau)R[t]^m), t)$.

By this trick, without loss of generosity, assume

$$(Q', \nu') = (R[t]^m/\alpha' R[t]^m, t)$$
$$(Q'', \nu'') = (R[t]^m/\alpha'' R[t]^m, t)$$
$$(Q_1, \nu_1) = (R[t]^m/\alpha_1 R[t]^m, t)$$
$$(Q_2, \nu_2) = (R[t]^m/\alpha_2 R[t]^m, t)$$

then equations (3.6), (3.7) can be written as

$$0 \longrightarrow Q' \longrightarrow Q_1 \longrightarrow Q'' \longrightarrow 0$$
$$0 \longrightarrow Q' \longrightarrow Q_2 \longrightarrow Q'' \longrightarrow 0$$

where all homomorphisms are $R[t]$-module homomorphisms.

Equation 3.8 can be written as

$$(R[t]^{n+m}/(t^k\alpha \oplus \alpha_1)R[t]^{n+m}, t) \cong (R[t]^{n+m}/(t^k \oplus \alpha_2)R[t]^{n+m}, t)$$

or equivalently,

$$R[t]^{n+m}/(t^k\alpha \oplus \alpha_1)R[t]^{n+m} \cong R[t]^{n+m}/(t^k \oplus \alpha_2)R[t]^{n+m} \tag{3.9}$$

as a $R[t]$-module isomorphism.

We are going to show $[\alpha_1] = [\alpha_2]$ in $K_1(R[t, t^{-1}])$. Actually, we have the following commutative diagrams:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & R[t]^m & \xrightarrow{\alpha'} & R[t]^m & \longrightarrow & Q' & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & Q_i & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & R[t]^m & \xrightarrow{\alpha''} & R[t]^m & \longrightarrow & Q'' & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

for $i = 1, 2$, with the two horizontal sequences exact.

By Horseshoe Lemma, there are two commutative diagrams:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R[t]^m & \xrightarrow{\alpha'} & R[t]^m & \longrightarrow & Q' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R[t]^{2m} & \xrightarrow{\alpha_i} & R[t]^{2m} & \longrightarrow & Q_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R[t]^m & \xrightarrow{\alpha''} & R[t]^m & \longrightarrow & Q'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

for $i = 1, 2$, with all horizontal and vertical sequences exact.

Since the first two vertical sequences from the left are exact, so $[\alpha_1] = [\alpha'] + [\alpha''] = [\alpha_2]$ in $K_1(R[t, t^{-1}])$.

By equation 3.9, and Proposition 3.17, we see $[\alpha]$ lies in the embedding image of $K_1(R[t]) \longrightarrow K_1(R[t, t^{-1}])$.

In addition, because ∂ has a right inverse as we proved in Lemma 3.14, so this short exact sequence splits, and by $K_0(\text{Nil } R) \cong K_0(R) \oplus NK_1(R)$, $K_1(R[t]) \cong K_1(R) \oplus NK_1(R)$, we get the conclusion:

$$K_1(R[t, t^{-1}]) \cong K_0(R) \oplus K_1(R) \oplus NK_1(R) \oplus NK_1(R).$$

□

3.2 Propagation Control

In this section, we investigate the propagation control of the boundary map ∂. First, we give the exact meaning of propagation:

Definition 3.19. *A filtered algebra over commutative ring R, is a R-algebra A with a family of R-submodules (A_r), $r \in \mathbb{R}$, such that*

(1) $A_r \subseteq A_{r'}$, *if* $r \leq r'$

(2) $A_r A_{r'} \subseteq A_{r+r'}$

(3) $A = \bigcup_r A_r$

where the family (A_r), $r \in \mathbb{R}$ is called a filtration of A. Every elements of A_r is said to have propagation $\leq r$.

If no other specification, we assign the propagation of an element a to be the least number r such that $a \in A_r$.

We are going to prove that for a group ring[6] RG, where R is a ring and G is a (multiplicative) group, we can give RG a filtration.

Definition 3.20. *For a length function on (multiplicative) group G, we mean a function $\ell : G \longrightarrow \mathbb{N}$ such that*

(1) $\ell(g) = 0$ *if and only if* $g = 1$*;*

(2) $\ell(gg') \leq \ell(g) + \ell(g')$ *for all* $g, g' \in G$*;*

(3) $\ell(g) = \ell(g^{-1})$ *for all* $g \in G$*.*

For a group G, select a generating set S of G, then we can define a length function $|\cdot|_S$ on G, by setting $|g|_S$ to be the shortest presentation of g as a word in $S \cup S^{-1}$.

[6]Group ring is also called group algebra for its natural R-algebra structure.

For the group ring RG, we can give RG a filtration by letting A_n to be the free R-submodule which is generated by

$$\{g \in G : |g|_S \leq n\},$$

then we see RG becomes a filtered algebra over R.

By Lemma 3.10 and Remark 3.11, we see for any $X \in GL(RG[t, t^{-1}])$, the image of $[X]$ under ∂ is $[(RG[t]^n/t^k XRG[t]^n, t)] - [(RG[t]^n/t^k RG[t]^n, t)] \in K_0(\text{Nil } RG)$, where $[RG[t]^n/t^k XRG[t]^n] = [PR^m]$, for some idempotent matrix P. So, we can track the propagation by considering the algorithm in Lemma 3.10 that make X into P. Before doing this, we need some preparations:

First, we see $RG[t, t^{-1}] \cong R[t, t^{-1}]G$, where the isomorphism is induced by

$$(rg)t^i \longrightarrow (rt^i)g$$
$$(rg)t^{-i} \longrightarrow (rt^{-i})g,$$

while $r \in R$, $g \in G$, $i \in \mathbb{Z}$.

For convenience, define a propagation function $\Pr : M(R[t, t^{-1}]G) \longrightarrow \mathbb{N}$ by letting $\Pr(X)$ to be the largest propagation of entries of X.

Define

$$D_n := \{X \in M(R[t, t^{-1}]G) : \Pr(X) \leq n\}$$

then we see $M(R[t, t^{-1}]G)$ becomes a filtered algebra over $R[t, t^{-1}]$, with filtration (D_n), $n = 0, 1, \cdots$.

Now, we are ready to consider the algorithm in (b) of Lemma 3.10. Assume

$$X = t^{-k}(X_0 + tX_1 + \cdots + t^n X_n) \in GL(R[t, t^{-1}]G),$$

then the entries from different X_i cannot be concelled out, so we have

$$\Pr(X) = \max_i \{\Pr(X_i)\}.$$

Assume $\Pr(X) \leq r_0$ and $\Pr(X^{-1}) \leq r_0$ for some integer r_0.

The algorithm that makes X into P can be stated into four steps:

(1) $X \longrightarrow t^k X$;

(2) $t^k X \longrightarrow B_0 + tB_1$;

(3) $B_0 + B_1 t \longrightarrow (B_0 + B_1)^{-1}(B_0 + tB_1) = 1 + (t-1)B$;

(4) $B \longrightarrow p(B)B^n = P$,

where

$$p(x) = \sum_{k=0}^{n-1} \binom{2n-1}{k} (1-x)^k x^{n-k-1}.$$

For (1), by our definition, X and $t^k X$ have the same propagation. For (2), we see

$$t^k X \longrightarrow \begin{pmatrix} t^k X & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} t^k X & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & t^{n-1}X_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^k X & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & t^{n-1}X_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^k X & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} = \begin{pmatrix} t^k X - t^{n-1}X_n & t^{n-1}X_n \\ -t & 1 \end{pmatrix},$$

do not change the propagation. We get $B_0 + tB_1$ by continuing this process. Also, by induction, we see $\Pr(B_0 + tB_1) \leq r_0$.

For step (3), we claim that $\Pr(1 + (t-1)B) \leq (2^{n-1} + 1)r_0$. Actually, by conducting corresponding "inverse operations" on X^{-1}, namely

$$X^{-1} \longrightarrow t^{-k}X^{-1},$$

$$t^{-k}X^{-1} \longrightarrow \begin{pmatrix} t^{-k}X^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} t^{-k}X^{-1} & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} t^{-k}X^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -t^{n-1}X_n \\ 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} t^{-k}X^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -t^{n-1}X_n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{-k}X^{-1} & -t^{n-k-1}X^{-1}X_n \\ t^{-k+1}X^{-1} & -t^{n-k}X^{-1}X_n + 1 \end{pmatrix},$$

we get the inverse of $B_0 + tB_1$. Because the last operation as above doubles the upper bound of the propagation, also by induction, we have $\Pr((B_0 + tB_1)^{-1}) \leq 2^{n-1}r_0$. Because $(B_0 + tB_1)^{-1}$ is also a Laurent polynomial on $M(R[t, t^{-1}]G)$, so the propagation of $(B_0 + tB_1)^{-1}$ is bounded by the largest propagation of the coefficients of Laurent polynomial $(B_0 + tB_1)^{-1}$, which implies

$$\Pr((B_0 + B_1)^{-1}) \leq \Pr((B_0 + tB_1)^{-1}) \leq 2^{n-1}r_0.$$

As a consequence, we see

$$\Pr(1 + (t-1)B) \leq \Pr((B_0 + B_1)^{-1}) + \Pr(B_0 + tB_1) \leq (2^{n-1} + 1)r_0.$$

For the final step (4), we have

$$\Pr(B) = \max\{\Pr(1 - B), \Pr(B)\} = \Pr(1 - B + tB) = \Pr(1 + (t-1)B)$$

therefore

$$\Pr(P) = \Pr(p(B)B^n) \leq \Pr(p(B)) + \Pr(B^n) \leq (2n-1)\Pr(B) \leq (2n-1)(2^{n-1}+1)r_0.$$

That means, if X has propagation $\leq r_0$, then P has propagation $\leq (2n-1)(2^{n-1}+1)r_0$, where n is the degree of X as a polynomial of t.

4. NEGATIVE K-THEORY

In this section, we are going to construct negative K-theory. The difference are denoted as "-", especially in K_1-groups, we denote the difference of elements $[A], [B]$ as $[A] - [B]$ rather than $[A][B]^{-1}$.

Define the group homomorphism $K_1(R[t]) \oplus K_1(R[t^{-1}]) \xrightarrow{\pm} K_1(R[t, t^{-1}])$ as

$$([A], [B]) \longrightarrow [A] - [B],$$

then we have the following proposition:

Proposition 4.1. *There is an isomorphism:*

$$K_0(R) \cong \text{coker } \left(K_1(R[t]) \oplus K_1(R[t^{-1}]) \xrightarrow{\pm} K_1(R[t, t^{-1}])\right).$$

Proof. We are going to prove the image of $\pm$ is isomorphic to $K_1(R) \oplus NK_1(R) \oplus NK_1(R)$ which is a normal subgroup of $K_0(R) \oplus K_1(R) \oplus NK_1(R) \oplus NK_1(R)$.

First, we claim that the image of $\pm$ is contained in $K_1(R) \oplus NK_1(R) \oplus NK_1(R)$. To prove this, we only have to prove for any $[A] \in K_1(R[t])$, $[B] \in K_1(R[t^{-1}])$, $\partial'([A] - [B]) = 0$, where $\partial' := F \circ \partial$, F is the forgetful map from $K_0(\text{Nil } R)$ to $K_0(R)$. Actually, because $[A]$ is contained in the kernel of ∂, so $\partial'([A] - [B]) = -\partial'([B])$. Similar to the proof of Lemma 3.10, we see $[B] = [B_0 - B_1 t^{-1}]$ where $B_1 \in M(R)$ and $B_0 \in GL(R)$. If $B_1 = 0$, then it is obvious that $\partial'([B]) = 0$. Assume $B_1 \neq 0$, then $(B_0 - B_1 t^{-1})R[t]^n = (1 - B_1' t^{-1})B_0 R[t]^n = (1 - B_1' t^{-1})R[t]^n$, where B_1' is nilpotent as we proved in Lemma 3.10. As a consequence,

$$\partial'([B]) = \partial'([B_0 - B_1 t^{-1}]) = [R[t]^n / t^k (B_0 - B_1 t^{-1}) R[t]^n] - [R[t]^n / t^k R[t]^n] = \partial'(1 - B_1' t^{-1}) = 0,$$

which means $\partial'([A] - [B]) = 0$.

Second, we proved that every element of $K_1(R) \oplus NK_1(R) \oplus NK_1(R)$ has a preimage in

$K_1(R[t]) \oplus K_1(R[t^{-1}])$. By Proposition 3.16, $K_1(R[t])$ embeds into $K_1(R[t,t^{-1}])$ as a direct sum, so we only need check every element in $NK_1(R)$ has preimage[1].

For any generator $[(R^n,\nu)] - [(R^n,0)] \in \mathrm{Nil}_0(R) \cong NK_1(R)$, we have $\partial([1-\nu t^{-1}]) = [(R^n,\nu)] - [(R^n,0)]$, where $[1-\nu t^{-1}] \in K_1(R[t^{-1}])$. So, for any element in $NK_1(R)$, it has a preimage in $K_1(R[t]) \oplus K_1(R[t^{-1}])$.

□

Corollary 4.2. *There is an exact sequence:*

$$0 \longrightarrow K_1(R) \xrightarrow{\Delta} K_1(R[t]) \oplus K_1(R[t^{-1}]) \xrightarrow{\pm} K_1(R[t,t^{-1}]) \leftrightarrows K_0(R) \longrightarrow 0,$$

where the epimorphism $K_1(R[t,t^{-1}]) \longrightarrow K_0(R)$ splits.

This result inspires us to define the negative K-groups:

Definition 4.3. *Define*

$$K_{-n}(R) := \text{coker}\,(K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \xrightarrow{\pm} K_{-n+1}(R[t,t^{-1}]))$$
$$NK_{-n}(R) := \text{coker}\,(K_{-n}(R) \longrightarrow K_{-n}(R[t]))$$

for $n = 1, 2, 3, \cdots$.

We see

$$R[t] \xrightarrow{t \longrightarrow 1} R$$

also induces a splitting of $K_{-n}(R) \longrightarrow K_{-n}(R[t])$, therefore

$$K_{-n}(R[t]) \cong K_{-n}(R) \oplus NK_{-n}(R),$$

and similarly,

$$K_{-n}(R[t^{-1}]) \cong K_{-n}(R) \oplus NK_{-n}(R)$$

[1]This $NK_1(R)$ is a summand of $K_0(\text{Nil } R)$.

where $n = 1, 2, 3, \cdots$.

By our definition of K_{-n}, we have

$$NK_{-n}(R) := \text{coker}\,(NK_{-n+1}(R[t]) \oplus NK_{-n+1}(R[t^{-1}]) \xrightarrow{\pm} NK_{-n+1}(R[t, t^{-1}]))$$

for $n = 1, 2, 3, \cdots$.

There is a generalization of Theorem 3.18.

Theorem 4.4 (Fundamental Theorem of Algebraic K-Theory)**.** *For any ring R, we have*

$$K_{-n+1}(R[t, t^{-1}]) \cong K_{-n}(R) \oplus K_{-n+1}(R) \oplus NK_{-n+1}(R) \oplus NK_{-n+1}(R)$$

for $n = 1, 2, 3, \cdots$.

Proof. By Theorem 3.18, we have isomorphisms:

$$K_1(R[s, t, t^{-1}]) \cong K_0(R[s]) \oplus K_1(R[s]) \oplus NK_1(R[s]) \oplus NK_1(R[s])$$
$$K_1(R[s^{-1}, t, t^{-1}]) \cong K_0(R[s^{-1}]) \oplus K_1(R[s^{-1}]) \oplus NK_1(R[s^{-1}]) \oplus NK_1(R[s^{-1}])$$
$$K_1(R[s, s^{-1}, t, t^{-1}]) \cong K_0(R[s, s^{-1}]) \oplus K_1(R[s, s^{-1}]) \oplus NK_1(R[s, s^{-1}]) \oplus NK_1(R[s, s^{-1}])$$

in which $K_0(R[s]), K_0(R[s^{-1}]), K_0(R[s, s^{-1}])$ are embedded into $K_1(R[s, t, t^{-1}])$, $K_1(R[s^{-1}, t, t^{-1}])$, $K_1(R[s, s^{-1}, t, t^{-1}])$, respectively, as direct summands.

However, by what we discuss above, there is a homomorphism

$$K_1(R[s, t, t^{-1}]) \oplus K_1(R[s^{-1}, t, t^{-1}]) \xrightarrow{\pm} K_1(R[s, s^{-1}, t, t^{-1}])$$

whose cokernel is $K_0(R[t, t^{-1}])$. By Definition 4.3, we get

$$K_0(R[t, t^{-1}]) \cong K_{-1}(R) \oplus K_0(R) \oplus NK_0(R) \oplus NK_0(R).$$

Because we do not use specific meaning of K_1, K_0 in the proof, so we can actually continue doing this. By induction, we are done.

$\square$

Corollary 4.5. *There is an exact sequence:*

$$0 \longrightarrow K_{-n+1}(R) \xrightarrow{\Delta} K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \xrightarrow{\pm} K_{-n+1}(R[t,t^{-1}]) \leftrightarrows K_{-n}(R) \longrightarrow 0,$$

for $n = 1, 2, 3, \cdots$, *where the epimorphism* $K_{-n+1}(R[t,t^{-1}]) \longrightarrow K_{-n}(R)$ *splits.*

As an application, we can extend the exact sequence in Corollary 2.32:

Theorem 4.6. *Let R be a ring, and I is an ideal of R, then there is an extended exact sequence:*

$$\cdots \longrightarrow K_0(R) \xrightarrow{q_*} K_0(R/I) \xrightarrow{\partial} K_{-1}(I) \xrightarrow{\ell_*} K_{-1}(R) \xrightarrow{q_*} K_{-1}(R/I) \xrightarrow{\partial} K_{-2}(I) \longrightarrow \cdots$$

where ℓ is the inclusion $I \longrightarrow R$, q is the quotient map $R \longrightarrow R/I$, ∂ is the boundary map.

Proof. By Corollary 4.2, 4.5 and 2.32, we have the following commutative diagram with vertical and horizontal sequences exact:

$$
\begin{array}{ccccccccc}
0 \longrightarrow & K_1(R) & \longrightarrow & K_1(R[t]) \oplus K_1(R[t^{-1}]) & \longrightarrow & K_1(R[t,t^{-1}]) & \longrightarrow & K_0(R) & \longrightarrow 0 \\
& \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & & \\
0 \longrightarrow & K_1(R/I) & \longrightarrow & K_1((R/I)[t]) \oplus K_1((R/I)[t^{-1}]) & \longrightarrow & K_1((R/I)[t,t^{-1}]) & \longrightarrow & K_0(R/I) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & & \\
0 \longrightarrow & K_0(I) & \longrightarrow & K_0(I[t]) \oplus K_0(I[t^{-1}]) & \longrightarrow & K_0(I[t,t^{-1}]) & \longrightarrow & K_{-1}(I) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \ell_*} & & \downarrow{\scriptstyle \ell_*} & & \downarrow{\scriptstyle \ell_*} & & & \\
0 \longrightarrow & K_0(R) & \longrightarrow & K_0(R[t]) \oplus K_0(R[t^{-1}]) & \longrightarrow & K_0(R[t,t^{-1}]) & \longrightarrow & K_{-1}(R) & \longrightarrow 0 \\
& \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & & \\
0 \longrightarrow & K_0(R/I) & \longrightarrow & K_0((R/I)[t]) \oplus K_0((R/I)[t^{-1}]) & \longrightarrow & K_0((R/I)[t,t^{-1}]) & \longrightarrow & K_{-1}(R/I) & \longrightarrow 0
\end{array}
$$

where the second vertical line of epimorphisms[2] from the right are split exact, so an exact sequence

[2]Namely, $K_1(R[t,t^{-1}]) \longrightarrow K_0(R)$, $K_1((R/I)[t,t^{-1}]) \longrightarrow K_0(R/I)$, etc.

is induced[3]:

$$K_0(R) \xrightarrow{q_*} K_0(R/I) \xrightarrow{\partial} K_{-1}(I) \xrightarrow{\ell_*} K_{-1}(R) \xrightarrow{q_*} K_{-1}(R/I).$$

Continue doing this, by induction, we are done. □

[3]For example, $K_0(R) \xrightarrow{q_*} K_0(R/I)$ is given by

$$K_0(R) \longrightarrow K_1(R[t,t^{-1}]) \xrightarrow{q_*} K_1((R/I)[t,t^{-1}]) \longrightarrow K_0(R/I).$$

5. CONCLUSION

We have given an explicit proof of the Fundamental Theorem for Lower Algebraic K-Theory.

Higher algebraic K-theory was first given by D. Quillen (cf. [7]). In his approach, he defined K-group as homotopy groups of certain spaces. Also, the Fundamental Theorem of Algebraic K-Theory can be generalized to higher cases under Quillen's definitions (see [4], Fundamental Theorem 8.2.). But the proof involves many topological techniques.

In 2012, D. Grayson gave a purely algebraic description of higher algebraic K-groups (cf. [8]). Furthermore, T. Harris provided new proofs of the additivity, resolution, and cofinality theorems under Grayson's framework (cf. [9]).

It is natural question whether a purely algebraic and explicit proof of the Fundamental Theorem for Higher Algebraic K-Theory exists. Such a proof would provide important quantitative information of higher algebraic K-theory.

This will be the subject of further investigation.

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