ISOPERIMETRIC PROPERTIES OF THE UNIFORM INFINITE PLANAR TRIANGULATION

A Thesis

by

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ABSTRACT

We investigate some of the geometric properties of rooted uniform infinite planar triangulations or UIPT. We wish to establish certain isoperimetric properties of the UIPT - for example, to obtain some bounds on the boundary size of a connected subset of the UIPT containing the root. The results are contingent on some unproven results. We attempt to give some idea how these may be shown and why, in all likelihood, they are in fact true. Also, we will show a proof that if A is a simply connected subset of the plane consisting of a finite union of faces of the UIPT, then $|\partial A| \geq cn^{1/n}$ for some constant c depending on ϵ, and where $|A| \geq n$.

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Contributors

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CHAPTER I

INTRODUCTION

The seminal work on rooted infinite planar triangulations was published in 2003 by Omer Angel and Oded Schramm [1]. Since then much work has been done on random planar geometry. Uniform finite planar triangulations had been studied before the work by Angel and Schramm, but their work was novel in that it considered a distribution on infinite maps. They define finite triangulations of size n of the sphere as an embedding of a connected planar graph G with n vertices such that the boundary of each connected component of $S^2 \setminus G$ meets exactly three edges of the embedded graph. The types of triangulations considered are those of type II (where the underlying graph is allowed to have multiple edges between vertices, but no loops), and type III (where the underlying graph has no loops and multiple edges are not allowed).

The main result of this paper is to show that for appropriate embeddings of planar graphs into the sphere (called sphere triangulations or triangulations of the sphere) there is a probability measure, call it τ, supported on infinite planar triangulations such that $\tau^i = \lim_n \tau_n^i$, $i = 2, 3$. Here τ_n^2 is the uniform distribution on rooted triangulations of type II of the sphere having n vertices, and similarly for τ_n^3, and the limit is taken to mean that $\lim_n \int f\tau_n = \int f\tau$ for every continuous real-valued function on the space of locally finite embedded rooted triangulations. They also showed that the UIPT is almost surely one ended. Here, a graph G is called one-ended if $G \setminus H$ contains exactly one infinite connected component for every finite subgraph H of G. This means that the τ^i probability of a rooted infinite planar triangulation having more than one end is zero. Thus, we can consider the UIPT to be a triangulation of the plane.

Following its introduction, the UIPT has been extensively studied. Of particular interest is the geometry of the UIPT. If T is a triangulation with a root, we define the ball of radius r to be the union of all faces of T whose boundary contains at least one vertex at graph distance less than or equal to $r-1$. We denote the ball of radius r by $B_r(T)$. In general, the complement of $B_r(T)$ is not connected. However, since the UIPT is almost surely one-ended, we know that at most one of the connected components of the complement of $B_r(T)$ is infinite. The standard hull of radius r, which will be denoted by $\overline{B}_r(T)$ is taken to be the union of the ball of radius r together with the finite connected components of the complement of this ball.

Angel showed in [2] that for any $\epsilon > 0$, both

$$\limsup_{r\to\infty} \frac{|\overline{B}_r(T)|}{r^4 \log^{6+\epsilon} r} < \infty$$

almost surely, and also

$$\lim_{r\to\infty} \frac{|\overline{B}_r(T)| \log^{32/3} +\epsilon}{r^4} = \infty$$

almost surely. He also showed that almost surely,

$$\limsup_{r\to\infty} \frac{|\partial\overline{B}_r(T)|}{r^2 \log^3 r} < \infty,$$

and for any given $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{|\partial\overline{B}_r(T)| \log^{6+\epsilon} r}{r^2} = \infty.$$

Here, $\partial\overline{B}_r(T)$ is the boundary of the standard hull of radius r in the triangulation T. These results, however, are not quite good enough for the result we want to show. Angel conjectured that the random variables $\frac{|B_r(T)|}{r^4}$ and $\frac{|\overline{B}_r(T)|}{r^4}$ both converge in distribution. It should similarly be expected that $\frac{|\partial\overline{B}_r(T)|}{r^2}$ converges in distribution, which would suffice for our problem (as stated in the abstract).

A paper by Nicolas Curien and Jean-François Le Gall [3] showed that these conjectures are in fact the case. Indeed, let A be a conneced subgraph of the UIPT. Define $\partial A = \{\{x, y\} : x \in A,\ y \notin A\}$ to be the edge boundary of A in the UIPT, and $\partial^{in} A = \{x \in A : x \text{ has a neighbor in } A^c\}$. Then the results of Curien and Le Gall [3] show that the random variables $|\frac{\overline{B}_r}{r^4}|$ and $|\frac{\partial^{in}\overline{B}_r}{r^2}|$ converge in distribution. That $\frac{\overline{B}_r}{r^4}$ converges in distribution plays an important role for us.

Finally, somewhat later, a paper by Curien and Lehéricy gives several results about the boundary size of connected subsets of the uniform infinite planar quadrangulation (UIPQ) [4]. They remark in that paper that many of these results should translate to results about the UIPT. We aim here to show how their methods regarding the UIPQ can be applied to give us some analogues in the space of the UIPT. As mentioned above, there is a contour lying outside of $\overline{B}_r(T)$ that separates the root from infinity, and its expected length grows linearly in r as r grows without bound. Le Gall and Lehéricy use this idea (but for quadrangulations). More precisely, they show that for any integer $r \geq 1$, if $L(r)$ is the smallest length of cycle separating $B_r(P)$ from infinity ($B_r(P)$ here being the ball of radius r of a quadrangulation instead of a triangulation), then given any $\delta < 2$, there is a constant C_δ such that for every $\epsilon \in (0,1)$,

$$\mathbb{P}(L(r) \geq \epsilon r) \leq C_\delta \epsilon^\delta$$

and also, there is some constant C and $\lambda > 0$ such that for any $a > 0$,

$$\mathbb{P}(L(r) \geq ar) \leq Ce^{-\lambda a}.$$

From this they glean the following: given $\epsilon > 0$, there is some constant $c_\epsilon > 0$ such that for every integer $n \geq 1$,

$$|\partial A| \geq c_\epsilon n^{1/4}$$

for every A such that A is a simply connected compact subset of the plane that is a finite union of faces of the UIPQ, and $|A| \geq n$. Notice that the standard hull of radius r is a simply connected compact subset of the plane that is a finite union of faces either of the UIPQ or the UIPT depending on the space we are working with. The proof of this follows from the above result about the length of the smallest cycle separating the ball of radius r in a quadrangulation from infinity. We aim to show an analogous result for triangulations.

We also have results from Krikun [5] showing that there is a contour lying outside the standard hull of radius r that separates the root from infinity, and its expected length is linear in r as r approaches infinity. A contour in this case is just a path in the UIPT. The result from Curien and Lehéricy on the bounds for the smallest length of a separating cycle is essentially an argument showing the smallest separating cycle of the standard hull of radius r from infinity grows approximately linearly with r. So this similar result from Krikun should be of great use to us.

CHAPTER II

CONJECTURED RESULTS

For our desired result, which we introduce below, we need a result providing quantitative estimates for the probability on the length of the smallest cycles separating the ball of radius r centered at the root in the UIPT from "infinity," meaning separating it from the unique infinite connected component. The analogous result in [4] is the following:

Conjecture 2.1 For every integer $r \geq 1$, let $L(r)$ be the smallest length of a cycle separating $B_r(P$ from infinity (P being the uniform infinite planar quadrangulation). Then, (i) For every $\delta < 2$, there exists a constant C_δ such that for every $r \geq 1$, an for every $\epsilon \in (0, 1)$,

$$\mathbb{P}(L(r) \leq \epsilon r) \leq C_\delta \epsilon^\delta$$

and

(ii) There exists constants C and $\lambda > 0$ such that for every $a > 0$ and $r \geq 1$,

$$\mathbb{P}(L(r) \geq aR) \leq Ce^{-\lambda a}$$

The results for triangulations should be a little bit different, but some similar bounds should hold. This is because the idea of the above theorem, proved in [4], is that the length of the smallest cycle separating the ball of radius r from infinity grows linearly with r. As mentioned previously, Krikun showed that the same is true of what he calls a "contour" in the UIPT [5]. So it should be possible to find bounds similar to those shown above for the UIPT, although the constants and details will likely be somewhat different.

Let $\mathcal{K}$ be the set of all simply connected subsets of the plane consisting of a finite union of faces of the UIPT. For $A \in \mathcal{K}$, denote by $|A|$ the number of faces of the UIPT contained in A. The result we want is the following:

Conjecture 2.2 Given $\epsilon > 0$, there is a positive constant $c = c(\epsilon)$ such that for every $n \in \mathbb{N}$,

$$|\partial A| \geq cn^{1/n}$$

where $a \in \mathcal{K}$, and $|A| \geq n$, we have $\mathbb{P}(|\partial A| \geq cn^{1/4}) > 1 - \epsilon$.

The proof of this conjecture relies on two facts. The first is that the standard hull of radius r is such that the random variable $\frac{|\overline{B}_r|}{r^4}$ converges in distribution to some finite limit. The second, loosely speaking, is that the smallest separating loop of the standard hull of radius r grows linearly with r. We will give a heuristic argument under this assumption, and also include a conjecture of something more rigorous that can help us prove our result. We will include here a proof that is essentially that of the corresponding theorem from [4] to show that it should be essentially the same with some modifications.

Proof. Conjectured proof of Theorem 2.2: Let $\epsilon > 0$. We know (by [3]) that the random variable $\frac{|\overline{B}_r|}{r^4}$ converges in distribution. So, there is some N_1 such that

$$\mathbb{P}(|\overline{B}_r/r^4| < N_1) \geq 1 - \epsilon/2 \qquad *$$

for every $r \geq 1$. Let $L(r, N)$ be the event that the minimal length of a cycle separating B_r from the unique infinite connected component of the complement of B_r (recall that the UIPT is almost surely one-ended) is greater that $\frac{r}{N}$. Now, assume we have the result of the conjecture 2.1. Then we can find N large enough such that

$$\mathbb{P}(L(r, N)) > 1 - \epsilon/2 \qquad **$$

and fix this N. Since both N_1 and N are fixed, we can find a constant c such that $N_1(N+1)^4c^4 < 1$. Choose $n \in \mathbb{N}$ so that $cn^{1/4} \geq 1$, and also $N_1((N+1)[[cn^{1/4}]])^4 < n$. Consider the following event: $\{|\overline{B}_{N+1[[cn^{1/4}]]}|\} \cap L(N[[cn^{1/4}]], N)$. Then

$$\begin{aligned}\mathbb{P}(\{|\overline{B}_{N+1[[cn^{1/4}]]}|\} \cap L(N[[cn^{1/4}]], N)) &= \mathbb{P}(\{|\overline{B}_{N+1[[cn^{1/4}]]}|\}) + \mathbb{P}(L(N[[cn^{1/4}]], N)) \\ &\quad - \mathbb{P}(\{|\overline{B}_{N+1[[cn^{1/4}]]}|\} \cup L(N[[cn^{1/4}]], N)) \\ &> 1 - \epsilon/2 + 1 - \epsilon/2 - 1 \\ &= 1 - \epsilon\end{aligned}$$

by our choice of N_1, N, and c. So, the event $\{|\overline{B}_{N+1[[cn^{1/4}]]}|\} \cap L(N[[cn^{1/4}]], N)$ has probability at least $1-\epsilon$. If we now show that $|\partial A| \geq cn^{1/4}$ on this event when $|A| \geq n$ and A is simply connected, compact, and consists of a finite union of faces of the UIPT, we will be done.

Let T denote the UIPT, and ρ denote its vertex. Consider first the case where the graph distance from ρ to ∂A is greater than $N[[cn^{1/4}]] + 1$. Then the ball $B_{N[[cn^{1/4}]]}$ is separated from the infinite connected component of T in the complement of this ball by ∂A, which is a cycle. Since we are arguing on a subset of the event $L(N[[cn^{1/4}]], N)$, this means that (just by the definition of $L(N[[cn^{1/4}]], N)$) $|\partial A| \geq N[[cn^{1/4}]]/N \geq cn^{1/4}$.

On the other hand, suppose that the graph distance from ρ to ∂A is at most $N[[cn^{1/4}]] + 1$, and that $\partial A < cn^{1/4}$. That is, the greatest distance between any two vertices is at most $[[cn^{1/4}]] - 1$. Since the graph distance from ρ to the boundary of A is less than $N[[cn^{1/4}]] + 1$, any vertex of ∂A is at graph distance at most $(N + 1)[[cn^{1/4}]]$, and therefore any edge in ∂A is incident to a vertex at graph distance at most $(N + 1)[[cn^{1/4}]] - 1$, from which it follows that ∂A is contained in the hull $\overline{B}_{(N+1)[[cn^{1/4}]]}$, which means that all of A is in this hull. But then, $|A| \leq |\overline{B}_{(N+1)[[cn^{1/4}]]}| < n$, which contradicts our assumption that $|A| > n$.

$\square$

CHAPTER III

SUMMARY AND SUGGESTED RESEARCH

To summarize, the main ingredient missing from our problem is to give quantitative estimates showing that there exists a loop separating the standard hull of radius r from infinity is approximately linear. Krikun proves that this is essentially the case in [5], but without giving a quantitative result as they use in [4]. So bounding the probability of how large or small a separating loop can be in the case of triangulations should be the first order of business. As mentioned, Curien and Lehéricy give a method of how to find the quantitative bounds on such a loop for the case of quadrangulations. However, I have yet to successfully apply those methods to the space of triangulations. It may well be, and hopefully is, that their methods will work, and it just requires a little more time. If this result is found, there should be many other interesting results about the geometry of the UIPT that can be proved. We give an example here.

Using the notation from the introduction, let $\mathcal{K}$ be the set of all simply connected compact subset of the plane that are finite unions of faces of the UIPT. Then the following theorem should follow in a similar manner to what is done in [4]:

Conjecture 3.1 Given $\epsilon > 0$, there is some constant $c > 0$ such that the following holds

$$\inf_{A \in \mathcal{K}} \frac{|\partial A|}{|A|^{\frac{1}{4}(\log|A|)^{-c-\epsilon}}} > 0, \text{ a.s.}$$

Here, I use the exponent $-c$. In [4] they find c to equal $\frac{3}{4}$ but remark that it is likely to not be the optimal exponent. The proof is quite similar to that of the one that we used to show Theorem 2.2. Thus, provided we find quantitative bounds on the probability for the smallest length of a separating loop, this result should follow in a very similar manner to 2.2.

These would be the analogues to the main results for quadrangulations that were found in [4]. There are many other results that have yet to be shown that are also of considerable interest, although they might not necessarily follow easily from the work already done or the concepts presented in this thesis. We list them here.

From the outset of this project, we were interested in the isoperimetry of the UIPT. For example, we wanted to learn more about the asymptotic behavior of the isoperimatric profile. This is a function, call it Φ such that $\Phi : \mathbb{N} \to [0, \infty)$ and it is defined by

$$\Phi(n) = \min\{\frac{|\partial A|}{|A|} : \rho \in A \subset G \text{ with } A \text{ connected, and } |A| \leq n\},$$

where G is a connected infinite graph with root vertex ρ. Here, ∂A is defined slightly differently than above and it is $\partial A = \{\{x, y\} : x \in A, y \notin A\}$, or the edge boundary of A. Similarly, we can define $\partial^{in} A = \{x \in A : x \text{ has a neighbor in } T \setminus A\}$ (T being the UIPT), and $\Phi^{in}(n)$ to be as above but replacing ∂A with $\partial^{in} A$. As was mentioned above, we know that the random variables $\frac{\overline{B}_r}{r^4}$ and $\frac{\partial^{in} \overline{B}_r}{r^2}$ were shown to converge in distribution by Curien and Le Gall in [3]. So, if we take A in the definition of $\Phi(n)$ to be the standard hull of volume approximately n, then $\partial^{in} A$ would have size approximately $n^{-1/2}$. It follows that asymptotically, we have for any function $\psi(n)$ such that $\psi(n) \to \infty$ as $n \to \infty$,

$\Phi^{in}(n) \leq |\partial^{in}|/|A| < \psi(n)n^{-1/2}$ for n large enough. A more difficult question that should follow from the heuristic argument is that $\Phi(n)$ should behave like $1/n^{-1/2}$. Thus, we have the following conjectures:

Conjecture 3.2 There is some constant $C > 0$ such that almost surely,

$$\Phi(n) < Cn^{-1/2}$$

for n large enough.

Angel and Schramm showed in [1] that the degree of every vertex has an exponentially decaying tail (this follows from Lemma 4.1 of [1]). This means that we can apply a standard union bound argument and get that it is enough to show the conjecture for $\Phi^{in}(n)$ instead of $\Phi(n)$. This should make the problem a little easier, since $\partial^{in}A$ is in some sense a simpler structure than ∂A. Since the part of the function Φ^{in} that can cause problems for this conjecture is $|\partial^{in}A|$, if we want to improve on the upper bound of $\psi(n)n^{-1/2}$, we would have to consider the standard hull of the ball and in some way bound the areas on the boundary which are too dense. How to show this conjecture to be true has proved elusive, but intuition seems to indicate that the layering process used in [5] should be instructive. In particular, since the degrees of the vertices have an exponentially decaying tail, hopefully the number of areas on the boundary that are too dense decreases in proportion to the volume of the standard hull.

Another conjecture suggested by the argument preceding conjecure 3.2 is the following:

Conjecture 3.3 There is a constant $c > 0$ such that, almost surely,

$$\Phi(n) > cn^{-1/2}$$

for n large enough.

Here again, it suffices to prove the conjecture for $\Phi^{in}(n)$ in place of $\Phi(n)$. This is because $|\partial^{in} A| \leq |\partial A|$ for any $A \subset T$, whence $\Phi^{in}(n) \leq \Phi(n)$.

Finally, the previous two conjectures suggest the following more difficult one:

Conjecture 3.4 The limit $\lim_{n\to\infty} n^{1/2}\Phi(n)$ exists almost surely.

These conjectures are the ones most related to the present work. There are, however, other interesting questions to ask about the UIPT that were suggested in the original research proposal. We list them here for thoroughness. For example, the relation of the UIPT to planar circle packing has some unresolved questions. A circle packing is a collection of circles that are connected, but their interiors are disjoint. In other words, any two circles in a circle packing meet in at most one point. To any circle packing, denoted by $Q = (Q_i)_{i\in I}$, there is a graph whose vertex set is I, and for each $i, j \in I$ there is an edge between i and j if and only if Q_i and Q_j are tangent. This graph is called the nerve graph. Oded Schramm proved the following rigidity theorem [6].

Theorem 3.1 Let T be a planar infinite triangulation of the 2-dimensional sphere S^2 with a countable number of accumulation points and Q a circle packing whose nerve graph is T. Then every other circle packing Q' whose nerve is T is Möbius equivalent to Q.

This means that if we take any rooted planar infinite triangulation T or S^2 with one accumulation point, there is a unique circle packing (unique up to inflation and rotation) sending the root to the origin, and the accumulation point to infinity, an whose nerve graph is T. This process with the circle packing provides us with a canonical embedding of the UIPT into $\mathbb{R}^2$. Let us denote this canonical embedding by Ξ. A natural question to ask is about the existence of limiting shapes, and their properties. For example, the following two questions were posed in the original research proposal:

Problem 3.1 What is the growth rate of $|\Xi^{-1}(\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R\})|$ as $R \to \infty$?

And, what is the typical Euclidean diameter of $\Xi(\overline{B}(R))$ as $R \to \infty$, where $\overline{B}(R)$ is the standard hull of radius R.

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