ON UPPER-TRIANGULAR FORMS IN TRACIAL VON NEUMANN ALGEBRAS

A Dissertation

by

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ABSTRACT

A classical theorem of Issai Schur states that any n$\times$n matrix is unitarily equivalent to an upper-triangular matrix, and hence can be decomposed as the sum of a normal matrix and a nilpotent matrix. Dykema, Sukochev and Zanin generalized this decomposition to any operator in a von Neumann algebra with a normal, faithful, tracial state, replacing nilpotent with s.o.t.-quasinilpotent.

In this paper we study the decomposition described by Dykema, Sukochev and Zanin. We generalize the construction presented by Dykema, Sukochev and Zanin and introduce the idea of a spectral ordering, a function $\phi : [0, 1] \to \mathbb{C}$ which is suitable for construction of such a decomposition. We give sufficient conditions for a function to be a spectral ordering for an operator.

In the course of our investigation we develop the theory of SOT-quasinilpotent operators, and construct an operator Q which is SOT-quasinilpotent and has a spectrum which is a non-trivial interval of the real line; such an operator had not previously appeared in the literature.

We then restrict ourselves to operators with finitely supported Brown measure and investigate the properties of an operator T with quasinilotent upper-triangular part Q. We show this is equivalent to several conditions, including decomposability (in the sense of C. Foiaş) and having a finite spectrum.

DEDICATION

To the memory of Uffe Haagerup, one of the great mathematicians of our time, whose work inspires us.

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Contributors

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1. INTRODUCTION AND BACKGROUND

1.1 Introduction

We start with some fundamentals of von Neumann algebras. Let $\mathcal{H}$ be a Hilbert space, and let $B(\mathcal{H})$ be the *-algebra of all bounded linear operators on $\mathcal{H}$. A von Neumann algebra $\mathcal{M}$ of operators on $\mathcal{H}$ is a *-subalgebra of $B(\mathcal{H})$ which contains the identity operator and is closed in the strong operator topology, meaning that if $T \in \mathcal{M}$, then $T^* \in \mathcal{M}$, and if S_n is a sequence of operators in $\mathcal{M}$ which converges pointwise on $\mathcal{H}$ to an operator $S \in B(\mathcal{H})$, then $S \in \mathcal{M}$. $\mathcal{M}$ is automatically closed in the norm topology on $B(\mathcal{H})$.

A normal, faithful, tracial state on $\mathcal{M}$ is a function $\tau : \mathcal{M} \to \mathbb{C}$ which satisfies

1. $\tau(T^*T) \geq 0$ for all $T \in \mathcal{M}$, and if $\tau(T^*T) = 0$, then $T = 0$ (we will use scalars to represent scalar multiples of the identity operator in $\mathcal{M}$ when it will not cause confusion),
2. $\tau(1) = 1$,
3. for $a \in \mathbb{C}$ and $S, T \in \mathcal{M}$, $\tau(aS + T) = a\tau(S) + \tau(T)$,
4. for all $S, T \in \mathcal{M}$, $\tau(ST) = \tau(TS)$,
5. whenever T_α is a monotone increasing net of operators with least upper bound T, $\tau(T_\alpha) \to \tau(T)$.

In this paper we study properties of operators which belong to von Neumann algebras equipped with normal, faithful, tracial states. The simplest such algebras are the algebras of square matrices with complex entries, $M_n(\mathbb{C})$. In the next chapter we describe a generalization of the Schur upper-triangular form for matrices to tracial von Neumann algebras, and the generalization of ordering the spectrum on the diagonal of the upper-triangular form. In the final chapter we will study consequences of this generalization, and compare properties of operators with respect to it.

Many of the results in this paper are generalizations of properties of square matrices, and for intuition the reader may wish to contemplate the results on an algebra of square matrices.

L. Brown [2] introduced his spectral distribution measure for a not necessarily normal operator T in a tracial von Neumann algebra, by which we mean a von Neumann algebra $\mathcal{M}$ equipped with a normal, faithful, tracial state τ. This measure, now known as the Brown measure of T, generalizes the spectral counting measure (weighted according to algebraic multiplicity) on matrices. Brown proved an analogue of Lidskii's Theorem in tracial von Neumann algebras: letting ν_T denote the Brown measure of T, we have

$$\tau(T) = \int_{\mathbb{C}} z \, d\nu_T(z).$$

Haagerup and Schultz [8] proved existence of analogues of generalized eigenspaces for operators T in tracial von Neumann algebras. Given a Borel subset B of the complex numbers, they found a T-hyperinvariant projection $P(T, B)$ satisfying $\tau(P(T, B)) = \nu_T(B)$ and splitting Brown measure as described in Theorem 1.5 below.

In [4], Dykema, Sukochev, and Zanin constructed upper-triangular decompositions for operators in tracial von Neumann algebras. These decompositions are of the form $T = N + Q$, where N is normal, Q is s.o.t.-quasinilpotent, and T and N have the same Brown measure. The constructions generalize the Schur decomposition of an $n \times n$ matrix. The normal part N is constructed as the conditional expectation of T onto an abelian algebra generated by an increasing net of Haagerup-Schultz projections of T.

Recall, for a bounded operator A on a Hilbert space, the notation $|A| = (A^*A)^{1/2}$. A matrix $T \in M_n(\mathbb{C})$ is called *nilpotent* if there exists $m \in \mathbb{N}$ such that $T^m = 0$, or, equivalently, $|T^m| = 0$. This definition has motivated two generalizations to operator algebras which are of interest to us. An operator $T \in B(\mathcal{H})$ is called *quasinilpotent* if any one of the following three equivalent conditions holds:

(i) $\|T^n\|^{1/n} \to 0$ as $n \to \infty$,

(ii) $|T^n|^{1/n} \to 0$ in norm as $n \to \infty$,

(iii) $\sigma(0) = \{0\}$.

Also, a bounded operator Q on Hilbert space is said to be s.o.t.-quasinilpotent if $|Q^n|^{1/n}$ converges in strong-operator-topology to 0 as $n \to \infty$. A principal motivation for studying these operators is the characterization, proved by Haagerup and Schultz [8], that, for elements of a tracial von Neumann algebra, being s.o.t.-quasinilpotent is equivalent to having Brown measure concentrated at 0.

1.2 Preliminaries and notation

Throughout the paper, the following notation and language will be used: the word trace will refer to a normal, faithful, tracial state. $\mathcal{M}$ will be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and having a trace τ. Unless otherwise specified, T will be an element of $\mathcal{M}$ and then $\sigma(T)$ will denote the spectrum of T. Finally, we use the standard notations: $\mathbb{C}$ is the complex plane, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$ centered at the origin, and $\mathbb{T}$ is the unit circle, namely, the boundary of $\mathbb{D}$.

Definition 1.1. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then there exists a unique trace-preserving faithful normal completely positive linear map $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. $\mathbb{E}_{\mathcal{N}}$ satisfies the properties

1. $\mathbb{E}_{\mathcal{N}}$ is completely positive and unital
2. For any $T_1, T_2 \in \mathcal{N}$ and any $S \in \mathcal{M}$, $\mathbb{E}_{\mathcal{N}}(T_1 S T_2) = T_1 \mathbb{E}_{\mathcal{N}}(S) T_2$.

The map $\mathbb{E}_{\mathcal{N}}$ is called the *conditional expectation* of $\mathcal{M}$ onto $\mathcal{N}$.

1.2.1 Brown measure and Haagerup-Schultz projections

In [2], L. Brown introduced a generalization of the spectral distribution measure for not necessarily normal operators in tracial von Neumann algebras.

Theorem 1.2. *Let $T \in \mathcal{M}$. Then there exists a unique probability measure ν_T such that for every $\lambda \in \mathbb{C}$,*

$$\int_{[0,\infty)} \log(x)\, d\mu_{|T-\lambda|}(x) = \int_{\mathbb{C}} \log|z-\lambda|\, d\nu_T(z),$$

where for a positive operator S, μ_S denotes the spectral distribution measure $\tau \circ E$, where E is the spectral measure of S.

The measure ν_T in Theorem 1.2 is called the *Brown measure* of T. If T is normal, then ν_T equals the spectral distribution of T. If $Mcal$ is finite dimensional, then ν_T is the spectral counting measure of T.

Brown also made the following observation [2].

Observation 1.3. *Let T be an element of a tracial von Neumann algebra and let B be a connected component of $\sigma(T)$. Then $\nu_T(B) > 0$.*

Dykema, Sukochev and Zanin proved the following result about the Brown measure of a conditional expectation in [5].

Lemma 1.4. *For any increasing, right-continuous family of T-invariant projections $(q_t)_{t\in[0,1]}$ with $q_0 = 0$ and $q_1 = 1$, letting $\mathcal{D}$ be the von Neumann algebra generated by the set of all the q_t and $\mathcal{D}'$ be the relative commutant of $\mathcal{D}$ in $\mathcal{M}$, and letting $\mathbb{E}_{\mathcal{D}'}$ be the τ preserving conditional expectation, the Fuglede–Kadison determinants of T and $\mathbb{E}_{\mathcal{D}'}(T)$ agree. Since the same is true for $T-\lambda$ and $\mathbb{E}_{\mathcal{D}'}(T)-\lambda$ for all complex numbers λ, we have that the Brown measures of T and $\mathbb{E}_{\mathcal{D}'}(T)$ agree.*

The following is from the main result (Theorem 1.1) of [8]. It provides projections that split the operator T according to the Brown measure.

Theorem 1.5. *For any Borel set $B \subseteq \mathbb{C}$, there exists a unique projection $p = P(T, B)$ such that*

(i) $Tp = pTp$,

(ii) $\tau(p) = \nu_T(B)$,

(iii) when $p \neq 0$, considering Tp as an element of $p\mathcal{M}p$, its Brown measure ν_{Tp} is concentrated in B,

(iv) when $p \neq 1$, considering $(1-p)T$ as an element of $(1-p)\mathcal{M}(1-p)$, $\nu_{(1-p)T}$ is concentrated in $\mathbb{C} \setminus B$.

Moreover, $P(T, B)$ is T-hyperinvariant and $B_1 \subseteq B_2$ implies $P(T, B_1) \leq P(T, B_2)$.

The projection $P(T, B)$ is called the *Haagerup-Schultz projection* of T associated with B. We may emphasize the algebra (and associated trace) with respect to which the projection is defined, by writing $P^{(\mathcal{M})}(T, B)$ instead of$P(T, B)$.

We will need the following characterization of the Haagerup-Schultz projection of T associated with the ball of radius r centered at 0 given in [8].

Characterization 1.6. *Suppose $\mathcal{M} \leq \mathcal{B}(\mathcal{H})$. Define a subspace $\mathcal{H}_r$ of $\mathcal{H}$ by*

$$\mathcal{H}_r = \{\xi \in \mathcal{H} : \exists \xi_n \to \xi, \text{ with } \limsup_{n\to\infty} \|T^n \xi_n\|^{1/n} \leq r\}.$$

Then the projection onto $\mathcal{H}_r$ is equal to $P(T, \overline{B_r})$.

The results about Brown measure and Haagerup-Schultz projections in the following lemma are basic and easy to prove except, perhaps, for the last of them, which is Corollary 7.27 of [8].

Lemma 1.7. *Let T be an element of a tracial von Neumann algebra $\mathcal{M}$. Then for any $\lambda \in \mathbb{C}$ and any Borel set $B \subseteq \mathbb{C}$, letting B^* denote the image of B under complex conjugation, we have*

(i) $\nu_{(T-\lambda)}(B) = \nu_T(B + \lambda)$

(ii) $\nu_{T^*}(B) = \nu_T(B^*)$

(iii) $P(T - \lambda, B) = P(T, B + \lambda)$

(iv) $P(T^*, B) = 1 - P(T, \mathbb{C} \setminus B^*)$.

In Theorem 8.1 of [8], Haagerup and Schultz also prove the following theorem.

Theorem 1.8. *Let T be an operator in a tracial von Neumann algebra. Then the sequence $|T^n|^{1/n}$ has a strong operator limit A, and for every $r \geq 0$, the spectral projection of A associated with the interval $[0, r]$ is $P(T, r\overline{\mathbb{D}})$.*

The above theorem gives us the previously mentioned result that T is s.o.t.-quasinilpotent if and only if the Brown measure of T is concentrated at 0 (see [8]). Hence s.o.t.-quasinilpotent operators are spectrally trivial with respect to Brown measure.

1.2.2 Upper-triangular forms in tracial von Neumann algebras

The following classical theorem of Schur allows for the upper-triangular decomposition of square matrices.

Theorem 1.9. *Let $A \in M_n(\mathbb{C})$ and let $a_1, a_2, \ldots, a_n$ be the eigenvalues of A, listed according to algebraic multiplicity and in any order. Then there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that U^*AU is an upper-triangular matrix and $[U^*AU]_{ii} = a_i$, for every $i \in \{1, \ldots, n\}$.*

We can then decompose the matrix A from Theorem 1.9 as $A = N + Q$, with N normal, Q nilpotent, and the Brown measure of N identical to the Brown measure of A, by letting $\hat{N}$ be the diagonal matrix with diagonal matching that of U^*AU, and setting $N = U\hat{N}U^*$.

This decomposition was generalized to tracial von Neumann algebras for continuous spectral orderings in Theorem 6 of [4]. The following theorem is the main result of Chapter 2. It further generalizes the result and does not require a continuous ordering of the spectrum.

Theorem 1.10. *Let $\mathcal{M}$ be a finite von Neumann algebra with a trace τ, and let $T \in \mathcal{M}$. Let $\psi : [0,1] \to \mathbb{C}$ be a Borel measurable function such that $\psi([0,t])$ is Borel for every $t \in [0,1]$, and the set $\{z \in \mathbb{C} \mid \psi^{-1}(z)$ has a minimum$\}$ is a full measure Borel set with respect to ν_T. Then there exists a spectral measure E satisfying*

(i) $E(\psi([0,t])) = P(T, \psi([0,t]))$, *and hence T and $N = \int_{\mathbb{C}} z\, dE(z)$ have the same Brown measure, and*

(ii) $Q = T - N$ *is s.o.t.-quasinilpotent.*

Many of the results below apply to decompositions of the form $T = N + Q$, where $N = \int_{\mathbb{C}} z dE(z)$ for a spectral measure E as described in Theorem 1.10. The term *upper-triangular decomposition* will refer to a decomposition of this type.

Let us briefly review some facts about the proof of the version of Theorem 1.10 found in [4], in the case of a continuous spectral ordering ψ. Here ψ is any continuous function from $[0,1]$ into the complex plane, whose image contains the support of ν_T. Letting $\mathcal{D}$ be the commutative von Neumann algebra generated by the set of projections $\{P(T, \psi([0,t])) \mid 0 \leq t \leq 1\}$, N is the conditional expectation onto $\mathcal{D}$.

Upper-triangular decompositions are compatible with spectral theory and Brown measure as described by the following two results. Though the first is well known, we include a proof for convenience.

Lemma 1.11. *Let T be an operator in a finite von Neumann algebra $\mathcal{M}$, and let p be a T-invariant projection with $p \notin \{0, 1\}$. Then T is invertible if and only if Tp and $(1-p)T$ are invertible in the algebras $p\mathcal{M}p$ and $(1-p)\mathcal{M}(1-p)$, respectively. It follows that $\sigma(T) = \sigma(Tp) \bigcup \sigma((1-p)T)$, where the spectra of Tp and $(1-p)T$ are for these operators considered as elements of the algebras $p\mathcal{M}p$ and $(1-p)\mathcal{M}(1-p)$, respectively.*

Proof. If T is invertible, then

$$(pT^{-1}p)(Tp) = p$$

and

$$((1-p)T)((1-p)T^{-1}(1-p)) = (1-p).$$

Hence Tp has a left inverse, and since $p\mathcal{M}p$ is finite, it follows that Tp is invertible. Additionally, $(1-p)T$ has a right inverse, and must be invertible.

If Tp and $(1-p)T$ are both invertible, we may write T in the form of a matrix

$$\begin{pmatrix} Tp & pT(1-p) \\ 0 & (1-p)T \end{pmatrix}$$

which has as an inverse

$$\begin{pmatrix} (Tp)^{-1} & -(Tp)^{-1}pT(1-p)((1-p)T)^{-1} \\ 0 & ((1-p)T)^{-1} \end{pmatrix}.$$

Note that if p is T-invariant then p is $(T-\lambda)$-invariant for every complex number λ. Thus $T-\lambda$ is invertible if and only if both $(T-\lambda)p$ and $(1-p)(T-\lambda)$ are both invertible. Thus $\sigma(T) = \sigma(Tp) \bigcup \sigma((1-p)T)$. $\square$

The following result is stated in Proposition 10 of [4], and is a consequence of Theorem 2.24 of [7].

Theorem 1.12. *Let T be an element of a finite von Neumann algebra $\mathcal{M}$ with a trace τ, and let p be a T-invariant projection. Then*

$$\nu_T = \tau(p)\nu_{Tp} + \tau(1-p)\nu_{(1-p)T},$$

where ν_S denotes the Brown measure of S, and Tp and $(1-p)T$ are considered as elements of $p\mathcal{M}p$ and $(1-p)\mathcal{M}(1-p)$, respectively.

We use Lemma 1.11 and Theorem 1.12 to give the following corollary.

Corollary 1.13. *Let T be an operator in a finite von Neumann algebra $\mathcal{M}$ with a trace τ and let $p \in \mathcal{M}$ be a T-invariant projection. Then the following two statements hold.*

(i) *T is quasinilpotent if and only if Tp and $(1-p)T$ are both quasinilpotent.*

(ii) *T is s.o.t.-quasinilpotent if and only if Tp and $(1-p)T$ are both s.o.t.-quasinilpotent.*

2. SPECTRAL ORDERINGS AND UPPER-TRIANGULAR FORMS

2.1 Introduction

In this chapter we investigate generalizations of Schur's Theorem 1.9 to tracial von Neumann algebras, and prove Theorem 1.10.

In [4], Dykema, Sukochev and Zanin use Haagerup-Schultz projections to construct upper-triangular decompositions of operators in tracial von Neumann algebras.

Theorem 2.1. *Let $\mathcal{M}$ be a diffuse, finite von Neumann algebra with normal, faithful, tracial state τ and let $T \in \mathcal{M}$. Then there exist $N, Q \in \mathcal{M}$ such that*

1. $T = N + Q$
2. *the operator N is normal and the Brown measure of N equals that of T*
3. *The operator Q is s.o.t.-quasinilpotent.*

The proof of Theorem 2.1 uses a Peano curve $\rho : [0,1] \to \overline{B_{\|T\|}}$. The normal operator N is created by taking the trace-preserving conditional expectation onto the von Neumann algebra generated by the Haagerup-Schultz projections of the operator T associated with the sets $\rho([0,t])$ for $t \in [0,1]$. These projections, along with the normal operator N, are determined by the ordering on the support of the Brown measure of T given by $z_1 \leq z_2$ if and only if $\min(\rho^{-1}(z_1)) \leq \min(\rho^{-1}(z_2))$. Theorem 2.1 generalizes the idea of using an ordering of the spectrum of the operator T to write it as an upper-triangular form. This raises the natural question: What functions act as orderings on the spectrum of an operator in a tracial von Neumann algebra to give such upper-triangular decompositions?

For the reader's convenience we restate Theorem 1.10.

Theorem 1.10. Let $\mathcal{M}$ be a finite von Neumann algebra with a trace τ, and let $T \in \mathcal{M}$. Let $\psi : [0,1] \to \mathbb{C}$ be a Borel measurable function such that $\psi([0,t])$ is Borel for every

$t \in [0,1]$, and the set $\{z \in \mathbb{C} \mid \psi^{-1}(z)$ has a minimum$\}$ is a full measure Borel set with respect to ν_T. Then there exists a spectral measure E satisfying

(i) $E(\psi([0,t])) = P(T, \psi([0,t]))$, and hence T and $N = \int_{\mathbb{C}} z\, dE(z)$ have the same Brown measure, and

(ii) $Q = T - N$ is s.o.t.-quasinilpotent.

In particular the conclusion holds if ψ is continuous or is a Borel isomorphism, or a composition of a continuous function and a Borel isomorphism. We leave open the following question: Given a function φ which satisfies the hypotheses of Theorem 1.10, does there exist a Borel ismorphism ψ such that φ and ψ generate the same spectral measure?

2.2 Construction of the spectral measure E

Throughout this section, ψ will be as described in Theorem 1.10, Z will denote $\{z \in \overline{B_{\|T\|}} : \psi^{-1}(z) \, \text{has a minimum}\}$ and Y will denote $\overline{B_{\|T\|}} \setminus Z$.

We first define a Borel measure on the unit interval which will be useful in later proofs.

Lemma 2.2. *Let $X = \{\min(\psi^{-1}(z)) : z \in \overline{B_{\|T\|}}\}$. If $b \subseteq [0,1]$ is Borel, then $\psi(b \cap X)$ is Borel.*

Proof. Note first that, for $t \in (0,1]$, we have $\psi([0,t] \cap X) = \psi([0,t]) \setminus Y$ and $\psi([0,t) \cap X) = \psi([0,t)) \setminus Y$, and these sets are Borel. Now, since ψ restricted to X is an injection, we have $\psi((\alpha,\beta) \cap X) = \psi([0,\beta) \cap X) \setminus \psi([0,\alpha] \cap X)$ which is Borel. Since $[0,1]$ is second countable, an arbitrary open set $v = \bigcup_{n \in \mathbb{N}} u_n$ is the countable union of open intervals so that $\psi(v \cap X) = \psi(\bigcup_{n \in \mathbb{N}}(u_n \cap X)) = \bigcup_{n \in \mathbb{N}}(\psi(u_n \cap X))$ is Borel.

To complete the proof, we show that the collection of sets

$$S = \{b \subseteq [0,1] : \psi(b \cap X) \, \text{is Borel}\}$$

forms a σ-algebra. Suppose that $\psi(b \cap X)$ is Borel. Then $\psi(b^c \cap X) = \psi(X \setminus (b \cap X)) = Z \setminus \psi(b \cap X)$ is Borel. Now suppose that $(b_n)_{n\in\mathbb{N}} \subseteq S$. Then $\bigcup_{n\in\mathbb{N}} b_n \in S$ by the same argument used for open sets, and we are done. ☐

We now define $\mu(b) = \nu_T(\psi(b \cap X))$ for any Borel set $b \subset [0,1]$. It is clear that μ is countably additive, and hence a Borel probability measure on $[0,1]$. That μ is a regular measure follows from Theorem 1.1 of [1].

Observation 2.3. *For any Borel set $B \subseteq \overline{B_{\|T\|}}$, $\mu(\psi^{-1}(B)) = \nu_T(B)$.*

Proof. Since ψ is a bijection from X to Z we have

$$\mu(\psi^{-1}(B)) = \nu_T(\psi(\psi^{-1}(B) \cap X)) = \nu_T(B \cap Z) = \nu_T(B)$$

☐

Prior to constructing the spectral measure, we will need a map from the open subsets of the closed unit interval to the set of projections in $\mathcal{M}$. For an open interval, define

$$\begin{aligned}
&F(\emptyset) = 0\\
&F((\alpha,\beta)) = P(T,\psi([0,\beta))) - P(T,\psi([0,\alpha]))\\
&F([0,\beta)) = P(T,\psi([0,\beta)))\\
&F((\alpha,1]) = 1 - P(T,\psi([0,\alpha])).
\end{aligned}$$

Since $P(T,\psi([0,t]))$ and $P(T,\psi([0,t)))$ are increasing in t, it follows that $F(u)$ is increasing in u, and $F(u_1)F(u_2) = 0$ if $u_1 \cap u_2 = \emptyset$. For $u_1 = (\alpha_1,\beta_1)$ and $u_2 = (\alpha_2,\beta_2)$

with $\alpha_1 \le \alpha_2 \le \beta_1 \le \beta_2$,

$$
\begin{aligned}
F(u_1)F(u_2) &= (P(T,\psi([0,\beta_1])) - P(T,\psi([0,\alpha_1])))(P(T,\psi([0,\beta_2])) - P(T,\psi([0,\alpha_2]))) \\
&= P(T,\psi([0,\beta_1])) - P(T,\psi([0,\alpha_2])) - P(T,\psi([0,\alpha_1])) + P(T,\psi([0,\alpha_1])) \\
&= F(u_1 \cap u_2).
\end{aligned}
$$

Hence for any open intervals u_1 and u_2, $F(u_1)F(u_2) = F(u_1 \cap u_2)$.

For an arbitrary open set $v \subset [0,1]$, we first write $v = \bigcup_{n\in\mathbb{N}} u_n$, where the u_n are pairwise disjoint, and all nonempty u_n are open intervals. Then $\sum_{n\in\mathbb{N}} F(u_n)$ converges to a projection in the strong operator topology. We define $F(v) = \sum_{n\in\mathbb{N}} F(u_n)$. Multiplication of the series and application of the corresponding result for intervals gives us $F(v_1)F(v_2) = F(v_1 \cap v_2)$ for open sets $v_1, v_2 \subset [0,1]$.

Observation 2.4. *For any open set $v \subseteq [0,1]$, $\tau(F(v)) = \mu(v)$.*

Proof. For an open interval $u = (\alpha, \beta)$, we have

$$
\begin{aligned}
\tau(F(u)) &= \tau(P(T,\psi([0,\beta))) - P(T,\psi([0,\alpha])) \\
&= \nu_T(\psi([0,\beta))) - \nu_T(\psi([0,\alpha])) \\
&= \mu([0,\beta)) - \mu([0,\alpha]) \\
&= \mu(u).
\end{aligned}
$$

The observation follows from additivity of μ, F and τ. $\square$

We are now ready to define the spectral measure E. For any Borel set $B \subseteq \overline{B_{\|T\|}}$, define

$$E(B) = \bigwedge\{F(v) : v\, is\, open\, and\, \psi^{-1}(B) \subseteq v\}.$$

Note that E is increasing and that the range of E is contained in the von Neumann algebra generated by the projections $P(T, \psi([0,t]))$ for $t \in [0,1]$, which is commutative. We will prove later that E defines a spectral measure.

Proposition 2.5. *For any Borel set $B \subset \overline{B_{\|T\|}}$, $\tau(E(B)) = \nu_T(B)$.*

Proof. Let $\epsilon > 0$ be given. There exist open sets $v_1, v_2 \subseteq [0,1]$ such that

1. $\psi^{-1}(B) \subset v_1$ and $\mu(v_1) - \mu(\psi^{-1}(B)) < \epsilon$, and
2. $\psi^{-1}(B) \subset v_2$ and $\tau(F(v_2)) - \tau(E(B)) < \epsilon$.

Applying Observations 2.3 and 2.4 to (1), we have

$$\tau(E(B)) - \nu_T(B) \leq \tau(F(v_1)) - \nu_T(B) = \mu(v_1) - \mu(\psi^{-1}(B)) < \epsilon.$$

Applying Observations 2.3 and 2.4 to (2) gives

$$\nu_T(B) - \tau(E(B)) = \mu(\psi^{-1}(B)) - \tau(E(B))$$

$$\leq \mu(v_2) - \tau(E(B)) = \tau(F(v_2)) - \tau(E(B)) < \epsilon.$$

Hence we have $|\tau(E(B)) - \nu_T(B)| < \epsilon$, and we are done. □

Lemma 2.6. *If B_1 and B_2 are Borel subsets of $\overline{B_{\|T\|}}$, then $E(B_1)E(B_2) = E(B_1 \cap B_2)$.*

Proof. Noting that whenever v_1 is an open set containing $\psi^{-1}(B_1)$ and v_2 is an open set containing $\psi^{-1}(B_2)$, $v_1 \cap v_2$ is an open set containing $\psi^{-1}(B_1) \cap \psi^{-1}(B_2)$, we have

$$E(B_1 \cap B_2) = \bigwedge\{F(v) : v\, open, \psi^{-1}(B_1 \cap B_2) \subseteq v\}$$
$$= \bigwedge\{F(v) : v\, open, \psi^{-1}(B_1) \cap \psi^{-1}(B_2) \subseteq v\}$$
$$\leq \bigwedge\{F(v_1 \cap v_2) : v_1, v_2\, open, \psi^{-1}(B_1) \subseteq v_1, \psi^{-1}(B_2) \subseteq v_2\}$$
$$= \bigwedge\{F(v_1)F(v_2) : v_1, v_2\, open, \psi^{-1}(B_1) \subseteq v_1, \psi^{-1}(B_2) \subseteq v_2\}$$
$$= \bigwedge\{F(v_1) : v_1\, open, \psi^{-1}(B_1) \subseteq v_1\} \bigwedge\{F(v_2) : v_2\, open, \psi^{-1}(B_2) \subseteq v_2\}$$
$$= E(B_1)E(B_2).$$

Now let $\epsilon > 0$ be given. There exist open subsets $v, \tilde{v_1}, \tilde{v_2}$ of $[0, 1]$ such that

1. $\psi^{-1}(B_1 \cap B_2) \subseteq v$ and $\mu(v \setminus \psi^{-1}(B_1 \cap B_2)) < \epsilon$,
2. $a_1 = \psi^{-1}(B_1) \setminus \psi^{-1}(B_1 \cap B_2) \subseteq \tilde{v_1}$ and $\mu(\tilde{v_1} \setminus a_1) < \epsilon$, and
3. $a_2 = \psi^{-1}(B_2) \setminus \psi^{-1}(B_1 \cap B_2) \subseteq \tilde{v_2}$ and $\mu(\tilde{v_2} \setminus a_2) < \epsilon$.

Let $v_i = \tilde{v_i} \bigcup v$ for $i = 1, 2$. Then v_1 is an open set containing $\psi^{-1}(B_1)$ and v_2 is an open set containing $\psi^{-1}(B_2)$. We have

$$\mu(v_1 \cap v_2 \setminus \psi^{-1}(B_1 \cap B_2)) \leq \mu(v \setminus \psi^{-1}(B_1 \cap B_2)) + \mu(\tilde{v_1} \cap \tilde{v_2} \setminus \psi^{-1}(B_1 \cap B_2)).$$

Observing that $a_1 \cap a_2 = \emptyset$ and

$$\tilde{v_1} \cap \tilde{v_2} = (a_1 \cap a_2) \bigcup ((\tilde{v_1} \setminus a_1) \cap a_2) \bigcup ((\tilde{v_2} \setminus a_2) \cap a_1) \bigcup ((\tilde{v_1} \setminus a_1) \cap (\tilde{v_2} \setminus a_2))$$

we have

$$\mu((v_1 \cap v_2) \setminus \psi^{-1}(B_1 \cap B_2)) < 4\epsilon.$$

Applying Observations 2.3 and 2.4 and Proposition 2.5, we have

$$\begin{aligned}\tau(E(B_1)E(B_2)) - \tau(E(B_1 \cap B_2)) &\leq \tau(F(v_1)F(v_2)) - \tau(E(B_1 \cap B_2)) \\ &= \tau(F(v_1 \cap v_2)) - \tau(E(B_1 \cap B_2)) \\ &< 4\epsilon,\end{aligned}$$

and we conclude $E(B_1)E(B_2) = E(B_1 \cap B_2)$. □

Lemma 2.7. *E is countably additive on disjoint sets, where convergence of the series is in the strong operator topology.*

Proof. Suppose $(B_n)_{n\in\mathbb{N}}$ is a countable collection of disjoint Borel subsets of $\overline{B_{\|T\|}}$. By Lemma 2.6, $E(B_i)E(B_j) = 0$ if $i \neq j$. Then $E(\bigcup_{n\in\mathbb{N}} B_n)$ is a superprojection of each $E(B_n)$, and hence a superprojection of $\sum_{n\in\mathbb{N}} E(B_n)$. Also, $\tau(E(\bigcup_{n\in\mathbb{N}} B_n)) = \nu_T(\bigcup_{n\in\mathbb{N}} B_n) = \tau(\sum_{n\in\mathbb{N}} E(B_n))$. We conclude $E(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} E(B_n)$. □

We are now ready to show that E is a spectral measure supported on $\mathrm{supp}(\nu_T)$.

Proof. We must prove three statements:

1. $E(\emptyset) = 0$ and $E(\mathrm{supp}(\nu_T)) = 1$
2. $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for Borel sets B_1, B_2, and
3. if $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$, and $x, y \in \mathcal{H}$, then $\eta(B) = \langle E(B)x, y\rangle$ defines a regular Borel measure on $\mathbb{C}$.

Statement 1 follows from Proposition 2.5, since $\tau(E(\emptyset)) = 0$ and $\tau(E(\mathrm{supp}(\nu_T))) = 1$.

Statement 2 was proven as Lemma 2.6.

To prove statement 3, note that η is countably additive on disjoint sets follows from Lemma 2.7. Regularity of η follows from Theorem 1.1 of [1]. □

2.3 Proof of Theorem 1.10

We first establish several results which will be used in the proof. Throughout this section, ψ is as described in Theorem 1.10, and μ, E and E_v are as defined in Section 2.2. $\mathcal{M}$ acts on a Hilbert space H.

We now show that $\int_{\mathbb{C}} zdE$ is the norm limit of conditional expectations onto an increasing sequence of abelian von Neumann algebras. For each n, divide the $3\|T\|$ by $3\|T\|$ square centered at 0 into 2^n by 2^n squares of equal size indexed $(A_{n,k})_{k=1}^{2^{2n}}$, k increasing to the right then down. Include in each $A_{n,k}$ the top and left edge, excluding the bottom-left and top-right corners, so that for each n, $A_{n,k} \bigcap A_{n,j} = \emptyset$ whenever $j \neq k$ and $\overline{B_{\|T\|}} \subset \bigcup_{k=1}^{2^{2n}} A_{n,k}$. Let D_n be the von Neumann algebra generated by the (orthogonal) projections $(E(A_{n,k}))_{k=1}^{2^{2n}}$.

Proposition 2.8. *Let $\mathbb{E}_{D_n}(T)$ denote the conditional expectation of T onto D_n. Then $\mathbb{E}_{D_n}(T)$ converges in norm as $n \to \infty$ to $\int_{\mathbb{C}} zdE$.*

Proof. Observe that

$$\mathbb{E}_{D_n}(T) = \sum_{\substack{1 \leq k \leq 2^{2n} \\ \tau(E(A_{n,k})) \neq 0}} \frac{\tau(E(A_{n,k})TE(A_{n,k}))}{\tau(E(A_{n,k}))} E(A_{n,k}).$$

Applying Brown's analog of Lidskii's theorem (see [2]) gives

$$\mathbb{E}_{D_n}(T) = \sum_{\substack{1 \leq k \leq 2^{2n} \\ \nu_T(A_{n,k}) \neq 0}} \frac{\int_{A_{n,k}} zd\nu_T(z)}{\nu_T(A_{n,k})} E(A_{n,k}).$$

For each n, define

$$f_n(w) = \sum_{\substack{1 \leq k \leq 2^{2n} \\ \nu_T(A_{n,k}) \neq 0}} \frac{\int_{A_{n,k}} zd\nu_T(z)}{\nu_T(A_{n,k})} \chi_{A_{n,k}}(w) + \sum_{\substack{1 \leq k \leq 2^{2n} \\ \nu_T(A_{n,k}) = 0}} \frac{\int_{A_{n,k}} zdm(z)}{m(A_{n,k})} \chi_{A_{n,k}}(w),$$

where m is the Lebesgue measure on $\mathbb{C}$.

Since $\nu_T(A_{n,k}) = 0$ implies $E(A_{n,k}) = 0$, $\int_{\mathbb{C}} f_n dE = \mathbb{E}_{D_n}(T)$. Note that f_n converges uniformly on $\operatorname{supp}(E)$ to the inclusion function $f(z) = z$. Hence $\int_{\mathbb{C}} f_n dE$ converges in norm to $\int_{\mathbb{C}} z dE$, and we are done. □

Let D be the von Neumann algebra generated by $(E(\psi([0,t])))_{t\in[0,1]}$ (or equivalently by $\bigcup_{n=1}^{\infty} D_n$).

Proposition 2.9. *Suppose that $T \in D'$ and $B \subset \overline{B_{\|T\|}}$ is Borel with $\nu_T(B) \neq 0$. Then the Brown measure of $E(B)TE(B)$, considered as an element of $E(B)\mathcal{M}E(B)$, is concentrated in B.*

Proof. We begin by observing that for any open $v \subset [0,1]$, with $\tau(F(v)) \neq 0$, $F(v) \in D$ and if $v = (\alpha, \beta)$ is an open interval, then $\nu_{TF(v)}$ is concentrated in $\psi([0,\beta)) \setminus \psi([0,\alpha])$, and hence is also concentrated in $\psi((\alpha,\beta)) \cap Z$, where Z is as described in Section 2.2. Thus $\nu_{TF(v)}$ is concentrated in $\psi((\alpha,\beta) \cap X)$.

Now suppose that $v = \bigcup_{n=1}^{\infty} u_n$ where all nonempty u_n are pairwise disjoint open intervals. Let $\epsilon > 0$ be given. Let N be so large that

$$\tau\left(\sum_{n=1}^{N} F(u_n)\right) > \tau(F(v))(1-\epsilon).$$

Then, since each $F(u_n)$ commutes with T, Theorem 1.12 gives

$$\nu_{TF(v)} = \frac{1}{\tau(F(v))}\left(\sum_{n=1}^{N} \tau(F(u_n))\nu_{TF(u_n)} + \tau\left(\sum_{n=N+1}^{\infty} F(u_n)\right)\nu_{(\sum_{n=N+1}^{\infty} F(u_n))T}\right).$$

Hence, since each $\nu_{TF(u_n)}$ is concentrated in $\psi(u_n \cap X) \subset \psi(v \cap X)$, we have

$$\nu_{TF(v)}(\psi(v \cap X)) \geq \frac{1}{\tau(F(v))}\left(\sum_{n=1}^{N} \tau(F(u_n)\right)\nu_{TF(u_n)}(\psi(v \cap X)) > 1-\epsilon,$$

so that $\nu_{TF(v)}$ is concentrated in $\psi(v \cap X)$.

Now observe that when v is an open set containing $\psi^{-1}(B)$, since

$$\nu_{TF(v)} = \frac{1}{\tau(F(v))}(\tau(E(B))\nu_{TE(B)} + \tau(F(v) - E(B))\nu_{(F(v)-E(B))T}),$$

$\nu_{TE(B)}$ is concentrated in $\psi(v \cap X)$.

Choose an open set $v \subset [0,1]$ such that $\psi^{-1}(B) \subset v$ and $\mu(v) - \mu(\psi^{-1}(B)) < \epsilon$. Then using Theorem 1.12 and Proposition 2.5,

$$\begin{aligned}
&\epsilon > \nu_T(\psi(v \cap X)) - \nu_T(B) \\
&\quad = \tau(E(B))\nu_{TE(B)}(\psi(v \cap X) \setminus B) + (1 - \tau(E(B)))\nu_{(1-E(B))T}(\psi(v \cap X) \setminus B) \\
&\quad \geq \tau(E(B))\nu_{TE(B)}(\psi(v \cap X) \setminus B).
\end{aligned}$$

Hence

$$\tau(E(B)) - \epsilon < \tau(E(B))(1 - \nu_{TE(B)}(\psi(v \cap X) \setminus B)) = \tau(E(B))(\nu_{TE(B)}(B)).$$

Thus

$$1 - \frac{\epsilon}{\tau(E(B))} < \nu_{TE(B)}(B).$$

Letting ϵ tend to 0 gives the desired result. □

Lemma 2.10. *If $T \in D'$, then the Brown measure of $T - \mathbb{E}_{D_n}(T)$ is supported in the ball of radius $\frac{6\sqrt{2}\|T\|}{2^n}$.*

Proof. The key observation is that for any $\alpha \in \mathbb{C}$, if $\nu_{T-\alpha}$ is the Brown measure of $T - \alpha$, then for any Borel set $B \subset \mathbb{C}$, $\nu_{T-\alpha}(B) = \nu_T(B - \alpha)$. Since whenever $E(A_{n,k}) \neq 0$ the

Brown measure of $TE(A_{n,k})$ is supported in $A_{n,k}$, the Brown measure of $\left(T - \frac{\tau(TE(A_{n,k}))}{\tau(E(A_{n,k}))}\right)$ $E(A_{n,k})$ is supported in the square centered at 0 with edge length $\frac{6\|T\|}{2^n}$. We complete the proof by observing that $T - \mathbb{E}_{D_n}(T) = \sum_{k=1}^{2^{2n}} \left(T - \frac{\tau(TE(A_{n,k}))}{\tau(E(A_{n,k}))}\right) E(A_{n,k})$ and applying Theorem 8 to compute the Brown measure of the sum. □

We now are ready to prove Theorem 1.10.

Proof. We claim that the spectral measure E constructed above satisfies the conclusions of Theorem 1.10. To show this, we must prove three things:

(i) For any $t \in [0,1]$, $P(T, \psi([0,t])) = E(\psi([0,t]))$

(ii) For any Borel set $B \subseteq \mathbb{C}$, $\tau(E(B)) = \nu_T(B)$

(iii) $T - \int_{\mathbb{C}} z dE(z)$ is s.o.t.-quasinilpotent.

To prove (i), let $t \in [0,1]$. Whenever v is an open set containing $\psi^{-1}(\psi([0,t]))$, there exists $\epsilon > 0$ such that $[0, t+\epsilon) \subset v$ so we see that

$$P(T, \psi([0,t])) \leq F([0, t+\epsilon)) \leq F(v).$$

Hence we have

$$P(T, \psi([0,t])) \leq E(\psi([0,t])).$$

By Proposition 2.5 and Theorem 1.5,

$$\tau(P(T, \psi([0,t]))) = \tau(E(\psi([0,t])))$$

so that

$$P(T, \psi([0,t])) = E(\psi([0,t])),$$

which proves (i).

Statement (ii) was proven as Proposition 2.5.

We now show (iii). We will prove that the Brown measure of $T - \int_{\mathbb{C}} z dE(z)$ is concentrated at 0, which proves the statement. We show it first in the case that $T \in D'$. Observe from the proof of Proposition 2.8 that $\|\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T)\| \leq \frac{3\sqrt{2}\|T\|}{2^n}$. We now follow the model of the proof of Lemma 24 in [4].

We assume without loss of generality that $\|T\| \leq 1/2$. Fix $n \in \mathbb{N}$ and a unit vector $\xi \in H$. By assumption $T \in D'$, so we have

$$(T - \mathbb{E}_D(T))^{2m} = \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} (\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T))^{2m-k} (T - \mathbb{E}_{D_n}(T))^k.$$

Since $\|T\| \leq 1/2$, both $\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T)$ and $T - \mathbb{E}_{D_n}(T)$ are contractions. For $k \leq m$ and any $\eta \in H$, we have

$$\|(\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T))^{2m-k}(T - \mathbb{E}_{D_n}(T))^k \eta\|_H \leq \|\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T)\|^m.$$

For $k > m$ and any $\eta \in H$ we have

$$\|(\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T))^{2m-k}(T - \mathbb{E}_{D_n}(T))^k \eta\|_H \leq \|(T - \mathbb{E}_{D_n}(T))^m \eta\|_H.$$

Hence for any $\eta \in H$,

$$\|(T - \mathbb{E}_D(T))^{2m} \eta\|_H \leq 2^{2m} \max\left\{ \left(\frac{3\sqrt{2}\|T\|}{2^n} \right)^m, \|(T - \mathbb{E}_{D_n}(T))^m \eta\|_H \right\}. \tag{2.1}$$

By Proposition 2.9, the Brown measure of $T - \mathbb{E}_{D_n}(T)$ is supported in the ball of radius $\frac{6\sqrt{2}\|T\|}{2^n}$ centered at 0. By the Haagerup-Schultz characterization (1.6), there exists a sequence $\xi_m \to \xi$ such that $\|\xi_m\|_H = 1$ and

$$\limsup_{m\to\infty} \|(T - \mathbb{E}_{D_n}(T))^m \xi_m\|_H^{1/m} \leq \frac{6\sqrt{2}\|T\|}{2^n}.$$

Hence there exists M (depending on n) such that

$$\|(T - \mathbb{E}_{D_n}(T))^m \xi_m\|_H \leq \left(\frac{7\sqrt{2}\|T\|}{2^n}\right)^m, \quad m > M.$$

Taking $\eta = \xi_m$ in Equation (1), we have

$$\|(T - \mathbb{E}_D(T))^{2m} \xi_m\|_H^{1/m} \leq \frac{28\sqrt{2}\|T\|}{2^n}, \quad m > M.$$

Since ξ was arbitrary, it follows from characterization (1.6) that the Brown measure of $(T - \mathbb{E}_D(T))^2$ is supported in the ball of radius $\frac{28\sqrt{2}\|T\|}{2^n}$ centered at 0. Letting $n \to \infty$, we obtain that the Brown measure of $T - \mathbb{E}_D(T)$ is δ_0.

For $T \notin D'$, we first show that $P(T, \psi([0,t])) = P(\mathbb{E}_{D'}(T), \psi([0,t]))$ for all $t \in [0,1]$. For any t, $P(T, \psi([0,t])) \in D$, so

$$\mathbb{E}_{D'}(T) P(T, \psi([0,t])) = P(T, \psi([0,t])) \mathbb{E}_{D'}(T) P(T, \psi([0,t])).$$

By Lemma 1.4, T and $\mathbb{E}_{D'}(T)$ have the same Brown measure, so we have for all t

$$\tau(P(T, \psi([0,t]))) = \nu_T(\psi([0,t])) = \nu_{\mathbb{E}_{D'}(T)}(\psi([0,t])).$$

For any $s, t \in [0,1]$ $P(T, \psi([0,s]))$ is $TP(T, \psi([0,t]))$ invariant, so by Lemma 1.4, the operators $TP(T, \psi([0,t]))$ and $\mathbb{E}_{D'}(TP(T, \psi([0,t])))$ have the same Brown measure for any t, so whenever $P(T,\psi([0,t])) \neq 0$ we have

$$\nu_{\mathbb{E}_{D'}(T)P(T,\psi([0,t]))} = \nu_{\mathbb{E}_{D'}(TP(T,\psi([0,t])))} = \nu_{TP(T,\psi([0,t]))}$$

is supported in $\psi([0,t])$. Similarly $P(T,\psi([0,s]))$ is $(1-P(T,\psi([0,t])))T$ invariant for all $s,t \in [0,1]$, so

$$\nu_{(1-P(T,\psi([0,t])))T} = \nu_{(1-P(T,\psi([0,t])))\mathbb{E}_{D'}(T)}$$

which is supported in $\mathbb{C} \setminus \psi([0,t])$ whenever $P(T,\psi([0,t])) \neq 1$. Hence by Theorem 1.5 $P(T,\psi([0,t]))$ is the Haagerup-Schultz projection of $\mathbb{E}_{D'}(T)$ associated with the set $\psi([0,t])$.

Since $P(T,\psi([0,t])) = P(\mathbb{E}_{D'}(T),\psi([0,t]))$ for all $t \in [0,1]$, we see that ψ generates the same spectral measure E and abelian subalgebra D for both T and $\mathbb{E}_{D'}(T)$. Applying Lemma 7 we have $T - \int_{\mathbb{C}} z dE$ and $\mathbb{E}_{D'}(T) - \int_{\mathbb{C}} z dE$ have the same Brown measure, which we have shown is δ_0, as desired. □

3. CONCLUSIONS: ON CONVERGENCE PROPERTIES AND UPPER-TRIANGULAR FORMS

3.1 Introduction

The work presented in this chapter is joint work with Ken Dykema and Dmitriy Zanin.

In this chapter we investigate how properties of the upper-triangular part Q of an operator T with upper-triangular decomposition $T = N + Q$ affect the properties of T. The principal results of this chapter are around the following question.

Question 3.1. *Given an element T in a tracial von Neumann algebra, under what circumstances is the s.o.t.-quasinilpotent operator Q in an upper-triangular decomposition $T = N + Q$ actually quasinilpotent.*

We confine ourselves to upper-triangular decompositions of operators with Brown measures supported at finitely many points. In the course of these investigations, we also gain knowledge about operators that are s.o.t.-quasinilpotent but not quasinilpotent.

Recall that in [8], Haagerup and Schultz show that whenever T is an operator in a tracial von Neumann algebra, the sequence $|T^n|^{1/n}$ has a strong operator limit as $n \to \infty$, and that the limit is determined by the Haagerup-Schultz projections of T associated with discs centered at 0. This result motivates our next definition, which generalizes the property of being quasinilpotent.

Definition 3.2. An operator T in a tracial von Neumann algebra has the *norm convergence property* if the sequence $|T^n|^{1/n}$ is norm-convergent. We say that T has the *shifted norm convergence property* if $T - \lambda I$ has the norm convergence property for every complex number λ.

A naive guess is that the answer to Question 3.1 is: Q is quasinilpotent if and only if

T has the norm convergence property. However, this is not correct, as we show by explicit construction of a counter-example in Example 3.12.

A less naive guess is that Q is quasinilpotent if and only if T has the shifted norm convergence property. This may be true and, in Theorem 3.17, we prove it is true when the Brown measure of T has finite support.

Question 3.1 is related to decomposability of the operator T. Decomposability was introduced in the realm of local spectral theory by Foiaş [6] and was later extensively developed. See the book [10] for an exposition. We show in Theorem 3.17 that for operators with finitely supported Brown measure, decomposability is equivalent to the shifted norm convergence property.

We now turn to the topic of the spectrum of an s.o.t.-quasinilpotent operator. It follows from Remark 4.4 of [2] that, for a general element of a tracial von Neumann algebra, every connected component of the spectrum must meet the support of the Brown measure; thus, the spectrum of an s.o.t.-quasinilpotent operator must be a closed, connected set containing 0.

A natural example of an s.o.t.-quasinilpotent operator that is not quasinilpotent is provided by the direct sum

$$Q = \oplus_{n=1}^{\infty} J_n \in \bigoplus_{n=1}^{\infty} M_n(\mathbb{C}),$$

where J_n is the $n \times n$ Jordan block. Note that this can be realized inside the hyperfinite II_1-factor. Since Q has the same $*$-distribution as $e^{i\theta}Q$ for every real θ (i.e., its $*$-distribution is invariant under rotation), and since the spectral radius of Q is easily computed to be 1, we have that the spectrum of Q is the unit disk centered at the origin.

Also the examples of s.o.t.-quasinilpotent operators found in [3] that are not quasinilpotent clearly have $*$-distributions that are invariant under rotations and, thus, have spectra that are disks centered at the origin. Prior to this writing, every example of such an operator which has appeared in the literature or could be constructed therefrom using holomorphic functional calculus, has had a spectrum with non-empty interior. In Theorem 3.4, we construct an s.o.t.-quasinilpotent, non-quasinilpotent operator having thin spectrum (i.e., contained in an interval).

The contents of the rest of this chapter are as follows. Section 3.2 contains the construction of an s.o.t.-quasinilpotent operator with thin spectrum. In Section 3.3 we investigate the norm convergence properties and decomposability for operators with finitely supported Brown measure.

3.2 An s.o.t.-quasinilpotent operator with thin spectrum

Our main purpose in this section is to construct an s.o.t.-quasinilpotent operator with spectrum that is equal to a nondegenerate interval in $\mathbb{R}$.

Let $f \in L^\infty(\mathbb{T})$ and let $(a_k)_{k\in\mathbb{Z}}$ be its Fourier coefficients:

$$f(e^{i\theta}) \sim \sum_{k\in\mathbb{Z}} a_k e^{ik\theta}.$$

Let M_f denote the multiplication operator on $L^2(\mathbb{T})$ given by $M_f h(e^{i\theta}) = f(e^{i\theta})h(e^{i\theta})$. Consider the usual orthonormal basis $(v_k)_{k\in\mathbb{Z}}$ for $L^2(\mathbb{T})$ where $v_k(e^{i\theta}) = e^{ik\theta}$. Writing M_f with respect to this orthonormal basis, we have the Laurent operator $L(f) := (a_{\ell-k})_{k,\ell\in\mathbb{Z}}$. Let P be the projection of $L^2(\mathbb{T})$ onto $\overline{\operatorname{span}}\,\{v_k \mid k \geq 0\}$ and for $n \geq 0$, let P_n be the projection of $L^2(\mathbb{T})$ onto $\operatorname{span}\{v_k \mid 0 \leq k \leq n\}$. Then $T(f) := PL(f)P$ is the Toeplitz operator $T(f) = (a_{\ell-k})_{k,\ell\geq 0}$ and $T_n(f) := P_nL(f)P_n$ is the Toeplitz matrix $T_n(f) = (a_{\ell-k})_{0\leq k,\ell\leq n}$.

The next result contains well known facts about norms of Toeplitz matrices; for convenience we give a brief proof of them. For more refined results in the self-adjoint case, see [9].

Proposition 3.3. *(a)* $\|L(f)\| = \|f\|_\infty$.

(b) For every $n \geq 0$, $\|T_n(f)\| \leq \|T_{n+1}(f)\| \leq \|T(f)\|$.

(c) $\|L(f)\| = \|T(f)\| = \lim_{n\to\infty} \|T_n(f)\|$.

Proof. Parts (a) and (b) are immediate from the definitions, as is the inequality $\|T(f)\| \leq \|L(f)\|$. Let $\varepsilon > 0$. There exist $x, y \in \operatorname{span}\{v_k \mid k \in \mathbb{Z}\}$ such that $\|x\| = \|y\| = 1$ and $|\langle L(f)x, y\rangle| \geq \|L(f)\| - \varepsilon$. Since $L(f)$ commutes with the bilateral shift operator, we may without loss of generality assume $x, y \in \operatorname{span}\{v_k \mid 0 \leq k \leq n\}$ for some $n \geq 0$. Thus,

$$\|T_n(f)\| \geq |\langle L(f)x, y\rangle| \geq \|L(f)\| - \varepsilon.$$

This implies (c). □

The mappings $f \mapsto L(f)$, $f \mapsto T(f)$ and $f \mapsto T_n(f)$ are, of course, linear and $*$-preserving, so we have

$$T_n(\Re f) = \Re T_n(f), \qquad T_n(\Im f) = \Im T_n(f).$$

Theorem 3.4. *In the hyperfinite II_1-factor, there exists an s.o.t.-quasinilpotent operator whose spectrum is a nondegenerate interval in the real line.*

Proof. Let f_n be a conformal mapping from the unit disk onto

$$\left\{ a + ib \,\middle|\, -1 < a < 1,\, -\frac{1}{n} < b < \frac{1}{n} \right\}.$$

that satifies $f_n(0) = 0$. Then, of course, for all n, we have $\|\Re f_n\|_\infty = 1$ and $\|\Im f_n\|_\infty = \frac{1}{n}$, so $\lim_{n\to\infty} \|f_n\|_\infty = 1$. Since f_n is holomorphic and $f_n(0) = 0$, its Fourier coefficients a_k vanish for $k \leq 0$. Thus, the Toeplitz matrix $T_k(f_n)$ is strictly upper triangular and, hence, nilpotent, for each $k \geq 1$. Applying Proposition 3.3, we get a sequence $(k(n))_{n=1}^{\infty}$ of positive integers such that

$$\lim_{n\to\infty} \|T_{k(n)}(f_n)\| = 1 = \lim_{n\to\infty} \|\Re T_{k(n)}(f_n)\|.$$

By a standard construction, we can realize the von Neumann algebra direct sum

$$\mathcal{M} = \bigoplus_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$$

as a von Neumann subalgebra of the hyperfinite II_1-factor. Let

$$Q = \oplus_{n=1}^{\infty} T_{k(n)}(f_n) \in \mathcal{M}.$$

Let $A_n = \Re(T_{k(n)}(f_n))$ and $B_n = \Im(T_{k(n)}(f_n))$.

Claim 3.4.1. Q is s.o.t.-quasinilpotent.

Since for each n, $T_{k(n)}(f_n)$ is nilpotent and since in $\mathcal{M}$, the projection $0^{\oplus n} \oplus 1 \oplus 1 \oplus \cdots$ converges in strong operator topology to 0 as $n \to \infty$, this is clear.

Claim 3.4.2. The spectral radius of Q is at least 1.

For each $m \in \mathbb{N}$, we have $\|Q^m\| \geq \limsup_{n\to\infty} \|T_{k(n)}^m\|$ and for each n, we have

$$\|T_{k(n)}^m\| \geq \|A_n^m\| - \sum_{k=1}^{m} \binom{m}{k} \|A_n\|^{m-k} \|B_n\|^k.$$

Since $\|A_n^m\| = \|A_n\|^m$ and $\lim_{n\to\infty} \|A_n\| = 1$, while $\lim_{n\to\infty} \|B_n\| = 0$, we get $\|Q^m\| \geq 1$. This proves Claim 3.4.2.

Claim 3.4.3. The spectrum of Q lies in $\mathbb{R}$.

Suppose $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We have $\lambda - Q = \bigoplus_{n=1}^{\infty}(\lambda - T_{k(n)}(f_n))$. Since for each n, $T_{k(n)}(f_n)$

is nilpotent, each $\lambda - T_{k(n)}(f_n)$ is invertible. To show $\lambda - Q$ is invertible, it will suffice to show

$$\sup_{n\geq 1} \|(\lambda - T_{k(n)}(f_n))^{-1}\| < \infty.$$

As soon as $n > 2/|\Im\lambda|$, we have $\|B_n\| \leq |\Im\lambda|/2$ and, thus, $\|(\Im\lambda - B_n)^{-1}\| \leq 2/|\Im\lambda|$. We also have

$$\begin{aligned}&(\lambda - T_{k(n)}(f_n))^{-1} = \big((\Im\lambda - B_n)i + (\Re\lambda - A_n)\big)^{-1}\\ &= |\Im\lambda - B_n|^{-1/2}\big(\pm i + |\Im\lambda - B_n|^{-1/2}(\Re\lambda - A_n)|\Im\lambda - B_n|^{-1/2}\big)^{-1}|\Im\lambda - B_n|^{-1/2},\end{aligned}$$

where the sign in $\pm i$ is the sign of $\Im\lambda$. Since the operator

$$|\Im\lambda - B_n|^{-1/2}(\Re\lambda - A_n)|\Im\lambda - B_n|^{-1/2}$$

is self-adjoint, the operator

$$\pm i + |\Im\lambda - B_n|^{-1/2}(\Re\lambda - A_n)|\Im\lambda - B_n|^{-1/2}$$

is normal and has inverse of norm ≤ 1, so we get

$$\big\|(\lambda - T_{k(n)}(f_n))^{-1}\big\| \leq \frac{2}{|\Im\lambda|}.$$

This shows that $\lambda - Q$ is invertible, and Claim 3.4.3 is proved.

From Claims 3.4.1, 3.4.2 and 3.4.3 and the fact that the spectrum of an s.o.t.-quasinilpotent operator must be connected and contain the point 0, it follows that Q is an s.o.t.-quasinilpotent operator in the hyperfinite II_1-factor whose spectrum is an interval in $\mathbb{R}$ containing 0 and at least one of the points ± 1. □

3.3 The norm convergence properties for operators with finitely supported Brown measure

We begin by proving two lemmas for quasinilpotent operators.

Lemma 3.5. *Let Q be quasinilpotent. Then the series $\sum_{k=1}^{\infty}\|Q^k\|$ converges and, hence, the series $\sum_{k=1}^{\infty}Q^k$ converges in norm to an operator that is quasinilpotent.*

Proof. For any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$, we have $\|Q^k\| \leq \delta^k$. Thus, we see that the series $\sum_{k=1}^{\infty}\|Q^k\|$ converges. From this, we get that the series $\sum_{k=0}^{\infty}Q^k$ converges in norm to a bounded operator R that commutes with Q, and the series $\sum_{k=1}^{\infty}Q^k$ converges in norm to the bounded operator RQ. Now standard estimates show that RQ is quasinilpotent. □

Lemma 3.6. *Let Q be quasinilpotent. Then $1+Q$ is invertible and $(1+Q)^{-1} = 1+S$, where S is quasinilpotent.*

Proof. By Lemma 3.5, the series $\sum_{k=1}^{\infty}(-1)^k Q^k$ converges to a quasinilpotent operator S. We easily see that $1+S = (1+Q)^{-1}$. □

Proposition 3.7. *Let Q be quasinilpotent. Then*

$$\lim_{n\to\infty}\left\||(1+Q)^n|^{1/n} - 1\right\| = 0.$$

Thus, $1+Q$ has the norm convergence property and Q has the shifted norm convergence property.

Proof. We must show that for all $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$, we have

$$(1-\varepsilon)^{2n} \leq (1+Q^*)^n(1+Q)^n \leq (1+\varepsilon)^{2n}.$$

Since $1+Q$ is invertible, it will suffice to show that for all $n \geq N_0$ we have

$$\|(1+Q)^n\| \leq (1+\varepsilon)^n \tag{3.1}$$

$$\|(1+Q)^{-n}\| \leq (1-\varepsilon)^{-n}. \tag{3.2}$$

To show (3.1), let N_1 be such that $\|Q\|^n \leq \left(\frac{\varepsilon}{2}\right)^n$ for every $n \geq N_1$. Using the binomial formula, for $n \geq N_1$ we have

$$\begin{aligned}\|(1+Q)^n\| &\leq \sum_{k=0}^{n} \binom{n}{k} \|Q^k\| \leq \sum_{k=0}^{N_1-1} \binom{n}{k} \|Q^k\| + \sum_{k=N_1}^{n} \binom{n}{k} \left(\frac{\varepsilon}{2}\right)^k \\ &\leq \left(1+\frac{\varepsilon}{2}\right)^n + \sum_{k=0}^{N_1-1} n^k \|Q\|^k \leq \left(1+\frac{\varepsilon}{2}\right)^n + N_1(1+n\|Q\|)^{N_1}.\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{\log\left((1+\frac{\varepsilon}{2})^n + N_1(1+n\|Q\|)^{N_1}\right)}{n} = \log\left(1+\frac{\varepsilon}{2}\right) < \log(1+\varepsilon),$$

we get that (3.1) holds for n large enough.

By Lemma 3.6, $(1+Q)^{-1} = 1+S$ for some quasinilpotent S. Hence applying (3.1) in the case of this operator S implies that (3.2) holds for n large enough. □

If the Brown measure of the operator T has more than one point in its support, then it is possible to construct distinct upper-triangular decompositions of T. Theorem 1.8 tells us that the strong operator limit, A, of the sequence $|T^n|^{1/n}$ has as spectral projections $P(T, r\overline{\mathbb{D}})$ for $r \in [0, \|T\|]$. Thus if $a_1, \ldots, a_m$ is an ordering of the support of the Brown measure of T satisfying $|a_1| \leq |a_2| \leq \cdots \leq |a_m|$, then the corresponding upper-triangular decomposition $T = N + Q$ is upper-triangular with respect to the spectral projections of A for disks centered at the origin. Theorem 1.8 thus implies that $A = |N|$. We use this fact to show that if the upper-triangular part Q for such an ordering is quasinilpotent, then T has the norm convergence property.

Proposition 3.8. *Let $\mathcal{M}$ be a finite von Neumann algebra and $Q \in \mathcal{M}$ be quasinilpotent. Let $(P_i)_{1\le i\le m}$ be projections such that $\sum_{i=1}^m P_i = 1$ and such that, for every $1 \le k \le m$, the projection $\sum_{i=1}^k P_i$ is Q-invariant. Let $a_1, \dots, a_m$ be complex numbers with $0 \le |a_1| \le |a_2| \le \cdots \le |a_m|$. Let $T = \sum_{i=1}^m a_i P_i + Q$. Then $|T^n|^{1/n}$ converges in norm as $n \to \infty$ to $\sum_{i=1}^m |a_i| P_i$. In particular, T has the norm convergence property.*

Proof. If $a_m = 0$, the result is clear, so assume $a_m \neq 0$. The proof proceeds by induction. The case $m = 1$ follows from Proposition 3.7. For $m > 1$, suppose the result holds for all $1 \le k < m$.

Let

$$N = \sum_{i=1}^{m-1} a_i P_i, \quad Q_{11} = Q \sum_{i=1}^{m-1} P_i, \quad Q_{12} = \sum_{i=1}^{m-1} P_i Q P_m, \quad Q_{22} = P_m Q.$$

Note that by Lemma 1.11, Q_{11} and Q_{22} are quasinilpotent.

We may now write T as the 2×2 matrix,

$$T = \begin{pmatrix} N + Q_{11} & Q_{12} \\ 0 & a_m + Q_{22} \end{pmatrix}.$$

We now write

$$A = N + Q_{11}, \qquad C = a_m + Q_{22}, \quad B_n = \sum_{k=0}^{n-1} A^k Q_{12} C^{n-k-1},$$
$$R_n = \begin{pmatrix} 1 & B_n C^{-n} \\ 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},$$

so that $T^n = R_n G^n$. Note that R_n is invertible in $\mathcal{M}$ and C is invertible in $P_m \mathcal{M} P_m$, with

$$C^{-1} = a_m^{-1} + S$$

for a quasinilpotent operator S (by Lemma 3.6), and

$$R_n^{-1} = \begin{pmatrix} 1 & -B_n C^{-n} \\ 0 & 1 \end{pmatrix}.$$

By the inductive hypothesis, for sufficiently small $\varepsilon > 0$, there exists $K_0 \in \mathbb{N}$ such that for any $n \geq K_0$, letting $\alpha = \max\{\|N\|, |a_m^{-1}|\} + 1$,

$$\begin{aligned} \|A^n\| < \left(\|N\| + \frac{\varepsilon}{3\alpha}\right)^n = \left(|a_{m-1}| + \frac{\varepsilon}{3\alpha}\right)^n, \\ (|a_m| - \varepsilon)^n < \|C^n\| < (|a_m| + \varepsilon)^n, \\ (|a_m^{-1}| - \varepsilon)^n < \|C^{-n}\| < \left(|a_m^{-1}| + \frac{\varepsilon}{3\alpha}\right)^n. \end{aligned}$$

Hence for $n > K_0$,

$$\begin{aligned} \|B_n C^{-n}\| &= \left\|\sum_{k=0}^{K_0} A^k Q_{12} C^{-k-1} + \sum_{K_0+1}^{n-1} A^k Q_{12} C^{-k-1}\right\| \\ &\leq \left\|\sum_{k=0}^{K_0} A^k Q_{12} C^{-k-1}\right\| + \sum_{K_0+1}^{n-1} \|A^k\| \|Q_{12} C^{-1}\| \|C^{-k}\| \\ &< \left\|\sum_{k=0}^{K_0} A^k Q_{12} C^{-k-1}\right\| + \|Q_{12} C^{-1}\| \sum_{K_0+1}^{n-1} \left(|a_{m-1}| + \frac{\varepsilon}{3\alpha}\right)^k \left(|a_m|^{-1} + \frac{\varepsilon}{3\alpha}\right)^k \\ &< \left\|\sum_{k=0}^{K_0} A^k Q_{12} C^{-k-1}\right\| + n \|Q_{12} C^{-1}\| (1+\varepsilon)^n, \end{aligned}$$

so that for $n > K_0$, we have

$$\|R_n\| < 1 + \left\| \sum_{k=0}^{K_0} A^k Q_{12} C^{-k-1} \right\| + n\|Q_{12}C^{-1}\|(1+\varepsilon)^n$$
$$\|R_n^{-1}\| < 1 + \left\| \sum_{k=0}^{K_0} A^k Q_{12} C^{-k-1} \right\| + n\|Q_{12}C^{-1}\|(1+\varepsilon)^n.$$

Therefore there exists $K_1 \geq K_0$ such that for $n \geq K_1$, we have

$$\|R_n\| < (1+2\varepsilon)^n \quad \text{and} \quad \|R_n^{-1}\| < (1+2\varepsilon)^n.$$

Since

$$|T^n|^2 = (G^*)^n R_n^* R_n G^n$$

and

$$(G^*)^n R_n^* \frac{(R_n^*)^{-1} R_n^{-1}}{\|(R_n^*)^{-1} R_n^{-1}\|} R_n G^n \leq |T^n|^2 \leq (G^*)^n \|R_n^* R_n\| G^n,$$

we get

$$(1+2\varepsilon)^{-1} |G^n|^{1/n} < |T^n|^{1/n} < (1+2\varepsilon)|G^n|^{1/n}. \tag{3.3}$$

The inductive hypothesis implies that $|G^n|^{1/n}$ converges in norm to

$$\begin{pmatrix} |N| & 0 \\ 0 & |a_m| \end{pmatrix}.$$

Call this operator $\hat{N}$. There exists $K_2 \geq K_1$ such that for any $n \geq K_2$,

$$\hat{N} - \varepsilon < |G^n|^{1/n} < \hat{N} + \varepsilon.$$

Combining this with equation (3.3) gives us

$$(1+2\varepsilon)^{-1}(\hat{N}-\varepsilon) < |T^n|^{1/n} < (1+2\varepsilon)(\hat{N}+\varepsilon).$$

Hence $|T^n|^{1/n}$ is norm-convergent and its limit is $\hat{N}$, as desired. □

We now show that for an operator with finitely supported Brown measure, the spectral ordering used to construct an upper-triangular decomposition $T = N + Q$ does not affect whether s.o.t.-quasinilpotent part Q is quasinilpotent.

Proposition 3.9. *Suppose $T \in \mathcal{M}$ has finitely supported Brown measure. Suppose that there exists an upper-triangular decomposition $T = N + Q$ such that Q is quasinilpotent. If $T = \hat{N} + \hat{Q}$ is another upper-triangular decomposition, then also $\hat{Q}$ is quasinilpotent.*

Proof. Writing $N = \sum_{k=1}^{n} a_k P_k$, where for $m \leq n$, $\sum_{k=1}^{m} P_k$ is the Haagerup-Schultz projection of T associated with the set $\{a_1, a_2, \ldots, a_m\}$, Lemma 1.11 and Corollary 1.13 imply that $\sigma(T) = \{a_1, a_2, \ldots, a_n\}$.

If $\{b_1, b_2, \ldots, b_n\}$ is any reordering of $\{a_1, a_2, \ldots, a_n\}$, and $T = \hat{N} + \hat{Q}$ is the corresponding decomposition of T, with $\hat{N} = \sum_{k=1}^{n} b_k R_k$ and with $\sum_{k=1}^{m} R_k$ the Haagerup-Schultz projection of T associated with the set $\{b_1, b_2, \ldots, b_m\}$ for all $m \leq n$, then Lemma 1.11 implies that $\sigma(R_k T R_k)$ contains finitely many points for each k. Combining this with Observation 1.3, we see that $\operatorname{supp}(\nu_{R_k T R_k}) = \sigma(R_k T R_k)$ for each k. Since $R_k \hat{Q} R_k$ is s.o.t.-quasinilpotent, we have for every k,

$$\sigma(R_k T R_k) = \operatorname{supp}(\nu_{R_k T R_k}) = \{b_k\}.$$

As $R_k T R_k = b_k + R_k \hat{Q} R_k$, this implies that $R_k \hat{Q} R_k$ is quasinilpotent. Since this is true for all k, Corollary 1.13 implies that $\hat{Q}$ is quasinilpotent, completing the proof. □

Proposition 3.10. *Let $Q \in \mathcal{M}$ be s.o.t.-quasinilpotent. Then the following are equivalent:*

(a) Q has the norm convergence property,

(b) Q has the shifted norm convergence property,

(c) Q is quasinilpotent.

Proof. The equivalence of (a) and (c) is clear. The implication (b) $\Longrightarrow$ (a) is also immediate, while the implication (c) $\Longrightarrow$ (b) is from Proposition 3.7. □

We will show, using the following proposition in Example 3.12 below, that the equivalence of (a) and (b) in Proposition 3.10 does not extend even to operators having Brown measure supported at exactly one (nonzero) point.

Proposition 3.11. *Let $\mathcal{M}$ be a finite von Neumann and let $Q \in \mathcal{M}$ be s.o.t.-quasinilpotent. Then $1 + Q$ has the norm convergence property if and only if $\sigma(1 + Q) \subseteq \mathbb{T}$.*

Proof. The proof is a straightforward application of the spectral radius formula. Suppose first that $\sigma(1 + Q) \subseteq \mathbb{T}$. Since the spectral radii of $1 + Q$ and $(1 + Q)^{-1}$ are both 1, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\|(1 + Q)^n\| < (1 + \varepsilon)^n$ and $\|(1 + Q)^{-n}\| < (1 + \varepsilon)^n$. Thus we have

$$(1 + Q^*)^n(1 + Q)^n \leq \|(1 + Q)^n\|^2 \leq (1 + \varepsilon)^{2n}$$

and

$$(1 + Q^*)^n(1 + Q)^n \geq (1 + Q^*)^n \frac{(1 + Q^*)^{-n}(1 + Q)^{-n}}{\|(1 + Q^*)^{-n}(1 + Q)^{-n}\|}(1 + Q)^n \geq (1 + \varepsilon)^{-2n}.$$

Hence,

$$(1 + \varepsilon)^{-1} \leq |(1 + Q)^n|^{1/n} \leq 1 + \varepsilon.$$

It follows that the sequence $|(1+Q)^n|^{1/n}$ converges in norm to 1.

Now assume that $1+Q$ has the norm convergence property. Since $|(1+Q)^n|^{1/n}$ must converge in norm to 1, it follows that for sufficiently large n, $|(1+Q)^n|^{1/n}$ is invertible, so that also $(1+Q)^n$ and $1+Q$ are invertible. We observe now that for large n we have

$$(1-\varepsilon)^{2n} \leq (1+Q^*)^n(1+Q)^n \leq (1+\varepsilon)^{2n}$$

so that

$$(1+\varepsilon)^{-2n} \leq (1+Q)^{-n}(1+Q^*)^{-n} \leq (1-\varepsilon)^{-2n}.$$

Hence, $(1+Q^*)^{-1}$ has the norm convergence property.

In addition, since $|(1+Q)^n|^{1/n}$ must converge in norm to 1, for large n we have $\|(1+Q)^n\| < (1+\varepsilon)^n$, so $\sigma(1+Q) \subseteq \overline{\mathbb{D}}$. Applying the same argument to $(1+Q^*)^{-1}$ we see that $\sigma(1+Q^*) \subseteq \mathbb{C} \setminus \mathbb{D}$. Thus,

$$\sigma(1+Q) \subseteq \overline{\mathbb{D}} \bigcap (\mathbb{C} \setminus \mathbb{D}) = \mathbb{T},$$

as desired. □

Here is the promised example, that serves both to show that Proposition 3.10 does not extend, and to show that the naive guess at an answer to Question 3.1 is wrong.

Example 3.12. *Let Q be the s.o.t.-quasinilpotent operator constructed in Section 3.2, where $\sigma(Q)$ is a nondegenerate interval in the real line. Using either the holomorphic functional calculus and the main result of* [5]*, or arguing more directly with power series, we have that the operator $\exp(iQ)$ is of the form $1+S$, where S is s.o.t.-quasinilpotent and $\exp(iQ)$ has spectrum contained in $\mathbb{T}$. Hence, by Proposition 3.11, $\exp(iQ)$ has the norm convergence property. However, $\sigma(\exp(iQ)-1) \neq \{0\}$, so $S = \exp(iQ)-1$ is* *not quasinilpotent and does not have the norm convergence property. Therefore,* $\exp(iQ)$ *does not have the shifted norm convergence property.*

Lemma 3.13. *Suppose that* T *has the shifted norm convergence property. Then* $\mathrm{supp}(\nu_T) = \sigma(T)$.

Proof. Let $\lambda \in \mathbb{C} \setminus \mathrm{supp}(\nu_T)$. Then by Theorem 1.8, $|(T-\lambda)^n|^{1/n}$ converges in the strong operator topology to an invertible operator, and by assumption this convergence is in norm. Hence for sufficiently large n, $|(T-\lambda)^n|^{1/n}$ is invertible, and consequently $T-\lambda$ is also invertible. Thus $\lambda \notin \sigma(T)$. Hence $\sigma(T) \subseteq \mathrm{supp}(\nu_T)$. Since the reverse inclusion is always true, the result is proven. □

Proposition 3.14. *Let* $T \in \mathcal{M}$. *Suppose that the support of the Brown measure of* T *is finite and that* T *has the shifted norm convergence property. Let* $T = N + Q$ *be an upper-triangular decomposition of* T. *Then* Q *is quasinilpotent.*

Proof. We let

$$\begin{pmatrix} a_1 + Q_{1,1} & Q_{1,2} & \dots & Q_{1,m} \\ 0 & a_2 + Q_{2,2} & \dots & Q_{2,m} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & a_m + Q_{m,m} \end{pmatrix}$$

be the matrix representation of T with respect to any upper-triangular decomposition.

By Lemma 3.13 and Observation 1.3, for each i, $\sigma(a_i + Q_{i,i})$ is a connected subset of a finite set, and hence is a singleton. It follows that $\sigma(Q_{i,i}) = \{0\}$ as it is a singleton and contains 0. The result is now a consequence of Corollary 1.13. □

We shift our attention now to decomposable operators. Recall that an operator T on a Hilbert space $\mathcal{H}$ is said to be *decomposable* if, for every pair (U, V) of open sets in the complex plane whose union is the whole complex plane, there are closed, T-invariant subspaces $\mathcal{H}'$ and $\mathcal{H}''$ such that the restrictions of T to these have spectra contained in U and V, respectively and such that $\mathcal{H}' + \mathcal{H}'' = \mathcal{H}$.

Given an operator T on a Hilbert space $\mathcal{H}$, a *spectral capacity* for T is a mapping E from the collection of closed subsets of $\mathbb{C}$ into the set of all closed T-invariant subspaces of $\mathcal{H}$ such that

1. $E(\emptyset) = \{0\}$ and $E(\mathbb{C}) = \mathcal{H}$,
2. $E(\overline{U_1}) + E(\overline{U_2}) + ... + E(\overline{U_n}) = \mathcal{H}$ for every finite open cover $\{U_1, U_2, ..., U_n\}$ of $\mathbb{C}$,
3. $E\left(\bigcap_{k=1}^{\infty} F_k\right) = \bigwedge_{k=1}^{\infty} E(F_k)$ for every countable family $(F_k)_{k=1}^{\infty}$ of closed subsets of $\mathbb{C}$,
4. letting $P_{E(F)}$ denote the projection with range $E(F)$, and calculating the spectrum in the algebra $P_{E(F)}\mathcal{M}P_{E(F)}$, $\sigma(TP_{E(F)}) \subseteq F$, with the convention that if T is an operator on the 0 Hilbert space, $\sigma(T) = \emptyset$.

The following, Proposition 1.2.23 of [10], gives conditions equivalent to decomposability for an operator.

Proposition 3.15. *Let T be a bounded operator on a Hilbert space H. Then the following are equivalent:*

1. *T is decomposable,*
2. *T has a spectral capacity,*
3. *for every closed subset F of $\mathbb{C}$, $\mathcal{H}_T(F)$ is closed and*

$$\sigma((1 - P_T(F))T) \subseteq \overline{\sigma(T) \setminus F}$$

where $\mathcal{H}_T(F)$ is the local spectral subspace of T corresponding to F, $P_T(F)$ is the projection onto $\mathcal{H}_T(F)$, and the spectrum is calculated in the algebra $(1 - P_T(F))\mathcal{H}_T(F)(1 - P_T(F))$.

See, for example, [10] for more on local spectral theory and decomposability.

In Section 9 of [8], Haagerup and Schultz proved that for decomposable operators $T \in \mathcal{M}$, the Haagerup-Schultz projection $P(T, F)$ equals the projection onto the local spectral subspace $\mathcal{H}_T(F)$ of T for F, whenever F is a closed subset of the complex plane.

Proposition 3.16. *Suppose T is an operator with finitely supported Brown measure. If T is decomposable, then T has the shifted norm convergence property.*

Proof. Since decomposabiliity and the shifted norm convergence property are both translation independent, we may assume without loss of generality that $0 \notin \text{supp}(\nu_T)$ and for any two elements λ_1 and λ_2 in the support of ν_T, $|\lambda_1| \neq |\lambda_2|$.

Our method for this proof is to choose a specific ordering, $\alpha_1, \alpha_2, ..., \alpha_n$, for the support of the Brown measure of T. We then consider the upper-triangular decomposition $T = N + Q$ corresponding to this ordering, and show that Q must be quasinilpotent.

Note also that by Corollary 9.3 of [8], $\text{supp}(\nu_T) = \sigma(T)$.

Let P_s denote the Haagerup–Schultz projection $P(T, s\overline{\mathbb{D}})$. By Proposition 9.2 of [8], P_s is the projection onto the closure of the local spectral subspace $\mathcal{H}_T(s\overline{\mathbb{D}})$ and, since T is decomposable, this local spectral subspace is itself closed. Since the map $F \mapsto \mathcal{H}_T(F)$ (for F an arbitrary closed subset of $\mathbb{C}$) is a spectral capacity, the spectrum of TP_s is contained in $s\overline{\mathbb{D}}$, while when $P_s \neq 1$, since $\sigma(T)$ is finite, the spectrum of $(1 - P_s)T$, considered as and element of $(1 - P_s)\mathcal{M}(1 - P_s)$, is contained in $\mathbb{C} \setminus s\mathbb{D}$.

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the elements of $\text{supp}(\nu_T)$, ordered by increasing absolute value, let $T = N + Q$ be the upper-triangular decomposition given by this ordering, and let $\alpha_0 = 0$. For any $0 < \varepsilon < |\alpha_1|$ and any $k \in \{1, 2, ..., n\}$, we have $\sigma(TP_{|\alpha_k|}) \subseteq |\alpha_k|\overline{\mathbb{D}}$

and $\sigma((1-P_{|\alpha_k|-\varepsilon})T) \subseteq \mathbb{C} \setminus (|\alpha_k|-\varepsilon)\mathbb{D}$, where the spectra are computed in the algebras $P_{|\alpha_k|}\mathcal{M}P_{|\alpha_k|}$, and $(1-P_{|\alpha_k|-\varepsilon})\mathcal{M}(1-P_{|\alpha_k|-\varepsilon})$, respectively.

Since the support of ν_T is finite we have for $k \in \{1, 2, ..., n\}$ (and for sufficiently small ε),

$$P_{|\alpha_k|-\varepsilon} = P_{|\alpha_{k-1}|}$$

and

$$P_{|\alpha_k|+\varepsilon} = P_{|\alpha_k|}.$$

Hence we have

$$\sigma((1-P_{|\alpha_{k-1}|})T) \subseteq \mathbb{C} \setminus (|\alpha_k|-\varepsilon)\mathbb{D},$$

with the spectrum computed in the algebra $(1-P_{|\alpha_{k-1}|})\mathcal{M}(1-P_{|\alpha_{k-1}|})$. Since the left side of this expression has no dependence on ε,

$$\sigma((1-P_{|\alpha_{k-1}|})T) \subseteq \mathbb{C} \setminus (|\alpha_k|)\mathbb{D}.$$

Therefore, by Lemma 1.11,

$$\sigma((1-P_{|\alpha_{k-1}|})TP_{|\alpha_k|}) \subseteq |\alpha_k|\mathbb{T},$$

where the spectrum is computed in the algebra $(P_{|\alpha_k|}-P_{|\alpha_{k-1}|})\mathcal{M}(P_{|\alpha_k|}-P_{|\alpha_{k-1}|})$. Since $N(P_{|\alpha_k|}-P_{|\alpha_{k-1}|}) = \alpha_k(P_{|\alpha_k|}-P_{|\alpha_{k-1}|})$, we know that

$$(1-P_{|\alpha_{k-1}|})QP_{|\alpha_k|} = (1-P_{|\alpha_{k-1}|})TP_{|\alpha_k|} - \alpha_k(P_{|\alpha_k|}-P_{|\alpha_{k-1}|})$$

and

$$\sigma((1-P_{|\alpha_{k-1}|})QP_{|\alpha_k|}) \subseteq |\alpha_k|\mathbb{T} - \alpha_k,$$

again computing the spectrum in the algebra $(P_{|\alpha_k|} - P_{|\alpha_{k-1}|})\mathcal{M}(P_{|\alpha_k|} - P_{|\alpha_{k-1}|})$.

Fix $j \in \{1, 2, ..., n\}$. Let $\rho > 0$ satisfy

$$\alpha_1 + \rho\alpha_j \neq 0$$

and for $2 \leq k \leq n$,

$$|\alpha_{k-1} + \rho\alpha_j| < |\alpha_k + \rho\alpha_j|.$$

A repetition of the previous argument, replacing T with $T + \rho\alpha_j$, gives

$$\sigma((1 - P_{|\alpha_{j-1}|})QP_{|\alpha_j|}) \subseteq |(1 + \rho)\alpha_j|\mathbb{T} - (1 + \rho)\alpha_j,$$

where the spectrum is computed in the algebra $(P_{|\alpha_j|} - P_{|\alpha_{j-1}|})\mathcal{M}(P_{|\alpha_j|} - P_{|\alpha_{j-1}|})$.

It follows that the spectrum of $(1 - P_{|\alpha_{j-1}|})QP_{|\alpha_j|})$, computed in the algebra $(P_{|\alpha_j|} - P_{|\alpha_{j-1}|})\mathcal{M}(P_{|\alpha_j|} - P_{|\alpha_{j-1}|})$, lies in the intersection of the two circles $|(1+\rho)\alpha_j|\mathbb{T} - (1+\rho)\alpha_j$ and $|\alpha_j|\mathbb{T} - \alpha_j$, which is precisely $\{0\}$. Since this argument holds for every j, it follows from Lemma 1.11 that Q is quasinilpotent, which completes our proof. □

We collect all of the results of this chapter into one final theorem:

Theorem 3.17. *Let $T \in \mathcal{M}$. Suppose the Brown measure of T is finitely supported. Then the following are equivalent:*

(i) *$\sigma(T)$ is a finite set*

(ii) *$\sigma(T) = \mathrm{supp}(\nu_T)$*

(iii) *T has the shifted norm convergence property*

(iv) *T is decomposable*

(v) *There exists an upper-triangular decomposition $T = N + Q$ such that Q is quasinilpotent*

(vi) For any upper-triangular decomposition $T = N + Q$*,* Q *is quasinilpotent*

(vii) T^* *has the shifted norm convergence property.*

Proof. We showed that $(iv) \implies (iii)$ in Proposition 3.16 and that $(iii) \implies (ii)$ in Lemma 3.13. That $(ii) \implies (i)$ is trivial.

That $(i) \implies (iv)$ follows from a general result that operators with finite spectra are decomposable. See, for example, Proposition 1.4.5 of [10] for a proof of a stronger result.

Equivalence of (v) and (vi) is a consequence of Proposition 3.9. We proved that $(iii) \implies (vi)$ as Proposition 3.14. The converse is a consequence of Proposition 3.8.

To prove equivalence of (i) and (vii), recall that $\sigma(T^*) = (\sigma(T))^*$, where the $*$ on the right refers to complex conjugation. Assuming (i), we note that $\sigma(T^*)$ is a finite set, and hence (vii) holds by the equivalence of (i) and (iii), applied to T^*. Assuming (vii), we have that $\sigma(T^*)$ is a finite set by equivalence of (iii) and (i) applied to T^*. Hence (i) holds, as desired.

We have thus shown equivalence of all seven statements, completing the proof. $\square$

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