SOLVING CONFORMAL FIELD THEORIES USING CONFORMAL BOOTSTRAP

A Thesis

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2017

Major Subject: Physics

ABSTRACT

Conformal field theories (CFTs) play central roles in modern theoretical physics. Many CFTs are strongly coupled and cannot be studied using perturbative method. Conformal bootstrap provides a non-perturbative approach to study CFTs, which only employs the crossing symmetry and unitarity while not depending on classical Lagrangian description. This method has been successfully applied to solve the 3D Ising model and O(N) vector model. In this thesis, the conformal bootstrap approach will be applied to study the $5D$ $O(N)$ vector model and $4D$ $N = 1$ supersymmetric conformal field theories (SCFT).

For the $5D$ $O(N)$ vector model, we bootstrap the mixed four-point correlators of the leading $O(N)$ vector ϕ_i and the $O(N)$ singlet σ. By imposing mild gaps in the spectra, we are able to isolate the scaling dimensions $(\Delta_\phi, \Delta_\sigma)$ in a small island for large $N = 500$, which is highly consistent with the results obtained from large N expansion. For smaller $N \leqslant 100$, the islands disappear after increasing Λ which suggests a lower bound on the critical value $N_c > 100$, below which the interacting $O(N)$ CFTs turn into nonunitary.

To bootstrap SCFTs, it needs analytical expression of the superconformal block function, which is the summation of several conformal block functions with coefficients determined by supersymmetry. We have calculated the most general $4D$ $N = 1$ superconformal block function of scalar operators based on the supershadow approach and superembedding formalism. In superembedding space the $4D$ $N = 1$ superconformal transformation $\mathcal{T} \in SU(2, 2|1)$ is realized linearly and the superconformal covariant three-point correlator functions can be constructed directly. Based on these results, the minimal SCFT with lowest c central charge among all the known $4D$ $N = 1$ SCFTs has been studied through bootstrapping the mixed correlators of the chiral operator Φ and the

operator $X \sim \Phi\Phi^\dagger$. The scaling dimensions (Δ_Φ, Δ_X) have been isolated into a small island! The results further confirm the existence of the promising minimal SCFT and reveal several interesting properties of this theory.

DEDICATION

To my family.

ACKNOWLEDGMENTS

I greatly appreciate my advisor Christopher Pope for his support, guide, encouragement and the patience on helping me to solve various kinds of problems, especially in the last year of my Ph.D. studies. I learned physics and mathematics both from his graduate courses and the stimulative discussions with him. There were unexpected problems during my Ph.D. studies, however, the conversions with him were always friendly, inspiring and constructive. I am also benefited from his abroad and in depth understanding on the correlation between high energy theory and geometry. I am grateful to Bhaskar Dutta for being my committee member and his encouragements and supports during my difficult period. I have learned a lot from his inspiring courses on quantum field theory, cosmology and particle physics and also the suggestive discussions with him throughout my Ph.D. studies. I also would like to thank Robert Webb for being my committee member and it was my pleasure to work for him as a teaching assistant for years. I appreciate Stephen Fulling for being my committee member and his valuable suggestions on studying modern mathematics. Besides, I would like to thank Tianjun Li and Dimitri Nanopoulos for the collaborations in the early stage of my Ph.D. studies. I am also grateful to Katrin Becker, Melanie Becker and Ergin Sezgin for their advanced courses on high energy theory and helpful discussions.

I would like to thank the graduate students and posdocs in TAMU high energy theory group including Arash Azizi, Yaniel Cabrera, Sunny Guha, Esteban Jimenez, Balakrishnan Nagaraj, Juncheng Ron, Ning Su, Sean Wu, Yaodong Zhu and Sebastian Guttenberg, William D. Linch, Ilarion Melnikov, Jakob Palmkvist , Daniel Robbins, Andrew Royston. I also appreciate our department head Peter McIntyre and the academic advisors, especially Sherree Kessler for their consistent assistances during my Ph.D. studies.

Finally, I deeply appreciate my family, for their consistent support and love.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supervised by a thesis committee consisting of Professor Christopher Pope, Professor Bhaskar Dutta and Professor Robert Webb of the Department of Physics and Professor Stephen Fulling of the Department of Mathematics.

The numerical calculations were performed on the Mac Lab cluster in the physics department.

Funding Sources

Graduate study was supported by the Department of Energy under grant DE-FG02-13ER42020 and a Teaching Assistantship from the Department of Physics.

NOMENCLATURE

AdS	Anti de Sitter
CB	Conformal block
CFT	Conformal field theory
CPW	Conformal partial wave
IR	Infrared
OPE	Operator product expansion
QED	Quantum electrodynamics
QFT	Quantum field theory
RG	Renormalization group
SCFT	Superconformal field theory
SDP	Semi-definite polynomial
SYM	Supersymmetric Yang-Mills
UV	Ultraviolet

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1. INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

Conformal field theories (CFTs) play crucial roles in modern theoretical physics. They can be used to describe various kinds of phenomena in nature, from the vaporization of water to the physical processes occurring near the horizon of a black hole. In quantum field theory (QFT), the evolution of the theory from Ultraviolet (UV) region to Infrared (IR) region is described by the renormalization group (RG) flow. For a large variety of QFTs, the RG flows of the coupling constants (beta functions) vanish at the fixed points in the IR, and the theories become scale invariant. The scale invariant symmetry of the theory is believed to be enhanced to conformal symmetry. At the IR fixed point, the theory could be rather different from its UV description given by the Lagrangian. In fact different UV Lagrangians (but with the same global symmetry) can lead to the same theory at the fixed point! Such a phenomenon is known as a duality between the Lagrangian theories, and the “common" fixed point is described as a critical universality. The “universality" classes unify the core contents in almost all the major directions of physics, such as statistical physics, condensed matter physics, particle physics, and quantum gravity.

When the QFT is weakly coupled, the RG flow can be evaluated perturbatively. The fixed point is obtained by solving the equation of vanishing beta function $\beta(\lambda) = 0$. The solutions can be expanded to higher orders from which we can get better and better understanding on the critical universality. However, for many interesting QFTs the fixed points are of strongly coupled. A well-known example is the 3D Ising model with UV Lagrangian

$$S = \int d^d x [\frac{(\partial\phi)^2}{2} + \frac{g\phi^4}{4}], \tag{1.1}$$

In dimension $d = 4 - \epsilon$ the one-loop beta function reads

$$\beta(\lambda) = -\epsilon\lambda + \frac{9}{8\pi^2}\lambda^2. \tag{1.2}$$

Set $\beta(\lambda) = 0$, the coupling constant g at the fixed point can be solved at the first order $g_* = \frac{8\pi^2}{9}\epsilon$. Take $\epsilon = 1$, this formula gives a large coupling coefficient $g_* > 1$ for $3D$ Ising model. Since the theory is strongly coupled one cannot expand the loop corrections in terms of g^n. This theory has been effectively studied in $D = 4-\epsilon$ using the ϵ expansion approach and finally taking $\epsilon = 1$. However, it is questionable if the ϵ expansion remains valid when $\epsilon \sim O(1)$.

The RG flow approach is considered to be non-essential to study CFTs due to the reason that the conformal symmetry is not directly evolved in the analysis. The perturbative approach cannot get access to the theories with strong couplings. Moreover, it requires a classical Lagrangian description of the theory in the UV side, while as we know now there are lots of interesting CFTs without the UV Lagrangian description. Therefore a more fundamental approach based on conformal symmetry is needed to completely classify the CFT landscape. Conformal bootstrap provides a promising non-perturbative approach on this purpose. Conformal bootstrap was first proposed in [2] which aims to solve the conformal field theories based on the conformal symmetry even without the knowledge of Lagrangian. It leads to remarkable success in 2D in solving the rational CFTs [4]. In 2D the global conformal symmetry enlarges to Virasoro symmetry which is of infinite dimension and makes the theory solvable. However, in higher dimensional spacetime $D > 2$, there is no such symmetry enhancement and one cannot directly reproduce the success on rational CFTs in higher dimensions. In consequence, this non-perturbative approach did not attract much attention until the breakthrough work [5].

The modern conformal bootstrap, initiated from [5], only employs consistent conditions on unitarity and crossing symmetry. Crossing symmetry requires the four-point correlator is invariant when evaluated in different channels, which also relates to the associativity of conformal operator algebra. Surprisingly, it has been shown that these simple constraints can generate highly non-trivial bounds on the CFT data, including operator scaling dimensions and OPE coefficients. Following this discovery, conformal bootstrap has been revived in the past years and there are lots of exciting developments in these directions. Specifically, we have the following interesting directions in which the modern conformal bootstrap plays a key role:

- **Classify CFT landscape** Classification of CFTs has been partially successful in 2D [4], and due to the developments in past years, it is possible to classify CFTs in higher dimensions as well. According to the modern numerical conformal bootstrap, it only needs to specify the global symmetry of the theory as well as the mild assumptions on the gap of scaling dimensions between the first few operators to isolate CFTs in the parameter space. The 3D Ising model and O(N) vector model have been isolated in small and closed regions using numerical conformal bootstrap, which give the most precise results on the CFT data up to now [46, 58]. Similar studies based on conformal bootstrap has also been fulfilled on the 5D O(N) vector model with cubic couplings [69]. The fundamental fields in these models are scalars which admit rather simple four-point functions [24, 25]. They provide ideal examples to show the feasibility of numerical conformal bootstrap. Besides the CFTs constructed by scalars, it is known that in 3D, the QED admits IR fixed points and shows interesting duality webs [7–10]. The 4D QED does not run to the IR fixed point, instead the IR theory is composed of free particles. However, the 4D

Yang-Mills theory does possess IR fixed point in the so-called “conformal window":

$$N_{f0} < N_f < \frac{11}{2} N_c, \tag{1.3}$$

where the N_{f0} is the critical value of flavors below which the IR theory has no fixed point. In principle the numerical conformal bootstrap can also solve these CFTs although there are more work to do.

- **Solve supersymmetric conformal field theories** Supersymmetric quantum field theories possess several interesting properties. Supersymmetry plays an important role to protect from the quantum loop corrections which makes the supersymmetric theories perform more elegantly at quantum level. In particular the space of SCFT-s can be described by geometrical structures which nicely relate to the algebraic curves or vertex algebra [12]. Similar to the supersymmetric field theories which are not too difficult to study but sufficiently interesting to learn the properties of QFTs, SCFT could be an ideal laboratory to learn the CFT classifications based on conformal bootstrap. Moreover, it has been shown that there are special SCFT families which have no classical Lagrangian description. These SCFTs are actually isolated theories in the SCFT landscape! The most famous non-Lagrangian SCFT is the $6D$ $(2,0)$ SCFT. It describes the IR dynamics of a stack of M5 branes, moreover, this theory plays a key role in several interesting duality phenomena of QFTs in lower dimensions. Our understanding on this theory is limited to some special cases with large central charge or supersymmetric-protected quantities. In the large central charge approximation the theory can be studied through AdS/CFT. Another interesting example are the $4D$ T_N theories, which perform as fundamental blocks to construct $4D$ S-classes [13]. Fortunately conformal bootstrap does not rely on classical Lagrangian description therefore it becomes the only technique that works to study the general physical quantities. Moreover, it is quite possible that by using conformal bootstrap, we can obtain certain isolated SCFTs whose existences were unnoticed since they cannot be accessed by classical Lagrangian description.

For the SCFTs with classical Lagrangian descriptions, usually the coupling is not uniquely determined at the fixed point, instead, the coupling coefficient can be varied continuously (4D $N = 2, 4$ SCFTs) or discretely (3D $N = 6$ ABJM theory). The strong-weak duality can help us to solve the theory perturbatively giving the coupling is sufficient strong. Similarly one may also employ the AdS/CFT correspondence, which relates the strongly coupled CFT to the weakly coupled holographic theory. However, there is a subtle range interpolating from weak coupling to strong couple, and in this case, the analytically perturbative approach, even with S-duality does not work properly. Conformal bootstrap therefore becomes the only reliable approach to evaluate the CFT data in this transition region.

- **Analytical structures of CFT** The numerical conformal bootstrap itself is a numerical laboratory to “test" the power of conformal bootstrap. The numerical conformal bootstrap has lead to remarkable successes in solving CFTs, however, we actually do not know the analytical reason why this approach is so powerful. More than providing the best numerical estimation on the CFT data, the numerical conformal bootstrap results are rather inspiring that they suggest there are hidden structures/symmetries in CFTs which make the theory solvable. It is a pivotal problem to understand the analytical reasons beyond the numerical conformal bootstrap.

  In the example of 3D Ising model conformal bootstrap, it needs to bootstrap the mixed four-point correlators of two scalars with opposite Z_2 charges, besides, one needs to apply mild assumptions on the spectrum that there are only two relevant scalars in the theory. An analytical study is expected to show why it is sufficient to isolate out the 3D Ising model just based on these conditions. Moreover, to isolate out extra fixed points, what are the new conditions needed in the conformal bootstrap? Hopefully the analytical studies will explain the sufficient bootstrap conditions and provide guidance on the specific bootstrap conditions for different CFTs. Furthermore, our current numerical conformal bootstrap can be significantly improved by combining with the analytical properties of CFTs.

A first step to study the possible hidden structures/symmetries beyond the numerical conformal bootstrap is to solve the crossing equation analytically. Crossing equation is one of the constraints used in conformal bootstrap*, and its properties are crucial in conformal bootstrap. By studying the crossing equation analytically one can obtain some primary but still non-trivial results on the CFT data. For example, by analyzing the crossing equation in the light-cone coordinates, the authors in [14, 15] have obtained spectrum and OPE coefficients of operators with large spin. The Wilson-Fisher fixed point of Ising model in $4-\epsilon$ dimension spacetime can be reproduced at the lower orders purely based on the conformal symmetry without calculating the Feynman diagrams [22]. A thorough study on the crossing equation is expected to introduce a more complete understanding on the CFTs.

The ultimate object of conformal bootstrap is to provide a mathematically rigorous definition of CFT by providing the operators and the their algebra. This is the way used to define 2D rational CFTs with finite-dimensional operator algebra. Currently it is not sure how to generalize this method to higher dimensions but the successes of numerical conformal bootstrap strongly suggest this is possible in higher dimensions. Moreover, it is hoped to generalize the conformal bootstrap to general QFTs, the so-called S-matrix bootstrap [23], which aims to describe and classify general

*There is another approach developed recently based on Mellin space [16], in which the crossing equation is replaced by different consistency equations in the bootstrap condition.

QFTs based on certain consistent conditions independent of the Lagrangian.

Besides the directions proposed above, there are extra interesting directions closely related to conformal bootstrap, like the S-matrix bootstrap [23] and the projects to understand quantum gravity emergence from the bootstrap through AdS/CFT correspondence [17, 21]. However, these problems are not directly relevant to my Ph.D. research so they will not be discussed in this thesis.

In this thesis, we will mainly focus on the first two directions discussed above. Specifically, we will present the bootstrap studies on the generalization of $O(N)$ vector model to higher dimension $D = 5$ and the $4D$ $N = 1$ minimal SCFT:

1.2 Bootstrapping $5D$ $O(N)$ Vector Model

The $3D$ $O(N)$ vector model has been known to play an important role both in condensed matter physics and the AdS_4/CFT_3 correspondence. It is easy to show that the classical $O(N)$ free scalar theories perturbed by ϕ^4 interaction have no stable IR fixed point in the spacetime $D > 4$. In [76] a new cubic model with $O(N)$ vector scalars and an extra singlet scalar has been proposed

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma\phi_i\phi_i + \frac{g_2}{3!}\sigma^3 \tag{1.4}$$

where the fields ϕ_i constructs an $O(N)$ vector representation while σ is an $O(N)$ singlet. In $D = 6 - \epsilon$ spacetime the cubic scalar interactions are relevant and the theory (1.4) is shown to admit an IR interacting fixed point from Feynman diagram calculations [76]. The fixed point is suggested to be stable and unitary for sufficient large $N > N_c$, where N_c is the critical value below which the theory ceases to be unitary. Such theory can play an interesting role in the AdS_6/CFT_5 correspondence. Their studies are based on the perturbative approaches: ϵ or large N expansion and the evaluations on the parameters are largely affected by the higher loop corrections. Studies based on the nonperturbative functional renormalization group equations lead to different conclusion on this theory. Therefore as a nonperturbative and Lagrangian-independent approach, the conformal bootstrap can be used to study the $5D$ $O(N)$ fixed point and the critical value N_c.

We will show that for sufficient large $N = 500$, the scaling dimensions of $(\Delta_\phi, \Delta_\sigma)$ can be isolated in a small island using conformal bootstrap, and the results are highly consistent with those obtained from perturbative approaches, which suggest that the fixed point for sufficient large N is unitary and physical. For smaller $N \leqslant 100$, the fixed point is still obtained in the bootstrap but it is nonunitary, indicating a rather large critical value $N_c > 100$.

1.3 Bootstrapping the Minimal $4D$ $N = 1$ SCFT

Evidence of the minimal $4D$ $N = 1$ SCFT has already been revealed in the study of four-point correlator $\langle \Phi\Phi^\dagger\Phi\Phi^\dagger \rangle$ [32, 62]. Here the scalar operator Φ is chiral which satisfies the shorted condition: $\bar{\mathcal{Q}}\Phi = 0$. Results from conformal bootstrap show that bound of the scaling dimension Δ_X has a kink around $(\Delta_\phi, \Delta_X) \approx (1.41, 3.14)$. This phenomenon reminds us the conformal bootstrap of 3D Ising model and $O(N)$ vector model, in which the sharp kinks appear at where the physical CFTs locate. Thus it is tempting to guess that the kink around $(\Delta_\phi, \Delta_X) \approx (1.41, 3.14)$ indicates a new $N = 1$ SCFT. This theory is dubbed "*minimal*" since its c central charge, which indicates the degree of freedom of the theory is extremely small $\approx 1/9$. It is interesting to study the mixed correlators of this theory. If the minimal SCFT is true, then the scaling dimensions of the fundamental scalar operator Φ (Δ_Φ) and the composite operator X appearing in the OPE of $X \sim \Phi\Phi^\dagger$ (Δ_X) can be isolated into a small region, out of which the scaling dimensions cannot satisfy the crossing symmetry so are unphysical.

The mixed correlators of chiral operator Φ and real operator X consist of the following three correlators

$$\langle \Phi(x_1)\bar{\Phi}(x_2)\Phi(x_3)\bar{\Phi}(x_4)\rangle, \tag{1.5}$$

$$\langle X(x_1)X(x_2)X(x_3)X(x_4)\rangle, \tag{1.6}$$

$$\langle \Phi(x_1)X(x_2)X(x_3)\bar{\Phi}(x_4)\rangle. \tag{1.7}$$

All other four-point correlators vanish due to $U(1)_R$ R-symmetry. To bootstrap the mixed correlators, the crucial ingredients are the conformal blocks of general scalar four-point function. For the first two four-point correlators, their conformal blocks have been calculated in [28, 108]. However, the conformal block of the third correlator was not solved until our work [109]. In which the $4D$ $N = 1$ conformal blocks for the most general scalars have been calculated. These results are expected to be applied in the mixed correlator conformal bootstrap in order to obtain the physically allowed isolated regions of the conformal scaling dimensions (Δ_Φ, Δ_X). Interesting results on this minimal SCFT can be obtained from modified single correlator bootstrap that the potential region of (Δ_Φ, Δ_X) is located at a sharp tip of upper bound of the scaling dimensions. Through bootstrapping above mixed correlators we do find that more regions are excluded with suitable assumptions and the sharp tip can form an isolated island! Our results support the very existence of the minimal SCFT, which has no classical Lagrangian description and can only be studied through conformal bootstrap.

On the other side, there is no known candidate UV Lagrangian description of the minimal SCFT. Alternatively one may guess that such theory can be constructed through geometric approach, for example, the $6D$ $(2, 0)$ SCFT compactified on certain 2-dimensional manifold, like the geometric configurations used to construct $4D$ $N = 2$ S-class SCFTs. We will work on this problem in future.

This thesis is organized as follows. In chapter 2, we discuss the technical details on numerical conformal bootstrap which can lead to unexpectedly strong constraints on the CFT data. In chapter 3 the numerical conformal bootstrap will be applied to study the $5D$ $O(N)$ vector model and estimate the range of the critical value N_c. In chapter 4 we calculate the superconformal block function of scalar operators in $4D$ $N = 1$ SCFTs using the superembedding formalism and supershadow approach. In chapter 5 we will apply the results on $4D$ $N = 1$ superconformal block function obtained in chapter 4 to bootstrap the $4D$ $N = 1$ minimal SCFT. We will provide further strong evidence on the existence of this theory and its properties. Further discussions on conformal bootstrap will be provided in chapter 6, including the analytical approach to solve crossing equation perturbatively, bootstrapping the the $3D$ QED IR fixed point and other interesting SCFTs in various dimensions and the application of Mellin space, etc. My future research plan will also be discussed in the last chapter.

2. REVIEW ON CONFORMAL BOOTSTRAP

2.1 Numerical Conformal Bootstrap

Conformal bootstrap employs the unitarity condition and crossing symmetry. Consider the following four point function of four scalar ϕ_is [*]:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle. \tag{2.1}$$

This four-point correlator can be calculated using OPEs:

$$\begin{aligned} \phi_1(x_1)\phi_2(x_2) &= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} C_{\mathcal{O}}(x_{12}, \partial_{x_2})\mathcal{O}(x_2), & (2.2)\\ \phi_3(x_3)\phi_4(x_4) &= \sum_{\mathcal{O}'} \lambda_{34\mathcal{O}'} C_{\mathcal{O}'}(x_{34}, \partial_{x_4})\mathcal{O}'(x_4), & (2.3) \end{aligned}$$

in which $x_{ij} = x_i - x_j$ and $\mathcal{O}$s are conformal primaries with non-zero spin in general. The contributions from the descendants of conformal primary fields are included in the functions $C(x_{ij}, \partial_x)$. Applying the OPEs of $\phi(x_i)\phi(x_j)$, the four-point correlator turns into

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}[C_{\mathcal{O}}(x_{12}, \partial_y)C_{\mathcal{O}}(x_{34}, \partial_z)\langle \mathcal{O}(x_2)\mathcal{O}(x_4)\rangle], \tag{2.4}$$

where we have normalized the operators so that

$$\langle \mathcal{O}(x_1)\mathcal{O}'(x_2)\rangle = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \delta_{\mathcal{O}\mathcal{O}'}. \tag{2.5}$$

[*] Here we have assumed the scaling dimensions of the four scalars are the same for simplicity. For the more general case, the differences among the scaling dimensions of four operators will slightly affect the function but not the main gist.

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}$$

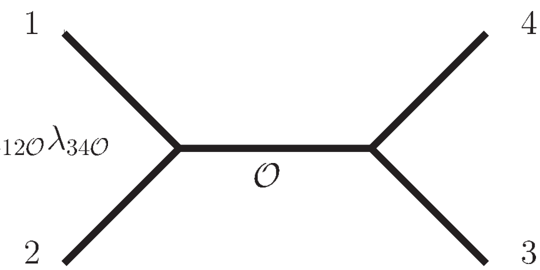

Figure 2.1: Conformal partial wave expansion of the four-point correlator. Note that the graph denotes the conformal partial wave expansion instead of the Feynman diagram.

In above formula the coefficients λs are free parameters of CFTs while the functions $C_{\mathcal{O}}(x_{ij}, \partial_x)$ are completely determined by the conformal symmetry. Besides, the two point function $\langle \mathcal{O}\mathcal{O}\rangle$ can be uniquely fixed according to conformal symmetry. Contributions on the four-point function from each conformal primary family are the conformal partial waves [†], which is almost fixed by the conformal symmetry weighted by the OPE coefficients. The four-point functions can be expanded in terms of the conformal partial waves, as graphically shown in the Figure 2.1.

Due to the associativity of the operator algebra, we can also evaluate the four-point correlator using the OPEs of $\langle \phi_2(x_2)\phi_3(x_3)\rangle$ and $\langle \phi_1(x_1)\phi_4(x_4)\rangle$, or $\langle \phi_1(x_1)\phi_3(x_3)\rangle$ and $\langle \phi_2(x_2)\phi_4(x_4)\rangle$. In any channel one should get the same four-point correlator, i.e., we have the equation on crossing symmetry shown in Figure 2.2.

The crossing symmetry provides highly non-trivial consistency constraint on the CFT data. However, it is an equation related to infinitely many parameters so it is extremely difficult to solve it analytically. In fact it is not sure how to get meaningful information on CFTs in $D > 2$ dimension spacetime from crossing symmetry equation for decades.

[†]There is an abuse in the terminology. In many papers the CPW is also called conformal block function. In this chapter, we will give the definition of conformal block function later, which is different from the conformal partial wave by a factor constructed from the four coordinates x_{ij}.

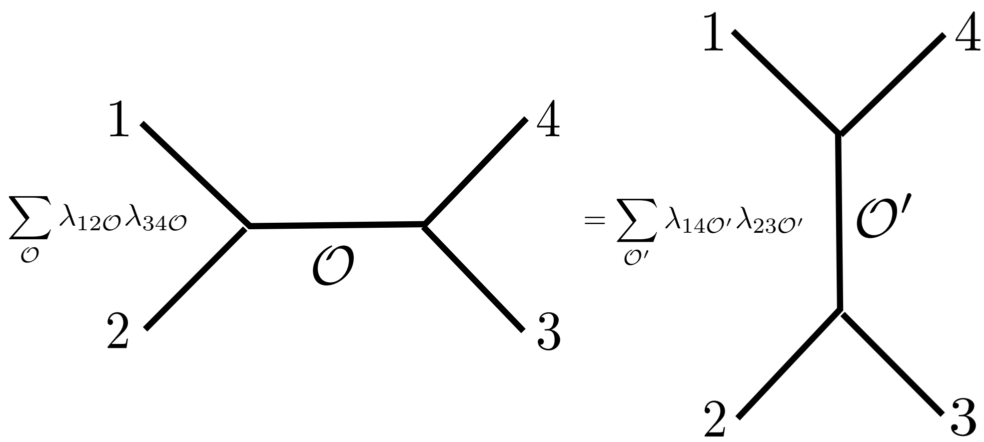

Figure 2.2: Crossing equation of the four-point correlator evaluated in $12-34$ channel and $13-24$ channel.

Let us consider the four-point function with four identical scalars:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle. \tag{2.6}$$

In the $12-34$ channel, the conformal partial wave expansion of four-point correlator reads

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \frac{g_{\Delta,\ell}(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}, \tag{2.7}$$

where Δ_ϕ denotes the scaling dimension of scalar ϕ, and Δ, ℓ are the scaling dimension and spin of the primary operator $\mathcal{O}$, respectively. The cross-ratios $u,\ v$ are defined as

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{2.8}$$

The compact term $g_{\Delta,\ell}(u,v)$ in (2.7) is the conformal block which is fixed by conformal symmetry. Similarly, in the $13-24$ channel, the cross-ratios $(u,v) \to (v,u)$ under the coordinate transformation $1 \leftrightarrow 3$, and the four-point correlator turns into

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \frac{g_{\Delta,\ell}(v,u)}{x_{13}^{2\Delta_\phi} x_{24}^{2\Delta_\phi}}. \tag{2.9}$$

Then the crossing symmetry requires

$$v^{\Delta_\phi} - u^{\Delta_\phi} + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 [v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u)] = 0. \tag{2.10}$$

In the LHS, the term $v^{\Delta_\phi} - u^{\Delta_\phi}$ is from the contribution of the unit operator. For any CFT, its spectra and the OPE coefficients need to be adjusted subtly so that they satisfy the crossing equation (3.12).

2.2 Conformal Block Function

Conformal block relates to the contribution on four-point function from a conformal primary operators and its descendants. It is fully determined by the conformal symmetry instead of the details on any specific CFT. According to the OPEs given in (2.2,5.15), the conformal block is

$$g_{\Delta,\ell}(x_i) = x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi} C_{\mathcal{O}}(x_{12}, \partial_{x_2}) C_{\mathcal{O}}(x_{34}, \partial_{x_4}) \langle\mathcal{O}(x_2)\mathcal{O}(x_4)\rangle. \tag{2.11}$$

There are several methods to evaluate the conformal block functions analytically or numerically. In [25] a rather simple method has been proposed based on the conformal Casimir operator C of the conformal algebra $SO(d+1,1)$

$$C = -\frac{1}{2} L_{ab} L^{ab}, \tag{2.12}$$

where the operator L_{ab} are the generators of $SO(d+1,1)$ group. Any conformal operators $\mathcal{O}$ in the irreducible representation (Δ,ℓ) are the eigenstate of the Casimir operator C with eigenvalue

$$C|\mathcal{O}\rangle = (\Delta(\Delta-d)+\ell(\ell+d-2))|\mathcal{O}\rangle. \tag{2.13}$$

Use L_{iab} to denote the action of $SO(d+1,1)$ generators on operator $\phi(x_i)$ and apply the equation (2.13) in the four-point correlator with propagating operator $\mathcal{O}$, we have

$$C\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle|_{\mathcal{O}} \propto -L_{0ab}L_0^{ab}\langle\phi(x_1)\phi(x_2)\mathcal{O}(x_0)\rangle\langle\mathcal{O}(x_0)\phi(x_3)\phi(x_3)\rangle \tag{2.14}$$

$$= -(L_{1ab}+L_{2ab})(L_1^{ab}+L_2^{ab})\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle|_{\mathcal{O}}, \tag{2.15}$$

where we have applied the following relationship among L_{iab}s

$$\sum_i L_{iab}\langle\phi(x_1)\phi(x_2)\mathcal{O}(x_0)\rangle = 0, \tag{2.16}$$

which is necessarily true to possess conformal invariance.

The conformal block is defined as

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle|_{\mathcal{O}} = \lambda_{\mathcal{O}}^2\frac{g_{\Delta,\ell}(u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}. \tag{2.17}$$

Multiply the Casimir operator on both sides we get the differential equation for the conformal block

$$\mathcal{D}g_{\Delta,\ell}(u,v) = (\Delta(\Delta-d)+\ell(\ell+d-2))g_{\Delta,\ell}(u,v), \tag{2.18}$$

in which the differential operator $\mathcal{D}$ is given by

$$\mathcal{D} = 2(z^2(1-z)\partial_z^2 - z^2\partial_z) + 2(\bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}) + 2(d-2)\frac{z\bar{z}}{z-\bar{z}}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}) \tag{2.19}$$

and the variables $z, \bar{z}$ are related to the cross-ratios u, v through

$$u = z\bar{z}, \;\; v = (1-z)(1-\bar{z}). \tag{2.20}$$

The Casimir equation has been solved analytically in even dimensions [25],

$$g_{\Delta,\ell}(u,v)|_{d=2} \;\; = \;\; k_{\Delta+\ell}(z)k_{\Delta-\ell}(\bar{z}) + k_{\Delta-\ell}(z)k_{\Delta+\ell}(\bar{z}), \tag{2.21}$$

$$g_{\Delta,\ell}(u,v)|_{d=4} \;\; = \;\; \frac{z\bar{z}}{z-\bar{z}}(k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - k_{\Delta-\ell-2}(z)k_{\Delta+\ell}(\bar{z})), \tag{2.22}$$

in which

$$k_a(x) = x^{\frac{a}{2}}\,{}_2F_1(\frac{a}{2}, \frac{a}{2}, a, x). \tag{2.23}$$

Unfortunately, in odd dimensions there is no such compact results on the conformal block. In practical application, the conformal block can be calculated using the series expansion obtained from the Casimir equation or recursion relations. In [24] a series expansion for the conformal blocks has been provided

$$g_{\Delta,\ell} = u^{\frac{\Delta-\ell}{2}} \sum_{m=0}^{\infty} u^m h_m(v), \tag{2.24}$$

in which the function h_m can be calculated recursively. The first term $h_0(v)$ reads

$$h_0(v) = (\frac{v-1}{2})^{\ell}\,{}_2F_1(\frac{\Delta+\ell}{2}, \frac{\Delta+\ell}{2}, \Delta+\ell, 1-v). \tag{2.25}$$

The series expansion holds in any spacetime dimension and will be quite helpful for analytical conformal bootstrap.

2.3 Numerical Solution of the Crossing Equation

As the conformal block functions can be calculated analytically or numerically, the undetermined factors in the crossing equation (3.12) are the CFT data, including the OPE coefficients and the scaling dimensions. This is an equation with infinite variables and infinite degrees, so mathematically the solution set is undetermined. There could be no solution at all, or finite (even infinite) solutions. Physically we know that there is at least one solution of the system, the free field theory. However, it is quite obscure to extract information on interacting CFTs from the crossing equation. In [5] a numerical method has been proposed which initiates a new stage on conformal bootstrap.

In [5], the authors proposed a numerical approach to test whether the possible CFT data can satisfy the consistency conditions, including the unitary condition and the crossing equation. Surprisingly they showed that a large part of the possible CFT data cannot pass the consistency test so is excluded for any physical CFTs! This approach has been refined further and now it is sufficiently powerful to exclude almost all the regions of the possible CFT data except a small isolated region, in which the physical CFT is believed to locate.

Let us consider the four-point correlator of four identical scalars ϕ with OPE

$$\phi\phi \sim 1 + \lambda_\sigma \sigma + \cdots \, . \tag{2.26}$$

Here the scalar σ is the first non-trivial conformal primary in the OPE. Its scaling dimension Δ_σ, in the free theory limit is $\Delta_\sigma = 2\Delta_\phi$. This relation is modified in an interacting CFT. Conformal bootstrap finds that there is a strong constraint on the $(\Delta_\phi, \Delta_\sigma)$ relation. Any physical scaling dimensions $(\Delta_\phi, \Delta_\sigma)$ should be consistent with the crossing equation

(3.12)

$$\sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 F_{\Delta,\ell}(u,v) \equiv v^{\Delta_\phi} - u^{\Delta_\phi} + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 [v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u)] = 0. \quad (2.27)$$

However, if there is a linear functional α satisfying

$$\begin{aligned}
\alpha(F_{0,0}(u,v)) &> 0, \text{ unitary operator,} \\
\alpha(F_{\Delta,0}(u,v)) &\geqslant 0, \text{ for any scalar with scaling } \mathcal{O} \text{ dimension } \Delta \geqslant \Delta_\sigma, \\
\alpha(F_{\Delta,\ell}(u,v)) &\geqslant 0, \text{ for any spinning operators } (\ell \text{ is even}) \ \ell \leqslant L,
\end{aligned}$$

then the LHS of the crossing equation (2.27) is strictly positive, and the equation can never be satisfied, which means the assumed scaling dimensions $(\Delta_\phi, \Delta_\sigma)$ are not physical. The crossing equation (2.27) contains infinite many spinning operators. In practice, the spectra are truncated to a finite spin L, which is sufficient large to capture the numerical behavior of the whole spectra. The linear functional α can be chosen freely. A convenient choice is the derivatives on the cross-ratios u, v

$$\alpha = \sum_{m+n \leqslant \Lambda} a_{mn} \partial_u^m \partial_v^n |_{u=v=\frac{1}{2}}. \quad (2.28)$$

In the numerical program the derivatives are taken up to the order Λ. The program can generate more precise results with larger Λ. According to the previous studies, significant results can be obtained around $\Lambda = 15$. The subscript indicates that the derivatives are taken at the symmetrical point $u = v = 1/2$. If we do find such a linear functional α, then the crossing equation (2.27) can never be satisfied and initial assumption on the spectra $(\Delta_\phi, \Delta_\sigma)$ have to be abandoned as unphysical.

By slightly adjusting the bootstrap conditions we can also get constraints on the OPE coefficients $\lambda_{\mathcal{O}}$ in the crossing equation (2.27). To bound the coefficient $\lambda_{\mathcal{O}_0}$ of operator $\mathcal{O}_0$ with scaling dimension Δ_0 and spin ℓ_0, we firstly rewrite the crossing equation as follows

$$\lambda^2_{\mathcal{O}_0} F_{\Delta_0,\ell_0}(u,v) = -F_{0,0}(u,v) - \sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} F_{\Delta,\ell}(u,v). \tag{2.29}$$

If we can find a linear functional α which satisfies

$$\alpha(F_{\Delta_0,\ell_0}(u,v)) \;=\; 1, \;\; \text{unitary operator}, \tag{2.30}$$

$$\alpha(F_{\Delta,\ell}(u,v)) \;\geqslant\; 0, \;\; \text{for any other non-unit operators}. \tag{2.31}$$

Then the OPE coefficient $\lambda_{\mathcal{O}_0}$ admits an upper bound

$$\lambda^2_{\mathcal{O}_0} = -\alpha(F_{0,0}(u,v)) - \alpha(\sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} F_{\Delta,\ell}(u,v)) \leqslant -\alpha(F_{0,0}(u,v)). \tag{2.32}$$

A strict bound on the OPE coefficient $\lambda_{\mathcal{O}}$ can be obtained by minimizing $-\alpha(F_{0,0}(u,v))$ in a set of α which satisfies the constraints (2.30-2.31).

3. BOOTSTRAPPING $5D$ $O(N)$ VECTOR MODEL *

3.1 Introduction

The conformal bootstrap [1–4] provides a non-perturbative approach to solve CFTs using general consistency conditions. It has led to great successes in 2D, such as the seminal work [4] on solving 2D rational CFTs. In recent years the conformal bootstrap has been revived since the breakthrough discovery in [5], which shows that the crossing symmetry and the unitary conditions can provide strong constraints on the operator scaling dimensions without an explicit form of Lagrangian. The crossing symmetry of four-point correlator leads to an infinite set of constraints on the CFT data. These constraints are difficult to be solved analytically, instead, they are truncated to a finite set and reformulated as a convex optimization problem so that they can be solved numerically. Here the convexity of conformal block functions [24, 25] plays a crucial role. Since then the conformal bootstrap has been significantly developed and it becomes a remarkably powerful technique to obtain CFT data, including operator scaling dimensions and operator product expansion (OPE) coefficients in $D > 2$ dimensions [26–39, 106, 41–56, 58–68]. Review of previous developments on conformal bootstrap is provided in [71].

From conformal bootstrap with single correlator $\langle\phi\phi\phi\phi\rangle$, one can obtain bounds on the conformal dimension or OPE coefficient of objective operator. The bounds may exhibit singular behaviors, such as kinks which are believed to be related to unitary CFTs. One can expect to obtain more information on CFTs through bootstrapping mixed correlators like $\langle\phi\phi\phi^2\phi^2\rangle$. Conformal bootstrap with mixed operators has been fulfilled in [46, 58] for $3D$ Ising model and critical $O(N)$ vector models and the results are quite impressive—the

*This work has been published as an open access article [69] in Journal of High Energy Physics.

allowed scaling dimensions are isolated in small islands. The accuracy can be improved further by refining the numerical techniques [55, 68]. Studies on the $3D$ $O(N)$ vector models are strongly motivated by their special importance in physics. For small $N \leqslant 3$ they describe second-order phase transitions occurring in real physical systems [72]. Besides, its $O(N)$-singlet sector is proposed to be dual to higher spin quantum gravity in AdS_4 with Dirichlet boundary conditions [73]. In the UV side, the $3D$ $O(N)$ vector model contains N free scalars $\phi_i,\ i = 1, \cdots, N$ perturbed by quartic coupling $(\phi_i\phi_i)^2$. The RG flows to an IR fixed point which is strongly coupled. For the critical $O(N)$ vector models with large N or in $D = 4 - \epsilon, \epsilon \ll 1$ dimensions, one can obtain reliable results using large N expansion or the well-known Wilson-Fisher ϵ expansion. Actually these analytical results have been used as consistency checks of conformal bootstrap in [36, 37]. Nevertheless, for the $3D$ ($\epsilon = 1$) critical $O(N)$ vector models with small N which are more physically attractive, these perturbative methods turn into less effective. In contrast, conformal bootstrap remains useful and has provided the most accurate results up to date [68].

Following the success of conformal bootstrap in critical $3D$ $O(N)$ vector models, one may expect to generalize the results to critical $O(N)$ vector models in higher dimensions. These models, if exist, are expected to provide examples on $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence in higher dimensions. In 4D there is no critical $O(N)$ CFT, while in $D > 4$, the interaction term $(\phi_i\phi_i)^2$ is irrelevant in the free $O(N)$ theory so the UV free $O(N)$ theory perturbed by the quartic interaction does not lead to an interacting fixed point in the IR, instead, the theory admits a Gaussian fixed point in the IR which flows to an interacting UV fixed point under $(\phi_i\phi_i)^2$ perturbation [74, 75]. In $D = 4 + \epsilon$ such UV fixed point theory is weakly coupled for sufficient small ϵ and it requires a negative quartic coupling coefficient, which may introduce the problem of instability even though the scaling dimensions of the operators are above the unitary bound. A UV-completed formulation of the $O(N)$ model in $D > 4$ dimensions has been proposed in [76, 77]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{1}{2}(\partial_i \sigma)^2 + \frac{1}{2} g\sigma\phi_i^2 + \frac{1}{6}\lambda\sigma^3, \tag{3.1}$$

in which the ϕ_i constructs fundamental representation of $O(N)$ and the $O(N)$ singlet σ performs as composite field ϕ_i^2 in the UV side. The theory contains cubic interaction terms which are relevant in space with dimension $D < 6$. Using the combination of ϵ and large N expansion it has been shown that this theory admits an interacting IR fixed point [76, 77], which is unitary for $N > N_c$, while below the critical value $N < N_c$ the coupling turns into complex and the IR fixed point theory is nonunitary. At one-loop level the critical value N_c is about $N_c \approx 1038$. For $5D$ ($\epsilon = 1$) critical $O(N)$ theories, the small ϵ condition for ϵ expansion approach breaks down so the results obtained from ϵ expansion should be treated carefully. Actually the critical value decreases to $N_c \approx 64$ at three-loop level. In [78] the author has obtained a critical value $N_c \approx 400$ at four-loop level based on resummation methods. A non-perturbative method is desirable to determine the critical value N_c in $5D$. The $5D$ critical $O(N)$ models have been studied using the nonperturbative functional renormalization group equations [79–84]. In these works the $5D$ interacting $O(N)$ fixed points have been obtained while the effective potential is metastalbe. Specifically the analysis in [84] agrees with the results from the $D = 6 - \epsilon$ perturbative approach when $\epsilon \ll 1$ and predicts the $5D$ critical value $N_c = 1$.

The conformal bootstrap approach has been employed to study $5D$ critical $O(N)$ models in [43, 53, 54] following the proposal of the cubic model [76, 77]. In [43] the $5D$ critical $O(N)$ models have been assumed to saturate the minimum of the $O(N)$ current central charge c_J for large N and the existence of $5D$ critical $O(N)$ models are indicated from these minimums obtained from conformal bootstrap. The authors focused on bootstrapping the OPE coefficients rather than the scaling dimensions of conformal primary operators. In $3D$ conformal bootstrap the interacting $O(N)$ CFTs have been found to lie at the kinks of the bounds for the scaling dimension Δ_σ of the $O(N)$ singlet σ, which appears as lowest dimension operator in the S channel of the correlator $\langle\phi_i\phi_j\phi_k\phi_l\rangle$. However, in $5D$ cubic model the lowest dimension $O(N)$ singlet operator σ performs as ϕ_i^2, $\Delta_\sigma = 2\Delta_\phi = 3$ at the UV Gaussian fixed point which reduces to $\Delta_\sigma = 2 + O(1/N)$ near the IR fixed point. The IR fixed point is below the upper bound of scaling dimensions Δ_σ so there is no clue on the fixed point theory in the bound of scaling dimensions. This problem has been overcome in [53, 54] by imposing a gap on the scaling dimensions of σ and the second lowest $O(N)$ singlet conformal primary scalar. With a reasonable assumption on the gap, the allowed region of the scaling dimensions $(\Delta_\phi, \Delta_\sigma)$ can be carved out and forms two sharp kinks. The UV Gaussian fixed point lies at the higher kink while the lower kink agrees with the large N expansion predictions on IR interacting fixed point theories. Furthermore, the kink disappears for small $N \approx 15$ which may indicate a small critical value N_c [54].

However, one should be careful to consider the kinks in conformal dimension bound or the minimum of central charges as unitary CFTs. From perturbative methods it is known that in $D = 6 - \epsilon,\ \epsilon \ll 1$ the IR fixed point of cubic $O(N)$ models is endowed with complex critical couplings for $N \leqslant 1000$ *. Nevertheless, in [54] a sharp kink is still generated from conformal bootstrap for $D = 5.95$, $N = 600$ which is much lower than the threshold value and should be nonunitary. The reason seems to be that the precision adopted in [54] is not high enough to detect the small violation of unitary. A more powerful bootstrap approach is needed to study the $5D$ $O(N)$ models, especially on its critical value

*CFTs in fractional dimensions are known to be nonunitary even with real couplings [85, 86]. However, the unitarity is violated by operators with high scaling dimensions so they are more difficult to be tested through conformal bootstrap approach.

N_c.

In this work, we will study the conformal bootstrap with multiple correlators of conformal primaries ϕ_i and σ: $\langle\phi_i\phi_j\phi_k\phi_l\rangle$, $\langle\phi_i\phi_j\sigma\sigma\rangle$, $\langle\sigma\sigma\sigma\sigma\rangle$. Since there are more operators involved in the bootstrap program, it is expected that the results will provide more rigid restrictions on the scaling dimensions of $(\Delta_\phi, \Delta_\sigma)$. Actually we find that the scaling dimensions $(\Delta_\phi, \Delta_\sigma)$ obtained from bootstrapping multiple correlators of $5D$ $O(500)$ model is isolated in a rather small island, which is nicely compatible with the perturbative results. We also study the critical value N_c in $5D$. In preliminary numerical calculations we find small islands on the allowed scaling dimensions $(\Delta_\phi, \Delta_\sigma)$ for all $N \geqslant 1$. However, these islands disappear after improving the bootstrapping precisions. Taking $N = 100$ for example, it shows an apparent kink in the bound from bootstrapping single correlator $\langle\phi_i\phi_j\phi_k\phi_l\rangle$. Using multiple correlator conformal bootstrap with small Λ, we obtain an island on $(\Delta_\phi, \Delta_\sigma)$ plane close to the kink from single correlator conformal bootstrap, while it vanishes after increasing Λ even though we relax the conformal dimension gap to $\Delta_{\phi_i'} \geqslant 5.0$ and $\Delta_{\sigma'} \geqslant 3.3$ †. Therefore our results suggest a rather large critical value $N_c > 100$ unless the perturbative methods have drastically overestimated the scaling dimensions.

This chapter is organized as follows. In section 2 we briefly review the cubic model of $O(N)$ vector model in $4 < D < 6$ and the perturbative results on scaling dimensions of lowest primary scalars. The scaling dimensions $(\Delta_\phi, \Delta_\sigma)$ obtained from large N and ϵ expansions provide consistency checks for the results from conformal bootstrap. In section 3 we introduce the numerical conformal bootstrap equations for $5D$ $O(N)$ vector models and their numerical implementation. Our results are presented in section 4. We show that through bootstrapping the multiple correlators the scaling dimensions $(\Delta_\phi, \Delta_\sigma)$ are

†As a comparison, $\Delta_{\phi_i'} \approx 5.39$ up to the order $1/N$ and $\Delta_{\sigma'} \approx 3.68$ up to the order $1/N^2$ from perturbative methods.

isolated in a small island for large $N = 500$, while disappear with larger Λ for $N \leqslant 100$. Conclusions are made in section 5.

3.2 Perturbative Results for $5D$ Critical $O(N)$ Models

The critical $O(N)$ vector model with quartic interaction in arbitrary dimensions $D = 4 - \epsilon$ has been analyzed using the large N expansions [87–93, 95–98]. In $2 < D < 4$ ($\epsilon > 0$), the quartic interaction is relevant and the RG flows from UV Gaussian fixed point to interacting IR fixed point perturbed by this coupling. The quartic interaction is irrelevant in $4 < D < 6$ ($\epsilon < 0$) so the long-range physics is described by free field theory. The quartic coupling generates RG flow from the IR Gaussian fixed point to an interacting UV fixed point. The perturbative result for small ϵ shows the interaction coupling is negative at interacting UV fixed point which may lead to the stability problem. However, the scaling dimensions of scalar operators obtained from the large N expansion are still above unitary bound and the unitary conditions remain unbroken for sufficient large N. One may expect the interacting UV fixed point from quartic model describes a universality class with $O(N)$ global symmetry in $4 < D < 6$ whose stable or metastable formulation may be realized in different model.

In $D = 5$ spacetime, the conformal dimensions of ϕ_i and σ have been evaluated at three-loop level

$$\Delta_\phi = \frac{3}{2} + \frac{0.216152}{N} - \frac{4.342}{N^2} - \frac{121.673}{N^3} + \cdots \tag{3.2}$$

$$\Delta_\sigma = 2 + \frac{10.3753}{N} + \frac{206.542}{N^2} + \cdots \tag{3.3}$$

$$\Delta_{\sigma^2} = 4 - \frac{13.8337}{N} - \frac{1819.66}{N^2} + \cdots \tag{3.4}$$

According to above $1/N$ expansion, the conformal dimension of ϕ_i is above the unitary bound ($\Delta_\phi > 3/2$ for scalar fields) given $N > 35$. The critical value $N_c = 35$ can be significantly modified by by higher order corrections. Actually the $5D$ $1/N$ expansions converge much slower than those in $3D$ [76].

Alternatively, the $5D$ quartic theory can also be studied using ϵ expansion [99]. Conformal dimensions of ϕ_i and ϕ^2 (σ) have been calculated up to five-loop [100]:

$$\Delta_\phi = 1 - \frac{\epsilon}{2} + \frac{N+2}{4(N+8)^2}\epsilon^2(1 + a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3) + \cdots , \tag{3.5}$$

where

$$\begin{aligned}
a_1 &= \frac{-N^2 + 56N + 272}{4(N+8)^2}, \\
a_2 &= -\frac{1}{16(N+8)^4}(5N^4 + 230N^3 - 1124N^2 - 17920N - 46144 \\
&\quad +384\zeta(3)(N+8)(5N+22)), \\
a_3 &= -\frac{1}{64(N+8)^6}\left(13N^6 + 946N^5 + 27620N^4 + 121472N^3 - 262528N^2\right. \\
&\quad -2912768N - 5655552 - 16\zeta(3)(N+8)(N^5 + 10N^4 + 1220N^3 - 1136N^2 \\
&\quad -68672N - 171264) + 1152\zeta(4)(N+8)^3(5N+22) \\
&\quad \left.-5120\zeta(5)(N+8)^2(2N^2 + 55N + 186)\right),
\end{aligned} \tag{3.6}$$

and

$$\Delta_\sigma = 2 - \epsilon + \frac{N+2}{N+8}\epsilon\left(1 + c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3 + c_4\epsilon^4\right) + \cdots , \tag{3.7}$$

where

$$
\begin{aligned}
c_1 &= \frac{13N+44}{2(N+8)^2}, \\
c_2 &= -\frac{1}{8(N+8)^4}(3N^3-452N^2-2672N-5312+96\zeta(3)(N+8)(5N+22)), \\
c_3 &= -\frac{1}{32(N+8)^6}\big(3N^5+398N^4-12900N^3-81552N^2-219968N-357120 \\
&\quad +16\zeta(3)(N+8)(3N^4-194N^3+148N^2+9472N+19488) \\
&\quad +288\zeta(4)(N+8)^3(5N+22)-1280\zeta(5)(N+8)^2(2N^2+55N+186)\big), \\
c_4 &= -\frac{1}{128(N+8)^8}\times\big(3N^7-1198N^6-27484N^5-1055344N^4-5242112N^3 \\
&\quad -5256704N^2+6999040N-626688-16\zeta(3)(N+8)\times \\
&\quad (19004N^4+102400N^3+13N^6-310N^5-381536N^2-2792576N-4240640) \\
&\quad -1024\zeta(3)^2(N+8)^2(2N^4+18N^3+981N^2+6994N+11688) \\
&\quad +48\zeta(4)(N+8)^3(148N^2+3N^4-194N^3+9472N+19488) \\
&\quad +256\zeta(5)(N+8)^2(155N^4+3026N^3+989N^2-66018N-130608) \\
&\quad -6400\zeta(6)(2N^2+55N+186)(N+8)^4 \\
&\quad +56448\zeta(7)(14N^2+189N+526)(N+8)^3\big). \qquad (3.8)
\end{aligned}
$$

Besides, for the next $O(N)$ vector operator $\phi_i' \equiv \phi^4\phi_i$, its scaling dimension has been provided in [94] at the first order

$$\Delta_{\phi_i'} = 5-\frac{\epsilon}{2}+\frac{12}{N+8}\epsilon+\cdots. \qquad (3.9)$$

Taking $\epsilon=-1$ the results can be interpolated to $5D$. For large N the higher order coefficients c_is are of order $1/N$. In this case the ϵ expansion performs worse asymptotically in $5D$ comparing with the large N expansion. While for small N it is not clear at this stage which approach can provide better estimation. These perturbative results will be useful to estimate the conformal dimension gap which can be applied in the conformal bootstrap to improve the numerical efficiency.

Both the large N expansion and the ϵ expansion contain negative terms at higher loop level. For small Ns these negative contributions may play dominating roles in the perturbative expansion and result in negative anomalous dimension. Specifically the five-loop result (3.5) shows the conformal dimension $\Delta_\phi < 3/2$ for $N \leqslant 14$ [76]. In [54] the conformal bootstrap with single correlator has been applied to generate bound on Δ_σ. Interestingly the bounds are featured with kinks which are expected to relate to certain unitary fixed point theories while the kinks disappear near $N \approx 15$, close to the critical value estimated from ϵ expansion. However, as in the large N expansion, the ϵ expansion in $5D$ is not converged up to fifth order and the contributions from higher loops are likely to modify the threshold value N_c significantly.

The cubic $O(N)$ model (3.1) provides an approach to realize stable interacting $O(N)$ fixed point in $5D$ [76, 77]. The authors show that at one of the IR fixed point the cubic $O(N)$ model shares the same relevant critical exponents with the quartic $O(N)$ model so the two models are expected to describe the same universality class.[‡] Like the quartic $O(N)$ model, the cubic $O(N)$ model also requires a critical value N_c from unitarity constraint. In the cubic model, the unitarity is violated in the way that the coupling coefficients acquire imaginary part when $N < N_c$. In [76, 77] the critical value N_c is evaluated up to order ϵ^2 in arbitrary dimension $D = 6 - \epsilon$. Four-loop results which include corrections on N_c at order ϵ^3 have been calculated in [78]

$$N_c = 1038.26605 - 609.83980\epsilon - 364.17333\epsilon^2 + 452.71060\epsilon^3 + O(\epsilon^4). \tag{3.10}$$

[‡]The renormalization group approach suggests the cubic model admits an extra RG relevant direction with positive critical exponent at the IR fixed point [84]. In this sense the universality class of the quartic $O(N)$ model is a subset of that of cubic $O(N)$ model.

As usual, above perturbative result is not sufficient to make a solid estimation on $5D$ ($\epsilon = 1$) N_c due to its asymptotic performance. It is tempting to evaluate the critical value N_c using non-perturbative method. Besides the above interacting IR fixed point, the cubic model also admits extra fixed points with different critical value N_c'; however, they are not corresponding to the classical interacting quartic fixed point and will not be studied in this chapter.

3.3 Conformal Bootstrap with Multiple Correlators

Conformal bootstrap with multiple correlators has been developed in [46, 58] which aimed to solve the $3D$ Ising model and $O(N)$ vector model. This approach has obtained the most accurate solutions on $3D$ Ising model and $O(N)$ vector model up to date [68]. Here we briefly introduce the conformal bootstrap program for $5D$ $O(N)$ vector model analogous to that for $3D$ $O(N)$ vector model [58]. More details on this program are provided in [55].

3.3.1 Bootstrap Equations from Crossing Symmetry

Conformal partial wave function is the crucial ingredient for conformal bootstrap. In even dimensions $D = 2, 4, 6$, the conformal partial wave functions have been solved analytically [24, 25]. In odd dimensions there is no analytical expression for conformal partial wave function; however, it can be calculated recursively with arbitrary precision [36, 46, 101]. [§] The general four-point function of scalar operators can be expanded in

[§]Details on calculating conformal block function in arbitrary dimensions are provided in [51] as part of an open-source numerical conformal bootstrap program JuliBootS. In this work we will use the JuliBoots code to calculate the conformal block functions of scalar operators in $5D$.

terms of conformal partial waves

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle = \frac{1}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}}\left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}}\left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}}\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,v), \quad (3.11)$$

where σ_is are scalar operators with conformal dimension Δ_i ($\Delta_{ij}=\Delta_i-\Delta_j$) and $\mathcal{O}$ is the conformal primary operator appears in the OPE expansion of $\sigma_1\sigma_2\sim\lambda_{12\mathcal{O}}\mathcal{O}$ (and also $\sigma_3\sigma_4\sim\lambda_{34\mathcal{O}}\mathcal{O}$), whose conformal dimension and spin are $(\Delta,\ \ell)$. The conformal invariant cross ratios $u,\ v$ are of the standard form $u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}$ and $v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$, $x_{ij}=|x_i-x_j|$.

The four-point function can be evaluated equivalently in different channels, as suggested by crossing symmetry, and it leads to the following equations

$$\sum_{\mathcal{O}}\left(\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}F_{\mp,\Delta,\ell}^{12,34}(u,v)\pm\lambda_{32\mathcal{O}}\lambda_{14\mathcal{O}}F_{\mp,\Delta,\ell}^{32,14}(u,v)\right)=0, \quad (3.12)$$

in which

$$F_{\mp,\Delta,\ell}^{12,34}(u,v)=v^{\frac{\Delta_2+\Delta_3}{2}}g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,v)\mp u^{\frac{\Delta_2+\Delta_3}{2}}g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(v,u). \quad (3.13)$$

To study the $5D$ $O(N)$ vector model, we apply the crossing relations for correlators $\langle\phi_i\phi_j\phi_k\phi_l\rangle$, $\langle\sigma\sigma\sigma\sigma\rangle$ and $\langle\phi_i\phi_j\sigma\sigma\rangle$. The $O(N)$ indices in the correlators are decomposed into three irreducible structures: the $O(N)$ invariant, traceless symmetric and antisymmetric tensors. The conformal primaries appearing in the OPE of $O(N)$ vector representations ϕ_i can be classified into three irreducible representations:

$$\phi_i\times\phi_j\sim\sum_S\lambda_{\phi\phi\mathcal{O}_S}\mathcal{O}\delta_{ij}+\sum_T\lambda_{\phi\phi\mathcal{O}_T}\mathcal{O}_{(ij)}+\sum_A\lambda_{\phi\phi\mathcal{O}_A}\mathcal{O}_{[ij]}, \quad (3.14)$$

in which $S,\ T$ and A denote $O(N)$ singlet, traceless symmetric tensor and anti-symmetric tensor representations. Consequently, the four-point correlator $\langle\phi_i\phi_j\phi_k\phi_l\rangle$ and its crossing symmetric partners are separated into three channels: S,T,A. For the mixed four-point correlator $\langle\phi_i\sigma\phi_j\sigma\rangle$, one needs to consider the OPE $\phi_i\sigma \sim \sum_V \lambda_{\phi_i\sigma\mathcal{O}_i}\mathcal{O}_i$ which introduces the vector representations (denoted by V) as propagating operators in the mixed four-point correlator and its crossing symmetric partner.

The crossing relations for bootstrapping $5D$ $O(N)$ critical theories are essentially the same as those for $3D$ $O(N)$ vector model [58]. These equations can be written in a compact form [58] which are presented below for later reference

$$\begin{aligned} 0 \;&=\; \sum_{\mathcal{O}_S} \begin{pmatrix} \lambda_{\phi\phi\mathcal{O}_S} & \lambda_{\sigma\sigma\mathcal{O}_S} \end{pmatrix} \vec{V}_S \begin{pmatrix} \lambda_{\phi\phi\mathcal{O}_S} \\ \lambda_{\sigma\sigma\mathcal{O}_S} \end{pmatrix} \\ &\quad + \sum_{\mathcal{O}_T} \lambda^2_{\phi\phi\mathcal{O}_T} \vec{V}_T + \sum_{\mathcal{O}_A} \lambda^2_{\phi\phi\mathcal{O}_A} \vec{V}_{A,\Delta,\ell} + \sum_{\mathcal{O}_V} \lambda^2_{\phi\sigma\mathcal{O}_V} \vec{V}_V. \end{aligned} \tag{3.15}$$

Explicit forms of the 7-component vectors $\vec{V}_S, \vec{V}_T, \vec{V}_A, \vec{V}_V$ are:

$$\vec{V}_T = \begin{pmatrix} F^{\phi\phi,\phi\phi}_{-,\Delta,\ell} \\ \left(1-\frac{2}{N}\right) F^{\phi\phi,\phi\phi}_{-,\Delta,\ell} \\ -\left(1+\frac{2}{N}\right) F^{\phi\phi,\phi\phi}_{+,\Delta,\ell} \\ \mathbf{0}_{4\times 1} \end{pmatrix}, \; \vec{V}_A = \begin{pmatrix} -F^{\phi\phi,\phi\phi}_{-,\Delta,\ell} \\ F^{\phi\phi,\phi\phi}_{-,\Delta,\ell} \\ -F^{\phi\phi,\phi\phi}_{+,\Delta,\ell} \\ \mathbf{0}_{4\times 1} \end{pmatrix}, \; \vec{V}_V = \begin{pmatrix} \mathbf{0}_{4\times 1} \\ (-1)^{\ell} F^{\phi\sigma,\phi\sigma}_{-,\Delta,\ell} \\ F^{\sigma\phi,\phi\sigma}_{-,\Delta,\ell} \\ -F^{\sigma\phi,\phi\sigma}_{+,\Delta,\ell} \end{pmatrix},$$

$$\vec{V}_S = \begin{pmatrix} \mathbf{0}_{2\times 2} \\ \begin{pmatrix} F^{\phi\phi,\phi\phi}_{-,\Delta,\ell}(u,v) & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} F^{\phi\phi,\phi\phi}_{+,\Delta,\ell}(u,v) & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & F^{\sigma\sigma,\sigma\sigma}_{-,\Delta,\ell}(u,v) \end{pmatrix} \\ \mathbf{0}_{2\times 2} \\ \begin{pmatrix} 0 & \frac{1}{2}F^{\phi\phi,\sigma\sigma}_{-,\Delta,\ell}(u,v) \\ \frac{1}{2}F^{\phi\phi,\sigma\sigma}_{-,\Delta,\ell}(u,v) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}F^{\phi\phi,\sigma\sigma}_{+,\Delta,\ell}(u,v) \\ \frac{1}{2}F^{\phi\phi,\sigma\sigma}_{+,\Delta,\ell}(u,v) & 0 \end{pmatrix} \end{pmatrix} . \tag{3.16}$$

Here our convention differs from [58] by a factor $(-1)^{\ell}$.

3.3.2 Bounds from Crossing Relations

The equations from crossing symmetry (3.15) provide nontrivial constraints on the CFT data. The numerical approach to study these equations was first proposed in [5] and the following developments show this method is extremely powerful. The logic of numerical conformal bootstrap is firstly to make assumptions on the CFT spectra. If the assumptions are physical they are required to satisfy the crossing relations (3.15) and the unitary condition. Numerical conformal bootstrap provides a systematical way to check the consistency between the assumptions and general constraints on CFTs. Bounds on the CFT data, including conformal dimensions of primary operators and OPE coefficients can be obtained by falsifying possible assumptions on the CFT spectra.

Specifically for any hypothetical spectra $(\Delta_\phi, \Delta_\sigma)$ above the unitary bounds, they should be consistent with the crossing relations (3.15). However, if there are linear functionals $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_7)$ satisfying

$$
\begin{aligned}
\begin{pmatrix} 1 & 1 \end{pmatrix} \vec{\alpha} &\cdot \vec{V}_{S,0,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1, \\
\vec{\alpha} &\cdot \vec{V}_{S,\Delta,\ell} \succeq 0, \qquad \Delta \geqslant \Delta^*_{S,0} \text{ for the } O(N) \text{ singlet scalars except } \sigma, \\
\vec{\alpha} &\cdot \vec{V}_{T,\Delta,\ell} \geqslant 0, \\
\vec{\alpha} &\cdot \vec{V}_{A,\Delta,\ell} \geqslant 0, \\
\vec{\alpha} &\cdot \vec{V}_{V,\Delta,\ell} \geqslant 0, \qquad \Delta \geqslant \Delta^*_{V,0} \text{ for the } O(N) \text{ vector scalars except } \phi_i, \\
\vec{\alpha} &\cdot \left(\vec{V}_{S,\Delta_\sigma,0} + \vec{V}_{V,\Delta_\phi,0} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \succeq 0,
\end{aligned}
\tag{3.17}
$$

then the crossing relations (3.15) can never be satisfied and initial assumption on the spectra $(\Delta_\phi, \Delta_\sigma)$ have to be abandoned as unphysical. In the bootstrap conditions (3.17), we have required the $O(N)$ singlet scalars (except σ) have conformal dimensions above a lower bound $\Delta^*_{S,0}$, and similarly a lower bound for $\Delta^*_{V,0}$ for $O(N)$ vector scalars in addition to ϕ_i. Besides, we have implicitly assumed that all the extra operators accord with the unitary bound. In the last equation of (3.17), it is the summation of contributions in S (from σ) and V channels (from ϕ) that is required to be positive-semidefinite due to the equality of OPE coefficients $\lambda_{\phi\phi\sigma} = \lambda_{\phi\sigma\phi}$.

The bootstrap conditions in (3.17) are not the only way to break the crossing relations (3.15). In particular, to bootstrap certain OPE coefficient of operator (Δ_0, ℓ_0) in channel X, one may set $\vec{\alpha} \cdot \vec{V}_{X,\Delta_0,\ell_0} = 1$ instead of choosing the unit operator as in (3.17).

The bootstrap conditions are further refined in [68]. The lower bounds $\Delta^*_{S,0}$ and $\Delta^*_{V,0}$ introduced in (3.17) are necessary to isolate the conformal dimensions $(\Delta_\phi, \Delta_\sigma)$ in a small island. A higher but remaining physical lower bound can improve the numerical efficiency to carve out the allowed parameter space. For sufficient large N, these lower bounds can be justified from perturbative expansions. The $O(N)$ singlet scalar next to ϕ^2 is ϕ^4 in the quartic model, and its conformal dimension can be evaluated through the large N expansion (3.4). In the cubic theory (3.1) this is given by a mixing of σ^2 and ϕ^2. One of the linear combination of σ^2 and ϕ^2 is actually the descendent of σ, while another orthogonal mixing constructs a primary $O(N)$ singlet that shares the same conformal dimension as obtained from quartic theory [76, 77]. The candidate of next $O(N)$ vector scalar ϕ'_i is $\phi^2\phi_i$ (or $\sigma\phi_i$ in the cubic theory). However, as argued in [58] for the $3D$ theories, in $D = 6-\epsilon,\ \epsilon \ll 1$ dimension the quartic theory generates the following equation of motion for ϕ_i:

$$\partial^2\phi_i \propto \phi^2\phi_i, \tag{3.18}$$

which suggests that the operator $\phi^2\phi_i$ is a descendent of ϕ_i rather than a conformal primary scalar. One can get the same conclusion in cubic theory (3.1) with replacement $\phi^2 \to \sigma$. The next candidate is $\phi^4\phi_i$ (in $D = 6-\epsilon,\ \epsilon \ll 1$ dimension operators with derivatives, like $\phi^2\partial^2\phi_i, (\partial_\mu\phi)^2\phi_i$ have different bare conformal dimensions given $\epsilon \neq 0$ so they do not mix with $\phi^4\phi_i$). At the interacting fixed point, the conformal dimension of $\phi^4\phi_i$ has been studied in [94]. At tree level the conformal dimension of ϕ^2 near the interacting fixed point is 2, so the conformal dimension of $\phi^4\phi_i$ is 5.5 with $1/N$ corrections, as shown in (3.9). In the cubic theory the potential second $O(N)$ vector scalar is a mixing of $\sigma^2\phi_i$ and $\phi^2\phi_i$, which has not been explicitly studied yet. One can expect that one of the mixing is actually a descendent of ϕ_i while another primary mixing has the same conformal dimension as $\phi^4\phi_i$ in quartic theory, like the quadratic and cubic $O(N)$ singlet operators

[76, 77]. The lower bound of the $\phi^4\phi_i$ conformal dimension would be rather subtle for small N. Fortunately we will show that a unitary interacting fixed point disappears even for $N = 100$ (corresponding to $\Delta_{\phi_i'} = 5.39$ at order $1/N$), indicating a large critical value N_c.

3.3.3 Numerical Implementation of Conformal Bootstrap

Equations from crossing symmetry (3.15) provide an infinite set of constraints (3.17) on the CFT data. For the numerical implementation the constraints need to be truncated to a large but finite set. In (3.17) the constraints are parameterized by (Δ, ℓ). The spins ℓ construct an infinite tower of spectra while in conformal bootstrap only these spectra with small ℓ will be considered. Contributions from operators with large spin are exponentially suppressed. The linear functionals $\vec{\alpha}$ can be expanded as

$$\alpha_i = \sum_{m+n\leqslant\Lambda} a_{imn}\partial_z^m\partial_{\bar{z}}^n, \tag{3.19}$$

where $(z, \bar{z})$ are defined in terms of (u, v) through: $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$. Moreover, for the linear functional α_i, the number of derivatives is also truncated up to Λ. Taking higher order of derivatives in (3.19), we have more chances to find the linear function satisfying (3.17). As a result, the conformal bootstrap program can exclude larger regions in parameter space. In practice the parameter Λ is restricted by computation power. The setups of parameter Λ and spins used in this work are as follows

$$\begin{aligned}
S_{\Lambda=19} &= \{0, 1, \cdots, 30\} \cup \{49, 50\}, \\
S_{\Lambda=21} &= \{0, 1, \cdots, 30\} \cup \{47, 48, 49, 50, 51, 52\}, \\
S_{\Lambda=23} &= \{0, 1, \cdots, 30\} \cup \{47, 48, 49, 50, 51, 52, 53, 54\}, \\
S_{\Lambda=25} &= \{0, 1, \cdots, 30\} \cup \{47, 48, 49, 50, 51, 52, 53, 54, 55, 56\}.
\end{aligned} \tag{3.20}$$

The problem to find the linear functions $\vec{\alpha}$ under truncated constraints can be solved with SDPB program [55].

3.4 Results

3.4.1 Bootstrapping $5D$ $O(500)$ Vector Model

The $5D$ $O(500)$ vector model has been studied in [43, 53, 54] using conformal bootstrap with single correlator $\langle\phi_i\phi_j\phi_k\phi_l\rangle$. At the fixed point the conformal dimensions $(\Delta_\phi, \Delta_\sigma)$ of the lowest $O(N)$ vector ϕ_i and $O(N)$ singlet σ can be evaluated from the large N expansion in (3.2, 3.3) or the ϵ expansion in (3.5, 3.7). Taking $N = 500$, we get $(\Delta_\phi, \Delta_\sigma) = (1.500414, 2.02158)$ from 3-loop large N expansion and $(\Delta_\phi, \Delta_\sigma) = (1.500400, 2.02156)$ from 5-loop ϵ expansion. These predictions will be compared with the results obtained from conformal bootstrap.

In Figure 3.1 we present the bounds on $(\Delta_\phi, \Delta_\sigma)$ obtained through bootstrapping the single correlator $\langle\phi_i\phi_j\phi_k\phi_l\rangle$ (light blue region) and the multiple correlators (dark blue island). To bootstrap the single correlator we have assumed that the next $O(N)$ singlet scalar has dimension above the gap $\Delta^*_{S,0} = 3.965$, which can be justified from the large N expansion result (3.4): $\Delta_{\sigma^2} \approx 3.972$. This gap is also employed in [54]. The upper part of light blue region is similar to the bound provided in [54]. Besides, there is an extra kink in the lower region and the whole region actually forms a sharp tip like presented in [53], although a much larger gap was used in that work. Results of perturbative methods are also shown in Figure 3.1. Prediction from the large N expansion (denoted by the black cross) lies in the allowed region while prediction from the ϵ expansion (denoted by the black dot) is outside of the bound so is excluded. According to the conformal bootstrap results, the large N expansion does provide a better estimation on the conformal dimensions for large $N = 500$. Difference between the two perturbative approaches appears at the order

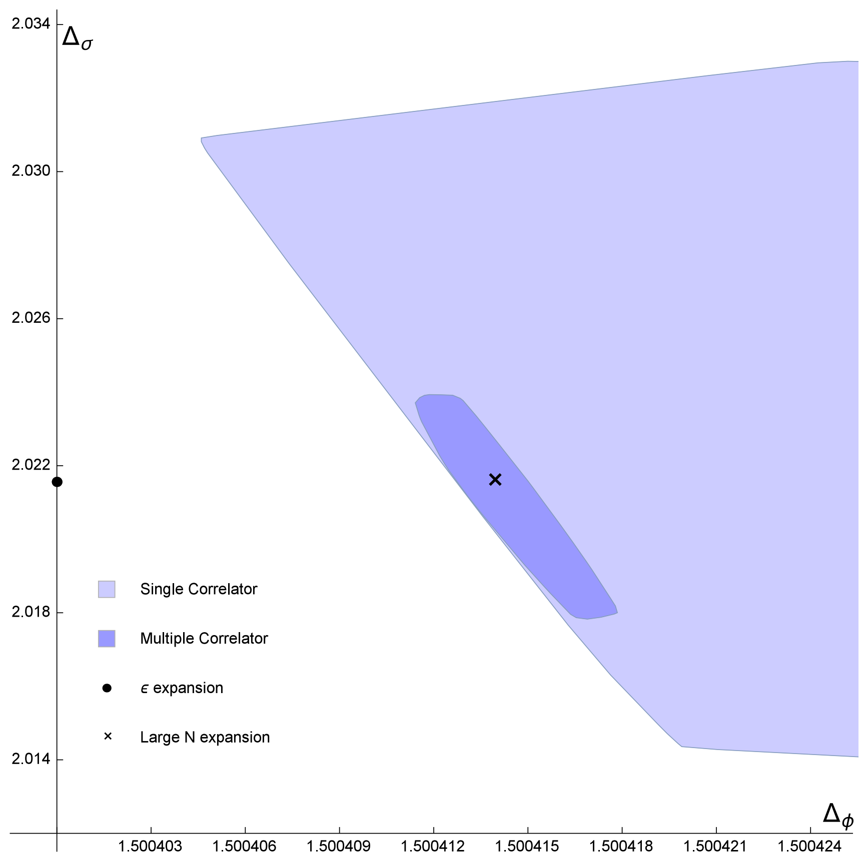

Figure 3.1: Bounds on the conformal dimensions $(\Delta_\phi, \Delta_\sigma)$ in the interacting $5D$ $O(500)$ CFT. The colored regions represent the conformal dimensions allowed by conformal bootstrap. Specifically the light blue region is obtained from single correlator bootstrap, while the dark blue island is isolated through bootstrapping the multiple correlators. We used the derivative at order $\Lambda = 19$ and spins $S_{\Lambda=19}$ in the numerical calculations. Besides, we assumed a gap $\Delta^*_{S,0} = 3.965$ in the S-channel. An extra gap $\Delta^*_{V,0} = 5$ has been used in the V-channel for bootstrapping multiple correlators. The black dot and cross relate to the predictions from ϵ expansion and large N expansion, respectively.

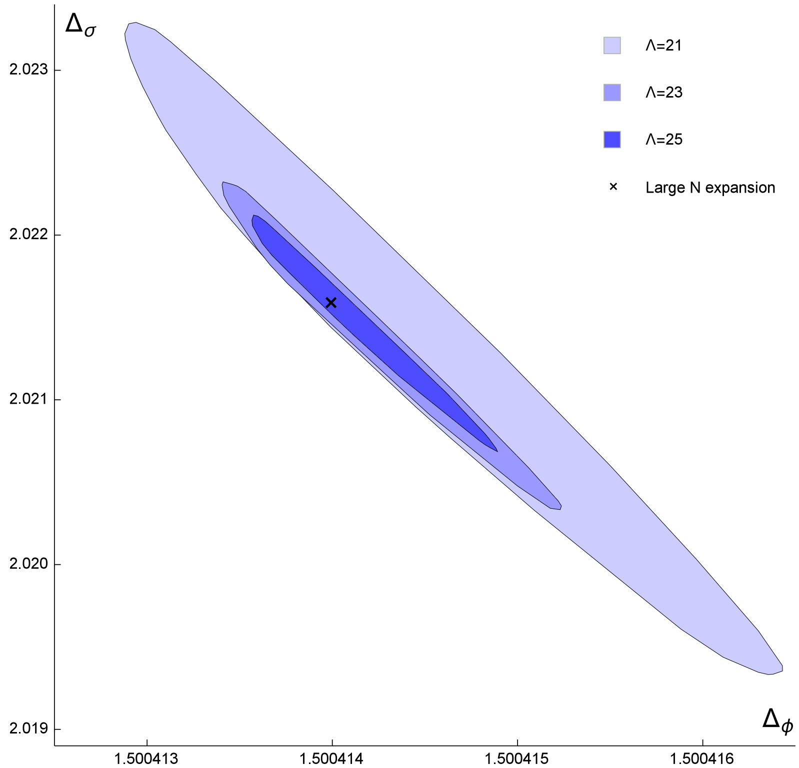

Figure 3.2: Isolated regions for the conformal dimensions $(\Delta_\phi, \Delta_\sigma)$ in $5D$ $O(500)$ vector model. The light, medium and dark blue regions are corresponding to the results from multiple correlator conformal bootstrap with $\Lambda = 21,\ 23,\ 25$, respectively. In the graph we have used the dimension gaps $\Delta^*_{S,0} = 3.965$ and $\Delta^*_{V,0} = 5$. The black cross denotes the prediction from large N expansion.

$10^{-5} \approx O(1/N^2)$, as discussed before.

Remarkably, the allowed region of $(\Delta_\phi, \Delta_\sigma)$ obtained from the multiple correlator bootstrap is enclosed in a small island, which is colored in dark blue in Figure 3.1. Besides the dimension gap $\Delta^*_{S,0} = 3.965$ in S-channel, we have employed another dimension gap $\Delta^*_{V,0} = 5$ in V-channel that the next primary $O(N)$ vector scalar has dimension $\Delta \geqslant 5$. The dark blue island lies in the center of the tip, and the black cross denoting the large N prediction is rather close to the center of this island. Such a high coincidence is extraordinary in view of only crossing symmetry and unitary condition are applied to carve out the island. On the other hand, the conformal bootstrap result also shows that the large N expansion is reliable at third order.¶

However, it should be careful to make statement based on results from conformal bootstrap with lower order of derivatives. Actually in preliminary study we have obtained isolated islands even for $N = 1$ with $\Lambda \sim 15$; however, they disappear after increasing Λ. As to the model with $N = 500$, we have checked the performance of the island with larger Λ. The results are provided in Figure 3.2. The allowed regions shrink notably from $\Lambda = 21$ to $\Lambda = 25$. Interestingly, the fixed point predicted by large N expansion remains located in the center of the small island even though the allowed region has contracted significantly.

3.4.2 Bootstrapping $5D$ $O(N)$ ($N \leqslant 100$) Vector Models and the Critical N_c

In $5D$ there is an interesting problem on the unitarity of the interacting $O(N)$ CFTs, that there is a threshold value N_c below which the CFTs become nonunitary [76, 77]. In contrast, the interacting $O(N)$ CFTs in $3D$ are unitary for any integer $N \geqslant 1$. Prior to our work, there are several evidences from conformal bootstrap which prefer to small N_c [43, 53, 54]. There are also some clues from perturbative results that the critical value $N_c < 100$. In this part we apply the conformal bootstrap with multiple correlators to study the $5D$ $O(N)$ vector model for small Ns. The multiple correlator conformal bootstrap involves in more $O(N)$ sectors and provides stronger constraints on the CFT data comparing with the conformal bootstrap with single correlator only.

We have searched the allowed regions on $(\Delta_\phi, \Delta_\sigma)$ plane for $N \leqslant 100$. The isolated islands can be obtained for small Ns with assumptions on the dimension gaps $(\Delta^*_{S,0}, \Delta^*_{V,0})$.

¶Strictly speaking, such consistency check is not completely self-contained since we have already used the large N expansion result in setting the dimension gaps.

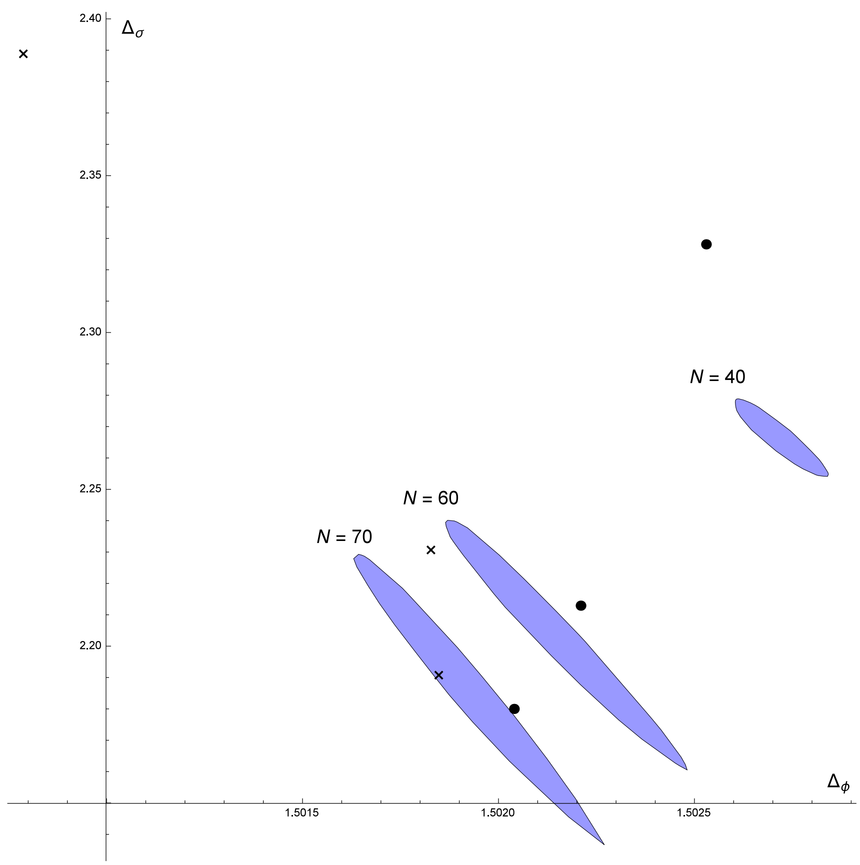

Figure 3.3: From top to bottom, the islands represent the allowed regions of $(\Delta_\phi, \Delta_\sigma)$ in the $5D$ $O(N)$ $N = 40, 60, 70$ vector models. The results are obtained from conformal bootstrap with $\Lambda = 19$ and spins $S_{\Lambda=19}$. The black dots and crosses denote predictions from ϵ expansion and large N expansions, respectively. The dimension gaps used in conformal bootstrap program are: $(\Delta^*_{S,0}, \Delta^*_{V,0}) = (3.4, 4.1)$ for $N = 40$, $(\Delta^*_{S,0}, \Delta^*_{V,0}) = (3.5, 4.3)$ for $N = 60, 70$. The perturbative methods, especially the large N expansion get abnormal and stay away from the region allowed by conformal bootstrap at $N = 40$.

However, these islands disappear after increasing the number of derivatives Λ. For $N \sim O(10)$ or smaller, the perturbative approaches cannot provide an approximate estimation on the conformal dimension Δ_{σ^2}. One may argue that the islands disappear due to the reason of the unphysical dimension gaps ($\Delta^*_{S,0}$, $\Delta^*_{V,0}$) used in the bootstrap program instead of the nonunitarity of the CFTs. While for sufficient large Ns the perturbative predictions are expected to provide rough estimations on the fixed point. This can be seen from the fact that the isolated islands obtained from conformal bootstrap are close to the perturbative predictions before vanishing. In Figure 3.3 we present the isolated regions for $N = 40, 60, 70$ from conformal bootstrap [‖]. At derivative order $\Lambda = 19$, the conformal bootstrap program generates closed regions on the $(\Delta_\phi, \Delta_\sigma)$ plane, which disappear for larger $\Lambda \geqslant 23$. According to the results from conformal bootstrap, for $N = 60, 70$ the perturbative approaches can still provide approximate estimations on the conformal dimensions at the interacting fixed points, although the theories are likely to be nonunitary. While for $N = 40$, the perturbative approaches, especially the large N expansion cannot provide reliable estimations on the interacting fixed point. One may note that the island corresponding to $N = 40$ shown in Figure 3.3 is rather close to the kink from single correlator bootstrap presented in [54], where the kink was considered to indicate a unitary CFT. However, our studies based on multiple correlator bootstrap show that bootstrap results from single correlator or mixed correlators with low derivatives can be significantly modified in a more precise evaluation.

In fact there is no stable island from conformal bootstrap even at $N = 100$. The perturbative methods predict that the interacting $O(100)$ fixed point locates in the position with conformal dimensions $(\Delta_\phi, \Delta_\sigma) = (1.50161, 2.124)$ from large N expansion and

[‖]The perturbative results on conformal dimension gaps are subtle for not so large Ns, and it is possible that the gaps used in Figure 3.3 are unphysical, however, in this figure we are more interested in the comparison with the predictions from perturbative approach. We will adopt more “safe" scaling dimension gaps for $N = 100$ to study the unitarity problem.

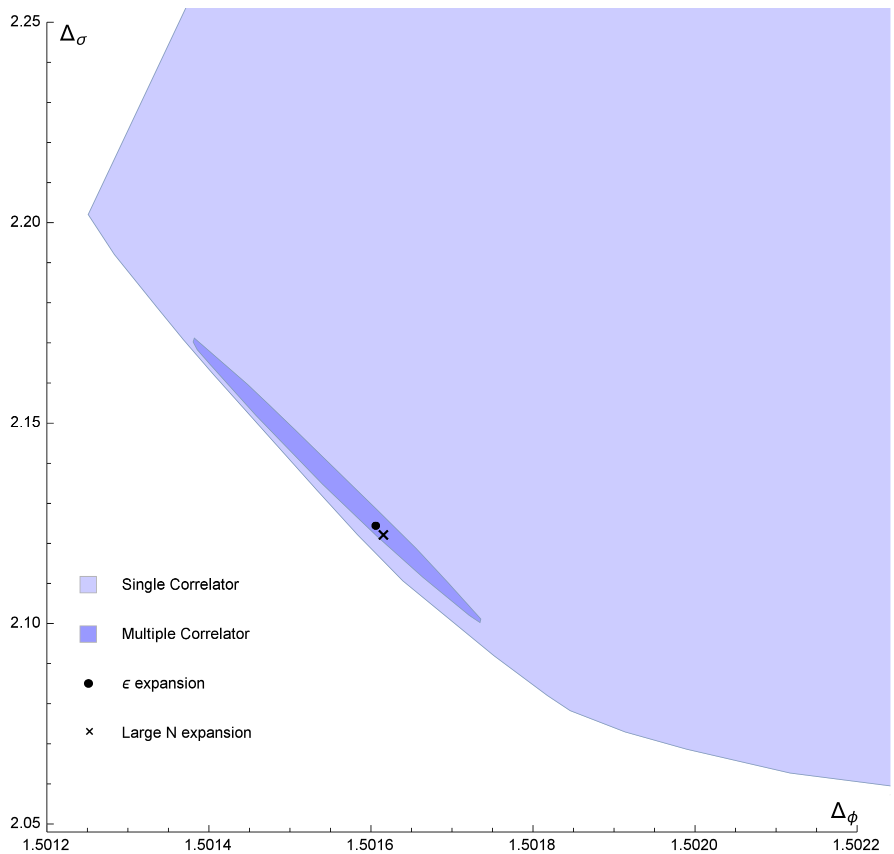

Figure 3.4: Bounds on the conformal dimensions $(\Delta_\phi, \Delta_\sigma)$ in $5D$ $O(100)$ vector model. The light blue region is obtained from single correlator bootstrap. The multiple correlators bootstrap leads to a small island colored in dark blue. In the bootstrap program we adopt the setup with $\Lambda = 19$ and the correspond spins provided in (3.20). We apply a dimension gap $\Delta^*_{S,0} = 3.6$ in the S-channel. Besides, an extra dimension gap $\Delta^*_{V,0} = 5$ has been used in the V-channel for bootstrapping multiple correlators. The black dot and cross relate to the predictions from ϵ expansion and large N expansion, respectively.

$(\Delta_\phi, \Delta_\sigma) = (1.50162, 2.122)$ from ϵ expansion. In Figure 3.4 we show the conformal bootstrap results of $O(100)$ vector model with $\Lambda = 19$. The single correlator conformal bootstrap generates a kinked bound similar to that of $O(500)$ vector model. The isolated region from multiple correlator conformal bootstrap lies in the middle of the tip. Here we have assumed a dimension gap $\Delta^*_{S,0} = 3.6$ in the S-channel, lower than the large N prediction $\Delta_{\sigma^2} \approx 3.68$. Besides, in the V-channel a dimension gap $\Delta^*_{V,0} = 5$ has been used, much lower than the one loop prediction $\Delta_{\phi'_i} \approx 5.39$. Predictions from large N and ϵ expansions are presented in the graph, both of which are nicely consistent with the conformal bootstrap bounds. In particular they locate in the isolated small island. All these features indicate a promising fixed point satisfying the crossing symmetry and unitarity constraints. However, the island disappears by taking higher order of derivatives $\Lambda = 23$! No isolated region can be found at $\Lambda = 25$ even we relax the S-channel gap to $\Delta^*_{S,0} = 3.3$. Unless the "true" island shrinks so drastically at certain order of Λ that it is hardly to be detected by scanning the parameter space, our bootstrap results disprove a unitary $5D$ $O(N)$ vector model even with $N = 100$!

Vanishing of the "allowed region" for $N \leqslant 100$ suggests that the theories with small N actually are not unitary. The violation of unitarity is rather small so that it cannot be uncovered by the bootstrap program with smaller Λ. This reminds us other examples on "pseudo" unitarity in conformal bootstrap. In [37] the $O(N)$ vector models in fractional dimensions $2 < D < 4$ have been studied using conformal bootstrap. In the work pronounced kinks are obtained in the bounds of conformal dimension of the lowest $O(N)$ singlet σ and are well consistent with the results obtained from extra approaches. However, careful studies in [85, 86] have shown that the CFTs in fractional dimensions are necessarily to be nonunitary, which are too subtle to be discovered in numerical conformal bootstrap. In the $5D$ $O(N)$ single correlator conformal bootstrap [54], sharp kinks are also generated in the fractional dimension $D = 5.95$ with $N = 600$, notably lower than the critical value $N_c \approx 1000$. We have studied this model through bootstrapping multiple correlators. There remains isolated allowed region even at $\Lambda = 21$, though it is quite small. The uncertainty on Δ_σ shown in the island is about 2×10^{-3}, while as shown in [54], the magnitude of imaginary part in Δ_σ is of the same order $\sim 1.5 \times 10^{-3}$ so it is expected that current conformal bootstrap program cannot capture the tiny unitarity violation unless the numerical accuracy can be improved significantly.

To summarize, the numerical conformal bootstrap provides a powerful approach to falsify assumptions on unitary CFTs. However, it is premature to validate the unitary CFTs using conformal bootstrap due to these “pseudo" unitary solutions. As to the $5D$ $O(500)$ model, although our results have provided strong evidence, they are still not sufficient to make a strict conclusion on its unitarity. On the other hand, it is surprising that the $5D$ $O(N)$ vector model is nonunitary even for $N = 100$. Consequently, the critical value $N_c > 100$, which is considerably larger than the value estimated before.

3.5 Conclusions

In this work, we have studied the interacting $5D$ CFTs with global $O(N)$ symmetry using the conformal bootstrap with multiple correlators. The multiple correlator conformal bootstrap has been developed in [46, 58] and obtained remarkable successes in $3D$ Ising and $O(N)$ vector models. The approach employs the correlators of the $O(N)$ vector scalar ϕ_i as well as the $O(N)$ singlet scalar σ. Since there are more operators involved in the crossing symmetry relations, the new method is expected to generate more strong constraints on the CFT data. Indeed the allowed regions on $(\Delta_\phi, \Delta_\sigma)$ plane is limited in a small island under reasonable assumptions on the dimension gaps.

Specifically, in this study we have shown that by bootstrapping multiple correlators from the interacting $5D$ CFTs with $O(N)$ symmetry ($N = 500$), the allowed conformal dimensions $(\Delta_\phi, \Delta_\sigma)$ are strongly limited in a closed region, which is highly consistent with predictions from large N expansion. In order to uncover the isolated region we also applied assumptions on the dimension gaps both in the $O(N)$ singlet sector and the $O(N)$ vector sector. Our results suggest that the interacting fixed point of $O(N)$ vector model is unitary for sufficient large N and support the asymptotic free $5D$ $O(N)$ cubic model proposed in [76, 77]. Evidence of such fixed point has already been shown in the single correlator conformal bootstrap studied in [43, 53, 54]. The island obtained in this work is rather close to the kink in the bound of conformal dimension Δ_σ obtained from bootstrapping correlator of four ϕ_is [54]. We have studied the performance of the island under higher order of derivatives Λ. The island shrinks notably from $\Lambda = 19$ to $\Lambda = 25$, while the large N expansion predictions remain staying in the center of the allowed region. Such coincidence is surprising in considering of that only crossing symmetry and unitary conditions are employed to generate the allowed region. Besides we only input the $O(N)$ global symmetry for this model while even did not use its Lagrangian at all.

We are particularly interested in the critical value N_c of $5D$ $O(N)$ vector model below which the interacting fixed point theory loses unitarity. The problem on the critical value N_c can also be seen from the perturbative expansions of conformal dimension Δ_ϕ, that below the critical value the scalar ϕ_i acquires conformal dimension smaller than the unitary bound and breaks the unitary condition. However, in $5D$ the perturbative expansions converges much slower comparing with these of $3D$. In [76, 77] the critical value N_c has been evaluated based on large N expansion in $D = 6 - \epsilon$ spacetime. The critical value $N_c \simeq 1038$ at one-loop level; however, it oscillates drastically order by order. Conformal bootstrap provides a nonperturbative approach to study CFTs, and it has been applied to estimate N_c in [54]. The authors found that the pronounced kink in the bound of Δ_σ

disappears near $N \sim 15$, which may suggest $N_c \sim 15$ in view of the observation that the singular behaviors, like kink in the dimension bound usually relate to unitary CFTs. In $3D$ such observation has helped to numerically solve the Ising model [41] and $O(N)$ vector model [36]. However, the unitarity condition becomes subtle for $5D$ CFTs and the unitarity violation may be too small to be detected by the bootstrap program with low order of derivatives. Therefore a kink does not necessarily guarantee unitarity, instead, it may relate to an interacting but nonunitary CFTs.

We have searched the allowed regions using multiple correlator conformal bootstrap for $1 \leqslant N \leqslant 100$. The isolated regions on the $(\Delta_\phi, \Delta_\sigma)$ plan can be obtained from conformal bootstrap program with lower order of derivatives. Moreover, the islands actually locate in the position close to the predictions from perturbative approaches given the Ns are not too small. However, the islands disappear after increasing the number of derivatives in bootstrap program. We believe these islands relate to interacting while nonunitary CFTs and the violation of unitarity can not be observed unless the program is equipped with sufficient high precision. In particular, our results suggest the critical value $N_c > 100$, much larger than the value estimated before. The bounds of N_c is expected to be improved further using conformal bootstrap. However, for larger N the unitarity violation in $O(N)$ fixed point theory gets smaller and more difficult to be detected. It requires higher accuracy in the bootstrap program to determine the critical value N_c and we leave this problem for future work. On the other hand, for a sufficient large N_c, the large N expansion approach is validated. The critical value N_c can be effectively studied based on this perturbative approach as well. Due to the asymptotic behavior of perturbative expansions in $5D$, probably one needs to calculate several orders higher than in [76–78] to get a sufficient good estimation.

4. SUPERCONFORMAL BLOCK FUNCTION OF SCALAR OPERATORS IN $4D$ $N = 1$ SCFT *

4.1 Introduction

The conformal bootstrap program, which was initially proposed for two dimensional conformal field theories (CFTs)[1–3], has been found to be a remarkably powerful tool to study CFTs in higher dimensional spacetime [5]. The crossing symmetry and unitarity condition can provide strong constraints on the operator scaling dimensions, coefficients in operator product expansion (OPE), and the central charges [26–39, 106, 41–56, 58–68]. The most striking results are obtained in [46, 58], in which the classical $3D$ Ising and $O(N)$ vector models are studied through bootstrapping the mixed correlators. It has been shown that by imposing certain reasonable assumptions on the spectrum, the CFT data can be isolated to small islands. These results are expected to be generalized to supersymmetric theories. Supersymmetry provides strong constraints on quantum dynamics and leads to abundant conformal theories. The supersymmetric conformal bootstrap is especially important for $4\mathcal{D}$ theories since most of the known $4\mathcal{D}$ CFTs are of supersymmetric conformal field theories (SCFTs).

The critical ingredient utilized in conformal bootstrap is the convexity of conformal blocks [5]. The four-point function can be decomposed into conformal partial waves which describe the exchange of primary operators together with their descendants. It can be shown from superconformal algebra that a superconformal primary multiplet can be decomposed into several conformal primary multiplets. Consequently, the superconformal block is the summation of several conformal blocks with coefficients restricted by

*This work has been published as an open access article [109] in Journal of High Energy Physics.

supersymmetry. Previous results on $4\mathcal{D}$ superconformal blocks have been presented in [28, 102–106] based on the superconformal Casimir approach. These studies are primarily focused on the four-point function of chiral-antichiral fields or conserved currents, which are protected by short-conditions or symmetries. Superconformal invariants appearing in the superconformal blocks cause the traditional superconformal Casimir approach to become less helpful for the four-point functions of more general fields. Recently, a new covariant approach based on the supershadow formalism has been proposed in [107] and applied in [108] for $\mathcal{N} = 1$ superconformal blocks corresponding to exchange of operators neutral under the $U(1)_R$ symmetry.

The new covariant approach generalizes the embedding and shadow formalisms proposed for CFTs and applies it to supersymmetric theories. The embedding formalism [110–116] realizes conformal transformations linearly and provides a convenient way to construct conformally covariant correlation functions. Specifically, the conformal covariance of correlation function is mapped into the Lorentz covariance of the correlation function in embedding space. Recently, the embedding formalism has been widely used to study the conformal blocks of spinor and tensor operators [117, 118, 61, 119**?** –121]. The $SU(2,2|\mathcal{N})$ superconformal symmetry transformations can be linearly realized in the supersymmetric generalization–superembedding space [122–126]. The shadow formalism was first proposed in [127–129] and has been recently applied to computing conformal blocks [116]. Using the shadow operators, one can construct projectors of the four-point function which decomposes the four-point function into conformal blocks represented by the exchanged primary operator. This provides an analytical method to compute the conformal blocks. Its supersymmetric generalization gives a systematic method to study the $\mathcal{N} = 1$ superconformal blocks.

In this work we will apply the supershadow formalism to study the most general $\mathcal{N} =$

1 four-point functions of scalars, $\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle$, where the scalars Φ_i have independent scaling dimensions and R-charges. The only constraint is from the vanishing net R-charges of the four scalars so that the $U(1)_R$ symmetry is preserved. Through partial wave decomposition the four-point function gives rise to the most general superconformal blocks, which provide crucial ingredients for $\mathcal{N}=1$ superconformal bootstrap. Our results are especially important for bootstrapping mixed correlators of scalars with arbitrary scaling dimensions and R-charges, which are beyond previous results on $\mathcal{N}=1$ superconformal blocks. An interesting problem is to bootstrap the mixed correlators between chiral and real scalars which appear in the minimal $4\mathcal{D}$ $\mathcal{N}=1$ SCFT [32, 62, 131].

The structure of this chapter is as follows: in section 2 we briefly review the superembedding space, supershadow formalism and their roles in computing $\mathcal{N}=1$ superconformal partial waves; in section 3 we study the most general three-point correlators consisting of two scalars and a spin-ℓ operator with arbitrary scaling dimensions and $U(1)$ R-charges; in section 4 we compute the superconformal partial waves, which are the supershadow projection of the four-point function and obtained from products of two three-point functions; in section 5 we present the final results on superconformal blocks, and compare our general superconformal blocks with known examples as a non-trivial consistent check. Conclusions are made in section 6. We will follow the conventions used in [107, 108] throughout this thesis.

4.2 Brief Review of Superembedding Space and Supershadow Formalism

We briefly review the superembedding space and supershadow formalism, especially for the techniques needed in our computation. More details on these topics are presented in [122, 123, 116, 107, 108].

4.2.1 Superembedding Space

There are two equivalent ways to construct superspace in which the $4\mathcal{D}$ $\mathcal{N} = 1$ superconformal group $SU(2,2|1)$ acts linearly. A natural choice is to construct (anti-) fundamental representation of $SU(2,2|1)$, the (dual) supertwistor $\mathcal{Y}_A \in \mathbb{C}^{4|1}$ ($\bar{\mathcal{Y}}^A$):

$$\mathcal{Y}_A = \begin{pmatrix} Y_\alpha \\ Y^{\dot\alpha} \\ Y_5 \end{pmatrix}, \qquad \bar{\mathcal{Y}}^A = \begin{pmatrix} \bar{Y}^\alpha & \bar{Y}_{\dot\alpha} & \bar{Y}^5 \end{pmatrix}, \tag{4.1}$$

where Y_α and $Y^{\dot\alpha}$ are bosonic complex components while Y_5 is fermionic. Representation for extended supersymmetry $\mathcal{N} > 1$ can be realized with more fermionic components in the supertwistors.

The well-known $4\mathcal{D}$ $\mathcal{N} = 1$ chiral superspace $(x_+^{\dot\alpha\alpha}, \theta_i^\alpha)$ can be reproduced from a pair of supertwistors $\mathcal{Y}_i^m$, $m = 1, 2$, with following constraints

$$\bar{\mathcal{Y}}^{nA}\mathcal{Y}_A^m = 0, \quad m, n = 1, 2. \tag{4.2}$$

Here one needs to fix the $\mathrm{GL}(2,\mathbb{C})$ gauge redundancy arising from the rotation of the two supertwistors, and similarly for the dual supertwistors. Taking the gauge named “Poincaré section", the supertwistor and its dual are simplified into

$$\mathcal{Y}_A^m = \begin{pmatrix} \delta_\alpha{}^m \\ ix_+^{\dot\alpha m} \\ 2\theta^m \end{pmatrix}, \qquad \bar{\mathcal{Y}}^{nA} = \begin{pmatrix} -ix_-^{n\alpha} & \delta_{\dot\alpha}^n & 2\bar\theta^n \end{pmatrix}. \tag{4.3}$$

In the “Poincaré section" the constraints (4.2) turn into $x_+ - x_- - 4i\bar\theta\theta = 0$ and can be solved by the chiral-antichiral coordinates of $4\mathcal{D}\ \mathcal{N}=1$ superspace.

The superembedding space provides another way to realize superconformal transformations linearly. Its coordinates are bi-supertwistors $(\mathcal{X}, \bar{\mathcal{X}})$

$$\mathcal{X}_{AB} \equiv \mathcal{Y}_A^m \mathcal{Y}_B^n \epsilon_{mn}, \qquad \bar{\mathcal{X}}^{AB} \equiv \bar{\mathcal{Y}}^{iA} \bar{\mathcal{Y}}^{jB} \epsilon_{ij}, \tag{4.4}$$

By construction, the bi-supertwistors are invariant under $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ and significantly reduce the gauge redundancies of supertwistors, besides, they satisfy the “null" conditions

$$\bar{\mathcal{X}}^{AB} \mathcal{X}_{BC} = 0. \tag{4.5}$$

Superconformal invariants are obtained from superstraces of successive products of $\mathcal{X}$'s and $\bar{\mathcal{X}}$'s. For example, the two-point invariant $\langle \bar{2}1 \rangle \equiv \mathrm{Tr}(\bar{\mathcal{X}}_2 \mathcal{X}_1)$ * is

$$\langle \bar{2}1 \rangle \equiv \bar{\mathcal{X}}_2^{AB} \mathcal{X}_{1BA} = -2(x_{2-} - x_{1+} + 2i\theta_1 \sigma \bar{\theta}_2)^2, \tag{4.6}$$

where the last step is evaluated in the Poincaré section and it is easy to show that

$$\langle \bar{2}1 \rangle^\dagger = \langle \bar{1}2 \rangle. \tag{4.7}$$

The $\mathcal{N}=1$ superconformal multiplets can be directly lifted to superembedding space. There are four parameters to characterize a $4\mathcal{D}\ \mathcal{N}=1$ superconformal primary superfield $\mathcal{O}$: the $\mathrm{SL}(2,\mathbb{C})$ Lorentz quantum numbers $(\frac{\ell}{2}, \frac{\bar{\ell}}{2})$, the scaling dimension Δ and $U(1)_R$ charge $R_\mathcal{O}$. For SCFTs, usually it is more convenient to use superconformal weights $q, \bar{q}$

$$q \equiv \frac{1}{2}\left(\Delta + \frac{3}{2} R_\mathcal{O}\right), \qquad \bar{q} \equiv \frac{1}{2}\left(\Delta - \frac{3}{2} R_\mathcal{O}\right) \tag{4.8}$$

*Here and after the indices $(j, \bar{k}, \cdots)$ denote the superembedding variables $(\mathcal{X}_j, \bar{\mathcal{X}}_k, \cdots)$.

rather than the scaling dimension Δ. Given a superfield $\phi_{\alpha_1\cdots\alpha_\ell}^{\dot\beta_1\cdots\dot\beta_{\bar\ell}}:(\frac{\ell}{2},\frac{\bar\ell}{2},q,\bar q)$, its map in superembedding space is a multi-twistor $\Phi_{B_1\cdots B_{\bar\ell}}{}^{A_1\cdots A_\ell}(\mathcal{X},\bar{\mathcal{X}})$ with homogeneity

$$\Phi(\lambda\mathcal{X},\bar\lambda\mathcal{X})=\lambda^{-q-\frac{\ell}{2}}\bar\lambda^{-\bar q-\frac{\bar\ell}{2}}\Phi(\mathcal{X},\bar{\mathcal{X}}). \tag{4.9}$$

The twistor indices make the computations cumbersome, especially for operators with large spin ℓ. Such difficulty is overcome in [114] based on an index-free notation for non-supersymmetric CFTs. The index-free notation is further generalized for $\mathcal{N}=1$ $4D$ SCFTs in [107]. The authors introduced pairs of null auxiliary twistors $\mathcal{S}_A,\bar{\mathcal{S}}^A:\ \bar{\mathcal{S}}^A\mathcal{S}_A=0$, which are used to contract with twistor indices of lifted fields

$$\Phi(\mathcal{X},\bar{\mathcal{X}},\mathcal{S},\bar{\mathcal{S}})\equiv\bar{\mathcal{S}}^{B_{\bar\ell}}\cdots\bar{\mathcal{S}}^{B_1}\Phi_{B_1\cdots B_{\bar\ell}}{}^{A_1\cdots A_\ell}\mathcal{S}_{A_\ell}\cdots\mathcal{S}_{A_1}. \tag{4.10}$$

By construction, $\Phi(\mathcal{X},\bar{\mathcal{X}},\mathcal{S},\bar{\mathcal{S}})$ is a polynomial of $\mathcal{S}_A,\bar{\mathcal{S}}^A$ while with no tensor index, and conversely, one can reproduce the initial superfield from the index-free superembedding fields $\Phi(\mathcal{X},\bar{\mathcal{X}},\mathcal{S},\bar{\mathcal{S}})$ through

$$\phi_{\alpha_1\cdots\alpha_\ell}^{\dot\beta_1\cdots\dot\beta_{\bar\ell}}=\frac{1}{\ell!}\frac{1}{\bar\ell!}\left(\bar{\mathcal{X}}\overrightarrow{\partial_{\bar{\mathcal{S}}}}\right)^{\dot\beta_1}\cdots\left(\bar{\mathcal{X}}\overrightarrow{\partial_{\bar{\mathcal{S}}}}\right)^{\dot\beta_{\bar\ell}}\Phi(\mathcal{X},\bar{\mathcal{X}},\mathcal{S},\bar{\mathcal{S}})\left(\overleftarrow{\partial_{\mathcal{S}}}\mathcal{X}\right)_{\alpha_1}\cdots\left(\overleftarrow{\partial_{\mathcal{S}}}\mathcal{X}\right)_{\alpha_\ell}\Bigg|_{\text{Poincaré}}. \tag{4.11}$$

To fix gauge redundancies in the lifted fields the auxiliary fields are set to be transverse $\bar{\mathcal{X}}\mathcal{S}=0,\ \bar{\mathcal{S}}\mathcal{X}=0.$

Strings with auxiliary fields, like $\bar{\mathcal{S}}_i j\bar{k}l\cdots\bar{m}\mathcal{S}_n$ are superconformal invariant so they provide a new type of superconformal invariants besides the supertraces of superembedding coordinates. Correlation functions are built from the two kinds of superconformal invariants. In particular, the two-point function can be completely determined by imposing homogeneity conditions.

It gets more difficult to evaluate three-point functions $\langle\Phi_1(1,\bar{1})\Phi_2(2,\bar{2})\Phi_3(3,\bar{3})\rangle$. For nonsupersymmetric CFTs, conformal symmetry and homogeneities of lifted fields are sufficient to fix three-point functions up to a constant. For SCFTs, the degree of freedoms of superembedding coordinates are notably enlarged by fermionic components, and it is possible to construct superconformal invariant cross ratio even for three-point correlator. In contrast, in CFTs it is impossible to construct conformal invariant cross ratio with fields less than 4. The invariant cross ratio is built from supertraces [132, 133, 123]

$$u=\frac{\langle 1\bar{2}\rangle\langle 2\bar{3}\rangle\langle 3\bar{1}\rangle}{\langle 2\bar{1}\rangle\langle 3\bar{2}\rangle\langle 1\bar{3}\rangle}, \tag{4.12}$$

which has no contribution on the homogeneity. As a consequence, the three-point function can be arbitrary function of the cross ratio u. Denoting

$$z=\frac{1-u}{1+u}, \tag{4.13}$$

one can show that z is proportional to the fermionic components $\theta_i,\ \bar{\theta}_i$ and satisfies

$$z^3=0, \qquad z|_{1\leftrightarrow 2}=z^\dagger=-z. \tag{4.14}$$

Therefore the most general function of z appearing in the three-point function is up to the second order. Considering its symmetry property under permutation $1\leftrightarrow 2$, there are four free parameters in the general three-point functions [108]. Additional restrictions like chirality can provide strong constraints on the parameters and simplify the three-point functions drastically. More details on the three-point correlators of general scalars will be studied in Section 3.

4.2.2 Supershadow Formalism

The supershadow approach is based on the observation that two operators $\mathcal{O} : (\frac{\ell}{2}, \frac{\bar{\ell}}{2}, q, \bar{q})$ and $\tilde{\mathcal{O}} : (\frac{\bar{\ell}}{2}, \frac{\ell}{2}, 1-q, 1-\bar{q})$ share the same superconformal Casimir so have non-vanishing two-point function. Then the operator $\tilde{\mathcal{O}}$, which is referred to shadow operator of $\mathcal{O}$, can be used to project the correlation functions onto irreducible representation of $\mathcal{O}$, i.e., the superconformal partial wave corresponding to exchange primary field $\mathcal{O}$ and its descendants.

The shadow operator $\tilde{\mathcal{O}}$ can be constructed from $\mathcal{O}$ through

$$\tilde{\mathcal{O}}(1, \bar{1}, \mathcal{S}, \bar{\mathcal{S}}) \equiv \int D[2, \bar{2}] \frac{\mathcal{O}^{\dagger}(2, \bar{2}, 2\bar{\mathcal{S}}, \bar{2}\mathcal{S})}{\langle 1\bar{2}\rangle^{1-q+\frac{\ell}{2}} \langle \bar{1}2\rangle^{1-\bar{q}+\frac{\bar{\ell}}{2}}}, \tag{4.15}$$

where $D[2, \bar{2}]$ gives the superconformal measure. One can show that the operator obtained from (4.15) has the expected quantum numbers of shadow operator $\tilde{\mathcal{O}}$. Then it is straightforward to write down the projector

$$|\mathcal{O}| = \frac{1}{\ell!^2 \bar{\ell}!^2} \int_M D[1, \bar{1}] \mathcal{O}(1, \bar{1}, \mathcal{S}, \bar{\mathcal{S}})\rangle \left(\overleftarrow{\partial}_{\mathcal{S}} 1 \overrightarrow{\partial}_{\mathcal{T}}\right)^{\ell} \left(\overleftarrow{\partial}_{\bar{\mathcal{S}}} \bar{1} \overrightarrow{\partial}_{\bar{\mathcal{T}}}\right)^{\bar{\ell}} \langle \tilde{\mathcal{O}}(1, \bar{1}, \mathcal{T}, \bar{\mathcal{T}}) \ , \tag{4.16}$$

in which the denotation M indicates "monodromy projection" [116]. By inserting the projector $|\mathcal{O}|$ into the four-point function $\langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle$ one can get the superconformal partial wave $\mathcal{W}_{\mathcal{O}}$

$$\mathcal{W}_{\mathcal{O}} \propto \langle \Phi_1 \Phi_2 \, |\mathcal{O}| \, \Phi_3 \Phi_4 \rangle, \tag{4.17}$$

which corresponds to exchange $\mathcal{O}$ and its descendants. Here the supershadow projector reduces the four-point function into a product of two three-point functions $\langle \Phi_1 \Phi_2 \mathcal{O} \rangle$ and $\langle \tilde{\mathcal{O}} \Phi_3 \Phi_4 \rangle$, which, as discussed before, can be easily obtained from superembedding formalism.

The remaining problem is to evaluate the integration in superembedding space. Normally the integrations involve both bosonic and fermionic components and are rather complex, while for the scalar four-point functions, where the external fermionic components of Φ_i are vanished $\theta_i \equiv \theta_{\rm ext} = 0$, it was proved in [107] that the integrations can be simplified into non-supersymmetric cases

$$\int D[\mathcal{Y}, \bar{\mathcal{Y}}] g(\mathcal{X}, \bar{\mathcal{X}})|_{\theta_{\rm ext}=\bar{\theta}_{\rm ext}=0} = \int D^4 X \partial_{\bar{X}}^2 g(X, \bar{X})|_{\bar{X}=X}, \tag{4.18}$$

where the embedding coordinates X's are the bosonic part of superembedding coordinates $\mathcal{X}$'s. Right hand side integration in embedding space has been comprehensively studied in [116].

Combining all these materials together one can study the $\mathcal{N} = 1$ superconformal blocks analytically, and the results can be expressed in a compact form. Superconformal partial wave $\mathcal{W}_{\mathcal{O}}$ for real ($U(1)_R$ neutral) $\mathcal{O}$ has been studied in [108]. In the following part we will apply this method to solve the most general superconformal partial waves.

4.3 General Three-Point Functions

In this section we analyze the most general three-point function $\langle \Phi_1(1, \bar{1}) \Phi_2(2, \bar{2}) \mathcal{O}(0, \bar{0}) \rangle$. The scalars $\Phi_1, \ \Phi_2$ have independent superconformal weights $(q_1, \bar{q}_1)$ and $(q_2, \bar{q}_2)$, respectively. The exchanged superprimary operator $\mathcal{O}$ has quantum numbers $(\frac{\ell}{2}, \frac{\ell}{2}, \Delta, R_{\mathcal{O}})$, where its $U(1)_R$ charge is $R_{\mathcal{O}} = \frac{2}{3} R \equiv \frac{2}{3}(\bar{q}_1 + \bar{q}_2 - q_1 - q_2)$. From superembedding coordinates, we can construct superconformal invariants $\langle i \bar{j} \rangle$ with $i, j \in 0, 1, 2$, two elementary tensor structures

$$S \equiv \frac{\bar{\mathcal{S}} 1 \bar{2} \mathcal{S}}{\langle 1 \bar{2} \rangle}, \qquad S|_{1 \leftrightarrow 2} = S^\dagger \equiv \frac{\bar{\mathcal{S}} 2 \bar{1} \mathcal{S}}{\langle 2 \bar{1} \rangle}, \tag{4.19}$$

and also the invariant cross ratio z. For superprimary operators $\mathcal{O}$ with spin-ℓ, it is useful to construct following "eigen" tensor structures with parity $\pm(-1)^\ell$ under coordinate interchange $1 \leftrightarrow 2$:

$$\begin{aligned} S_-^\ell &= \frac{1}{2}\left(S^\ell + (-1)^\ell (1 \leftrightarrow 2)\right), \\ S_+ S_-^{\ell-1} &= \frac{1}{2\ell}\left(S^\ell - (-1)^\ell (1 \leftrightarrow 2)\right). \end{aligned} \tag{4.20}$$

All the spin-ℓ tensor structures $S_+^m S_-^{\ell-m}$ with $m \geqslant 2$ vanish due to the null condition of S_+.

The most general three-point function is constructed in terms of supertraces, invariant cross ratio and tensor structures as follows:

$$\begin{aligned} \langle \Phi_1(1,\bar{1})\Phi_2(2,\bar{2})\mathcal{O}(0,\bar{0},\mathcal{S},\bar{\mathcal{S}})\rangle &= \\ & \frac{\left(\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}} + \lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} z + \lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}} z^2\right) S_-^\ell + \lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}} S_+ S_-^{\ell-1}}{\left(\langle 1\bar{0}\rangle\langle 2\bar{0}\rangle\right)^\delta \langle 1\bar{2}\rangle^{q_1-\delta}\langle 2\bar{1}\rangle^{q_2-\delta}\langle 0\bar{2}\rangle^{(\bar{q}_2-q_1)+\delta}\langle 0\bar{1}\rangle^{(\bar{q}_1-q_2)+\delta}}, \end{aligned} \tag{4.21}$$

where $\delta \equiv \frac{1}{4}(\Delta + \ell - R)$. The numerator contains four free coefficients according to the properties of spin-ℓ tensor structures and invariant cross ratio z. It is straightforward to show that the denominator satisfies the homogeneity conditions of the three operators, but this is not the only choice. The homogeneity conditions can only fix the powers of supertraces $\langle i\bar{j}\rangle$ up to a free parameter. Specifically, one can adjust the powers of supertraces through the identity

$$\left(\frac{\langle 1\bar{2}\rangle}{\langle 1\bar{0}\rangle\langle 0\bar{2}\rangle}\right)^{2a} = \left(\frac{\langle 1\bar{2}\rangle\langle 2\bar{1}\rangle}{\langle 1\bar{0}\rangle\langle 0\bar{2}\rangle\langle 0\bar{1}\rangle\langle 2\bar{0}\rangle}\right)^{a}(1 - 2az + 2a^2z^2), \tag{4.22}$$

in the meanwhile, the coefficients $\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}}$ will be transformed linearly. In (4.21) we have adopted a particular gauge that the supertraces $\langle 1\bar{0}\rangle$ and $\langle 2\bar{0}\rangle$ have identical power. It will be more convenient to compute superconformal integration in this gauge.

4.3.1 Remarks on the Complex Coefficients

For the three-point correlator of scalars with arbitrary superconformal weights, it needs to clarify the relationship between $(\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}})^*$ and $\lambda^{(i)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger}$.

Let us evaluate three-point correlator $\langle\Phi_2^\dagger(1,\bar{1})\Phi_1^\dagger(2,\bar{2})\mathcal{O}^\dagger(0,\bar{0})\rangle$. We can directly apply Eq. (4.21) with the quantum numbers $(0,\ 0,\ \bar{q}_2, q_2), (0,\ 0,\ \bar{q}_1, q_1), (\frac{\ell}{2}, \frac{\ell}{2}, \Delta, -R_{\mathcal{O}})$:

$$\begin{aligned}\langle\Phi_2^\dagger(1,\bar{1})\Phi_1^\dagger(2,\bar{2})\mathcal{O}^\dagger(0,\bar{0})\rangle &= \\ &\frac{\left(\lambda^{(0)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger} + \lambda^{(1)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger} z + \lambda^{(2)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger} z^2\right) S_-^\ell + \lambda^{(3)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger} S_+ S_-^{\ell-1}}{\left(\langle 1\bar{0}\rangle\langle 2\bar{0}\rangle\right)^{\delta'} \langle 1\bar{2}\rangle^{\bar{q}_2-\delta'}\langle 2\bar{1}\rangle^{\bar{q}_1-\delta'}\langle 0\bar{2}\rangle^{(q_1-\bar{q}_2)+\delta'}\langle 0\bar{1}\rangle^{(q_2-\bar{q}_1)+\delta'}},\end{aligned} \tag{4.23}$$

where $\delta' \equiv \frac{1}{4}(\Delta+\ell+R)$.

Alternatively, we can also solve above three-point correlator by taking Hermitian conjugate on (4.21) and then permuting coordinates $1 \leftrightarrow 2$. Both the invariant cross ratio z and the spin-ℓ tensor structure S are invariant under the combination actions of Hermitian conjugate and coordinate permutation $1 \leftrightarrow 2$, the new three-point function turns into

$$\begin{aligned}\langle\Phi_1^\dagger(2,\bar{2})\Phi_2^\dagger(1,\bar{1})\mathcal{O}^\dagger(0,\bar{0})\rangle &= \\ &\frac{\left((\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}})^* + (\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}})^* z + (\lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}})^* z^2\right) S_-^\ell + (\lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}})^* S_+ S_-^{\ell-1}}{\left(\langle 0\bar{1}\rangle\langle 0\bar{2}\rangle\right)^{\delta} \langle 1\bar{2}\rangle^{q_1-\delta}\langle 2\bar{1}\rangle^{q_2-\delta}\langle 1\bar{0}\rangle^{(\bar{q}_2-q_1)+\delta}\langle 2\bar{0}\rangle^{(\bar{q}_1-q_2)+\delta}}.\end{aligned} \tag{4.24}$$

To compare Eq. (4.24) with Eq. (4.23), we need to make a transformation (4.22) in Eq. (4.24) with parameter

$$a = \frac{q_2+\bar{q}_2-q_1-\bar{q}_1}{2} = -\frac{r}{2}, \tag{4.25}$$

then the two equations share exactly the same denominator. Identifying the tensor structures in their numerators, we obtain following linear relationships among the complex coefficients

$$
\begin{aligned}
\lambda^{(0)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger} &= (\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}})^*, \\
\lambda^{(1)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger} &= r(\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}})^* + (\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}})^*, \\
\lambda^{(2)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger} &= \frac{1}{2}r^2(\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}})^* + r(\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}})^* + (\lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}})^* + \frac{1}{2}r(\lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}})^*, \\
\lambda^{(3)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger} &= (\lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}})^*.
\end{aligned}
\tag{4.26}
$$

By taking above complex conjugate transformation of the coefficients twice, we go back to the original coefficients, as expected. The linear transformation turns into trivial $(\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}})^* = \lambda^{(i)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger}$ given $r = 0$, i.e., scalars Φ_1 and Φ_2 share the same scaling dimension.

4.3.2 Three-point Functions with Chiral Operator

Three-point function can be significantly simplified if there is a chiral or anti-chiral operator. Results obtained from these short multiplets will provide key elements to compute the most general superconformal blocks.

Let us consider the three-point correlator $\langle\Phi(1)X(2,\bar{2})\mathcal{O}(0,\bar{0})\rangle$ which will be needed to compute the shadow coefficients. The three-point correlator contains a chiral field Φ : $(0,\ 0,\ q_1,\ 0)$, a general field X : $(0,\ 0,\ q_2,\ \bar{q}_2)$ and a spin-ℓ operator $\mathcal{O}$: $(\frac{\ell}{2},\ \frac{\ell}{2},\ \frac{\Delta+R}{2},\ \frac{\Delta-R}{2})$, where $R = \bar{q}_2 - q_1 - q_2$. From the chirality of Φ, we can obtain the simplified three-point function

$$
\langle\Phi(1)X(2,\bar{2})\mathcal{O}(0,\bar{0},\mathcal{S},\bar{\mathcal{S}})\rangle = \frac{\lambda_{\Phi X\mathcal{O}}\ S^\ell}{\langle 1\bar{2}\rangle^{\frac{1}{2}(q_1+q_2+\bar{q}_2-\Delta-\ell)}\langle 1\bar{0}\rangle^{\frac{1}{2}(q_1-q_2-\bar{q}_2+\Delta+\ell)}\langle 2\bar{0}\rangle^{q_2}\langle 0\bar{2}\rangle^{\frac{1}{2}(-q_1-q_2+\bar{q}_2+\Delta+\ell)}}. \tag{4.27}
$$

Taking the transformation (4.22) with $a = \frac{1}{4}(\Delta + \ell + 2r + R)$, where $r = q_1 - q_2 - \bar{q}_2$, the above equation turns into

$$\begin{aligned}\langle \Phi(1) X(2,\bar{2}) \mathcal{O}(0,\bar{0},\mathcal{S},\bar{\mathcal{S}})\rangle &= \\ &= \lambda_{\Phi X \mathcal{O}} \frac{(1 - 2az + a\,(2a-\ell) z^2) S_-^{\ell} + \ell S_+ S_-^{\ell-1}}{\langle 1\bar{2}\rangle^{q_1 - q_2 - a} \langle 2\bar{1}\rangle^{-a} (\langle 1\bar{0}\rangle \langle 2\bar{0}\rangle)^{a+q_2} \langle 0\bar{1}\rangle^{a} \langle 0\bar{2}\rangle^{a+q_2+\bar{q}_2-q_1}},\end{aligned} \tag{4.28}$$

which is consistent with the general three-point function (4.21) given $\bar{q}_1 = 0,\ \delta = a + q_2$. The four free coefficients are fixed by the chirality condition up to an overall constant. Such a three-point function with real X appears in bootstrapping the mixed correlator of minimal $4D$ $N = 1$ SCFT. In the theory, the scalar X appears in OPE $\Phi \times \Phi^\dagger$ so is real: $q_2 = \bar{q}_2$.

Similarly, one can use anti-chirality condition to partially fix the coefficients in three-point function $\langle \Phi(\bar{1})^\dagger X(2,\bar{2}) \mathcal{O}(0,\bar{0})\rangle$:

$$(\lambda^{(0)}_{\Phi^\dagger X \mathcal{O}},\ \lambda^{(2)}_{\Phi^\dagger X \mathcal{O}},\ \lambda^{(1)}_{\Phi^\dagger X \mathcal{O}},\ \lambda^{(3)}_{\Phi^\dagger X \mathcal{O}}) = \lambda_{\Phi^\dagger X \mathcal{O}}(1,\ a'(2a'-\ell),\ -2a',\ \ell), \tag{4.29}$$

where $a' = \frac{1}{4}(\Delta + \ell - R)$, $R = \bar{q}_1 + \bar{q}_2 - q_2$.

4.4 Superconformal Partial Waves

Now we are ready to study the most general four-point correlator

$$\langle \Phi_1(1,\bar{1}) \Phi_2(2,\bar{2}) \Phi_3(3,\bar{3}) \Phi_4(4,\bar{4})\rangle, \tag{4.30}$$

where Φ_i have arbitrary superconformal weights $(q_i, \bar{q}_i)$ constrained by vanishing net R-charges

$$\sum_i q_i - \sum_i \bar{q}_i = 0. \tag{4.31}$$

Here we are interested in the superconformal partial wave which gives the amplitude of exchanging an irreducible representation of the $\mathcal{N} = 1$ superconformal group. Let us denote such irreducible representation by its superprimary field $\mathcal{O} : (\frac{\ell}{2}, \frac{\ell}{2}, \Delta, R_{\mathcal{O}})$. By inserting the projector constructed from $\mathcal{O}$ and its shadow operator $\tilde{\mathcal{O}}$ into the four-point correlator, the superconformal partial wave $\mathcal{W}_{\mathcal{O}}$ becomes

$$
\begin{aligned}
\mathcal{W}_{\mathcal{O}} &\propto \langle \Phi_1 \Phi_2 \left| \mathcal{O} \right| \Phi_3 \Phi_4 \rangle = \int D[0, \bar{0}] \langle \Phi_1 \Phi_2 \mathcal{O}(0, \bar{0}, \mathcal{S}, \bar{\mathcal{S}}) \rangle \overleftrightarrow{\mathcal{D}_\ell} \langle \tilde{\mathcal{O}}(0, \bar{0}, \mathcal{T}, \bar{\mathcal{T}}) \Phi_3 \Phi_4 \rangle \\
&= \frac{1}{\langle 1\bar{2}\rangle^{q_1-\delta} \langle 2\bar{1}\rangle^{q_2-\delta} \langle 3\bar{4}\rangle^{q_3-\delta'} \langle 4\bar{3}\rangle^{q_4-\delta'}} \int D[0, \bar{0}] \\
&\quad \frac{\mathcal{N}_\ell^f}{(\langle 1\bar{0}\rangle\langle 2\bar{0}\rangle)^{\delta} (\langle 3\bar{0}\rangle\langle 4\bar{0}\rangle)^{\delta'} \langle 0\bar{2}\rangle^{\delta+\bar{q}_2-q_1} \langle 0\bar{1}\rangle^{\delta+\bar{q}_1-q_2} \langle 0\bar{4}\rangle^{\delta'+\bar{q}_4-q_3} \langle 0\bar{3}\rangle^{\delta'+\bar{q}_3-q_4}},
\end{aligned}
\tag{4.32}
$$

where $\delta = \frac{\Delta+\ell-R}{4}$, $\delta' = \frac{2+R+\ell-\Delta}{4}$ and $\overleftrightarrow{\mathcal{D}_\ell} \equiv \frac{1}{\ell!^4}(\partial_{\mathcal{S}} 0 \partial_{\mathcal{T}})^\ell (\partial_{\bar{\mathcal{S}}} \bar{0} \partial_{\bar{\mathcal{T}}})^\ell$. $\mathcal{N}_\ell^f$ represents the tensor structures as defined in [108]:

$$
\begin{aligned}
\mathcal{N}_\ell^f &= \left((\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}} + \lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} z + \lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}} z^2) S_-^\ell + \lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}} S_+ S_-^{\ell-1} \right) \\
&\quad \overleftrightarrow{\mathcal{D}_\ell} \left((\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \lambda^{(1)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} \tilde{z} + \lambda^{(2)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} \tilde{z}^2) T_-^\ell + \lambda^{(3)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} T_+ T_-^{\ell-1} \right).
\end{aligned}
\tag{4.33}
$$

In (4.33) we have applied the three-point function

$$
\begin{aligned}
&\langle \Phi_3(3, \bar{3}) \Phi_4(4, \bar{4}) \tilde{\mathcal{O}}(0, \bar{0}, \mathcal{T}, \bar{\mathcal{T}}) \rangle = \\
&\qquad \frac{\left(\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \lambda^{(1)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} \tilde{z} + \lambda^{(2)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} \tilde{z}^2 \right) T_-^\ell + \lambda^{(3)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} T_+ T_-^{\ell-1}}{(\langle 3\bar{0}\rangle\langle 4\bar{0}\rangle)^{\delta'} \langle 3\bar{4}\rangle^{q_3-\delta'} \langle 4\bar{3}\rangle^{q_4-\delta'} \langle 0\bar{4}\rangle^{(\bar{q}_4-q_3)+\delta'} \langle 0\bar{3}\rangle^{(\bar{q}_3-q_4)+\delta'}},
\end{aligned}
\tag{4.34}
$$

where $(\tilde{z},\ T_\pm^\ell)$, like $(z,\ S_\pm^\ell)$ in (4.21), are invariant cross ratio and tensor structures. Tensor structures in $\mathcal{N}_\ell^f$ consist of the polynomial $\mathcal{N}_\ell$

$$
\mathcal{N}_\ell \equiv (\bar{\mathcal{S}} 1 \bar{2} \mathcal{S})^\ell \overleftrightarrow{\mathcal{D}_\ell} (\bar{\mathcal{T}} 3 \bar{4} \mathcal{T})^\ell
\tag{4.35}
$$

and its coordinate exchanges. Giving $\theta_{\rm ext}=\bar\theta_{\rm ext}=0$ and $\mathcal{X}_0=\bar{\mathcal{X}}_0$, $\mathcal{N}_\ell$ reduces to

$$N_\ell = y_0^{\frac{\ell}{2}} C_\ell^{(1)}(y_0), \tag{4.36}$$

where $C_\ell^{(\lambda)}(y)$ are the Gegenbauer polynomials and

$$\begin{aligned} x_0 &\equiv -\frac{X_{13}X_{20}X_{40}}{2\sqrt{X_{10}X_{20}X_{30}X_{40}X_{12}X_{34}}} - (1\leftrightarrow 2) - (3\leftrightarrow 4)\,, && (4.37)\\ y_0 &\equiv \frac{1}{2^{12}}X_{10}X_{20}X_{30}X_{40}X_{12}X_{34}. && (4.38)\end{aligned}$$

For the four-point function of scalars, we are only interested in the lowest component of a supermultiplet. To throw away irrelevant higher dimensional components, we set the fermionic coordinates $\theta_{\rm ext}=\bar\theta_{\rm ext}=0$. The bi-supertwistors $\mathcal{X}_{AB}$ and $\tilde{\mathcal{X}}^{AB}$ degenerate into twistors $X_{\alpha\beta}$ and $X^{\alpha\beta}$ which are equivalent to the six dimensional vector representations of $\mathrm{SU}(2,2)\cong\mathrm{SO}(4,2)$, and the supertraces $\langle i\bar{j}\rangle$ become inner products of vectors $X_{ij}\equiv -2X_i\cdot X_j$. Moreover, under the restriction $\theta_{\rm ext}=\bar\theta_{\rm ext}=0$ the superconformal integration (4.33) can be simplified into nonsupersymmetric conformal integration, as suggested in (4.18). To summarize, the superconformal partial wave $\mathcal{W}_{\mathcal{O}}$ is

$$\mathcal{W}_{\mathcal{O}}\big|_{\theta_{\rm ext}=0} \propto \frac{1}{X_{12}^{q_1+q_2-2\delta}X_{34}^{q_3+q_4-2\delta'}}\int D^4X_0\ \partial_{\bar 0}^2\frac{\mathcal{N}_\ell^f}{D_\ell}\bigg|_{\bar 0=0}, \tag{4.39}$$

and D_ℓ denotes the products of supertraces containing $\mathcal{X}_0$ or $\bar{\mathcal{X}}_0$

$$D_\ell \equiv (X_{1\bar 0}X_{2\bar 0})^{\delta}(X_{3\bar 0}X_{4\bar 0})^{\delta'}X_{0\bar 2}^{\delta+\bar q_2-q_1}X_{0\bar 1}^{\delta+\bar q_1-q_2}X_{0\bar 4}^{\delta'+\bar q_4-q_3}X_{0\bar 3}^{\delta'+\bar q_3-q_4}. \tag{4.40}$$

As shown in (4.39), there are only two steps to accomplish the superconformal integration for $\mathcal{W}_{\mathcal{O}}$: partial derivatives on $\mathcal{N}_\ell^f/D_\ell$ and conformal integration. The partial derivatives are straightforward to evaluate. The conformal integration related to Gegenbauer polynomial $C_\ell^{(1)}(x_0)$ has been detailedly studied in [24, 116]. Since the result is fundamental for our study we repreat it here for convenience

$$\int_M D^4 X_0 \frac{(-1)^\ell C_\ell^{(1)}(x_0)}{X_{10}^{\frac{\Delta+r}{2}} X_{20}^{\frac{\Delta-r}{2}} X_{30}^{\frac{\tilde{\Delta}+\tilde{r}}{2}} X_{40}^{\frac{\tilde{\Delta}-\tilde{r}}{2}}} = \xi_{\Delta,\tilde{\Delta},\tilde{r},\ell} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} X_{12}^{-\frac{\Delta}{2}} X_{34}^{-\frac{\tilde{\Delta}}{2}} g_{\Delta,\ell}^{r,\tilde{r}}(u,v), \tag{4.41}$$

in which $r \equiv \Delta_1 - \Delta_2$, $\tilde{r} \equiv \Delta_3 - \Delta_4$ and

$$\xi_{\Delta,\tilde{\Delta},\tilde{r},\ell} \equiv \frac{\pi^2 \Gamma(\tilde{\Delta}+\ell-1)\Gamma(\frac{\Delta-\tilde{r}+\ell}{2})\Gamma(\frac{\Delta+\tilde{r}+\ell}{2})}{(2-\Delta)\Gamma(\Delta+\ell)\Gamma(\frac{\tilde{\Delta}-\tilde{r}+\ell}{2})\Gamma(\frac{\tilde{\Delta}+\tilde{r}+\ell}{2})}. \tag{4.42}$$

The conformal blocks $g_{\Delta,\ell}^{r,\tilde{r}}(u,v)$ are defined as usual

$$\begin{aligned} g_{\Delta,\ell}^{r,\tilde{r}}(u,v) &= \frac{\rho\bar{\rho}}{\rho-\bar{\rho}} \left[k_{\Delta+\ell}(\rho) k_{\Delta-\ell-2}(\bar{\rho}) - (\rho \leftrightarrow \bar{\rho})\right], \\ k_\beta(x) &= x^{\frac{\beta}{2}} {}_2F_1\left(\frac{\beta-r}{2}, \frac{\beta+\tilde{r}}{2}, \beta, x\right), \end{aligned} \tag{4.43}$$

where $u,\ v$ are the standard conformal invariants and $u = \rho\bar{\rho},\ v = (1-\rho)(1-\bar{\rho})$.

To apply the above results on conformal integrations, it is crucial to write the integrand into a compact form in terms of Gegenbauer polynomials.

Giving $\theta_{\text{ext}} = \bar{\theta}_{\text{ext}} = 0$, the only non-vanishing fermionic coordinates are θ_0 and $\bar{\theta}_0$ from bisupertwistors $\mathcal{X}_0$ and $\bar{\mathcal{X}}_0$. Superconformal invariants proportional to the fermionic coordinates therefore vanish at third and higher orders. Moreover, as shown in [108], the tensor structure terms in $\mathcal{N}_\ell^f$ can be separated into symmetric ($\mathcal{N}_\ell^+$) or antisymmetric ($\mathcal{N}_\ell^-$) parts according to their performances under coordinate interchange $1 \leftrightarrow 3,\ 2 \leftrightarrow 4$:

$$\mathcal{N}_\ell^f = \mathcal{N}_\ell^+ + \mathcal{N}_\ell^-, \tag{4.44}$$

in which

$$\begin{aligned}\mathcal{N}_\ell^+ &= S_-^\ell \overleftrightarrow{\mathcal{D}_\ell} T_-^\ell \Big(\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} z^2 + \lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(2)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} \tilde{z}^2 \\ &\quad + \lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(1)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} z\tilde{z}\Big) + S_-^\ell \overleftrightarrow{\mathcal{D}_\ell} T_+ T_-^{\ell-1} \lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(3)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} z \\ &\quad + S_+ S_-^{\ell-1} \overleftrightarrow{\mathcal{D}_\ell} T_-^\ell \lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(1)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} \tilde{z} + S_+ S_-^{\ell-1} \overleftrightarrow{\mathcal{D}_\ell} T_+ T_-^{\ell-1} \lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(3)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}, \end{aligned} \tag{4.45}$$

and

$$\begin{aligned}\mathcal{N}_\ell^- &= z S_-^\ell \overleftrightarrow{\mathcal{D}_\ell} T_-^\ell \lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \tilde{z} S_-^\ell \overleftrightarrow{\mathcal{D}_\ell} T_-^\ell \lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(1)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} \\ &\quad + S_-^\ell \overleftrightarrow{\mathcal{D}_\ell} T_+ T_-^{\ell-1} \lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(3)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + S_+ S_-^{\ell-1} \overleftrightarrow{\mathcal{D}_\ell} T_-^\ell \lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}. \end{aligned} \tag{4.46}$$

Contributions of the symmetric terms $\mathcal{N}_\ell^+$ on the superconformal partial wave $\mathcal{W}_\mathcal{O}$ have been studied in detail in [108] under the restrictions

$$q_1 = \bar{q}_2,\ q_2 = \bar{q}_1,\ q_3 = \bar{q}_4,\ q_4 = \bar{q}_3. \tag{4.47}$$

Under above restrictions, the coordinate interchange symmetry in $\mathcal{N}_\ell^+$ is further realized in the whole integrand of superconformal partial wave $\mathcal{W}_\mathcal{O}$. Due to this symmetry, it gets much simpler to evaluate contributions on $\mathcal{W}_\mathcal{O}$ from the symmetric terms. For the most general superconformal partial waves we do not have such restrictions on the superconformal weights, nevertheless, there is a free parameter related to the transformation (4.22), and we can choose the gauge in which $X_{1\bar{0}}$ $(X_{3\bar{0}})$ and $X_{2\bar{0}}$ $(X_{4\bar{0}})$ have the same power, then it is straightforward to calculate contributions of these terms on $\mathcal{W}_\mathcal{O}$. More details on the calculations are provided in Appendix B.

The major challenge comes from the four terms in $\mathcal{N}_\ell^-$ which are anti-symmetric under the coordinate interchange $1 \leftrightarrow 3,\ 2 \leftrightarrow 4$ (anti-symmetric terms). For the cases studied in [108], due to the restrictions (4.47), D_ℓ is invariant under coordinate interchange $1 \leftrightarrow 3,\ 2 \leftrightarrow 4$, and contributions from anti-symmetric terms are cancelled automatically. For general four-point functions there is no such coordinate interchange symmetry in D_ℓ, and contributions from terms in (4.46) are proportional to the differences of scaling dimensions $r,\ \tilde{r}$.

4.4.1 Superconformal Integrations of Anti-symmetric Terms

In this section we evaluate superconformal integrations of the anti-symmetric terms in (4.46) following the strategy discussed before. However, to apply the conformal integration formulas in (4.41), we need to figure out relationships between tensor structures in $\mathcal{N}_\ell^-$ and the Gegenbauer polynomials. For tensor structures in $\mathcal{N}_\ell^+$, the polynomials satisfy coordinate interchange symmetry and can be simplified using Clifford algebra. Nevertheless, for tensor structures in $\mathcal{N}_\ell^-$, the polynomials are anti-symmetric under coordinate permutation and the Clifford algebra cannot help to simplify the polynomials directly. Instead, we show that these polynomials possesses recursion relations which can be used to determine the superconformal integrations.

The anti-symmetric terms in (4.46) consist of $\frac{zN_\ell}{D_\ell}$, $\frac{\tilde{z}N_\ell}{D_\ell}$, $\frac{N_\ell}{D_\ell}$ and their coordinate exchanges. The partial differentiations are

$$\begin{aligned}
\partial_{\bar{0}}^2 \frac{zN_\ell}{D_\ell}|_{\bar{0}=0} &= 2\delta' \frac{N_\ell}{D_\ell}\left[\frac{X_{13}}{X_{10}X_{30}} - \frac{X_{23}}{X_{20}X_{30}} + \frac{X_{14}}{X_{10}X_{40}} - \frac{X_{24}}{X_{20}X_{40}}\right] \\
&\quad + \frac{1}{2}\frac{1}{D_\ell}\frac{\ell}{\ell!^2}\left(\partial_S 0 \partial_T\right)^\ell \left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left[\frac{X_{12}}{X_{10}X_{20}}X_{10}\left(S\bar{2}3\bar{4}T\right)\right], \qquad (4.48)\\
\partial_{\bar{0}}^2 \frac{\tilde{z}N_\ell}{D_\ell}\bigg|_{\bar{0}=0} &= 2\delta \frac{N_\ell}{D_\ell}\left[\frac{X_{13}}{X_{10}X_{30}} + \frac{X_{23}}{X_{20}X_{30}} - \frac{X_{14}}{X_{10}X_{40}} - \frac{X_{24}}{X_{20}X_{40}}\right] \\
&\quad + \frac{1}{2}\frac{1}{D_\ell}\frac{\ell}{\ell!^2}\left(\partial_S 0 \partial_T\right)^\ell \left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left[\frac{X_{34}}{X_{30}X_{40}}X_{30}\left(S\bar{2}1\bar{4}T\right)\right], \qquad (4.49)\\
\partial_{\bar{0}}^2 \frac{N_\ell}{D_\ell}\bigg|_{\bar{0}=0} &= -\frac{N_\ell}{D_\ell}\left[4\delta^2\frac{X_{12}}{X_{10}X_{20}} + 4\delta'^2\frac{X_{34}}{X_{30}X_{40}} + 4\delta\delta'\left(\frac{X_{13}}{X_{10}X_{30}} + \frac{X_{23}}{X_{20}X_{30}} + \frac{X_{14}}{X_{10}X_{40}}\right.\right. \\
&\quad \left.\left. + \frac{X_{24}}{X_{20}X_{40}}\right)\right] + \frac{1}{2}\frac{1}{D_\ell}\frac{\ell}{\ell!^2}\left(\partial_S 0 \partial_T\right)^\ell \left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left[2\delta\frac{X_{12}}{X_{10}X_{20}}\left(X_{10}S\bar{2}3\bar{4}T\right)\right. \\
&\quad \left. + 2\delta'\frac{X_{34}}{X_{30}X_{40}}\left(X_{30}S\bar{2}1\bar{4}T\right)\right]. \qquad (4.50)
\end{aligned}$$

For the terms proportional to N_ℓ, their conformal integrations can be evaluated directly using Eq. (4.41). The results are provided in Appendix B. While for extra terms, we need to find their relationships with Gegenbauer polynomials before we can apply Eq. (4.41). Tensor structures in (4.46) can be expanded in terms of $\mathcal{N}_\ell$ and its coordinate exchanges as

$$\begin{aligned}
S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell &= \frac{\mathcal{N}_\ell}{4\langle 1\bar{2}\rangle^\ell \langle 3\bar{4}\rangle^\ell} + (-1)^\ell (1 \leftrightarrow 2) + (-1)^\ell (3 \leftrightarrow 4), \qquad (4.51)\\
S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_+ T_-^{\ell-1} &= \frac{\mathcal{N}_\ell}{4\ell\langle 1\bar{2}\rangle^\ell \langle 3\bar{4}\rangle^\ell} + (-1)^\ell (1 \leftrightarrow 2) - (-1)^\ell (3 \leftrightarrow 4), \qquad (4.52)\\
S_+ S_-^{\ell-1} \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell &= \frac{\mathcal{N}_\ell}{4\ell\langle 1\bar{2}\rangle^\ell \langle 3\bar{4}\rangle^\ell} - (-1)^\ell (1 \leftrightarrow 2) + (-1)^\ell (3 \leftrightarrow 4), \qquad (4.53)
\end{aligned}$$

which lead to following polynomial terms in the conformal integrand

$$
\begin{aligned}
R_\ell \equiv \frac{\ell}{\ell!^2}\left(\partial_S 0 \partial_T\right)^\ell \left(S\bar{2}\bar{1}\bar{0}\bar{3}\bar{4}T\right)^{\ell-1} \times& \\
\left(X_{10} S\bar{2}\bar{3}\bar{4}T + X_{20} S\bar{1}\bar{3}\bar{4}T - X_{10} S\bar{2}\bar{4}\bar{3}T - X_{20} S\bar{1}\bar{4}\bar{3}T\right)&, \quad (4.54)\\
P_\ell \equiv \frac{\ell}{\ell!^2}\left(\partial_S 0 \partial_T\right)^\ell \left(S\bar{2}\bar{1}\bar{0}\bar{3}\bar{4}T\right)^{\ell-1} \times& \\
\left(X_{30} S\bar{2}\bar{1}\bar{4}T + X_{40} S\bar{2}\bar{1}\bar{3}T - X_{30} S\bar{1}\bar{2}\bar{4}T - X_{40} S\bar{1}\bar{2}\bar{3}T\right)&. \quad (4.55)
\end{aligned}
$$

Appendix A shows that the above polynomials satisfy the recursion relations

$$
\begin{aligned}
R_\ell &= \ell \Delta_A N_{\ell-1} + \frac{1}{2^6}(\ell-1) X_{10} X_{20} X_{34} \Delta_B N_{\ell-2} + y_0 R_{\ell-2}, && (4.56)\\
P_\ell &= \ell \Delta_B N_{\ell-1} + \frac{1}{2^6}(\ell-1) X_{30} X_{40} X_{12} \Delta_A N_{\ell-2} + y_0 P_{\ell-2}. && (4.57)
\end{aligned}
$$

The conformal integrations related to R_ℓ and P_ℓ are

$$
\begin{aligned}
\int D^4 X_0 \, \frac{X_{12}}{X_{10}X_{20}} \frac{R_\ell}{D_\ell}\bigg|_{\bar{0}=0} = \;& \frac{8\, c_\ell\, \xi_{\Delta+2,2-\Delta,1+\tilde{r},\ell-1}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} \times \\
& \left[-\frac{4\tilde{r}\Delta(\ell+1)(\Delta-\ell)}{(\Delta-1)(\Delta+\tilde{r}-\ell)(\Delta+\tilde{r}+\ell)} g^{r,\tilde{r}}_{\Delta+1,\ell-1}\right. \\
& \left. + \frac{r\ell(\Delta-\ell)(\Delta-\tilde{r}+\ell)}{(\Delta+\ell)(\Delta+\ell+1)(\Delta+\tilde{r}-\ell)} g^{r,\tilde{r}}_{\Delta+2,\ell}\right], \qquad (4.58)\\
\int D^4 X_0 \, \frac{X_{34}}{X_{30}X_{40}} \frac{P_\ell}{D_\ell}\bigg|_{\bar{0}=0} = \;& \frac{8\, c_\ell\, \xi_{\Delta,4-\Delta,1+\tilde{r},\ell-1}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} \times \\
& \left[-\frac{r(\Delta-2)(\ell+1)(-\Delta+\tilde{r}+\ell+2)(\Delta-\tilde{r}+\ell-2)}{4(\Delta-1)(-\Delta+\ell+1)(-\Delta+\ell+2)(\Delta+\ell-1)} g^{r,\tilde{r}}_{\Delta+1,\ell-1}\right. \\
& \left. - \frac{\tilde{r}\ell(\Delta-\tilde{r}+\ell-2)}{(\Delta+\ell-1)(\Delta+\tilde{r}-\ell-2)} g^{r,\tilde{r}}_{\Delta,\ell}\right], \qquad (4.59)
\end{aligned}
$$

where $c_\ell = 2^{-6\ell}$. The above equations can be proven using mathematical induction based on the recursion relations (4.56) and (4.57). Conformal integrations in (4.58) and (4.59), together with the results presented in Appendix B, provide all the necessary materials to compute the superconformal partial waves $\mathcal{W}_{\mathcal{O}}$ for general scalars Φ_i. Here we present the final results of superconformal partial wave (4.39):

$$\mathcal{W}_{\mathcal{O}} \propto \frac{1}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} \times \left(a_1\, g^{r,\tilde{r}}_{\Delta,\ell} + a_2\, g^{r,\tilde{r}}_{\Delta+1,\ell+1} + a_3\, g^{r,\tilde{r}}_{\Delta+1,\ell-1} + a_4\, g^{r,\tilde{r}}_{\Delta+2,\ell}\right), \tag{4.60}$$

in which the coefficients a_i are the abbreviations of following long expressions:

$$\begin{aligned} a_1 &= 2\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}} \left[-\delta'\left(1+2\delta\frac{(2-\Delta)\tilde{r}^2-(\ell+2-\Delta)(\Delta+\ell)}{(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)}\right)\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \lambda^{(2)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} \right. \\ &\left. + \frac{\tilde{r}((\Delta-2)R+(-\Delta+\ell+2)(\Delta+\ell))}{2(-\Delta+\ell+2)(\Delta+\ell)}\lambda^{(1)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \frac{\tilde{r}(R+\ell+2-\Delta)}{4(-\Delta+\ell+2)}\lambda^{(3)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}\right], \end{aligned} \tag{4.61}$$

$$\begin{aligned} a_2 &= -\frac{(\Delta-2)(\Delta-\tilde{r}+\ell)(\Delta+\tilde{r}+\ell)}{4(\Delta-1)(\Delta+\ell)(\Delta+\ell+1)} \\ &\times\left(\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(\Delta-R+\ell)}{2(\Delta+\ell)}\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\right)\left(\lambda^{(1)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \frac{\tilde{r}(R+\ell+2-\Delta)}{2(-\Delta+\ell+2)}\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}\right), \end{aligned} \tag{4.62}$$

$$\begin{aligned} a_3 &= -\frac{(\Delta-2)(\Delta-\tilde{r}-\ell-2)(\Delta+\tilde{r}-\ell-2)}{4(\Delta-1)(-\Delta+\ell+1)(-\Delta+\ell+2)} \\ &\times\left(\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{\ell+1}{\ell}\lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(-\Delta+R+\ell+2)}{2(-\Delta+\ell+2)}\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\right) \\ &\times\left(\lambda^{(1)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \frac{\ell+1}{\ell}\lambda^{(3)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}} + \frac{\tilde{r}(\Delta-R+\ell)}{2(\Delta+\ell)}\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}\right), \end{aligned} \tag{4.63}$$

$$\begin{aligned} a_4 &= 2\lambda^{(0)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}\frac{(\Delta-2)(-\Delta-\tilde{r}+\ell+2)(-\Delta+\tilde{r}+\ell+2)(\Delta-\tilde{r}+\ell)(\Delta+\tilde{r}+\ell)}{16\Delta(-\Delta+\ell+1)(-\Delta+\ell+2)(\Delta+\ell)(\Delta+\ell+1)} \\ &\times\left[-\delta\left(1-2\delta'\frac{(r^2\Delta-(\Delta+\ell)(-\Delta+\ell+2))}{(\Delta-1)(-\Delta+\ell+2)(\Delta+\ell)}\right)\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}} + \lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}}\right. \\ &\left. + \frac{r(\Delta(-\Delta+R+2)+\ell(\ell+2))}{2(-\Delta+\ell+2)(\Delta+\ell)}\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(\Delta-R+\ell)}{4(\Delta+\ell)}\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}}\right]. \end{aligned} \tag{4.64}$$

Several interesting properties appear in the above long expressions of coefficients a_i.

Ignoring the constant term, a_1 and a_4 are related to each other through a transformation

$$\Delta \leftrightarrow 2-\Delta,\ \ r \leftrightarrow \tilde{r},\ \ R \leftrightarrow -R,\ \ \lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}} \leftrightarrow \lambda^{(i)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}, \tag{4.65}$$

while a_2 and a_3 are invariant under this transformation. Such symmetry is expected since it corresponds to exchange the roles of operator $\mathcal{O}$ and its supershadow operator $\tilde{\mathcal{O}}$.

4.5 Superconformal Blocks

Conformal blocks are obtained from conformal partial waves by dropping some trivial factors. The $\mathcal{N}=1$ superconformal block $\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell}$ is related to the superconformal partial wave $\mathcal{W}_{\mathcal{O}}$ through

$$\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell} = X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}} \left(\frac{X_{24}}{X_{14}}\right)^{-\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{-\frac{\tilde{r}}{2}} \mathcal{W}_{\mathcal{O}}. \tag{4.66}$$

By applying the results on $\mathcal{W}_{\mathcal{O}}$ (4.60-4.64), one can get the superconformal block in terms of $\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}}$ and $\lambda^{(i)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}$. The supershadow coefficients $\lambda^{(i)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}$ need to be transformed into the normal coefficients $\lambda^{(i)}_{\Phi_3\Phi_4\mathcal{O}^\dagger}$. In principle, one can solve the transformation between the two types of coefficients by inserting the integral expression of the supershadow operator $\tilde{\mathcal{O}}$ (4.15) in the three-point function $\langle\Phi_3(3,\bar{3})\Phi_4(4,\bar{4})\tilde{\mathcal{O}}(0,\bar{0},\mathcal{T},\bar{\mathcal{T}})\rangle$ (4.34). However it needs to evaluate a complex superconformal integration to obtain the results. A simpler method is proposed in [108] which applies the unitarity of SCFTs. In this work, the unitarity of SCFTs is also employed to solve the transformation of supershadow coefficients.

Giving $\Phi_3=\Phi_2^\dagger$ and $\Phi_4=\Phi_1^\dagger$, the unitarity of the four-point function $\langle\Phi_1\Phi_2\Phi_2^\dagger\Phi_1^\dagger\rangle$ requires the coefficients a_i (4.61-4.64) of four conformal blocks in $\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell}$ to be positive. To apply the unitary condition we need to go back to the coefficients $(\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}})^*$ rather than use $\lambda^{(i)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger}$ directly. At first it is not clear whether there is a linear map connecting $\lambda^{(i)}_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}}$

with $(\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}})^*$. Possible transformations among the three types of coefficients are shown in graph as below

$$\begin{tikzcd}
\lambda^{(i)}_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}} \arrow[r, "H_1"] \arrow[dr, "H_0"'] & (\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}})^* \arrow[d, "H_2"] \\
 & \lambda^{(i)}_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger}
\end{tikzcd}$$

in which H_2 has already been solved in (4.26). Since both H_0 and H_2 are linear transformations, $H_1 = H_0 \cdot H_2^{-1}$ is linear as well. In practice, we will firstly calculate H_1 based on the unitarity of superconformal partial waves and then solve H_0 in terms of H_1 and H_2.

The transformation H_1 has been solved in Appendix C, and the most general $\mathcal{N}=1$ superconformal block $\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell}$ is written in terms of $\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}}$ and $(\lambda^{(i)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*$. Transformation from $(\lambda^{(i)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*$ to $\lambda^{(i)}_{\Phi_3\Phi_4\mathcal{O}^\dagger}$ has been solved in (4.26), its inverse map gives $H_2(\tilde{r})$:

$$\begin{pmatrix} (\lambda^{(0)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* \\ (\lambda^{(2)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* \\ (\lambda^{(1)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* \\ (\lambda^{(3)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}\tilde{r}^2 & 1 & \tilde{r} & \frac{1}{2}\tilde{r} \\ \tilde{r} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{(0)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} \\ \lambda^{(2)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} \\ \lambda^{(1)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} \\ \lambda^{(3)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} \end{pmatrix}, \tag{4.67}$$

and it satisfies

$$H_2(r) \cdot H_2(-r) = I_{4\times 4}, \tag{4.68}$$

which is expected since the coefficients are invariant by taking complex conjugate twice.

It is straightforward to get transformation H_0 by combining the results of H_1 and H_2.

Here we do not present the explicit expression of H_0. The $\mathcal{N}=1$ superconformal block is

$$\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell} = a_1\, g^{r,\tilde{r}}_{\Delta,\ell} + a_2\, g^{r,\tilde{r}}_{\Delta+1,\ell+1} + a_3\, g^{r,\tilde{r}}_{\Delta+1,\ell-1} + a_4\, g^{r,\tilde{r}}_{\Delta+2,\ell}, \tag{4.69}$$

in which the coefficients of individual conformal blocks a_i are written in terms of $\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}}$ and $\lambda^{(i)}_{\Phi_3\Phi_4\mathcal{O}^\dagger}$

$$a_1 = \lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\lambda^{(0)}_{\Phi_3\Phi_4\mathcal{O}^\dagger}, \tag{4.70}$$

$$\begin{aligned} a_2 = {} & \frac{\Delta+\ell}{(\Delta+\ell+1)(\Delta-R+\ell)(\Delta+R+\ell)}\left(\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(\Delta-R+\ell)}{2(\Delta+\ell)}\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\right) \\ & \times\left(\lambda^{(1)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} + \frac{\tilde{r}(\Delta+R+\ell)}{2(\Delta+\ell)}\lambda^{(0)}_{\Phi_3\Phi_4\mathcal{O}^\dagger}\right), \end{aligned} \tag{4.71}$$

$$\begin{aligned} a_3 = {} & \frac{\ell+2-\Delta}{(-\Delta+\ell+1)(-\Delta-R+\ell+2)(-\Delta+R+\ell+2)} \\ & \times\left(\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{\ell+1}{\ell}\lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(-\Delta+R+\ell+2)}{2(-\Delta+\ell+2)}\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\right) \\ & \times\left((\lambda^{(1)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} + \frac{\ell+1}{\ell}\lambda^{(3)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} + \frac{\tilde{r}(-\Delta-R+\ell+2)}{2(-\Delta+\ell+2)}\lambda^{(0)}_{\Phi_3\Phi_4\mathcal{O}^\dagger}\right), \end{aligned} \tag{4.72}$$

$$\begin{aligned} a_4 = {} & \frac{4(\Delta-1)^2(-\Delta+\ell+2)(\Delta+\ell)}{\Delta^2(\ell+1-\Delta)(\Delta+\ell+1)(\ell+2-R-\Delta)(\ell+2+R-\Delta)(\Delta-R+\ell)(\Delta+R+\ell)}\times \\ & \left[-\frac{(\Delta-R+\ell)\left(R\left(\ell(\ell+2)-\Delta\left(\Delta+r^2-2\right)\right)+(\ell+2-\Delta)\left((\Delta+\ell)^2-\Delta r^2\right)\right)}{8(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)}\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\right. \\ & \left.+\lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(\Delta(R+2-\Delta)+\ell(\ell+2))}{2(\ell+2-\Delta)(\Delta+\ell)}\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(\Delta-R+\ell)}{4(\Delta+\ell)}\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}}\right]\times \\ & \left[\frac{(\Delta+R+\ell)\left(R\left(\ell(\ell+2)-\Delta\left(\Delta+\tilde{r}^2-2\right)\right)-(\ell+2-\Delta)\left((\Delta+\ell)^2-\Delta\tilde{r}^2\right)\right)}{8(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)}\lambda^{(0)}_{\Phi_3\Phi_4\mathcal{O}^\dagger}\right. \\ & \left.+\lambda^{(2)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} + \frac{\tilde{r}(\Delta(-R+2-\Delta)+\ell(\ell+2))}{2(\ell+2-\Delta)(\Delta+\ell)}\lambda^{(1)}_{\Phi_3\Phi_4\mathcal{O}^\dagger} + \frac{\tilde{r}(\Delta+R+\ell)}{4(\Delta+\ell)}\lambda^{(3)}_{\Phi_3\Phi_4\mathcal{O}^\dagger}\right]. \end{aligned} \tag{4.73}$$

Comparing with the superconformal blocks (C.23-C.26) in terms of $(\lambda^{(i)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*$, above superconformal blocks show improved symmetry that terms appear in pairs with correspondences

$$\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}} \leftrightarrow \lambda^{(i)}_{\Phi_3\Phi_4\mathcal{O}^\dagger},\ \ r\leftrightarrow\tilde{r},\ \ R\leftrightarrow -R. \tag{4.74}$$

Taking $r=\tilde{r}=R=0$, the coefficients a_i presented in (4.70-4.73) reduce to the results obtained in [108]. Under the chirality or current conservation conditions, the results in [108] can reproduce the four-point function of chiral scalars $\langle\phi\phi^\dagger\phi\phi^\dagger\rangle$ [28] and the four-point function of scalars in the global symmetry current multiplets $\langle J_1J_2J_3J_4\rangle$ [105, 106]. For non-vanishing $r,\ \tilde{r}$, and R, if certain fields Φ's in four-point function satisfy shortening conditions, like chirality, the tensor structures can be simplified and there will be strong constraints on the coefficients $\lambda^{(i)}_{\Phi_i\Phi_2\mathcal{O}}$. In this case the superconformal blocks can be conveniently solved through superconformal Casimir approach [107, 57, 63]. As a non-trivial check, we compare our work with previous results on $\mathcal{N}=1$ superconformal blocks obtained from superconformal Casimir approach [57, 63].

In [57] superconformal blocks in SCFTs with four supercharges have been studied. The authors considered four-point function $\langle\Phi_1(1)X_1(2,\bar{2})\Phi_2(3)X_2(4,\bar{4})\rangle$, in which $\Phi_{1,2}$ are chiral, while $X_{1,2}$ are scalars with arbitrary superconformal weights. As shown in (4.28), chirality conditions of Φ_1 and Φ_2 lead to following constraints on the coefficients

$$\begin{aligned}
(\lambda^{(0)}_{\Phi_1X_1\mathcal{O}},\ \lambda^{(2)}_{\Phi_1X_1\mathcal{O}},\ \lambda^{(1)}_{\Phi_1X_1\mathcal{O}},\ \lambda^{(3)}_{\Phi_1X_1\mathcal{O}}) &= \lambda_{\Phi_1X_1\mathcal{O}}(1,\ e_1(2e_1-\ell),\ -2e_1,\ \ell), &(4.75)\\
(\lambda^{(0)}_{\Phi_2X_2\mathcal{O}^\dagger},\ \lambda^{(2)}_{\Phi_2X_2\mathcal{O}^\dagger},\ \lambda^{(1)}_{\Phi_2X_2\mathcal{O}^\dagger},\ \lambda^{(3)}_{\Phi_2X_2\mathcal{O}^\dagger}) &= \lambda_{\Phi_2X_2\mathcal{O}^\dagger}(1,\ e_2(2e_2-\ell),\ -2e_2,\ \ell), &(4.76)
\end{aligned}$$

where parameters e_1 and e_2 are

$$e_1=\frac{1}{4}(\Delta+\ell+2r+R),\quad e_2=\frac{1}{4}(2-\Delta+\ell+2\tilde{r}-R), \tag{4.77}$$

and here the scaling dimension differences r and $\tilde{r}$ become $r=\Delta_{\Phi_1}-\Delta_{X_1},\ \tilde{r}=\Delta_{\Phi_2}-\Delta_{X_2}$. Plugging these constraints in (4.70-4.73), coefficients of conformal blocks in $\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell}$

turn into

$$a_1 = \lambda_{\Phi_1 X_1 \mathcal{O}} \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}, \tag{4.78}$$

$$a_2 = \frac{(\Delta + r + \ell)(\Delta + \tilde{r} + \ell)}{4(\Delta + \ell)(\Delta + \ell + 1)} \lambda_{\Phi_1 X_1 \mathcal{O}} \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}, \tag{4.79}$$

$$a_3 = \frac{(\Delta + r - \ell - 2)(\Delta + \tilde{r} - \ell - 2)}{4(-\Delta + \ell + 1)(-\Delta + \ell + 2)} \lambda_{\Phi_1 X_1 \mathcal{O}} \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}, \tag{4.80}$$

$$a_4 = \frac{(\Delta + r - \ell - 2)(\Delta + \tilde{r} - \ell - 2)(\Delta + r + \ell)(\Delta + \tilde{r} + \ell)}{16(-\Delta + \ell + 1)(-\Delta + \ell + 2)(\Delta + \ell)(\Delta + \ell + 1)} \lambda_{\Phi_1 X_1 \mathcal{O}} \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}, \tag{4.81}$$

which are in agreement with the results obtained in [57]. $\mathcal{N} = 1,\ 2$ superconformal blocks of chiral-antichiral scalars are also presented in [63], in which the four-point correlator $\langle \Phi_1 \bar{\Phi}_2 \Phi_2 \bar{\Phi}_1 \rangle$ consists of chiral-antichiral scalars with arbitrary $U(1)$ R-charges. For the $\mathcal{N} = 1$ case, the superconformal blocks are related to above coefficients a_i with the constraint $\tilde{r} = -r$ and are well consistent with our results.

4.6 Discussion

In this work we have computed the most general $\mathcal{N} = 1$ superconformal partial waves $\mathcal{W}_{\mathcal{O}} \propto \langle \Phi_1 \Phi_2 | \mathcal{O} | \Phi_3 \Phi_4 \rangle$, in which the scalars Φ_i have arbitrary scaling dimensions and $U(1)$ R-charges. Our computations are based on the superembedding space formalism and supershadow approach, which provide a systematic way to study $\mathcal{N} = 1$ superconformal blocks. Unitarity of SCFTs has been used to evaluate the coefficients in the three-point function of supershadow operator. Besides, it shows deep connections between conformal field theories and mathematical properties of hypergeometric functions throughout the computations. Our results nicely reproduce all the known results on the $\mathcal{N} = 1$ superconformal blocks under certain restrictions.

The superconformal blocks of operators with arbitrary scaling dimensions and R-charges are crucial ingredients for the mixed operator conformal bootstrap, and our results provide necessary materials for bootstrapping any $\mathcal{N} = 1$ SCFTs. An attractive problem is the $4D\,\mathcal{N} = 1$ minimal SCFT, which has no Lagrangian description and its existence is only revealed in superconformal bootstrap [32, 62]. More details of the theory are expected to be studied through bootstrapping the mixed operator correlators [131]. Our current results on the SCFTs are limited to $4D\,\mathcal{N} = 1$ scalars, and obviously it can be generalized from three aspects: dimension of spacetime, number of supercharges and spin of the fields in four-point correlator. The supershadow approach has impressive successes in solving $4D\,\mathcal{N} = 1$ scalar superconformal blocks, we hope this method, and its generalizations can be used to obtain the superconformal blocks of spinning operators in other dimensional spacetime with different supercharges.

5. BOOTSTRAPPING $4D$ $N = 1$ MINIMAL SCFT

5.1 Introduction

Conformal bootstrap has been shown to be powerful to study CFTs in higher dimensions $D > 2$. Up to now most of the works on conformal bootstrap are targeted on these CFTs whose existences and properties have already been partially known, such as the 3D Ising model, $O(N)$ vector model [46, 58] and $5D$ cubic model [69]. The conformal bootstrap approach is also expected to uncover new CFTs which are not known before based on the classical approaches. Actually we do have such an example, the so-called $4D$ $N = 1$ minimal SCFT. Evidence on this theory was first shown in [32] by bootstrapping the four-point correlator $\langle \Phi\Phi^\dagger\Phi\Phi^\dagger \rangle$. The authors noticed the boundary of the scaling dimension Δ_X ($X \sim \Phi\Phi^\dagger$) shows clear kink, which is expected to correspond to an interacting SCFT. This theory has been studied further in [62], in which it suggests the theory admits the chiral ring condition $\Phi^2 = 0$. The OPE coefficient $c_{\Phi\Phi\Phi^2}$ admits lower bound as a function of Δ_Φ and it disappears for a special value of $\Delta_\Phi = \Delta_\Phi^*$ where the kink locates. Moreover, the scaling dimension of the chiral operator Φ is estimated to be $\Delta_\Phi = \frac{10}{7}$ (we will show that this estimation on Δ_Φ actually is not precise.) while the c central charge is about $c_m = \frac{1}{9}$ [70]. In [70] the authors bootstrapped the mixed correlators with operators Φ and X in order to isolate the physically allowed region of (Δ_Φ, Δ_X) into a small island, nevertheless the attempt fails to obtain an isolated region. The superconformal bootstrap is more difficult than these of non-supersymmetric theories since the superconformal block function and also the crossing equations are more complex.

In this chapter, we will bootstrap the mixed correlators including both the chiral operator Φ and the first non-unitary scalar operator X from the OPE $\Phi\Phi^\dagger$. The results on superconformal block function obtained in chapter 3 are crucial ingredients for this study. Specifically, the mixed correlator $\langle \Phi X \Phi^\dagger X \rangle$ involves the superconformal block function corresponding to propagating operators with non-vanishing $U(1)_R$ charge and its analytical expression was not known before our work [109]. We will write down the crossing equations corresponding to the mixed correlators with Φ and X, which form a 7×1 vector equation. By applying certain mild assumptions on the spectra we will show that the physically allowed region of (Δ_Φ, Δ_X) can be isolated into a rather small island, or in other word, we are able to solve this minimal SCFT approximately!

5.2 Crossing equations of the $4D$ $N = 1$ minimal SCFT

We assume there is one chiral superfield Φ in the minimal SCFT, its OPE reads

$$\Phi\Phi^\dagger \sim 1 + \lambda_{\Phi\Phi^\dagger X} X + \cdots \,, \tag{5.1}$$

where the superfield X is real, besides, the superfield Φ is expected to satisfy the the chiral ring relation

$$\Phi^2 \sim 0, \tag{5.2}$$

i.e., there is no extra chiral superprimary multiplet appearing in the OPE of Φ, neither in the minimal SCFT.

To bootstrap the minimal SCFT with superprimary fields Φ and X, we study the following four point correlation functions

$$\begin{aligned}
&\langle\Phi(1)\Phi^\dagger(\bar{2})\Phi(3)\Phi^\dagger(\bar{4})\rangle\\
&\langle\Phi(1)\Phi^\dagger(\bar{2})X(3,\bar{3})X(4,\bar{4})\rangle\\
&\langle X(1,\bar{1})X(2,\bar{2})X(3,\bar{3})X(4,\bar{4})\rangle\\
&\langle\Phi(1)X(2,\bar{2})X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle\\
&\langle X(1,\bar{1})\Phi(2)X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle.
\end{aligned}$$

The bootstrap equations for each correlation function are listed as below.

5.2.1 Crossing equations for $\langle\Phi(1)\Phi^\dagger(\bar{2})\Phi(3)\Phi^\dagger(\bar{4})\rangle$

The four point correlation function of Φ has already been solved before [28], the results are

$$\sum_{\mathcal{O}\in\Phi\Phi^\dagger}\lambda^2_{\Phi\Phi^\dagger\mathcal{O}}\begin{pmatrix}\mathcal{F}_{\Delta,\ell}\\ \tilde{\mathcal{F}}_{\Delta,\ell}\\ \tilde{\mathcal{H}}_{\Delta,\ell}\end{pmatrix}+\sum_{\mathcal{O}\in\Phi\Phi}^{\text{BPS}}\lambda^2_{\Phi\Phi\mathcal{O}}\begin{pmatrix}0\\ F_{2d+\ell,\ell}\\ -H_{2d+\ell,\ell}\end{pmatrix}+\sum_{\mathcal{O}\in\Phi\Phi}^{\text{non-BPS}}\lambda^2_{\Phi\Phi\mathcal{O}}\begin{pmatrix}0\\ F_{\Delta,\ell}\\ -H_{\Delta,\ell}\end{pmatrix}=0,$$

where

$$\begin{aligned}
\mathcal{F}_{\Delta,\ell} \equiv\;& F_{\Delta,\ell}+\frac{(\Delta+\ell)}{4(\Delta+\ell+1)}F_{\Delta+1,\ell+1}+\frac{(\Delta-\ell-2)}{4(\Delta-\ell-1)}F_{\Delta+1,\ell-1}\\
&+\frac{(\Delta+\ell)(\Delta-\ell-2)}{16(\Delta+\ell+1)(\Delta-\ell-1)}F_{\Delta+2,\ell}.
\end{aligned}\tag{5.3}$$

Expressions for $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{H}}$ are similar but with an extra factor $(-1)^\ell$.

5.2.2 Crossing equations for $\langle X(1,\bar{1})X(2,\bar{2})X(3,\bar{3})X(4,\bar{4})\rangle$

For this four point correlation function we can directly apply the results presented in [108], by taking $\mathcal{A}_i = \mathcal{B}_i = X$. Firstly, we need to figure out the constraints on the OPE coefficients λ_i.

Since each operator is identical with superconformal weights $q_2, \bar{q}_2 = q_2$, the three point function of $\langle X(1,\bar{1})X(2,\bar{2})\mathcal{O}(0,\bar{0})\rangle$ reads

$$\langle X(1,\bar{1})X(2,\bar{2})\mathcal{O}(0,\bar{0},\mathcal{S},\bar{\mathcal{S}})\rangle = \frac{\left(\lambda^{(0)}_{XX\mathcal{O}} + \lambda^{(1)}_{XX\mathcal{O}} z + \lambda^{(2)}_{XX\mathcal{O}} z^2\right) S_-^{\ell} + \lambda^{(3)}_{XX\mathcal{O}} S_+ S_-^{\ell-1}}{\langle 1\bar{2}\rangle^{q_2-\frac{1}{4}(\Delta+\ell)}\langle 2\bar{1}\rangle^{q_2-\frac{1}{4}(\Delta+\ell)}(\langle 0\bar{1}\rangle\langle 1\bar{0}\rangle\langle 0\bar{2}\rangle\langle 2\bar{0}\rangle)^{\frac{1}{4}(\Delta+\ell)}}, \tag{5.4}$$

above three point function should be invariant under transformation $1 \leftrightarrow 2$. Both z and S_- are antisymmetric under this transformation, while S_+ is symmetric. So we have

$$\lambda^{(1)}_{XX\mathcal{O}} = \lambda^{(3)}_{XX\mathcal{O}} = 0 \quad \text{for even } \ell \tag{5.5}$$

$$\lambda^{(0)}_{XX\mathcal{O}} = \lambda^{(2)}_{XX\mathcal{O}} = 0 \quad \text{for odd } \ell. \tag{5.6}$$

There is no limitation on the non-vanishing OPE coefficients.

Plug the constraints on the OPE coefficients in Eq. (5.2) of [108], we get the conformal block

$$\begin{aligned}\mathcal{G}^{\mathcal{N}=1|XX;XX}_{\Delta,\ell} &= \left|\lambda^{(0)}_{XX\mathcal{O}}\right|^2 g_{\Delta,\ell} + \\ &+ \frac{\left|(\Delta+\ell)^2\lambda^{(0)}_{XX\mathcal{O}} - 8(\Delta-1)\lambda^{(2)}_{XX\mathcal{O}}\right|^2}{16\Delta^2(\Delta-\ell-1)(\Delta-\ell-2)(\Delta+\ell)(\Delta+\ell+1)} g_{\Delta+2,\ell},\end{aligned} \tag{5.7}$$

for even ℓ, and

$$\mathcal{G}_{\Delta,\ell}^{\mathcal{N}=1|XX;XX} = \frac{\left|\lambda^{(1)}_{XX\mathcal{O}}\right|^2}{(\Delta+\ell)(\Delta+\ell+1)} g_{\Delta+1,\ell+1} + \frac{\left|\lambda^{(1)}_{XX\mathcal{O}} + \frac{\ell+1}{\ell}\lambda^{(3)}_{XX\mathcal{O}}\right|^2}{(\Delta-\ell-1)(\Delta-\ell-2)} g_{\Delta+1,\ell-1} \quad (5.8)$$

for odd ℓ. Note that for odd ℓ it is the conformal block function with even spin ($\ell \pm 1$) $g_{\Delta+1,\ell\pm1}$ involving in the bootstrap, the same as the cases with even ℓ.

Switching $1 \leftrightarrow 2$ in the superconformal block expansion gives the crossing relation

$$\sum_{\mathcal{O}\in X\times X} \mathcal{G}_{\Delta,\ell}(u,v) = (\frac{u}{v})^{2q_2} \sum_{\mathcal{O}\in X\times X} \mathcal{G}_{\Delta,\ell}(v,u). \quad (5.9)$$

Note that the OPE coefficient $\lambda_{XX\mathcal{O}}$ has already been absorbed in the conformal block $\mathcal{G}_{\Delta,\ell}(u,v)$. The crossing relation can be further modified as

$$\sum_{\mathcal{O}\in X\times X} \left|\lambda^{(0)}_{XX\mathcal{O}}\right|^2 E_{\Delta,\ell}(u,v) + \sum_{\mathcal{O}\in X\times X} \frac{\left|(\Delta+\ell)^2\lambda^{(0)}_{XX\mathcal{O}} - 8(\Delta-1)\lambda^{(2)}_{XX\mathcal{O}}\right|^2}{16\Delta^2(\Delta-\ell-1)(\Delta-\ell-2)(\Delta+\ell)(\Delta+\ell+1)} E_{\Delta+2,\ell}(u,v) = 0, \quad (5.10)$$

for even ℓ, and

$$\sum_{\mathcal{O}\in X\times X} \frac{\left|\lambda^{(1)}_{XX\mathcal{O}}\right|^2}{(\Delta+\ell)(\Delta+\ell+1)} E_{\Delta+1,\ell+1} + \sum_{\mathcal{O}\in X\times X} \frac{\left|\lambda^{(1)}_{XX\mathcal{O}} + \frac{\ell+1}{\ell}\lambda^{(3)}_{XX\mathcal{O}}\right|^2}{(\Delta-\ell-1)(\Delta-\ell-2)} E_{\Delta+1,\ell-1} = 0 \quad (5.11)$$

for odd ℓ, where

$$
\begin{aligned}
E_{\Delta,\ell}(u,v) &\equiv \frac{v^{2q_2} g_{\Delta,\ell}(u,v) - u^{2q_2} g_{\Delta,\ell}(v,u)}{u^{2q_2} - v^{2q_2}}, &(5.12)\\
E_{0,0}(u,v) &= -1. &(5.13)
\end{aligned}
$$

Here we only get one bootstrap equation, while for chiral superprimary field Φ, we get three bootstrap equations instead. Such difference is reasonable since for Φ, they three equations are essentially from the different symmetry factors of the $U(1) \simeq SO(2)$, while the real field X is just the singlet of the $U(1)$ symmetry.

5.2.3 Crossing equations for $\langle \Phi(1)\Phi^\dagger(\bar{2})X(3,\bar{3})X(4,\bar{4})\rangle$

Due to the mixing of two different superprimary fields Φ and X, we need to solve three different types of four point correlation functions, which are $\langle \Phi(1)\Phi^\dagger(\bar{2})X(3,\bar{3})X(4,\bar{4})\rangle$, $\langle \Phi(1)X(2,\bar{2})X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle$ and $\langle X(1,\bar{1})\Phi(2)X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle$.

5.2.3.1 $\langle \Phi(1)\Phi^\dagger(\bar{2})X(3,\bar{3})X(4,\bar{4})\rangle$*:*

This channel still belongs to the case already solved in [108] by taking $\mathcal{A}_i = \Phi$, $\mathcal{B}_i = X$. In consequence, the OPE coefficients λ_i admit the constraints

$$
\begin{aligned}
\lambda^{(1)}_{\Phi\Phi^\dagger\mathcal{O}} &= -\frac{1}{2}(\Delta+\ell)\lambda^{(0)}_{\Phi\Phi^\dagger\mathcal{O}},\\
\lambda^{(2)}_{\Phi\Phi^\dagger\mathcal{O}} &= \frac{1}{8}(\Delta+\ell)(\Delta-\ell)\lambda^{(0)}_{\Phi\Phi^\dagger\mathcal{O}},\\
\lambda^{(1)}_{\Phi\Phi^\dagger\mathcal{O}} &= \ell\,\lambda^{(0)}_{\Phi\Phi^\dagger\mathcal{O}},\\
\lambda^{(1)}_{XX\mathcal{O}} &= \lambda^{(3)}_{XX\mathcal{O}} = 0 \quad \text{for even } \ \ell,\\
\lambda^{(0)}_{XX\mathcal{O}} &= \lambda^{(2)}_{XX\mathcal{O}} = 0 \quad \text{for odd } \ \ell. &(5.14)
\end{aligned}
$$

Denote $\lambda^{(0)}_{\Phi\Phi^\dagger\mathcal{O}} = \lambda_{\Phi\Phi^\dagger\mathcal{O}}$, the conformal block functions for the correlation function are

$$\begin{aligned} \mathbf{G}^{\mathcal{N}=1|\Phi\Phi^\dagger;XX}_{\Delta,\ell} &= \lambda_{\Phi\Phi^\dagger\mathcal{O}}\lambda^{(0)}_{XX\mathcal{O}}\, g_{\Delta,\ell} \\ &\quad -\lambda_{\Phi\Phi^\dagger\mathcal{O}}\frac{(\Delta+\ell)^2\lambda^{(0)}_{XX\mathcal{O}} - 8(\Delta-1)\lambda^{(2)}_{XX\mathcal{O}}}{16\Delta(\Delta-\ell-1)(\Delta+\ell+1)} g_{\Delta+2,\ell} \end{aligned} \tag{5.15}$$

for even ℓ and

$$\begin{aligned} \mathbf{G}^{\mathcal{N}=1|\Phi\Phi^\dagger;XX}_{\Delta,\ell} &= -\frac{\lambda_{\Phi\Phi^\dagger\mathcal{O}}\lambda^{(1)}_{XX\mathcal{O}}}{2(\Delta+\ell+1)} g_{\Delta+1,\ell+1} \\ &\quad -\frac{\lambda_{\Phi\Phi^\dagger\mathcal{O}}\left(\lambda^{(1)}_{XX\mathcal{O}} + \frac{(\ell+1)}{\ell}\lambda^{(3)}_{XX\mathcal{O}}\right)}{2(\Delta-\ell-1)} g_{\Delta+1,\ell-1} \end{aligned} \tag{5.16}$$

for odd ℓ.

5.2.3.2 $\langle\Phi(1)X(2,\bar{2})X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle$*:*

As we calculated in the previous note, the conformal block of this four point correlation function is

$$\mathcal{G}^{\mathcal{N}=1|\Phi X;X\Phi^\dagger}_{\Delta,\ell} \equiv \mathcal{G}^{r,-r}_{\Delta,\ell} = g^{r,-r}_{\Delta,\ell} + a_1 g^{r,-r}_{\Delta+1,\ell+1} + a_2 g^{r,-r}_{\Delta+1,\ell-1} + a_3 g^{r,-r}_{\Delta+2,\ell}, \tag{5.17}$$

where $\lambda_{X\Phi^\dagger\mathcal{O}^\dagger} = \lambda^*_{\Phi X\mathcal{O}}$ and

$$\begin{aligned} a_1 &= \frac{(\Delta-r+\ell)(\Delta+r+\ell)}{4(\Delta+\ell)(\Delta+\ell+1)}, \\ a_2 &= -\frac{(-\Delta+r+\ell+2)(\Delta+r-\ell-2)}{4(-\Delta+\ell+1)(-\Delta+\ell+2)}, \\ a_3 &= \frac{(-\Delta+r-\ell)(-\Delta+r+\ell+2)(\Delta+r-\ell-2)(\Delta+r+\ell)}{16(-\Delta+\ell+1)(-\Delta+\ell+2)(\Delta+\ell)(\Delta+\ell+1)}, \end{aligned} \tag{5.18}$$

here q is the conformal weight of chiral field Φ, $r = q - 2q_X$.

5.2.3.3 $\langle X(1,\bar{1})\Phi(2)X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle$*:*

It is rather easy to get the conformal block $\tilde{\mathcal{G}}$ for this correlation function based on the last four point function. One can follow the same procedure presented in previous note by switching $1 \leftrightarrow 2$ in the whole calculation, the results are unchanged except the minus sign in $(S\bar{2}1\bar{0}3\bar{4}S) \to (S\bar{1}2\bar{0}3\bar{4}S)$, which accounts an extra sign factor $(-1)^\ell$

$$\begin{aligned}\tilde{\mathcal{G}}_{\Delta,\ell}^{\mathcal{N}=1|X\Phi;X\Phi^\dagger} &\equiv \tilde{\mathcal{G}}_{\Delta,\ell}^{-r,-r} \\ &= \lambda_{X\Phi\mathcal{O}}\lambda_{X\Phi^\dagger\mathcal{O}^\dagger}(-1)^\ell\left(g_{\Delta,\ell}^{-r,-r} - a_1 g_{\Delta+1,\ell+1}^{-r,-r} - a_2 g_{\Delta+1,\ell-1}^{-r,-r} + a_3 g_{\Delta+2,\ell}^{-r,-r}\right) \\ &= \lambda_{\Phi X\mathcal{O}}\lambda_{X\Phi^\dagger\mathcal{O}^\dagger}\left(g_{\Delta,\ell}^{-r,-r} - a_1 g_{\Delta+1,\ell+1}^{-r,-r} - a_2 g_{\Delta+1,\ell-1}^{-r,-r} + a_3 g_{\Delta+2,\ell}^{-r,-r}\right),\end{aligned} \tag{5.19}$$

where we have applied the condition $\lambda_{X\Phi\mathcal{O}} = \lambda_{\Phi X\mathcal{O}}(-1)^\ell$.

Based on the above conformal blocks, it is straightforward to show the bootstrap equations. The cross relation of four point function $\langle\Phi(1)X(2,\bar{2})X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle$ is

$$\begin{aligned}\langle\Phi(1)X(2,\bar{2})X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle &= \\ &\frac{1}{X_{12}^{\frac{q_1+2q_2}{2}}X_{34}^{\frac{q_1+2q_2}{2}}}\left(\frac{X_{14}}{X_{13}}\right)^{-\frac{r}{2}}\left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}}\sum_{\mathcal{O}\in\Phi X}\lambda_{\Phi X\mathcal{O}}\lambda_{X\Phi^\dagger\mathcal{O}^\dagger}\mathcal{G}_{\Delta,\ell}^{r,-r}(u,v)\end{aligned} \tag{5.20}$$

$$\langle\Phi(1)\Phi^\dagger(\bar{4})X(3,\bar{3})X(2,\bar{2})\rangle = \frac{1}{X_{12}^{q_1}X_{34}^{2q_2}}\sum_{\mathcal{O}\in\Phi\Phi^\dagger}\lambda_{\Phi\Phi^\dagger\mathcal{O}}\lambda_{XX\mathcal{O}}G_{\Delta,\ell}^{0,0}(v,u), \tag{5.21}$$

where the conformal blocks $\mathcal{G}_{\Delta,\ell}^{r,-r}$ and $G_{\Delta,\ell}^{0,0}$ are from Eq. (5.17) and Eq. (5.15), respectively. Cross symmetry of CFT requires the two formulas in Eq. (5.20) and Eq. (5.21) identical, therefore we have

$$\sum_{\mathcal{O}\in\Phi X}\lambda_{\Phi X\mathcal{O}}\lambda_{X\Phi^\dagger\mathcal{O}^\dagger}\, v^{2q_2}\,\mathcal{G}_{\Delta,\ell}^{r,-r}(u,v) = \sum_{\mathcal{O}\in\Phi\Phi^\dagger}\lambda_{\Phi\Phi^\dagger\mathcal{O}}\lambda_{XX\mathcal{O}}\, u^{q_2+\frac{q_1}{2}}\, G_{\Delta,\ell}^{0,0}(v,u). \tag{5.22}$$

The cross relation of four point function $\langle \Phi(1) X(2,\bar{2}) X(3,\bar{3}) \Phi^\dagger(\bar{4})\rangle$ is

$$\begin{aligned}
\langle X(1,\bar{1})\Phi(2)X(3,\bar{3})\Phi^\dagger(\bar{4})\rangle &= \\
&\frac{1}{X_{12}^{\frac{q_1+2q_2}{2}} X_{34}^{\frac{q_1+2q_2}{2}}} \left(\frac{X_{14}}{X_{13}}\right)^{-\frac{r}{2}} \left(\frac{X_{24}}{X_{14}}\right)^{-\frac{r}{2}} \sum_{\mathcal{O}\in X\Phi} \lambda_{\Phi X \mathcal{O}} \lambda_{X\Phi^\dagger \mathcal{O}^\dagger} \tilde{\mathcal{G}}^{-r,-r}_{\Delta,\ell}(u,v) \\
\langle X(1,\bar{1})\Phi^\dagger(4)X(3,\bar{3})\Phi(2)\rangle &= \\
&\frac{1}{X_{14}^{\frac{q_1+2q_2}{2}} X_{23}^{\frac{q_1+2q_2}{2}}} \left(\frac{X_{12}}{X_{13}}\right)^{-\frac{r}{2}} \left(\frac{X_{24}}{X_{12}}\right)^{-\frac{r}{2}} \sum_{\mathcal{O}\in X\Phi} \lambda_{\Phi X \mathcal{O}} \lambda_{X\Phi^\dagger \mathcal{O}^\dagger} \tilde{\mathcal{G}}^{-r,-r}_{\Delta,\ell}(v,u).
\end{aligned} \tag{5.23}$$

The equation from cross symmetry gives

$$\sum_{\mathcal{O}\in X\Phi} \lambda_{\Phi X \mathcal{O}} \lambda_{X\Phi^\dagger \mathcal{O}^\dagger}\, v^{q_2+\frac{q_1}{2}}\, \tilde{\mathcal{G}}^{-r,-r}_{\Delta,\ell}(u,v) = \sum_{\mathcal{O}\in X\Phi} \lambda_{\Phi X \mathcal{O}} \lambda_{X\Phi^\dagger \mathcal{O}^\dagger}\, u^{q_2+\frac{q_1}{2}}\, \tilde{\mathcal{G}}^{-r,-r}_{\Delta,\ell}(v,u). \tag{5.24}$$

Let's define

$$\begin{aligned}
F^{r,s}_{\pm,\Delta,\ell}(u,v) &= v^{q_2+\frac{q_1}{2}}\, G^{r,s}_{\Delta,\ell}(u,v) \pm u^{q_2+\frac{q_1}{2}}\, G^{r,s}_{\Delta,\ell}(v,u) \\
\mathcal{F}^{r,s}_{\pm,\Delta,\ell}(u,v) &= v^{2q_2}\, \mathcal{G}^{r,s}_{\Delta,\ell}(u,v) \pm u^{2q_2}\, \mathcal{G}^{r,s}_{\Delta,\ell}(v,u) \\
\tilde{\mathcal{F}}^{r,s}_{\pm,\Delta,\ell}(u,v) &= v^{q_2+\frac{q_1}{2}}\, \tilde{\mathcal{G}}^{r,s}_{\Delta,\ell}(u,v) \pm u^{q_2+\frac{q_1}{2}}\, \tilde{\mathcal{G}}^{r,s}_{\Delta,\ell}(v,u).
\end{aligned}$$

In particular, the conformal block $F^{0,0}_{\pm,\Delta,\ell}(u,v)$ (note the difference between $F^{0,0}_{\pm,\Delta,\ell}(u,v)$ and the classical conformal block $F_{\pm,\Delta,\ell}(u,v)$ used in (5.3)) contains two terms

$$\begin{aligned}
F^{0,0}_{\pm,\Delta,\ell}(u,v) &= \lambda_{\Phi\Phi^\dagger\mathcal{O}} \lambda^{(0)}_{XX\mathcal{O}}\, D_{\pm,\Delta,\ell} \\
&\quad -\lambda_{\Phi\Phi^\dagger\mathcal{O}} \frac{(\Delta+\ell)^2\lambda^{(0)}_{XX\mathcal{O}} - 8(\Delta-1)\lambda^{(2)}_{XX\mathcal{O}}}{16\Delta(\Delta-\ell-1)(\Delta+\ell+1)} D_{\pm,\Delta+2,\ell}
\end{aligned} \tag{5.25}$$

for even ℓ and

$$F^{0,0}_{\pm,\Delta,\ell}(u,v) = -\frac{\lambda_{\Phi\Phi^\dagger\mathcal{O}}\lambda^{(1)}_{XX\mathcal{O}}}{2(\Delta+\ell+1)}D_{\pm,\Delta+1,\ell+1} - \frac{\lambda_{\Phi\Phi^\dagger\mathcal{O}}\left(\lambda^{(1)}_{XX\mathcal{O}}+\frac{(\ell+1)}{\ell}\lambda^{(3)}_{XX\mathcal{O}}\right)}{2(\Delta-\ell-1)}D_{\pm,\Delta+1,\ell-1} \tag{5.26}$$

for odd ℓ, in which

$$D_{\pm,\Delta,\ell}(u,v) \equiv v^{q_2+\frac{q_1}{2}}g_{\Delta,\ell}(u,v) - u^{q_2+\frac{q_1}{2}}g_{\Delta,\ell}(v,u). \tag{5.27}$$

Note the only difference between $D_{-,\Delta,\ell}$ and $E_{\Delta,\ell}$ is the power of u, v. From the cross symmetry equation (5.22), add or subtract its $u \leftrightarrow v$ exchanged form, we can get two equations

$$\sum_{\mathcal{O}\in\Phi X}\lambda_{\Phi X\mathcal{O}}\lambda_{X\Phi^\dagger\mathcal{O}^\dagger}\ \mathcal{F}^{r,-r}_{-,\Delta,\ell}(u,v) + \sum_{\mathcal{O}\in\Phi\Phi^\dagger} F^{0,0}_{-,\Delta,\ell}(u,v) = 0, \tag{5.28}$$

$$\sum_{\mathcal{O}\in\Phi X}\lambda_{\Phi X\mathcal{O}}\lambda_{X\Phi^\dagger\mathcal{O}^\dagger}\ \mathcal{F}^{r,-r}_{+,\Delta,\ell}(u,v) - \sum_{\mathcal{O}\in\Phi\Phi^\dagger} F^{0,0}_{+,\Delta,\ell}(u,v) = 0. \tag{5.29}$$

For $F^{0,0}_{\pm,\Delta,\ell}(u,v)$ the OPE coefficients $\lambda_{\Phi\Phi^\dagger\mathcal{O}}\lambda^{(i)}_{XX\mathcal{O}}$ is contained in its expression so not shown in above bootstrap equations. While for the cross symmetry equation (5.24), we get one equation

$$\sum_{\mathcal{O}\in X\Phi}\lambda_{\Phi X\mathcal{O}}\lambda_{X\Phi^\dagger\mathcal{O}^\dagger}\ \tilde{\mathcal{F}}^{-r,-r}_{-,\Delta,\ell}(u,v) = 0. \tag{5.30}$$

As discussed before, the OPE coefficients are not contained in the conformal blocks to show their roles explicitly.

The three equations (5.28, 5.29, 5.30) are the bootstrap equations from mixed operators $\langle\Phi(1)\Phi^\dagger(\bar{2})X(3,\bar{3})X(4,\bar{4})\rangle$. Now we are ready to write down all the bootstrap equations from three different type of four point correlation functions.

For even ℓ we have the following bootstrap equation group

$$0 = \sum_{\mathcal{O}\in\Phi\Phi}^{\text{BPS}} |\lambda_{\Phi\Phi\mathcal{O}}|^2 \begin{bmatrix} 0 \\ F_{2q_1+\ell,\ell}(u,v) \\ -H_{2q_1+\ell,\ell}(u,v) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sum_{\mathcal{O}\in\Phi\Phi}^{\text{nonBPS}} |\lambda_{\Phi\Phi\mathcal{O}}|^2 \begin{bmatrix} 0 \\ F_{\Delta,\ell}(u,v) \\ -H_{\Delta,\ell}(u,v) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.31}$$

$$+ \sum_{\mathcal{O}\in\Phi X} 2|\lambda_{\Phi X\mathcal{O}}|^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{\mathcal{F}}^{-r,-r}_{-,\Delta,\ell}(u,v) \\ \mathcal{F}^{r,-r}_{+,\Delta,\ell}(u,v) \\ \mathcal{F}^{r,-r}_{-,\Delta,\ell}(u,v) \end{bmatrix} +$$

$$\sum_{\mathcal{O}\in\Phi X}\begin{pmatrix}\lambda_{\Phi\Phi^\dagger\mathcal{O}} & \lambda^{(0)}_{XX\mathcal{O}} & \lambda^{(x)}_{XX\mathcal{O}}\end{pmatrix}\left[\begin{matrix}\begin{pmatrix}\mathcal{F}_{\Delta,\ell}(u,v) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ \begin{pmatrix}\tilde{\mathcal{F}}_{\Delta,\ell}(u,v) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ \begin{pmatrix}\tilde{\mathcal{H}}_{\Delta,\ell}(u,v) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ \begin{pmatrix}0 & 0 & 0\\ 0 & E_{\Delta,\ell}(u,v) & 0\\ 0 & 0 & E_{\Delta+2,\ell}(u,v)\end{pmatrix}\\ \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ \begin{pmatrix}0 & -D_{+,\Delta,\ell} & c_1D_{+,\Delta+2,\ell}\\ -D_{+,\Delta,\ell} & 0 & 0\\ c_1D_{+,\Delta+2,\ell} & 0 & 0\end{pmatrix}\\ \begin{pmatrix}0 & D_{-,\Delta,\ell} & -c_1D_{-,\Delta+2,\ell}\\ D_{-,\Delta,\ell} & 0 & 0\\ -c_1D_{-,\Delta+2,\ell} & 0 & 0\end{pmatrix}\end{matrix}\right]\begin{pmatrix}\lambda_{\Phi\Phi^\dagger\mathcal{O}}\\ \lambda^{(0)}_{XX\mathcal{O}}\\ \lambda^{(x)}_{XX\mathcal{O}}\end{pmatrix},$$

in which we have denoted

$$\lambda^{(x)}_{XX\mathcal{O}} = \frac{(\Delta+\ell)^2\lambda^{(0)}_{XX\mathcal{O}} - 8(\Delta-1)\lambda^{(2)}_{XX\mathcal{O}}}{4\Delta\sqrt{(\Delta-\ell-1)(\Delta-\ell-2)(\Delta+\ell)(\Delta+\ell+1)}}, \tag{5.32}$$

and also

$$c_1 = \frac{1}{4}\sqrt{\frac{(\Delta+\ell)(\Delta-\ell-2)}{(\Delta-\ell-1)(\Delta+\ell+1)}}. \tag{5.33}$$

While for odd ℓ, the bootstrap equations remain the same except the last part, which turns into

$$\sum_{\mathcal{O}\in\Phi X} \left(\lambda_{\Phi\Phi^\dagger\mathcal{O}}\ \lambda^{(a)}_{XX\mathcal{O}}\ \lambda^{(b)}_{XX\mathcal{O}}\right) M \begin{pmatrix} \lambda_{\Phi\Phi^\dagger\mathcal{O}} \\ \\ \lambda^{(a)}_{XX\mathcal{O}} \\ \\ \lambda^{(b)}_{XX\mathcal{O}} \end{pmatrix}, \tag{5.34}$$

and the matrix M is

$$M = \begin{bmatrix} \begin{pmatrix} \mathcal{F}_{\Delta,\ell}(u,v) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \tilde{\mathcal{F}}_{\Delta,\ell}(u,v) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \tilde{\mathcal{H}}_{\Delta,\ell}(u,v) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{\Delta+1,\ell+1}(u,v) & 0 \\ 0 & 0 & E_{\Delta+1,\ell-1}(u,v) \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & c_2 D_{+,\Delta+1,\ell+1} & c_3 D_{+,\Delta+1,\ell-1} \\ c_2 D_{+,\Delta+1,\ell+1} & 0 & 0 \\ c_3 D_{+,\Delta+1,\ell-1} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -c_2 D_{-,\Delta+1,\ell+1} & -c_3 D_{-,\Delta+1,\ell-1} \\ -c_2 D_{-,\Delta+1,\ell+1} & 0 & 0 \\ -c_3 D_{-,\Delta+1,\ell-1} & 0 & 0 \end{pmatrix} \end{bmatrix}.$$

In (5.34) we have denoted

$$\lambda^{(a)}_{XX\mathcal{O}} = \frac{\lambda^{(1)}_{XX\mathcal{O}}}{\sqrt{(\Delta+\ell)(\Delta+\ell+1)}}, \tag{5.35}$$

$$\lambda^{(b)}_{XX\mathcal{O}} = \frac{\lambda^{(1)}_{XX\mathcal{O}} + \frac{\ell+1}{\ell}\lambda^{(3)}_{XX\mathcal{O}}}{\sqrt{(\Delta-\ell-1)(\Delta-\ell-2)}}, \tag{5.36}$$

and also

$$c_2 = \frac{1}{2}\sqrt{\frac{\Delta+\ell}{\Delta+\ell+1}}, \tag{5.37}$$

$$c_3 = \frac{1}{2}\sqrt{\frac{\Delta-\ell-2}{\Delta-\ell-1}}, \tag{5.38}$$

in (5.35).

5.2.4 Extra conditions on the bootstrap equations:

Let's denote the bootstrap equation group (5.31) for simplicity as follows

$$\begin{aligned} 0 \;=\; & \sum_{\mathcal{O}\in\Phi\Phi}^{\text{BPS}} |\lambda_{\Phi\Phi\mathcal{O}}|^2 \vec{V}_{B,\Delta,\ell} + \sum_{\mathcal{O}\in\Phi\Phi}^{\text{nonBPS}} |\lambda_{\Phi\Phi\mathcal{O}}|^2 \vec{V}_{T,\Delta,\ell} + \sum_{\mathcal{O}\in\Phi X} 2|\lambda_{\Phi X\mathcal{O}}|^2 \vec{V}_{M,\Delta,\ell} \\ & + \sum_{\mathcal{O}\in\Phi X, XX} \left(\lambda_{\Phi\Phi^\dagger\mathcal{O}}\ \lambda^{(a)}_{XX\mathcal{O}}\ \lambda^{(b)}_{XX\mathcal{O}}\right) \vec{V}_{S,\Delta,\ell} \begin{pmatrix} \lambda_{\Phi\Phi^\dagger\mathcal{O}} \\ \lambda^{(a)}_{XX\mathcal{O}} \\ \lambda^{(b)}_{XX\mathcal{O}} \end{pmatrix}. \end{aligned} \tag{5.39}$$

The bootstrap equations presented in (5.31) are for the general cases, here we need to apply following constraints on the equations for minimal SCFT:

- For the minimal SCFT we have an assumption on the chiral ring relation $\Phi^2 \simeq 0$, which means the BPS chiral superprimary field Φ^2 does not appear in the OPE of $\Phi(1)\Phi(2)$, in consequence, there is no contribution from scalar BPS operator on the conformal block/bootstrap equations, i.e., we do not have the first term in the right hand side of Eq. (5.31) with $\ell = 0$, or $\vec{V}_{B,\Delta,0} = 0$ in (5.39).

- The unitary bound for the remaining non-BPS operators $\mathcal{O} \in \Phi\Phi$ satisfy $\Delta \geqslant |2q_1 - 3| + 3 + \ell$, here $q_1 = \Delta_\Phi$ is the conformal weight of Φ. For the operators $\mathcal{O} \in \Phi X,\ \Phi\Phi^\dagger,\ XX,\ \cdots$, they satisfy the unitary bound $\Delta \geqslant 2 + \ell$. Moreover, similar to the 3D Ising model and O(N) vector model, we have a crucial assumption on the spectrum of the minimal SCFT: there are only three relevant scalar operators Φ, X and a short multiplet saturating the unitary bound in $\vec{V}_T$ channel, while all the other scalars have dimension $\Delta \geqslant 4$ (but for the operators with $\ell > 0$, they could be relevant).

- Since $X \in \Phi\Phi^\dagger$ with spin $\ell = 0$, its dimension should satisfy $\Delta_X \geqslant 2$, i.e., $q_2 \geqslant 1$. Actually according to the results in [62], the dimension of operator X is about $3.2 (< 4$ so is relevant).

- Since both Φ and X are scalars, therefore the OPE coefficient $\lambda_{\Phi\Phi^\dagger X} = \lambda_{\Phi X\Phi^\dagger}$, we need to apply this condition in our bootstrap equations, this will slightly modify the bootstrap equations.

Specifically, in the contributions from mixed operators $\sum\limits_{\mathcal{O}\in\Phi X} 2|\lambda_{\Phi X\mathcal{O}}|^2\vec{V}_{M,\Delta,\ell}$ there contains a term $2|\lambda_{\Phi X\Phi^\dagger}|^2\vec{V}_{M,q_1,0}$, which can be rewritten as

$$
\left(\lambda_{\Phi\Phi^\dagger X}\ \lambda^{(a)}_{XXX}\ \lambda^{(b)}_{XXX}\right) 2\vec{V}_{M,q_1,0} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_{\Phi\Phi^\dagger\mathcal{O}} \\ \lambda^{(a)}_{XX\mathcal{O}} \\ \lambda^{(b)}_{XX\mathcal{O}} \end{pmatrix},
$$

therefore this term can be added on the last part $\propto \vec{V}_{S,\Delta_X,0}$ (only for $\mathcal{O} = \Phi^\dagger$, for all the other $\mathcal{O}$, there is no such addition), and gives

$$\begin{pmatrix}\lambda_{\Phi\Phi^\dagger X} \; \lambda^{(a)}_{XXX} \; \lambda^{(b)}_{XXX}\end{pmatrix} \left(\vec{V}_{S,\Delta_X,0} + 2\vec{V}_{M,q_1,0} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \begin{pmatrix}\lambda_{\Phi\Phi^\dagger X} \\ \lambda^{(a)}_{XXX} \\ \lambda^{(b)}_{XXX}\end{pmatrix}.$$

Bootstrap program is to find linear functionals $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_7)$ with properties $\vec{\alpha} \cdot \vec{V}_{x,\Delta,\ell} \geqslant 0$ to rule out the hypothetical SCFT spectrum. In particular ,we want the functionals satisfying

$$(1\;1\;0)\,\vec{\alpha}\cdot\vec{V}_{S,0,0}\begin{pmatrix}1\\1\\0\end{pmatrix} > 0\;(\text{or} = 1), \qquad \text{unit operator: } \Delta = 0, \ell = 0; \tag{5.40}$$

$$\vec{\alpha}\cdot\vec{V}_{B,\Delta,\ell} \geqslant 0, \qquad \Delta = 2\Delta_\Phi + \ell,\; \ell = 2,4,\cdots; \tag{5.41}$$

$$\vec{\alpha}\cdot\vec{V}_{T,\Delta,\ell} \geqslant 0, \qquad \begin{aligned}&\Delta = |2q_1 - 3| + 3,\; \ell = 0,\\ &\Delta \geqslant \Delta_T^*,\; \ell = 0,\\ &\Delta \geqslant |2q_1 - 3| + 3 + \ell,\; \ell > 0;\end{aligned} \tag{5.42}$$

$$\vec{\alpha}\cdot\vec{V}_{M,\Delta,\ell} \geqslant 0, \qquad \begin{aligned}&\Delta \geqslant \Delta_M^*,\;\; \ell = 0,\\ &\Delta \geqslant 2 + \ell,\;\; \ell > 0;\end{aligned} \tag{5.43}$$

$$\vec{\alpha}\cdot\vec{V}_{S,\Delta,\ell} \succeq 00, \qquad \begin{aligned}&\Delta \geqslant \Delta_S^*,\;\; \ell = 0,\\ &\Delta \geqslant 2 + \ell,\;\; \ell > 0;\end{aligned} \tag{5.44}$$

$$\vec{\alpha}\cdot\left(\vec{V}_{S,\Delta_X,0} + 2\vec{V}_{M,q_1,0}\otimes\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\right) \succeq 0, \qquad \text{terms proportional to } \lambda_{\Phi\Phi^\dagger X}. \tag{5.45}$$

The bootstrap conditions are mainly based on the unitarity constraint, moreover, we have also made certain mild assumptions on the spectra of the CFT:

- In Eq. (5.40), we have adopted the fact $\lambda^{(2)}_{XX\mathcal{O}} = 0$ when $\mathcal{O}$ is the unit operator.
- In Eq. (5.42), we have applied the condition that the unitary bound $\Delta \geqslant |2q_1 - 3| + 3$ is saturated by a short multiplet. Evidence of this short multiplet can be uncovered

from single correlator bootstrap. We also assume that the next scalar operator has scaling dimension above a threshold value Δ_T^*.

- In Eq. (5.43) and Eq. (5.44), we have assumed scaling dimension gaps Δ_M^* and Δ_S^* in the $\vec{V}_M$ and $\vec{V}_S$ channels, respectively. From single correlator conformal bootstrap, we can obtain evidence on the scaling dimension gap in $\vec{V}_S$ channel but not $\vec{V}_M$ channel, since operators in $\vec{V}_M$ channel never appear in the $\Phi\Phi^\dagger$ or $\Phi\Phi$ OPEs. However, the final results is not sensitive to the specific value of Δ_M^* so we may simply adopt the relevant condition $\Delta_M^* = 4$, which indicates there is no relevant scalar in this channel besides $\Phi^\dagger$.

- The Eq. (5.45) is the semi-definite problem for a 7×1 matrix with 3×3 matrices as its elements. Actually as pointed out in [46] for the 3D Ising model, here the condition (5.45) is too strong for bootstrap and can be relaxed. A more efficient way is to introduce extra θ factors which measure the ratios among the coefficients. Since we have three independent OPE coefficients here it needs two angle factors (θ_1, θ_2). Then we can sample all the possible θs and require

$$(1 \ \tan\theta_1 \ \tan\theta_2)\, \vec{\alpha} \cdot \left(\vec{V}_{S,q_1,0} + 2\vec{V}_{M,q_1,0} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ \tan\theta_1 \\ \tan\theta_2 \end{pmatrix} \geqslant 0. \quad (5.46)$$

The physically allowed region of (Δ_Φ, Δ_X) is the union of the results from each pair of θs. In this thesis, we will not introduce the θ factors to optimize our results. We expect to improve our results using this method in our future studies.

5.3 Results

5.3.1 Bound on the scaling dimension Δ_X from single correlator

Evidence of the $4D$ $N = 1$ minimal SCFT has been shown in [32] through bootstrapping the single correlator $\langle \Phi\Phi^\dagger\Phi\Phi^\dagger \rangle$, where the operator Φ ($\Phi^\dagger$) is a chiral (anti-chiral) scalar satisfying the short condition $\mathcal{Q}\Phi = 0$. The results provide upper bound on the scaling dimension Δ_X of real operator X, which is shown in Figure 5.1. To obtain this figure, we have adopted the derivatives up to $\Lambda = 17$ in the numerical calculation. The bound shows an apparent kink around $\Delta_\Phi = 1.4, \Delta_X = 3.2$, indicating a non-trivial solution of the crossing equation from the single correlator $\langle \Phi\Phi^\dagger\Phi\Phi^\dagger \rangle$. By taking higher order of derivatives, the bound and kink can be slightly shifted and converge. In technical one can trace the shift of the kink under increasing Λ and extrapolate to $\Lambda \to \infty$ to find the converge position. This method has been used in [62] and it was suggested that the position of the minimal SCFT in this plan locates at $\Delta_\Phi = \frac{10}{7}$. Nevertheless, the extrapolation is rather ad hoc and a much better estimation on the (Δ_Φ, Δ_X) of the minimal SCFT can be obtained by bootstrapping mixed correlators, which leads to both upper and lower range of the scaling dimensions, as will be shown later.

5.3.2 Bounds on the scaling dimensions (Δ_Φ, Δ_X) from mixed correlators

Bound from single correlator is quite limited. It indicates a nontrivial solution of the crossing equation but cannot provide more information about the theory. It is hard to estimate the error of scaling dimensions of the operators. It would be more striking if the kink can be further isolated into a small island, like the remarkable results for $3D$ Ising model [46] and $3D/5D$ $O(N)$ vector models [58, 69]. However, the attempt to obtain isolated island for this minimal $4D$ $N = 1$ minimal SCFT has been failed [70]. For

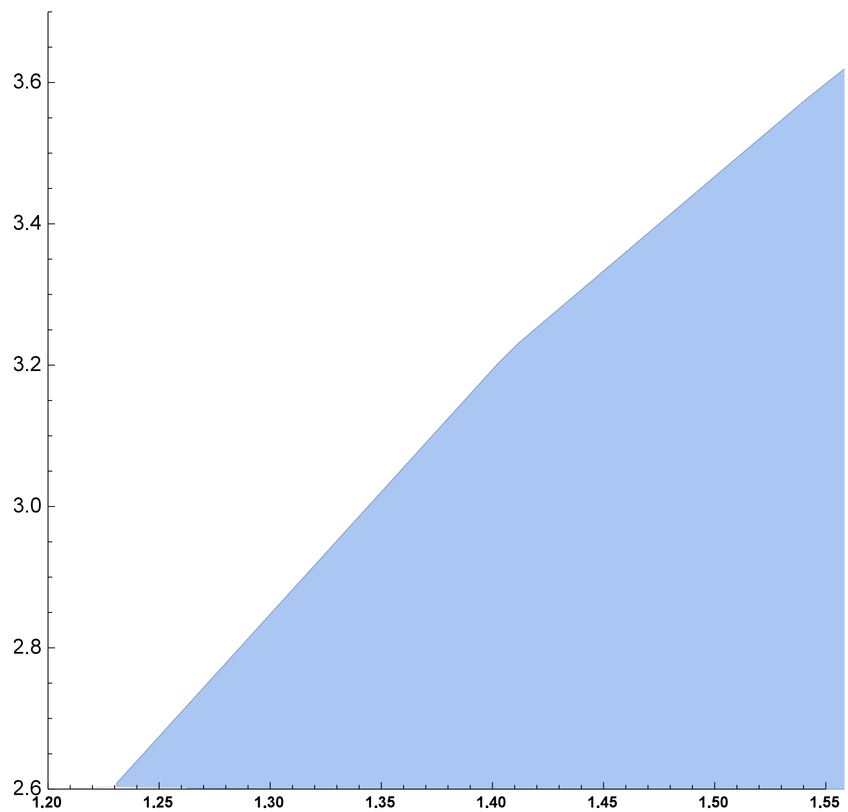

Figure 5.1: Bounds on the scaling dimensions (Δ_ϕ, Δ_X) of operators in the $4D$ $N = 1$ SCFT. The colored regions represent the conformal dimensions allowed by conformal bootstrap. A kink appears around $\Delta_\Phi = 1.4, \Delta_X = 3.2$. In this figure the derivative $\Lambda = 17$. The position of the kink can be slightly shifted with higher Λ.

supersymmetric theories, the spectra and crossing equations are more complex than the non-supersymmetric theories and it is rather difficult to control the bootstrap conditions to find the solutions.

Using the crossing equations presented in the last section, the islands are obtained with mild assumptions on the spectra. The results are shown in Figure 5.2. The islands shrink notably with increasing Λ. For $\Lambda = 21$, we find the scaling dimensions of Φ and X are about $\Delta_\Phi = 1.412(7)$ and $\Delta_X = 3.174(23)$. Apparently, previous conclusion on the scaling dimension $\Delta_\Phi = \frac{10}{7}$ [62] lies outside of the island so is falsified.

In the bootstrap setup, we have assumed that in the $\vec{V}_T$ channel, there is a $\bar{\mathcal{Q}}$ exact term that saturates the unitary bound $\Delta \geqslant |2q_1 - 3| + 3$. The operator $\mathcal{O}$ appears in the none BPS channel of the OPE

$$\Phi \times \Phi \sim \Phi^2 + \sum_{\ell=2,4,\cdots} \bar{\mathcal{Q}}\mathcal{O}_\ell + \sum_{\mathcal{O}} \bar{\mathcal{Q}}^2\mathcal{O}. \tag{5.47}$$

Besides, in the superconformal partial wave expansion of the mixed correlator functions, we only consider the contributions from supermultiplets whose lowest components appear in the $\Phi \times X$ OPE. In principle there could be contributions from supermultiplets whose lowest components have no contribution to the $\Phi \times X$ OPE, however, their supersymmetric descendents ($\mathcal{Q}$ or $\bar{\mathcal{Q}}$ exact terms) may appear in the $\Phi \times X$ OPE. Taking the operator $\mathcal{O}^{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_\ell;\alpha_1\cdots\alpha_\ell}$ with $q = \frac{1}{2}(\Delta - \Delta_\Phi + \frac{3}{2})$ and $\bar{q} = \frac{1}{2}(\Delta + \Delta_\Phi - \frac{3}{2})$ for example, its lowest component is fermionic so has vanishing three point function with Φ and X. However, its $\bar{\mathcal{Q}}$ descendent $\bar{\mathcal{Q}}_{\dot{\alpha}}\mathcal{O}^{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_\ell;\alpha_1\cdots\alpha_\ell}$ is a conformal primary operator with integer spin ℓ. Its $U(1)_R$ charge is $-\frac{2}{3}\Delta$, just opposite to the $U(1)_R$ charge of the chiral operator Φ so can have non-vanishing three point function $\langle \Phi X(\bar{\mathcal{Q}}\mathcal{O})\rangle$. The superconformal block functions of these supermultiplets have been calculated in [70]. We also studied the contributions from these operators while they do not introduce notable difference in the results. The islands remain the same as presented in Figure 5.2. This is not surprise since for the “minimal" it is totally possible that no relevant spectra in this channel.

5.3.3 Bootstrapping the c central charge of the minimal SCFT

In this part we will show that the crossing equation (5.39) can also lead to strong constraints on the OPE coefficients. Let us take the coefficient of Φ^2 in the $\Phi \times \Phi$ OPE for example. As we have mentioned, due to the chiral ring condition $\Phi^2 = 0$, this OPE coefficient λ_{Φ^2} should vanish at the fixed point, and this fact can be verified from the bounds of λ_{Φ^2} obtained from bootstrap [32, 62]. To get the bound of λ_{Φ^2}, we just need to slightly adjust the setup of bootstrap conditions for scaling dimensions presented in (5.40-5.45):

$$\begin{aligned} \vec{\alpha} \cdot \vec{V}_{B,\, \Delta_{\Phi^2},\, 0} &= 1; && (5.48) \\ \vec{\alpha} \cdot \vec{V}_{X,\, \Delta,\, \ell} &\leqslant 0, \qquad X \in \{B, T, M, S\} \text{ and } \Delta \notin \{0, \Delta_{\Phi^2}\}. && (5.49) \end{aligned}$$

In the above formulas we have implicitly adopted the same spectra conditions as in (5.40-5.45). Then the crossing equation (5.39) suggests

$$\begin{aligned} \lambda_{\Phi^2}^2 &= -(1\ 1\ 0) \times \vec{\alpha} \cdot \vec{V}_{S,\, 0,\, 0} \times (1\ 1\ 0)^T - \sum_{\mathcal{O} \neq \mathcal{O}_0} \lambda_{\mathcal{O}}^2 \times \vec{\alpha} \cdot \vec{V}_{X,\, \Delta,\, \ell} && (5.50) \\ &\geqslant -(1\ 1\ 0) \times \vec{\alpha} \cdot \vec{V}_{S,\, 0,\, 0} \times (1\ 1\ 0)^T, && (5.51) \end{aligned}$$

which gives a lower bound on $\lambda_{\Phi^2}^2$. Similarly, we may reverse the inequalities in (5.49) then the crossing equation (5.39) leads to an upper bound on $\lambda_{\Phi^2}^2$.

Here we are more interested in bootstrapping the c central charge. The c central charge

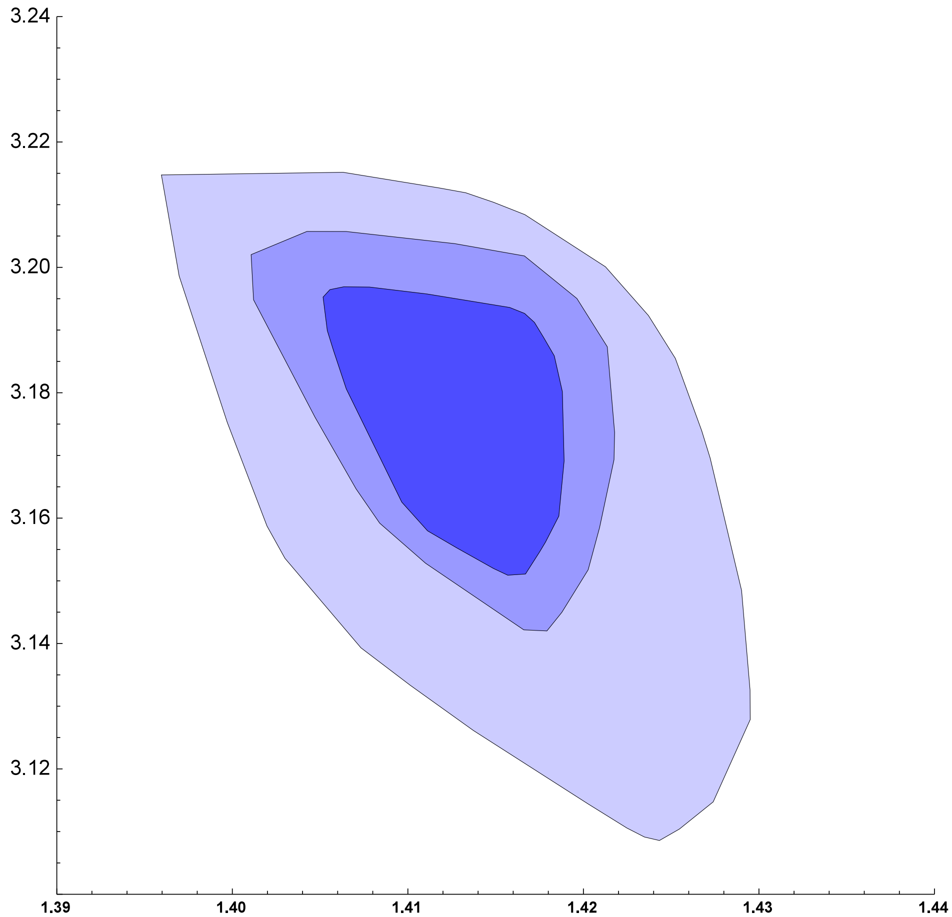

Figure 5.2: Bounds on the scaling dimensions (Δ_ϕ, Δ_X) from mixed correlators. The light, medium and dark blue regions correspond to the bounds from mixed correlators with $\Lambda = 15, 17, 21$, respectively. To obtain the outer island with light blue, we have applied the scaling dimension gaps $\Delta_S^* = 6, \Delta_T^* = 6$ and $\Delta_M^* = 4$. We noticed the island is not sensitive by replacing the $\vec{V}_M$ channel gap from $\Delta_M^* = 4$ to $\Delta_M^* = 6$. The inner two islands with medium and dark blue colors relate to gaps $\Delta_S^* = 6, \Delta_T^* = 6$ and $\Delta_M^* = 5.5$.

corresponds to the Weyl anomaly when the theory is on a curved background:

$$\langle T^{\mu}_{\mu} \rangle = \frac{c}{16\pi^2} C_{\alpha\beta\rho\sigma} C^{\alpha\beta\rho\sigma} - \frac{a}{16\pi^2} \tilde{R}_{\alpha\beta\rho\sigma} \tilde{R}^{\alpha\beta\rho\sigma}, \tag{5.52}$$

in which $C_{\alpha\beta\rho\sigma}$ is the Weyl tensor and $\tilde{R}^{\alpha\beta\rho\sigma}$ is the dual Riemann tensor. Both a and c central charges relate to the degree of freedom of the theory. The c central charge appears as the coefficient in the two point function of stress tensor $T^{\mu\nu}$:

$$\langle T^{\mu\nu}(x) T^{\alpha\beta}(0) \rangle = \frac{40c}{\pi^4} \frac{I^{\mu\alpha}(x) I^{\nu\beta}(x)}{x^8}, \tag{5.53}$$

where the covariant tensor $I^{\mu\nu}(x) = \eta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2}$. The stress tensor performs as the local current that generates the dilatation charge $D = -\int_{S^3} d\Omega\, \hat{x}_\mu \hat{x}_\nu T^{\mu\nu}$ and $D\phi(0) = \Delta_\phi \phi(0)$ for conformal primary operator ϕ. To fulfill the dilatation action, the OPE with stress tensor has to be [130]

$$T^{\mu\nu}\phi(0) \;\sim\; -\frac{2\Delta_\phi}{3\pi^2 x^6}(x^\mu x^\nu - \frac{1}{4}\eta^{\mu\nu} x^2)\phi(0) + \text{less singular terms}, \tag{5.54}$$

and the conformal partial wave of four point function $\langle\phi\phi\phi\phi\rangle$ related to the stress tensor $T^{\mu\nu}$ is

$$\langle\phi\phi\phi\phi\rangle_T = \sum_{T,PT,\cdots} \frac{\langle\phi\phi\mathcal{O}\rangle\langle\mathcal{O}\phi\phi\rangle}{\langle\mathcal{O}\mathcal{O}\rangle} \sim \frac{\Delta_\phi^2}{360\,c} \frac{1}{(x_{12}x_{34})^{2\Delta_\phi}} g_{4,2}(u,v). \tag{5.55}$$

Therefore the bound on the c central charge can be obtained through the bound of OPE coefficient λ_T in the crossing equation.

In the $4D$ $N = 1$ SCFTs, the stress tensor is the spin-2 supercomponent in the $U(1)_R$ supercurrent supermultiplet J_R: $T \sim \mathcal{Q}\bar{\mathcal{Q}} J_R$, therefore the OPE coefficient λ_T is determined by the OPE coefficient of superconformal primary operator J_R: λ_{J_R}. The $U(1)_R$ current is neutral under $U(1)_R$ symmetry so it does not appear in the superconformal partial wave expansion of the mixed correlator $\langle \Phi X \Phi^\dagger X \rangle$, but it does appear in the correlators $\langle \Phi\Phi^\dagger\Phi\Phi^\dagger \rangle$ and $\langle XXXX \rangle$.

The OPE of chiral/anti-chiral operators is

$$\Phi^\dagger \sim \sum_{\mathcal{O}} \left(\mathcal{O} + (\mathcal{Q}\bar{\mathcal{Q}}\mathcal{O})_{\ell-1} + (\mathcal{Q}\bar{\mathcal{Q}}\mathcal{O})_{\ell+1} + \mathcal{Q}^2\bar{\mathcal{Q}}^2\mathcal{O} \right). \tag{5.56}$$

The $U(1)_R$ supercurrent J_R satisfies the conserved conditions

$$\mathcal{Q}^2 J_R(x) = \bar{\mathcal{Q}}^2 J_R(x) = 0, \tag{5.57}$$

so only the lowest component J_R and stress tensor $T^{\mu\nu}$ has non-vanishing contributions. Their OPE coefficients are fixed by supersymmetry

$$\lambda^2_{\Phi\Phi^\dagger T} = \frac{1}{5}\lambda^2_{\Phi\Phi^\dagger J_R}. \tag{5.58}$$

As solved in (5.4), the three point function of real operator X could have more independent OPE coefficients. Since $U(1)_R$ supercurrent has spin 1, there are two nonzero OPE coefficients $\lambda^{(1)}_{XXJ}$ and $\lambda^{(3)}_{XXJ}$. Due to the conservation conditions (5.57) these two OPE coefficients are not independent. Instead, they satisfy [105]

$$\lambda^{(1)}_{XXJ} = -2\lambda^{(3)}_{XXJ}. \tag{5.59}$$

According to above relation, the superconformal block function of four Xs correlator function reduces to

$$\mathcal{G}^{N=1|XX;XX}_{\Delta,\ell} = \frac{|\lambda^{(1)}_{XXJ}|^2}{20} g_{4,2}(u,v) \tag{5.60}$$

while the second term corresponds to $g_{4,0}$ vanishes. This is reasonable as there is no scalar conformal primary component in $U(1)_R$ current due to the conservation conditions and the superconformal block function turns into non-supersymmetric case. Comparing with the conformal partial wave from stress tensor (5.55) one can directly relate the OPE coefficient $\lambda^{(1)}_{XXJ}$ to the c central charge.

To bootstrap the c central charge, we firstly use the supercurrent operator J_R in the normalization condition:

$$\left(\Delta_\Phi \ \ \frac{1}{5}\Delta_X \ \ 0\right) \times \vec{\alpha} \cdot \vec{V}_{S,\,3,\,1} \times \left(\Delta_\Phi \ \ \frac{1}{5}\Delta_X \ \ 0\right)^T = 1. \tag{5.61}$$

For all the other non-unit operators, we set their functional action to be negative (positive) to generate a lower (upper) bound of the coefficients

$$\frac{1}{72c} \geqslant -\,(1\ 1\ 0) \times \vec{\alpha} \cdot \vec{V}_{S,\,0,\,0} \times (1\ 1\ 0)^T. \tag{5.62}$$

For a given point (Δ_Φ, Δ_X) on the island in Figure 5.2, we can apply this procedure to generate both upper and lower bounds on the c central charge. It would be quite interesting to obtain a specific range of this value and compare it with the previous estimation $c \approx \frac{1}{9}$.

In [134, 135] it has been proposed that this minimal SCFT could be obtained by the $N = 2$ A_2 Argyres-Douglas theory [136] perturbed by $N = 1$ superpotential. Specifically, the $4D$ $N = 2$ chiral supermultiplet can be decomposed to two scalar $N = 1$ chiral multiplet and one $N = 1$ spinor chiral multiplet (Φ, Ψ, F). Then the $N = 1$ deformation with superpotential $W \sim \lambda\Phi^2$ leads to the chiral ring relation

$$\Phi^2 = 0, \tag{5.63}$$

as expected for the $4D$ $N = 1$ minimal SCFT. However, in this construction the scaling dimension of the chiral operator $\Delta_\Phi = \frac{3}{2}$ and the c central charge is about 0.35. The scaling dimension Δ_Φ lies apart from the isolated island shown in Figure 5.2. As to the c central charge, it is also notably higher than the lower bound of the c central charge obtained from conformal bootstrap. We are still working on the upper bound of c central charge and it is interesting to see how the constraints it will provide on the candidate theories. The theoretical construction of the minimal SCFT is still mysterious. It would be quite interesting to see if we can realize this theory through certain geometrical construction, like by putting the $6D$ $(2,0)$ theory on a punctured Riemann surface. This approach is quite successful in building $4D$ $N=2$ SCFTs, and hopefully, it can be suggestive for us to construct the $N=1$ SCFT.

6. SUMMARY AND CONCLUSIONS

6.1 CFT Landscape from Conformal Bootstrap

One of the ambitious object of conformal bootstrap is to classify the CFT landscape. The first step toward this aim is to bootstrap the simplest CFT constructed by scalar operators, such as the $3D$ Ising model, which has been numerically solved in [46] and its generalized form with $O(N)$ global symmetry has been solved in [58]. Following these landmark works, the next targets are the CFTs constructed by scalars in higher dimensions $D > 3$ and the CFTs in $D = 3$ spacetime constructed by spinning operators. Besides, it is an interesting question that can we find $3D$ CFTs beyond the classical Wilson-Fisher fixed point using conformal bootstrap? In this part I am going to explain my research plan on these problems.

6.1.1 3D self-dual QED with flavor $N_f = 2$

The $2 + 1$ d interacting conformal field theories can be constructed using supersymmetry or gauge theories with sufficient large flavor symmetry. Recent studies on gauged topological insulator have shown that the $(2+1)$ d QED (without supersymmetry) actually admits interacting IR fixed point even with small flavor symmetry ($N_f = 1$) [7]. In the UV side the $N = 1$ QED is described by the following Lagrangian

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - ia_\mu)\psi + \frac{1}{e^2}f_{\mu\nu}^2 + \frac{i}{4\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu A_\rho. \tag{6.1}$$

The theory is expected to run to the IR fixed point which is also described by a free Dirac fermion, therefore the above theory is actually the “dual" theory of the free Dirac fermion theory. This model provides a fermionic version of the classical particle (boson)-vortex duality in $2+1$ dimensions. A larger "web" of 3D particle (fermion)-vortex dualities has been proposed recently [8, 9]. In particular, the following $2+1$ d QED with $N_f = 2$ admits self-duality [9, 10]:

$$\begin{aligned}\mathcal{L} =& \bar{\psi_1}\gamma_\mu(\partial_\mu - ika_\mu - i2n_B B_\mu)\psi_1 + \bar{\psi_2}\gamma_\mu(\partial_\mu - ia_\mu)\psi_2 \\ &+ \frac{1}{e^2}f_{\mu\nu}^2 + \frac{in_A}{4\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu A_\rho + \cdots\end{aligned} \tag{6.2}$$

where the vector fields A_μ, B_μ are external background $U(1)$ gauge fields while a_μ is the dynamical $U(1)$ gauge field. At the IR fixed point the theory is strongly coupled. Since the theory is dual to itself, there is no free fermionic description of this theory as we have for the $N_f = 1$ theory. Moreover, due to the small flavor number $N_f = 2$ one cannot study the theory analytically based on the large-N perturbative approach. For the theory with large k (the $U(1)$ charge of ψ_1), after taking the self-duality, one of the dual fermion has $U(1)$ charge $1/k$ in the dual theory so is decouple from the theory by taking $k \to \infty$, and we go back to the $2+1$ d QED with $N_f = 1$, which is dual to a free fermion. Therefore the theory can still be studied in terms of $1/k$ expansion through duality. While for small k, the results from $1/k$ expansion is questionable. The theory near IR fixed point has been numerically studied using Lattice method [11]. The results suggest the theory indeed runs into a scale-invariant theory and the anomalous dimensions of certain operators have been studied.

Conformal bootstrap provides a nonperturbative approach to study the theory at IR fixed point. In the UV side the theory is gauged and currently, it is difficult to solve the correlators of operators with gauge charges, however, the gauge invariant fermion bilinears have power-law correlators so can be studied through bootstrap approach. We will be particularly interested in the theory with $k = 1$, which cannot be studied using $1/k$ expansion, and more importantly, the global symmetry of the theory (6.2) is enhanced to $O(4) = SU(2) \times SU(2) \times Z_2$, where the two $SU(2)$s are the flavor symmetry of the two dual theories and the Z_2 is their exchange symmetry. It is possible to add an extra mass term $m\bar{\psi}_i\psi_i$ in the Lagrangian (6.2), which in $SO(4)$ invariant while breaks the Z_2 symmetry. By adding a mass term the theory runs into a fixed point with $SO(4)$ symmetry and one relevant $SO(4)$ singlet. This CFT is expected to be fundamentally different from the classical $O(4)$ Wilson-Fisher fixed point theory due to the non-trivial contributions from the topological term. The $O(4)$ Wilson-Fisher fixed point theory has been perfectly solved by numerical conformal bootstrap in [58]. It would be extremely interesting to see whether the conformal bootstrap can solve the novel $SO(4)$ fixed point from the self-dual QED.

6.2 Solve Supersymmetric Conformal Field Theories

Supersymmetry can be quite helpful in constructing CFTs. For the supersymmetric theory, its properties under quantum loop corrections are protected by supersymmetry which makes the theory partially or completely integrable*. Not surprisingly, the SCFTs are more likely to be solved by conformal bootstrap. An interesting example has been proposed in [35], in which the four-point function of the stress tensor supermultiplet of $4D$ $N = 4$ SCFT

$$\langle \mathcal{O}^I_{\mathbf{20'}} \mathcal{O}^J_{\mathbf{20'}} \mathcal{O}^K_{\mathbf{20'}} \mathcal{O}^L_{\mathbf{20'}} \rangle = \frac{\mathcal{G}^{IJKL}}{x_{12}^4 x_{34}^4} \tag{6.3}$$

has been bootstrapped†. In RHS of (6.3) the denominator has power 4 since $\Delta_{\mathcal{O}^I_{\mathbf{20'}}} = 2$. Sharp kinks have been observed in the physical bound of the scaling dimensions with different central charges. The results are nicely fit with the predictions from $4D$ $N = 4$ SYMs. Undoubtedly more interesting phenomena on SCFTs can be discovered through

*Here the “partially integrable" indicates that a subsectors of the theory, like the chiral algebra are subject to the integrable equations.

†The stress tensor is a spin-2 component of a half-BPS supermultiplet whose superconformal primary $\mathcal{O}^I_{\mathbf{20'}}$ constructs $\mathbf{20'}$ representation of the $SU(4)_R$ R-symmetry.

conformal bootstrap.

6.2.1 $4D$ $N = 2$ S-duality from Conformal Bootstrap

The $4D$ $N = 2$ SYMs are an affluent topic to study QFTs since the historical discovery in [137]. For the $4D$ $N = 4$ SYM, the supersymmetry is so strong that many complex phenomena in general QFTs are uniquely fixed, while in the $4D$ $N = 2$ SYMs the supersymmetry is reduced and many interesting field theory phenomena, like the non-perturbative effect, electric-magnetic duality and confinement appear while the theories are not too complex to solve due to the remaining supersymmetry. It is well-known that the $N = 4$ SYMs admit S-duality with $SL(2, Z)$ symmetry. Interestingly the S-duality also appears in the $N = 2$ $SU(N_c)$ SYMs with $N_f = 2N_c$ fundamental hypermultiplets and it acts on the gauge coupling with symmetry $\Gamma_0(2) \subset SL(2, Z)$. The S-duality has been developed further in [138] and [139], which lead to a large class of strongly coupled $N = 2$ SCFTs. The fundamental elements in constructing these SCFTs are the so-called T_N theories, which are $4D$ $N = 2$ SCFTs with $SU(N)^3$ global symmetries. It can be obtained from the $6D$ $(2, 0)$ SCFTs of type $SU(N)$ compactified on a sphere with three full punctures. The T_N theories do not have classical Lagrangian description so provide perfect examples for conformal bootstrap study.

To bootstrap the T_N theory, one may expect to apply the method used for $N = 4$ SCFTs in [35], i.e. bootstrap the four-point correlator of the scalar in the stress tensor supermultiplet $\mathcal{J}$:

$$\mathcal{J}(x, \theta, \bar{\theta})| = J(x), \;\; \mathcal{J}(x, \theta, \bar{\theta})|_{\theta\bar{\theta}} = J^{ij}_{\mu}(x), \;\; \mathcal{J}(x, \theta, \bar{\theta})|_{\theta^2\bar{\theta}^2} = T_{\mu\nu}(x). \tag{6.4}$$

The scalar component J is a singlet of the $SU(2)_R \times U(1)_r$ R-symmetry therefore its four-point function $\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle$ does not contain extra global indices. This four-point function seems to be as simple as we have in (2.7) for four identical scalars non-supersymmetric theories. Unfortunately, this four-point function is actually much more complex than the non-supersymmetric theories since there are several conformal primaries contained in each superprimary multiplets. In consequence one needs to account contributions from different components in the superprimary multiplets. There are also extra invariant cross-ratios besides u, v. Different from the $N = 4$ SCFTs, the superconformal block function of this four-point correlator is still unknown yet. We need to calculate the four-point function of stress tensor supermultiplet before we can get any information from bootstrapping the correlator.

The difficulty in calculating the superconformal block function is that we need to evaluate the contributions from each component in the supermultiplet, which belongs to different conformal primaries and there are several tensor structures related to these contributions. Therefore it is rather cumbersome to calculate the superconformal block functions directly. We actually have very powerful techniques based on the supershadow formalism and superembedding approach in calculating $4D$ $N = 1$ superconformal block functions. These two techniques can be directly generalized for $N = 2$ SCFTs, however, they can not reproduce all the $N = 2$ tensor structures due to the non-Abelian $SU(2) \times U(1)$ R-symmetry. For these extra components, it is possible to do the calculation in brute force method.

After solving the conformal blocks of the four-point correlator

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^4 x_{34}^4}, \tag{6.5}$$

it can be subjected to the conformal bootstrap study directly. Hopefully we can get some strong results on the existence of the T_N theories by detecting the kink phenomena in the bound of the next scalar operator. Furthermore, we can also bootstrap the OPE coefficients and get the bounds on the central charges of the theories. More details on the theory can be solved through extremal functional method.

6.3 Analytical Properties of Crossing Equation

Successes of the numerical conformal bootstrap indicate that crucial information on CFTs has been encoded in crossing equation. A particularly interesting question is, can we decode the results on CFTs from the crossing equation analytically? Or alternatively, can we calculate the CFT data by employing conformal symmetry? For the CFTs with classical Lagrangian descriptions, the CFT data can be calculated perturbatively using Feynman diagrams. However, the approach mainly relies on the RG flow while does not take advantage of the conformal symmetry. In [22] the anomalous dimension of $3D$ $O(N)$ vector has been estimated up to the order $O(\epsilon^2)$ only based on the conformal symmetry and primary multiplet recombination. Nevertheless, this approach can not give higher order corrections. This is expected since only three-point function has been used and crossing symmetry plays no role in the calculation, therefore it cannot get access to the results obtained by numerical conformal bootstrap. A more complete analytical solution of CFT requires careful observation and application of the properties of four-point correlator functions and the crossing symmetry.

6.3.1 Perturbative Solution of the Crossing Equation

For the CFTs with a small parameter, such as the $1/N$ expansion in the large N approximation, or ϵ expansion in $4-\epsilon$ dimensional spacetime, the CFT data can be perturbatively expanded with this small parameter. Let us take the $3D$ Ising model to illustrate the idea of this method. The operators $\mathcal{O}$ appearing in the crossing equation (3.12) are constructed by ϕs. For example, the “double trace" operators with spin ℓ are given by

$$\mathcal{O} \propto \phi \Box^n \underbrace{\partial_\mu \partial_\nu \dots \partial_\gamma}_{\ell} \phi. \tag{6.6}$$

In the free theory limit, its scaling dimension is $\Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell$. Operators with more traces may also appear in the $\phi\phi$ OPE, like $\phi\Box^n\partial^l\phi\Box^m\partial^r \cdots \phi$. However, note that in Ising model we have Z_2 symmetry therefore only operators with even number of ϕs can appear in the $\phi\phi$ OPE. These multiple trace operators appear in the crossing equation at higher order of ϵ expansion. At the first order $O(\epsilon)$, we only have contributions from double trace operators, which will be denoted as $\mathcal{O}(n,\ell)$ below.

For an interacting CFT, the scaling dimension of the fundamental scalar ϕ and the OPE coefficients $\lambda_{n,\ell}$ can be expanded in the ϵ expansion

$$\Delta_\phi = \Delta_0 + \delta^1_\phi \epsilon + \delta^2_\phi \epsilon^2 + \cdots, \tag{6.7}$$

$$\Delta_{n,\ell} = 2\Delta_0 + 2n + \ell + \delta^1_{n,\ell}\,\epsilon + \delta^2_{n,\ell}\,\epsilon^2 + \cdots, \tag{6.8}$$

$$\lambda_{n,\ell} = \lambda^0_{n,\ell} + \lambda^1_{n,\ell}\epsilon + \lambda^2_{n,\ell}\epsilon^2 + \cdots, \tag{6.9}$$

where the parameter $\lambda^0_{n,\ell}$ is the OPE coefficient in the $4D$ free theory

$$\phi\phi \sim 1 + \sum_{n,\ell} \lambda_{n,\ell}\, \mathcal{O}(n,\ell). \tag{6.10}$$

Plugging the ϵ expansion of above CFT data in the crossing equation

$$v^{\Delta_\phi} + \sum_{\mathcal{O}} \lambda^2_{\mathcal{O}}\, v^{\Delta_\phi} g_{\Delta,\ell}(u,v) = u^{\Delta_\phi} + \sum_{\mathcal{O}} \lambda^2_{\mathcal{O}}\, u^{\Delta_\phi} g_{\Delta,\ell}(v,u). \tag{6.11}$$

We expect this crossing equation can be satisfied order by order in the ϵ expansion! At the lowest order the ϵ expansion reproduces the CFT data of free theory, therefore it automatically satisfies above crossing equation. At order $O(\epsilon)$, the double trace operators involve in the crossing equation. The crossing equation is a function of cross-ratios u, v (or equivalently $z, \bar{z}$) indexed by two integers. The unknown parameters are provided in (6.7-6.9) also indexed by two integers (n, ℓ). It is unclear right now will this infinite equation system leads to zero, finite or infinite solutions.

It needs to be noted that in the crossing equation (6.11),the conformal block functions from non-unit operators $g_{\Delta,\ell}(u, v)$ are infinite series of variables u, v with integer powers. However, for the contributions from the unite operator u^{Δ_ϕ}, v^{Δ_ϕ}, the power Δ_ϕ has non-integer part from ϵ contributions

$$v^{\Delta_\phi} = v^{\Delta_0+\delta^1_\phi\epsilon+\cdots} = v^{\Delta_0}(1 + \delta^1_\phi \ln(v)\epsilon) + O(\epsilon^2), \tag{6.12}$$

which contains singularity $\ln(v)$ in the limit $v \to 0$, and a similar singular term proportional to $\ln(u)$ when $u \to 0$. The RHS of equation (6.11) should reproduce the term v^{Δ_ϕ} with singularity, however, this cannot be fulfilled by regular terms v^n, $n \in Z$ unless this is a sum of an infinite series and the OPE coefficients $\lambda_{\mathcal{O}}$ will have a specific asymptotic behavior to reproduce the right power of v^{Δ_ϕ}. It would be quite interesting to find an effective approach to solve the ϵ expansion equation at higher orders as well!

6.3.2 Conformal Bootstrap in the Mellin Space

Mellin transformation can drastically simplify the scatter amplitudes in AdS spacetime [18]. It provides an alternative way to express the conformal partial waves besides the conformal block functions. The conformal partial waves have rather simple form in the s-channel. In the Mellin space the crossing symmetry of four-point correlator is automatically fulfilled, however, there are spurious terms in the four-point function which need to be canceled. The cancellation of these spurious terms lead to non-trivial constraints on the

CFT data and it plays a fundamental role for conformal bootstrap in Mellin space, similar to the role of crossing symmetry in the conformal bootstrap based on conformal block functions.

In [16] the four-point function of $3D$ Ising model has been studied in Mellin space. In Mellin space, the crossing symmetry of the four-point correlator function is automatically fulfilled while there are spurious terms u^{Δ_ϕ} and $u^{\Delta_\phi}\ln(u)$ which should not appear in the physical amplitudes. By requiring the cancellation of these spurious terms the authors obtained the CFT data up to the order ϵ^3. However, it is rather difficult to generalize this method to higher order corrections due to the complex calculation in evaluating the poles of intermediate operators with spin. With or without Mellin space, one of the major difficulty in the analytical approach is to treat with the the conformal block function. It requires a concise way to evaluate the function approximately while sufficient to capture the core information on the CFT. We are working on the perturbative approach to reach this objective so that we can reproduce the perturbative results on $3D$ $O(N)$ vector model without Feymann diagram. The analytical solution is expected to help us to answer the question: why the conformal bootstrap is so effective?

6.4 Future Research on Conformal Bootstrap

In above sections I have discussed several projects on conformal bootstrap. They are expected to improve our understanding on CFTs and even QFTs significantly in the near future. In the last part of this thesis, I would like to briefly discuss my future research on conformal bootstrap.

Fruitful results have been obtained using analytical properties of the conformal block and crossing equation. In [14, 15] the crossing equation has been studied under the limitation with small conformal cross ratios. The spectrum and anomalous dimensions of operators with sufficiently large spin have been obtained. The results have interesting application in the higher spin AdS/CFT dualities. Moreover, since the conformal block and crossing equation encode essential dynamics of CFT, their analytical properties can help us to uncover some profound problems in the dual bulk theory through AdS/CFT duality. A typical example is provided in [17] that the constraints on the boundary CFT obtained from conformal symmetry, crossing equation and unitarity agree with the bulk locality. As suggested in [19, 20], the well-known black hole information loss problem corresponds to the singularity structure of conformal block in the large central charge limitation, whose properties may lead to a better understanding on the problem. In [21] the authors have studied the correlators containing the stress tensor operator and the results show that constraints from causality and unitarity leads to a unique solution which corresponds to the Einstein gravity in AdS_5. A thorough study on the analytical properties of the conformal block and crossing equation will significantly improve our understanding on the bulk dual physics.

The ultimate objective of conformal bootstrap is to find a powerful approach to solve the CFTs analytically. [‡] It is expected that conformal bootstrap provides a new definition of CFTs without Lagrangian[§] and classification of CFT landscape would be possible based on conformal bootstrap. Analytical solution of conformal field theory is a fantastic topic in theoretical physics, which can only be done for certain 2D CFTs with the famous BPZ method [4]. The infinite dimensional Virasoro algebra is the key to solve these rational CFTs. The Virasoro algebra is absent for CFTs in $D > 2$ so there is no similar solutions in higher dimensions. About 30 years later a numerical bootstrap approach has been shown that the CFTs in higher dimensions $D = 3$ or $D = 5$ can be solved even without the Virasoro algebra! This is the first light of the coming analytical approach to solve CFTs in

[‡] Here "solve" means to calculate the CFT data, including the spectra and OPE coefficients.

[§] Probably it needs to introduce certain extra constraint, like the modular invariance for 2D CFTs.

higher dimensions. The numerical results have shown that the secrets of CFTs are encoded in the conformal symmetries and the conformal partial wave functions. It is likely that there are certain hidden rules in the higher dimensional CFTs which can help us to decode the CFT data with very few assumptions.

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APPENDIX A

GEGENBAUER POLYNOMIAL AND SOME IDENTITIES

It has been shown in [24, 116, 107, 108] that $\mathcal{N}_\ell$ appearing in the superconformal and conformal partial wave integration directly relates to Gegenbauer polynomial $C_\ell^{(\lambda)}(x)$

$$\mathcal{N}_\ell \equiv \left(\bar{\mathcal{S}}1\bar{2}\mathcal{S}\right)^\ell \overleftrightarrow{\mathcal{D}_\ell} \left(\bar{\mathcal{T}}3\bar{4}\mathcal{T}\right)^\ell = \frac{1}{\ell!^2}\left(\partial_{\mathcal{S}}0\partial_{\mathcal{T}}\right)^\ell \left(\mathcal{S}\bar{2}1\bar{0}3\bar{4}\mathcal{T}\right)^\ell = (-1)^\ell y^{\frac{\ell}{2}} C_\ell^{(1)}(x), \quad \text{(A.1)}$$

in which

$$x \equiv \frac{\langle\bar{2}1\bar{0}3\bar{4}0\rangle}{2\sqrt{y}}, \quad y \equiv \frac{1}{2^6}\langle\bar{0}1\rangle\langle\bar{2}0\rangle\langle\bar{0}3\rangle\langle\bar{4}0\rangle\langle\bar{2}1\rangle\langle\bar{4}3\rangle. \quad \text{(A.2)}$$

Giving $\theta_{\text{ext}} = 0$, variables x and y turn into

$$x \quad \longrightarrow \quad x_0 \equiv -\frac{X_{13}X_{20}X_{40}}{2\sqrt{X_{10}X_{20}X_{30}X_{40}X_{12}X_{34}}} - (1 \leftrightarrow 2) - (3 \leftrightarrow 4)\,, \quad \text{(A.3)}$$

$$y \quad \longrightarrow \quad y_0 \equiv \frac{1}{2^{12}}X_{10}X_{20}X_{30}X_{40}X_{12}X_{34}, \quad \text{(A.4)}$$

in which the supertraces $\langle i\bar{j}\rangle$ have been reduced to inner products of six dimensional vectors X_{ij}. We follow the conventions used in [108] that the super-parameters are replaced by

$$\mathcal{S} \to S,\ \bar{\mathcal{S}} \to \bar{S},\ \mathcal{N}_\ell \to N_\ell \quad \text{(A.5)}$$

after setting $\theta_{\text{ext}} = 0$, and the Gegenbauer polynomial N_ℓ reads

$$N_\ell = (\bar{S}1\bar{2}S)^\ell \overleftrightarrow{\mathcal{D}_\ell}(\bar{T}3\bar{4}T)^\ell = \frac{1}{\ell!^2}(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^\ell. \quad \text{(A.6)}$$

Giving $0=\bar{0}$, one can show

$$S\bar{2}1\bar{0}3\bar{4}T=\frac{1}{4}X_{10}S\bar{2}3\bar{4}T-\frac{1}{4}X_{20}S\bar{1}3\bar{4}T=\frac{1}{4}X_{30}S\bar{2}1\bar{4}T-\frac{1}{4}X_{40}S\bar{2}1\bar{3}T \tag{A.7}$$

based on the Clifford algebra and the transverse conditions of auxiliary fields $S\bar{0}=\bar{0}T=0$. It clearly shows that $S\bar{2}1\bar{0}3\bar{4}T$ is antisymmetric under $1\leftrightarrow 2$ or $3\leftrightarrow 4$.

Let us consider following formulas related to the Gegenbauer polynomials

$$\left(\partial_S 0\partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left(X_{10}S\bar{2}3\bar{4}T+X_{20}S\bar{1}3\bar{4}T+X_{10}S\bar{2}4\bar{3}T+X_{20}S\bar{1}4\bar{3}T\right), \tag{A.8}$$

$$\left(\partial_S 0\partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left(X_{10}S\bar{2}3\bar{4}T+X_{20}S\bar{1}3\bar{4}T-X_{10}S\bar{2}4\bar{3}T-X_{20}S\bar{1}4\bar{3}T\right), \tag{A.9}$$

$$\left(\partial_S 0\partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left(X_{10}S\bar{2}3\bar{4}T-X_{20}S\bar{1}3\bar{4}T+X_{10}S\bar{2}4\bar{3}T-X_{20}S\bar{1}4\bar{3}T\right), \tag{A.10}$$

$$\left(\partial_S 0\partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left(X_{10}S\bar{2}3\bar{4}T-X_{20}S\bar{1}3\bar{4}T-X_{10}S\bar{2}4\bar{3}T+X_{20}S\bar{1}4\bar{3}T\right), \tag{A.11}$$

which are symmetric or anti-symmetric under coordinate interchanges $1\leftrightarrow 2$ and $3\leftrightarrow 4$. These polynomials appear in the conformal integral (4.18) from differentiations $(\partial_{\bar{0}}z)\cdot(\partial_{\bar{0}}N_\ell)$ or $(\partial_{\bar{0}}\frac{1}{D_\ell})\cdot(\partial_{\bar{0}}N_\ell)$ and inherit the symmetry properties from tensor structure terms in (4.45). We need to find their close relationships with Gegenbauer polynomials to accomplish the conformal integration (4.18).

Formulas in (A.8) and (A.11) are invariant under simultaneous coordinate interchange $1\leftrightarrow 2,\ 3\leftrightarrow 4$, and they can be easily simplified into compact form N_ℓ. Specifically, the formula (A.11) gives

$$8\left(\partial_S 0\partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell}\propto N_\ell, \tag{A.12}$$

while for (A.8), one can show that it reduces to

$$\begin{aligned}
\frac{1}{4}\left(\partial_S 0 \partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}&\left(X_{10}X_{34}S\bar{2}T+X_{20}X_{34}S\bar{1}T\right)\\
&=\frac{\ell(\ell+1)}{8}X_{10}X_{20}X_{34}\left(\partial_S 0 \partial_T\right)^{\ell-1}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\\
&\propto X_{10}X_{20}X_{34}N_{\ell-1}.
\end{aligned} \tag{A.13}$$

In contrast, formulas in (A.9) and (A.10) are antisymmetric under $1 \leftrightarrow 2,\ 3 \leftrightarrow 4$. It is easy to show that formula (A.10) vanishes.

Similarly, we can reduce following formulas to compact forms proportional to N_ℓ:

$$\left(\partial_S 0 \partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left(X_{30}S\bar{2}1\bar{4}T+X_{40}S\bar{2}1\bar{3}T+X_{30}S\bar{1}2\bar{4}T+X_{40}S\bar{1}2\bar{3}T\right), \tag{A.14}$$

$$\left(\partial_S 0 \partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left(X_{30}S\bar{2}1\bar{4}T+X_{40}S\bar{2}1\bar{3}T-X_{30}S\bar{1}2\bar{4}T-X_{40}S\bar{1}2\bar{3}T\right), \tag{A.15}$$

$$\left(\partial_S 0 \partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left(X_{30}S\bar{2}1\bar{4}T-X_{40}S\bar{2}1\bar{3}T+X_{30}S\bar{1}2\bar{4}T-X_{40}S\bar{1}2\bar{3}T\right), \tag{A.16}$$

$$\left(\partial_S 0 \partial_T\right)^{\ell}\left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1}\left(X_{30}S\bar{2}1\bar{4}T-X_{40}S\bar{2}1\bar{3}T-X_{30}S\bar{1}2\bar{4}T+X_{40}S\bar{1}2\bar{3}T\right), \tag{A.17}$$

except (A.15).

The formulas (A.9) and (A.15) can not be simply written in terms of N_ℓ. Nevertheless, their relationships with the Gegenbauer polynomials are given in the recursion equations, which can be used to obtain the final results of conformal integrations they involve in.

Denote

$$R_\ell \equiv \frac{\ell}{\ell!^2}\left(\partial_S 0 \partial_T\right)^\ell \left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1} \times$$
$$\left(X_{10}S\bar{2}3\bar{4}T + X_{20}S\bar{1}3\bar{4}T - X_{10}S\bar{2}4\bar{3}T - X_{20}S\bar{1}4\bar{3}T\right), \quad \text{(A.18)}$$

$$P_\ell \equiv \frac{\ell}{\ell!^2}\left(\partial_S 0 \partial_T\right)^\ell \left(S\bar{2}1\bar{0}3\bar{4}T\right)^{\ell-1} \times$$
$$\left(X_{30}S\bar{2}1\bar{4}T + X_{40}S\bar{2}1\bar{3}T - X_{30}S\bar{1}2\bar{4}T - X_{40}S\bar{1}2\bar{3}T\right), \quad \text{(A.19)}$$

and

$$\Delta_A \equiv \frac{1}{8}(X_{20}X_{40}X_{13} - X_{20}X_{30}X_{14} + X_{10}X_{40}X_{23} - X_{10}X_{30}X_{24}), \quad \text{(A.20)}$$
$$\Delta_B \equiv \frac{1}{8}(X_{20}X_{40}X_{13} + X_{20}X_{30}X_{14} - X_{10}X_{40}X_{23} - X_{10}X_{30}X_{24}). \quad \text{(A.21)}$$

Note the sign differences among x_0, Δ_A and Δ_B. The crucial properties of R_ℓ and P_ℓ are that they satisfy the following mutual recursion relations:

$$R_\ell = \ell\Delta_A N_{\ell-1} + \frac{1}{2^6}X_{10}X_{20}X_{34}P_{\ell-1}, \quad \text{(A.22)}$$
$$P_\ell = \ell\Delta_B N_{\ell-1} + \frac{1}{2^6}X_{30}X_{40}X_{12}R_{\ell-1}, \quad \text{(A.23)}$$

which leads to the independent recursion relations of R_ℓ and P_ℓ:

$$R_\ell = \ell\Delta_A N_{\ell-1} + \frac{1}{2^6}(\ell-1)X_{10}X_{20}X_{34}\Delta_B N_{\ell-2} + y_0 R_{\ell-2}, \quad \text{(A.24)}$$
$$P_\ell = \ell\Delta_B N_{\ell-1} + \frac{1}{2^6}(\ell-1)X_{30}X_{40}X_{12}\Delta_A N_{\ell-2} + y_0 P_{\ell-2}. \quad \text{(A.25)}$$

Above two recursion equations are needed to determine the conformal integrations of the antisymmetric terms in (4.46).

APPENDIX B

SUPERCONFORMAL INTEGRATIONS OF SYMMETRIC SUPEREMBEDDING TERMS

The superconformal partial waves $\mathcal{W}_\mathcal{O}$ are largely determined by the tensor structures in (4.45). These terms are separated into two parts: invariant and antisymmetric terms according to their transformations under coordinate interchange $1 \leftrightarrow 2,\ 3 \leftrightarrow 4$. Here we show the main steps toward contributions of invariant terms on $\mathcal{W}_\mathcal{O}$. Due to the gauge adopted in (4.21), wit is straightforward to obtain the results, similar to the steps used in [108] but generalized to Φ_i's with arbitrary superconformal weights.

As discussed before, there are two steps to accomplish the superconformal integrations for $\mathcal{W}_\mathcal{O}$: partial derivatives and conformal integration. The partial derivatives can be obtained by the same steps provided in [108] with coefficients replacements

$$\frac{\ell+\Delta}{2} \to 2\delta, \qquad \frac{2+\ell-\Delta}{2} \to 2\delta'. \tag{B.1}$$

The conformal integrations are modified accordingly, specifically there are new terms proportional to the scaling dimension differences $r, \tilde{r}$:

$$\int D^4X_0\, \frac{X_{12}}{X_{10}X_{20}} \frac{N_\ell}{D_\ell}\bigg|_{\bar{0}=0} = \frac{c_\ell\, \xi_{\Delta+2,2-\Delta,\tilde{r},\ell}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} g^{r,\tilde{r}}_{\Delta+2,\ell}(u,v), \tag{B.2}$$

$$\int D^4X_0\, \frac{X_{34}}{X_{30}X_{40}} \frac{N_\ell}{D_\ell}\bigg|_{\bar{0}=0} = \frac{c_\ell\, \xi_{\Delta,4-\Delta,\tilde{r},\ell}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} g^{r,\tilde{r}}_{\Delta,\ell}(u,v), \tag{B.3}$$

$$\int D^4X_0\, X_{12}X_{34} \frac{N_{\ell-1}}{D_\ell}\bigg|_{\bar{0}=0} = \frac{c_{\ell-1}\, \xi_{\Delta+1,3-\Delta,\tilde{r},\ell-1}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} g^{r,\tilde{r}}_{\Delta+1,\ell-1}(u,v), \tag{B.4}$$

$$\begin{aligned}
&\int D^4X_0 \left[\frac{X_{13}}{X_{10}X_{30}} + \frac{X_{23}}{X_{20}X_{30}} + \frac{X_{14}}{X_{10}X_{40}} + \frac{X_{24}}{X_{20}X_{40}}\right] \frac{N_\ell}{D_\ell}\bigg|_{\bar{0}=0} = \\
&\quad \frac{c_\ell\, \xi_{\Delta+1,3-\Delta,1+\tilde{r},\ell}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} \left[\frac{4\,(\tilde{r}^2 + (\Delta-\ell-2)(\Delta+\ell))}{(\tilde{r}+\Delta-\ell-2)\,(\tilde{r}+\Delta+\ell)} g^{r,\tilde{r}}_{\Delta,\ell}\right. \\
&\qquad + \frac{(r^2 + (\Delta-\ell-2)(\Delta+\ell))\,(\tilde{r}-\Delta-\ell)\,(\tilde{r}-\Delta+\ell+2)}{4(\Delta-\ell-2)(\Delta-\ell-1)(\Delta+\ell)(\Delta+\ell+1)} g^{r,\tilde{r}}_{\Delta+2,\ell} \\
&\qquad + \frac{r\tilde{r}\,(\tilde{r}-\Delta-\ell)}{(\Delta+\ell)(\Delta+\ell+1)\,(\tilde{r}+\Delta-\ell-2)} g^{r,\tilde{r}}_{\Delta+1,\ell+1} \\
&\qquad \left. + \frac{r\tilde{r}\,(\tilde{r}-\Delta+\ell+2)}{(\Delta-\ell-2)(\Delta-\ell-1)\,(\tilde{r}+\Delta+\ell)} g^{r,\tilde{r}}_{\Delta+1,\ell-1}\right],
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
&\int D^4X_0 \left[\frac{X_{13}}{X_{10}X_{30}} - \frac{X_{23}}{X_{20}X_{30}} - \frac{X_{14}}{X_{10}X_{40}} + \frac{X_{24}}{X_{20}X_{40}}\right] \frac{N_\ell}{D_\ell}\bigg|_{\bar{0}=0} = \\
&\quad \frac{c_\ell\, \xi_{\Delta+1,3-\Delta,1+\tilde{r},\ell}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\tilde{r}}{2}} \left[\frac{(\Delta-\ell-2)\,(-\tilde{r}+\Delta+\ell)}{(\Delta+\ell+1)\,(\tilde{r}+\Delta-\ell-2)} \times \right. \\
&\qquad \left. g^{r,\tilde{r}}_{\Delta+1,\ell+1} + \frac{(\Delta+\ell)\,(-\tilde{r}+\Delta-\ell-2)}{(\Delta-\ell-1)\,(\tilde{r}+\Delta+\ell)} g^{r,\tilde{r}}_{\Delta+1,\ell-1}\right].
\end{aligned} \tag{B.6}$$

APPENDIX C

SOLUTION OF THE SHADOW COEFFICIENTS TRANSFORMATION

Here we solve the linear transformation H_1 between the supershadow coefficients $\lambda^{(i)}_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}}$ and $(\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}})^*$. As proposed in [108], the unitarity of superconformal partial wave plays a crucial role in determining H_1.

The linear transformation H_1 is described by a 4×4 matrix

$$\begin{pmatrix} \lambda^{(0)}_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}} \\ \lambda^{(2)}_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}} \\ \lambda^{(1)}_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}} \\ \lambda^{(3)}_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}} \end{pmatrix} = \begin{pmatrix} a & b & e & g \\ c & d & f & h \\ u & v & p & k \\ w & t & q & s \end{pmatrix} \begin{pmatrix} (\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}})^* \\ (\lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}})^* \\ (\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}})^* \\ (\lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}})^* \end{pmatrix}. \tag{C.1}$$

Note that in [108] the 4×4 matrix is block diagonal protected by the parity of coefficients under coordinate exchange in the three-point function. For the three-point function with general superconformal weights, the coordinate exchange symmetry is broken by arbitrary superconformal weights. Therefore in our case the 4×4 matrix is not simply block diagonal. Unitarity, together with the extra constraint is still useful to solve the transformation H_1.

Giving $\Phi_3 = \Phi_2^\dagger$ and $\Phi_4 = \Phi_1^\dagger$, unitarity requires that the four coefficients a_i of conformal blocks appearing in the superconformal blocks $\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell}$ are positive. By transforming coefficients $\lambda^{(i)}_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}}$ to $(\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}})^*$, this is equivalent to the following equations:

$$\begin{aligned}
&\left(-\delta'\left[\frac{(2-\Delta)\tilde{r}^2-(\ell+2-\Delta)(\Delta+\ell)}{(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)}2\delta+1\right],\ 1,\ \frac{\tilde{r}((\Delta-2)R+(-\Delta+\ell+2)(\Delta+\ell))}{2(-\Delta+\ell+2)(\Delta+\ell)},\right.\\
&\qquad\left.\frac{\tilde{r}(R+\ell+2-\Delta)}{4(-\Delta+\ell+2)}\right)\cdot H_1 \quad\propto\quad (1,\ 0,\ 0,\ 0), \qquad\text{(C.2)}\\
&\left(\frac{\tilde{r}(R+\ell+2-\Delta)}{2(-\Delta+\ell+2)},\ 0,\ 1,\ 0\right)\cdot H_1 \quad\propto\quad \left(\frac{r(\Delta-R+\ell)}{2(\Delta+\ell)},\ 0,\ 1,\ 0\right), \qquad\text{(C.3)}\\
&\left(\frac{\tilde{r}(\Delta-R+\ell)}{2(\Delta+\ell)},\ 0,\ 1,\ \frac{\ell+1}{\ell}\right)\cdot H_1 \quad\propto\quad \left(\frac{r(-\Delta+R+\ell+2)}{2(-\Delta+\ell+2)},\ 0,\ 1,\ \frac{\ell+1}{\ell}\right), \qquad\text{(C.4)}\\
&(1,\ 0,\ 0,\ 0)\cdot H_1 \quad\propto\quad \left(-\delta\left(1-2\delta'\frac{r^2\Delta-(\Delta+\ell)(-\Delta+\ell+2)}{(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)}\right),\right.\\
&\qquad\left.1,\ \frac{r(\Delta(-\Delta+R+2)+\ell(\ell+2))}{2(-\Delta+\ell+2)(\Delta+\ell)},\ \frac{r(\Delta-R+\ell)}{4(\Delta+\ell)}\right), \qquad\text{(C.5)}
\end{aligned}$$

in which $\tilde{r}=-r$. From above equation groups we can solve 15 out of 16 H_1's elements (except c) up to three re-scaling coefficients.

Then we consider two three-point functions $\langle\Phi X\mathcal{O}\rangle$ and $\langle X\Phi^\dagger\tilde{\mathcal{O}}\rangle$, in which Φ : $(0,\ 0,\ q_1,\ 0)$ is a chiral field and X : $(0,\ 0,\ q_2,\ q_2)$ is real *. Such kind of three-point function has been studied in (4.28). Due to the chirality of Φ, the four coefficients satisfy the constraint

$$(\lambda^{(0)}_{\Phi X\mathcal{O}},\ \lambda^{(2)}_{\Phi X\mathcal{O}},\ \lambda^{(1)}_{\Phi X\mathcal{O}},\ \lambda^{(3)}_{\Phi X\mathcal{O}}) \;=\; \lambda_{\Phi X\mathcal{O}}(1,\ \delta(2\delta-\ell),\ -2\delta,\ \ell), \qquad\text{(C.6)}$$

$$(\lambda^{(0)}_{X\Phi^\dagger\tilde{\mathcal{O}}},\ \lambda^{(2)}_{X\Phi^\dagger\tilde{\mathcal{O}}},\ \lambda^{(1)}_{X\Phi^\dagger\tilde{\mathcal{O}}},\ \lambda^{(3)}_{X\Phi^\dagger\tilde{\mathcal{O}}}) \;=\; \lambda_{X\Phi^\dagger\tilde{\mathcal{O}}}(1,\ \delta'(2\delta'-\ell),\ -2\delta',\ \ell), \qquad\text{(C.7)}$$

in which $\delta=\frac{\Delta+\ell+R+2r}{4}$ and $\delta'=\frac{2-\Delta+\ell+R}{4}$ with $R=-q_1$, $r=q_1-2q_2$. Then the

*X could be any scalar and the results will be the same, here we set X as real for convenience.

transformation between coefficients in (C.7) and the complex conjugate of (C.6) gives

$$\begin{pmatrix} 1 \\ \delta'(2\delta'-\ell) \\ -2\delta' \\ \ell \end{pmatrix} \propto \begin{pmatrix} a & b & e & g \\ c & d & f & h \\ u & v & p & k \\ w & t & q & s \end{pmatrix} \begin{pmatrix} 1 \\ \delta(2\delta-\ell) \\ -2\delta \\ \ell \end{pmatrix}. \tag{C.8}$$

Plugging the solutions of equation groups (C.2-C.5) into (C.8), we can solve all of the 16 elements of H_1 and the three re-scaling coefficients up to the re-scaling coefficient of (C.8), denoted as z_*. The results are

$$\alpha_* \times \begin{pmatrix} a_* & -\frac{8(\ell-\Delta+2)(\ell+\Delta)}{\ell+\Delta-R} & -\frac{4r(\ell(\ell+2)+(R-\Delta+2)\Delta)}{\ell+\Delta-R} & -2r(\ell-\Delta+2) \\ c_* & d_* & f_* & h_* \\ u_* & -\frac{4r(R+\ell-\Delta+2)(\ell+\Delta)}{\ell+\Delta-R} & p_* & -r^2(R+\ell-\Delta+2) \\ w_* & \frac{8rR\ell}{\ell+\Delta-R} & \frac{4\ell\left(Rr^2+(\ell-\Delta+2)\Delta(\ell+\Delta)\right)}{\ell+\Delta-R} & s_* \end{pmatrix} \tag{C.9}$$

where the elements with long expressions are abbreviated as

$$\alpha_* = \frac{z_*(\Delta-1)(\Delta-R+\ell)}{\Delta(-\Delta-r+\ell+2)(\Delta+r+\ell)(-\Delta-R+\ell+2)(\Delta+R+\ell)}, \tag{C.10}$$

$$a_* = \frac{R\left(\ell(\ell+2)-\Delta\left(\Delta+r^2-2\right)\right)+(-\Delta+\ell+2)\left((\Delta+\ell)^2-\Delta r^2\right)}{\Delta-1}, \tag{C.11}$$

$$d_* = \frac{R+\ell+2-\Delta}{(\Delta-1)(R-\ell-\Delta)}\left(\Delta^2\left(r^2-R+\ell+4\right)-\Delta^3+\Delta\left(\left(2-r^2\right)R+\ell\left(r^2+\ell\right)-4\right)+\ell(\ell+2)(R-\ell-2)\right), \tag{C.12}$$

$$h_* = \frac{r(R+\ell+2-\Delta)}{4(\Delta-1)}\left(-\Delta\left(\Delta^2+\Delta-2r^2-4\right)-(\Delta-1)R(\Delta+\ell+2)-(\Delta+1)\ell^2-2((\Delta-1)\Delta+2)\ell-4\right), \tag{C.13}$$

$$u_* = -\frac{r(R+\ell+2-\Delta)}{2(-1+\Delta)}(-\Delta\left((\Delta-3)\Delta-2r^2+4\right)+(\Delta-1)R(\ell-\Delta)-(\Delta+1)\ell^2+2(\Delta-3)\Delta\ell), \tag{C.14}$$

$$p_* = \frac{R+\ell+2-\Delta}{(\Delta-1)(R-\ell-\Delta)}\left(r^2(\Delta(3\Delta-R-2)+(3\Delta-2)\ell)+\Delta(-\Delta+\ell+2)(\Delta+\ell)(\Delta+R-\ell-2)\right), \tag{C.15}$$

$$s_* = \frac{1}{\Delta-1}\left(r^2((\Delta-2)R+\Delta(\Delta-\ell-2))+\Delta(-\Delta+\ell+2)(\Delta+\ell)(\Delta+R+\ell)\right), \tag{C.16}$$

$$w_* = r\ell(4\Delta+(R+\ell-\Delta)(R+2\Delta)), \tag{C.17}$$

and

$$
\begin{aligned}
c_* \quad = \quad & \frac{\Delta + R - \ell - 2}{8(\Delta-1)(\ell+2-\Delta-R)(\Delta-R+\ell)} \left(4(\Delta-1)\Delta r^2 R^3 + (\Delta-1)R^4(\ell+2-\Delta)(\Delta+\ell) - 4(\Delta-1)\Delta r^2 R\right. \\
& \left((\Delta-4)\Delta - 2r^2 + 3\ell(\ell+2) + 6\right) + 2R^2(\ell(\ell+2)(\ell(\ell+2)+2) + \Delta^5 - 5\Delta^4 - 2\Delta^3\left(r^2-5\right) \\
& +2\Delta^2\left(r^2-5\right) - \Delta\left(2r^4 - 2r^2(\ell+1)^2 + \ell(\ell+2)(\ell(\ell+2)+2) - 4\right) + (\ell+2-\Delta)(\Delta+\ell)\left(\Delta^5 - 5\Delta^4\right. \\
& -2\Delta^3\left(2r^2 + \ell(\ell+2) - 4\right) + 2\Delta^2\left(6r^2 + 3\ell(\ell+2) - 2\right) + \Delta\left(4r^4 - 4r^2(\ell(\ell+2)+3) + \ell^3(\ell+4) - 8\ell\right) \\
& \left.\left. -\ell^2(\ell+2)^2\right)\right), \qquad \text{(C.18)} \\
f_* \quad = \quad & \frac{R+\ell+2-\Delta}{2(\Delta-1)(\Delta-R+\ell)} \left(\ell(\ell+2)(-R+\ell+2) + \Delta\left(R\left(2r^2 - \ell(\ell+3) - 4\right) + \ell\left(-2r^2 + \ell^2 + \ell + 4\right) + R^2 + 4\right)\right. \\
& \left. +\Delta^2\left(-2r^2 - R^2 + R(\ell+2) + (\ell-3)\ell\right) + \Delta^3(\ell-3) + \Delta^4\right). \qquad \text{(C.19)}
\end{aligned}
$$

The transformation H_1 presented above seems to be rather cumbersome, however it does satisfy following simple relation

$$
H_1(\Delta,\ R,\ r) \cdot H_1(\Delta \to 2-\Delta,\ R \to -R,\ r \to -r) \propto I_{4\times 4}, \qquad \text{(C.20)}
$$

which is expected since by applying the supershadow transformation twice we go back to the original coefficients. Setting the Eq. (C.20) to be strictly equal, the overall coefficient z_* can be fixed up to a factor z_x satisfying

$$
z_x(\Delta,\ R,\ r) \cdot z_x(\Delta \to 2-\Delta,\ R \to -R,\ r \to -r) = 1, \qquad \text{(C.21)}
$$

which has no effect on the superconformal block functions.

Besides the three-point correlators $\langle \Phi X \mathcal{O} \rangle$ and $\langle X \Phi^\dagger \mathcal{O} \rangle$, we can also partially fix the coefficients in the three-point correlators like $\langle \Phi^\dagger X \mathcal{O} \rangle$ and $\langle X \Phi \mathcal{O} \rangle$ and their supershadow duals. Their coefficients are expected to be related to the shadow coefficients by H_1 with proper redefinitions of the parameters r and R. One can show that indeed above solution of H_1 can realize the transformation of shadow coefficients with parameters $R \to -R$ and $r \to -r$, respectively.

Under transformation H_1, the coefficients $\lambda^{(i)}_{\Phi_3\Phi_4\tilde{\mathcal{O}}}$ in (4.61-4.64) can be mapped to $(\lambda^{(i)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*$, and now we are ready to write down the most general $\mathcal{N}=1$ superconformal block $\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell}$ in terms of three-point coefficients $\lambda^{(i)}_{\Phi_1\Phi_2\mathcal{O}}$ and $(\lambda^{(i)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*$:

$$\mathcal{G}^{r,\tilde{r}}_{\Delta,\ell} = a_1\, g^{r,\tilde{r}}_{\Delta,\ell} + a_2\, g^{r,\tilde{r}}_{\Delta+1,\ell+1} + a_3\, g^{r,\tilde{r}}_{\Delta+1,\ell-1} + a_4\, g^{r,\tilde{r}}_{\Delta+2,\ell}, \tag{C.22}$$

where the coefficients of individual conformal blocks a_i are

$$a_1 = \lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}(\lambda^{(0)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*, \tag{C.23}$$

$$\begin{aligned} a_2 = {} & \frac{\Delta+\ell}{(\Delta+\ell+1)(\Delta-R+\ell)(\Delta+R+\ell)}\left(\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(\Delta-R+\ell)}{2(\Delta+\ell)}\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\right) \\ & \times\left((\lambda^{(1)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* - \frac{\tilde{r}(\Delta-R+\ell)}{2(\Delta+\ell)}(\lambda^{(0)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*\right), \end{aligned} \tag{C.24}$$

$$\begin{aligned} a_3 = {} & \frac{\ell+2-\Delta}{(-\Delta+\ell+1)(-\Delta-R+\ell+2)(-\Delta+R+\ell+2)} \\ & \times\left(\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{\ell+1}{\ell}\lambda^{(3)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(-\Delta+R+\ell+2)}{2(-\Delta+\ell+2)}\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}}\right) \\ & \times\left((\lambda^{(1)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* + \frac{\ell+1}{\ell}(\lambda^{(3)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* - \frac{\tilde{r}(-\Delta+R+\ell+2)}{2(-\Delta+\ell+2)}(\lambda^{(0)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*\right), \end{aligned} \tag{C.25}$$

$$\begin{aligned} a_4 = {} & \frac{4(\Delta-1)^2(-\Delta+\ell+2)(\Delta+\ell)}{\Delta^2(\ell+1-\Delta)(\Delta+\ell+1)(\ell+2-R-\Delta)(\ell+2+R-\Delta)(\Delta-R+\ell)(\Delta+R+\ell)} \\ & \left[-\frac{(\Delta-R+\ell)\left(R\left(\ell(\ell+2)-\Delta\left(\Delta+r^2-2\right)\right)+(\ell+2-\Delta)\left((\Delta+\ell)^2-\Delta r^2\right)\right)}{8(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)}\right. \\ & \left.\lambda^{(0)}_{\Phi_1\Phi_2\mathcal{O}} + \lambda^{(2)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(\Delta(R+2-\Delta)+\ell(\ell+2))}{2(\ell+2-\Delta)(\Delta+\ell)}\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}} + \frac{r(\Delta-R+\ell)}{4(\Delta+\ell)}\lambda^{(1)}_{\Phi_1\Phi_2\mathcal{O}}\right] \\ & \left[-\frac{(\Delta-R+\ell)\left(R\left(\ell(\ell+2)-\Delta\left(\Delta+r^2-2\right)\right)+(\ell+2-\Delta)\left((\Delta+\ell)^2-\Delta r^2\right)\right)}{8(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)}\right. \\ & (\lambda^{(0)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* + (\lambda^{(2)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* - \frac{\tilde{r}(\Delta(R+2-\Delta)+\ell(\ell+2))}{2(\ell+2-\Delta)(\Delta+\ell)}(\lambda^{(1)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^* \\ & \left.-\frac{\tilde{r}(\Delta-R+\ell)}{4(\Delta+\ell)}(\lambda^{(3)}_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}})^*\right]. \end{aligned} \tag{C.26}$$